Ultrasonic propagation across a thin layer between two bulk media: Theory, computer simulation and experiment.

Jeffrey Sadler
University of Windsor

Follow this and additional works at: https://scholar.uwindsor.ca/etd

Recommended Citation
https://scholar.uwindsor.ca/etd/4450

This online database contains the full-text of PhD dissertations and Masters' theses of University of Windsor students from 1954 forward. These documents are made available for personal study and research purposes only, in accordance with the Canadian Copyright Act and the Creative Commons license—CC BY-NC-ND (Attribution, Non-Commercial, No Derivative Works). Under this license, works must always be attributed to the copyright holder (original author), cannot be used for any commercial purposes, and may not be altered. Any other use would require the permission of the copyright holder. Students may inquire about withdrawing their dissertation and/or thesis from this database. For additional inquiries, please contact the repository administrator via email (scholarship@uwindsor.ca) or by telephone at 519-253-3000ext. 3208.
Ultrasonic Propagation Across a Thin Layer Between Two Bulk Media: Theory, Computer Simulation and Experiment

By
Jeffrey Sadler

A Thesis
Submitted to the Faculty of Graduate Studies and Research
Through the Department of Physics
In Partial Fulfillment of the Requirements for
The Degree of Master of Science at the
University of Windsor

Windsor, Ontario, Canada
September, 2003
The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

L'auteur conserve la propriété du droit d'auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.
Abstract

This thesis describes the changes in the reflected and transmitted acoustic waves due to the addition of a thin layer (thickness much less than an acoustic wavelength) between two half-spaces of material. The restriction of the thin layer allows one to combine perturbation methods with the standard fully bonded boundary conditions at each interface and derive a set of equations where one no longer is concerned with the acoustic waves internal to the thin layer. This approach allows one to further examine the cases of a nonlinear thin layer, an anisotropic thin layer, and a thin layer with combined anisotropy and nonlinearity.

A simulation is created to explore the various combinations of materials for the half-spaces and thin layers. In the simulation it is found that the case of half-spaces made with identical materials presents the most accurate results, and creates significant changes in the reflected and transmitted acoustic waves. The addition of anisotropy and nonlinearity also creates significant changes in certain acoustic polarizations.

Finally, a qualitative experimental verification is attempted for the specific case of two identical half-spaces using a common adhesive, and anisotropic material for the thin layer. The experiment is found to yield qualitatively similar results to what is predicted by the simulation.
Acknowledgements

I would like to take this opportunity to thank the entire CIRAMC team for their assistance and support, especially Brian O'Neill for the basis of this theory, Ron Kumon for his proofreading skills, Fedar Severin for his experimental expertise and of course my supervisor Dr. R. Maev. I would also like to thank Dr. W. Kedzierski, Dr. R. Barron, and Dr. G. Drake for their roles as my committee members. Finally I wish to thank my family for their support through the years.
Table of Contents

Abstract ........................................................................................................ iv
Acknowledgements ..................................................................................... v
List of Tables ................................................................................................. viii
List of Figures ................................................................................................. ix
List of Symbols and Notation ....................................................................... xiv

Chapter 1: Introduction

1.1: Overview ................................................................................................. 1
1.2: Specific Literature .................................................................................. 2
  1.2.1: General Use of Boundary Conditions in Literature ...................... 2
  1.2.2: General Information on Nonlinearity ......................................... 3
  1.2.3: General Information on Anisotropy .......................................... 4
  1.2.4: Anisotropic Layers in Literature ............................................ 4
  1.2.5: Nonlinear Layers in Literature .............................................. 5
1.3: Linearity vs. Nonlinearity ...................................................................... 6
1.4: Isotropy vs. Anisotropy ....................................................................... 8
1.5: Physical Effects to Consider ................................................................ 9
  1.5.1: Propagation .............................................................................. 9
  1.5.2: Types of Interfaces and Boundary Conditions ......................... 10
1.6: Uniqueness and Scope of Thesis ...................................................... 12

Chapter 2: Theory

2.1: Overview ............................................................................................... 14
2.2: Set up of the Problem and Definitions ........................................... 18
2.3: Perturbation Due to the Thin Layer ............................................... 21
2.4: First Order Approximation: Two Half-Spaces ............................... 23
2.5: Second Order Approximation: Adding the Thin Layer ................ 27
2.6: First Extension of Simulation: Adding Nonlinearity to the Thin Layer .................................................. 31
2.7: Second Extension of Simulation: Adding Anisotropy to the Thin Layer .................................................................................. 33

Chapter 3: Simulation

3.1: Introduction .......................................................................................... 35
3.2: Two Half-Spaces (No Layer) ............................................................ 36
  3.2.1: Two Different Bulk Media ..................................................... 36
  3.2.2: Two Identical Bulk Media .................................................... 39
3.3: A Thin Linear Isotropic Layer Between Two Half-Spaces ............. 39
  3.3.1: Two Different Bulk Media ..................................................... 39
  3.3.2: Two Identical Bulk Media .................................................... 47
Chapter 4: Experiment

4.1: Purpose ................................................................. 87
4.2: Materials ............................................................... 87
  4.2.1: General Requirements ........................................ 87
  4.2.2: Materials Used .................................................. 89
  4.2.3: Electronics and Transducers ............................... 89
  4.2.4: Material Preparation ........................................ 90
4.3: Procedure ............................................................ 92
4.4: Results and Discussion ........................................... 93
  4.4.1: Control Sample ............................................... 93
  4.4.2: Cyanoacrylate Layer ....................................... 94
  4.4.3: Fibreglass Cloth Layer ................................... 96
  4.4.4: Boron Fibre Layer .......................................... 98
4.5: Conclusions ......................................................... 100

Chapter 5: Summary and Conclusions .................................. 101

References ......................................................................... 103

Appendices
  Appendix A: Material Parameters .................................. 105
  Appendix B: Energy Conservation .................................. 107
  Appendix C: Multilayer System Theory ......................... 108

Vita Auctoris ..................................................................... 110
## List of Tables

<table>
<thead>
<tr>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5.1: Summary of frequency multiples added due to various classical</td>
<td>66</td>
</tr>
<tr>
<td>nonlinear functions.</td>
<td></td>
</tr>
<tr>
<td>3.5.2: Summary of frequency multiples created due to combination A' F(T),</td>
<td>68</td>
</tr>
<tr>
<td>where A' is the cosine function.</td>
<td></td>
</tr>
<tr>
<td>4.4.1: Displacement amplitude measurements in a control sample.</td>
<td>94</td>
</tr>
</tbody>
</table>
List of Figures

<table>
<thead>
<tr>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1.1: Thin layer between two half-spaces</td>
<td>17</td>
</tr>
<tr>
<td>2.1.2: Thin layer between two half-spaces</td>
<td>18</td>
</tr>
<tr>
<td>3.2.1: Time domain output from an incident longitudinal wave (frequency 0.1 MHz, angle of incidence 40 degrees) in an aluminum-brass sample with no thin layer.</td>
<td>37</td>
</tr>
<tr>
<td>3.2.2: Change in amplitude of the various polarized waves with angle of incidence in an aluminum-brass sample. Note that the amplitude is taken as an absolute measure ignoring the phase of the waves.</td>
<td>38</td>
</tr>
<tr>
<td>3.2.3: Change in the relative intensity of the various polarized waves with angle of incidence in an aluminum-brass sample.</td>
<td>38</td>
</tr>
<tr>
<td>3.3.1: Time domain output from an incident longitudinal wave (frequency 0.1 MHz, angle of incidence 40 degrees) in an aluminum--brass sample where the thin layer of epoxy is 50 μm.</td>
<td>43</td>
</tr>
<tr>
<td>3.3.2: A closer examination of the time domain output in Figure 3.3.1 showing the delay in phase reflected and transmitted wave with respect to the incident wave in the aluminum-brass sample.</td>
<td>44</td>
</tr>
<tr>
<td>3.3.3: Change of the phase terms with angle of incidence in the aluminum-brass sample.</td>
<td>44</td>
</tr>
<tr>
<td>3.3.4: Comparison of the amplitude in the case of no layer to the case of a thin layer of epoxy (thickness 50 μm) of the aluminum-brass sample. (a) Amplitude of reflected longitudinal wave (b) Amplitude of reflected shear vertical wave (c) Amplitude of transmitted shear vertical wave</td>
<td>45</td>
</tr>
<tr>
<td>3.3.5: Total relative intensity in the aluminum-epoxy-brass sample.</td>
<td>46</td>
</tr>
<tr>
<td>3.3.6: Transmitted longitudinal phase factor versus various incident angles for a system of three identical materials, showing an exact fit for the phase expansion.</td>
<td>50</td>
</tr>
<tr>
<td>3.3.7: Transmitted longitudinal amplitude versus various incident angles for a system of three identical materials.</td>
<td>50</td>
</tr>
</tbody>
</table>
3.3.8: Characteristic curves for the reflected longitudinal wave in a sample with two identical bulk media. The density and impedance have been altered by 10 and 20% relative to the bulk media.
(a) Density increased, impedance decreased
(b) Density and impedance increased
(c) Density and impedance decreased
(d) Density decreased, impedance increased

3.3.9: Characteristic curves for the reflected vertical wave in a sample with two identical bulk media. The density and impedance have been altered by 20% relative to the bulk media.
(a) Density increased, impedance decreased
(b) Density and impedance increased
(c) Density and impedance decreased
(d) Density decreased, impedance increased

3.3.10: Characteristic curves for the transmitted vertical wave in a sample with two identical bulk media. The density and impedance have been altered by 20% relative to the bulk media.
(a) Density increased, impedance decreased
(b) Density and impedance increased
(c) Density and impedance decreased
(d) Density decreased, impedance increased

3.3.11: Transmitted longitudinal wave in a sample with two identical bulk media.

3.4.1: Beginning microsecond of the perturbation and multilayer simulation time domain data.

3.4.2: Final microsecond of the perturbation and multilayer simulation time domain data.

3.4.3: Difference between the perturbation and multilayer time domain outputs.

3.4.4: Comparison of the amplitude of the reflected longitudinal wave in the multilayer and perturbation theories as the size of the thin layer is increased in an aluminum-epoxy-brass system.

3.4.5: Comparison of the amplitude of the transmitted longitudinal wave in the multilayer and perturbation theories as the size of the thin layer is increased in an aluminum-epoxy-brass system.

3.4.6: Percent difference of the amplitude of the reflected and transmitted longitudinal wave between the multilayer and perturbation theories as the size of the thin layer is increased in an aluminum-epoxy-brass system.
3.4.7: Comparison of the amplitude of the reflected longitudinal wave between the multilayer and perturbation theories as the size of the thin layer is increased in an acrylic-epoxy-acrylic system.

3.4.8: Percent difference of the amplitude of the transmitted longitudinal wave between the multilayer and perturbation theories as the size of the thin layer is increased in an acrylic-epoxy-acrylic system.

3.5.1: Fourier transform from acrylic-epoxy-acrylic sample with classical nonlinear thin layer
   (a) Classical nonlinear function $\beta \cdot T_{kl}$
   (b) Classical nonlinear function $\chi \cdot T_{kl}^2$
   (c) Classical nonlinear function $\beta \cdot T_{kl} + \chi \cdot T_{kl}^2$

3.5.2: Nonlinear waveform for a classical nonlinear function in the acrylic-epoxy-acrylic sample.
   (a) Classical nonlinear function $\beta \cdot T_{kl}$
   (b) Classical nonlinear function $\chi \cdot T_{kl}^2$
   (c) Classical nonlinear function $\beta \cdot T_{kl} + \chi \cdot T_{kl}^2$

3.5.3: Time dependence of phase factor for a classical nonlinear function in the acrylic-epoxy-acrylic sample.
   (a) Classical nonlinear function $\beta \cdot T_{kl}$
   (b) Classical nonlinear function $\chi \cdot T_{kl}^2$
   (c) Classical nonlinear function $\beta \cdot T_{kl} + \chi \cdot T_{kl}^2$

3.5.4: Change in harmonic with angle of incidence for classical nonlinear functions in the acrylic-epoxy-acrylic sample.
   (a) Second harmonic for classical nonlinear functions $\beta \cdot T_{kl}$ and
      $\beta \cdot T_{kl} + \chi \cdot T_{kl}^2$
   (b) Third harmonic for classical nonlinear functions $\chi \cdot T_{kl}^2$ and
      $\beta \cdot T_{kl} + \chi \cdot T_{kl}^2$

3.5.5: Change in the linear phase factor with angle of incidence in the acrylic-epoxy-acrylic sample.

3.6.1: Rotation of a hexagonal anisotropic system about the $x$-axis, so that the fibres lay in the $x$-$y$ plane.
3.6.2: Change in amplitude of the reflected longitudinal wave
(a) Change with angle of incidence for various polar angles, and a thin layer with hexagonal anisotropy. Each of the various curves is separated by 10 degrees of polar angle beginning at zero degrees.
(b) Change with polar angle for various incident angles, and a thin layer with hexagonal anisotropy. Each of the curves is separated by 10 degrees of incident angle (beginning at normal incidence), except the highest curve which is an incident angle of 85 degrees.

3.6.3: Change in amplitude of the transmitted longitudinal wave
(a) Change with angle of incidence for various polar angles, and a thin layer with hexagonal anisotropy. Each of the various curves is separated by 10 degrees of polar angle beginning at zero degrees.
(b) Change with polar angle for various incident angles, and a thin layer with hexagonal anisotropy. Each of the curves is separated by 10 degrees of incident angle (beginning at normal incidence), except the highest curve which is an incident angle of 85 degrees.

3.6.4: Change in amplitude of the reflected shear vertical wave with angle of incidence for various polar angles, and a thin layer with hexagonal anisotropy. Each of the various curves is separated by 10 degrees of polar angle beginning at zero degrees.

3.6.5: Surface plot of the change in amplitude of the transmitted shear vertical wave with both polar and incident angles, and a thin layer with hexagonal anisotropy.

3.6.6: Surface plot of the change in amplitude of the reflected shear horizontal wave with both polar and incident angles, and a thin layer with hexagonal anisotropy.

3.6.7: Surface plot of the change in amplitude of the transmitted shear horizontal wave with both polar and incident angles, and a thin layer with hexagonal anisotropy.

3.6.8: Surface plot of the change in amplitude of the transmitted shear vertical wave with both polar and incident angles, and a thin layer with orthogonal anisotropy.

3.7.1: Change in second harmonic with polar and incident angle of the reflected horizontal wave for the acrylic-carbon fibre-acrylic sample with nonlinearity $\beta \cdot T_{kl}$.

4.2.1: Transducer adhered to a small piece of acrylic using phenyl salicylate. The acrylic surface is curved to match the curved edge of the sample.
4.3.1: General overview of experimental set up.  
4.3.2: Boron rod sample with transducers connected.  
4.4.1: Reflected displacement at 12 degrees incident angle.  
4.4.2: Reflected amplitude for a thin layer of cyanoacrylate adhesive.  
4.4.3: Fibreglass cloth reflected amplitude with changing angle of incidence; error represents the smallest error needed to make the normal incidence points equal within each others error.  
4.4.4: Various trials of transmitted amplitude for the boron fibre layer.  
4.4.5: Proposal for a sample to measure amplitude changes with polar angle.  
B.1: Set up of the problem for multilayer system
### List of Symbols and Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Representation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>I, II, Int</td>
<td>Material notation (I = First, II = Second, Int = Thin layer)</td>
<td></td>
</tr>
<tr>
<td>(\alpha)</td>
<td>Polarization variable for first material</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>Incident acoustic wave</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>Reflected acoustic wave</td>
<td></td>
</tr>
<tr>
<td>(\beta)</td>
<td>Polarization variable for second material</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>Transmitted acoustic wave</td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>Thickness of the thin layer</td>
<td></td>
</tr>
<tr>
<td>(\delta_{ij})</td>
<td>Dirac delta function</td>
<td></td>
</tr>
<tr>
<td>(s_{ij})</td>
<td>Strain tensor</td>
<td></td>
</tr>
<tr>
<td>f</td>
<td>Frequency</td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>Shear horizontal polarization</td>
<td></td>
</tr>
<tr>
<td>(K^\alpha)</td>
<td>Wave vector with polarization (\alpha)</td>
<td></td>
</tr>
<tr>
<td>(k_i^\alpha)</td>
<td>Wave number with polarization (\alpha) on axis (i)</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>Longitudinal polarization</td>
<td></td>
</tr>
<tr>
<td>(\lambda^i)</td>
<td>Lamé constant in material I</td>
<td></td>
</tr>
<tr>
<td>P</td>
<td>Specifically defined tensor</td>
<td></td>
</tr>
<tr>
<td>(p_{1\alpha}^i)</td>
<td>Polarization of type (\alpha) on axis (i)</td>
<td></td>
</tr>
<tr>
<td>(\rho)</td>
<td>Density</td>
<td></td>
</tr>
<tr>
<td>(\Phi)</td>
<td>Polar angle</td>
<td></td>
</tr>
<tr>
<td>(\phi_\alpha)</td>
<td>Phase coefficient</td>
<td></td>
</tr>
<tr>
<td>(R_\alpha)</td>
<td>Reflection coefficient with polarization (\alpha)</td>
<td></td>
</tr>
<tr>
<td>(s_{ijkl})</td>
<td>Compliance matrix</td>
<td></td>
</tr>
<tr>
<td>t</td>
<td>Time</td>
<td></td>
</tr>
<tr>
<td>(T_\beta)</td>
<td>Transmission coefficient with polarization (\beta)</td>
<td></td>
</tr>
<tr>
<td>(T_{ij})</td>
<td>Stress tensor</td>
<td></td>
</tr>
<tr>
<td>(\theta)</td>
<td>Angle of incidence</td>
<td></td>
</tr>
<tr>
<td>(u_{i}^j)</td>
<td>Displacement in material I on axis (i)</td>
<td></td>
</tr>
<tr>
<td>(\mu^i)</td>
<td>Lamé constant in material I</td>
<td></td>
</tr>
<tr>
<td>v</td>
<td>Velocity</td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>Shear vertical polarization</td>
<td></td>
</tr>
<tr>
<td>(\omega)</td>
<td>Angular velocity</td>
<td></td>
</tr>
<tr>
<td>((x,y,z))</td>
<td>Cartesian coordinate system</td>
<td></td>
</tr>
</tbody>
</table>

\(A \cdot B\) Small dot signifies multiplication (for clarity)  
\(A \ast B\) A large dot signifies dot product (not used)  
\(\mathbf{x}_j \cdot \mathbf{k}_j = \sum_j \mathbf{x}_j \cdot \mathbf{k}_j\) Repeated roman lettered indices imply summation
Chapter 1: Introduction

1.1: Overview

The goal of this research is to examine the changes in the reflected and transmitted ultrasonic plane waves due to the addition of a thin layer between two media, and the resulting properties as this thin layer is allowed to be isotropic or anisotropic, and linear or nonlinear. The thin layer is restricted to have a thickness much less than an acoustic wavelength so that a perturbation approach can be taken to create an overall set of boundary equations dealing with the layer itself and needing only some basic information about the attributes of the thin layer and surrounding media. This approach also has the advantage of being able to ignore the propagation of the acoustic waves inside the thin layer, thereby simplifying the process of adding an incident wave with a non-normal angle of incidence and also calculating the effects of nonlinearity and anisotropy in the layer. In contrast the multilayer approach requires the boundary conditions at each interface of the thin layer to be satisfied, and must account for multiple reflections of each wave mode inside the thin layer. The addition of anisotropy to this type of system would create more acoustic waves inside the thin layer to keep track of, and if nonlinearity were added, each wave mode inside the thin layer would need additional alterations.

In this introduction some of the theoretical and experimental work done in the areas of boundary conditions, nonlinearity, anisotropy, and anisotropic or nonlinear thin layers is examined in the context of acoustic waves. This review does not provide an extensive overview on each of these subjects, but instead provides selected details on
work done in these areas, including the approaches that are used in the theory, simulations, and experiment. In some cases factors that have not been included by this thesis are also discussed.

1.2: Specific Literature

The work that will be presented in this thesis encompasses many different areas which each have multiple uses outside of this context. As a result, some references are given which are outside the scope of this thesis simply to show the versatility of the topic, or to present references to general information that may be useful to understand each topic in more detail. Furthermore, the information presented here only briefly describes the contents of the particular references; more detailed descriptions, definitions, and properties of each of the areas will be discussed in later sections of this chapter.

1.2.1: General Use of Boundary Conditions in Literature

The use of boundary conditions to describe a simple interface between two media is not a new idea, and the displacement and traction conditions have already been found to be able to describe a fully bonded interface (or perfectly welded interface) between two solids or a liquid and a solid [1]. However, the use of boundary conditions is not limited to bonded interfaces. It also can be used for situations where the two media are unbonded, this allows variations where the two materials are allowed to clap [2, 3] (connect and disconnect over time), remain connected but slide at a frictionless interface [2], or remain connected and slide and stick at an interface with friction [4]. In particular Comninou and Dundurs [3, 4] use the boundary method to describe a separated interface, first only considering separation by ignoring friction [3] and later including friction [4].
O'Neill, Sadler, Severin, and Maev [2] use a similar approach in one dimension to show that boundary conditions can describe the clapping and slipping interface as well as an interface with a nonlinear thin layer. The solutions for the clapping and slipping interfaces are not limited to boundary condition method only. Richardson [5] has developed a solution to the unbonded clapping interface using the acoustic equation of motion with the boundary conditions used to determine the initial conditions of the open and closed state. Even though these clapping and sliding cases, and many others like them, exploit the same idea of using boundary conditions, the base conditions are different from the situation that will be discussed in this thesis. The situation discussed in this thesis will use the fully bonded interface conditions, and perturbation techniques described by O'Neill et al. [2, 6] and Sadler et al. [7] and, at the most basic level, are thus incompatible with the unbonded systems.

1.2.2: General Information on Nonlinearity

Several reviews and books have been published to deal with the subject of nonlinearity. Overviews by Solodov and Maev [8] and Zheng et al. [9] that discuss the theory of nonlinearity and give a discussion on many of the experimental techniques for material characterization have been published. An overview with special emphasis on the nonlinearity in rocks and other geological materials has been given by Ostrovsky and Johnson [10]. In addition, work by Hamilton [11], Hamilton et al. [12] and Norris [13] have been collected in a text edited by Blackstock and Hamilton and contains some useful information on nonlinearity in solids [13]; the remainder of the papers in this text deal mostly with nonlinearity in liquids, thus only the specific chapters dealing with solids have been referenced. Delsanto et al. [14] provide an explanation of some of the
various classifications of nonlinearity, with respect to both where the nonlinearity exists in the material and in the mathematics, in a paper dealing with the modeling of ultrasonic waves in the interface region between two bonded elements.

1.2.3: General Information on Anisotropy

An introduction to the theory of anisotropic materials, including references to more detailed reviews and also to specific areas of interest, is given by O'Neill and Maev [15], which deals in the mathematical methods for characterization of anisotropic materials using ultrasound. In addition, Rokhlin and Huang [16] provides a theoretical background and discusses the fundamentals of an anisotropic layer and wave propagation through that layer.

1.2.4: Anisotropic Layers in Literature

Rokhlin and Huang have published two papers [16, 17] involving plane acoustic waves traveling through an anisotropic medium placed between two other anisotropic mediums comparing a variety of methods. In the first paper [16] they use a combination of boundary conditions, Christoffel equations, and scattering matrices to form an exact solution for a layer of any size. The specific case of a thin layer is also examined using a transfer matrix already developed for a general system. In the second paper [17] they expand on this approach using a finite difference approximation creating second order boundary conditions and are able to satisfy an energy balance. In both papers they conclude that the use of boundary conditions is simple compared to the exact solution because there is no need to describe the waves in the interface layer. Niklasson et al. also have published two papers [18, 19] dealing with thin anisotropic layers. In these papers
they consider guided waves in plates made with anisotropic thin layers. One of the papers examines thin anisotropic coatings on a bulk material [18], while the other models a thin anisotropic layer between two isotropic layers [19]. In both of these papers, the authors use the perfectly welded boundary conditions in the stress-strain relation for the middle layer to obtain an approximate dispersion relation. They also compare the results from using the approximation of a thin layer with the results from the more complex exact solution and obtain very similar results. The possibility of guided waves in the thin layer, and the dispersion relation will not be considered by the theory used in this thesis.

1.2.5: Nonlinear Layers in Literature

Rothenfusser et al. [20] uses the finite element method to examine the specific situation of a thin nonlinear epoxy layer between two bulk aluminum layers. They choose a one-dimensional approach and examine the changes that occur as the thickness of the layer increases. In the same paper [20] they also report on results from an experiment, and demonstrate qualitatively similar results from both the simulation and experiment. Hedberg and Rudenko [21] also look at a one-dimensional system containing a nonlinear layer between two identical half-spaces. While Rothenfusser et al. [20] chose to use a nonlinear stress-strain relation, Hedberg and Rudenko [21] chose to incorporate the nonlinearity in a relation between density and pressure (equation of state) and use a set of boundary conditions that involve the continuity of pressure and velocity. In the latter paper [21] they also look at both a thin layer with weak nonlinearity and a strong nonlinear layer, using the same definition of "thin" that will be used in this thesis, namely, the thickness of the layer is much less than the acoustic wavelength. O'Neil et al. [2, 6] and Sadler et al. [7] also consider the possibility of a thin nonlinear layer by
using a combination of boundary conditions and perturbation theory to describe the thin layer. They begin by describing a thin nonlinear layer in one dimension [2] and then expand on this idea by taking it into three dimensions [6, 7].

1.3: Linearity vs. Nonlinearity

The linear theory of acoustics assumes that the material will produce a linear response between the strength of the input and output signals [9]. In the linear theory the amplitude of the displacement of the media by the acoustic wave does not alter the properties of the material such as velocity, density, and elasticity [8, 9]. The linear approach is suitable as long as the amplitude remains small enough that the changes to strain, pressure, and density in the medium remain small [8, 9]. As the acoustic signal increases in amplitude the amplitude-dependent effects can no longer be neglected, and the waves in the nonlinear medium must now be described with nonlinear equations [8, 9, 10].

The nonlinearity in a system of materials is typically classified by either the nonlinear response, or by a classification of the material causing the nonlinearity. Classical nonlinearity is typically used to describe the class of materials known as Atomic Elastic Materials (AEM); typically this class is made up of most fluids and monocrystalline solids [10] and the nonlinearity is created by the elastic forces in the crystal lattice at the atomic or molecular level [10, 14]. In the classical theory, the nonlinearity is most often represented using an expansion of strain terms in the stress-strain relation [10, 14, 20]

\[ T_{ij} = e_{ijkl} \cdot e_{kl} \cdot \left( 1 + \beta \cdot e_{kl} + \chi \cdot e_{kl}^2 + \ldots \right), \quad (1.3.1) \]
where \( T \) is the stress, \( \varepsilon \) is the strain, \( c_{ijkl} \) are the stiffness coefficients and \( \beta \) and \( \chi \) are nonlinearity coefficients. At this point one should note that the implied summation convention is being used for repeated Roman lettered indices in this thesis. It should be noted that the implied summation convention is being used for the repeated Roman lettered indices. Because the higher harmonics that are generated in the reflected and transmitted wave are created due to nonlinear bonding forces, the nonlinear effects are often necessary to yield information about the bonding of adhesives [14]. Non-classical nonlinearity is typically used to describe the Nonlinear Mesoscopically Elastic Materials (NME) (sometimes also referred to as Structural Nonlinear Elastic Materials, or Nanoscale Elastic Materials). Typically the materials included in this class are of interest in seismology (e.g. rock, sand, soil), and civil engineering [10, 14]. Typically, NME materials feature large nonlinear responses in comparison to the classical theory, and also may produce a hysteretic nonlinear response, with discrete memory (a memory of the previous maximum strain state). It has also been discovered that damaged AEM have a far greater nonlinear response than perfect materials, causing them to behave as mesoscopic materials, and leading to the hypothesis that the nonlinearity is highly correlated to damage and micro-cracks in the materials. In this case the nonlinearity is represented by adding an additional function \( (\Lambda) \) to the stress-strain relation

\[
T_{ij} = s_{ijkl} \cdot \varepsilon_{kl} \left( 1 + \beta \cdot \varepsilon_{kl} + \chi \cdot \varepsilon_{kl}^2 \right) + \Lambda(\varepsilon, u),
\]

(1.3.2)

where \( u \) is the displacement. The unbonded contact of two materials has also been found to create larger nonlinear response than expected. This type of nonlinearity has been dubbed Contact Acoustic Nonlinearity (CAN) [8], and typically includes any non-perfect contact type defect such as cracking, disbonding, delamination and microstructural
1.4: Isotropy vs. Anisotropy

As a general definition, an isotropic medium is said to posses identical properties when examined in any direction. This uniformity can be a result of an ideal case where the medium is truly symmetric or one in which the medium is so random that on average one direction cannot be discerned from another even though there may be localized anisotropic areas. In the specific case of acoustics, this property of being identical in any direction refers to the material's acoustic properties (elasticity, velocity) being uniform in any direction the acoustic wave will travel. In contrast, the acoustic wave's velocity in an anisotropic material is dependent on the direction of propagation though the medium. Isotropic and anisotropic media can also be distinguished in terms of the coupling between the three possible polarizations of an acoustic wave (the polarization is the direction of displacement with respect to the wave's propagation). In the case of an isotropic medium the longitudinal and shear vertical polarizations are coupled together, while the shear horizontal polarization is left uncoupled. This coupling causes a longitudinal or shear vertical wave, which is incident upon an isotropic medium, to produce both longitudinal and shear vertical polarizations. In the case of an anisotropic layer all three polarizations are coupled (in this case two quasi-transverse and one quasi-longitudinal polarizations). Thus, in general, for any acoustic wave entering an anisotropic material produces three differently polarized acoustic waves. In the anisotropic medium the polarizations are no longer purely longitudinal, or purely shear polarizations, and the direction of the polarizations, the velocities of each polarized wave, and the direction of the propagation of the wave is different depending upon the direction
which the incident wave enters the isotropic material. In the case of an anisotropic material there also exists special directions of propagation called acoustic axes where the two transverse waves have the same velocity and the transverse waves can be arbitrarily polarized.

1.5: Physical Effects to Consider

1.5.1: Propagation

The velocity at which the acoustic wave propagates though the medium is material dependent; in addition the velocity in each material is dependent on the polarization of the acoustic wave. For an isotropic material this dependence is limited to a difference between longitudinal polarized waves and shear polarized (transversely polarized) waves, while in the case of an anisotropic material the velocity changes depending on the direction of propagation through the material, as well as changing with respect to three possible polarizations.

The propagation of an acoustic wave can be altered by attenuation and also dispersion. The attenuation in a material causes the amplitude of an acoustic wave to dissipate as the wave travels through the medium. The attenuation is a combination of the factors of scattering (microscopic reflections of the acoustic wave in other directions than its propagation) and absorption (conversion of the energy of the acoustic waves into other forms). For a plane wave the amplitude \((A)\) due to attenuation is expressed as an exponential decay

\[
A = A_o \cdot e^{-\alpha x},
\]

(1.5.1)

where \(A_o\) is the initial amplitude, \(\alpha\) is the attenuation factor (attenuation coefficient), and
x is the distance the acoustic wave has travelled. The attenuation factor is very dependent on the manufacturing of the material, and is roughly proportional to the square of the frequency of the acoustic wave. Dispersion is the dependence of phase speed on frequency and can possibly alter the time of propagation of an acoustic wave across a medium, or the angle at which the acoustic wave travels through the medium as the wavelength or frequency is changed. In the case of unbounded homogenous media the dispersion is a weak effect and can often be neglected, while in a bubbly liquid dispersion becomes a strong effect and must be considered [12].

1.5.2: Types of Interfaces and Boundary Conditions

The propagation of an acoustic wave is also altered due to changes in the material's impedance (impedance is the product of density and velocity). The location of the change between the two different impedances is referred to as an interface or boundary. This boundary could be a slight alteration in the material, or a connection between two different materials. As an acoustic wave travels from the first medium to the second, the direction of the propagation of the wave changes in relation to the incident angle and in relation of the acoustic velocities of the media. The amplitude of the waves propagating from the interface is dependent on the relative impedances of the two media as well as the angle of incidence. In addition acoustic waves of different polarizations may also be created at the boundary depending on the angle of incidence, the polarization of the incident wave, the state of the material (liquid or solid) on each side of the boundary, and the symmetry of the material (isotropic or anisotropic).

The type of contact between media and the structure of the interface determine the
exact boundary conditions that must be considered. The contact formed during solidification, metallurgical solid-state bonding, dry contact, or lubricated contact, change the boundary conditions and thus the conditions of a perfect bond are not adequate to describe the wave interaction of all interfaces [16, 17]. O’Neill et al. [2], Rokhlin and Wang [22], Rokhlin and Marom [23] agree on two different categories of boundaries and boundary conditions for an interface connecting two solids. The first type of boundary is referred to as the perfectly welded, or perfectly bonded interface, where the boundary conditions are written so that both the physical displacement and stress caused by the acoustic wave are continuous across the interface. These perfectly bonded conditions are the set of conditions chosen to describe the structure considered in this thesis. The second type of boundary is not bonded and the two pieces of material creating the boundary may connect and disconnect (a kissing or clapping interface), or move in a sliding motion (a slipping or sliding) interface. In the case of the clapping interface, O’Neill et al. [2] use two sets of boundary conditions to describe the conditions when the materials are connected and disconnected. They again use two sets of boundary conditions to describe a set of materials that alternate between a stuck state (fully connected) and a sliding state. This sliding state is identical to the slipping boundary conditions of Rokhlin and Wang [22], Rokhlin and Marom [23], where only the normal components of stress and displacement are continuous, the shear traction components vanish and the displacement components are allowed to be discontinuous. Rokhlin and Wang [22], Rokhlin and Marom [23] both note that this slipping interface is an ideal form for two solids with a thin layer of ideal liquid between them [21].
1.6: Uniqueness and Scope of Thesis

The basis of the theory in this thesis is that the interface layer has a thickness much less than an acoustic wavelength; this assumption will be required later in order to use a perturbation expansion. The connection of thin layer to the bulk media will be described using fully bonded boundary conditions, thereby eliminating the possibility to look at any contact type of interface. In addition, the incident wave is an acoustic plane wave in the ultrasonic frequency range (typically defined as 20 kHz or more), and the theory will only predict transmitted and reflected plane waves, ignoring the possibility of plate waves inside the interface. The ultrasonic frequency range is typically used in material imaging and characterization, as the reduced size of the acoustic wavelength is able to more accurately probe smaller areas as compared to lower frequency acoustic waves. The theory and simulations explore the case where a layer of thickness much less than an ultrasonic wavelength is to be examined. It is noted that in most cases the simulation uses only a mid-range ultrasonic frequency (usually 1.0 MHz or 0.1 MHz) simply to be able to express the thin layer in convenient units. The theory also allows for the thin layer to be nonlinear, isotropic, or possibly both, where the nonlinearity of the thin layer is allowed to be a generic function and is included in the stress-strain relation. While the theory supports the possibility of a critical angle between the two bulk media, the simulation does not currently support such a situation. The theory only concerns itself with the density and velocity of the materials (or in the case of an anisotropic thin layer the stiffness or compliance matrix) and does not consider changes due to other material properties such as attenuation and dispersion.
Compared to the current work, other authors working with a thin layer tend to be concerned with either anisotropy [16, 17, 18, 19] or nonlinearity [20, 21] alone, while this theory is able to use both. As mentioned previously, this theory is concerned with plane waves [16, 17], ignoring the possibility of plate waves and dispersion [18, 19]. In addition this theory and simulation use equations based in the time domain, uses a perturbation method, and produces displacement-based output, or frequency-based output. Using the time domain allows for a simpler simulation than some of the more complex methods, and in addition the theory's development will find that most of the parameters to be found are independent of time, allowing the time portions to be ignored until needed. Other common approaches include using the finite element method [16], using equations written in terms of pressure (or energy) [16, 17], or using equations written in the frequency domain such as for the cases investigating nonlinearity [20, 21].
Chapter 2: Theory

2.1: Overview

In this section we examine the theory behind the propagation of a bulk acoustic plane wave as it travels from a half-space of homogenous initial isotropic material, through a thin layer of material, which may be isotropic or anisotropic, and linear or nonlinear, into a second half-space of homogenous isotropic material (Figure 2.1.1). As an acoustic wave travels through a material, it creates localized displacements, as well as creating localized stress or pressure changes in the medium. The size of the localized displacements and pressure changes are microscopically large because the dimensions of the volume of material displaced are large compared to the acoustic wavelength, but at the same time are not on the microscopic scale because the area changed contains a large number of molecules and is larger than the molecular mean free path (the distance a molecule can travel without a collision). When the acoustic wave reaches an interface between two materials these changes in displacement and pressure are partially transferred to the second medium and partially reflected, producing various possible reflected and transmitted bulk acoustic waves. The theory uses a set of boundary equations to describe the necessary conditions the displacement and stress must satisfy at a boundary between two materials. These boundary conditions that will be used in this work are typically referred to as the perfectly bounded interface conditions. The first condition in the set is that the displacement of the material by the acoustic wave will be continuous at the boundary. In particular the displacement continuity will enforce the condition that the materials will not separate, and also that they will not overlap. Mathematically, the displacement continuity equation is written as follows:
Displacement Continuity: \[ u_i^{l}(x, t) - u_i^{s}(x, t) = 0, \] (2.1.1)

where \( u_i^{l}(x, t) \) and \( u_i^{s}(x, t) \) represent the components of displacement (\( i = 1,2,3 \)) in the first medium and second medium respectively, at time \( t \), at the location in space \( x \) where the two materials intersect. The second condition is that of traction continuity, which ensures that the pressure at the interface is equal in both media. Mathematically, the traction continuity equation is written as follows (where the stress is labelled \( T_{ij} \)):

Traction Continuity: \[ T_{iz}^{l}(x, t) - T_{iz}^{s}(x, t) = 0, \] (2.1.2)

where the \( z \)-axis has been defined to be the axis perpendicular to the interface. Equations (2.1.1) and (2.1.2) need to be applied at each set of interfaces, the first interface is located at the position where the first infinite medium connects with the thin layer (\( x = x_0 \)), and the second interface is located where the thin layer of thickness \( d \) connects with the second infinite medium (\( x = x_0 + d \)). In the case of a very thin layer, O’Neill et al. [2, 6] and Sadler et al. [7] that after a perturbation expansion about the thickness \( d \), the two sets of boundary conditions could become one single set incorporating the information of the thin layer. With the \( z \)-axis set to be normal to the interface (Figure 2.1.1) the thin layer boundary conditions are

\[
u_i^{l}(x_0 + d, t) - u_i^{s}(x_0, t) = \left( 2 \cdot \varepsilon_{iz}^{\text{int}} - \varepsilon_{zz}^{\text{int}} \cdot d_{iz} - \frac{\partial u_i^{l}}{\partial x_1} \cdot d_{i1} - \frac{\partial u_i^{s}}{\partial x_2} \cdot d_{i2} \right) \cdot d, \] (2.1.3)

\[
T_{iz}^{l}(x_0 + d, t) - T_{iz}^{s}(x_0, t) = \left( \rho \cdot \frac{\partial^2 u_i^{l}}{\partial t^2} - \frac{\partial T_{i1}^{l}}{\partial x_1} - \frac{\partial T_{i2}^{l}}{\partial x_2} \right) \cdot d, \] (2.1.4)
where the notation 1 and 2 denotes two arbitrary axes forming a left handed Cartesian co-ordinate system with the already defined z-axis, \( \rho^\text{int} \) is the density of the interface material, and \( \varepsilon_{ij} \) are the components of the linear strain defined by

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial \xi_i} + \frac{\partial u_i}{\partial \xi_j} \right).
\]

(2.1.5)

One should note that in the case of no thin layer \( (d = 0) \) these equations simplify to the original continuity equations (2.1.1) and (2.1.2). One very important factor, which has yet to be mentioned, is wave polarization. The polarization of the wave is a unit vector in the direction that the wave physically displaces the medium. The incident wave may have any one of three possible orthogonal polarizations: longitudinal (L), shear vertical (V), or shear horizontal (H) (Figure 2.1.2). This incident wave may create reflected or transmitted waves with the same polarization, and in addition it is possible that the additional polarizations will be created via a process known as mode conversion (Figure 2.1.2). Each of these possible mode converted waves can have a different angle of reflection \( (\theta_R^\alpha) \) or angle of transmission \( (\theta_T^\beta) \), and thus also has a different wave vector \( (\kappa^\alpha \text{ and } \kappa^\beta) \). Here \( \alpha \) and \( \beta \) represent the possible polarizations in the first and second medium respectively. The actual polarizations that will be created can be predetermined knowing the type of incident wave and the crystalline symmetry of the thin layer. However, as the boundary conditions will determine which are mathematically possible, this predetermination is not necessary. There also exists a boundary condition to determine the angle of reflection and angle of transmission of the various polarized waves in the form of Snell's Law.
\[ K \cdot \sin(\theta_1) = |K^\alpha| \cdot \sin(\theta_\alpha) = K^\beta \cdot \sin(\theta_\beta). \]  

(2.1.6)

The generalized form of Snell's Law [equation (2.1.6)] becomes very useful in removing the references to the various incident angles of reflection and transmission because it is equivalent to making

\[ k_\psi^\beta = |k_\psi^\alpha| = K \cdot \sin(\theta_1), \]  

(2.1.7)

as will be seen in equation (2.2.5), where \( k \) is the wave vector and \( \psi \) is a defined axis as seen in Figure 2.2.1.

---

**Figure 2.1.1: Thin layer between two half-spaces**

The overall setup of the problem with an interface layer between two half-spaces (marked I and II). The incident, reflected and transmitted waves all travel in the same plane of incidence which may be at some polar angle (\( \Phi \)) relative to the material's axis.
2.2: Set up of the Problem and Definitions

This simulation considers the propagation of bulk acoustic plane waves though two half-spaces joined together in the x-y plane by an interface made up of a thin layer of material (Figure 2.1.1) of thickness d, where this thin layer is restricted to having a thickness much less than an acoustic wavelength. Each material can be made of different materials, but the properties are assumed to be known. The first material is defined by the following parameters:

Longitudinal Velocity in material I: \( v_L^1 \)
Shear Velocity in material I: \( v_S^1 \)
Density of material I: \( \rho^1 \)  

(2.2.1)
The second material and thin layer also are defined by these parameters but with the notation "I" and "Int" respectively. It is known that the set of possible incident, reflected and transmitted waves all travel within the same plane of incidence that is rotated at some polar angle \( \Phi \), measured relative to an arbitrary x-axis in the thin layer of material (Figure 2.1.1). As all the waves travel in the same plane it is useful to define a \((\psi, \eta, z)\) coordinate system with respect to the plane of incidence (Figure 2.1.1 and 2.1.2). Each individual wave enters, or leaves the thin layer at some angle of incidence \( (\theta_I) \), reflection \( (\theta_R^{\alpha}) \) or transmission \( (\theta_T^{\beta}) \), measured relative to the z-axis. The incident wave \( (A_0) \) is a sinusoidal plane wave, with a single angular frequency \( \omega \), incident on the thin layer in the +z direction at any valid angle of incidence \( (\theta_I) \), and any polar angle \( (\Phi) \). The incident wave is expressed as an arbitrary sinusoidal function of the form

\[
A_0(\omega \cdot t - k_0^0 \cdot x_j),
\]

where the 0 is a placeholder for the incident wave's polarization (longitudinal, shear vertical or shear horizontal), and the summation convention is being used for repeated Roman lettered subscripts. With the incident wave's direction defined the, reflected \( (B) \) and transmitted \( (C) \) waves take the form

\[
B_\alpha(\omega \cdot t + k_j^\alpha \cdot x_j),
\]

\[
C_\beta(\omega \cdot t - k_j^\beta \cdot x_j),
\]

where \( \alpha \) and \( \beta \) are placeholders for the wave's polarization. This particular arrangement of equations (2.2.3) and (2.2.4) was decided upon simply because it gives the incident and transmitted waves uniform signs for all components. The reflected wave will need to
be given uniform signs by defining the $k_\psi$ component to be negative. Apart from the sign change, the wave vectors along the $\psi$-axis (defined in Figure 2.1.1 and 2.1.2) will still be required to satisfy Snell's Law [equation (2.1.6)], making

$$k_\psi^0 = K^0 \cdot \sin(\theta_1), \quad k_\psi^\alpha = -K^0 \cdot \sin(\theta_1), \quad k_\psi^\beta = K^0 \cdot \sin(\theta_1),$$

(2.2.5)

where $K^0$ is the wave number of the incident wave. As the wave vector is contained in the plane of incidence, the $\eta$ components are zero, leaving the $z$ component of the wave vectors defined by the Pythagorean theorem:

$$k_z^0 = \sqrt{(K^0)^2 - (k_\psi^0)^2}, \quad k_z^\alpha = \sqrt{(K^\alpha)^2 - (k_\psi^\alpha)^2}, \quad k_z^\beta = \sqrt{(K^\beta)^2 - (k_\psi^\beta)^2},$$

(2.2.6)

or written in terms of angles as

$$k_z^0 = K^0 \cdot \cos(\theta_1), \quad k_z^\alpha = K^\alpha \cdot \cos(\theta_1), \quad k_z^\beta = K^\beta \cdot \cos(\theta_1).$$

(2.2.7)

It is also useful to define the possible polarizations that the incident, reflected and transmitted waves may have (Figure 2.1.2). The longitudinal polarization (L) is defined to be the same direction as the wave vector. As the wave vector for the reflected waves was defined to be in the direction opposite to propagation (Figure 2.1.2), the longitudinal polarization of the reflected wave will be aligned in the $+z$ direction with the longitudinal polarization for the incident and transmitted waves. The longitudinal polarization components are

$$p_i^0 = \frac{k_i^0}{K^0}, \quad p_i^L = \frac{k_i^L}{K^L}, \quad p_i^\|L = \frac{k_i^\|L}{K^\|L}.$$  

(2.2.8)

The shear vertical polarization (V) is perpendicular to the longitudinal polarization and also in the same plane as the incident reflected and transmitted waves (Figure 2.1.2); its individual components are
\[
\begin{align*}
    p_\psi^0 &= \frac{k_z^0}{K^0}, & p_\psi^{l_v} &= \frac{k_z^{l_v}}{K^{l_v}}, & p_\psi^{ll_v} &= \frac{k_z^{ll_v}}{K^{ll_v}}, \\
    p_z^0 &= \frac{k_\psi^0}{K^0}, & p_z^{l_v} &= \frac{k_\psi^{l_v}}{K^{l_v}}, & p_z^{ll_v} &= \frac{k_\psi^{ll_v}}{K^{ll_v}}.
\end{align*}
\] (2.2.9) (2.2.10)

It is important to recall that the sign of \( k_\psi^a \) is negative and thus the \( p_z^{l_v} \) component is also negative as seen in Figure 2.1.2. The \( \eta \) components for the shear vertical waves are zero. The final possible polarization is the shear horizontal (H) component, which is defined to be perpendicular to the longitudinal and shear vertical polarizations (Figure 2.1.2). The components of the shear horizontal polarization are

\[
    p_\eta^0 = 1, \quad p_\eta^{l_H} = 1, \quad p_\eta^{ll_H} = 1.
\] (2.2.11)

The \( z \) and \( \psi \) components for the horizontal wave are both zero. It should be noted that the incident wave is either longitudinal, shear vertical, or shear horizontal and that the \( p^0 \) vector is defined only with one of the sets of equations (2.2.8) though (2.2.11).

2.3: Perturbation Due to the Thin Layer

In addition to causing the perturbation of the fully bonded boundary conditions of equations (2.1.1) and (2.1.2), the thin layer will also cause a phase change \(( \phi_\alpha \cdot k_z^\alpha \cdot d, \) and \( \phi_\beta \cdot k_z^\beta \cdot d \) in the reflected and transmitted waves. Incorporating the phase change into the reflected [equation (2.2.3)] and transmitted waveforms [equation (2.2.4)] results in

\[
    B_\alpha (\omega \cdot t + k_j^\alpha \cdot x_j) \rightarrow B_\alpha (\omega \cdot t + k_j^\alpha \cdot x_j + \phi_\alpha \cdot k_z^\alpha \cdot d),
\] (2.3.1)
\[ C_\beta \left( \omega \cdot t - k_j^\beta \cdot x_j \right) \rightarrow C_\beta \left( \omega \cdot t - k_j^\beta \cdot x_j + \phi_\beta \cdot k_z^\beta \cdot d \right). \] (2.3.2)

It should be noted that the implied summation convention is being used only for the repeated Roman lettered subscripts, and not the Greek letters. Because the layer is very thin, the phase change will also be small and the waveforms can be expressed as a perturbation expansion (with respect to \( \pm k_z \cdot z \))

\[ B_\alpha = B_0^\alpha + \phi_\alpha \cdot k_z^\alpha \cdot d \cdot B_\alpha', \] (2.3.4)

\[ C_\beta = C_0^\beta + \phi_\beta \cdot k_z^\beta \cdot d \cdot C_\beta', \] (2.3.5)

where the values \( B_0^\alpha \) and \( C_0^\beta \) are the solutions with no thin layer (that is the case when \( d = 0 \)). When the waveforms in equations (2.3.4) and (2.3.5) are examined at the boundary position \( x = 0 \), the derivatives \( B' \) and \( C' \) can be changed from derivatives with respect to \( \pm k_z \cdot z \), to derivatives with respect to \( \omega \cdot t \) by using the properties

\[ \frac{d}{d(k_z \cdot z)} B \left( \omega \cdot t + k_j^\alpha \cdot x_j \right) \bigg|_{x_j = 0} = \frac{d}{d(\omega \cdot t)} B(\omega \cdot t), \] (2.3.6)

\[ \frac{d}{d(k_z \cdot z)} C \left( \omega \cdot t - k_j^\beta \cdot x_j \right) \bigg|_{x_j = 0} = \frac{d}{d(\omega \cdot t)} C(\omega \cdot t). \] (2.3.7)

With this alteration equations (2.3.4) and (2.3.5) can now be written in terms of reflection (\( R_\alpha \)) and transmission (\( T_\beta \)) coefficients as

\[ B_\alpha = R_\alpha \cdot A_0 + \phi_\alpha \cdot k_z^\alpha \cdot d \cdot R_\alpha \cdot A_0', \] (2.3.8)

\[ C_\beta = T_\beta \cdot A_0 + \phi_\beta \cdot k_z^\beta \cdot d \cdot T_\beta \cdot A_0'. \] (2.3.9)
where the prime denotes the derivative with respect to \( \omega \cdot t \). For example, in the case that the incidence wave is a pure sine wave, one would have

\[
A'_0(\omega \cdot t) = \omega \cdot \frac{d}{d(\omega \cdot t)} A_0(\omega \cdot t) .
\]

By using the notation in equation (2.3.10) one has made the factors from derivation fully visible in the theory.

2.4: First Order Approximation: Two Half-Spaces

The first level of approximation follows standard perturbation techniques and sets \( d = 0 \) thus creating a case identical to when the bulk materials are directly connected. The solution begins by using the requirements of displacement and traction continuity at the boundary, i.e. equations (2.1.1) and (2.1.2)

Displacement Continuity: \( u_{i}^{ll}(x_0, t) - u_{i}^{r}(x_0, t) = 0 \),

Traction Continuity: \( T_{iz}^{ll}(x_0, t) - T_{iz}^{r}(x_0, t) = 0 \).

The displacement in the first medium is the sum of the incident wave and each polarization of the reflected waves, while in the second medium the displacement is the sum of the variously polarized transmitted waves,

\[
u_{i}^{ll}(x_0 = 0, t) = p_{i}^{0} \cdot A_{0}(\omega \cdot t) + \sum_{\alpha} p_{i}^{\alpha} \cdot B_{0}^{\alpha}(\omega \cdot t) , \quad (2.4.1)\]

\[
u_{i}^{rr}(x_0 = 0, t) = \sum_{\beta} p_{i}^{\beta} \cdot C_{0}^{\beta}(\omega \cdot t) , \quad (2.4.2)\]

where the boundary between the two media is set so that \( x_0 = 0 \). The displacements can be more conveniently written using the reflection and transmission coefficients \( R_{\alpha} \) and \( T_{\beta} \), respectively, as
\[ u_i^\prime(0,t) = p_i^0 \cdot A_0(\omega \cdot t) + \sum_\alpha p_i^\alpha \cdot R_\alpha \cdot A_0(\omega \cdot t), \quad (2.4.3) \]

\[ u_i^{\prime\prime}(0,t) = \sum_\beta p_i^\beta \cdot T_\beta \cdot A_0(\omega \cdot t). \quad (2.4.4) \]

The displacement continuity equation (2.1.1) can now be written in terms of the sum of the individual polarized waves

\[ \sum_\beta p_i^\beta \cdot T_\beta \cdot A_0(\omega \cdot t) - \sum_\alpha p_i^\alpha \cdot R_\alpha \cdot A_0(\omega \cdot t) = p_i^0 \cdot A_0(\omega \cdot t), \quad (2.4.5) \]

where equation (2.4.5) has been rearranged so that the known wave \( A_0 \) is on the right side. To determine the stress one utilizes the stress-strain relation for an acoustic wave travelling though an isotropic material, and the definition of strain for small acoustic displacements:

**Stress-Strain Relation:**

\[ T_{ij} = \sum_k \lambda \cdot \varepsilon_{kk} \cdot \delta_{ij} + 2 \cdot \mu \cdot \varepsilon_{ij}, \quad (2.4.6) \]

**Strain Definition:**

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right). \quad (2.4.7) \]

It should be noted that in equation (2.4.6) the summation convention for the repeated letter \( k \) has be written explicitly ….. Using the displacement definitions in equations (2.4.3) and (2.4.4) in (2.4.7) results in the strain in each medium as

\[ \varepsilon_{ij}^\prime = \frac{1}{2} \cdot \left( p_i^0 \cdot k_j^0 + p_j^0 \cdot k_i^0 \right) \cdot A_0 \cdot (\omega \cdot t - k_m^0 \cdot x_m) \]

\[ + \sum_\alpha \frac{1}{2} \cdot \left( p_i^\alpha \cdot k_j^\alpha + p_j^\alpha \cdot k_i^\alpha \right) \cdot B_\alpha (\omega \cdot t + k_m^\alpha \cdot x_m), \quad (2.4.8) \]

\[ \varepsilon_{ij}^{\prime\prime} = \sum_\alpha \frac{1}{2} \cdot \left( p_i^\beta \cdot k_j^\beta + p_j^\beta \cdot k_i^\beta \right) \cdot C_\alpha (\omega \cdot t - k_m^\beta \cdot x_m). \quad (2.4.9) \]
where, as one recalls from section 2.3, the prime denotes the derivative with respect to the necessary argument has been taken. From (2.4.8) and (2.4.9) the stress in each of the bulk media can be written out explicitly using (2.4.6)

\[
\begin{align*}
T_{ij}^1 &= \left[ \lambda \cdot p_n^0 \cdot k_n^0 \cdot \delta_{ij} + \mu \cdot \left( p_i^0 \cdot k_j^0 + p_j^0 \cdot k_i^0 \right) \right] \cdot A'_0\left( \omega \cdot t - k_m^0 \cdot \chi_m \right) \\
&\quad + \sum_{\alpha} \left[ \lambda \cdot p_n^\alpha \cdot k_n^\alpha \cdot \delta_{ij} + \mu \cdot \left( p_i^\alpha \cdot k_j^\alpha + p_j^\alpha \cdot k_i^\alpha \right) \right] \cdot B'_0\left( \omega \cdot t - k_m^\alpha \cdot \chi_m \right),
\end{align*}
\]  
(2.4.10)

\[
T_{ij}^2 = \left[ \lambda \cdot p_n^\beta \cdot k_n^\beta \cdot \delta_{ij} + \mu \cdot \left( p_i^\beta \cdot k_j^\beta + p_j^\beta \cdot k_i^\beta \right) \right] \cdot C'_0\left( \omega \cdot t - k_m^\beta \cdot \chi_m \right),
\]  
(2.4.11)

To simplify the expressions for stress, a new variable is defined:

\[
P_{ij}^\alpha = \lambda \cdot p_k^\alpha \cdot k_k^\alpha \cdot \delta_{ij} + \mu \cdot \left( p_i^\alpha \cdot k_j^\alpha + p_j^\alpha \cdot k_i^\alpha \right),
\]  
(2.4.12)

\[
P_{ij}^\beta = \lambda \cdot p_k^\beta \cdot k_k^\beta \cdot \delta_{ij} + \mu \cdot \left( p_i^\beta \cdot k_j^\beta + p_j^\beta \cdot k_i^\beta \right).
\]  
(2.4.13)

Using equations (2.4.12) and (2.4.13) in equations (2.4.10) and (2.4.11) respectively, the traction boundary condition equation (2.1.2) becomes

\[
\sum_{\beta} P_{iz}^\beta \cdot C'_0(\omega \cdot t) + \sum_{\alpha} P_{iz}^\alpha \cdot B'_0(\omega \cdot t) = P_{iz}^0 \cdot A'_0(\omega \cdot t),
\]  
(2.4.14)

where equation (2.4.14) has been rearranged so that the known wave is on the right side and the boundary \( \chi_0 = 0 \) is used. As written, equations (2.4.5) and (2.4.14) contain the function A and the derivatives A', B' and C' respectively. In order to solve equations (2.4.5) and (2.4.14) the traction continuity equation (2.4.14) should be integrated to remove the derivative. To do this, one integrates over time using the property

\[
\int A(\omega \cdot t) \, dt = \frac{1}{\omega} \cdot A(\omega \cdot t).
\]  
(2.4.15)

Using equation (2.4.15) on the traction continuity equation (2.4.14), and writing in terms of reflection and transmission coefficients yields
\[
\sum_{\beta} p_{iz}^\beta \cdot T\beta \cdot A_0(\omega \cdot t) + \sum_{\alpha} p_{iz}^{\alpha} \cdot R_{\alpha} \cdot A_0(\omega \cdot t) = p_{iz}^0 \cdot A_0(\omega \cdot t),
\]

(2.4.16)

It is noted that the factor \( A_0(\omega \cdot t) \) can be eliminated from the set of equations (2.4.5) and (2.4.16) making them totally time independent. To be able to solve equations (2.4.5) and (2.4.16) in a computer simulation, it is necessary to examine the set of equations (2.4.5) and (2.4.16) further. These equations can either be written using the \((x, y, z)\) set of components, or the \((\psi, \eta, z)\) set. The second set is chosen because then the set of equations (2.4.5) and (2.4.16) can be split into two separate sets of independent equations. The separability is easier to see by writing out equations (2.4.5) and (2.4.16) in matrix form and explicitly writing in the polarizations that are zero according to the definitions in equations (2.2.8) through (2.2.11),

\[
\begin{pmatrix}
-p_{\psi}^{L} & -p_{\psi}^{V} & p_{\psi}^{L} & p_{\psi}^{V} & 0 & 0 \\
-p_{\eta}^{L} & -p_{\eta}^{V} & p_{\eta}^{L} & p_{\eta}^{V} & 0 & 0 \\
p_{\omega}^{L} & p_{\omega}^{L} & p_{\omega}^{V} & p_{\omega}^{V} & 0 & 0 \\
p_{\omega}^{L} & p_{\omega}^{L} & p_{\omega}^{V} & p_{\omega}^{V} & 0 & 0 \\
0 & 0 & 0 & 0 & -p_{\eta} & p_{\eta} \\
0 & 0 & 0 & 0 & p_{\eta} & p_{\eta}
\end{pmatrix}
\begin{pmatrix}
R_{L} \\
R_{V} \\
T_{L} \\
T_{V} \\
R_{H} \\
T_{H}
\end{pmatrix}
= \begin{pmatrix}
p_{\psi}^{0} \\
p_{\eta}^{0} \\
p_{\omega}^{0} \\
p_{\omega}^{0} \\
p_{\eta}^{0} \\
p_{\eta}^{0}
\end{pmatrix}.
\]

(2.4.17)

Equation (2.4.17) has been rearranged in matrix form to show the separability more easily. The displacement continuity equations (2.4.5) are placed on rows 1, 2 and 5, while the traction continuity equations (2.4.16) are placed on rows 3, 4 and 6. Examining equation (2.4.17) of equations (2.4.5) and (2.4.16) shows that the boundary equations can be separated into two sets of equations, the first set using the \(\psi\) and \(z\) axes to find the coefficients for the longitudinal (L) and shear vertical waves (V):
\[ p_i \mathbf{l}_i \cdot T_L + p_i \mathbf{l}_i \cdot T_V - p_i \mathbf{h}_i \cdot R_L - p_i \mathbf{h}_i \cdot R_V = p_i^0, \quad (i = \psi, z) \] (2.4.18)

\[ p_{iz} \mathbf{l}_{iz} \cdot T_L + p_{iz} \mathbf{l}_{iz} \cdot T_V + p_{iz} \mathbf{h}_{iz} \cdot R_L + p_{iz} \mathbf{h}_{iz} \cdot R_V = p_{iz}^0, \] (2.4.19)

and the second set using the \( \eta \) axis to find the coefficients for the shear horizontal (H) waves:

\[ p_{\eta} \mathbf{l}_{\eta} \cdot T_H - p_{\eta} \mathbf{h}_{\eta} \cdot R_H = p_{\eta}^0, \quad (i = \eta) \] (2.4.20)

\[ p_{\eta z} \mathbf{l}_{\eta z} \cdot T_H + p_{\eta z} \mathbf{h}_{\eta z} \cdot R_H = p_{\eta z}^0. \] (2.4.21)

2.5: Second Order Approximation: Adding the Thin Layer

The second order of the perturbation adds in the changes in the acoustical properties of the overall structure due to the addition of the thin layer of material. The development of the equations will follow much of the same reasoning as in section 2.4. In this case now one must use the boundary equations (2.1.3) and (2.1.4) and the entire expanded waveform from equation (2.3.8) and (2.3.9). Using the expanded waveforms in equations (2.3.8) and (2.3.9) for the displacement, the left side of the displacement boundary value equations (2.1.3) becomes

\[ \sum_{\beta} p_i^\beta \cdot (T_{\beta} \cdot A_0 + \phi_\beta \cdot k_\beta \cdot d \cdot T_{\beta} \cdot A_0') - \sum_{\alpha} p_i^\alpha \cdot (R_{\alpha} \cdot A_0 + \phi_\alpha \cdot k_\alpha \cdot d \cdot R_{\alpha} \cdot A_0') - p_i^0 \cdot A_0 = \text{RHS}_{-1}, \]  

(2.5.1)

where RHS\_1 is the right hand side of equation (2.1.3) and will be dealt with later.

Because the \( d = 0 \) case is valid for the values of \( T_{\beta} \) and \( R_{\alpha} \), one also knows from rearranging equation (2.4.5):
\[ \sum_{\beta} p_{i}^{\beta} \cdot T_{\beta} - \sum_{\alpha} p_{i}^{\alpha} \cdot R_{\alpha} - p_{i}^{0} = 0. \]  \hspace{1cm} (2.5.2)

Equation (2.5.2) allows the first level of the perturbation to be removed from the displacement equation (2.5.1) producing a displacement equation made fully of second order factors

\[ \sum_{\beta} p_{i}^{\beta} \cdot T_{\beta} \cdot \phi_{\beta} \cdot k_{z}^{\beta} - \sum_{\alpha} p_{i}^{\alpha} \cdot R_{\alpha} \cdot \phi_{\alpha} \cdot k_{z}^{\alpha} = \text{RHS}_{2i} \cdot \frac{1}{d \cdot \Lambda''(\omega \cdot t)}. \]  \hspace{1cm} (2.5.3)

The definition of traction needed for equation (2.1.4) follows the same derivation as in equations (2.4.6) through (2.4.14), but now using the expanded wave forms of equations (2.3.8) and (2.3.9). One must remember though that during the process the traction equation was integrated with respect to time using equation (2.4.15), and that also there was an overall sign change due to the negative sign in each of the stress components. Taking care of all this, and also removing the first order solution using to equation (2.4.16), the left hand side of the second order traction equation is

\[ \sum_{\beta} p_{iz}^{\beta} \cdot T_{\beta} \cdot \phi_{\beta} \cdot k_{z}^{\beta} + \sum_{\alpha} p_{iz}^{\alpha} \cdot R_{\alpha} \cdot \phi_{\alpha} \cdot k_{z}^{\beta} = - \int \text{RHS}_{2i} d(\omega \cdot t) \cdot \frac{1}{d \cdot \Lambda''(\omega \cdot t)}, \]  \hspace{1cm} (2.5.4)

where \( \text{RHS}_{2i} \) is the right hand side of equation (2.1.4). With the left hand side of the boundary equations (2.1.3) and (2.1.4) computed to form equations (2.5.3) and (2.5.4), the right hand side of the equations (2.5.3) and (2.5.4) need to be determined:

\[ \frac{\text{RHS}_{1i}}{d} = 2 \cdot e_{iz}^{\text{int}} - e_{zz}^{\text{int}} \cdot \delta_{iz} \cdot \frac{d}{dx_{1}} u_{z}^{l} \cdot \delta_{i1} + \frac{d}{dx_{2}} u_{z}^{l} \cdot \delta_{i2}, \]  \hspace{1cm} (2.5.5)

\[ \int \frac{\text{RHS}_{2i}}{d} d(\omega \cdot t) = \int \left( \rho \cdot \frac{d^{2}u_{i}^{l}}{dt^{2}} + \frac{d}{dx_{1}} T_{i1}^{l} + \frac{d}{dx_{2}} T_{i2}^{l} \right) d(\omega \cdot t), \]  \hspace{1cm} (2.5.6)
where equations (2.5.5) and (2.5.6) contain the right hand sides of the second order boundary equations (2.1.3) and (2.1.4) respectively. In equation (2.5.5) the strain of the interface can be computed using the acoustic stress-strain relation for an anisotropic material

$$

\varepsilon_{ij}^{\text{int}} = s_{ijkl}^{\text{int}} \cdot T_{kl}^{\text{int}},

$$

(2.5.7)

where $s_{ijkl}$ is the compliance tensor and contains the necessary information to incorporate the anisotropic properties of the thin layer when needed. For an isotropic system equation (2.5.7) simplifies to

$$

\varepsilon_{ij}^{\text{int}} = -\lambda^{\text{int}} \cdot T_{kk}^{\text{int}} \cdot \delta_{ij} + \frac{(2 \cdot \mu^{\text{int}} + 3 \cdot \lambda^{\text{int}}) \cdot T_{ij}^{\text{int}}}{2 \cdot \mu^{\text{int}} \cdot (2 \cdot \mu^{\text{int}} + 3 \cdot \lambda^{\text{int}})}

$$

(2.5.8)

Because the stress of the internal layer is unknown the approximation $T_{kl}^{l} = T_{kl}^{\text{int}}$ for non $T_{iz}$ components must be used to obtain

$$

\varepsilon_{ij}^{\text{int}} = -\lambda^{\text{int}} \cdot T_{kk}^{l} \cdot \delta_{ij} + \frac{(2 \cdot \mu^{\text{int}} + 3 \cdot \lambda^{\text{int}}) \cdot T_{ij}^{l}}{2 \cdot \mu^{\text{int}} \cdot (2 \cdot \mu^{\text{int}} + 3 \cdot \lambda^{\text{int}})}.

$$

(2.5.9)

Note that equation (2.5.9) implicitly uses components of the stress, which are discontinuous at the interface (the components other than $T_{iz}$). The stress in the first medium can be found using the first order solutions in equation (2.4.10), leaving the last two terms of equation (2.5.5) to be computed. The solution for the derivatives in equation (2.5.5) is

$$

\frac{d}{dx} u_{z}^{l} = k_{j}^{0} \cdot \left( P_{z}^{0} \cdot \sum_{\alpha} P_{z}^{\alpha} \cdot R_{\alpha} \right) \cdot A_{0}^{l}.

$$

(2.5.10)
Next, one needs to compute the various portions of the right hand side of the traction equation (2.5.6), looking at each piece in turn:

\[
\frac{d^2 u_i}{dt^2} q_i = \omega \cdot \frac{du_i}{dt} = \omega^2 \cdot \left( p_i + \sum_{\alpha} p_{i\alpha} \cdot R_{\alpha} \right) \cdot A'_0,
\]

(2.5.11.a)

\[
\int \frac{d}{dx_j} T_{ij} q_j = \int k_j \cdot \left( p_{ij} + \sum_{\alpha} p_{ij\alpha} \cdot R_{\alpha} \right) A'_0 \cdot q_i = k_j \cdot \left( p_{ij} + \sum_{\alpha} p_{ij\alpha} \cdot R_{\alpha} \right) \cdot A'_0.
\]

(2.5.11.b)

It is noted the properties in equations (2.3.6) and (2.3.7) have been used in equations (2.5.10) and (2.5.11b), respectively, thus making $A'$ a derivative with respect to $\omega \cdot t$.

Combining the equations (2.5.10) and (2.5.11) with (2.5.3) and (2.5.4) respectively, and along including the necessary sign in the right hand side of the traction equation (2.5.3) yields the second order boundary equations

\[
\sum_{\beta} p_{i\beta} \cdot T_{\beta} \cdot \phi_\beta \cdot k_z^\beta - \sum_{\alpha} p_{i\alpha} \cdot R_{\alpha} \cdot \phi_\alpha \cdot k_z^\alpha =
\]

(2.5.12)

\[
\frac{2 \cdot \epsilon_{iz}^{\text{int}} - \epsilon_{zz}^{\text{int}} \cdot \delta_{iz}}{A'_0} - k_i \cdot \left( -p_{iz}^0 + \sum_{\alpha} p_{iz\alpha} \cdot R_{\alpha} \right) \cdot (1 - \delta_{iz}),
\]

\[
\sum_{\beta} p_{iz\beta} \cdot T_{\beta} \cdot \phi_\beta \cdot k_z^\beta - \sum_{\alpha} p_{iz\alpha} \cdot R_{\alpha} \cdot \phi_\alpha \cdot k_z^\alpha =
\]

(2.5.13)

\[-p^{\text{int}} \cdot \omega^2 \cdot \left( p_i^0 + \sum_{\alpha} p_{i\alpha} \cdot R_{\alpha} \right) + k_1 \cdot \left( p_{i1}^0 + \sum_{\alpha} p_{i1\alpha} \cdot R_{\alpha} \right) + k_2 \cdot \left( p_{i2}^0 + \sum_{\alpha} p_{i2\alpha} \cdot R_{\alpha} \right).
\]

The notation 1, 2 in the set of equations (2.5.12) and (2.5.13) can refer to either the x, y components from the (x, y, z) set of axes defined by orientation of the thin layer, or $\psi, \eta$ components in the ($\psi, \eta, z$) system defined by the plane of incidence (Figure 2.1.1).
Comparing the left side of the second order boundary condition equations (2.5.12) and (2.5.13) with the first order boundary condition equations (2.4.5) and (2.4.6) shows that the reflection ($R_{\alpha}$) and transmission ($T_{\beta}$) coefficients have essentially been replaced with the phase factors $T_{\beta} \cdot \phi_{\beta} \cdot k_z^\beta$ and $R_{\alpha} \cdot \phi_{\alpha} \cdot k_z^\alpha$. Thus, solving for the phase factors $T_{\beta} \cdot \phi_{\beta} \cdot k_z^\beta$ and $R_{\alpha} \cdot \phi_{\alpha} \cdot k_z^\alpha$ will yield the same 8x8 matrix on the left hand side as produced in the first order solution (equation 2.4.17). Equations (2.5.12) and (2.5.13) can then again be split into two sets of two equations. The first set describes the longitudinal and shear vertical waves using the $\psi$ and $z$ indices. The second set describes the shear horizontal wave using the $\eta$ index. Solving for the phase factors $T_{\beta} \cdot \phi_{\beta} \cdot k_z^\beta$ and $R_{\alpha} \cdot \phi_{\alpha} \cdot k_z^\alpha$ also yields the entire factor that needs to be known to calculate the waveform in equations (2.3.8) and (2.3.9). If one solved for just the parameter $\phi_{\alpha}$ or $\phi_{\beta}$, the reflection and transmission coefficients would still need to be multiplied back into the solution. Also there exist some cases, like that of two identical bulk media, where when calculated separately the reflection coefficient ($R_{\alpha}$) is zero, the phase coefficient ($\phi_{\alpha}$) is infinitely large, but when calculated together the overall phase factor ($R_{\alpha} \cdot \phi_{\alpha} \cdot k_z^\alpha$) is a non-zero value.

2.6: First Extension of Simulation: Adding Nonlinearity to the Thin Layer

The acoustic nonlinearity of the system is added to the thin layer of material between the two half-spaces. Because the nonlinearity is in the thin layer, the first approximation (where there is no layer) will remain totally unchanged. The second level of the approximation does include information about the thin layer, and thus will have to
be altered. Overall, the parameters of the second order equations (2.5.12) and (2.5.13), and expanded waveforms of equations (2.3.8) and (2.3.9), that have to do with the thin layer are

\[ d, \varepsilon_{ij}^{\text{Int}}, T_{ij}^{\text{Int}}, \rho^{\text{Int}}, \]  

(2.6.1)

where the strain in the layer is related to the stress of the surrounding media using the approximation \( T_{kl}^{l} = T_{kl}^{\text{Int}} \) (for non \( T_{iz} \) components) and the compliance coefficients

\[ \varepsilon_{ij}^{\text{Int}} = s_{ijkl}^{\text{Int}} \cdot T_{kl}^{l}. \]  

(2.6.2)

To correspond to the classical nonlinear theory (for use with most atomic elastic materials) additional terms can be added to the stress-strain relation in terms of a power series

\[ \varepsilon_{ij}^{\text{Int}} = s_{ijkl}^{\text{Int}} \cdot T_{kl}^{l} \cdot \left(1 + \beta \cdot T_{kl} + \chi \cdot T_{kl}^2 + \ldots\right). \]  

(2.6.3)

To correspond to non-classical nonlinearity (typically corresponding to nonlinear mesoscopic elastic materials [NME]) an additional term is added

\[ \varepsilon_{ij}^{\text{Int}} = s_{ijkl}^{\text{Int}} \cdot T_{kl}^{l} \cdot \left(1 + \beta \cdot T_{kl} + \chi \cdot T_{kl}^2 + \ldots\right) + A(\varepsilon, u). \]  

(2.6.4)

One notes that typically the nonlinear equations (2.6.3) and (2.6.4) would be expressed as a solution for the stress, but due to the development of the theory the inverse of the relation is necessary.

Both the classical [equation (2.6.3)] and non-classical [equation (2.6.4)] nonlinearities add an additional time dependent factor in equation (2.5.12). If one separates the phase coefficient (\( \phi \)) into a linear and nonlinear solution

\[ \phi = \phi_{\text{Linear}} + \phi_{\text{Nonlinear}(t)}, \]  

(2.6.5)
where the linear solution ($\phi_{\text{linear}}$) is still a valid solution for the phase ($\phi$) in equations (2.5.12) and (2.5.13), and thus a set of equations for the nonlinear phase coefficients ($\phi_{\text{nonlinear}}$) can be obtained. In the case of classical nonlinearity this remaining nonlinear coefficient to be found by using

$$
\sum_{\beta} p_{i}^{\beta} \cdot T_{\beta} \cdot \phi_{\beta} \cdot k_{z}^{\beta} - \sum_{\alpha} p_{i}^{\alpha} \cdot R_{\alpha} \cdot \phi_{\alpha} \cdot k_{z}^{\alpha} = \frac{2 \cdot s_{izkl}^{\text{int}} - s_{zzkl}^{\text{int}}}{A'_{0}} \cdot T_{kl}^{l} \cdot (1 + \beta \cdot T_{kl} + \chi \cdot T_{kl}) \cdot (2.6.6)
$$

Because the only time dependent part is the nonlinear addition (note that the factor $A'$ removes the time dependence from $T_{kl}^{l}$), this can be factored out of equation (2.6.6), and one should expect a solution for the nonlinear phase factor in terms of

$$
\phi_{\text{nonlinear}}(t) = \phi_{NL} \cdot F(NL),
$$

where ($\phi_{NL}$) is a constant value, and $F(NL)$ is a nonlinear function. One can now write the waveform as a three-level expansion

$$
C_{\beta}(\omega \cdot t) = T_{\beta} \cdot A_{0}(\omega \cdot t) + T_{\beta} \cdot \phi_{L}^{\beta} \cdot k_{z}^{\beta} \cdot d \cdot A'_{0}(\omega \cdot t) + T_{\beta} \cdot \phi_{L}^{\beta} \cdot k_{z}^{\beta} \cdot d \cdot F(NL) \cdot A'_{0}(\omega \cdot t). \quad (2.6.8)
$$

Written out verbally, the physical relevance of each section in equation (2.6.8) is more apparent:

Waveform = Two Media Solution + Thin Layer Phase Correction + Nonlinear Term. \quad (2.6.9)

### 2.7 Second Extension of Simulation: Adding Anisotropy to the Thin Layer

As previously mentioned in section 2.5 the information about the anisotropic properties of the thin layer is contained in the compliance tensor ($s_{ijkl}$) used to relate the acoustical properties of stress and stain.
\[ \varepsilon_{ij}^{\text{int}} = s_{ijkl}^{\text{int}} \cdot T_{kl}^{\text{int}}. \]

The compliance coefficients contain the information about the material structure and determine the velocity of each polarized wave and the direction of the polarization in the axis of the material. By setting the z-axis perpendicular to the thin layer and using this orientation to produce the boundary value equations (2.1.2) through (2.1.4), the cut of any crystal structure used in the thin layer naturally corresponds to a Z-cut material. It is possible to use materials with other cuts (e.g. X-cut, Y-cut), but the compliance matrix will need to be rotated into this particular frame of reference. In addition some anisotropic materials are isotropic about the z-axis (transversely isotropic) and will not show any anisotropic changes such as mode conversion when placed in the configuration defined in Figure 2.1.1. However, the material can be orientated on a different set of axes and in turn the compliance matrix can again be rotated and the anisotropic changes will appear. This particular situation will be used in section 3.6 when a hexagonal system, which is symmetric about the z-axis, is rotated so that the symmetry is in the x-y plane. It is noted that the anisotropic material is defined on the (x, y, z) system (Figure 2.1.1) and the final solution to first order and second order boundary equations, (2.4.18) through (2.4.21), and (2.5.12) and (2.5.13), respectively, were placed in the (\(\psi, \eta, z\)) system in order to be solvable via computer simulation. To transform between the two different coordinate systems so that the anisotropy can be accounted for, one uses the rotations

\[ Q_y = Q_x \cdot \cos(\phi) + Q_y \cdot \sin(\phi), \]  
\[ Q_\eta = -Q_x \cdot \sin(\phi) + Q_y \cdot \cos(\phi), \] 

(2.7.1)  
(2.7.2)
Chapter 3: Simulation

3.1: Introduction

Simulations based on the theory of Chapter 2 allow for a wide variation of situations to be examined, including one to examine a variety of bulk material configurations as well as changes due to incident and polar angle, frequency, thin layer size, and the relative properties of the thin layer with respect to the bulk material. Limitations of the simulation include requiring an incident ultrasonic plane wave, homogeneous materials, and a thin layer of thickness much less than an acoustic wavelength. However, even with these limitations the simulations still apply to many practical scenarios.

Two possible configurations exist with respect to the bulk materials. In the first case, the bulk materials possess different velocities, densities, or impedances. If the second material has greater impedance than the first, the reflected wave is phase inverted in displacement. If the second material has greater velocity than the first, critical angles and evanescent waves can occur, but these phenomena will not be considered in these simulations. In the second case the two bulk materials are identical. When there is no layer (or if the layer is identical to the media), the incident wave passes through the media unchanged and becomes a transmitted wave of the same polarization. When the thin layer of a different substance is added to this case, the change in the sample will be easier to detect because any additional waves such as reflected or mode converted waves will solely be due to the existence of the thin layer.
3.2: Two Half-Spaces (No Layer)

Before presenting the results where a thin layer is included, it is advantageous to consider the limiting case in which there is no layer. This case provides a base set of results to refer back to when the more complex cases (e.g. thin layer, nonlinearity, anisotropy) are considered. The results also verify that the simulation is functioning correctly in the first order term of the perturbation expansion.

3.2.1: Two Different Bulk Media

For the case of two different materials an aluminum-brass sample is chosen because the second material (brass) has higher impedance but lower velocity than that of the aluminum (See Appendix A for each material's physical properties). This arrangement allows for phase inversion on the reflected waves but eliminates the possibility of critical angles for incident longitudinal waves. The simulation is set with an incident wave of longitudinal polarization with a frequency of 0.1 MHz. Examining the time domain output (displacement versus time) from the incident longitudinal wave at 40 degrees incidence (Figure 3.2.1), the phase inversion of the reflected wave can be seen as well as all the possible mode converted shear vertical waves. The amplitude change with incident angle (Figure 3.2.2) can also be examined; note that in Figure 3.2.2 the maximum amplitude is taken and the phase inversion is ignored. As expected, there is no mode conversion at normal incidence, but mode converted waves are possible at every other angle of incidence, reaching maximum amplitude at about 55 degrees of incidence. At about this same angle of incidence the reflected longitudinal wave crosses the point where it is no longer phase inverted. It is useful, for the sake of verification of the data, to also examine the relative intensity of each reflected and transmitted wave and ensure the
total intensity is constant as required by conservation of energy. Using the details found in Appendix B, the relative intensities can be found and are plotted in Figure 3.2.3. As expected the total is a constant unit value indicating energy is conserved. Finally it is also possible to take the Fourier transform of the time domain data to present a frequency spectrum change with angle of incidence, but in this case its waveform is made up of only the first harmonic of size proportional to the wave's amplitude. Thus the frequency spectrum is only of the first harmonic that changes identically to Figure 3.2.2.

Figure 3.2.1: Time domain output from an incident longitudinal wave (frequency 0.1 MHz, angle of incidence 40 degrees) in an aluminum-brass sample with no thin layer.
Figure 3.2.2: Change in amplitude of the various polarized waves with angle of incidence in an aluminum-brass sample. Note that the amplitude is taken as an absolute measure ignoring the phase of the waves.

Figure 3.2.3: Change in the relative intensity of the various polarized waves with angle of incidence in an aluminum-brass sample.
3.2.2: Two Identical Bulk Media

For the case of two identical materials, an acrylic sample is chosen (See Appendix A for the material's physical properties), as acrylic is the type of material used in the experimental work in Chapter 4. As expected, the simulation shows no surprising results, producing only a transmitted wave with the same amplitude as the incident wave for all angles of incidence.

3.3: A Thin Linear Isotropic Layer Between Two Half-Spaces

Next the addition of a thin layer between the two bulk media is considered. In reality this could be a thin layer of adhesive between two bonded materials, the bonding region between a bulk material and the bulk of adhesive in a larger structure, an interface between two metals, or a small flawed area in a bulk material. The same bulk materials as used in the previous section are used here, with the additional materials used for the thin layer. As previously mentioned the various materials' physical properties can be found in Appendix A.

3.3.1: Two Different Bulk Media

Upon adding a thin layer of epoxy (thickness 50 micrometers) between the two metals, and running the simulation again using the same longitudinal polarized incident wave with frequency 0.1 MHz, a small phase shift is seen in the reflected and transmitted waves with respect to the incident wave (angle of incidence 40 degrees) in the time domain output (displacement amplitude versus time) (Figure 3.3.1). With closer inspection of the time domain output (Figure 3.3.2), the phase change can be seen more clearly. The reflected and transmitted waves no longer match up perfectly with the
incident wave as it goes through an area of zero displacement (compare with Figure 3.2.1). Physically, this phase shift is due to the time delay caused by the acoustic wave propagating through the thin layer of epoxy. Mathematically, this phase change is due to the second term in the perturbation expansion in equations (2.3.8) and (2.3.9),

\[ B_\alpha = R_\alpha \cdot A_0 + \phi_\alpha \cdot k_z^\alpha \cdot d \cdot R_\alpha \cdot A_0', \]

\[ C_\beta = T_\beta \cdot A_0 + \phi_\beta \cdot k_z^\beta \cdot d \cdot T_\beta \cdot A_0'. \]

The second order terms \( T_\beta \cdot \phi_\beta \cdot k_z^\beta \cdot d \cdot A_0' \) and \( R_\alpha \cdot \phi_\alpha \cdot k_z^\alpha \cdot d \cdot A_0' \) change due to the thickness of the layer \((d)\), the angle of incidence, and the relative material properties (this information is contained within \( \phi \)). Plotting only the change in the phase terms \( R_\alpha \cdot \phi_\alpha \cdot k_z^\alpha \cdot d \) and \( T_\beta \cdot \phi_\beta \cdot k_z^\beta \cdot d \) with respect to the angle of incidence (Figure 3.3.3), it is seen that all but the reflected longitudinal phase terms qualitatively follow a similar curve as in Figure 3.2.2. The similarity between the shape of the phase term curves (Figure 3.3.3) and the shape of the amplitude curves (Figure 3.2.2) with angle of incidence, means the dependence due to angle of incidence in the reflection and transmission coefficients from the two bulk media is stronger than the angular dependence located in the phase coefficient \((\phi)\). The dependence on angle of incidence in the phase coefficient is contained within the stress or strain of the thin layer added in with the second order boundary conditions [equations (2.1.3) and (2.1.4)]. One notes that there is also the possibility that the reflection and transmission coefficients, and the phase coefficient have dependence on angle of incidence. The phase term for the reflected longitudinal wave (Figure 3.3.3) does not follow a similar curve when compared to the reflected longitudinal coefficient (Figure 3.2.2). The curve begins to differ significantly at
approximately 55 degrees, where it changes the sign of its concavity and approaches zero instead of continuing to increase in size. As the other phase terms also approach zero for large angles of incidence, it is possible that at large angles the changes due to the thin layer are able to overcome the strong angular dependence between the two bulk media.

The small values of the phase factors seen in Figure 3.3.3 are not able to make significant changes to the size of the amplitudes of the reflected and transmitted waves. The amplitude change with incident angle varies little in comparison to the case when there is no layer (Figure 3.3.4). The most noticeable changes are at the positions of maximum amplitude for the shear vertical waves (Figures 3.3.4b and 3.3.4c), and in the normal incidence case for the reflected longitudinal wave (Figure 3.3.4a). In the case of the transmitted longitudinal wave (not shown), the change is too small to appear when plotted. Although the maximum amplitude does vary in size, its position does not, the combination of the small phase factors and high angular dependence of the bulk media overwhelm the changes due to the addition of the thin layer. As in Section 3.2.1, the change in the amplitude of the frequency spectrum for various angles of incidence will be identical to the amplitude change with incident angle (Figure 3.3.3), because thus far the only time dependent portions of the perturbation expansion in equations (2.3.8) and (2.3.9) are the waveform A and its derivative A'. Both have the same frequency, and thus no harmonics will be created.

Finally it is useful to investigate the relative intensities of the various reflected and transmitted waves, as well as the total intensity of the acoustic waves. By examining the amount of deviation of the total intensity (equivalently total energy) according to
perturbation theory from the state of energy conservation one can obtain a measurement of the simulations deviation from the exact solution. The change in relative intensities of the acoustic waves are found to be affected in a similar fashion to their amplitude counterparts producing results nearly identical to those found in Figure 3.2.3. Examining the total intensity (Figure 3.3.5) the small changes in each polarization are found to cause the total intensity to deviate approximately 0.9% from an energy conserving system. It is interesting to note that at any possible angle of incidence the energy is no longer conserved in the simulation, and that the perturbation theory in this case is adding energy to the system. In the next section (Section 3.3.2) it will be discovered that the lack of energy conservation, although small, is present in even the simplest case. This lack of energy conservation is expected from the theory the perturbation of the boundary conditions [equations (2.5.12) and (2.5.13)] do not explicitly constrict energy to be conserved. The thin layer is also expected to produce additional polarizations that are not created when the thin layer is not present, as well as producing the expected polarizations due to the two bulk media. As energy conservation has not explicitly been required it is highly unlikely that the amplitudes of the expected polarizations would balance with the amplitudes of the additional polarizations to produce energy conservation.

Although this particular case dealt with a thin layer whose density, velocity and impedance were lower than the bulk media (the density and velocity of aluminum is roughly 2.5 times that of epoxy), a thin layer with density, velocity and impedance higher than the bulk media also yields similar results due to the high dependencies in the angle of incidence. In fact in testing the simulation with an artificial material with density and velocity roughly twice as large as the density and velocity of aluminum, the change in the
results were even smaller than with a thin layer of epoxy. In addition, allowing the velocity and density of the thin layer to lie between those of the aluminum and brass produces the expected result of a nearly negligible phase term, as the thin layer at this point can be approximately considered as one of the bulk materials.

Figure 3.3.1: Time domain output from an incident longitudinal wave (frequency 0.1 MHz, angle of incidence 40 degrees) in an aluminum-brass sample where the thin layer of epoxy is 50 μm.
Figure 3.3.2: A closer examination of the time domain output in Figure 3.3.1 showing the delay in phase reflected and transmitted wave with respect to the incident wave in the aluminum-brass sample.

Figure 3.3.3: Change of the phase terms with angle of incidence in the aluminum-epoxy-brass sample.
Figure 3.3.4: Comparison of the amplitude in the case of no layer to the case of a thin layer of epoxy (thickness 50 μm) of the aluminum-brass sample.

(a) Amplitude of reflected longitudinal wave

(b) Amplitude of reflected shear vertical wave
Figure 3.3.4: Comparison of the amplitude in the case of no layer to the case of a thin layer of epoxy (thickness 50 \( \mu \text{m} \)) of the aluminum-brass sample.
(c) Amplitude of transmitted shear vertical wave

Figure 3.3.5: Total relative intensity in the aluminum-epoxy-brass sample.
3.3.2: Two Identical Bulk Media

One of the more interesting situations to examine is the case in which the two bulk media are identical. In this situation the first order results (results where there is no thin layer) call for full transmission of the incident wave, no reflected wave, and no mode conversion. When the thin layer is considered, any change in the reflected wave or any mode conversion will be fully caused by the addition of the thin layer.

Consider first a system where the middle layer is identical to the bulk media. This case, while trivial, provides an opportunity to verify the output of the simulation, the perturbation expansion term, and yields facts about the errors introduced by the perturbation approach. Using an incident longitudinal wave of frequency 1.0 MHz propagating through a fully acrylic sample with a thin layer of 50 micrometers, one obtains the information about the amplitude and phase factor in the transmitted longitudinal wave. The phase factor $T_\beta \cdot \phi_\beta \cdot k_z^\beta$ can be shown to exactly fit a cosine curve of the form $-k^0 \cdot \cos(\theta_1)$ (Figure 3.3.6). This fit corresponds to the overall property of $k_z^\beta = k^\beta \cdot \cos(\theta_\beta) = k^0 \cdot \cos(\theta_1)$, found by combining equation (2.2.7) with the fact both media are identical and will have the same wave number and angles of incidence and transmission, a transmission amplitude coefficient of $T_L = 1$, and results in the factor $\phi_L^{ll} = -1$ for all angles of incidence. The factor $\phi_L^{ll}$ being constant for all angles of incidence in the case of three identical media is expected, because changing the angle of incidence should produce no other alteration than the phase delay due to the thin layer.
Thus the result should be consistent with the case of normal incidence for two identical media, where it can be calculated that $\phi_{\parallel} = -1$.

One might expect the amplitude of the transmitted longitudinal wave over the various angles (Figure 3.3.7) to follow a curve proportional to $k_z$ from a quick examination of the perturbation expansion [equation (2.3.9)]

$$C_{\beta} = T_{\beta} \cdot A_0 + \phi_{\beta} \cdot k_z^\beta \cdot d \cdot T_{\beta} \cdot A'_0.$$  

But this ignores the fact that the phase component's angular dependence alters the phase delay of the transmitted wave as the angle of incidence changes. These different phase delays cause the maximum amplitude to be located at different times with respect to different angles of incidence, in other words the time of the maximum amplitude is dependent on angle of incidence. This dependence can be shown mathematically by examining the location of the maximum, or minimum of the transmitted wave in time, obtained by solving

$$\frac{d}{d(\omega \cdot t)} C_{\beta} = T_{\beta} \cdot A'_0 + \phi_{\beta} \cdot k_z^\beta \cdot d \cdot T_{\beta} \cdot A''_0 = 0,$$  \hspace{1cm} (3.3.1)

For sinusoidal waveforms there are no point of inflections, thus $C_{\beta}$ is a maximum, or minimum when

$$\frac{A'_0}{A''_0} = -\phi_{\beta} \cdot k_z^\beta \cdot d.$$

In the case of an incident wave proportional to $\sin(\omega \cdot t)$, equation (3.3.2) is

$$\cot(\omega \cdot t) = \phi_{\beta} \cdot k_z^\beta \cdot d.$$  \hspace{1cm} (3.3.3)
Thus the location in time of maximum amplitude is dependent on the phase factor of the thin layer. As the phase change is dependent on angle of incidence, the location in time of the maximum amplitude in turn is dependent on angle of incidence. Thus one cannot simply consider the values of $A$ and $A'$ constant as the angle of incidence changes.

The transmitted amplitude is found to deviate from the actual theoretical result by a small margin. These deviations are of course due to the approximate nature of perturbation theory, if the third order of the perturbation expansion was explored it is expected that this discrepancy would be reduced further. In fact the transmitted longitudinal wave's maximum amplitude (Figure 3.3.7) only deviates 0.65% from the expected solution of a constant value of one micrometer. It is expected that this 0.65% deviation will be the approximate limit of the accuracy of the simulations. By examining the transmitted amplitude wave it is apparent that energy is not conserved even in this most simple of case, though when plotted the maximum deviation is found to be only 1.3% from a conserving system. The failure of energy conservation at this most simple case indicates that energy conservation has not been taken into account by the perturbation expansion of the boundary conditions [equations (2.5.12) and (2.5.13)], or alternatively the waveforms []. Due to the small deviation from an energy conserving system, and the complexity of adding the perturbed information to the intensity conservation equation (see appendix C, equation C.1) the theory is still very useful as it continues to obtain results within 1% of the expected results.
Figure 3.3.6: Transmitted longitudinal phase factor versus various incident angles for a system of three identical materials, showing an exact fit for the phase expansion.

Figure 3.3.7: Transmitted longitudinal amplitude versus various incident angles for a system of three identical materials.
By changing the various properties of the thin layer (velocity, density, and impedance), with respect to the two identical bulk media and analyzing the results, it is possible to find a set of characteristic curves where each curve's overall shape can be associated with a set of relative parameters. Note that these are not universal curves, where all data fits exactly when the appropriate proportions are used. Instead these curves define overall characteristics such as the existence and approximate locations of maximums, minimums, zeros, slope and concavity. For the case of an incident longitudinal plane wave of frequency 1.0 MHz, the reflected longitudinal wave forms characteristic curves in terms of amplitude versus angle of incidence, and these various curves are most easily arranged with respect to the relative density and relative impedance of the thin layer to the bulk material (Figures 3.3.8(a) though (d)). Because impedance, velocity and density are all interrelated, the characteristic curves could also be arranged with respect to velocity, but doing so simply leads to a more complex arrangement. In Figures 3.3.8(a) though (d) a relative sign convention has been established by considering the sign of the first peaks of the reflected longitudinal waves considered. Of note is a sign inversion in the curve's shapes in Figures 3.3.8(a) relative to (d) and (b) relative to (c). The sign dependence in the relative impedance is expected and can be found from the examination of the reflection coefficient in both the multilayer theory and the reflected phase term from the theory presented in Chapter 2, where the result for the specific case of normal incidence and two identical media is given by [2]

\[
R = \frac{Z - Z^{\text{int}}}{2 \cdot Z},
\]

(3.3.4)
Reflected Phase \( \phi_{\alpha} \cdot R_{\alpha} \cdot k_{Z} = k_{Z}^{\text{Int}} \cdot \left[ \frac{Z^2 - (Z^{\text{Int}})^2}{Z^{\text{Int}} \cdot Z} \right] \). (3.3.5)

The fact that the sign dependence is also tied to density is likely explained by the interrelation between impedance, density, and velocity. If the phase term were fully calculated it is highly likely that some of the dependence on velocity would be removed due to the factor of the wave number \( k \). Also of note is the change in phase in Figure 3.3.8(b) and (c) creating a point where there is no reflected longitudinal wave. The position of this zero is found to be quite stable with changes in the relative parameters and is thus not very suitable for material characterization of the thin layer.

The amplitude versus angle of incidence curves for the reflected shear vertical (Figures 3.3.9(a) through (d)) and transmitted shear vertical (Figures 3.3.10 (a) through (d)) waves created from the same incident longitudinal wave also form a set of characteristic curves with respect to the density and impedance of the thin layer relative to the bulk material. The reflected shear wave data again features inversion of the curve's shapes (Figures 3.3.9(a) relative to (d) and (b) relative to (c)), and in addition the position of the maximum (or minimum) amplitude and the position of the zero point (if it exists) are now dependent on the relative material properties. The transmitted shear wave does not produce a clear inversion as seen in previous polarizations. Figure 3.3.10(c) is found to contain sign inversion while Figure 3.3.10(b) does not, and in addition Figure 3.3.10(a) possesses its maximum at a much smaller angle of incidence than the minimum in Figure 3.3.10(d) in the case of a 20% change in relative material parameters. This alteration in the positions of the maximum (or minimum) amplitude and the existence and position of a zero amplitude point with respect to the thin layers relative properties could potentially
be quite useful for characterizing an unknown thin layer.

Figure 3.3.8: Characteristic curves for the reflected longitudinal wave in a sample with two identical bulk media. The density and impedance have been altered by 10 and 20% relative to the bulk media
(a) Density increased, impedance decreased
(b) Density and impedance increased
(c) Density and impedance decreased
(d) Density decreased, impedance increased
Figure 3.3.9: Characteristic curves for the reflected vertical wave in a sample with two identical bulk media. The density and impedance have been altered by 20% relative to the bulk media.

(a) Density increased, impedance decreased
(b) Density and impedance increased
(c) Density and impedance decreased
(d) Density decreased, impedance increased
Figure 3.3.10: Characteristic curves for the transmitted vertical wave in a sample with two identical bulk media. The density and impedance have been altered by 20% relative to the bulk media.

(a) Density increased, impedance decreased  
(b) Density and impedance increased  
(c) Density and impedance decreased  
(d) Density decreased, impedance increased

Finally, one can also attempt to create a set of characteristic curves with the transmitted longitudinal wave. At first the transmitted waves seem to again create a set of characteristic curves that depend on the thin layer's impedance and density relative to the bulk material. But on closer inspection all curves follow the same curve shape as seen in Figure 3.3.11, where the curve begins with an amplitude value larger than unity for the transmitted wave, then approaches unity asymptotically in the region of 60 through 70 degrees of incidence, after which the amplitude proceeds to rise rapidly. This leaves the
only characteristic being the amplitude of the transmitted wave at normal incidence (or small angles). As it was found that the various transmitted longitudinal curves would only approach a level of unit amplitude, and never become less than the unit amplitude, it could be quite useful to examine this asymptotic condition further to discover whether it would reveal any interesting properties. Expressed as an inequality, the asymptotic approach to unit values means the perturbation expansion is required to satisfy the inequality

\[ T_0 \cdot A_m + \phi_\beta \cdot k_z^\beta \cdot d \cdot T_0 \cdot A'_m > 1, \] (3.3.6)

where \( A_m \) represents the point of maximum amplitude in the time domain. Because this simulation is a case of two identical media, \( T_0 = 1 \). Also, since the incident wave has been defined to have an amplitude of one unit, \( A_m \leq 1 \). Hence it follows from (equation 3.3.6) that

\[ \phi_\beta \cdot k_z^\beta \cdot A'_m > 0. \] (3.3.7)

In addition it is known that \( k_z > 1 \) because the angle of incidence is being restricted to the range: \(-90^\circ < \theta < 90^\circ\) (any larger angle results in the incident wave arriving in the second media). To satisfy equation 3.3.7, one possible case is

\[ \phi_\beta < 0, \ A'_m < 0. \] (3.3.8)

The inequalities in equation 3.3.8 make it so that the maximum amplitude of the transmitted wave will always be on the negatively sloped side of the incident wave when \( \phi_\beta < 0 \). This means that the transmitted wave will always lag in phase with respect to the incident wave. The other possible condition is
\( \phi_B > 0, \ A_n > 0. \)  \hspace{1cm} (3.3.9)

In this case, the transmitted wave leads the incident wave. Examining the phase factor it is found that \( \phi_L^{II} \), while generally negative, possesses both positive and negative values. The existence of the positive values means it is possible for the transmitted wave to lead the incident wave in phase, even though the time delay across the thin layer causes a lag in the transmission of the wave.

![Transmitted Longitudinal Wave](image)

**Figure 3.3.11:** Transmitted longitudinal wave in a sample with two identical bulk media.

### 3.4: Comparison With a Multilayered System

Here a special case is considered to compare the thin layer approximation to the exact result of a multilayered system. To simplify the mathematics of the multilayer system, (which still uses the boundary conditions, but requires conditions at each interface), this comparison will use only normal incidence and an incident longitudinal wave. A short overview of the mathematical details and the solution (in terms of a recursive function) for this multilayered system can be found in Appendix C. Simulations were performed in the time domain, using both the thin layer approximation and the
multilayered solution for the aluminum-epoxy-brass system of Section 3.3.1 with an incident longitudinal sinusoidal wave of frequency 0.1 MHz at normal incidence. The major difference between the domain outputs (displacement amplitude versus time) is in the beginning and ending of the waveform (Figures 3.4.1 and 3.4.2). The multilayer system correctly begins with zero displacement in the reflected and transmitted waves and each wave smoothly increases in amplitude during the beginning of the simulation (Figure 3.4.1). The reflected and transmitted wave's displacement in the multilayer system also smoothly reaches zero amplitude at the end of the simulation when the incident waveform has ended (Figure 3.4.2). The phase perturbation theory cannot predict this initial and final behaviour and begins with a non-zero value for the reflected and transmitted amplitudes, and both waves also suddenly change to a level of zero amplitude when the incident waveform ends. The lack of a proper initial and final behaviour by the perturbation theory is of course due to the fact that it is an approximation, while the multilayered system explicitly uses the time delay resulting from the thin layer to calculate the reflected and transmitted waves. As the behaviour is only an issue for less than a picosecond and this time is not of critical interest, the errors here can be safely ignored. Subtraction of the two time domain results (displacement amplitude versus time) (Figure 3.4.3) shows that the difference in time domain output of the two methods is very minimal, ignoring the large errors in the initial and final few seconds of the waveform. The difference for the amplitude of the reflected longitudinal wave oscillates between ±0.005 μm. Ignoring the sign change, which is caused by the phase inversion of the reflected wave with respect to the incident wave, the difference in the reflected longitudinal wave slightly lags the incident wave (shown with a significantly decreased
amplitude) in phase. Due to the close alignment of the maximum error with the maximum in the incident wave, this means the amplitude data is unfortunately where the error is largest, especially in waveforms that lag the incident wave in time. The difference in the transmitted longitudinal wave oscillates with a slightly lower error range and also slightly lags in phase with the incident wave causing the same increased error in the amplitude data as in the reflected data.

Figure 3.4.1: Beginning microsecond of the perturbation and multilayer simulation time domain data.
Figure 3.4.2: Final microsecond of the perturbation and multilayer simulation time domain data.

Figure 3.4.3: Difference between the perturbation and multilayer time domain outputs.
One can determine the expected range of validity of the thin layer approximation by reconsidering the mathematical basis behind the perturbation expansion from equations (2.3.8) and (2.3.9),

\[ B_{\alpha} = R_{\alpha} \cdot A_0 + \phi_{\alpha} \cdot k_{z}^{\alpha} \cdot d \cdot R_{\alpha} \cdot A'_0 \]

\[ C_{\beta} = T_{\beta} \cdot A_0 + \phi_{\beta} \cdot k_{z}^{\beta} \cdot d \cdot T_{\beta} \cdot A'_0 \]

This expansion is essentially the small angle approximation for the sine and cosine functions. If one allows for a 2% difference between the small angle approximation and true value, the corresponding maximum angle is roughly 20 degrees or 0.35 radians. Using this fact, the rough maximum can be determined for the thin layer. To satisfy the small angle approximation (to a 2% difference) the phase change must satisfy

\[ \phi \cdot k \cdot d < 0.35. \quad (3.4.1) \]

Examining the reflected longitudinal wave in the aluminum-epoxy-brass system yields a phase factor of \( R_L \cdot \phi_{RL} \cdot k_z^L = 0.00145 \mu \text{m}^{-1} \) and reflection coefficient of \( R_L = 0.636 \), thus requiring:

\[ d < 82 \mu \text{m}, \quad (3.4.2) \]

or in terms of a percentage of the interface layer's acoustic wavelength \( \frac{d}{\lambda} < 3\% \). To verify this limit, both the multilayer and perturbation simulations can be run using the same aluminum-epoxy-brass sample with a normal incidence longitudinal wave of frequency 0.1 MHz. A plot of the reflected and transmitted amplitude versus thin layer size can be created for both the multilayer and the phase perturbation approaches (Figures 3.4.4 and 3.4.5, respectively). In addition, the percent difference of the reflected and transmitted amplitudes between the two simulations can be taken, where the multilayer system is
considered as the true value (Figure 3.4.6). As expected from the comparison in the time domain output (Figure 3.4.3), both the reflected and transmitted longitudinal waves have larger amplitudes when calculated by the perturbation theory. Examining Figure 3.4.4 and 3.4.5, the difference in the reflected and transmitted amplitudes between the two simulations is found to gradually increase as the thickness of the layer increases and the perturbation theory becomes invalid. The increase in the resulting percent difference between the perturbation theory and multilayer theory (Figure 3.4.6) increases proportional to thickness squared (actually slightly less than a power of two). The reflected longitudinal wave is found to deviate from the expected solution quicker than the transmitted wave, and reaches a 2% error at roughly 70 µm. One notes that the deviation has risen faster than expected in this case indicating that the differences between the multilayer and perturbation theories cannot be described simply by the small angle approximation.

The same comparison can be done for the case of the acrylic-epoxy-acrylic sample that will be used in the sections on anisotropy and nonlinearity (Sections 3.5 and 3.6, respectively). Again an incident longitudinal wave with normal incidence is used, but now the frequency is increased to 1 MHz. The amplitude of the reflected data from perturbation theory is nearly a perfect fit to the multilayer data (Figure 3.4.7) even up to a layer with thickness of 6% of a wavelength where the percent difference between the two methods reaches a 2% error (Figure 3.4.8). The transmitted wave in this particular case deviates more quickly with increases in thickness than the reflected wave, reaching a 2% error at 3% of an acoustic wavelength in epoxy (roughly 75 µm at this frequency).
Figure 3.4.4: Comparison of the amplitude of the reflected longitudinal wave in the multilayer and perturbation theories as the size of the thin layer is increased in an aluminum-epoxy-brass system.

Figure 3.4.5: Comparison of the amplitude of the transmitted longitudinal wave in the multilayer and perturbation theories as the size of the thin layer is increased in an aluminum-epoxy-brass system.
Figure 3.4.6 Percent difference of the amplitude of the reflected and transmitted longitudinal wave between the multilayer and perturbation theories as the size of the thin layer is increased in an aluminum-epoxy-brass system.

Figure 3.4.7 Comparison of the amplitude of the reflected longitudinal wave between the multilayer and perturbation theories as the size of the thin layer is increased in an acrylic-epoxy-acrylic system.
3.5: A Nonlinear Thin Layer between Two Half-Spaces

In this section the changes due to adding nonlinearity to the thin layer is explored. The simulation utilizes an arbitrary function in the stress-strain relation so that different levels of expansion of the classical theory can be explored, in addition this allows for the possibility of non-classical nonlinearity to be explored with a simple alteration. The arbitrary function also allows for piecewise functions to be explored, this allows for cases where the nonlinearity may change; currently the function is only altered with respect to one variable, but by adding an additional variable, materials with nonlinear hysteresis can be explored. In the case of a nonlinear thin layer it is useful to look at which harmonics are created and the relative of the size of the harmonics, as well as how the additional harmonics amplitudes are altered with changes in angle of incidence. For the cases of a
nonlinear system the simulation will use an acrylic-epoxy-acrylic system (see Appendix A for material parameter), so that any changes are attributed to the thin layer and its nonlinearity. An incident longitudinal wave with a frequency of 1.0 MHz will be used, and the epoxy layer set to a thickness of 50 μm.

3.5.1: Classical Nonlinearity

To correspond with equation (2.6.4) the classical nonlinear stress-strain equation is

\[ \varepsilon_{ij} = s_{ijkl} \cdot T_{kl} \cdot (1 + F(NL)), \]

(3.5.1)

where the NL simply serves as a holding spot for the appropriate stress or strain terms. A summary of some nonlinear functions, and the harmonics created can be found in Table 3.5.1, and the corresponding Fourier transforms in Figures 3.5.1(a), (b) and (c). It should be noted that in these spectrums only the additional harmonics have be plotted, the size of the first harmonic is such that if it were plotted the additional harmonics would not be identifiable. The Fourier transforms are also scaled so that the size of the harmonic directly represents its corresponding amplitude, for example in these simulations the incident wave is set to have an amplitude of 1, and thus it would have a first harmonic with size of 1. One notes that the nonlinear function \( \beta \cdot T_{kl} + \chi \cdot T_{kl}^2 \) is simply a total of the effects of its individual sums.

<table>
<thead>
<tr>
<th>Function F(NL) Proportional To</th>
<th>Figure</th>
<th>Frequency Multiples added</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta \cdot T_{kl} )</td>
<td>3.5.1(a)</td>
<td>0, 2</td>
</tr>
<tr>
<td>( \chi \cdot T_{kl}^2 )</td>
<td>3.5.1(b)</td>
<td>3</td>
</tr>
<tr>
<td>( \beta \cdot T_{kl} + \chi \cdot T_{kl}^2 )</td>
<td>3.5.1(c)</td>
<td>0, 2, 3</td>
</tr>
</tbody>
</table>

Table 3.5.1: Summary of frequency multiples added due to various classical nonlinear functions.
Figures 3.5.1: Fourier transform from acrylic-epoxy-acrylic sample with classical nonlinear thin layer
(a) Classical nonlinear function $\beta \cdot T_{kl}$. 
(b) Classical nonlinear function $\chi \cdot T_{kl}^2$. 
(c) Classical nonlinear function $\beta \cdot T_{kl} + \chi \cdot T_{kl}^2$. 

67
The question of why each particular function summarized in Table 3.5.1 produces certain harmonics can be found by examining the nonlinear functions (Figures 3.5.2(a), (b) and (c)) and comparing them to the time domain change of the phase factors (Figures 3.5.3(a), (b) and (c)). From these figures it is discovered that as predicted in Section 2.6, the phase coefficient is made up of linear and nonlinear parts and solution for the waveform can be expressed in terms of equation (2.6.8),

\[ C_\beta(\omega \cdot t) = T_\beta \cdot A_0(\omega \cdot t) + T_\beta \cdot \phi L^\beta \cdot k_x^\beta \cdot d \cdot \phi L^\beta \cdot k_x^\beta \cdot d \cdot F(NL) \cdot A_0(\omega \cdot t). \]

As the nonlinear term of equation (2.6.8) is the only time dependent portion in these cases, the additional harmonics that are created are attributed to the combination

\[ A' \cdot F(NL). \quad (3.5.2) \]

The various functions made by the combination \( A' \cdot F(NL) \) and the frequencies that correspond to this function are tabulated in Table 3.5.2. Comparing Tables 3.5.1 and 3.5.2, the frequency multiples are found to match up exactly as expected for nonlinearity caused by the factor \( A' \cdot F(NL) \).

<table>
<thead>
<tr>
<th>Function F(NL) Proportional To</th>
<th>Nonlinear Factor A' F(NL) (Time domain portion)</th>
<th>Frequency Multiples</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta \cdot T_{kl} )</td>
<td>( \cos(\omega \cdot t)^2 )</td>
<td>0, 2</td>
</tr>
<tr>
<td>( \chi \cdot T_{kl}^2 )</td>
<td>( \cos(\omega \cdot t)^3 )</td>
<td>3</td>
</tr>
<tr>
<td>( \beta \cdot T_{kl} + \chi \cdot T_{kl}^2 )</td>
<td>( \cos(\omega \cdot t)^2 + \cos(\omega \cdot t)^3 )</td>
<td>0, 2, 3</td>
</tr>
</tbody>
</table>

Table 3.5.2: Summary of frequency multiples created due to combination \( A' F(T) \), where \( A' \) is the cosine function.
Figure 3.5.2: Nonlinear waveform for a classical nonlinear function in the acrylic-epoxy-acrylic sample.

(a) Classical nonlinear function $\beta \cdot T_{kl}$

(b) Classical nonlinear function $\chi \cdot T_{kl}^2$
Figure 3.5.2: Nonlinear waveform for a classical nonlinear function in the acrylic-epoxy-acrylic sample.

(c) Classical nonlinear function $\beta \cdot T_{kl} + \chi \cdot T_{kl}^2$

Figure 3.5.3: Time dependence of phase factor for a classical nonlinear function in the acrylic-epoxy-acrylic sample.

(a) Classical nonlinear function $\beta \cdot T_{kl}$
Figure 3.5.3: Time dependence of phase factor for a classical nonlinear function in the acrylic-epoxy-acrylic sample.

(b) Classical nonlinear function $\chi \cdot T_{kl}^2$

Figure 3.5.3: Time dependence of phase factor for a classical nonlinear function in the acrylic-epoxy-acrylic sample.

(c) Classical nonlinear function $\beta \cdot T_{kl} + \chi \cdot T_{kl}^2$
Finally, examining the changes in the harmonics versus angle of incidence (Figure 3.5.4(a) and (b)), it is found that the additional harmonics created from the various nonlinear functions follow roughly the same shape curves indicating that the nonlinear function has had little effect on the position of maximum amplitude. Examining the change in the linear phase term with angle of incidence (Figure 3.5.5) it is seen that the dependence on angle of incidence in the reflection and transmission coefficients dominate the curve’s shape.

![Graph showing FFT Amplitude vs Angle of Incidence](image)

**Figure 3.5.4**: Change in harmonic with angle of incidence for classical nonlinear functions in the acrylic-epoxy-acrylic sample.

(a) Second harmonic for classical nonlinear functions $\beta \cdot T_{kl}$ and $\beta \cdot T_{kl} + \chi \cdot T_{kl}^2$
Figure 3.5.4: Change in harmonic with angle of incidence for classical nonlinear functions in the acrylic-epoxy-acrylic sample.

(b) Third harmonic for classical nonlinear functions $\chi \cdot T_{kl}^2$ and $\beta \cdot T_{kl} + \chi \cdot T_{kl}^2$

Figure 3.5.5: Change in the linear phase factor with angle of incidence in the acrylic-epoxy-acrylic sample.
3.6: An Anisotropic Thin Layer between Two Half-Spaces

Next, consider a thin anisotropic layer embedded between two half-spaces. Typically the natural, or crystalline axes of the system of interest are not aligned with the orientation defined in Figure 2.1.1 and therefore must be transformed. In general the formula for the rotation of the compliance matrix is

\[ [s'] = [N][s][N]^T, \]

(3.6.1)

where \([s]\) and \([N]\) are, respectively, the compliance and rotation matrices in the reduced 6x6 matrix notation. The particular case that is used in the simulation is the rotation of a hexagonally anisotropic system 90° about the x-axis (see Figure 3.6.1) where

\[ N = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}. \]

(3.6.2)

The hexagonally anisotropic system is of interest because a structure made of longitudinally aligned fibres, or longitudinally aligned small metallic rods feature the same symmetry as hexagonal anisotropy, giving a clear correspondence to a physically manufacturable structure.
Figure 3.6.1: Rotation of a hexagonal anisotropic system about the x-axis, so that the fibres lay in the x-y plane.

A hexagonal carbon fibre layer (the compliance matrix for the carbon fibre layer used is found in Appendix A) is chosen for this simulation and is placed between two identical half-spaces, in this case acrylic. Two identical bulk media have been chosen again so that the angular dependence due to different materials will not overwhelm the anisotropic changes. The amplitudes of the reflected and transmitted waves are plotted for an incident wave of longitudinal polarization at various incident and polar angles. In the case of the reflected longitudinal wave (Figures 3.6.2(a) and (b)), the transmitted longitudinal wave (Figure 3.6.3(a) and (b)), and the reflected shear vertical wave (Figure 3.6.4), the dependence of amplitude on angle of incidence is large compared to the dependence on polar angle. When the amplitude curves (amplitude versus angle of incidence) are plotted for various polar angles, where each polar angle is graphed as a solid line, (Figure 3.6.2(a), Figure 3.6.3(a), and Figure 3.6.4), it is possible to notice that the amplitude does vary slightly with the change in polar angle, even though the various curves are difficult to follow in some places. The data can also be plotted showing the change in amplitude versus the polar angle for various angles of incidence, where each
angle of incidence is graphed as a solid line, the smallest angles of incidence resulting in the lowest amplitude (Figure 3.6.2(b), and Figure 3.6.3(b)), more clearly showing the small changes in amplitude with polar angle, especially at larger angles of incidence. If this same data were plotted as a surface graph, these small changes would be difficult to notice, although for the additional polarizations it will be possible to create surface plots as the changes with polar angle become more pronounced. In all three cases the amplitude curves all begin with the same initial amplitude at normal incidence as expected, and then generally begin to diverge. The anisotropic thin layer has also created two points of interest to examine in these three cases. In the case of the reflected longitudinal wave's amplitude (Figure 3.6.2(a)) it is found that the position of phase inversion with respect to the angle of incidence has changed in a range of 5 degrees of incident angle due to the anisotropy of the thin layer. Physically this point of phase inversion corresponds to a situation where no reflected longitudinal wave is produced. As the angle of incidence for this situation where there is no reflected wave is found to be dependant upon the velocity of the thin layer (or equivalently the impedance), this dependence could be very useful for characterizing the thin layer even in cases of an isotropic material. In the case of the reflected shear vertical wave's amplitude (Figure 3.6.4) a point of constant amplitude is found at an angle of incidence of 65 degrees, making the material essentially isotropic at this angle, but only for this particular polarization. That is to say, the effective reflection coefficient of the combined acrylic-epoxy-acrylic system is anisotropic at 65 degrees of incidence, and the combined slowness surface would be circular about the polar direction for the reflected shear vertical polarization. As previously mentioned for the remaining polarizations
(transmitted shear vertical, reflected shear horizontal and transmitted shear horizontal) the changes with polar angle are more significant and can be detected along side changes in the angle of incidence. In these three cases the absolute value of the amplitude is plotted with respect to both angle of incidence and polar angle, creating a surface plot of the maximum amplitude. In the case of the transmitted shear vertical wave (Figure 3.6.5) a change in the polar angle increases the size and location of the local maximum amplitude with respect to the angle of incidence and also changes the location of the phase inversion. Because the sign of the wave relative to the incident wave is not plotted, it is worthwhile to note that the transmitted shear vertical wave is phase inverted for the lower angles of incidence. The most dramatic changes are found in the case of the reflected shear horizontal (Figure 3.6.6) and transmitted shear horizontal (Figure 3.6.7) waves. In these cases, when either the polar angle or incident angle becomes an angle other than 0 or 90 degrees, mode conversion to the shear horizontal wave becomes possible creating a non-zero amplitude wave. It is also worthwhile to note that the reflected shear horizontal wave is actually inverted in phase with respect to the transmitted shear horizontal wave since the sign inversion is not plotted. It is noted that Figures 3.6.5 through 3.6.7 all feature mirror symmetry at 0, 90, 180, and 270 degrees in the polar angle, thus only one quarter of a full polar angle rotation has been used. Physically this symmetry correlates to the symmetry of the hexagonal anisotropic material in the x-y plane (see Figure 3.6.1)
Figure 3.6.2: Change in amplitude of the reflected longitudinal wave
(a) Change with angle of incidence for various polar angles, and a thin layer with hexagonal anisotropy. Each of the various curves is separated by 10 degrees of polar angle beginning at zero degrees.

Figure 3.6.2: Change in amplitude of the reflected longitudinal wave
(b) Change with polar angle for various incident angles, and a thin layer with hexagonal anisotropy. Each of the curves is separated by 10 degrees of incident angle (beginning at normal incidence), except the highest curve which is an incident angle of 85 degrees.
Figure 3.6.3: Change in amplitude of the transmitted longitudinal wave
(a) Change with angle of incidence for various polar angles, and a thin layer with hexagonal anisotropy. Each of the various curves is separated by 10 degrees of polar angle beginning at zero degrees.

Figure 3.6.3: Change in amplitude of the transmitted longitudinal wave
(b) Change with polar angle for various incident angles, and a thin layer with hexagonal anisotropy. Each of the curves is separated by 10 degrees of incident angle (beginning at normal incidence), except the highest curve which is an incident angle of 85 degrees.
Figure 3.6.4: Change in amplitude of the reflected shear vertical wave with angle of incidence for various polar angles, and a thin layer with hexagonal anisotropy. Each of the various curves is separated by 10 degrees of polar angle beginning at zero degrees.

Figure 3.6.5: Surface plot of the change in amplitude of the transmitted shear vertical wave with both polar and incident angles, and a thin layer with hexagonal anisotropy.
Figure 3.6.6: Surface plot of the change in amplitude of the reflected shear horizontal wave with both polar and incident angles, and a thin layer with hexagonal anisotropy.

Figure 3.6.7: Surface plot of the change in amplitude of the transmitted shear horizontal wave with both polar and incident angles, and a thin layer with hexagonal anisotropy.
Two other classes of anisotropic systems are typically of interest due to their prevalence among materials or physical representation. The first materials of interest are cubically anisotropic materials; these are of interest due to the large number of crystals with cubic symmetry. The second material of interest are orthogonally anisotropic materials (also called orthogonal or orthorhombic), in this case the symmetry is lower than the hexagonal case and physically could be represented by longitudinal fibres, but with the hexagonal symmetry being broken by fibre size or concentration [23].

Changing the carbon fibre layer to an orthogonally anisotropic material by altering three particular components of the hexagonally anisotropic compliance matrix ($s_{22}$, $s_{44}$, and $s_{66}$ of the non-rotated form) is found to change some of the curves' characteristics, but only very slightly in most cases. In general the change to an orthogonally anisotropic material primarily affects the size of the amplitude changes due to variations in the polar angle. These slight changes in the amplitudes are due to the effective change in the velocities along different directions through the crystal, while the increased dependence on the polar angle is likely due to the fact that the elements of the compliance matrix that were changed were increased in magnitude. One exception is that the characteristics of the transmitted shear vertical wave (Figure 3.6.8 compared with Figure 3.6.5) have changed significantly. The sudden change in this particular polarization is due to the particular changes that were made (Appendix A), and not because this one polarization will be extra sensitive to changes between anisotropic materials. The symmetry with the polar angle is still a feature in all cases and is the same symmetry as hexagonal anisotropy, this is due to the fact that the symmetry in the x-z and y-z planes has been maintained (the symmetry about the y axis has been decreased
instead). Finally, for the particular case of the reflected shear vertical wave, in the case of hexagonal anisotropy (Figure 3.6.4), a point of constant amplitude was found at an incident angle of 65 degrees. In the orthogonal anisotropic material the point of constant amplitude still remains and is again located at 65 degrees. But overall no other characteristics of the curves have changed either, thus the most that can be determined is that this strange characteristic probably related to factors other than the $s_{22}$, $s_{44}$, and $s_{66}$ compliance matrix elements.

Figure 3.6.8: Surface plot of the change in amplitude of the transmitted shear vertical wave with both polar and incident angles, and a thin layer with orthogonal anisotropy.
3.7: A Nonlinear and Anisotropic Layer

In this final simulation the effects of the nonlinearity (Section 3.5) and anisotropy (Section 3.6) combine producing harmonics that are now dependent on the change in polar angle. Again simulating the acrylic-epoxy-acrylic sample with layer thickness 50 μm and an incident longitudinal wave of frequency 1.0 MHz produces the same first harmonic results as seen in the amplitude curves in Section 3.5. As an example of the harmonics changing with respect to polar angle and incident angle using the nonlinearity $\beta \cdot T_{kl}$ in the thin layer creates the reflected horizontal wave's second harmonic seen in Figure 3.7.1.

![Graph showing change in second harmonic with polar and incident angle](image)

**Figure 3.7.1:** Change in second harmonic with polar and incident angle of the reflected horizontal wave for the acrylic-carbon fibre-acrylic sample with nonlinearity $\beta \cdot T_{kl}$. 
3.8: Conclusions

The changes due to adding the thin layer were found to be easiest to detect theoretically in case of two identical bulk media due to the addition of reflected and transmitted waves via mode conversion. The number of mode-converted waves was in addition found to increase when the thin layer became anisotropic. By simulating various thin layers between two identical media, a set of characteristic curves was found that differ with respect to the relative density and impedance of the thin layer to the bulk media. This could possibly be useful for applications dealing with material characterization of thin layers, or detecting changes in the characteristics of a thin layer. The simulation was also found to be most accurate in case of two identical bulk media when the perturbation and multilayer techniques are compared. Adding nonlinearity to the thin layer also presented additional opportunities to detect the thin layer in terms of additional harmonics.

In the case of two identical materials, the simulation was found to be less accurate and changes due to the thin layer are harder to detect in the case of two identical bulk media. This is especially true in the case where the properties (density, velocity and impedance) of the thin layer lie between those of the two bulk media. In the case of two different media joined together by a thin layer, a nonlinear or anisotropic layer will make detection of the layer simpler.

In all the simulations attempted, the perturbation theory allowed for the results to be obtained without calculating the acoustic waves internal to the thin layers, allowing for variation in the thin layer including anisotropy, nonlinearity, and combined anisotropy
and nonlinearity without creating additional complications in the simulation.
Chapter 4: Experiment

4.1: Purpose

The purpose of this experiment is to use ultrasonic waves to investigate an arrangement of materials similar to what has been discussed in the previous theoretical computer simulations, and compare the two results. This is achieved by collecting data in the form of the displacement amplitude of the reflected and transmitted acoustic waves created due to an ultrasonic wave incident on the artificial boundary layer inside the bulk media at various angles of incidence. The experimental sample will be made of two bulk media connected with a thin layer made of a different material than the bulk media. The same experiment can then be simulated using the theoretical methods in the previous sections and the predictions obtained can be compared to the experimental data. It will also be useful to explore the specific case of an anisotropic thin layer to prove that it is possible to practically detect the changes in amplitude due to the anisotropy in the thin layer.

4.2: Materials

4.2.1: General Requirements

The requirements for the materials used in this experiment are mostly due to the theoretical basis of the problem, or acoustic properties. As mentioned, the sample to be explored will be made up of two bulk media, connected by a thin layer of a third material. In section 3.4 it was discovered that examining a case where the two half-spaces were made of identical material had the advantage of clearly showing the existence of a change in the reflected and transmitted waves as compared to two different media. The
experiment will utilize this discovery and thus the two bulk media will be made of the same material. The bulk material itself will need to be of high enough acoustic quality to produce little or no reflection due to internal defects or pores. It would also be advantageous for the material to possess a side on which to place the transducers, which will propagate the acoustic signal into the medium, at various angles. As this bulk media corresponds to the theoretical half-space of material, the medium on each side of the thin layer should be large enough to support multiple acoustic wavelengths so that interference in the bulk media is not an issue. This requirement is not difficult to satisfy as the acoustic wave velocity ranges from 3 mm/s for a typical plastic to 6 mm/s for most metals. At an acoustic frequency of 1.0 MHz, a typical plastic would have a wavelength of 3 mm, thus even a fairly small 3 cm thick sample will be the size of 10 acoustic wavelengths. The materials with lower acoustic velocity though will cause problems with respect to the thickness of the thin layer, thus the material chosen for the thin layer will be limited to those which can be found in thickness of 5% of an acoustic wavelength or less. (This percentage is chosen to correspond with the limitations of the theory found in section 3.4). As this requirement would create a need for thin materials of roughly 150μm in size at a frequency of 1.0 MHz, it is necessary to use ultrasonic waves at 1.0 MHz or less so that the experiment will not become severely limited due to difficulty of operating with thin layers. The thin layer must be in good acoustic contact with the two bulk materials and create sufficient continuity in the materials so that the acoustic waves can propagate through the medium, thus the most likely material to begin with is an adhesive. As other materials to be investigated, specifically anisotropic materials, are typically found in the form of solids, these materials will need to be connected to the bulk media
using adhesive, or ultrasonic gel and sufficient pressure for material continuity and will be chosen due to their anisotropic properties. This does unfortunately change the material properties, but will serve as a basis to justify more advanced methods to create an appropriate structure in the future.

4.2.2: Materials Used

A suitable bulk material was determined to be acrylic plexiglas. The sample obtained had very good acoustical properties producing only negligible reflections due to internal defects. The acrylic was also available in the shape of a long cylinder, where the round sides of the cylinder nicely allowed for placement of acoustic transducers at various incident angles. Three possible thin layers were chosen to be examined, the first being the liquid adhesive cyanoacrylate (also known as superglue) used to create a thin isotropic thin layer. The second two materials were chosen for their anisotropic properties. The first anisotropic layer is formed of fibreglass cloth fibres, which needed to be hand-laid on the interface in the appropriate direction and glued in place. The second anisotropic layer is a sample made of long boron fibres, technically small rods, held together with a thin aluminum coating. Both anisotropic layers correspond to hexagonal anisotropic systems, or possibly orthogonal anisotropic systems since both systems are only one fibre layer thick.

4.2.3: Electronics and Transducers

The Ritec SNAP-1-30 was used as the waveform generator to drive one transducer from a pair of standard broadband transducers, as well as a receiver to receive the reflected or transmitted signal from the second transducer. The Ritec waveform
generator possesses the ability to produce a tone burst signal at a specific frequency (in the range of 0.1 to 30 MHz). This feature is highly beneficial in order to ensure that acoustic wavelength is much larger than the size of the thin layer. The Ritec also has the ability to control the tone burst duration so that interference is not an issue, as well as amplifying the received signal, and outputting the data to an oscilloscope. The signal of the Ritec however is in the form of a normalized sine wave, causing the beginning of the pulse to be almost impossible to detect, as well as making it difficult to determine if the received pulse has become inverted. The transducers used for the experiment are standard planar broadband transducers with a central frequency of 7.5 MHz, but are operating at lower frequency. The transducers can be either placed directly on the surface of the acrylic using ultrasonic gel as a couplant to transfer the acoustic signal, or temporarily adhered using phenyl salicylate to a small piece of acrylic moulded to match the curve of the bulk material (Figure 4.2.1). This transducer-acrylic combination is also be coupled with ultrasonic gel to the sample, in order to transfer the acoustic signal. The second option has the benefit of a more certain position of the transducer on the sample making the angle of incidence more certain. A digital oscilloscope (Tektronix TDS 5052) is used to collect and analyze the data received from the Ritec receiver channel. To reduce noise, this particular oscilloscope allows the possibility to average a defined number of signals. It will also directly measure the signal properties, as well as save the data for future use.

4.2.4: Material Preparation

As the acrylic source came in the shape of a long cylinder, the preparation of the bulk material consisted of cutting the tube into shorter cylinders (roughly 2.5 cm thick so the transducer could be placed on the edge), and then cutting each cylinder in half along
the diameter forming the necessary two identical bulk media for each sample. The first sample consisted of two acrylic pieces adhered together with a thin layer of cyanoacrylate glue. Measuring the adhesive layer under a light microscope found the thin layer of glue to be 0.11 mm thick. The next samples consisted of a thin layer of fibreglass cloth fibres, these were hand-laid in the interface and the two acrylic pieces held together again with cyanoacrylate glue. Two of these fibreglass cloth samples were formed, the first with the fibres orientated lengthwise along the radius of the interface, the second with fibres orientated across the interface, or perpendicular to the circular surface. Measuring the thin layers under a light microscope found the thin layers to be 0.14 mm for the lengthwise fibres and 0.20 mm thick for the perpendicular fibres. Another anisotropic sample consisted of the boron fibres placed between two bulk acrylic pieces. So that many orientations of the boron fibres could be examined, the boron fibres were simply coated with ultrasonic couplant and a clamp was used to hold the sample together and maintain overall continuity in the sample. Doing this, the boron could be easily reoriented and many individual samples with different polar angles were not needed.

Figure 4.2.1: Transducer adhered to a small piece of acrylic using phenyl salicylate. The acrylic surface is curved to match the curved edge of the sample.
4.3: Procedure

With the samples prepared and the electronics connected (Figure 4.3.1) and designed to continually obtain data, the procedure for data collection becomes a relatively straightforward procedure. The transducers are first placed on the sample typically being held in place with an elastic band resting on a small plastic transducer holder (Figure 4.3.2). Using a clear plastic protractor, the transducers could then be positioned at an
angle of interest, and with ultrasonic gel used to form a contact between the transducer and the sample. The acoustic signal received by the oscilloscope would then be maximized by slightly readjusting the position of the transducers. Finally the oscilloscope's digital features were used to determine the maximum, minimum, or peak-to-peak amplitude of the received electrical signal, which is proportional to the amplitude of the acoustic wave. In the case of the glue and fibreglass samples the transducers would then be moved to the next angle of interest. Due to the size of the sample being used, the amplitude data was taken at intervals of 10 degrees. In addition, due to the diameter of the transducers, data could only be acquired for angles between 0 through 60 or 70 degrees from the normal. In the case of the boron fibres, the transducers were not moved from their initial position, but instead the direction of the boron fibres was changed to effectively alter the polar angle of the transducers. Again, amplitude of the received waveform was taken as the polar angle was adjusted in 10-degree intervals ranging from 40 degrees through 140 degrees, with 90 degrees defined as when the rods were perpendicular to the circular surface of the sample.

4.4: Results and Discussion

4.4.1: Control Sample

A solid piece of the acrylic was used as a control to find an approximate error due to placing the transducers into different positions on the acrylic surface, using different amounts of ultrasonic gel, and also the change in pressure in which the elastic bands hold down the transducers. The Ritec is set to deliver an acoustic wave frequency of 1.0 MHz into the acrylic sample, and the oscilloscope is set to display the results from an average of 20 waveforms. The oscilloscope readings that were taken unfortunately did not
provide a measure of the deviation in the average amplitude readings, but it is known that
the individual readings from the oscilloscope are highly accurate, and typically the
reading would only fluctuate by roughly 1% when collecting data in real time. It should
be noted that when data was taken, the oscilloscope was not operating in real time,
instead the data collection was temporarily paused to obtain a definite reading. The
transducer was placed at various positions on the acrylic control sample and the largest
transmitted signal was obtained in order to measure the transmitted displacement. The
data obtained for these cases can be found in Table 4.4.1. Using standard statistical
calculations, the standard error for these measurements is ±2.4 mV. As the error from the
control sample exceeds the fluctuation in the oscilloscope, it will serve as the
experimental error unless a more appropriate error is discovered.

<table>
<thead>
<tr>
<th>Measurement Number</th>
<th>Displacement Amplitude (Maximum) (mV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>298.9</td>
</tr>
<tr>
<td>2</td>
<td>303.9</td>
</tr>
<tr>
<td>3</td>
<td>309.3</td>
</tr>
<tr>
<td>4</td>
<td>310.4</td>
</tr>
</tbody>
</table>

Table 4.4.1: Displacement amplitude measurements in a control sample.

4.4.2: Cyanoacrylate Layer

In this case, we are interested in recording the amplitude of the acoustic waves as
a function of the angle of incidence. The reflected longitudinal wave data will be
examined as it is expected to have a very detectible change with the angle of incidence
compared to the amplitude of the transmitted longitudinal wave which is expected to be
nearly constant. Examining the reflected wave will also ensure that the expected
reflection due to the existence of the thin layer can be detected experimentally. Setting
the Ritec to deliver an incident wave of frequency 1.0 MHz and collecting data for a variety of incident angles, where each data point represent an average of 20 waveforms, it is found that at even the smallest amplitude it is possible to discern the reflected waveform from the background noise (Figure 4.4.1). Plotting the data as maximum amplitude versus angle of incidence (Figure 4.4.2) reveals a curve similar to the characteristic curves of figures 3.3.8(a) and (d) once inverted. As it is believed the waveform has not been inverted, one expects the glue layer to have a higher density and lower impedance than the acrylic and correspond to a material with larger density and lower impedance than the bulk material. In addition, the near-zero value at normal incidence indicates that the cyanoacrylate glue is very similar to that of the bulk acrylic.

![Reflected Displacement](image)

**Figure 4.4.1:** Reflected displacement at 12 degrees incident angle.
4.4.3: Fibreglass Cloth Layer

In this case our goal was to detect a change in the amplitude of the acoustic wave between the two different orientations of the fibreglass cloth fibres. Again the amplitude of the reflected longitudinal wave will be used so that both the changes due to the orientation, and the angle of incidence can be examined. The generator is again set to deliver a 1.0 MHz incident wave into the sample, and the oscilloscope is set to display an average 20 waveforms. The raw reflected amplitude versus angle of incidence data is plotted in Figure 4.4.3 for both orientations of the fibres. All of the corresponding data points taken between the two materials are separated by a significantly larger voltage than what is attributed to the readings of the oscilloscope; in fact this error is smaller than the symbols used to plot the data points. This includes the data taken at normal incidence, which should, theoretically, be found to be the same. Instead, roughly 10 mV separates
the amplitude readings between the two orientations. The reason for this difference is attributed to the fact that two separate samples have been used to provide the data sets. It is highly likely that factors such as attenuation, and the reflection and transmission coefficients vary between the two materials. Increasing the error to ±5 mV, so that the data at normal incidence can be considered equal within error, shows that even with this amount of error considered the two orientations of fibres are still able to be distinguished (Figure 4.4.3) even at lower angles of incidence. Again the data indicates that the overall glue and fibreglass layer possesses larger density and lower impedance than the bulk acrylic (Figure 3.3.8(a)), and now contains a larger reflected signal at normal incidence when compared to the thin glue sample (Figure 4.4.2).

![Graph showing voltage or proportional amplitude vs. angle of incidence](image)

**Figure 4.4.3:** Fibreglass cloth reflected amplitude with changing angle of incidence, error represents the smallest error needed to make the normal incidence points equal within each others error.
4.4.4: Boron Fibre Layer

In this case we are interested in detecting a change in the amplitude of the acoustic wave as the orientation of the fibres is changed and the angle of incidence remains constant. The amplitude of the reflected longitudinal wave is expected to be very small in this case due to the low reflection coefficient as the acoustic wave travels from the acrylic to the thin boron layer. With a lower strength reflection signal than in the previous experiments, the changes in the reflected waves amplitude due to the anisotropic layer could possibly be confused with random changes due to noise, especially since both the reflected and transmitted amplitude data are expected to change only a small amount with changes in the polar angle. Thus, in this case, the amplitude of the transmitted wave is chosen to be examined. As the size of the thin layer has increased significantly in comparison to the previous cases, the frequency of the incident wave is reduced to 0.1 MHz to ensure the experiment is operating within the bounds of the thin layer approximation. The raw transmitted amplitude versus polar angle (actually change in fibre orientation) is plotted for two of the trials (Figure 4.4.4). The differences between the two trials can be explained in that additional ultrasonic gel was added to the interface in the second trial. A smaller amount of gel decreases the efficiency of transferring ultrasound between the layers, decreasing the overall transmitted amplitude. Apart from the amplitude change both trials are roughly symmetric about a polar angle of 90 degrees as is expected for a hexagonal or orthorhombic anisotropic interface. The symmetry though is not perfect, in fact in all the figures the amplitude readings for polar angles larger than 90 degrees are larger than the amplitude readings for polar angles less than 90 degrees. This is likely attributed to a shifting in the position of the boron layer as it is rotated. To obtain better
quality data it is proposed that the next trial of the experiment use a system where the fibres are not rotated, so that the various polar angles probe the same area of the boron layer. Figure 4.4.5 shows one current proposal for a new sample, which will fit the necessary requirements. Even with this new sample it would be difficult to perform a quantitative comparison between the experiment and the simulation since, in this case, one would need to fully characterize the boron layer to obtain the compliance matrix.

![Voltage vs Polar Angle or Fibre Orientation](image)

**Figure 4.4.4:** Various trials of transmitted amplitude for the boron fibre layer
4.5: Conclusions

The reflected amplitude due to the existence of the thin layer can be detected experimentally; in addition the variation of the reflected amplitude with angle of incidence was found to correspond to one of the possible characteristic curves found during investigation of the results of the simulation allowing for relative material characterization of the thin layer with respect to the bulk media. It was also possible to detect the changes in anisotropy with respect to either examining the amplitude for various polar angles (or fibre orientation) while maintaining a constant angle of incidences, or examining various angles of incidence for two different fibre orientations. The results from the experiment also compared well to the results from the simulation, although since the exact material properties of the thin layer are unknown it is impossible to know if the simulation has in fact characterized the thin layer.
Chapter 5: Summary and Conclusions

By using a computer simulation based on perturbation methods of fully bonded boundary conditions it was found that the changes due to adding a thin layer were easiest to detect in a situation where the media on either side were identical, and that the thin layer was able to be detected by experimental means in this case of two identical bulk media. Anisotropic thin layers were found to have an advantage of being detected due to the process of mode conversion to previously non-allowed polarizations, and the changes due to anisotropy were able to be detected experimentally. While a nonlinear experiment was not attempted, the results using a classical formulation were as expected, and the flexibility of a computer simulation allows the nonlinearity to be easily altered to attempt to discover nonlinearity due to changes in thin layer size, density, or test non-classical formulations. The simulation was also found to be able to simulate the case of a thin layer with both nonlinear and anisotropic characteristics, creating the expected combined additional harmonics with dependence on polar angle.

The error due to using perturbation theory does present some limitations, as one must ensure to use an appropriate acoustic frequency for the thin layers size. The case of two identical bulk media is again found to give more accurate results, and allow a wider range of frequencies or layer sizes than compared to the case of two different bulk media. Overall the boundary conditions method combined with perturbation theory has been a success, and while not an exact solution presents advantages in terms of altering the thin layer, without the complication of calculating acoustic waves internal to the thin layer. The simulation also allows for further extensions, such as additional nonlinearities and
cases dealing with critical angles in the bulk media. In addition, the same perturbation method could possibly be used for materials with alternative boundary conditions.
References


Appendix A: Material Parameters

Acrylic (Plexiglas)
Velocity (Longitudinal) 2750 m/s
Velocity (Shear or Transverse) 1375 m/s
Density 1190 kg/m³

Aluminum (Rolled)
Velocity (Longitudinal) 6420 m/s
Velocity (Shear or Transverse) 3040 m/s
Density 2700 kg/m³

Brass (70% Cu 30% Zn)
Velocity (Longitudinal) 4372 m/s
Velocity (Shear or Transverse) 2100 m/s
Density 8500 kg/m³

Epoxy
Velocity (Longitudinal) 2400 m/s
Velocity (Shear or Transverse) 1201 m/s
Density 1154 kg/m³

Carbon Fibre (Hexagonal Anisotropic)
Stiffness Matrix (Not Rotated)
\[
\begin{bmatrix}
13.5 & 6.3 & 5.5 & 0 & 0 & 0 \\
6.3 & 13.5 & 5.5 & 0 & 0 & 0 \\
5.5 & 5.5 & 125.9 & 0 & 0 & 0 \\
0 & 0 & 0 & 6.2 & 0 & 0 \\
0 & 0 & 0 & 0 & 6.2 & 0 \\
0 & 0 & 0 & 0 & 0 & 3.6
\end{bmatrix} \text{GN/m}^2
\]

Compliance Matrix Rotated (90° clockwise about the x-axis)
\[
\begin{bmatrix}
0.095 & -2.261 \times 10^{-3} & -0.044 & 0 & 0 & 0 \\
-2.261 \times 10^{-3} & 8.14 \times 10^{-3} & -2.261 \times 10^{-3} & 0 & 0 & 0 \\
-0.044 & -2.261 \times 10^{-3} & 0.095 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.161 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.278 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.161
\end{bmatrix} \text{m}^2/\text{GN}
\]

Density 1577 kg/m³
Orthogonal (also called orthogonal or orthorhombic) Anisotropic Material
Compliance Matrix Rotated (90° clockwise about the x-axis)

\[
\begin{pmatrix}
0.095 & -2.261 \times 10^{-3} & -0.044 & 0 & 0 & 0 \\
-2.261 \times 10^{-3} & 8.14 \times 10^{-3} & -2.261 \times 10^{-3} & 0 & 0 & 0 \\
-0.044 & -2.261 \times 10^{-3} & 0.065 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.261 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.378 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.161
\end{pmatrix} \text{ m}^2 \text{ GN}
\]

Density \hspace{1cm} 1577 \text{ kg/m}^3

Notes:


Carbon Fibre parameters supplied by B. O'Neill.

Unknown Orthogonal Anisotropic material parameters based on Carbon Fibre parameters.
Appendix B: Energy Conservation

Taking into account the relative area of each acoustic wave on the interface, the reflected and transmitted waves follow an overall energy conservation equation (technically intensity conservation)

\[ 1 = \sum_{\alpha} \frac{Z_{\alpha}}{Z_\alpha} \cdot r_\alpha^2 \cdot \frac{\cos(\theta_\alpha)}{\cos(\theta_i)} + \sum_{\beta} \frac{Z_{\beta}}{Z_\beta} \cdot t_\beta^2 \cdot \frac{\cos(\theta_\beta)}{\cos(\theta_i)}, \quad (C.1) \]

where the equation has been normalized with respect to the incident wave, and \( r \) and \( t \) are the reflected and transmitted pressure coefficients. To relate the pressure coefficients to the amplitude coefficients used in this thesis, one converts between the two using

\[ r_\alpha = \frac{-Z_{\alpha}}{Z_o} \cdot R_\alpha, \quad (C.2) \]

\[ t_\beta = \frac{Z_{\beta}}{Z_o} \cdot T_\beta, \quad (C.3) \]

Appendix C: Multilayer System Theory

Here we examine the specific case of normal incidence on the same thin layer system in Figure 2.1.1. In this case though, the waves internal to the thin layer are specifically found, and the displacement and traction continuity equations [equations (2.1.1) and (2.2.1), respectively] are taken at both boundaries (Figure B.1) with no approximation for the layer's size. At the first and second boundaries, respectively, the continuity equations in one dimension are

\[ l(x_0, t) + R(x_0, t) = A(x_0, t) + B(x_0, t), \]  
\[ -Z_1 \cdot l(x_0, t) + Z_1 \cdot R(x_0, t) = -Z_{\text{Int}} \cdot A(x_0, t) + Z_{\text{Int}} \cdot B(x_0, t), \]  
\[ A(x_0 + d, t) + B(x_0 + d, t) = T(x_0 + d, t), \]  
\[ -Z_{\text{Int}} \cdot A(x_0 + d, t) + Z_{\text{Int}} \cdot B(x_0 + d, t) = -Z_2 \cdot T(x_0 + d, t). \]

Because the thin layer is linear, isotropic, and attenuation is being neglected the internal waves \( A \) and \( B \) at each side of the interface are related via a phase change

\[ A(x_0 + d, t) = A \left( x_0, t - \frac{k}{\omega} d \right) = A(t - \delta), \]
\[ B(x_0, t) = B\left(x_0 + d, t - \frac{k}{\omega} \cdot d\right) = B(t - \delta), \quad (B.6) \]

Using the phase change and solving the boundary equations (B.1) through (B.4) for the reflected and transmitted waves yields

\[ R(t) = \frac{Z_1 - Z_{\text{Int}}}{Z_1 + Z_{\text{Int}}} \cdot I(t) + \frac{2 \cdot Z_{\text{Int}}}{Z_1 + Z_{\text{Int}}} \cdot B(t - \delta), \quad (B.7) \]

\[ T(t) = \frac{2 \cdot Z_{\text{Int}}}{Z_2 + Z_{\text{Int}}} \cdot A(t - \delta). \quad (B.8) \]

As the incident wave is known and the internal waves A and B need to be solved at this previous time \( t - \delta \), again equations (B.1) through (B.2) are used

\[ A(t) = \frac{2 \cdot Z_1}{Z_1 + Z_{\text{Int}}} \cdot I(t) - \frac{Z_1 - Z_{\text{Int}}}{(Z_1 + Z_{\text{Int}})} \cdot B(t - \delta), \quad (B.9) \]

\[ B(t) = \frac{Z_{\text{Int}} - Z_2}{Z_1 + Z_{\text{Int}}} \cdot A(t - \delta). \quad (B.10) \]

Equations (B.7) through (B.10) form a set of recursive formulas, and thus require an initial condition to be solved. The initial condition used is that before time \( t = 0 \) there are no acoustic waves. It may be possible to form a simpler solution from the equations (B.7) through (B.10), but for the purposes of a computer simulation one needs only to take time steps of size \( \frac{n \cdot d}{c} \) (where \( n \) is a positive integer) to produce an exact result.
Vita Auctoris

Jeff Sadler was born in 1976 in Chatham, Ontario. He graduated from Chatham Collegiate Institute in 1995. From there he went on to University of Guelph where he obtained a H. B. Sc. in Physics in 1999, and then obtained a B. Ed. at the University of Windsor in 2000. He then proceeded to obtain his Master's degree in Physics at the University of Windsor in 2003, and is currently a candidate for a Ph. D. in Physics at the University of Winsor.