Variable cutoff frequency 1-D and 2-D FIR and IIR filters.

Bruno Romanzin

University of Windsor

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VARIABLE CUTOFF FREQUENCY 1-D AND 2-D
FIR AND IIR FILTERS

by

Bruno Romanzin

A Thesis
submitted to the
Faculty of Graduate Studies and Research
through the Department of
Electrical Engineering in Partial Fulfillment
of the requirements for the Degree
of Master of Applied Science at
the University of Windsor

Windsor, Ontario, Canada

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To my parents Pietro and Pia nee Moro, my only living grandmother in Italy, Emma Moro nee Miculan, my brother Renato, and my many relatives and friends here and in Europe who I thank and love most dearly and who had and have to put up with me.
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ABSTRACT

Digital filtering has become one of the most advanced and the most indispensable means of processing analogue and digital data for electrical engineers today. Filters are used in an extremely wide range of practical everyday applications as in the medical field to very special uses as in space exploration. Each filter is used in a particular situation and for a specific use. Hence a special filter must be designed for each application. This can be costly when many different types of filters are needed on a particular apparatus. That is to say that each filter has a fixed cutoff frequency or frequencies.

This report emphasizes a technique by which digital filters, both 1-D and 2-D FIR and IIR filters can have a variable cutoff frequency. This tunability characteristic is developed for both the 1-D case and then expanded to the 2-D case. Many existing algorithms along with new algorithms have been incorporated into the development of these filters. Numerous examples are given to illustrate the usefulness of the proposed technique. Implementation issues for these filters are also presented, and tunability characteristics and maximum error are shown to analyze the quality of the filter through its different cutoff frequency values.

It is believed that by following the guidelines of this report, one can design fewer filters as compared to designing many fixed cutoff frequency filters and yet have greater versatility.
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\[ \omega_1 = 1.2 \text{ rad/s}, \quad \omega_2 = 2.9 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis

3.42 Three Dimensional Plot of Bandstop 2-D FIR Filter transformed to \( \omega_1 = 1.2 \text{ rad/s}, \quad \omega_2 = 2.9 \text{ rad/s} \)

3.43 Contour Plot of Bandstop 2-D FIR Filter transformed to \( \omega_1 = 1.2 \text{ rad/s}, \quad \omega_2 = 2.9 \text{ rad/s} \)

3.44 Lowpass 2-D FIR Filter with
\[ \omega_1 = 1.1 \text{ rad/s}, \quad \omega_2 = 1.1 \text{ rad/s} \]
along \( \omega_1 \) axis

3.45 Lowpass 2-D FIR Filter with
\[ \omega_1 = 1.1 \text{ rad/s}, \quad \omega_2 = 1.1 \text{ rad/s} \]
along \( \omega_2 \) axis

3.46 Lowpass 2-D FIR Filter with
\[ \omega_1 = 1.1 \text{ rad/s}, \quad \omega_2 = 1.1 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis

3.47 Three Dimensional Plot of Lowpass 2-D FIR Filter with \( \omega_1 = 1.1 \text{ rad/s}, \quad \omega_2 = 1.1 \text{ rad/s} \)

3.48 Contour Plot of Lowpass 2-D FIR Filter with
\[ \omega_1 = 1.1 \text{ rad/s}, \quad \omega_2 = 1.1 \text{ rad/s} \]
3.49 Lowpass 2-D FIR Filter transformed to
\[ \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \]
along \( \omega_1 \) axis

3.50 Lowpass 2-D FIR Filter transformed to
\[ \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \]
along \( \omega_2 \) axis

3.51 Lowpass 2-D FIR Filter transformed to
\[ \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis

3.52 Three Dimensional Plot of Lowpass 2-D FIR Filter
transformed to \( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \)

3.53 Contour Plot of Lowpass 2-D FIR Filter trans-
formed to \( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \)

3.54 Highpass 2-D FIR Filter transformed to
\[ \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \]
along \( \omega_1 \) axis

3.55 Highpass 2-D FIR Filter transformed to
\[ \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \]
along \( \omega_2 \) axis

3.56 Highpass 2-D FIR Filter transformed to
\[ \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis

3.57 Three Dimensional Plot of Highpass 2-D FIR Filter
transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \)

3.58 Contour Plot of Highpass 2-D FIR Filter trans-
formed to \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \)
3.59  Highpass 2-D FIR Filter transformed to 
      \( \omega_{c1} = 1.8 \text{ rad/s}, \omega_{c2} = 1.8 \text{ rad/s} \)
      along \( \omega_1 \) axis

3.60  Highpass 2-D FIR Filter transformed to 
      \( \omega_{c1} = 1.8 \text{ rad/s}, \omega_{c2} = 1.8 \text{ rad/s} \)
      along \( \omega_2 \) axis

3.61  Highpass 2-D FIR Filter transformed to 
      \( \omega_{c1} = 1.8 \text{ rad/s}, \omega_{c2} = 1.8 \text{ rad/s} \)
      along \( \omega_1 = \omega_2 \) axis

3.62  Three Dimensional Plot of Highpass 2-D FIR Filter 
      transformed to \( \omega_{c1} = 1.8 \text{ rad/s}, \omega_{c2} = 1.8 \text{ rad/s} \)

3.63  Contour Plot of Highpass 2-D FIR Filter trans-
      formed to \( \omega_{c1} = 1.8 \text{ rad/s}, \omega_{c2} = 1.8 \text{ rad/s} \)

3.64  Bandpass 2-D FIR Filter transformed to 
      \( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 2.5 \text{ rad/s} \)
      along \( \omega_1 \) axis

3.65  Bandpass 2-D FIR Filter transformed to 
      \( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 2.5 \text{ rad/s} \)
      along \( \omega_2 \) axis

3.66  Bandpass 2-D FIR Filter transformed to 
      \( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 2.5 \text{ rad/s} \)
      along \( \omega_1 = \omega_2 \) axis

3.67  Three Dimensional Plot of Bandpass 2-D FIR Filter 
      transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 2.5 \text{ rad/s} \)

3.68  Contour Plot of Bandpass 2-D FIR Filter trans-
      formed to \( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 2.5 \text{ rad/s} \)

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3.69 Bandpass 2-D FIR Filter transformed to
\( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 2.1 \text{ rad/s} \)
along \( \omega_1 \) axis

3.70 Bandpass 2-D FIR Filter transformed to
\( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 2.1 \text{ rad/s} \)
along \( \omega_2 \) axis

3.71 Bandpass 2-D FIR Filter transformed to
\( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 2.1 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis

3.72 Three Dimensional Plot of Bandpass 2-D FIR Filter
transformed to \( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 2.1 \text{ rad/s} \)

3.73 Contour Plot of Bandpass 2-D FIR Filter transformed to
\( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 2.1 \text{ rad/s} \)

3.74 Bandstop 2-D FIR Filter transformed to
\( \omega_{c1} = 1.3 \text{ rad/s}, \ \omega_{c2} = 2.3 \text{ rad/s} \)
along \( \omega_1 \) axis

3.75 Bandstop 2-D FIR Filter transformed to
\( \omega_{c1} = 1.3 \text{ rad/s}, \ \omega_{c2} = 2.3 \text{ rad/s} \)
along \( \omega_2 \) axis

3.76 Bandstop 2-D FIR Filter transformed to
\( \omega_{c1} = 1.3 \text{ rad/s}, \ \omega_{c2} = 2.3 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis

3.77 Three Dimensional Plot of Bandstop 2-D FIR Filter
transformed to \( \omega_{c1} = 1.3 \text{ rad/s}, \ \omega_{c2} = 2.3 \text{ rad/s} \)

3.78 Contour Plot of Bandstop 2-D FIR Filter transformed to
\( \omega_{c1} = 1.3 \text{ rad/s}, \ \omega_{c2} = 2.3 \text{ rad/s} \)
3.79 Bandstop 2-D FIR Filter transformed to
\[ \omega_{c1} = 1.2 \text{ rad/s}, \quad \omega_{c2} = 2.7 \text{ rad/s} \]
along \( \omega_1 \) axis

3.80 Bandstop 2-D FIR Filter transformed to
\[ \omega_{c1} = 1.2 \text{ rad/s}, \quad \omega_{c2} = 2.7 \text{ rad/s} \]
along \( \omega_2 \) axis

3.81 Bandstop 2-D FIR Filter transformed to
\[ \omega_{c1} = 1.2 \text{ rad/s}, \quad \omega_{c2} = 2.7 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis

3.82 Three Dimensional Plot of Bandstop 2-D FIR Filter transformed to \( \omega_{c1} = 1.2 \text{ rad/s}, \quad \omega_{c2} = 2.7 \text{ rad/s} \)

3.83 Contour Plot of Bandpass 2-D FIR Filter transformed to \( \omega_{c1} = 1.2 \text{ rad/s}, \quad \omega_{c2} = 2.7 \text{ rad/s} \)

3.84 Elliptical Lowpass 2-D FIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \quad \omega_{c2} = 1.7 \text{ rad/s} \)
along \( \omega_1 \) axis

3.85 Elliptical Lowpass 2-D FIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \quad \omega_{c2} = 1.7 \text{ rad/s} \)
along \( \omega_2 \) axis

3.86 Elliptical Lowpass 2-D FIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \quad \omega_{c2} = 1.7 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis

3.87 Three Dimensional Plot of Elliptical Lowpass 2-D FIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \quad \omega_{c2} = 1.7 \text{ rad/s} \)

3.88 Contour Plot of Elliptical Lowpass 2-D FIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \quad \omega_{c2} = 1.7 \text{ rad/s} \)
3.89  Elliptical Lowpass 2-D FIR Filter transformed to 
\[ \omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s} \] 
along \( \omega_1 \) axis

3.90  Elliptical Lowpass 2-D FIR Filter transformed to 
\[ \omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s} \] 
along \( \omega_2 \) axis

3.91  Elliptical Lowpass 2-D FIR Filter transformed to 
\[ \omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s} \] 
along \( \omega_1 = \omega_2 \) axis

3.92  Three Dimensional Plot of Elliptical Lowpass 2-D FIR Filter transformed to 
\[ \omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s} \]

3.93  Contour Plot of Elliptical Lowpass 2-D FIR Filter transformed to 
\[ \omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s} \]

3.94  Elliptical Highpass 2-D FIR Filter transformed to 
\[ \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.4 \text{ rad/s} \] 
along \( \omega_1 \) axis

3.95  Elliptical Highpass 2-D FIR Filter transformed to 
\[ \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.4 \text{ rad/s} \] 
along \( \omega_2 \) axis

3.96  Elliptical Highpass 2-D FIR Filter transformed to 
\[ \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.4 \text{ rad/s} \] 
along \( \omega_1 = \omega_2 \) axis

3.97  Three Dimensional Plot of Elliptical Highpass 2-D FIR Filter transformed to 
\[ \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.4 \text{ rad/s} \]

3.98  Contour Plot of Elliptical Highpass 2-D FIR Filter transformed to 
\[ \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.4 \text{ rad/s} \]
Elliptical Highpass 2-D FIR Filter transformed to
\( \omega_{c1} = 2.0 \text{ rad/s, } \omega_{c2} = 1.3 \text{ rad/s} \)
along \( \omega_1 \) axis

Elliptical Highpass 2-D FIR Filter transformed to
\( \omega_{c1} = 2.0 \text{ rad/s, } \omega_{c2} = 1.3 \text{ rad/s} \)
along \( \omega_2 \) axis

Elliptical Highpass 2-D FIR Filter transformed to
\( \omega_{c1} = 2.0 \text{ rad/s, } \omega_{c2} = 1.3 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis

Three Dimensional Plot of Elliptical Highpass 2-D FIR Filter transformed to
\( \omega_{c1} = 2.0 \text{ rad/s, } \omega_{c2} = 1.3 \text{ rad/s} \)

Contour Plot of Elliptical Highpass 2-D FIR Filter transformed to
\( \omega_{c1} = 2.0 \text{ rad/s, } \omega_{c2} = 1.3 \text{ rad/s} \)

Elliptical Bandpass 2-D FIR Filter transformed to
\( \omega_{c11} = 1.2 \text{ rad/s, } \omega_{c12} = 1.9 \text{ rad/s}, \)
\( \omega_{c21} = 1.6 \text{ rad/s, } \omega_{c22} = 2.3 \text{ rad/s}, \)
along \( \omega_1 \) axis

Elliptical Bandpass 2-D FIR Filter transformed to
\( \omega_{c11} = 1.2 \text{ rad/s, } \omega_{c12} = 1.9 \text{ rad/s}, \)
\( \omega_{c21} = 1.6 \text{ rad/s, } \omega_{c22} = 2.3 \text{ rad/s}, \)
along \( \omega_2 \) axis

Elliptical Bandpass 2-D FIR Filter transformed to
\( \omega_{c11} = 1.2 \text{ rad/s, } \omega_{c12} = 1.9 \text{ rad/s}, \)
\( \omega_{c21} = 1.6 \text{ rad/s, } \omega_{c22} = 2.3 \text{ rad/s}, \)
along \( \omega_1 = \omega_2 \) axis
Three Dimensional Plot of Elliptical Bandpass

2-D FIR Filter transformed to
\[ \omega_{c11} = 1.2 \text{ rad/s}, \; \omega_{c12} = 1.9 \text{ rad/s}, \]
\[ \omega_{c21} = 1.6 \text{ rad/s}, \; \omega_{c22} = 2.3 \text{ rad/s} \]

Contour Plot of Elliptical Bandpass

2-D FIR Filter transformed to
\[ \omega_{c11} = 1.2 \text{ rad/s}, \; \omega_{c12} = 1.9 \text{ rad/s}, \]
\[ \omega_{c21} = 1.6 \text{ rad/s}, \; \omega_{c22} = 2.3 \text{ rad/s} \]

Elliptical Bandpass 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.4 \text{ rad/s}, \; \omega_{c12} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.1 \text{ rad/s}, \; \omega_{c22} = 2.4 \text{ rad/s}, \]
along \( \omega_1 \) axis

Elliptical Bandpass 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.4 \text{ rad/s}, \; \omega_{c12} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.1 \text{ rad/s}, \; \omega_{c22} = 2.4 \text{ rad/s}, \]
along \( \omega_2 \) axis

Elliptical Bandpass 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.4 \text{ rad/s}, \; \omega_{c12} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.1 \text{ rad/s}, \; \omega_{c22} = 2.4 \text{ rad/s}, \]
along \( \omega_1 = \omega_2 \) axis

Three Dimensional Plot of Elliptical Bandpass

2-D FIR Filter transformed to
\[ \omega_{c11} = 1.4 \text{ rad/s}, \; \omega_{c12} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.1 \text{ rad/s}, \; \omega_{c22} = 2.4 \text{ rad/s} \]

Contour Plot of Elliptical Bandpass

2-D FIR Filter transformed to
\[ \omega_{c11} = 1.4 \text{ rad/s}, \; \omega_{c12} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.1 \text{ rad/s}, \; \omega_{c22} = 2.4 \text{ rad/s} \]
3.114 Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.1 \text{ rad/s}, \quad \omega_{c12} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.3 \text{ rad/s}, \quad \omega_{c22} = 2.0 \text{ rad/s}, \]
along \( \omega_1 \) axis

3.115 Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.1 \text{ rad/s}, \quad \omega_{c12} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.3 \text{ rad/s}, \quad \omega_{c22} = 2.0 \text{ rad/s}, \]
along \( \omega_2 \) axis

3.116 Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.1 \text{ rad/s}, \quad \omega_{c12} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.3 \text{ rad/s}, \quad \omega_{c22} = 2.0 \text{ rad/s}, \]
along \( \omega_1 = \omega_2 \) axis

3.117 Three Dimensional Plot of Elliptical Bandstop
2-D FIR Filter transformed to
\[ \omega_{c11} = 1.1 \text{ rad/s}, \quad \omega_{c12} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.3 \text{ rad/s}, \quad \omega_{c22} = 2.0 \text{ rad/s} \]

3.118 Contour Plot of Elliptical Bandstop
2-D FIR Filter transformed to
\[ \omega_{c11} = 1.1 \text{ rad/s}, \quad \omega_{c12} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.3 \text{ rad/s}, \quad \omega_{c22} = 2.0 \text{ rad/s} \]

3.119 Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.2 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.6 \text{ rad/s}, \]
along \( \omega_1 \) axis

3.120 Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.2 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.6 \text{ rad/s}, \]
along \( \omega_2 \) axis
3.121 Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.2 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.6 \text{ rad/s}, \]
along \( \omega_1 = \omega_2 \) axis

3.122 Three Dimensional Plot of Elliptical Bandstop
2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.2 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.6 \text{ rad/s} \]

3.123 Contour Plot of Elliptical Bandstop
2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.2 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.6 \text{ rad/s} \]

3.124 Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.5 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.9 \text{ rad/s}, \]
along \( \omega_1 \) axis

3.125 Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.5 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.9 \text{ rad/s}, \]
along \( \omega_2 \) axis

3.126 Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.5 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.9 \text{ rad/s}, \]
along \( \omega_1 = \omega_2 \) axis

3.127 Three Dimensional Plot of Elliptical Bandstop
2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.5 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.9 \text{ rad/s} \]
3.128 Contour Plot of Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \ \omega_{c12} = 2.5 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \ \omega_{c22} = 2.9 \text{ rad/s} \]

4.1 Mapping of \( \omega = \omega_c \) onto the \( \omega_1 - \omega_2 \) for different values of \( b \)

4.2 Third-order Lowpass Butterworth filter 3-D plot using the 2-Variable Reactance Function: 
\( a_1 = a_2 = 1, \ b = 0.1 \)

4.3 Third-order Lowpass Butterworth filter contour plot using the 2-Variable Reactance Function: 
\( a_1 = a_2 = 1, \ b = 0.1 \)

4.4 Third-order Lowpass Butterworth filter 3-D plot using the 2-Variable Reactance Function: 
\( a_1 = a_2 = 1, \ b = 0.01 \)

4.5 Third-order Lowpass Butterworth filter contour plot using the 2-Variable Reactance Function: 
\( a_1 = a_2 = 1, \ b = 0.01 \)

4.6 Third-order Lowpass Butterworth filter 3-D plot using the 2-Variable Reactance Function: 
\( a_1 = a_2 = 2, \ b = 0.01 \)

4.7 Third-order Lowpass Butterworth filter contour plot using the 2-Variable Reactance Function: 
\( a_1 = a_2 = 2, \ b = 0.01 \)

4.8 Third-order Lowpass Butterworth filter 3-D plot using the 2-Variable Reactance Function: 
\( a_1 = a_2 = 0.5, \ b = 0.01 \)

4.9 Third-order Lowpass Butterworth filter contour plot using the 2-Variable Reactance Function: 
\( a_1 = a_2 = 0.5, \ b = 0.01 \)
4.10 Third-order Lowpass Butterworth filter 3-D plot using the 2-Variable Reactance Function:
a_1 = a_2 = 1, b = 0.5

4.11 Third-order Lowpass Butterworth filter contour plot using the 2-Variable Reactance Function:
a_1 = a_2 = 1, b = 0.5

4.12 Third-order Lowpass Butterworth filter 3-D plot using the 2-Variable Reactance Function:
a_1 = a_2 = 1, b = 0.2

4.13 Third-order Lowpass Butterworth filter contour plot using the 2-Variable Reactance Function:
a_1 = a_2 = 1, b = 0.2

4.14 Special Case (square type) Lowpass 2-D IIR Filter with \( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s} \) along \( \omega_1 \) axis

4.15 Special Case (square type) Lowpass 2-D IIR Filter with \( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s} \) along \( \omega_2 \) axis

4.16 Special Case (square type) Lowpass 2-D IIR Filter with \( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s} \) along \( \omega_1 = \omega_2 \) axis

4.17 Three Dimensional Plot of Special Case (square type) Lowpass 2-D IIR Filter with \( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s} \)

4.18 Contour Plot of Special Case (square type) Lowpass 2-D IIR Filter with \( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s} \)
4.19 Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 1.5$ rad/s along $\omega_1$ axis

4.20 Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 1.5$ rad/s along $\omega_2$ axis

4.21 Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 1.5$ rad/s along $\omega_1 = \omega_2$ axis

4.22 Three Dimensional Plot of Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 1.5$ rad/s

4.23 Contour Plot of Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 1.5$ rad/s

4.24 Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 0.7$ rad/s, $\omega_{c2} = 0.7$ rad/s along $\omega_1$ axis

4.25 Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 0.7$ rad/s, $\omega_{c2} = 0.7$ rad/s along $\omega_2$ axis

4.26 Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 0.7$ rad/s, $\omega_{c2} = 0.7$ rad/s along $\omega_1 = \omega_2$ axis

4.27 Three Dimensional Plot of Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 0.7$ rad/s, $\omega_{c2} = 0.7$ rad/s
4.28 Contour Plot of Special Case (square type)  
Lowpass 2-D IIR Filter transformed to  
$\omega_{c1} = 0.7$ rad/s, $\omega_{c2} = 0.7$ rad/s  

4.29 Type 1 Lowpass 2-D IIR Filter with  
$\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s  
along $\omega_1$ axis  

4.30 Type 1 Lowpass 2-D IIR Filter with  
$\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s  
along $\omega_2$ axis  

4.31 Type 1 Lowpass 2-D IIR Filter with  
$\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s  
along $\omega_1 = \omega_2$ axis  

4.32 Three Dimensional Plot of Type 1 Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s  

4.33 Contour Plot of Type 1 Lowpass 2-D IIR Filter  
with $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s  

4.34 Type 1 Lowpass 2-D IIR Filter transformed to  
$\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 1.5$ rad/s  
along $\omega_1$ axis  

4.35 Type 1 Lowpass 2-D IIR Filter transformed to  
$\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 1.5$ rad/s  
along $\omega_2$ axis  

4.36 Type 1 Lowpass 2-D IIR Filter transformed to  
$\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 1.5$ rad/s  
along $\omega_1 = \omega_2$ axis  

4.37 Three Dimensional Plot of Type 1 Lowpass 2-D IIR Filter transformed to  
$\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 1.5$ rad/s  

xxx
4.38 Contour Plot of Type 1 Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 1.5$ rad/s

4.39 Type 1 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s along $\omega_1$ axis

4.40 Type 1 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s along $\omega_2$ axis

4.41 Type 1 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s along $\omega_1 = \omega_2$ axis

4.42 Three Dimensional Plot of Type 1 Highpass 2-D IIR Filter with $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s

4.43 Contour Plot of Type 1 Highpass 2-D IIR Filter with $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s

4.44 Type 1 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s along $\omega_1$ axis

4.45 Type 1 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s along $\omega_2$ axis

4.46 Type 1 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s along $\omega_1 = \omega_2$ axis

4.47 Three Dimensional Plot of Type 1 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s
4.48 Contour Plot of Type 1 Highpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.8 \text{ rad/s}, \ \omega_{c2} = 1.8 \text{ rad/s} \)

4.49 Type 1 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \) along \( \omega_1 \) axis

4.50 Type 1 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \) along \( \omega_2 \) axis

4.51 Type 1 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 3.5 \text{ rad/s} \) along \( \omega_1 = \omega_2 \) axis

4.52 Three Dimensional Plot of Type 1 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \)

4.53 Contour Plot of Type 1 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \)

4.54 Type 1 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 2.1 \text{ rad/s} \) along \( \omega_1 \) axis

4.55 Type 1 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 2.1 \text{ rad/s} \) along \( \omega_2 \) axis

4.56 Type 1 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 2.1 \text{ rad/s} \) along \( \omega_1 = \omega_2 \) axis

4.57 Three Dimensional Plot of Type 1 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 2.1 \text{ rad/s} \)
4.58 Contour Plot of Type 1 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.5 \text{ rad/s}, \; \omega_{c2} = 2.1 \text{ rad/s} 
\)

4.59 Type 1 Bandstop 2-D IIR Filter transformed to \( \omega_{c1} = 1.3 \text{ rad/s}, \; \omega_{c2} = 2.3 \text{ rad/s} 
\) along \( \omega_1 \) axis

4.60 Type 1 Bandstop 2-D IIR Filter transformed to \( \omega_{c1} = 1.3 \text{ rad/s}, \; \omega_{c2} = 2.3 \text{ rad/s} 
\) along \( \omega_2 \) axis

4.61 Type 1 Bandstop 2-D IIR Filter transformed to \( \omega_{c1} = 1.3 \text{ rad/s}, \; \omega_{c2} = 2.3 \text{ rad/s} 
\) along \( \omega_1 = \omega_2 \) axis

4.62 Three Dimensional Plot of Type 1 Bandstop 2-D IIR Filter transformed to \( \omega_{c1} = 1.3 \text{ rad/s}, \; \omega_{c2} = 2.3 \text{ rad/s} 
\)

4.63 Contour Plot of Type 1 Bandstop 2-D IIR Filter transformed to \( \omega_{c1} = 1.3 \text{ rad/s}, \; \omega_{c2} = 2.3 \text{ rad/s} 
\)

4.64 Type 1 Bandstop 2-D IIR Filter transformed to \( \omega_{c1} = 1.2 \text{ rad/s}, \; \omega_{c2} = 2.7 \text{ rad/s} 
\) along \( \omega_1 \) axis

4.65 Type 1 Bandstop 2-D IIR Filter transformed to \( \omega_{c1} = 1.2 \text{ rad/s}, \; \omega_{c2} = 2.7 \text{ rad/s} 
\) along \( \omega_2 \) axis

4.66 Type 1 Bandstop 2-D IIR Filter transformed to \( \omega_{c1} = 1.2 \text{ rad/s}, \; \omega_{c2} = 2.7 \text{ rad/s} 
\) along \( \omega_1 = \omega_2 \) axis

4.67 Three Dimensional Plot of Type 1 Bandstop 2-D IIR Filter transformed to \( \omega_{c1} = 1.2 \text{ rad/s}, \; \omega_{c2} = 2.7 \text{ rad/s} 
\)
4.68 Contour Plot of Type 1 Bandstop 2-D IIR Filter transformed to \( \omega_{c1} = 1.2 \text{ rad/s}, \ \omega_{c2} = 2.7 \text{ rad/s} \)

4.69 Type 2 Lowpass 2-D IIR Filter with \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \) along \( \omega_1 \) axis

4.70 Type 2 Lowpass 2-D IIR Filter with \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \) along \( \omega_2 \) axis

4.71 Type 2 Lowpass 2-D IIR Filter with \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \) along \( \omega_1 = \omega_2 \) axis

4.72 Three Dimensional Plot of Type 2 Lowpass 2-D IIR Filter with \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \)

4.73 Contour Plot of Type 2 Lowpass 2-D IIR Filter with \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \)

4.74 Type 2 Lowpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \) along \( \omega_1 \) axis

4.75 Type 2 Lowpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \) along \( \omega_2 \) axis

4.76 Type 2 Lowpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \) along \( \omega_1 = \omega_2 \) axis

4.77 Three Dimensional Plot of Type 2 Lowpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \)
4.78 Contour Plot of Type 2 Lowpass 2-D IIR Filter transformed to $\omega_c = 1.5$ rad/s, $\omega_c = 1.5$ rad/s

4.79 Type 2 Highpass 2-D IIR Filter transformed to $\omega_c = 1.1$ rad/s, $\omega_c = 1.1$ rad/s along $\omega_1$ axis

4.80 Type 2 Highpass 2-D IIR Filter transformed to $\omega_c = 1.1$ rad/s, $\omega_c = 1.1$ rad/s along $\omega_2$ axis

4.81 Type 2 Highpass 2-D IIR Filter transformed to $\omega_c = 1.1$ rad/s, $\omega_c = 1.1$ rad/s along $\omega_1 = \omega_2$ axis

4.82 Three Dimensional Plot of Type 2 Highpass 2-D IIR Filter with $\omega_c = 1.1$ rad/s, $\omega_c = 1.1$ rad/s

4.83 Contour Plot of Type 2 Highpass 2-D IIR Filter with $\omega_c = 1.1$ rad/s, $\omega_c = 1.1$ rad/s

4.84 Type 2 Highpass 2-D IIR Filter transformed to $\omega_c = 1.8$ rad/s, $\omega_c = 1.8$ rad/s along $\omega_1$ axis

4.85 Type 2 Highpass 2-D IIR Filter transformed to $\omega_c = 1.8$ rad/s, $\omega_c = 1.8$ rad/s along $\omega_2$ axis

4.86 Type 2 Highpass 2-D IIR Filter transformed to $\omega_c = 1.8$ rad/s, $\omega_c = 1.8$ rad/s along $\omega_1 = \omega_2$ axis

4.87 Three Dimensional Plot of Type 2 Highpass 2-D IIR Filter transformed to $\omega_c = 1.8$ rad/s, $\omega_c = 1.8$ rad/s
4.88 Contour Plot of Type 2 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s

4.89 Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 2.5$ rad/s along $\omega_1$ axis

4.90 Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 2.5$ rad/s along $\omega_2$ axis

4.91 Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 2.5$ rad/s along $\omega_1 = \omega_2$ axis

4.92 Three Dimensional Plot of Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 2.5$ rad/s

4.93 Contour Plot of Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 2.5$ rad/s

4.94 Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 2.6$ rad/s along $\omega_1$ axis

4.95 Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 2.6$ rad/s along $\omega_2$ axis

4.96 Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 2.6$ rad/s along $\omega_1 = \omega_2$ axis
4.97 Three Dimensional Plot of Type 2 Bandpass 2-D IIR Filter transformed to
$\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 2.6 \text{ rad/s}$

4.98 Contour Plot of Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 2.6 \text{ rad/s}$

4.99 Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 2.4 \text{ rad/s}$
along $\omega_1$ axis

4.100 Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 2.4 \text{ rad/s}$
along $\omega_2$ axis

4.101 Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 2.4 \text{ rad/s}$
along $\omega_1 = \omega_2$ axis

4.102 Three Dimensional Plot of Type 2 Bandstop 2-D IIR Filter transformed to
$\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 2.4 \text{ rad/s}$

4.103 Contour Plot of Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 2.4 \text{ rad/s}$

4.104 Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2 \text{ rad/s}, \omega_{c2} = 2.7 \text{ rad/s}$
along $\omega_1$ axis

4.105 Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2 \text{ rad/s}, \omega_{c2} = 2.7 \text{ rad/s}$
along $\omega_2$ axis

4.106 Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2 \text{ rad/s}, \omega_{c2} = 2.7 \text{ rad/s}$
along $\omega_1 = \omega_2$ axis
4.107 Three Dimensional Plot of Type 2 Bandstop 2-D IIR Filter transformed to
\[ \omega_{c1} = 1.2 \text{ rad/s}, \ \omega_{c2} = 2.7 \text{ rad/s} \]

4.108 Contour Plot of Type 2 Bandstop 2-D IIR Filter transformed to \[ \omega_{c1} = 1.2 \text{ rad/s}, \ \omega_{c2} = 2.7 \text{ rad/s} \]

4.109 3-D plot of the amplitude response of the designed 2-D lowpass filter as given by Dr. Ahmadi et al. [5]

4.110 Type 3 Lowpass 2-D IIR Filter with \[ \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \] along \( \omega_1 \) axis

4.111 Type 3 Lowpass 2-D IIR Filter with \[ \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \] along \( \omega_2 \) axis

4.112 Type 3 Lowpass 2-D IIR Filter with \[ \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \] along \( \omega_1 = \omega_2 \) axis

4.113 Three Dimensional Plot of Type 3 Lowpass 2-D IIR Filter with \[ \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \]

4.114 Contour Plot of Type 3 Lowpass 2-D IIR Filter with \[ \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \]

4.115 Type 3 Lowpass 2-D IIR Filter transformed to \[ \omega_{c1} = 3.5 \text{ rad/s}, \ \omega_{c2} = 3.5 \text{ rad/s} \] along \( \omega_1 \) axis

4.116 Type 3 Lowpass 2-D IIR Filter transformed to \[ \omega_{c1} = 3.5 \text{ rad/s}, \ \omega_{c2} = 3.5 \text{ rad/s} \] along \( \omega_2 \) axis
4.117 Type 3 Lowpass 2-D IIR Filter transformed to
\[ \omega_{c1} = 3.5 \text{ rad/s}, \ \omega_{c2} = 3.5 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis

4.118 Three Dimensional Plot of Type 3 Lowpass 2-D IIR Filter transformed to
\[ \omega_{c1} = 3.5 \text{ rad/s}, \ \omega_{c2} = 3.5 \text{ rad/s} \]

4.119 Contour Plot of Type 3 Lowpass 2-D IIR Filter transformed to \( \omega_{c1} = 3.5 \text{ rad/s}, \ \omega_{c2} = 3.5 \text{ rad/s} \)

4.120 Type 3 Highpass 2-D IIR Filter transformed to \( \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \)
along \( \omega_1 \) axis

4.121 Type 3 Highpass 2-D IIR Filter transformed to \( \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \)
along \( \omega_2 \) axis

4.122 Type 3 Highpass 2-D IIR Filter transformed to \( \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis

4.123 Three Dimensional Plot of Type 3 Highpass 2-D IIR Filter with \( \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \)

4.124 Contour Plot of Type 3 Highpass 2-D IIR Filter with \( \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \)

4.125 Type 3 Highpass 2-D IIR Filter transformed to \( \omega_{c1} = 3.8 \text{ rad/s}, \ \omega_{c2} = 3.8 \text{ rad/s} \)
along \( \omega_1 \) axis

4.126 Type 3 Highpass 2-D IIR Filter transformed to \( \omega_{c1} = 3.8 \text{ rad/s}, \ \omega_{c2} = 3.8 \text{ rad/s} \)
along \( \omega_2 \) axis
4.127 Type 3 Highpass 2-D IIR Filter transformed to \( \omega_{c1} = 3.8 \text{ rad/s}, \ \omega_{c2} = 3.8 \text{ rad/s} \) along \( \omega_1 = \omega_2 \) axis

4.128 Three Dimensional Plot of Type 3 Highpass 2-D IIR Filter transformed to \( \omega_{c1} = 3.8 \text{ rad/s}, \ \omega_{c2} = 3.8 \text{ rad/s} \)

4.129 Contour Plot of Type 3 Highpass 2-D IIR Filter transformed to \( \omega_{c1} = 3.8 \text{ rad/s}, \ \omega_{c2} = 3.8 \text{ rad/s} \)

4.130 Type 3 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 3.7 \text{ rad/s} \) along \( \omega_1 \) axis

4.131 Type 3 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 3.7 \text{ rad/s} \) along \( \omega_2 \) axis

4.132 Type 3 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 3.7 \text{ rad/s} \) along \( \omega_1 = \omega_2 \) axis

4.133 Three Dimensional Plot of Type 3 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 3.7 \text{ rad/s} \)

4.134 Contour Plot of Type 3 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 3.7 \text{ rad/s} \)

4.135 Type 3 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 3.0 \text{ rad/s}, \ \omega_{c2} = 4.2 \text{ rad/s} \) along \( \omega_1 \) axis

4.136 Type 3 Bandpass 2-D IIR Filter transformed to \( \omega_{c1} = 3.0 \text{ rad/s}, \ \omega_{c2} = 4.2 \text{ rad/s} \) along \( \omega_2 \) axis

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4.137 Type 3 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 3.0 \text{ rad/s}, \omega_{c2} = 4.2 \text{ rad/s} \quad 358$
along $\omega_1 = \omega_2$ axis

4.138 Three Dimensional Plot of Type 3 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 3.0 \text{ rad/s}, \omega_{c2} = 4.2 \text{ rad/s} \quad 359$

4.139 Contour Plot of Type 3 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 3.0 \text{ rad/s}, \omega_{c2} = 4.2 \text{ rad/s} \quad 360$

4.140 Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 4.1 \text{ rad/s} \quad 361$
along $\omega_1$ axis

4.141 Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 4.1 \text{ rad/s} \quad 362$
along $\omega_2$ axis

4.142 Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 4.1 \text{ rad/s} \quad 363$
along $\omega_1 = \omega_2$ axis

4.143 Three Dimensional Plot of Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 4.1 \text{ rad/s} \quad 364$

4.144 Contour Plot of Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 4.1 \text{ rad/s} \quad 365$

4.145 Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.7 \text{ rad/s}, \omega_{c2} = 3.9 \text{ rad/s} \quad 366$
along $\omega_1$ axis

4.146 Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.7 \text{ rad/s}, \omega_{c2} = 3.9 \text{ rad/s} \quad 367$
along $\omega_2$ axis
4.147 Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.7 \text{ rad/s}, \omega_{c2} = 3.9 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis

4.148 Three Dimensional Plot of Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.7 \text{ rad/s}, \omega_{c2} = 3.9 \text{ rad/s}$

4.149 Contour Plot of Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.7 \text{ rad/s}, \omega_{c2} = 3.9 \text{ rad/s}$

4.150 Type 4 Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}$ along $\omega_1$ axis

4.151 Type 4 Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}$ along $\omega_2$ axis

4.152 Type 4 Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis

4.153 Three Dimensional Plot of Type 4 Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}$

4.154 Contour Plot of Type 4 Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}$

4.155 Type 4 Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s}$ along $\omega_1$ axis

4.156 Type 4 Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s}$ along $\omega_2$ axis
4.157 Type 4 Lowpass 2-D IIR Filter transformed to
\[ \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis

4.158 Three Dimensional Plot of Type 4 Lowpass 2-D IIR Filter transformed to
\[ \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \]

4.159 Contour Plot of Type 4 Lowpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \)

4.160 Type 4 Highpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \)
along \( \omega_1 \) axis

4.161 Type 4 Highpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \)
along \( \omega_2 \) axis

4.162 Type 4 Highpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis

4.163 Three Dimensional Plot of Type 4 Highpass 2-D IIR Filter with \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \)

4.164 Contour Plot of Type 4 Highpass 2-D IIR Filter with \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \)

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I. INTRODUCTION

Since the onset of digital filtering, there has been a rapid rise in the development and widespread application of this technology. From the widely and well developed one-dimensional filtering to the recent development of two-dimensional filtering.

One dimensional filters have had many uses in signal processing. They have been utilized in such important and diverse areas as biomedical engineering, acoustics, music, sonar, radar, seismology, speech communication, data communication, nuclear science, digital filter synthesis, the design and synthesis of digital spectrum analyzers, as well as the field of computer design [1].

The ever so important and increasing recognition of 2-D filtering has found many practical applications of two-dimensional signals such as in the analysis and enhancement of computer images such as aerial photographs, satellite weather photos, and television transmissions, in lunar and deep-space probes [2,3,4], in geophysical and seismic data arrays [1-6], in medicine and biology such as X-ray enhancement [2,3,4,6], in nondestructive testing [3].

There are two types of filters which are separated as: those which have finite impulse response (FIR), also known as nonrecursive filters, and those which have infinite impulse response (IIR), also known as recursive filters. This applies both to the 1-D and the 2-D cases [3].
1.1 CHARACTERIZATION OF 1-D FIR AND IIR FILTERS

A 1-D FIR or 1-D nonrecursive digital filter of order \(2N+1\) can be characterized by the transfer function [7,8]

\[
H(z) = z^{-N}H_0(z)
\]  
(1.1a)

\[
N = \sum_{n=0}^{N} h_0(n)z^{-n}
\]  
(1.1b)

where

\[
H_0(z) = h_0(0) + \sum_{n=1}^{N} h_0(n)[z^{n} + z^{-n}]
\]  
(1.1c)

where \(h_0(n)\) are the impulse response coefficients of the digital filter.

For a 1-D IIR or 1-D recursive digital filter can be characterized by the transfer function [4,9]

\[
H(z) = \frac{\sum_{i=0}^{M} a_i z^{-i}}{\sum_{j=0}^{N} b_j z^{-j}}
\]  
(1.2)

where \(N\) and \(D\) are polynomials in \(z\) (= exp \(sT\)), and the polynomial coefficients \(a_i\) and \(b_j\) are found using any one of the design techniques which are discussed below.

1.2 DESIGN APPROACHES FOR 1-D FIR AND IIR FILTERS

There are a numerous variety of design techniques available for 1-D FIR and IIR filters [3]. These techniques range from the closed-form to the algorithmic techniques. The former involves only substitution of design specifications into design formulas, while the latter is used where a solution is obtained by an
iterative procedure.

1.2.1. Design Approaches for 1-D FIR Filters

Some of the most common design methods for the 1-D FIR filters is the Fourier Series method in conjunction with windows. The most common and frequently used are the Hann, Hamming, Blackman, and Kaiser windows [2,10]. Iterative techniques using linear or non-linear programming is another design method. These include computer-aided optimization techniques which are a very powerful method [10,11].

1.2.2. Design Approaches for 1-D IIR Filters

Just as for the case of 1-D FIR filter design, there also as many types of design techniques for the 1-D IIR filters. The most traditional approach involves the discretization of an analogue filter to a digital filter via spectral transformations [4,10]. The most useful analogue filters are the Butterworth, Chebyshev, Elliptical, and Bessel filters, for which there is vast knowledge and a large supply of tables readily available for these types of filters [4]. These analogue design formulas approximate the magnitude and/or the phase responses of the ideal filter specifications. The analogue filter is then converted to a digital filter by applying any one of these spectral transformations, such as the bilinear transformation which is one of the most commonly used [10]. Again, the iterative techniques using linear or non-linear programming can be employed as another design method for the 1-D IIR filters as for the 1-D FIR filters. This is an optimization technique where an iterative method is used in order to obtain a minimum error between the designed
filter and the ideal filter specifications. This technique can also be used in cases where more than one passband or stopband is needed.

Whenever a filter is designed, it is designed with certain specifications such as the cutoff frequency, or the passband and the stopband edges. The various required specifications are then implemented so that the final filter cannot be varied. That is if one requires a filter with a different cutoff frequency or different passband and stopband edges, then a whole filter must be designed for the new specifications. Anytime a different cutoff frequency is required, or certain different filter specifications are needed, then a new filter must be designed. That is, all the filter coefficients must be changed. This can be very labourious and costly in applications where one or any of the various specifications needs to be changed during the operation process of the filter. Thus some effective method of changing the cutoff frequency, the passband and stopband edges of the filter, or in other words, a method of filter 'tunability', would be desirable.

This thesis examines a method for designing 1-D FIR and 1-D IIR filters which requires fewer coefficient changes than with changing the entire filter coefficients in order to obtain the set specifications.
1.3 INTRODUCTION OF 2-D FIR AND IIR FILTERS

As it was mentioned above, the greatly increasing interest of 2-D filtering has found many practical applications of two-dimensional signals in such important areas as in the processing of computer image processing and enhancement, in geophysical and seismic data arrays, and in medicine and biology [1-6].

1.3.1. Characterization of 2-D FIR and IIR Filters

A 2-D digital FIR filter is characterized by the transfer function [3,12]

\[ H(z_1, z_2) = \sum_{i=0}^{M} \sum_{j=0}^{N} h(i,j)z_1^{-i}z_2^{-j} \]  

(1.3)

where the \( h(i,j) \)'s are the impulse response coefficients of the filter.

A 2-D analogue filter (IIR) is characterized by the transfer function [5,6,13,14]

\[ H(s_1, s_2) = \frac{N(s_1, s_2)}{D(s_1, s_2)} \]  

(1.4a)

\[ \sum_{i=0}^{M} \sum_{j=0}^{N} c(i,j)s_1^{-i}s_2^{-j} \]

(1.4b)

\[ \sum_{i=0}^{M} \sum_{j=0}^{N} d(i,j)s_1^{-i}s_2^{-j} \]

where \( N \) and \( D \) are polynomials in \( s_1 \) and \( s_2 \). The design problem is to obtain the polynomial coefficients \( c(i,j) \) and \( d(i,j) \) such that \( H(s_1, s_2) \) approximates a required response, and that the designed filter is stable. That is [5,6,13],
\[ D(s, s_\ast) \neq 0 \text{ for } \bigwedge_{i=1}^{2} \text{Re } s_i \geq 0 \] (1.5)

Similarly, a 2-D recursive digital filter (IIR) is characterized by the transfer function \([5,6,13]\)

\[ H(z, z_\ast) = \frac{N(z, z_\ast)}{D(z, z_\ast)} \] (1.6a)

\[ = \frac{\sum_{i=0}^{M} \sum_{j=0}^{N} a(i,j) z_i^{-1} z_\ast^{-j}}{\sum_{i=0}^{M} \sum_{j=0}^{N} b(i,j) z_i^{-1} z_\ast^{-j}} \] (1.6b)

where \(N\) and \(D\) are polynomials in
\[ z_i = e^{j\omega_i}, \quad z_\ast = e^{j\omega_\ast} \] (1.7)

and the design problem is to obtain polynomial coefficients \(a(i,j)\) and \(b(i,j)\) such that the \(z\)-transfer function evaluated on the unit circles in the \(z\) and \(z_\ast\) planes approximates the desired response of the filter, in addition to maintaining the stability of the filter. The latter condition requires \([5,6,13]\) that

\[ D(z, z_\ast) \neq 0 \text{ for } \bigwedge_{i=1}^{2} |z_i| \geq 1 \] (1.8)

The approximation can be carried out in the analogue \((s)\) or digital \((z)\) domain using any optimization method, while maintaining stability of the designed filter.

1.3.2. Design Approach for FIR Filters

The design approach for the 2-D FIR filter can be a direct technique or an optimization technique. For the direct technique, a 1-D FIR filter is designed using the Fourier Series in conjunction with a window as mentioned previously. This is
then converted to a 2-D FIR filter by using a Spectral transformation, which in this case is a 1-D to 2-D transformation.

The optimization approach is similar to the 1-D FIR filter design except it is expanded to the 2-D case. The 2-D FIR filter is optimized by minimizing the least-mean-square error between the ideal and designed magnitude and/or the group delay responses of the filter. Though the optimization technique is quite powerful and requires few mathematical manipulations, it is a computationally intensive approach.

1.3.3. Design Approach for IIR Filters

There are as many techniques in designing 2-D filters as there are 1-D filter techniques. Some of these techniques for 2-D IIR filters are the use of the spectral transformations and the optimization techniques. The first method has various approaches: the design consists of a 1-D analogue filter and transformed to a 2-D digital filter, or from a 2-D analogue filter transformed to a 2-D digital filter, or a 2-D digital filter transformed to another 2-D digital filter [3,12]. The second method involves an optimization technique which minimizes the least-mean-square error between the ideal and designed magnitude and/or the group delay responses of the filter [5,6]. The transformations for the first approach still require much work of efficient algorithms for the various contour approximations, while the optimization technique, though advantageous, requires some computational time. In both of these approaches, there lies the difficulty of maintaining stability of the
designed filter which is very important and critical in order to obtain a properly designed filter.

1.4 A NEW DESIGN METHODOLOGY: FILTER TUNABILITY

Whenever a filter is designed, it is designed with certain specifications such as the cutoff frequency, or the passband and the stopband edges. The various required specifications are then implemented so that the final filter cannot be varied. That is if one requires a filter with a different cutoff frequency or different passband and stopband edges, then a whole filter must be designed for the new specifications. Anytime a different cutoff frequency is required, or certain different filter specifications are needed, then a new filter must be designed. That is, all the filter coefficients must be changed. This can be very labourious and costly in applications where one or any of the various specifications needs to be changed during the operation process of the filter. Thus some effective method of changing the cutoff frequency, the passband and stopband edges of the filter, or in other words, a method of filter 'tunability', would be desirable.

It can be seen then that it would be advantageous to design a digital filter for which the cutoff frequency is variable [4]. The first possible method, which was mentioned above, is to vary all the filter coefficients so that the desired cutoff frequency is obtained. The filter coefficients would have to be calculated for every cutoff frequency desired. This would be very time consuming, especially for 2-D filters, and would limit the filter
to only the cutoff frequencies calculated.

Another method, which would be more desirable, is to vary the cutoff frequency with only a few parameters, or even only one or two parameters. This would be more practical in applications where the cutoff frequency is varied quite frequently, and also where many variations of different cutoff frequencies are needed. This last feature allows the filter to be fine tuned giving high selectivity. Another feature that this method allows is that it lends itself to simpler hardware realizations and thus more versatile. Also, this feature has many filters built into one design which is very beneficial and economical as in the filter size and cost.

This thesis deals with a method for designing both the 1-D FIR and IIR filters, and more importantly, the 2-D FIR and IIR filters. A technique by which only a few parameters are used is preferred over a complete redesign. Thus the objective of this thesis is to come up with a design method such that fewer coefficients are needed rather than changing the entire filter coefficients in order to obtain the desired specifications.

The second chapter is the development of the frequency transformations which explain the fundamental algorithms. The concept of 'tunability' is then demonstrated by applying these algorithms to the 1-D FIR and IIR filters.

The third chapter expands the 1-D FIR filter to the 2-D FIR filter by use of a transformation technique. A 2-D FIR filter optimization technique is also presented. Various filters displaying cross sections, 2-D plots and contours will be given
for both of these techniques to see the effect of tunability. Also included in this chapter are various elliptical filters using the variable cutoff frequency transformations. In order to limit the size of the thesis report, these elliptical filters are done using the optimized 2-D FIR filter designed in this chapter. However these elliptical filters can be designed with any of the 2-D filters designed in this report.

The fourth chapter deals with the 2-D IIR variable cutoff frequency filter expanded from the 1-D IIR filters given in chapter two. Various existing 1-D to 2-D transformations are discussed showing their advantages and disadvantages. Here optimization techniques are used to do three types of filter, and then an existing filter will be used on which the variable cutoff frequency algorithms are performed. Different types of filters will be demonstrated. Numerous examples will then be shown to illustrate the effect of tunability.

The fifth chapter is an introduction of various FIR and IIR filter structures which allow the implementation of the variable cutoff frequency filters discussed in the previous chapters.

The sixth chapter analyzes the tunability and maximum error characteristics for the various 2-D prototype filters designed in the previous chapters. For the tunability characteristics a comparison is made between the desired cutoff frequency selected to the actual cutoff frequency obtained. This is done for a range of cutoff frequency values. This chapter also examines the maximum error obtained in the passband and stopband for the same range of cutoff frequencies mentioned above.
Since the introduction of variable cutoff frequency filters introduces a large number of different possible filters and to the limited size of the report, this report has been developed for the variable cutoff frequencies greater than the prototype cutoff frequency for the 2-D filters, but the algorithms are also just as valid for cutoff frequencies below the prototype cutoff frequency. In order to demonstrate this however, the 1-D filters of chapter two, will illustrate the tunability above and below the prototype cutoff frequency.
II. FREQUENCY TRANSFORMATIONS FOR 1-D FIR AND IIR FILTERS

This chapter deals with the development of the frequency transformations to obtain the desired variable cutoff frequency filters. Various methods will be examined for designing variable cutoff frequency 1-D FIR and IIR filters. The theory and algorithms of the types of filters will be discussed and examples will be given to demonstrate these concepts.

2.1 VARIABLE CUTOFF FREQUENCY 1-D FIR FILTERS

This section examines the method for designing variable cutoff frequency 1-D FIR filters.

2.1.1. Generation of the 1-D FIR Transfer Function

The method described below is based on a technique first presented by Oppenheim, Mecklenbrauker, and Mersereau [7].

In a causal linear phase FIR filter with an impulse response \( h(n) \) of length \( 2N+1 \), is given by

\[
h(n) = h_o(n-N) \tag{2.1}
\]

where, \( h_o(n) \) represents the impulse of a zero phase nonrecursive filter which is symmetric, that is

\[
h_o(n) = h_o(-n) \tag{2.2}
\]

From these two equations, the transfer function, \( H(z) \), can be expressed as

\[
H(z) = z^{-N}H_o(z) \tag{2.3a}
\]

where
\[ H_0(z) = h_0(0) + \sum_{n=1}^{N} h_0(n)(z^n + z^{-n}) \] (2.3b)

\[ N = \sum_{n=0}^{N} h_0(n) \cos(n\omega) \] (2.3c)

which alternately can be expressed as

\[ H_0(z) = h_0(0) + \sum_{n=1}^{N} \hat{h}_0(n)(\cos\omega)^n \] (2.4)

It should be noted that \( \cos\omega = \frac{z + z^{-1}}{2} \) is related to \( z^n + z^{-n} \) through the Chebyshev polynomials as follows

\[ z^n + z^{-n} = 2T_n\left[ \frac{z + z^{-1}}{2} \right] \] (2.5)

where \( T_n(x) \) is a Chebyshev polynomial of \( n \)th order. Therefore \( H_0(z) \) can be expressed as

\[ H_0(z) = \sum_{n=0}^{N} a(n) \left[ \frac{z + z^{-1}}{2} \right]^n \] (2.6)

where the coefficients \( a(n) \) are related to \( \hat{h}_0(n) \) by the Chebyshev polynomial coefficients. The frequency response of the linear phase filter is thus

\[ H(e^{j\omega}) = e^{-j\omega N} H_0(e^{j\omega}) \] (2.7a)

where

\[ H_0(e^{j\omega}) = \sum_{n=0}^{N} a(n)(\cos\omega)^n \] (2.7b)

and \( z = \exp(j\omega) \).

To obtain a variable cutoff filter, a transformation is
applied to \( H_0(e^{j\omega}) \) which the frequency response is preserved while the frequency axis is distorted. This transformation is of the form:

\[
\cos \omega = \sum_{n=0}^{P} A_k (\cos \theta)^k
\]

(2.8)

By substituting (2.8) in (2.7b), the following equation is obtained:

\[
H_0(e^{j\Omega}) = \sum_{n=0}^{N} a(n) \left[ \sum_{k=0}^{P} A_k (\cos \theta)^k \right]^n
\]

(2.9)

From this, it can be seen that \( H_0(e^{j\Omega}) \) is still expressed as a cosine polynomial. Hence the corresponding impulse is symmetrical, and thus the transformed filter has zero phase. It is now of length \( 2NP+1 \) which is expressed as

\[
\hat{H}(z) = z^{-NP} \hat{H}_0(z)
\]

(2.10)

By changing the coefficients \( A_k \) in the transformation, Equation 2.8, the relationship between the frequency response of the transformed filter will change. The cutoff frequency can be varied by appropriately choosing \( A_k \). In order to guarantee that the transformation of (2.8) represents a mapping of \( H_0(z) \) for \( z \) on the unit circle, the coefficients in (2.8) must be constrained such that \( |\cos \omega| \leq 1 \) for \(-\pi \leq \theta \leq \pi\).

If \( P=1 \) in (2.8), the first-order transformation, the following form is obtained:

\[
\cos \omega = A_0 + A_1 \cos \omega
\]

(2.11)

The resulting mapping is illustrated in Figure 2.1 for a lowpass
Figure 2.1: First-order frequency transformation.
prototype filter.

For the case of a variable cutoff lowpass filter, it may be desired to constrain the transformation so that

$$H_o(e^{j\Omega})|_{\Omega=0} = H(e^{j\omega})|_{\omega=0}$$  \hspace{1cm} (2.12)$$

in which case it is required that $A_o + A_t = 1$. Hence, the substitution into (2.12) becomes

$$\cos \omega = A_o + (1 - A_o) \cos \Omega$$  \hspace{1cm} (2.13a)$$

where

$$0 \leq A_o < 1$$  \hspace{1cm} (2.13b)$$
in order to ensure that $|\cos \omega| \leq 1$. If the prototype filter is lowpass with a cutoff frequency of $\omega_c$, then the transformed filter will have a cutoff frequency $\Omega_c$ where

$$\Omega_c = \cos^{-1}\left(\frac{\cos \omega_c - A_o}{1 - A_o}\right)$$  \hspace{1cm} (2.14)$$

The relationship between the lowpass prototype filter and the transformed filter with the constraint of (2.12) is illustrated in Figure 2.2.

It can be observed that as $A_o$ is varied between zero and unity, the cutoff frequency of the transformed filter is always greater than or equal to the cutoff frequency of the prototype filter. The reverse situation can be achieved by choosing $A_t = (1 + A_o)$ so that

$$\hat{H}_o(e^{j\Omega})|_{\Omega=\pi} = H(e^{j\omega})|_{\omega=\pi}$$  \hspace{1cm} (2.15)$$

In this case the transformation is given by
Figure 2.2: First-order frequency transformation with $A_0 + A_1 = 1$. 
\[ \cos \omega = A_o + (1 + A_o) \cos \Omega \quad (2.16a) \]

where

\[ -1 < A_o < 1 \quad (2.16b) \]

Hence the transformed filter will have a cutoff frequency \( \Omega_c \) of

\[ \Omega_c = \cos^{-1} \left[ \frac{\cos \omega_c - A_o}{1 + A_o} \right] \quad (2.17) \]

In this way, the cutoff frequency of the transformed filter can be less than or equal to the cutoff frequency of the prototype. In more general situations, it may be desired to specify a different relationship between the parameters \( A_o \) and \( \omega_c \). Thus from this technique, the cutoff frequency of the FIR filter can be varied using one parameter, \( A_o \).

The parameters \( a(n) \) in Equation 2.9 can be found either by optimizing the filter to desired frequency response, or by designing the filter using any of the familiar windows. In this particular case, the filter is designed using the popular Kaiser window.

2.1.2. A 1-D Lowpass FIR Filter design using Kaiser Window

A lowpass 1-D FIR filter is designed using the Kaiser window. A prototype filter is desired with the following specifications:

- Passband ripple in frequency range: 0 to 0.7 rad/s ≤ 0.1dB
- Minimum stopband attenuation in frequency range: 1.5 to \( \pi \) (Pi) rad/s ≥ 40dB
- Sampling frequency: \( 2\pi \) (6.28) rad/s.
The desired frequency response is seen in Figure 2.3. A 23rd order filter is thus required and the cutoff frequency corresponds to $\omega_c = 1.1$ rad/s. The values of the coefficients of the designed prototype filter (Equation 2.7b) are given in Table 2.1. The response can be seen in Figure 2.4. The frequency responses for $\omega_c = 1.5$ and $\omega_c = 0.7$ rad/s are shown in Figure 2.5 and Figure 2.6 respectively. A comparison of all these can be seen in Figure 2.7.

2.2 VARIABLE CUTOFF FREQUENCY 1-D IIR FILTERS

This section examines the method for designing variable cutoff frequency 1-D IIR filters.

2.2.1. Generation of the 1-D IIR Transfer Function

A 1-D analogue filter (IIR) is characterized by its transfer function [4,9]

$$H(s) = \frac{A(s)}{B(s)} = \frac{\sum_{i=0}^{M} a_i s^i}{\sum_{j=0}^{N} b_j s^j} \tag{2.18a}$$

where the coefficients of the filter $a_i$ and $b_j$ are real while $s = j\omega$ is a complex Laplace variable and generally for analogue filters $M \geq N$. Such a filter is stable if

$$B(s) \neq 0 \text{ for } \text{Re } s \geq 0 \tag{2.18b}$$

For the case of a 1-D recursive digital filter (IIR), the transfer function can be written in the form [4,9]
Figure 2.3: Specifications of Lowpass FIR Filter.
<table>
<thead>
<tr>
<th>Values for the Prototype Lowpass Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>a( 0) = -5.17072849E-03</td>
</tr>
<tr>
<td>a( 1) = -5.68555234E-03</td>
</tr>
<tr>
<td>a( 2) =  7.31987283E-01</td>
</tr>
<tr>
<td>a( 3) =  3.94657028E+00</td>
</tr>
<tr>
<td>a( 4) =  5.00978322E+00</td>
</tr>
<tr>
<td>a( 5) = -6.32480416E+00</td>
</tr>
<tr>
<td>a( 6) = -1.54902227E+01</td>
</tr>
<tr>
<td>a( 7) =  7.01622060E-01</td>
</tr>
<tr>
<td>a( 8) =  1.56659795E+01</td>
</tr>
<tr>
<td>a( 9) =  4.63823504E+00</td>
</tr>
<tr>
<td>a(10) = -5.41388159E+00</td>
</tr>
<tr>
<td>a(11) = -2.45671780E+00</td>
</tr>
</tbody>
</table>

Table 2.1: The values of the coefficients of the designed prototype Lowpass 1-D FIR Filter using Kaiser Window.
Figure 2.4: Lowpass FIR Filter with $\omega_c = 1.1$ rad/s.
Figure 2.5: Lowpass FIR Filter with $\omega_c = 1.5$ rad/s.
Figure 2.6: Lowpass FIR Filter with $\omega_c = 0.7$ rad/s.
Figure 2.7: A comparison of all 3 lowpass FIR filters.
\[ H(z) = \frac{\sum_{i=0}^{M} n_i z^{-i}}{\sum_{j=0}^{N} d_j z^{-j}} \]  \hfill (2.19a)

where the coefficients \( n_i \) and \( d_j \) are real and \( z = \exp sT \). The stability of the recursive filter is assured if

\[ D(z) \neq 0 \text{ for } |z| < 1 \]  \hfill (2.19b)

The coefficients of either the \( z \)-domain or the \( s \)-domain can be determined by optimization. If the filter is designed in the \( s \)-domain, then the filter can be discretized to the \( z \)-domain by use of the bilinear transformation. This transfer function is given by

\[ H(z) = H(s) \bigg|_{s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}} \]  \hfill (2.20)

In order to obtain a variable cutoff frequency filter, both the numerator and the denominator need to have a frequency transformation. Thus the Constantinides transformations can be used which are done in the \( z \)-domain and can be applied to the transfer function of Equation 2.19a. The Constantinides transformations can be seen in Table 2.2.

where

- \( \omega_p \) is the prototype passband edge
- \( \omega'_p \) is the transformed passband edge
- \( T \) is the sampling time interval.

For the lowpass and the highpass filter,

- \( z \) is in the prototype filter \( z \)-domain
- \( \bar{z} \) is in the transformed filter \( \bar{z} \)-domain.
<table>
<thead>
<tr>
<th>Type</th>
<th>Transformation</th>
<th>( \alpha, k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP to LP</td>
<td>( z = \frac{z - \alpha}{1 - z^2} )</td>
<td>( \alpha = \frac{\sin \left( (\Omega_p - \omega_p)T/2 \right)}{\sin \left( (\Omega_p + \omega_p)T/2 \right)} )</td>
</tr>
<tr>
<td>LP to HP</td>
<td>( z = -\frac{z - \alpha}{1 - z^2} )</td>
<td>( \alpha = \frac{\cos \left( (\Omega_p - \omega_p)T/2 \right)}{\cos \left( (\Omega_p + \omega_p)T/2 \right)} )</td>
</tr>
</tbody>
</table>
| LP to BP| \( z = -\frac{z^2 - \frac{2\alpha k}{k + 1} z + \frac{k - 1}{k + 1}}{1 - \frac{2\alpha k}{k + 1} z + \frac{k - 1}{k + 1} z^2} \) | \( \alpha = \frac{\cos \left( (\omega_{p_2} + \omega_{p_1})T/2 \right)}{\cos \left( (\omega_{p_2} - \omega_{p_1})T/2 \right)} \)  
|         |                                                                              | \( k = \tan \frac{\Omega_p T}{2} \cot \left( \frac{(\omega_{p_2} - \omega_{p_1})T}{2} \right) \) |
| LP to BS| \( z = -\frac{z^2 - \frac{2\alpha}{1 + k} z + \frac{1 - k}{1 + k}}{1 - \frac{2\alpha}{1 + k} z + \frac{1 - k}{1 + k} z^2} \) | \( \alpha = \frac{\cos \left( (\omega_{p_2} + \omega_{p_1})T/2 \right)}{\cos \left( (\omega_{p_2} - \omega_{p_1})T/2 \right)} \)  
|         |                                                                              | \( k = \tan \frac{\Omega_p T}{2} \tan \left( \frac{(\omega_{p_2} - \omega_{p_1})T}{2} \right) \) |

Table 2.2: Constantinides transformations.
and for the bandpass and the bandstop filter,

\[ \omega_{p1} \text{ is the lower passband edge} \]
\[ \omega_{p2} \text{ is the upper passband edge.} \]

From this, it is possible to vary the prototype filter to various cutoff frequencies by varying just one parameter, mainly \( \alpha \), and just two parameters, \( \alpha \) and \( \kappa \), for both the bandpass and the bandstop filters.

The prototype filter uses a technique which is proposed by M. Ahmadi et al. [9]. In this technique a 1-D stable analogue polynomial is generated and discretized by the application of the bilinear transformation and assigned to the denominator of the 1-D filter, Equation 2.18a. Then the parameters of the 1-D filter are calculated through optimization by minimizing some desired cost function. The details of the design approach along with the method of generating the 1-D stable polynomials is discussed in the next section.

2.2.2. Hurwitz Polynomials obtained by the application of Positive definite matrices and resistive matrices and their application in 1-D filter design

A symmetric positive definite (or positive semi-definite) matrix is always physically realizable (Ahmadi et al. [5,6,9]). Also, any positive definite matrix \( P \) can always be rewritten as a product of two matrices \( QQ^T \), where \( Q \) is either an upper-triangular or a lower-triangular matrix. Therefore, the positive definite and resistor matrix is

\[ D = AR^T + G + R \Delta R^T \]  \hspace{1cm} (2.21)

where \( A \) and \( R \) are upper-triangular matrices, \( \Gamma \) and \( \Delta \) are diagonal matrices, and \( G \) is a skew-symmetric matrix, and is realizable.
as a one-variable reactance network. Therefore, the determinant of \( D \), \( \det D \), forms a Hurwitz Polynomial (HP). The elements of \( r \) and \( \Delta \) can be zero.

For a 2 by 2 positive definite and resistor matrix, \( A \), \( R \), \( r \), \( \Delta \), and \( C \) can be written as
\[
D = \begin{bmatrix}
1 & a \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\gamma_i^2 & 0 \\
0 & \gamma_z^2
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
a & 1
\end{bmatrix}
\begin{bmatrix}
s & 0 \\
-g & 0
\end{bmatrix}
\]
\[
+ \begin{bmatrix}
1 & r \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\sigma_i^2 & 0 \\
0 & \sigma_z^2
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
r & 1
\end{bmatrix}
\]
\[
D = \begin{bmatrix}
(\gamma_i^2 + a^2 \gamma_z^2) s + (\sigma_i^2 + r \sigma_z^2) & (a \gamma_i^2 s + g + r \sigma_z^2) \\
(a \gamma_i^2 s - g + r \sigma_z^2) & \gamma_z^2 s + \sigma_z^2
\end{bmatrix}
\] (2.22)

The \( \det D \) gives the following
\[
\det D = D(s) = As^2 + Bs + C
\] (2.24)

where
\[
A = \gamma_i^2 \gamma_z^2
\] (2.25a)
\[
B = \gamma_i^2 \gamma_z^2 + r \sigma_z^2 \gamma_z^2 + \gamma_i^2 \sigma_z^2 + \alpha^2 \gamma_i^2 \sigma_z^2 - 2a \gamma_i^2 \sigma_z^2
\] (2.25b)
\[
C = \sigma_i^2 \sigma_z^2 + g^2
\] (2.25c)

Thus this second order polynomial provides a Hurwitz Polynomial which forms a stable, realizable network. The bilinear transformation provides the discrete version, \( D(z) \). The parameters of the discrete positive definite and resistor matrix HP are used as variables of an optimization technique. The polynomial may be any dimensional square matrix, or any number of lower order polynomials may be cascaded together to obtain a
higher order polynomial.

2.2.3. Formulation of the Design Problem

The previously discussed Hurwitz Polynomials obtained by the application of Positive definite and resistive matrices, will be assigned to the denominator of the 1-D analogue filter given by the transfer function of Equation 2.18a, while the numerator is left unchanged. The discrete version of the filter is obtained by the application of the bilinear transformation (2.20).

The error between the ideal and the designed magnitude response of the 1-D filter is calculated using the relationship

\[ E_M(j\omega_m) = |H_I(\exp(j\omega_mT))| - |H_D(\exp(j\omega_mT))| \quad (2.26) \]

where \( E_M \) is the error in the magnitude response and \( |H_I| \) and \( |H_D| \) are the magnitude responses of the ideal and the designed filter respectively. The general mean square error \( E_G \) is introduced as a cost function for formulation of the design problem for approximation of the magnitude response of Equation 2.26 as

\[ E_G(j\omega_m, \psi) = \sum_{m \in I_{ps}} E_M^2(j\omega_m) \quad (2.27) \]

where \( I_{ps} \) is the set of all discrete frequency points along the \( \omega \) axis in the passband and stopband regions and \( \psi \) is the coefficient vector.

To design a 1-D filter satisfying a prescribed magnitude specification, the parameter vector \( \psi \) should be calculated in such a way so that \( E_G \) in Equation 2.27 is minimized subject to the constraint that the elements of the \( R \) and \( \Delta \) matrices in Equation 2.21 are non-negative. This is a simple constrained
nonlinear optimization procedure where the constraints can be removed by the following variable substitution technique \( r = r^2 \) and \( \Delta = \Delta^2 \). In this case, the Hooke and Jeeves optimization technique [15] is used.

2.2.4. **Examples of Variable Cutoff 1-D IIR Filters using Positive Definite and Resistive Matrices**

Various 1-D IIR filters with variable cutoff frequency are designed using the transfer function of Equation 2.19a and the various frequency transformations of Table 2.2.

For the prototype lowpass filter, the following specifications are desired:

\[
H(j\omega) = 1 \quad \text{for} \quad 0 \leq \omega \leq 0.7 \text{ rad/s}
\]

\[
H(j\omega) = 0 \quad \text{for} \quad 1.5 \leq \omega \leq \pi \text{ rad/s}
\]

Sampling frequency, \( \omega_s = 2\pi (6.28) \text{ rad/s} \)

where \( H(j\omega) = \frac{1}{D(j\omega)} \)

The values of the coefficients of the designed prototype filter are given in Table 2.3. The desired frequency response is seen in Figure 2.8. The filter is designed using the optimization technique which minimizes the least-mean-squared error between the desired and designed magnitude responses discussed in the previous section. In this example, the denominator consists of two cascaded second order networks given by Equation 2.24, resulting in a fourth order polynomial. The filter is then discretized to obtain the digital form using the bilinear transformation (Equation 2.20). The cutoff frequency corresponds to \( \omega_c = 1.1 \text{ rad/s} \). This magnitude response along with \( \omega_c = 1.5 \)
and $\omega_c = 0.7$ rad/s are shown in Figures 2.9 and 2.10. A comparison of all these can be seen in Figure 2.11.

This can also be done for the case of the highpass, bandpass, and bandstop filters by using the transformations in Table 2.2. The prototype lowpass filter designed above can be transformed into other new prototype filters with the following specifications:

For highpass, \[ H(j\omega) = \begin{cases} 0 & \text{for } 0 \leq \omega \leq 0.7 \text{ rad/s} \\ 1 & \text{for } 1.5 \leq \omega \leq \pi \text{ rad/s} \end{cases} \]

For bandpass, \[ H(j\omega) = \begin{cases} 0 & \text{for } 0 \leq \omega \leq 0.7 \text{ rad/s} \\ 1 & \text{for } 1.1 \leq \omega \leq 2.1 \text{ rad/s} \\ 0 & \text{for } 2.4 \leq \omega \leq \pi \text{ rad/s} \end{cases} \]

For bandstop, \[ H(j\omega) = \begin{cases} 1 & \text{for } 0 \leq \omega \leq 1.1 \text{ rad/s} \\ 0 & \text{for } 1.5 \leq \omega \leq 1.7 \text{ rad/s} \\ 1 & \text{for } 2.1 \leq \omega \leq \pi \text{ rad/s} \end{cases} \]

These filters are then varied with different cutoff and passband frequencies to obtain the desired variable cutoff frequency filters.

The various magnitude responses for $\omega_c = 1.1$, 1.5, and 0.7 rad/s for the highpass filter can be seen in Figures 2.12 to 2.15 respectively, and for the bandpass and bandstop filters, the lower passband frequencies are $\omega_p = 1.1$, 1.5, 0.7 rad/s and the corresponding upper passband frequencies are $\omega_p = 2.1$, 2.5, 1.7 rad/s. These can be seen in Figures 2.16 to 2.19 for the bandpass filter and Figures 2.20 to 2.23 for the bandstop filter respectively.
Values of the denominator coefficients

<table>
<thead>
<tr>
<th>Section 1</th>
<th>Section 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{11} = 1.4150390625000E+00$</td>
<td>$\gamma_{12} = 5.9765625000000E-01$</td>
</tr>
<tr>
<td>$\gamma_{21} = 2.5834960937500E+00$</td>
<td>$\gamma_{22} = 1.0556640625000E+00$</td>
</tr>
<tr>
<td>$\sigma_{11} = 9.7753906250000E-01$</td>
<td>$\sigma_{12} = 3.1347656250000E-01$</td>
</tr>
<tr>
<td>$\sigma_{22} = 1.53125000000000E+00$</td>
<td>$\sigma_{22} = 2.9687500000000E-01$</td>
</tr>
<tr>
<td>$a_1 = 3.4472656250000E-01$</td>
<td>$a_2 = 7.6318359375000E-01$</td>
</tr>
<tr>
<td>$r_1 = 5.14648437500000E-01$</td>
<td>$r_2 = 7.3681640625000E-01$</td>
</tr>
<tr>
<td>$g_1 = 1.28466796875000E+00$</td>
<td>$g_2 = 5.0000000000000E-01$</td>
</tr>
</tbody>
</table>

Table 2.3: The values of the coefficients of the designed prototype Lowpass 1-D IIR Filter.
Figure 2.8: Specifications of Lowpass IIR Filter.
Figure 2.9: Lowpass IIR Filter transformed to $\omega_c = 1.5 \text{ rad/s}$. 
Figure 2.10: Lowpass IIR Filter transformed $\omega_c = 0.7$ rad/s.
Figure 2.11: A comparison of all 3 lowpass IIR filters.
Figure 2.12: Highpass IIR Filter transformed to $\omega_c = 1.1 \text{ rad/s}$. 
Figure 2.13: Highpass IIR Filter transformed to $\omega_c = 1.5$ rad/s.
Figure 2.14: Highpass IIR Filter transformed to $\omega_c = 0.7$ rad/s.
Figure 2.15: A comparison of all 3 highpass IIR filters.
Figure 2.16: Bandpass IIR Filter transformed to 
$\omega_{p1} = 1.1 \text{ rad/s}, \ \omega_{p2} = 2.1 \text{ rad/s}.$
Figure 2.17: Bandpass IIR Filter transformed to
\[ \omega_{p1} = 1.5 \text{ rad/s}, \omega_{p2} = 2.5 \text{ rad/s}. \]
Figure 2.18: Bandpass IIR Filter transformed to
\[ \omega_{pl} = 0.7 \text{ rad/s}, \quad \omega_p = 1.7 \text{ rad/s}. \]
Figure 2.19: A comparison of all 3 bandpass IIR filters.
Figure 2.20: Bandstop IIR Filter transformed to
\[ \omega_p = 1.1 \text{ rad/s}, \] \[ \omega_p = 2.1 \text{ rad/s}. \]
Figure 2.21: Bandstop IIR Filter transformed to
\[ \omega_{pl} = 1.5 \text{ rad/s}, \quad \omega_{p2} = 2.5 \text{ rad/s}. \]
Figure 2.22: Bandstop IIR Filter transformed to
\[ \omega_p = 0.7 \text{ rad/s}, \quad \omega_{p2} = 1.7 \text{ rad/s}. \]
Figure 2.23: A comparison of all 3 bandstop IIR filters.
III. 2-D VARIABLE CUTOFF FREQUENCY FIR FILTERS

The concepts of the 1-D FIR Filters are now extended to the 2-D case. There are various ways of designing 2-D FIR filters which include direct transformations from 1-D to 2-D, and by the use of optimization techniques such as linear and non-linear programming. These methods have their advantages and disadvantages. The methods used here include the transformation of 1-D zero phase FIR filter to a 2-D zero phase FIR filter using the McClellan transformation and an optimization technique which calculates the parameters of a 2-D FIR filter transfer function. The optimized 2-D FIR filter will be used to illustrate various elliptical filter designs.

3.1 DIRECT DESIGN APPROACH FOR 2-D FIR FILTERS

3.1.1. Transformation Technique

A very effective method of transforming a 1-D zero-phase FIR filter into a corresponding 2-D zero-phase FIR filter is by the McClellan transformation [2].

The generalized McClellan transformation converts 1-D filters of the form given in Equation 2.3c by

\[
H_0(e^{j\omega}) = h_0(0) + \sum_{n=1}^{N} 2h_0(n)\cos(n\omega)
\]

into 2-D filters which have frequency responses of the form:

\[
H_0(e^{j\omega_1}, e^{j\omega_2}) = \sum_{m=0}^{M} \sum_{n=0}^{N} a(m,n)\cos(m\omega_1)\cos(n\omega_2)
\]

by using the substitution
\[
\cos \omega = \sum_{p=0}^{P} \sum_{q=0}^{Q} t(p,q) \cos(p \omega_i) \cos(q \omega_z) \quad (3.3)
\]

In order to use this transformation (3.3), the transfer function of Equation 3.1 needs to be written as it was done in the 1-D case. This is given by Equation 2.7b

\[
H_0(e^{j\omega}) = \sum_{n=0}^{N} a(n) (\cos \omega)^n \quad (2.7b)
\]

and thus substituting (3.3) into (2.7b), the following equation is obtained

\[
H_0(e^{j\omega}, e^{j\omega_z}) = \sum_{n=0}^{N} a(n) \left[ \sum_{p=0}^{P} \sum_{q=0}^{Q} t(p,q) \cos(p \omega_i) \cos(q \omega_z) \right]^n \quad (3.4)
\]

By considering the case where \(P=Q=1\) and

\[-t(0,0) = t(1,0) = t(0,1) = t(1,1) = 1/2 \quad (3.5)\]

this produces nearly circular contours for low values of \(\omega\) and then becomes increasingly squared for larger values of \(\omega\). This is observed in Figures 3.1 to 3.3. This situation is ideal for lowpass filters. The resulting equation then becomes

\[
H_0(e^{j\omega}, e^{j\omega_z}) = \sum_{n=0}^{N} a(n) \left[ t_{00} + t_0 \cos \omega_z + t_{10} \cos \omega_i + t_{11} \cos \omega_i \cos \omega_z \right]^n \quad (3.6)
\]

Hence a lowpass 1-D FIR filter can be transformed to a 2-D FIR filter using the McClellan transformation. It should be noted that even though this transformation yields very good results, it is a suboptimal design.
Figure 3.1: McClellan lowpass transformation contours: 0.1 increments.
Figure 3.2: McClellan lowpass transformation contours: 0.05 increments.
Figure 3.3: McClellan lowpass transformation plot.
3.1.2. The 2-D Lowpass to Lowpass Transformation

In Equation 3.6, $\cos \omega$, and $\cos \omega_z$ result from the 1-D to 2-D McClellan transformation. The first-order variable cutoff frequency transformation of Equation 2.13a or 2.16a can be used

$$\cos \omega = A_0 + (1-A_0) \cos \Omega \quad \Omega \geq \Omega_c \quad (2.13a)$$

and

$$\cos \omega = A_0 + (1+A_0) \cos \Omega \quad \Omega < \Omega_c \quad (2.16a)$$

Using the situation where $\Omega \geq \Omega_c$ in Equation 2.13a, the Oppenheim transformation can be expanded for the 2-D case as

$$\cos \omega_x = A_0 + (1-A_0) \cos \Omega_x \quad (3.7)$$

and

$$\cos \omega_z = B_0 + (1-B_0) \cos \Omega_z \quad (3.8)$$

which corresponds to the variable cutoff frequency in both the $\omega_x$ and $\omega_z$ direction respectively. The introduction of the variable $B_0$ thus allows elliptical lowpass filters, and for the special case of $A_0 = B_0$, circular lowpass filters. Evidently, if circular filters are desired only, then only one parameter, $A_0$, is needed to obtain the desired cutoff frequency for the 2-D filter. The same can be done for $\Omega_x, \Omega < \Omega_c$. Thus Equation 2.16a will yield for the 2-D filter, the following equations

$$\cos \omega_x = A_0 + (1+A_0) \cos \Omega_x \quad (3.9)$$

and

$$\cos \omega_z = B_0 + (1+B_0) \cos \Omega_z \quad (3.10)$$

From these equations, the 2-D Lowpass to Lowpass FIR filter transformations have been established. Thus both these variable cutoff transformations along with the McClellan transformation allows a 2-D variable cutoff frequency lowpass filter to be designed from a prototype 1-D FIR filter.
3.1.3. **Examples of Variable Cutoff 2-D FIR Filters using Kaiser Window**

Various examples using this transformation technique are demonstrated by using the lowpass FIR filter specifications given in Figure 2.3 of Chapter 2 for the 1-D case, along with the Kaiser window, which will then become a 2-D filter. Therefore the desired prototype filter is designed with the following specifications:

Passband ripple in frequency range:

\[ 0 \leq \left( \omega_1^2 + \omega_2^2 \right)^{1/2} \leq 0.7 \text{ rad/s} \leq 0.1 \text{dB} \]

Minimum stopband attenuation in frequency range:

\[ 1.5 \leq \left( \omega_1^2 + \omega_2^2 \right)^{1/2} \leq \pi \text{ (Pi) rad/s} \geq 40 \text{dB} \]

Sampling frequency:

\[ 2\pi \times (6.28) \text{ rad/s}. \]

Thus the prototype 2-D FIR filter has a cutoff frequency \( \omega_c = 1.1 \text{ rad/s}. \) This first designed prototype filter is given by the first set of values in Table 3.1. These values of coefficients correspond exactly to those of Table 2.1 since this filter is expanded from a 1-D filter to a 2-D filter via the McClellan transformation. Three major cross sections are taken along the \( \omega_1, \omega_z, \) and \( \omega_2 = \omega_z \) axes which are shown in Figures 3.4, 3.5, and 3.6, the actual 3-D plot in Figure 3.7, and the contour plot in Figure 3.8.

This lowpass filter is then varied to \( \omega_c = 1.5 \text{ rad/s}. \) The various cross sections for this filter are shown in Figures 3.9, 3.10, and 3.11, the actual 3-D plot in Figure 3.12, and the contour plot in Figure 3.13.

A 2-D FIR highpass filter can be created be a simple
change in the transfer function of the 1-D FIR lowpass filter using the Kaiser window which was done above and the lowpass to lowpass transformation can then be applied. A lowpass to highpass transformation can also be used, but in this particular case the former technique will be used.

The design of a 2-D highpass FIR filter has a cutoff frequency $\omega_C = 1.9$ rad/s. The various cross sections are taken along the $\omega_i$, $\omega_j$, and $\omega_i = \omega_j$ axes which are shown in Figures 3.14, 3.15, and 3.16, the actual 3-D plot in Figure 3.17, and the contour plot in Figure 3.18.

This highpass filter is then varied to $\omega_C = 2.3$ rad/s. The various cross sections for this filter are shown in Figures 3.19, 3.20, and 3.21, the actual 3-D plot in Figure 3.22, and the contour plot in Figure 3.23.

To obtain a bandpass filter, a second filter is required. This involves adding a lowpass filter with a highpass filter. This latter filter, the highpass filter, is a prototype designed 1-D lowpass filter transformed to a 1-D highpass filter which is then transformed to a 2-D highpass filter via the McClellan transformation and hence the required 2-D highpass filter is obtained. This second prototype 2-D FIR lowpass filter has a cutoff frequency $\omega_C = 1.9$ rad/s. The specifications for this second prototype filter are as follows:

Passband ripple in frequency range:

$$0 \leq (\omega_i^2 + \omega_j^2)^{1/2} \leq 1.5 \text{ rad/s} \leq 0.1 \text{ dB}$$

Minimum stopband attenuation in frequency range:

$$2.3 \leq (\omega_i^2 + \omega_j^2)^{1/2} \leq \pi \text{ (Pi) rad/s} \geq 40 \text{ dB}$$
Sampling frequency:
\[ 2\pi \ (5.28) \text{ rad/s}. \]

The second set of values of the coefficients for this second designed prototype filter are given in Table 3.1. To obtain this bandpass filter, this second designed prototype 2-D FIR lowpass filter is added with a 2-D FIR highpass filter. This highpass filter is the transformed filter of the first designed prototype 2-D FIR lowpass filter designed above.

Thus a 2-D FIR bandpass filter is designed with \( \omega_{c1} = 1.1 \text{ rad/s} \) and \( \omega_{c2} = 2.1 \text{ rad/s} \). The various cross sections for this filter are shown in Figures 3.24, 3.25, and 3.26, the actual 3-D plot in Figure 3.27, and the contour plot in Figure 3.28.

This bandpass filter is then varied to \( \omega_{c1} = 1.5 \text{ rad/s} \) and \( \omega_{c2} = 1.9 \text{ rad/s} \). The various cross sections for this filter are shown in Figures 3.29, 3.30, and 3.31, the actual 3-D plot in Figure 3.32, and the contour plot in Figure 3.33.

The design of a 2-D FIR bandstop filter is designed with \( \omega_{c1} = 1.1 \text{ rad/s} \) and \( \omega_{c2} = 2.4 \text{ rad/s} \). This filter is designed using the first prototype designed 2-D FIR lowpass filter added with a 2-D FIR highpass filter. This highpass filter is the transformed filter of the second designed prototype 2-D FIR lowpass filter designed previously. Thus the various cross sections for this bandstop filter are shown in Figures 3.34, 3.35, and 3.36, the actual 3-D plot in Figure 3.37, and the contour plot in Figure 3.38.

This bandstop filter is then varied to \( \omega_{c1} = 1.2 \)
rad/s and \( \omega_{c2} = 2.9 \) rad/s. The various cross sections for this filter are shown in Figures 3.39, 3.40, and 3.41, the actual 3-D plot in Figure 3.42, and the contour plot in Figure 3.43.

3.2 OPTIMIZATION DESIGN APPROACH FOR 2-D FIR FILTERS

3.2.1. Optimization Technique

Again extending the 1-D transfer function given in Equation 2.3c, the following 2-D transfer function is obtained

\[
H_0(e^{j\omega_1}, e^{j\omega_2}) = \sum_{m=0}^{M} \sum_{n=0}^{N} a(m,n) \cos(m \omega_1) \cos(n \omega_2) \quad (3.2)
\]

As it was done also for the 1-D case, Equation 3.2 can be rewritten as

\[
H_0(e^{j\omega_1}, e^{j\omega_2}) = \sum_{m=0}^{M} \sum_{n=0}^{N} a(m,n) (\cos \omega_1)^m (\cos \omega_2)^n \quad (3.11)
\]

Thus the variable cutoff transformations of Equations 3.7 and 3.8 can be substituted into (3.11) and hence obtain a 2-D variable cutoff filter. The parameters \( \hat{a}(m,n) \) are hence found by optimizing the filter to the desired frequency response by minimizing some desired cost function.

3.2.2. Formulation of the Design Problem

The various cutoff frequency transformations will be then applied to these prototype filters. These filters will then be tuned to different cutoff frequencies thus providing tunability.

The parameters of the 2-D filter are calculated through optimization by minimizing some desired cost function. The error between the ideal and the designed magnitude response of the 1-D filter is calculated using the relationship
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Table 3.1: The values of the coefficients for the two designed prototype Lowpass 2-D FIR Filters using Kaiser Window.
Figure 3.4: Lowpass 2-D FIR Filter with
\[ \omega_{c_1} = 1.1 \text{ rad/s}, \quad \omega_{c_2} = 1.1 \text{ rad/s} \]
along the \( \omega_1 \) axis.
Figure 3.5: Lowpass 2-D FIR Filter with
\( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \)
along \( \omega_2 \) axis.
Figure 3.6: Lowpass 2-D FIR Filter with
\[ \omega_1 = 1.1 \text{ rad/s}, \ \omega_2 = 1.1 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis.
Figure 3.7: Three Dimensional Plot of Lowpass 2-D FIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}$.
Figure 3.8: Contour Plot of Lowpass 2-D FIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}.$
Figure 3.9: Lowpass 2-D FIR Filter transformed to
\[ \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \]
along \( \omega_1 \) axis.
Figure 3.10: Lowpass 2-D FIR Filter transformed to
$\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s}$
along $\omega_2$ axis.
Figure 3.11: Lowpass 2-D FIR Filter transformed to
$\omega_c^1 = 1.5 \text{ rad/s, } \omega_c^2 = 1.5 \text{ rad/s}$
along $\omega_1 = \omega_2$ axis.
Figure 3.12: Three Dimensional Plot of Lowpass 2-D FIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s}$. 
Figure 3.13: Contour Plot of Lowpass 2-D FIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 1.5 \text{ rad/s}$.
Figure 3.14: Highpass 2-D FIR Filter transformed to
\[ \omega_{c1} = 1.9 \text{ rad/s}, \quad \omega_{c2} = 1.9 \text{ rad/s} \]
along \( \omega_1 \) axis.
Figure 3.15: Highpass 2-D FIR Filter transformed to 
\( \omega_{c1} = 1.9 \text{ rad/s}, \omega_{c2} = 1.9 \text{ rad/s} \) 
along \( \omega_2 \) axis.
Figure 3.16: Highpass 2-D FIR Filter transformed to
\( \omega_{c1} = 1.9 \text{ rad/s}, \omega_{c2} = 1.9 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis.
Figure 3.17: Three Dimensional Plot of Highpass 2-D FIR Filter transformed to $\omega_{c1} = 1.9$ rad/s, $\omega_{c2} = 1.9$ rad/s.
Figure 3.18: Contour Plot of Highpass 2-D FIR Filter transformed to $\omega_{c1} = 1.9$ rad/s, $\omega_{c2} = 1.9$ rad/s.
Figure 3.19: Highpass 2-D FIR Filter transformed to
$\omega_{c1} = 2.3 \text{ rad/s}, \omega_{c2} = 2.3 \text{ rad/s}$
along $\omega_1$ axis.
Figure 3.20: Highpass 2-D FIR Filter transformed to 
$\omega_{c1} = 2.3 \text{ rad/s}, \omega_{c2} = 2.3 \text{ rad/s}$
along $\omega_2$ axis.
Figure 3.21: Highpass 2-D FIR Filter transformed to
$\omega_{c1} = 2.3 \text{ rad/s}$, $\omega_{c2} = 2.3 \text{ rad/s}$
along $\omega_1 = \omega_2$ axis.
Figure 3.22: Three Dimensional Plot of Highpass 2-D FIR Filter transformed to $\omega_{c1} = 2.3$ rad/s, $\omega_{c2} = 2.3$ rad/s.
Figure 3.23: Contour Plot of Highpass 2-D FIR Filter transformed to $\omega_{c1} = 2.3$ rad/s, $\omega_{c2} = 2.3$ rad/s.
Figure 3.24: Bandpass 2-D FIR Filter transformed to

\[ \omega_{c1} = 1.1 \text{ rad/s}, \quad \omega_{c2} = 2.1 \text{ rad/s} \]

along \( \omega_1 \) axis.
Figure 3.25: Bandpass 2-D FIR Filter transformed to
$\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 2.1 \text{ rad/s}
along $\omega_2$ axis.
Figure 3.26: Bandpass 2-D FIR Filter transformed to
\( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 2.1 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis.
Figure 3.27: Three Dimensional Plot of Bandpass 2-D FIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 2.1 \text{ rad/s}$.
Figure 3.28: Contour Plot of Bandpass 2-D FIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 2.1 \text{ rad/s}$. 
Figure 3.29: Bandpass 2-D FIR Filter transformed to 
\( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.9 \text{ rad/s} \) 
along \( \omega_1 \) axis.
Figure 3.30: Bandpass 2-D FIR Filter transformed to
\[ \omega_1 = 1.5 \text{ rad/s}, \ \omega_2 = 1.9 \text{ rad/s} \]
along \( \omega_2 \) axis.
Figure 3.31: Bandpass 2-D FIR Filter transformed to
\( \omega_c1 = 1.5 \text{ rad/s}, \omega_c2 = 1.9 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis.
Figure 3.32: Three Dimensional Plot of Bandpass 2-D FIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 1.9$ rad/s.
Figure 3.33: Contour Plot of Bandpass 2-D FIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 1.9 \text{ rad/s}$. 
Figure 3.34: Bandstop 2-D FIR Filter transformed to 
\( \omega_c_1 = 1.1 \text{ rad/s}, \ \omega_c_2 = 2.4 \text{ rad/s} \)
along \( \omega_1 \) axis.
Figure 3.35: Bandstop 2-D FIR Filter transformed to
\( \omega_c^1 = 1.1 \text{ rad/s}, \ \omega_c^2 = 2.4 \text{ rad/s} \)
along \( \omega_2 \) axis.
Figure 3.36: Bandstop 2-D FIR Filter transformed to 
$\omega_{c1} = 1.1 \text{ rad/s, } \omega_{c2} = 2.4 \text{ rad/s}$
along $\omega_1 = \omega_2$ axis.
Figure 3.37: Three Dimensional Plot of Bandstop 2-D FIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 2.4 \text{ rad/s}$. 
Figure 3.38: Contour Plot of Bandstop 2-D FIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 2.4$ rad/s.
Figure 3.39: Bandstop 2-D FIR Filter transformed to
\( \omega_{c1} = 1.2 \text{ rad/s, } \omega_{c2} = 2.9 \text{ rad/s} \)
along \( \omega_1 \) axis.
Figure 3.40: Bandstop 2-D FIR Filter transformed to
\( \omega_{c1} = 1.2 \text{ rad/s}, \ \omega_{c2} = 2.9 \text{ rad/s} \)
along \( \omega_2 \) axis.
Figure 3.41: Bandstop 2-D FIR Filter transformed to
\( \omega_{c1} = 1.2 \text{ rad/s}, \ \omega_{c2} = 2.9 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis.
Figure 3.42: Three Dimensional Plot of Bandstop 2-D FIR Filter transformed to $\omega_{c1} = 1.2$ rad/s, $\omega_{c2} = 2.9$ rad/s.
Figure 3.43: Contour Plot of Bandstop 2-D FIR Filter transformed to $\omega_{c1} = 1.2 \text{ rad/s}$, $\omega_{c2} = 2.9 \text{ rad/s}$. 
\[ E_M(j \omega_m, j \omega_n) = |H_I(\exp(j \omega_m T, j \omega_n T))| - |H_D(\exp(j \omega_m T, j \omega_n T))| \]  

(3.11)

where \( E_M \) is the error in the magnitude response and \( |H_I| \) and \( |H_D| \) are the magnitude responses of the ideal and the designed filter respectively. The general mean square error \( E_G \) is introduced as a cost function for formulation of the design problem for approximation of the magnitude response of Equation 3.11 as

\[ E_G(j \omega_m, j \omega_n, \psi) = \sum_{m,n \in I_{ps}} E_M^2(j \omega_m, j \omega_n) \]  

(3.12)

where \( I_{ps} \) is the set of all discrete frequency points along the \( \omega_i \) and \( \omega_s \) axis in the passband and stopband regions and \( \psi \) is the coefficient vector.

To design a 2-D filter satisfying a prescribed magnitude specification, the coefficient vector \( \psi \) should be calculated in such a way so that \( E_G \) in Equation 3.12 is minimized. In this case the Hooke and Jeeves optimization [15] is utilized for all the filters designed in this thesis just as it is in Chapter Two for the 1-D IIR filter case.

3.2.3. **Examples of Variable Cutoff 2-D FIR Filters using Optimized Filters**

Various examples using this optimization technique are demonstrated by using the lowpass 2-D FIR filter specifications given above for the Direct Approach. Therefore the desired prototype filter is designed with the following specifications:

\[ H(j \omega, T, j \omega_s T) = 1 \text{ for } 0 \leq (\omega_i^2 + \omega_s^2)^{1/2} \leq 0.7 \text{ rad/s} \]

\[ H(j \omega, T, j \omega_s T) = 0 \text{ for } 1.5 \leq (\omega_i^2 + \omega_s^2)^{1/2} \leq \pi \text{ rad/s} \]
Sampling frequency, $\omega_s = 2\pi (6.28) \text{ rad/s}$.

Thus the prototype 2-D FIR filter has a cutoff frequency $\omega_c = 1.1 \text{ rad/s}$. This first designed prototype filter is given by the first set of values in Table 3.2. Three major cross sections are taken along the $\omega_x$, $\omega_y$, and $\omega_z$ axes which are shown in Figures 3.44, 3.45, and 3.46, the actual 3-D plot in Figure 3.47, and the contour plot in Figure 3.48.

This lowpass filter is then varied to $\omega_c = 1.5 \text{ rad/s}$. The various cross sections for this filter are shown in Figures 3.49, 3.50, and 3.51, the actual 3-D plot in Figure 3.52, and the contour plot in Figure 3.53.

To create the highpass filter from this normalized 2-D FIR filter requires a simple subtracting from one which results in the desired response.

The design of a 2-D highpass FIR filter has a cutoff frequency $\omega_c = 1.1 \text{ rad/s}$. The various cross sections are taken along the $\omega_x$, $\omega_y$, and $\omega_z$ axes which are shown in Figures 3.54, 3.55, and 3.56, the actual 3-D plot in Figure 3.57, and the contour plot in Figure 3.58.

This highpass filter is then varied to $\omega_c = 1.8 \text{ rad/s}$. The various cross sections for this filter are shown in Figures 3.59, 3.60, and 3.61, the actual 3-D plot in Figure 3.62, and the contour plot in Figure 3.63.

To obtain a bandpass filter, a second filter is required. This involves adding a lowpass filter with a highpass filter. This latter filter, the highpass filter, is a prototype designed 2-D lowpass filter which is then subtracted from one.
This second prototype 2-D FIR lowpass filter has a cutoff frequency $\omega_c = 1.9 \text{ rad/s}$. The specifications for this second prototype filter are as follows:

$$H(j\omega, T, j\omega, T) = 1 \text{ for } 0 \leq (\omega_i^2 + \omega_s^2)^{1/2} \leq 1.5 \text{ rad/s}$$

$$H(j\omega, T, j\omega, T) = 0 \text{ for } 2.3 \leq (\omega_i^2 + \omega_s^2)^{1/2} \leq \pi \text{ rad/s}$$

Sampling frequency, $\omega_s = 2\pi (6.28) \text{ rad/s}$.

The second set of values of the coefficients for this second designed prototype filter are given in Table 3.3. To obtain this bandpass filter, this second designed prototype 2-D FIR lowpass filter is added with a 2-D FIR highpass filter. This highpass filter is the transformed filter of the first designed prototype 2-D FIR lowpass filter designed above.

Thus a 2-D FIR bandpass filter is designed with $\omega_{c1} = 1.1 \text{ rad/s}$ and $\omega_{c2} = 2.5 \text{ rad/s}$. The various cross sections for this filter are shown in Figures 3.64, 3.65, and 3.66, the actual 3-D plot in Figure 3.67, and the contour plot in Figure 3.68.

This bandpass filter is then varied to $\omega_{c1} = 1.5 \text{ rad/s}$ and $\omega_{c2} = 2.1 \text{ rad/s}$. The various cross sections for this filter are shown in Figures 3.69, 3.70, and 3.71, the actual 3-D plot in Figure 3.72, and the contour plot in Figure 3.73.

The design of a 2-D FIR bandstop filter is designed with $\omega_{c1} = 1.3 \text{ rad/s}$ and $\omega_{c2} = 2.3 \text{ rad/s}$. This filter is designed using the first prototype designed 2-D FIR lowpass filter added with a 2-D FIR highpass filter. This highpass filter is the transformed filter of the second designed prototype.
2-D FIR lowpass filter designed previously. Thus the various cross sections for this bandstop filter are shown in Figures 3.74, 3.75, and 3.76, the actual 3-D plot in Figure 3.77, and the contour plot in Figure 3.78.

This bandstop filter is then varied to $\omega_{c1} = 1.2$ rad/s and $\omega_{c2} = 2.7$ rad/s. The various cross sections for this filter are shown in Figures 3.79, 3.80, and 3.81, the actual 3-D plot in Figure 3.82, and the contour plot in Figure 3.83.

3.3 ELLIPTICAL FILTERS USING THE OPTIMIZED 2-D FIR FILTERS

3.3.1. Elliptical Filter Technique

In order to obtain elliptical filters, the transformations given by Equations 3.7 and 3.8,

$$\cos \omega_1 = A_0 + (1-A_0)\cos \omega, \quad (3.7)$$

and

$$\cos \omega_2 = B_0 + (1-B_0)\cos \omega, \quad (3.8)$$

which correspond to the variable cutoff frequency in both the $\omega_1$ and $\omega_2$ direction respectively, are used. The circular filters designed above included the same transformations where both the parameters $A_0$ and $B_0$ were equal (i.e. $A_0 = B_0$). Therefore only one parameter is needed for circular filters. In the case of elliptical filters, $A_0$ and $B_0$ are not equal. This is achieved by simply selecting different values of the cutoff frequencies in both the $\omega_1$ and $\omega_2$ directions.

In order to demonstrate the effect of the different cutoff frequencies, various examples are given using the optimized 2-D FIR filters given above. These elliptical filters can be designed using any of the 2-D filters designed in this
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Figure 3.44: Lowpass 2-D FIR Filter with
\[ \omega_c = 1.1 \text{ rad/s}, \quad \omega_{c2} = 1.1 \text{ rad/s} \]
along \( \omega_1 \) axis.
Figure 3.45: Lowpass 2-D FIR Filter with
\[ \omega_c = 1.1 \text{ rad/s}, \quad \omega_c = 1.1 \text{ rad/s} \]
along \( \omega_2 \) axis.
Figure 3.46: Lowpass 2-D FIR Filter with
\[ \omega_{c1} = 1.1 \text{ rad/s}, \quad \omega_{c2} = 1.1 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis.
Figure 3.47: Three Dimensional Plot of Lowpass 2-D FIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}$. 
Figure 3.48: Contour Plot of Lowpass 2-D FIR Filter with $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s.
Figure 3.49: Lowpass 2-D FIR Filter transformed to 
\( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \) 
along \( \omega_{1} \) axis.
Figure 3.50: Lowpass 2-D FIR Filter transformed to
$\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 1.5 \text{ rad/s}$
along $\omega_2$ axis.
Figure 3.51: Lowpass 2-D FIR Filter transformed to
\[ \omega_{c1} = 1.5 \text{ rad/s}, \quad \omega_{c2} = 1.5 \text{ rad/s} \]
along \[ \omega_1 = \omega_2 \text{ axis.} \]
Figure 3.52: Three Dimensional Plot of Lowpass 2-D FIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 1.5 \text{ rad/s}$. 
Figure 3.53: Contour Plot of Lowpass 2-D FIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 1.5$ rad/s.
Figure 3.54: Highpass 2-D FIR Filter transformed to
\[ \omega_c^1 = 1.1 \text{ rad/s}, \ \omega_c^2 = 1.1 \text{ rad/s} \]
along \( \omega_1 \) axis.
Figure 3.55: Highpass 2-D FIR Filter transformed to 
\( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s} \)
along \( \omega_2 \) axis.
Figure 3.56: Highpass 2-D FIR Filter transformed to 
\[ \omega_{c1} = 1.1 \text{ rad/s}, \quad \omega_{c2} = 1.1 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis.
Figure 3.57: Three Dimensional Plot of Highpass 2-D FIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$. 
Figure 3.58: Contour Plot of Highpass 2-D FIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}$. 
Figure 3.59: Highpass 2-D FIR Filter transformed to
\( \omega_{c1} = 1.8 \text{ rad/s}, \ \omega_{c2} = 1.8 \text{ rad/s} \)
along \( \omega_1 \) axis.
Figure 3.60: Highpass 2-D FIR Filter transformed to
\( \omega_{c1} = 1.8 \text{ rad/s}, \ \omega_{c2} = 1.8 \text{ rad/s} \)
along \( \omega_2 \) axis.
Figure 3.61: Highpass 2-D FIR Filter transformed to 
$\omega_{c1} = 1.8 \text{ rad/s}, \; \omega_{c2} = 1.8 \text{ rad/s}$
along $\omega_1 = \omega_2$ axis.
Figure 3.62: Three Dimensional Plot of Highpass 2-D FIR Filter transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s.
Figure 3.63: Contour Plot of Highpass 2-D FIR Filter transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s.
### Values for the Prototype Lowpass Filter

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**Table 3.3:** The values of the coefficients for the second of two designed Optimized Prototype Lowpass 2-D IIR Filters.
Figure 3.64: Bandpass 2-D FIR Filter transformed to
\( \omega_c = 1.1 \text{ rad/s}, \ \omega_c = 2.5 \text{ rad/s} \)
along \( \omega_1 \) axis.
Figure 3.65: Bandpass 2-D FIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 2.5 \text{ rad/s}$ along $\omega_2$ axis.
Figure 3.66: Bandpass 2-D FIR Filter transformed to
\[ \omega_c = 1.1 \text{ rad/s}, \quad \omega_c = 2.5 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis.
Figure 3.67: Three Dimensional Plot of Bandpass 2-D FIR Filter transformed to \( \omega_c^1 = 1.1 \text{ rad/s}, \omega_c^2 = 2.5 \text{ rad/s} \).
Figure 3.68: Contour Plot of Bandpass 2-D FIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 2.5 \text{ rad/s}$. 
Figure 3.69: Bandpass 2-D FIR Filter transformed to 
\[ \omega_c1 = 1.5 \text{ rad/s}, \quad \omega_c2 = 2.1 \text{ rad/s} \]
along \( \omega_1 \) axis.
Figure 3.70: Bandpass 2-D FIR Filter transformed to
$\omega_c1 = 1.5 \text{ rad/s, } \omega_c2 = 2.1 \text{ rad/s}$
along $\omega_2$ axis.
Figure 3.71: Bandpass 2-D FIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 2.1 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 3.72: Three Dimensional Plot of Bandpass 2-D FIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 2.1 \text{ rad/s}$. 
Figure 3.73: Contour Plot of Bandpass 2-D FIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 2.1 \text{ rad/s}$. 
Figure 3.74: Bandstop 2-D FIR Filter transformed to
$\omega_{c1} = 1.3 \text{ rad/s}, \omega_{c2} = 2.3 \text{ rad/s}
along \omega_1 \text{ axis.}
Figure 3.75: Bandstop 2-D FIR Filter transformed to $\omega_{c1} = 1.3$ rad/s, $\omega_{c2} = 2.3$ rad/s along $\omega_2$ axis.
Figure 3.76: Bandstop 2-D FIR Filter transformed to
\( \omega_{c1} = 1.3 \text{ rad/s}, \omega_{c2} = 2.3 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis.
Figure 3.77: Three Dimensional Plot of Bandstop 2-D FIR Filter transformed to $\omega_{c1} = 1.3 \text{ rad/s}$, $\omega_{c2} = 2.3 \text{ rad/s}$. 
Figure 3.78: Contour Plot of Bandstop 2-D FIR Filter transformed to $\omega_{c1} = 1.3$ rad/s, $\omega_{c2} = 2.3$ rad/s.
Figure 3.79: Windstop 2-D FIR Filter transformed to
\( \omega_{c1} = 1.2 \text{ rad/s, } \omega_{c2} = 2.7 \text{ rad/s} \)
along \( \omega_1 \) axis.
Figure 3.80: Bandstop 2-D FIR Filter transformed to
$\omega_{c1} = 1.2 \text{ rad/s}, \omega_{c2} = 2.7 \text{ rad/s}$
along $\omega_2$ axis.
Figure 3.81: Bandstop 2-D FIR Filter transformed to
\[ \omega_{c1} = 1.2 \text{ rad/s}, \quad \omega_{c2} = 2.7 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis.
Figure 3.82: Three Dimensional Plot of Bandstop 2-D FIR Filter transformed to $\omega_{c1} = 1.2 \text{ rad/s}$, $\omega_{c2} = 2.7 \text{ rad/s}$.
Figure 3.83: Contour Plot of Bandpass 2-D FIR Filter transformed to $\omega_c^1 = 1.2 \text{ rad/s}$, $\omega_c^2 = 2.7 \text{ rad/s}$.
thesis, however they will be limited to the following examples.

3.3.2. **Examples of Variable Cutoff Elliptical 2-D FIR Filters using Optimized Filters**

Various examples of variable cutoff frequency elliptical filters are demonstrated by using the lowpass 2-D FIR filter specifications given above using the optimization technique.

This lowpass prototype 2-D FIR filter is varied to \( \omega_{c1} = 1.1 \text{ rad/s} \) and \( \omega_{c2} = 1.7 \text{ rad/s} \). Three major cross sections are taken along the \( \omega_1, \omega_2, \) and \( \omega_1 = \omega_2 \) axes which are shown in Figures 3.84, 3.85, and 3.86, the actual 3-D plot in Figure 3.87, and the contour plot in Figure 3.88.

This lowpass filter is then varied to \( \omega_{c1} = 2.5 \text{ rad/s} \) and \( \omega_{c2} = 1.5 \text{ rad/s} \). The various cross sections for this filter are shown in Figures 3.89, 3.90, and 3.91, the actual 3-D plot in Figure 3.92, and the contour plot in Figure 3.93.

The design of a 2-D highpass FIR filter has a cutoff frequency \( \omega_{c1} = 1.1 \text{ rad/s} \) and \( \omega_{c2} = 1.4 \text{ rad/s} \). The various cross sections are taken along the \( \omega_1, \omega_2, \) and \( \omega_1 = \omega_2 \) axes which are shown in Figures 3.94, 3.95, and 3.96, the actual 3-D plot in Figure 3.97, and the contour plot in Figure 3.98.

This highpass filter is then varied to \( \omega_{c1} = 2.0 \text{ rad/s} \) and \( \omega_{c2} = 1.3 \text{ rad/s} \). The various cross sections for this filter are shown in Figures 3.99, 3.100, and 3.101, the actual 3-D plot in Figure 3.102, and the contour plot in Figure 3.103.

Thus a 2-D FIR bandpass filter is designed with \( \omega_{c11} = 1.2 \text{ rad/s}, \omega_{c12} = 1.9 \text{ rad/s}, \omega_{c21} = 1.6 \text{ rad/s}, \) and \( \omega_{c22} = 2.3 \text{ rad/s} \). The various cross sections for this filter are shown in Figures 3.104, 3.105, and 3.106, the actual 3-D plot
in Figure 3.107, and the contour plot in Figure 3.108.

This bandpass filter is then varied to $\omega_{c11} = 1.4$ rad/s, $\omega_{c12} = 2.0$ rad/s, $\omega_{c21} = 1.1$ rad/s, and $\omega_{c22} = 2.4$ rad/s. The various cross sections for this filter are shown in Figures 3.109, 3.110, and 3.111, the actual 3-D plot in Figure 3.112, and the contour plot in Figure 3.113.

The design of a 2-D FIR bandstop filter is designed with $\omega_{c11} = 1.1$ rad/s, $\omega_{c12} = 2.0$ rad/s, $\omega_{c21} = 1.3$ rad/s, and $\omega_{c22} = 2.0$ rad/s. Thus the various cross sections for this bandstop filter are shown in Figures 3.114, 3.115, and 3.116, the actual 3-D plot in Figure 3.117, and the contour plot in Figure 3.118.

This bandstop filter is then varied to $\omega_{c11} = 1.3$ rad/s, $\omega_{c12} = 2.2$ rad/s, $\omega_{c21} = 1.7$ rad/s, and $\omega_{c22} = 2.6$ rad/s. The various cross sections for this filter are shown in Figures 3.119, 3.120, and 3.121, the actual 3-D plot in Figure 3.122, and the contour plot in Figure 3.123.

This bandstop filter is then varied to $\omega_{c11} = 1.3$ rad/s, $\omega_{c12} = 2.5$ rad/s, $\omega_{c21} = 1.7$ rad/s, and $\omega_{c22} = 2.9$ rad/s. The various cross sections for this filter are shown in Figures 3.124, 3.125, and 3.126, the actual 3-D plot in Figure 3.127, and the contour plot in Figure 3.128.
Figure 3.84: Elliptical Lowpass 2-D FIR Filter transformed to $\omega_c_1 = 1.1 \text{ rad/s}$, $\omega_c_2 = 1.7 \text{ rad/s}$ along $\omega_1$ axis.
Figure 3.85: Elliptical Lowpass 2-D FIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.7$ rad/s along $\omega_2$ axis.
Figure 3.86: Elliptical Lowpass 2-D FIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.7 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 3.87: Three Dimensional Plot of Elliptical Lowpass 2-D FIR Filter transformed to

\[ \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.7 \text{ rad/s}. \]
Figure 3.88: Contour Plot of Elliptical Lowpass 2-D FIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.7$ rad/s.
Figure 3.89: Elliptical Lowpass 2-D FIR Filter transformed to 
\( \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \)
along \( \omega_1 \) axis.
Figure 3.90: Elliptical Lowpass 2-D FIR Filter transformed to

\[ \omega_{c1} = 2.5 \text{ rad/s}, \quad \omega_{c2} = 1.5 \text{ rad/s} \]

along \( \omega_2 \) axis.
Figure 3.91: Elliptical Lowpass 2-D FIR Filter transformed to 
\[ \omega_c^1 = 2.5 \text{ rad/s}, \quad \omega_c^2 = 1.5 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis.
Figure 3.92: Three Dimensional Plot of Elliptical Lowpass
2-D FIR Filter transformed to
\( \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \).
Figure 3.93: Contour Plot of Elliptical Lowpass 2-D FIR Filter transformed to $\omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s}.$
Figure 3.94: Elliptical Highpass 2-D FIR Filter transformed to
$\omega_c^1 = 1.1$ rad/s, $\omega_c^2 = 1.4$ rad/s
along $\omega_1$ axis.
Figure 3.95: Elliptical Highpass 2-D FIR Filter transformed to

$\omega_c^1 = 1.1 \text{ rad/s}, \omega_c^2 = 1.4 \text{ rad/s}$

along $\omega_2$ axis.
Figure 3.96: Elliptical Highpass 2-D FIR Filter transformed to
\[ \omega_c^1 = 1.1 \text{ rad/s}, \quad \omega_c^2 = 1.4 \text{ rad/s} \]
along \( \omega_1 = \omega_2 \) axis.
Figure 3.97: Three Dimensional Plot of Elliptical Highpass 2-D FIR Filter transformed to
$\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.4 \text{ rad/s}$.
Figure 3.98: Contour Plot of Elliptical Highpass 2-D FIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.4 \text{ rad/s}$. 
Figure 3.99: Elliptical Highpass 2-D FIR Filter transformed to $\omega_{c1} = 2.0$ rad/s, $\omega_{c2} = 1.3$ rad/s along $\omega_1$ axis.
Figure 3.100: Elliptical Highpass 2-D FIR Filter transformed to
\( \omega_{c1} = 2.0 \text{ rad/s}, \ \omega_{c2} = 1.3 \text{ rad/s} \)
along \( \omega_2 \) axis.
Figure 3.101: Elliptical Highpass 2-D FIR Filter transformed to
\( \omega_{c1} = 2.0 \text{ rad/s, } \omega_{c2} = 1.3 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis.
Figure 3.102: Three Dimensional Plot of Elliptical Highpass 2-D FIR Filter transformed to 
\[ \omega_{c1} = 2.0 \text{ rad/s}, \ \omega_{c2} = 1.3 \text{ rad/s}. \]
Figure 3.103: Contour Plot of Elliptical Highpass 2-D FIR Filter transformed to $\omega_{c1} = 2.0 \, \text{rad/s}, \, \omega_{c2} = 1.3 \, \text{rad/s}$. 
Figure 3.104: Elliptical Bandpass 2-D FIR Filter transformed to
$\omega_{c11} = 1.2 \text{ rad/s}$, $\omega_{c12} = 1.9 \text{ rad/s}$,
$\omega_{c21} = 1.6 \text{ rad/s}$, $\omega_{c22} = 2.3 \text{ rad/s}$,
along $\omega_1$ axis.
Figure 3.105: Elliptical Bandpass 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.2 \text{ rad/s}, \ \omega_{c12} = 1.9 \text{ rad/s}, \]
\[ \omega_{c21} = 1.6 \text{ rad/s}, \ \omega_{c22} = 2.3 \text{ rad/s}, \]
along \( \omega_2 \) axis.
Figure 3.106: Elliptical Bandpass 2-D FIR Filter transformed to
\( \omega_{c11} = 1.2 \text{ rad/s}, \ \omega_{c12} = 1.9 \text{ rad/s}, \)
\( \omega_{c21} = 1.6 \text{ rad/s}, \ \omega_{c22} = 2.3 \text{ rad/s}, \)
along \( \omega_1 = \omega_2 \) axis.
Figure 3.107: Three Dimensional Plot of Elliptical Bandpass
2-D FIR Filter transformed to
\[ \omega_{c11} = 1.2 \text{ rad/s}, \omega_{c12} = 1.9 \text{ rad/s}, \]
\[ \omega_{c21} = 1.5 \text{ rad/s}, \omega_{c22} = 2.3 \text{ rad/s}. \]
Figure 3.108: Contour Plot of Elliptical Bandpass
2-D FIR Filter transformed to
\[ \omega_{c11} = 1.2 \text{ rad/s}, \ \omega_{c12} = 1.9 \text{ rad/s}, \]
\[ \omega_{c21} = 1.6 \text{ rad/s}, \ \omega_{c22} = 2.3 \text{ rad/s}. \]
Figure 3.109: Elliptical Bandpass 2-D FIR Filter transformed to

ω_{c11} = 1.4 \text{ rad/s}, \ \omega_{c12} = 2.0 \text{ rad/s},

ω_{c21} = 1.1 \text{ rad/s}, \ \omega_{c22} = 2.4 \text{ rad/s},

along \omega_1 \text{ axis.}
Figure 3.110: Elliptical Bandpass 2-D FIR Filter transformed to
\( \omega_{c11} = 1.4 \text{ rad/s}, \ \omega_{c12} = 2.0 \text{ rad/s}, \)
\( \omega_{c21} = 1.1 \text{ rad/s}, \ \omega_{c22} = 2.4 \text{ rad/s}, \)
along \( \omega_2 \) axis.
Figure 3.111: Elliptical Bandpass 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.4 \text{ rad/s}, \ \omega_{c12} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.1 \text{ rad/s}, \ \omega_{c22} = 2.4 \text{ rad/s}, \]
along \( \omega_1 = \omega_2 \) axis.
Figure 3.112: Three Dimensional Plot of Elliptical Bandpass 2-D FIR Filter transformed to

\[
\omega_c^{11} = 1.4 \text{ rad/s}, \quad \omega_c^{12} = 2.0 \text{ rad/s},
\]

\[
\omega_c^{21} = 1.1 \text{ rad/s}, \quad \omega_c^{22} = 2.4 \text{ rad/s}.
\]
Figure 3.113: Contour Plot of Elliptical Bandpass
2-D FIR Filter transformed to
\( \omega_{c11} = 1.4 \text{ rad/s}, \omega_{c12} = 2.0 \text{ rad/s}, \)
\( \omega_{c21} = 1.1 \text{ rad/s}, \omega_{c22} = 2.4 \text{ rad/s}. \)
Figure 3.114: Elliptical Bandstop 2-D FIR Filter transformed to
\( \omega_{c11} = 1.1 \text{ rad/s}, \; \omega_{c21} = 2.0 \text{ rad/s}, \)
\( \omega_{c22} = 1.3 \text{ rad/s}, \; \omega_{c22} = 2.0 \text{ rad/s}, \)
along \( \omega_1 \) axis.
Figure 3.115: Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{cl1} = 1.1 \text{ rad/s}, \quad \omega_{cl2} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.3 \text{ rad/s}, \quad \omega_{c22} = 2.0 \text{ rad/s}, \]
along \( \omega_2 \) axis.
Figure 3.116: Elliptical Bandstop 2-D FIR Filter transformed to
$\omega_{c11} = 1.1 \text{ rad/s, } \omega_{c12} = 2.0 \text{ rad/s,}$
$\omega_{c21} = 1.3 \text{ rad/s, } \omega_{c22} = 2.0 \text{ rad/s,}$
along $\omega_1 = \omega_2$ axis.
Figure 3.117: Three Dimensional Plot of Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.1 \text{ rad/s}, \quad \omega_{c12} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.3 \text{ rad/s}, \quad \omega_{c22} = 2.0 \text{ rad/s}. \]
Figure 3.118: Contour Plot of Elliptical Bandstop
2-D FIR Filter transformed to
\[ \omega_{c11} = 1.1 \text{ rad/s}, \ \omega_{c12} = 2.0 \text{ rad/s}, \]
\[ \omega_{c21} = 1.3 \text{ rad/s}, \ \omega_{c22} = 2.0 \text{ rad/s}. \]
Figure 3.119: Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.2 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.6 \text{ rad/s}, \]
along \( \omega_1 \) axis.
Figure 3.120: Elliptical Bandstop 2-D FIR Filter transformed to
\( \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.2 \text{ rad/s}, \)
\( \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.6 \text{ rad/s}, \)
along \( \omega_2 \) axis.
Figure 3.121: Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.2 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.6 \text{ rad/s}, \]
along \( \omega_1 = \omega_2 \) axis.
Figure 3.122: Three Dimensional Plot of Elliptical Bandstop 2-D FIR Filter transformed to
\( \omega_{c11} = 1.3 \text{ rad/s}, \ \omega_{c12} = 2.2 \text{ rad/s}, \)
\( \omega_{c21} = 1.7 \text{ rad/s}, \ \omega_{c22} = 2.6 \text{ rad/s}. \)
Figure 3.123: Contour Plot of Elliptical Bandstop
2-D FIR Filter transformed to
\( \omega_{c11} = 1.3 \text{ rad/s}, \omega_{c12} = 2.2 \text{ rad/s}, \)
\( \omega_{c21} = 1.7 \text{ rad/s}, \omega_{c22} = 2.6 \text{ rad/s}. \)
Figure 3.124: Elliptical Bandstop 2-D FIR Filter transformed to
$\omega_{c11} = 1.3$ rad/s, $\omega_{c12} = 2.5$ rad/s,
$\omega_{c21} = 1.7$ rad/s, $\omega_{c22} = 2.9$ rad/s,
along $\omega_1$ axis.
Figure 3.125: Elliptical Bandstop 2-D FIR Filter transformed to
\( \omega_{c11} = 1.3 \text{ rad/s}, \ \omega_{c12} = 2.5 \text{ rad/s}, \)
\( \omega_{c21} = 1.7 \text{ rad/s}, \ \omega_{c22} = 2.9 \text{ rad/s}, \)
along \( \omega_2 \) axis.
Figure 3.126: Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.5 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.9 \text{ rad/s}, \]
along \( \omega_1 = \omega_2 \) axis.
Figure 3.127: Three Dimensional Plot of Elliptical Bandstop 2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.5 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.9 \text{ rad/s}. \]
Figure 3.128: Contour Plot of Elliptical Bandstop
2-D FIR Filter transformed to
\[ \omega_{c11} = 1.3 \text{ rad/s}, \quad \omega_{c12} = 2.5 \text{ rad/s}, \]
\[ \omega_{c21} = 1.7 \text{ rad/s}, \quad \omega_{c22} = 2.9 \text{ rad/s}. \]
IV. 2-D IIR VARIABLE CUTOFF FREQUENCY FILTERS

The concepts of the 1-D IIR Filters are now extended to the
2-D case. There are various ways of designing 2-D filters which
include direct transformations from 1-D to 2-D, using one step or
two step optimization techniques, or possibly a combination of
these. Some of the more common of the existing techniques will
be reviewed describing all their advantages and disadvantages.
The proposed design methods and techniques will then be described
with examples to illustrate the effect of frequency variability.

4.1 TRANSFORMATIONS

4.1.1. Pendergrass Spectral Transformations

There are two categories of Pendergrass transforma-
tions [3]. They include the

1) Single Variable Transformations, and the
2) Two Variable Transformations

The former type has two individual transformations equations for
each variable $z_i$ and $z_j$. The latter transformation has one
equation with both $z_i$ and $z_j$. These two categories originate
from the general form which is given by $z_i$. All these transfor-
mations are given by the equations in Table 4.1.

The Single Variable Transformations can be completely
determined by the mapping of one or two points. These points are
shown in Table 4.2 which illustrate the effects of these trans-
formations.

For the Two Variable Transformations, the parameters of
Single Variable Transformations

\[ Z_i = G_i(z_i) = \pm z_i^N \frac{1 + \sum_{j=1}^{N} a_j^{(1)} z_i^{-j}}{1 + \sum_{j=1}^{N} a_j^{(1)} z_i^j}, \quad N = 1, 2 \]

\[ Z_2 = G_1(z_2) = \pm z_2^M \frac{1 + \sum_{j=1}^{M} a_j^{(2)} z_2^{-j}}{1 + \sum_{j=1}^{M} a_j^{(2)} z_2^j}, \quad M = 1, 2. \]

Two Variable Transformations

\[ Z_i = G_i(z_1, z_2) = \pm \frac{\gamma_i + \beta_i z_1 + z_1 z_2 + z_1 \bar{z}_2}{1 + \alpha_i z_1 + \beta_i z_2 + \gamma_i \bar{z}_2}, \quad i = 1, 2. \]

The General Form of the Spectral Transformation

\[ Z_i = G_i(z_1, z_2) = \pm \prod_{i=1}^{K} \left\{ \frac{z_1^{N_i^{(1)}} z_2^{M_i^{(1)}} \left( \sum_{n=0}^{N_i^{(1)}} \sum_{m=0}^{M_i^{(1)}} a_{nm}^{(1)} z_1^{-n} z_2^{-m} \right)}{\left( \sum_{n=0}^{N_i^{(1)}} \sum_{m=0}^{M_i^{(1)}} a_{nm}^{(1)} z_1^n z_2^m \right)} \right\}, \quad i = 1, 2 \]

with \(a_{00} = 1.\)

Table 4.1: The Pendergrass Spectral Transformations.
<table>
<thead>
<tr>
<th>Transformation equations</th>
<th>Transformation effect on</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_1 = \frac{a_1 + z_1}{1 + a_1 z_1}$</td>
<td>$a_1 = \frac{\sin \frac{1}{2}(\beta - \omega_c)}{\sin \frac{1}{2}(\beta + \omega_c)}$</td>
</tr>
<tr>
<td>$Z_2 = \frac{a_2 + z_2}{1 + a_2 z_2}$</td>
<td>$r = 1, 2$</td>
</tr>
</tbody>
</table>

| $Z_1 = \frac{a_1 + z_1}{1 + a_1 z_1}$ | $a_1 = \frac{\cos \frac{1}{2}(\beta - \omega_c)}{\cos \frac{1}{2}(\beta + \omega_c)}$ |
| $Z_2 = \frac{a_2 + z_2}{1 + a_2 z_2}$ | $a_2 = \frac{-\sin \frac{1}{2}(\beta - \omega_c)}{-\sin \frac{1}{2}(\beta + \omega_c)}$ |

| $Z_1 = \frac{a_1 + z_1}{1 + a_1 z_1}$ | $a_1 = \frac{\cos \frac{1}{2}(\beta - \omega_c)}{\cos \frac{1}{2}(\beta + \omega_c)}$ |
| $Z_2 = \frac{a_2 + z_2}{1 + a_2 z_2}$ | $a_2 = \frac{-\sin \frac{1}{2}(\beta - \omega_c)}{-\sin \frac{1}{2}(\beta + \omega_c)}$ |

| $Z_1 = \frac{1 - K_1}{1 + K_1} \cdot \frac{2a_1}{1 - K_1} z_1 + z_1^2$ | $a_1 = \frac{\cos \frac{1}{2}(\beta - \omega_c)}{\cos \frac{1}{2}(\beta + \omega_c)}$ |
| $Z_2 = \frac{1 - K_2}{1 + K_2} \cdot \frac{2a_2}{1 - K_2} z_2 + z_2^2$ | $K_1 = \tan \left(\frac{\beta - \omega_c}{2}\right) \tan \left(\frac{\beta}{2}\right)$ |

| $Z_1 = \frac{1 - K_1}{1 + K_1} \cdot \frac{2a_1}{1 - K_1} z_1 + z_1^2$ | $a_1, K_1$ chosen as in IV |
| $Z_2 = \frac{1 - K_2}{1 + K_2} \cdot \frac{2a_2}{1 - K_2} z_2 + z_2^2$ |

| $Z_1 = \frac{a_1 + z_1}{1 + a_1 z_1}$ | $a_1, K_1$ chosen as in IV |
| $Z_2 = \frac{a_2 + z_2}{1 + a_2 z_2}$ | $a_2$ chosen as in I |

Table 4.2: One Variable Spectral Transformations.
Table 4.2 (cont'd): One Variable Spectral Transformations.
the equation must be carefully chosen so that they are unique and produce stable transformations. The effect of changing the parameters are not easily related to the results obtained, thus the form of this transformation is not suitable for use in design. Different hybrids of this transformation can be found in order to develop predictable effects.

Both the Single Variable and the Two Variable Transformations have their advantages and disadvantages which can be summarized in Table 4.3.

**Pendergrass Transformation**

<table>
<thead>
<tr>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>-simple in structure</td>
<td>-limited type of filters can be designed</td>
</tr>
<tr>
<td>-two possible methods:</td>
<td>-trial and error technique in obtaining various magnitude responses</td>
</tr>
<tr>
<td>one variable and two variable structure</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.3: Advantages and Disadvantages for Pendergrass Spectral Transformations

4.1.2. **Transformation Using 2-Variable Reactance Function**

Another type of transformation that will transform a 1-D IIR filter to a 2-D IIR filter is the second order 2-Variable (2-V) reactance function by Ahmadi and Antoniou [13,14,16], which is given by

\[ s = F(s_1, s_2) = \frac{a_1 s_1 + a_2 s_2}{1 + b s_1 s_2} \]  \hspace{1cm} (4.1)

For a given 1-D analog filter with cutoff frequency \( \omega_c \),
Equation 4.1 becomes

\[ \omega_z = \frac{\omega_C - a_i \omega_i}{a_i - b \omega_C \omega_i} \]  \hspace{1cm} (4.2)

The mapping of \( \omega = \omega_C \) onto the \( \omega_z, \omega_i \) plane for various values of \( b \) is illustrated in Figure 4.1.

This transformation has its advantages and disadvantages which can be summarized in Table 4.4.

**Ahmadi transformation**

**advantages**  \hspace{2cm} **disadvantages**

- simple transformation  \hspace{1cm} - limited shapes possible of magnitude response
- variable cutoff frequency and convexity obtainable  \hspace{1cm} - added filter needed since a lowpass transforms to bandpass, thus a guard filter is needed
- with same transformation

Table 4.4: Advantages and Disadvantages for Ahmadi's Spectral Transformations

4.1.3. **Examples of the 2-Variable Reactance Function**

To illustrate the reactance function, a third order lowpass Butterworth filter is used

\[ B(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \]  \hspace{1cm} (4.3)

The cutoff frequency for this normalized filter is \( \omega_C = 1 \) rad/s. By varying \( a_i \) and \( a_i \), the cutoff frequency can be changed, and by varying \( b \), the shape of the filter can be changed. The effect of changing these various parameters can be seen in Figures 4.2
Increasing $b$ increases the convexity of the contour.

$b = 0$

Figure 4.1: Mapping of $\omega = \omega_c$ onto the $\omega_1 - \omega_2$ for different values of $b$. 
to 4.13 respectively.

4.2 DESIGN PROCEDURE FOR 2-D IIR FILTERS

As discussed above, there are many difficulties that must be considered in going from 1-D to 2-D filters, as with the Pendergrass transformation, as compared to the simpler 1-D to 1-D transformation design, such as the Constantinides transformations. One method to get around this is by using various other techniques that achieve the same result. One very effective technique is to use a separable denominator which allows the 1-D transformations to be used for IIR filter designs. Other techniques are by the use of the Oppenheim transform for a zero phase numerator, in conjunction with the Constantinides or Pendergrass transformations for the denominator. Thus the amount of optimization required is greatly reduced. Various combinations of this approach will be demonstrated to show its effectiveness in overcoming the design problem with IIR filters.

4.2.1. Characterization of 2-D IIR Filter Transfer Function

A 2-D analogue filter (IIR) is characterized by its transfer function [5]

\[
H_a(s_1, s_2) = \frac{\sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} s_1^i s_2^j}{\sum_{i=0}^{M} \sum_{j=0}^{N} b_{ij} s_1^i s_2^j}
\] (4.4a)

where \(A\) and \(B\) are polynomials in \(s_1\) and \(s_2\), and the coefficients of the filter \(a_{ij}\) and \(b_{ij}\) are real while \(s_i = j\omega_i\), \(i=1,2\) is a complex Laplace variable. Such a filter is stable if
Figure 4.2: Third-order Lowpass Butterworth filter 3-D plot using the 2-Variable Reactance Function:

\[ a_1 = a_4 = 1, \ b = 0.1 \]
Figure 4.3: Third-order Lowpass Butterworth filter contour plot using the 2-Variable Reactance Function:

\[ a_1 = a_2 = 1, \ b = 0.1 \]
Figure 4.4: Third-order Lowpass Butterworth filter 3-D plot using the 2-Variable Reactance Function:

\[ a_1 = a_2 = 1, \ b = 0.01 \]
Figure 4.5: Third-order Lowpass Butterworth filter contour plot using the 2-Variable Reactance Function:

\[ a_1 = a_2 = 1, \ b = 0.01 \]
Figure 4.6: Third-order Lowpass Butterworth filter 3-D plot using the 2-Variable Reactance Function:
\[ a_1 = a_2 = 2, \ b = 0.01 \]
Figure 4.7: Third-order Lowpass Butterworth filter contour plot using the 2-Variable Reactance Function:

\[
a_i = a_z = 2, \quad b = 0.01
\]
Figure 4.8: Third-order Lowpass Butterworth filter 3-D plot using the 2-Variable Reactance Function:

\[ a_1 = a_2 = 0.5, \ b = 0.01 \]
Figure 4.9: Third-order Lowpass Butterworth filter contour plot using the 2-Variable Reactance Function:

\[ a_1 = a_2 = 0.5, \quad b = 0.01 \]
Figure 4.10: Third-order Lowpass Butterworth filter 3-D plot using the 2-Variable Reactance Function:

\[ a_1 = a_2 = 1, \quad b = 0.5 \]
Figure 4.11: Third-order Lowpass Butterworth filter contour plot using the 2-Variable Reactance Function:
\[ a_1 = a_2 = 1, \ b = 0.5 \]
Figure 4.12: Third-order Lowpass Butterworth filter 3-D plot using the 2-Variable Reactance Function:

\[ a_1 = a_2 = 1, \quad b = 0.2 \]
Figure 4.13: Third-order Lowpass Butterworth filter contour plot using the 2-Variable Reactance Function:

\[ a_1 = a_2 = 1, \ b = 0.2 \]
\[ B(s_1, s_2) \neq 0 \quad \text{for} \quad \bigwedge_{i=1}^{2} \text{Re } s_i \geq 0 \quad (4.4b) \]

From Equation 4.4a, a 2-D digital transfer function can be obtained by the application of the double bilinear transformation to the \( s_i \), \( s_j \) variables which is of the form

\[ H_d(z_i, z_j) = \frac{N(z_i, z_j)}{D(z_i, z_j)} \quad (4.5a) \]

\[ = \frac{A(s_i, s_j)}{B(s_i, s_j)} \bigg|_{s_i = \frac{2z_i - 1}{Tz_i + 1}} \quad \text{for } i=1,2 \quad (4.5b) \]

\[ = \sum_{i=0}^{M} \sum_{j=0}^{N} n_{ij} z_i^i z_j^j \quad (4.5c) \]

\[ = \sum_{i=0}^{M} \sum_{j=0}^{N} d_{ij} z_i^i z_j^j \]

where \( N \) and \( D \) are polynomials in \( z_i \) and \( z_j \), and the coefficients of the filter \( n_{ij} \) and \( d_{ij} \) are real while \( z_i = \exp(j \omega_i T) \), \( i=1,2 \) is a complex variable. The stability of the recursive filter is assured if

\[ D(z_i, z_j) \neq 0 \quad \text{for} \quad \bigwedge_{i=1}^{2} |z_i| \leq 1 \quad (4.5d) \]

The coefficients of the filter can be calculated in either the \( z \)-domain or the \( s \)-domain by using any optimization technique by minimizing some desired cost function. Again the Hooke and Jeeves optimization [15] is utilized for all the filters designed in this chapter just as it is in Chapter Two for the 1-D IIR filter and Chapter 3 for the 2-D FIR filters.

In order to obtain a variable cutoff frequency filter...
both the numerator and the denominator of the transfer function (Equation 4.4 or 4.5) require some type of frequency transformation. These can be the Constantinides or Pendergrass transformations or, as it will be shown later when the transfer function can be altered, the Oppenheim transformations can be used.

Some of the prototype filters designed here use the Positive Definite and Resistor matrix Hurwitz Polynomials described in Chapter 2.

4.2.2. Formulation of the Design Problem

In the design method, different types of filters will be designed using different types of transfer functions similar to those of Equations 4.4 and 4.5. The last one will be a transfer function that already has been precalculated, thus this will serve as the prototype filter. The various cutoff frequency transformations will be then applied to these prototype filters. These filters will then be tuned to different cutoff frequencies thus providing tunability.

The parameters of the 2-D filter are calculated through optimization by minimizing some desired cost function. The error between the ideal and the designed magnitude response of the 1-D filter is calculated using the relationship

\[ E_M(j\omega_m, j\omega_n) = |H_I(\exp(j\omega_m T, j\omega_n T))| - |H_D(\exp(j\omega_m T, j\omega_n T))| \quad (4.6) \]

where \( E_M \) is the error in the magnitude response and \( |H_I| \) and \( |H_D| \) are the magnitude responses of the ideal and the designed filter respectively. The general mean square error \( E_G \) is introduced as a cost function for formulation of the design problem for approx-
mation of the magnitude response of Equation 4.6 as

\[ E_G(j\omega, m, j\omega, n, \psi) = \sum_{m, n \in I_{ps}} E_{_H}^2(j\omega, m, j\omega, n) \]  

(4.7)

where \( I_{ps} \) is the set of all discrete frequency points along the \( \omega, \) and \( \omega \), axis in the passband and stopband regions and \( \psi \) is the coefficient vector.

To design a 2-D filter satisfying a prescribed magnitude specification, the coefficient vector \( \psi \) should be calculated in such a way so that \( E_G \) in Equation 4.7 is minimized. For the filter designs incorporating the Positive Definite and Resistor matrices of Chapter 2, the same requirements apply here. The coefficient vector is calculated subject to the constraint that the elements of the \( \Gamma \) and \( \Delta \) matrices in Equation 2.21 are non-negative. This is a simple constrained nonlinear optimization procedure where the constraints can be removed by the following variable substitution technique \( \Gamma = \Gamma^2 \) and \( \Delta = \Delta^2 \).

4.2.3. **Special Case of a 2-D IIR Lowpass Filter**

From the 1-D case, the 1-D IIR filter of Equation 2.19a is of the form

\[ H_0(z) = \frac{\sum_{i=0}^{M} a_i z^{-i}}{\sum_{j=0}^{N} b_j z^{-j}} \]  

(4.8)

where \( a_i \) and \( b_j \) are the coefficients to be calculated by optimization. For the special case of this 2-D IIR filter, two of these sections can be cascaded together to obtain a separable product denominator of the form.
\[ H(z_1,z_2) = H_0(z_1) H_0(z_2) \]  \hspace{1cm} (4.9)

where \( H_0(z_1) \) and \( H_0(z_2) \) represents the transfer function in the \( \omega_1 \) and \( \omega_2 \) directions respectively. The various Constantinides transformations can then be applied. For the case of a lowpass 2-D filter, the lowpass 1-D IIR filter specifications given previously in Figure 2.8 of Chapter 2 are used, which results in a 2-D square lowpass filter. The numerator for both sections are set equal to one. The prototype 2-D square IIR filter has a cutoff frequency \( \omega_C = 1.1 \text{ rad/s} \). Hence the specifications for this will result as follows:

\begin{align*}
H(j \omega_1 T, j \omega_2 T) &= 1 \quad \text{for} \quad 0 \leq \omega_i \leq 0.7 \text{ rad/s}, \ i=1,2 \\
H(j \omega_1 T, j \omega_2 T) &= 0 \quad \text{for} \quad 1.5 \leq \omega_i \leq \pi \text{ rad/s}, \ i=1,2 \\
\text{Sampling frequency,} \quad \omega_s &= 2\pi (6.28) \text{ rad/s}
\end{align*}

where \( H(j \omega_1 T, j \omega_2 T) = \frac{1}{D_0(j \omega_1 T) D_0(j \omega_2 T)} \)

The denominator sections are the Definite Positive and Resistor Matrix Hurwitz Polynomials described in Chapter Two. The values of the coefficients of this filter are those of Table 2.3, but in this case there is a separable product 2-D filter corresponding to both the \( \omega_1 \) and \( \omega_2 \) directions. Therefore these values are used twice, once for each direction resulting in a fourth order 2-D filter. The filter is then discretized using the double bilinear transformation of Equation 4.5b. Three major cross sections are taken along the \( \omega_1 \), \( \omega_2 \), and \( \omega_1 = \omega_2 \) axes which are shown in Figures 4.14, 4.15, and 4.16, the actual 3-D plot in Figure 4.17, and the contour plot in Figure 4.18.
This lowpass filter is then varied to $\omega_c = 1.5 \text{ rad/s}$ and $\omega_c = 0.7 \text{ rad/s}$. The various cross sections for the former filter are shown in Figures 4.19, 4.20, and 4.21, the actual 3-D plot in Figure 4.22, and the contour plot in Figure 4.23. For the latter filter, the various cross sections for the latter filter are shown in Figures 4.24, 4.25, and 4.26, the actual 3-D plot in Figure 4.27, and the contour plot in Figure 4.28.

4.2.4. IIR Filter: Type I

This type of transfer function with a zero phase numerator for an IIR filter is given by

$$H(z_1, z_2) = \frac{\sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij}(\cos \omega_i)^i(\cos \omega_i)^j}{\sum_{i=0}^{M} \sum_{j=0}^{N} b_{ij}z^{-i}z^{-j}}$$

$$= \frac{N(z_1, z_2)}{D(z_1) D(z_2)} \quad (4.10a)$$

where

$$z_1 = e^{j\omega_1}, z_2 = e^{j\omega_2} \quad (4.10c)$$

and

$$\cos \omega_i = \frac{z_1 + z_2^{-1}}{2}, \quad \cos \omega_2 = \frac{z_1 + z_2^{-1}}{2} \quad (4.10d)$$

The numerator is substituted with Oppenheim's transformations, as stated in Chapter 3, given by the following equations

$$\cos \omega_1 = A_0 + (1-A_0)\cos \omega_i \quad (3.9)$$

and

$$\cos \omega_2 = B_0 + (1-B_0)\cos \omega_i \quad (3.10)$$

The denominator are separable and stable polynomials and
Figure 4.14: Special Case (square type) Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.15: Special Case (square type) Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.16: Special Case (square type) Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s along $\omega_1 = \omega_2$ axis.
Figure 4.17: Three Dimensional Plot of Special Case (square type) Lowpass 2-D IIR Filter with 
\( \omega_c = 1.1 \text{ rad/s}, \quad \omega_c = 1.1 \text{ rad/s}. \)
Figure 4.18: Contour Plot of Special Case (square type) Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$. 
Figure 4.19: Special Case (square type) Lowpass 2-D IIR Filter transformed to \( \omega_c1 = 1.5 \text{ rad/s} \), \( \omega_c2 = 1.5 \text{ rad/s} \) along \( \omega_1 \) axis.
Figure 4.20: Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.21:  Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 1.5 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.22: Three Dimensional Plot of Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s}.$
Figure 4.23: Contour Plot of Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 1.5 \text{ rad/s}$. 
Figure 4.24: Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 0.7 \text{ rad/s}$, $\omega_{c2} = 0.7 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.25: Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 0.7$ rad/s, $\omega_{c2} = 0.7$ rad/s along $\omega_2$ axis.
Figure 4.26: Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 0.7$ rad/s, $\omega_{c2} = 0.7$ rad/s along $\omega_1 = \omega_2$ axis.
Figure 4.27: Three Dimensional Plot of Special Case (square type) Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 0.7 \text{ rad/s}, \omega_{c2} = 0.7 \text{ rad/s}$.
Figure 4.28: Contour Plot of Special Case (square type)
Lowpass 2-D IIR Filter transformed to
\[ \omega_c^1 = 0.7 \text{ rad/s}, \quad \omega_c^2 = 0.7 \text{ rad/s}. \]
thus \( z_i \) and \( z_j \) are substituted with the Constantinides
transformations.

To create the highpass filter from this normalized
2-D IIR filter requires a simple subtracting from one which
results in the desired response. The bandpass and bandstop
filters are created by the addition of a lowpass and a highpass
filter just described. In the following examples, two designed
prototype filters will be used to obtain the bandpass and
bandstop filters.

4.2.5. Examples of Variable Cutoff 2-D IIR Filters: Type 1

Various examples using this type 1 transfer function
are demonstrated by using a two step procedure. The first is
done by using the lowpass IIR filter specifications given in
Figure 2.8 of Chapter 2 for the 1-D case. The portion \( 1/D(z) \) of
Equation 4.10b is used to obtain this filter through optimization
using the Positive Definite and Resistor Matrix Hurwitz Poly-
nomial given in Chapter Two. Two of these functions which will
then represent each of the two dimensions, mainly \( z_i \) and \( z_j \), are
then multiplied together to achieve the denominator portion,
\( 1/D(z_i)D(z_j) \), of the transfer function (4.10b). This is essen-
tially the Special Case filter designed in the previous section,
but now it is extended further to obtain the circular shape which
is now the next step in the procedure.

The next step is to optimize the numerator \( N(z_i, z_j) \)
using the denominator from the first step. Hence the two step
optimization will then become the desired 2-D filter of Equation
4.10b. The numerator, which acts a weighting function, provides
the circular contour shape for the 2-D filter.

The prototype 2-D IIR filter has a cutoff frequency \( \omega_c = 1.1 \text{ rad/s} \). The specifications for this designed prototype filter are as follows:

\[
\begin{align*}
H(j\omega_1, T, j\omega_2, T) &= 1 \text{ for } 0 \leq (\omega_1^2 + \omega_2^2)^{1/2} \leq 0.7 \text{ rad/s} \\
H(j\omega_1, T, j\omega_2, T) &= 0 \text{ for } 1.5 \leq (\omega_1^2 + \omega_2^2)^{1/2} \leq \pi \text{ rad/s}
\end{align*}
\]

Sampling frequency, \( \omega_s = 2\pi \times (6.28) \text{ rad/s} \)

The values of the coefficients for this first designed prototype filter are given in Table 4.5. Three major cross sections are taken along the \( \omega_1, \omega_2, \) and \( \omega_1 = \omega_2 \) axes which are shown in Figures 4.29, 4.30, and 4.31, the actual 3-D plot in Figure 4.32, and the contour plot in Figure 4.33.

This lowpass filter is then varied to \( \omega_c = 1.5 \text{ rad/s} \). The various cross sections for this filter are shown in Figures 4.34, 4.35, and 4.36, the actual 3-D plot in Figure 4.37, and the contour plot in Figure 4.38.

The design of a 2-D highpass IIR filter has a cutoff frequency \( \omega_c = 1.1 \text{ rad/s} \). The various cross sections are taken along the \( \omega_1, \omega_2, \) and \( \omega_1 = \omega_2 \) axes which are shown in Figures 4.39, 4.40, and 4.41, the actual 3-D plot in Figure 4.42, and the contour plot in Figure 4.43.

This highpass filter is then varied to \( \omega_c = 1.8 \text{ rad/s} \). The various cross sections for this filter are shown in Figures 4.44, 4.45, and 4.46, the actual 3-D plot in Figure 4.47, and the contour plot in Figure 4.48.
### Values of the numerator coefficients

\[
\begin{align*}
a(0,0) &= 4.9999982118607\times10^{-2} & a(2,3) &= 1.0000002384186\times10^{-1} \\
a(0,1) &= -1.999998807307\times10^{-1} & a(2,4) &= -1.9999928474433\times10^{-1} \\
a(0,2) &= -5.9604644775391\times10^{-8} & a(3,0) &= 1.0000002384186\times10^{-1} \\
a(0,3) &= -1.0000002384186\times10^{-1} & a(3,1) &= 1.0000002384186\times10^{-1} \\
a(0,4) &= -5.9604644775391\times10^{-8} & a(3,2) &= -9.9999964237213\times10^{-2} \\
a(1,0) &= -4.9999994039536\times10^{-1} & a(3,3) &= 3.0000001192093\times10^{-1} \\
a(1,1) &= 3.999997615814\times10^{-1} & a(3,4) &= 1.50000000596046\times10^{-1} \\
a(1,2) &= 3.999997615814\times10^{-1} & a(4,0) &= 1.0000002384186\times10^{-1} \\
a(1,3) &= -9.9999964237213\times10^{-2} & a(4,1) &= -9.9999964237213\times10^{-2} \\
a(1,4) &= 3.0000001192093\times10^{-1} & a(4,2) &= 3.0000001192093\times10^{-1} \\
a(2,0) &= -9.9999964237213\times10^{-2} & a(4,3) &= -9.9999964237213\times10^{-2} \\
a(2,1) &= 1.0000002384186\times10^{-1} & a(4,4) &= 1.0000002384186\times10^{-1} \\
a(2,2) &= 3.0000001192093\times10^{-1} \\
\end{align*}
\]

### Values of the denominator coefficients

**Section 1**

\[
\begin{align*}
\gamma_{11} &= 1.1290792614221E+00 \\
\gamma_{21} &= 3.2378990948200E-01 \\
\sigma_{11} &= 1.0891706049442E-01 \\
\sigma_{21} &= 1.2937499701977E+00 \\
a_1 &= 9.0193811953068E-01 \\
r_1 &= 7.0625112652779E-01 \\
g_1 &= 3.6992599666119E-01 \\
\end{align*}
\]

**Section 2**

\[
\begin{align*}
\gamma_{12} &= 1.0500000000000E+00 \\
\gamma_{22} &= 4.5624390542507E-01 \\
\sigma_{12} &= 9.1079878807067E-02 \\
\sigma_{22} &= 1.5439417719841E+00 \\
a_2 &= 2.0979929651218E-01 \\
r_2 &= -2.8070016130806E-01 \\
g_2 &= 7.7499786019325E-01 \\
\end{align*}
\]

**Table 4.5:** The values of the coefficients for the first of two designed Type 1 Prototype Lowpass 2-D IIR Filters.
Figure 4.29: Type 1 Lowpass 2-D IIR Filter with 
$\omega_{cl} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s 
along $\omega_1$ axis.
Figure 4.30: Type 1 Lowpass 2-D IIR Filter with \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \) along \( \omega_2 \) axis.
Figure 4.31: Type 1 Lowpass 2-D IIR Filter with 
$\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}$
along $\omega_1 = \omega_2$ axis.
Figure 4.32: Three Dimensional Plot of Type 1 Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}$.
Figure 4.33: Contour Plot of Type 1 Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$.
Figure 4.34: Type 1 Lowpass 2-D IIR Filter transformed to
$\omega_{c1} = 1.5 \text{ rad/s, } \omega_{c2} = 1.5 \text{ rad/s}$
along $\omega_1$ axis.

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Figure 4.35: Type 1 Lowpass 2-D IIR Filter transformed to
$\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s}$
along $\omega_2$ axis.
Figure 4.36: Type 1 Lowpass 2-D IIR Filter transformed to
$\omega_{c1} = 1.5 \text{ rad/s}, \; \omega_{c2} = 1.5 \text{ rad/s}$
along $\omega_1 = \omega_2$ axis.
Figure 4.37: Three Dimensional Plot of Type 1 Lowpass 2-D IIR Filter transformed to
\[ \omega_c^1 = 1.5 \text{ rad/s}, \ \omega_c^2 = 1.5 \text{ rad/s}. \]
Figure 4.38: Contour Plot of Type 1 Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 1.5 \text{ rad/s}$.
Figure 4.39: Type 1 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s along $\omega_1$ axis.
Figure 4.40: Type 1 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.41: Type 1 Highpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis.
Figure 4.42: Three Dimensional Plot of Type 1 Highpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$. 
Figure 4.43: Contour Plot of Type 1 Highpass 2-D IIR Filter with $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s.
Figure 4.44: Type 1 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8 \text{ rad/s}, \omega_{c2} = 1.8 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.45: Type 1 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s along $\omega_2$ axis.
Figure 4.46: Type 1 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s along $\omega_1 = \omega_2$ axis.
Figure 4.47: Three Dimensional Plot of Type 1 Highpass 2-D IIR Filter transformed to
\[ \omega_{c1} = 1.8 \text{ rad/s}, \quad \omega_{c2} = 1.8 \text{ rad/s.} \]
Figure 4.48: Contour Plot of Type 1 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8 \text{ rad/s}, \omega_{c2} = 1.8 \text{ rad/s}$.
The design of a 2-D IIR bandpass filter is designed with \( \omega_{c1} = 1.1 \text{ rad/s} \) and \( \omega_{c2} = 2.5 \text{ rad/s} \). Thus a second filter is required. This second prototype 2-D IIR filter has a cutoff frequency \( \omega_c = 1.9 \text{ rad/s} \). The specifications for this second prototype filter are as follows:

\[
H(j\omega, T, j\omega_c T) = 1 \quad \text{for} \quad 0 \leq \left( \omega_i^2 + \omega_c^2 \right)^{1/2} \leq 1.5 \text{ rad/s}
\]

\[
H(j\omega, T, j\omega_c T) = 0 \quad \text{for} \quad 2.3 \leq \left( \omega_i^2 + \omega_c^2 \right)^{1/2} \leq \pi \text{ rad/s}
\]

Sampling frequency, \( \omega_s = 2\pi (6.28) \text{ rad/s} \)

The values of the coefficients for this second designed prototype filter are given in Table 4.6. To obtain this bandpass filter, this second designed prototype 2-D IIR lowpass filter is added with a 2-D IIR highpass filter. This highpass filter is the transformed filter of the first designed prototype 2-D IIR lowpass filter designed above. Thus the various cross sections for this bandpass filter are shown in Figures 4.49, 4.50, and 4.51, the actual 3-D plot in Figure 4.52, and the contour plot in Figure 4.53.

This bandpass filter is then varied to \( \omega_{c1} = 1.5 \text{ rad/s} \) and \( \omega_{c2} = 2.1 \text{ rad/s} \). The various cross sections for this filter are shown in Figures 4.54, 4.55, and 4.56, the actual 3-D plot in Figure 4.57, and the contour plot in Figure 4.58.

The design of a 2-D IIR bandstop filter is designed with \( \omega_{c1} = 1.3 \text{ rad/s} \) and \( \omega_{c2} = 2.3 \text{ rad/s} \). This filter is designed using the first prototype designed 2-D IIR lowpass filter added with a 2-D IIR highpass filter. This highpass filter is the transformed filter of the second designed prototype.
2-D IIR lowpass filter designed previously. Thus the various cross sections for this bandpass filter are shown in Figures 4.59, 4.60, and 4.61, the actual 3-D plot in Figure 4.62, and the contour plot in Figure 4.63.

This bandstop filter is then varied to \( \omega_{c1} = 1.2 \) rad/s and \( \omega_{c2} = 2.7 \) rad/s. The various cross sections for this filter are shown in Figures 4.64, 4.65, and 4.66, the actual 3-D plot in Figure 4.67, and the contour plot in Figure 4.68.

4.2.6. IIR Filter: Type 2

This type of transfer function for an IIR filter is given by

\[
H(z_1, z_2) = \frac{\sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} z_1^{-i} z_2^{-j}}{\sum_{i=0}^{M} b_{i1} z_1^{-i}} \left[ \sum_{j=0}^{N} b_{i2} z_2^{-j} \right]
\]

\[
= \frac{N(z_1, z_2)}{D(z_1) D(z_2)} \tag{4.11a}
\]

where

\[
z_1 = e^{j\omega_1}, \quad z_2 = e^{j\omega_2} \tag{4.11c}
\]

The denominator consists of two separable and stable polynomials which are the same as those used in the type 1 IIR filter. In this particular design, both the numerator and denominator are substituted with Constantinides transformations.

The design of the highpass, the bandpass, and the bandstop filters from this normalized 2-D IIR filter are performed in the same manner as described for the type 1 IIR filter.
Values of the numerator coefficients

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<th>Value</th>
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<td>$a(2,3)$</td>
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<td>$a(4,4)$</td>
<td>4.0000000000000E-01</td>
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<tr>
<td>$a(2,2)$</td>
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Values of the denominator coefficients

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<td>$\gamma_{11}$</td>
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<td>$\gamma_{12}$</td>
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<tr>
<td>$\gamma_{21}$</td>
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Table 4.6: The values of the coefficients for the second of two designed Type 1 Prototype Lowpass 2-D IIR Filters.
Figure 4.49: Type I Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 2.5$ rad/s along $\omega_1$ axis.
Figure 4.50: Type 1 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.51: Type 1 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 2.5 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.52: Three Dimensional Plot of Type 1 Bandpass 2-D IIR Filter transformed to
\( \omega_{c1} = 1.1 \, \text{rad/s}, \ \omega_{c2} = 2.5 \, \text{rad/s}. \)
Figure 4.53: Contour Plot of Type I Bandpass 2-D IIR Filter
transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 2.5 \text{ rad/s}$. 
Figure 4.54: Type 1 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 2.1$ rad/s along $\omega_1$ axis.
Figure 4.55: Type 1 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 2.1 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.56: Type 1 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 2.1$ rad/s along $\omega_1 = \omega_2$ axis.
Figure 4.57:  Three Dimensional Plot of Type 1 Bandpass 2-D IIR Filter transformed to
\[ \omega_{c1} = 1.5 \text{ rad/s}, \quad \omega_{c2} = 2.1 \text{ rad/s}. \]
Figure 4.58: Contour Plot of Type 1 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 2.1$ rad/s.
Figure 4.59: Type 1 Bandstop 2-D IIR Filter transformed to $\omega_c = 1.3 \text{ rad/s}, \omega_c = 2.3 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.60: Type 1 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.3$ rad/s, $\omega_{c2} = 2.3$ rad/s along $\omega_2$ axis.
Figure 4.61: Type 1 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.3 \text{ rad/s, } \omega_{c2} = 2.3 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.62: Three Dimensional Plot of Type 1 Bandstop 2-D IIR Filter transformed to
\( \omega_c^1 = 1.3 \text{ rad/s}, \omega_c^2 = 2.3 \text{ rad/s}. \)
Figure 4.63: Contour Plot of Type 1 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.3$ rad/s, $\omega_{c2} = 2.3$ rad/s.
Figure 4.64: Type 1 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2 \text{ rad/s}, \omega_{c2} = 2.7 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.65: Type 1 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2 \text{ rad/s}, \ \omega_{c2} = 2.7 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.66: Type 1 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2 \text{ rad/s}$, $\omega_{c2} = 2.7 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.67: Three Dimensional Plot of Type 1 Bandstop 2-D IIR Filter transformed to
$\omega_{c1} = 1.2 \text{ rad/s}, \omega_{c2} = 2.7 \text{ rad/s}.$
Figure 4.68: Contour Plot of Type 1 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2 \text{ rad/s}, \omega_{c2} = 2.7 \text{ rad/s}$. 
In this case however, only one prototype filter will be designed, but two of the same filter will be used to design the bandpass and bandstop filters in the following examples.

4.2.7. **Examples of Variable Cutoff 2-D IIR Filters: Type 2**

Various examples using this type 2 transfer function are demonstrated by using a two step procedure which is exactly the same as the type 1 transfer function except in this case the numerator is in a different form. The first is done by using the lowpass IIR filter specifications given in Figure 2.8 of Chapter Two for the 1-D case. The portion 1/D(z) of Equation 4.11b is used to obtain this filter through optimization using the Positive Definite and Resistor Matrix Hurwitz Polynomial given in Chapter 2. Two of these functions which will then represent each of the two dimensions, mainly z_1 and z_2, are then multiplied together to achieve the denominator portion, 1/D(z_1)D(z_2), of the transfer function (4.11b). This step takes on essentially the same step as the type 1 IIR filter, up to this point.

The next step is to optimize the numerator N(z_1,z_2) using the denominator from the first step. Hence the two step optimization will then become the desired 2-D prototype filter of Equation 4.11b.

The designed prototype 2-D IIR lowpass filter has a cutoff frequency \( \omega_c = 1.1 \text{ rad/s} \). The values of the coefficients of the designed prototype filter are given in Table 4.7. Three major cross sections are taken along the \( \omega_1, \omega_2 \), and \( \omega_1 = \omega_2 \), axes which are shown in Figures 4.69, 4.70, and 4.71, the actual 3-D plot in Figure 4.72, and the contour plot in
Figure 4.73.

This lowpass filter is then varied to $\omega_c = 1.5$ rad/s. The various cross sections for this filter are shown in Figures 4.74, 4.75, and 4.76, the actual 3-D plot in Figure 4.77, and the contour plot in Figure 4.78.

The design of a 2-D highpass IIR filter has a cutoff frequency $\omega_c = 1.1$ rad/s. The various cross sections are taken along the ($\omega_1$, $\omega_2$, and $\omega_3 = \omega_i$) axes which are shown in Figures 4.79, 4.80, and 4.81, the actual 3-D plot in Figure 4.82, and the contour plot in Figure 4.83.

This highpass filter is then varied to $\omega_c = 1.8$ rad/s. The various cross sections for this filter are shown in Figures 4.84, 4.85, and 4.86, the actual 3-D plot in Figure 4.87, and the contour plot in Figure 4.88.

The design of a 2-D IIR bandpass filter is designed with $\omega_{c1} = 1.1$ rad/s and $\omega_{c2} = 2.5$ rad/s. In this case a second filter is required. This filter is the same as the designed prototype filter of Table 4.7, but this will be a second and separate filter from the first prototype filter. The design of the bandpass filter takes on the exact same procedure as that of the type 1 filter above. The various cross sections for this filter are shown in Figures 4.89, 4.90, and 4.91, the actual 3-D plot in Figure 4.92, and the contour plot in Figure 4.93.

This bandpass filter is then varied to $\omega_{c1} = 1.5$ rad/s and $\omega_{c2} = 2.6$ rad/s. The various cross sections for this filter are shown in Figures 4.94, 4.95, and 4.96, the actual 3-D plot in Figure 4.97, and the contour plot in Figure 4.98.

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The design of a 2-D IIR bandstop filter is designed with $\omega_{c1} = 1.1$ rad/s and $\omega_{c2} = 2.4$ rad/s. Here again a second filter is required. This second designed prototype filter is the same filter used for the bandpass filter above. The design of the bandstop filter takes on the exact same procedure as that of the type 1 filter above. The various cross sections for this filter are shown in Figures 4.99, 4.100, and 4.101, the actual 3-D plot in Figure 4.102, and the contour plot in Figure 4.103.

This bandstop filter is then varied to $\omega_{c1} = 1.2$ rad/s and $\omega_{c2} = 2.7$ rad/s. The various cross sections for this filter are shown in Figures 4.104, 4.105, and 4.106, the actual 3-D plot in Figure 4.107, and the contour plot in Figure 4.108.

4.2.8. **Applying transformations to an existing IIR Filter: Type 3**

The variable cutoff frequency transformations are applied to a 2-Variable Very Strict Hurwitz Polynomial (VSHP) in a 2-dimensional (2-D) stable recursive digital filter design developed by Dr. Ahmadi et al. [5]. This type of Hurwitz Polynomial does not involve the Resistor Matrix which is described in Chapter Two. The Constantinides or Pendergrass transformations are applied to this filter design to observe if the variable cutoff frequency is obtainable. The third order transfer function, and its various definitions are given in Table 4.8.

The original design specifications for this 2-D lowpass filter along with the resulting coefficients are given Table 4.9. The resulting amplitude response for this lowpass filter is given in Figure 4.109. This filter can be compared to the transformed lowpass filters obtained in the following examples,
Values of the numerator coefficients

| $a(0,0)$ | $-2.0000000000000E-01$ | $a(2,3)$ | $-2.0000000000000E-01$ |
| $a(0,1)$ | $-2.0000000000000E-01$ | $a(2,4)$ | $1.0000000000000E-01$ |
| $a(0,2)$ | $-1.3877787807814E-17$ | $a(3,0)$ | $-2.0000000000000E-01$ |
| $a(0,3)$ | $-2.0000000000000E-01$ | $a(3,1)$ | $1.0000000000000E-01$ |
| $a(0,4)$ | $-1.5000000000000E-01$ | $a(3,2)$ | $0.0000000000000E+00$ |
| $a(1,0)$ | $-1.0000000000000E-01$ | $a(3,3)$ | $-2.0000000000000E-01$ |
| $a(1,1)$ | $1.5000000000000E-01$ | $a(3,4)$ | $-1.0000000000000E-01$ |
| $a(1,2)$ | $1.0000000000000E-01$ | $a(4,0)$ | $-2.5000000000000E-01$ |
| $a(1,3)$ | $2.0000000000000E-01$ | $a(4,1)$ | $-1.0000000000000E-01$ |
| $a(1,4)$ | $0.0000000000000E+00$ | $a(4,2)$ | $1.0000000000000E-01$ |
| $a(2,0)$ | $0.0000000000000E+00$ | $a(4,3)$ | $-1.0000000000000E-01$ |
| $a(2,1)$ | $1.0000000000000E-01$ | $a(4,4)$ | $-5.0000000000000E-02$ |
| $a(2,2)$ | $1.0000000000000E-01$ |

Values of the denominator coefficients

<table>
<thead>
<tr>
<th>Section 1</th>
<th>Section 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{11}$</td>
<td>$1.41503906250000E+00$</td>
</tr>
<tr>
<td>$\gamma_{21}$</td>
<td>$2.58349609375000E+00$</td>
</tr>
<tr>
<td>$\sigma_{11}$</td>
<td>$9.77539062500000E-01$</td>
</tr>
<tr>
<td>$\sigma_{21}$</td>
<td>$1.53125000000000E+00$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$3.44726562500000E-01$</td>
</tr>
<tr>
<td>$r_1$</td>
<td>$5.14648437500000E-01$</td>
</tr>
<tr>
<td>$g_1$</td>
<td>$1.28466796875000E+00$</td>
</tr>
</tbody>
</table>

Table 4.7: The values of the coefficients for the designed Type 2 Prototype Lowpass 2-D IIR Filter.
Figure 4.69: Type 2 Lowpass 2-D IIR Filter with 
$\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$ 
along $\omega_1$ axis.
Figure 4.70: Type 2 Lowpass 2-D IIR Filter with
\( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s} \nolonger \omega_2 \text{ axis.} \)
Figure 4.71: Type 2 Lowpass 2-D IIR Filter with
\( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis.
Figure 4.72: Three Dimensional Plot of Type 2 Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}$. 
Figure 4.73: Contour Plot of Type 2 Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s.
Figure 4.74: Type 2 Lowpass 2-D IIR Filter transformed to
$\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 1.5$ rad/s
along $\omega_1$ axis.
Figure 4.75: Type 2 Lowpass 2-D IIR Filter transformed to 
$\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s}$
along $\omega_2$ axis.
Figure 4.76: Type 2 Lowpass 2-D IIR Filter transformed to
\( \omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis.
Figure 4.77: Three Dimensional Plot of Type 2 Lowpass 2-D IIR Filter transformed to
\[ \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s}. \]
Figure 4.78: Contour Plot of Type 2 Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s}$. 
Figure 4.79: Type 2 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.80: Type 2 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s along $\omega_2$ axis.
Figure 4.81: Type 2 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.82: Three Dimensional Plot of Type 2 Highpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$. 
Figure 4.83: Contour Plot of Type 2 Highpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}$. 
Figure 4.84: Type 2 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s along $\omega_1$ axis.
Figure 4.85: Type 2 Highpass 2-D IIR Filter transformed to $\omega_c1 = 1.8$ rad/s, $\omega_c2 = 1.8$ rad/s along $\omega_2$ axis.
Figure 4.86: Type 2 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s along $\omega_1 = \omega_2$ axis.
Figure 4.87: Three Dimensional Plot of Type 2 Highpass 2-D IIR Filter transformed to
\[ \omega_{c1} = 1.8 \text{ rad/s}, \omega_{c2} = 1.8 \text{ rad/s}. \]
Figure 4.88: Contour Plot of Type 2 Highpass 2-D IIR Filter
transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s.
Figure 4.89:  Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 2.5 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.90: Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 2.5 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.91: Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 2.5 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.92: Three Dimensional Plot of Type 2 Bandpass 2-D IIR Filter transformed to 
\[ \omega_{c1} = 1.1 \text{ rad/s}, \quad \omega_{c2} = 2.5 \text{ rad/s}. \]
Figure 4.93: Contour Plot of Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s, } \omega_{c2} = 2.5 \text{ rad/s.}$
Figure 4.94: Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 2.6 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.95: Type 2 Bandpass 2-D IIR Filter transformed to $\omega_c = 1.5 \text{ rad/s}$, $\omega_{c2} = 2.6 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.96: Type 2 Bandpass 2-D IIR Filter transformed to $\omega_c = 1.5 \text{ rad/s}$, $\omega_c = 2.6 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.97: Three Dimensional Plot of Type 2 Bandpass 2-D IIR Filter transformed to

\[ \omega_{c1} = 1.5 \text{ rad/s}, \quad \omega_{c2} = 2.6 \text{ rad/s}. \]
Figure 4.98: Contour Plot of Type 2 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 2.6 \text{ rad/s}.$
Figure 4.99: Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 2.4$ rad/s along $\omega_1$ axis.
Figure 4.100: Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 2.4 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.101: Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 2.4 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.102: Three Dimensional Plot of Type 2 Bandstop 2-D IIR Filter transformed to
\[ \omega_{c1} = 1.1 \text{ rad/s}, \quad \omega_{c2} = 2.4 \text{ rad/s}. \]
Figure 4.103: Contour Plot of Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 2.4 \text{ rad/s}$. 
Figure 4.104: Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2 \text{ rad/s}$, $\omega_{c2} = 2.7 \text{ rad/s}$ along $\omega_\perp$ axis.
Figure 4.105: Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2 \text{ rad/s}$, $\omega_{c2} = 2.7 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.106: Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2$ rad/s, $\omega_{c2} = 2.7$ rad/s along $\omega_1 = \omega_2$ axis.
Figure 4.107: Three Dimensional Plot of Type 2 Bandstop 2-D IIR Filter transformed to
\( \omega_{c1} = 1.2 \text{ rad/s}, \ \omega_{c2} = 2.7 \text{ rad/s}. \)
Figure 4.108: Contour Plot of Type 2 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2$ rad/s, $\omega_{c2} = 2.7$ rad/s.
and especially the first one since this is the prototype filter.

The design of the highpass, the bandpass, and the bandstop filters from this normalized 2-D IIR filter are performed in the same manner as described for the type 1 and type 2 IIR filter. In this case again, only one prototype filter will be designed as it is done for the type 2 2-D IIR filter, but two of the same filter will be used to design the bandpass and bandstop filters in the following examples.

4.2.9. Examples of Variable Cutoff 2-D IIR Filters: Type 3

Various examples using this type 3 transfer function are demonstrated by using the example given above. As just indicated, the lowpass prototype 2-D IIR filter specifications and the coefficients used in the transfer function of Table 4.8, are those given in Table 4.9.

The prototype 2-D IIR filter has a cutoff frequency $\omega_c = 2.5 \text{ rad/s}$. Three major cross sections are taken along the $\omega_1$, $\omega_2$, and $\omega_1 = \omega_2$ axes which are shown in Figures 4.110, 4.111 and 4.112, the actual 3-D plot in Figure 4.113, and the contour plot in Figure 4.114. One should note that the 3-D plot of Figure 4.109 should correspond to that of Figure 4.113.

This lowpass filter is then varied to $\omega_c = 3.5 \text{ rad/s}$. The various cross sections for this filter are shown in Figures 4.115, 4.116, and 4.117, the actual 3-D plot in Figure 4.118, and the contour plot in Figure 4.119.

The design of a 2-D highpass IIR filter has a cutoff frequency $\omega_c = 2.5 \text{ rad/s}$. The various cross sections are taken along the $\omega_1$, $\omega_2$, and $\omega_1 = \omega_2$ axes which are shown in
\[ H_d(s_1, s_2) = \frac{P(s_1, s_2, \mathbf{p})}{Q(s_1, s_2, \mathbf{q})} = \frac{\sum_{i_1=0}^{M_1} \sum_{i_2=0}^{M_2} p(i_1, i_2) s_1^{i_1} s_2^{i_2}}{\sum_{i_1=0}^{M_1} \sum_{i_2=0}^{M_2} q(i_1, i_2) s_1^{i_1} s_2^{i_2}} \quad (1) \]

\[ Q(s_1, s_2, \mathbf{q}) = 0 \quad \text{for} \quad \frac{1}{2} \sum_{i=1}^2 \text{Re} s_i \geq 0. \quad (2) \]

\[ H_d(z_1, z_2) = \frac{N(z_1, z_2, \mathbf{n})}{D(z_1, z_2, \mathbf{d})} = \left. \frac{P(s_1, s_2, \mathbf{p})}{Q(s_1, s_2, \mathbf{q})} \right|_{s_i = \frac{2z_i - 1}{2z_i + 1}} \quad \text{for} \quad i = 1, 2 \]

\[ = \frac{\sum_{i_1=0}^{M_1} \sum_{i_2=0}^{M_2} n(i_1, i_2) z_1^{i_1} z_2^{i_2}}{\sum_{i_1=0}^{M_1} \sum_{i_2=0}^{M_2} d(i_1, i_2) z_1^{i_1} z_2^{i_2}} \quad (3) \]

\[ Q_1(s_1, s_2, \mathbf{a}, \mathbf{g}) = (d_{11} - d_{12}) (s_1 + 1) - d_{13} (s_2 + 1) + d_{41} [s_1^2 (s_2 + 1) + 2s_1 s_2] \quad (4) \]

where

\[ d_{11} = (a_{11} g_{33} - a_{12} g_{13} + a_{13} g_{12})^2 \]
\[ d_{12} = (a_{11} g_{33} - a_{12} g_{13} + a_{13} g_{12})^2 \]
\[ d_{13} = (a_{11} g_{33} - a_{12} g_{13} + a_{13} g_{12})^2 \]
\[ d_{41} = (a_{11} d_{22} a_{33} - a_{12} a_{23} d_{32} + a_{13} a_{22} a_{31})^2 \]
\[ - a_{12} a_{21} a_{33} + a_{13} a_{22} a_{31} + a_{11} a_{23} a_{31} (a_{31} - a_{32} a_{21} a_{31})^2. \quad (5) \]

\[ Q_2(s_1, s_2, \mathbf{a}, \mathbf{g}) = d_{12} (s_2 - 1) - (d_{22} + d_{21} + d_{23}) (s_2 - 1) + d_{42} [s_1^2 (s_2 + 1) + 2s_1 s_2] \quad (6) \]

where

\[ d_{12} = (a_{11} g_{33} - a_{12} g_{13} + a_{13} g_{12})^2 \]
\[ d_{22} = (a_{21} g_{33} - a_{22} g_{13} + a_{23} g_{12})^2 \]
\[ d_{23} = (a_{32} g_{33} - a_{22} g_{13} + a_{33} g_{12})^2 \]
\[ d_{42} = (a_{11} a_{22} a_{33} - a_{12} a_{23} a_{32} - a_{13} a_{22} a_{31})^2, \quad (7) \]

By multiplying \( Q_1(s_1, s_2, \mathbf{a}, \mathbf{g}) \) and \( Q_2(s_1, s_2, \mathbf{a}, \mathbf{g}) \), we obtain

\[ Q(s_1, s_2, \mathbf{a}, \mathbf{g}) = Q_1(s_1, s_2, \mathbf{a}, \mathbf{g}) \cdot Q_2(s_1, s_2, \mathbf{a}, \mathbf{g}). \quad (8) \]

Equation (8) constitutes a 2-variable VSHP since \( Q_1 \) and \( Q_2 \) are both 2-variable VSHPs.

Table 4.8: The transfer function using the 2-variable Very Strict Hurwitz Polynomial (VSHP) developed by Dr. Ahmadi et al. [5].
A 2-D lowpass filter with the following specification

\[ H_1(e^{j\omega_1^x}, e^{j\omega_2^y}) = \begin{cases} 
1.0 & \text{for } 0 \leq \sqrt{\omega_1^x + \omega_2^y} \leq 2 \text{ rad s}^{-1} \\
0 & \text{for } 3 \leq \sqrt{\omega_1^x + \omega_2^y} \leq 5 \text{ rad s}^{-1}.
\end{cases} \]

The order of the filter chosen was to be equal to three

<table>
<thead>
<tr>
<th>Parameters of the numerator</th>
<th>Parameters of the denominator</th>
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<tbody>
<tr>
<td>( \rho_{01} = -14.928813 )</td>
<td>( a_{111} = 1.2738203 )</td>
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<tr>
<td>( \rho_{01} = 2.628499 )</td>
<td>( a_{121} = 0.8187158 )</td>
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<tr>
<td>( \rho_{02} = -0.0010808 )</td>
<td>( a_{131} = 0.05614251 )</td>
</tr>
<tr>
<td>( \rho_{03} = 0.04982763 )</td>
<td>( a_{221} = -0.3984208 )</td>
</tr>
<tr>
<td>( \rho_{04} = 3.685929 )</td>
<td>( a_{221} = -0.3984208 )</td>
</tr>
<tr>
<td>( \rho_{05} = -2.505096 )</td>
<td>( a_{231} = -0.4019452 )</td>
</tr>
<tr>
<td>( \rho_{06} = 0.269577 )</td>
<td>( a_{311} = 0.407831 )</td>
</tr>
<tr>
<td>( \rho_{07} = -0.078664 )</td>
<td>( a_{321} = -1.1632566 )</td>
</tr>
<tr>
<td>( \rho_{08} = 0.0595262 )</td>
<td>( a_{331} = -0.4197668 )</td>
</tr>
<tr>
<td>( \rho_{09} = 0.292546 )</td>
<td>( g_{111} = 1.5251108 )</td>
</tr>
<tr>
<td>( \rho_{10} = 0.1756626 )</td>
<td>( g_{121} = 1.3397384 )</td>
</tr>
<tr>
<td>( \rho_{11} = 0.002778 )</td>
<td>( g_{221} = 1.23212681 )</td>
</tr>
<tr>
<td>( \rho_{12} = 0.0644073 )</td>
<td>( g_{231} = 1.23212681 )</td>
</tr>
<tr>
<td>( \rho_{13} = -0.0777573 )</td>
<td>( g_{311} = 1.23212681 )</td>
</tr>
<tr>
<td>( \rho_{14} = 0.00015738 )</td>
<td>( g_{321} = 1.23212681 )</td>
</tr>
</tbody>
</table>

Table 4.9: Third order lowpass filter specifications and the resulting coefficients of the designed filter as given by Dr. Ahmadi et al. [5].
Figure 4.109: 3-D plot of the amplitude response of the designed 2-D lowpass filter as given by Dr. Ahmadi et al. [5].
Figures 4.120, 4.121, and 4.122, the actual 3-D plot in Figure 4.123, and the contour plot in Figure 4.124.

This highpass filter is then varied to $\omega_c = 3.8 \text{ rad/s}$. The various cross sections for this filter are shown in Figures 4.125, 4.126, and 4.127, the actual 3-D plot in Figure 4.128, and the contour plot in Figure 4.129.

The design of a 2-D IIR bandpass filter is designed with $\omega_{c1} = 2.5 \text{ rad/s}$ and $\omega_{c2} = 3.7 \text{ rad/s}$. In this case a second filter is required. This filter is the same as the designed prototype filter of Table 4.9, but this will be a second and separate filter from the first prototype filter. The design of the bandpass filter takes on the exact same procedure as that of the type 1 or type 2 filter described above. The various cross sections for this filter are shown in Figures 4.130, 4.131, and 4.132, the actual 3-D plot in Figure 4.133, and the contour plot in Figure 4.134.

This bandpass filter is then varied to $\omega_{c1} = 3.0 \text{ rad/s}$ and $\omega_{c2} = 4.2 \text{ rad/s}$. The various cross sections for this filter are shown in Figures 4.135, 4.136, and 4.137, the actual 3-D plot in Figure 4.138, and the contour plot in Figure 4.139.

The design of a 2-D IIR bandstop filter is designed with $\omega_{c1} = 2.5 \text{ rad/s}$ and $\omega_{c2} = 4.1 \text{ rad/s}$. Here again a second filter is required. This second designed prototype filter is the same filter used for the bandpass filter above. The design of the bandstop filter takes on the exact same procedure as that of the type 1 or type 2 filter above. The various cross sections for this filter are shown in Figures 4.140, 4.141, and 4.142,
the actual 3-D plot in Figure 4.143, and the contour plot in Figure 4.144.

This bandstop filter is then varied to \( \omega_{c1} = 2.7 \) rad/s and \( \omega_{c2} = 3.9 \) rad/s. The various cross sections for this filter are shown in Figures 4.145, 4.146, and 4.147, the actual 3-D plot in Figure 4.148, and the contour plot in Figure 4.149.

4.2.10. **Hurwitz Polynomials obtained by the application of Positive definite matrices and resistive matrices and their application in 2-D filter design**

Expanding the 1-D Hurwitz Polynomial obtained in Section 2.2.2. to the 2-D case using the positive definite and resistor matrix, the following equation is obtained [17]

\[
M = A \Delta A^T s + B \Delta B^T s_i + G + RER^T
\]  (4.12)

where \( A, B, \) and \( R \) are upper-triangular matrices, \( \Gamma, \Delta \) and \( \Sigma \) are diagonal matrices, and \( G \) is a skew-symmetric matrix. The determinant of \( M, \det M, \) can be shown to be a Very Strict Hurwitz Polynomial (VSHP). The elements of \( \Gamma \) and \( \Delta \) can be zero in order to obtain a VSHP. The cascade of two or more lower order VSHP's can be used to obtain a higher order VSHP.

For a 2 by 2 positive definite and resistor matrix, \( A, B, R, \Gamma, \Delta, \Sigma, \) and \( G \) can be written as

\[
M = \begin{bmatrix}
1 & a \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & \gamma^2
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
\alpha & 1
\end{bmatrix}
s_i
+ \begin{bmatrix}
0 & 0 \\
-g & 0
\end{bmatrix}
+ \begin{bmatrix}
1 & b \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & \delta^2
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
b & 1
\end{bmatrix}
s_i
\]
\[
M = \begin{bmatrix}
\gamma_i^2 s_i + b^2 \sigma_i^2 s_i + \sigma_i^2 + r^2 \sigma_i^2 & a \gamma_i^2 s_i + b \sigma_i \sigma_i + r \sigma_i^2 + g \\
\gamma_i^2 s_i + b \sigma_i \sigma_i + r \sigma_i^2 - g & \gamma_i^2 s_i + \sigma_i \sigma_i + \sigma_i^2
\end{bmatrix}
\]

(4.14)

The \(\text{det} \ M\) gives the following

\[
\text{det} \ M = M(s_i, s_i) = A s_i s_i + B s_i + C s_i + D
\]

(4.15)

where

\[
A = (a-b)^2 \gamma_i^2 s_i^2
\]

(4.16a)

\[
B = \gamma_i^2 \left[ (a-r)^2 \sigma_i^2 + \sigma_i^2 \right]
\]

(4.16b)

\[
C = \sigma_i^2 \sigma_i^2 + \sigma_i^2
\]

(4.16c)

\[
D = \sigma_i^2 \sigma_i^2 + \sigma_i^2
\]

(4.16d)

It can easily be shown that the resulting two-variable polynomial is indeed a VSHP. Thus this second order polynomial provides a Very Strict Hurwitz Polynomial which forms a stable, realizable network. The bilinear transformation provides the discrete version, \(M(z_i, z_i)\). The parameters of the discrete positive definite and resistor matrix VSHP are used as variables of an optimization technique. The polynomial may be any dimensional square matrix, or any number of lower order polynomials may be cascaded together to obtain a higher order polynomial. Hence a fourth order VSHP can be generated either by starting with fourth order matrices or by cascading two second order polynomials.
Figure 4.110: Type 3 Lowpass 2-D IIR Filter with $\omega_c1 = 2.5 \text{ rad/s}$, $\omega_c2 = 2.5 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.111: Type 3 Lowpass 2-D IIR Filter with
\( \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 2.5 \text{ rad/s} \)
along \( \omega_2 \) axis.
Figure 4.112: Type 3 Lowpass 2-D IIR Filter with
$\omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 2.5 \text{ rad/s}$
along $\omega_1 = \omega_2$ axis.
Figure 4.113: Three Dimensional Plot of Type 3 Lowpass 2-D IIR Filter with $\omega_{c1} = 2.5$ rad/s, $\omega_{c2} = 2.5$ rad/s.
Figure 4.114: Contour Plot of Type 3 Lowpass 2-D IIR Filter with \( \omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 2.5 \text{ rad/s}. \)
Figure 4.115: Type 3 Lowpass 2-D IIR Filter transformed to
$\omega_c = 3.5$ rad/s, $\omega_c = 3.5$ rad/s
along $\omega_1$ axis.
Figure 4.116: Type 3 Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 3.5$ rad/s, $\omega_{c2} = 3.5$ rad/s along $\omega_2$ axis.
Figure 4.117: Type 3 Lowpass 2-D IIR Filter transformed to
\( \omega_{c1} = 3.5 \text{ rad/s}, \; \omega_{c2} = 3.5 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis.
Figure 4.118: Three Dimensional Plot of Type 3 Lowpass 2-D IIR Filter transformed to
\(\omega_{c1} = 3.5 \text{ rad/s}, \; \omega_{c2} = 3.5 \text{ rad/s}.\)
Figure 4.119: Contour Plot of Type 3 Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 3.5 \text{ rad/s}, \omega_{c2} = 3.5 \text{ rad/s}$. 
Figure 4.120: Type 3 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 2.5$ rad/s, $\omega_{c2} = 2.5$ rad/s along $\omega_1$ axis.
Figure 4.121: Type 3 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 2.5 \text{ rad/s}$, $\omega_{c2} = 2.5 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.122: Type 3 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 2.5$ rad/s, $\omega_{c2} = 2.5$ rad/s along $\omega_1 = \omega_2$ axis.
Figure 4.123: Three Dimensional Plot of Type 3 Highpass 2-D IIR Filter with $\omega_{c1} = 2.5 \text{ rad/s}$, $\omega_{c2} = 2.5 \text{ rad/s}$. 
Figure 4.124: Contour Plot of Type 3 Highpass 2-D IIR Filter with $\omega_{c1} = 2.5$ rad/s, $\omega_{c2} = 2.5$ rad/s.
Figure 4.125: Type 3 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 3.8 \text{ rad/s}$, $\omega_{c2} = 3.8 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.126: Type 3 Highpass 2-D IIR Filter transformed to $\omega_c^1 = 3.8 \text{ rad/s}$, $\omega_c^2 = 3.8 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.127: Type 3 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 3.8 \text{ rad/s}$, $\omega_{c2} = 3.8 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.128: Three Dimensional Plot of Type 3 Highpass 2-D IIR Filter transformed to
\( \omega_{c1} = 3.8 \text{ rad/s}, \omega_{c2} = 3.8 \text{ rad/s}. \)
Figure 4.129: Contour Plot of Type 3 Highpass 2-D IIR Filter
transformed to $\omega_{c1} = 3.8$ rad/s, $\omega_{c2} = 3.8$ rad/s.
Figure 4.130: Type 3 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 2.5 \text{ rad/s}$, $\omega_{c2} = 3.7 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.131: Type 3 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 2.5$ rad/s, $\omega_{c2} = 3.7$ rad/s along $\omega_2$ axis.
Figure 4.132: Type 3 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 2.5 \text{ rad/s}$, $\omega_{c2} = 3.7 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.133: Three Dimensional Plot of Type 3 Bandpass 2-D IIR Filter transformed to
\[ \omega_{c1} = 2.5 \text{ rad/s}, \ \omega_{c2} = 3.7 \text{ rad/s}. \]
Figure 4.134: Contour Plot of Type 3 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 2.5 \text{ rad/s}$, $\omega_{c2} = 3.7 \text{ rad/s}$. 
Figure 4.135: Type 3 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 3.0 \text{ rad/s}$, $\omega_{c2} = 4.2 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.136: Type 3 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 3.0 \text{ rad/s}, \omega_{c2} = 4.2 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.137: Type 3 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 3.0 \text{ rad/s}, \omega_{c2} = 4.2 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.138: Three Dimensional Plot of Type 3 Bandpass 2-D IIR Filter transformed to
\( \omega_{c1} = 3.0 \text{ rad/s}, \omega_{c2} = 4.2 \text{ rad/s}. \)
Figure 4.139: Contour Plot of Type 3 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 3.0 \text{ rad/s}$, $\omega_{c2} = 4.2 \text{ rad/s}$. 
Figure 4.140: Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 4.1 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.141: Type 3 Bandstop 2-D IIR Filter transformed to $\omega_c = 2.5 \text{ rad/s}$, $\omega_c = 4.1 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.142: Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.5$ rad/s, $\omega_{c2} = 4.1$ rad/s along $\omega_1 = \omega_2$ axis.
Figure 4.143: Three Dimensional Plot of Type 3 Bandstop 2-D IIR Filter transformed to
\[ \omega_c^1 = 2.5 \text{ rad/s}, \omega_c^2 = 4.1 \text{ rad/s}. \]
Figure 4.144: Contour Plot of Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.5 \text{ rad/s}, \omega_{c2} = 4.1 \text{ rad/s}$. 
Figure 4.145: Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.7 \text{ rad/s}$, $\omega_{c2} = 3.9 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.146: Type 3 Bandstop 2-D IIR Filter transformed to $\omega_c = 2.7$ rad/s, $\omega_c = 3.9$ rad/s along $\omega_2$ axis.
Figure 4.147: Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.7$ rad/s, $\omega_{c2} = 3.9$ rad/s along $\omega_1 = \omega_2$ axis.
Figure 4.148: Three Dimensional Plot of Type 3 Bandstop 2-D IIR Filter transformed to

\[ \omega_{c1} = 2.7 \text{ rad/s}, \; \omega_{c2} = 3.9 \text{ rad/s}. \]
Figure 4.149: Contour Plot of Type 3 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 2.7$ rad/s, $\omega_{c2} = 3.9$ rad/s.
4.2.11. IIR Filter: Type 4

This type of transfer function with a zero phase numerator for an IIR filter is given by

\[ H(z_1, z_2) = \frac{\sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij}(\cos \omega_1)^i(\cos \omega_2)^j}{\sum_{i=0}^{M} \sum_{j=0}^{N} b_{ij}z_1^{-i}z_2^{-j}} \]  
(4.17a)

\[ = \frac{N(z_1, z_2)}{D(z_1, z_2)} \]  
(4.17b)

where

\[ z_1 = e^{j\omega_1}, \quad z_2 = e^{j\omega_2} \]  
(4.17c)

and

\[ \cos \omega_1 = \frac{z_1 + z_1^{-1}}{2} \quad \cos \omega_2 = \frac{z_2 + z_2^{-1}}{2} \]  
(4.17d)

The numerator is substituted with Oppenheim's transformations, as stated in Chapter 3, given by the following equations

\[ \cos \omega_1 = A_0 + (1-A_0)\cos \omega_1 \]  
(3.9)

and

\[ \cos \omega_2 = B_0 + (1-B_0)\cos \omega_2 \]  
(3.10)

The denominator used here is the VSHP described in the previous section which is then discretized using the double bilinear transformation of Equation 4.5b, and thus \( z_1 \) and \( z_2 \) are substituted with the Constantinides transformations.

To create the highpass filter from this normalized 2-D IIR filter requires a simple subtracting from one which results in the desired response. The bandpass and bandstop filters are created by the addition of a lowpass and a highpass
4.2.12. Examples of Variable Cutoff 2-D IIR Filters: Type 4

Various examples using this type 4 transfer function are demonstrated. The transfer function of Equation 4.17 is optimized in a single one step procedure.

The prototype 2-D IIR filter has a cutoff frequency \( \omega_c = 1.1 \text{ rad/s} \). The specifications for this designed prototype filter are as follows:

\[
H(j\omega, T, j\omega_z, T) = 1 \text{ for } 0 \leq (\omega^2 + \omega_z^2)^{1/2} \leq 0.7 \text{ rad/s}
\]

\[
H(j\omega, T, j\omega_z, T) = 0 \text{ for } 1.5 \leq (\omega^2 + \omega_z^2)^{1/2} \leq \pi \text{ rad/s}
\]

Sampling frequency, \( \omega_s = 2\pi (6.28) \text{ rad/s} \)

The values of the coefficients for this first designed prototype filter are given in Table 4.10. Three major cross sections are taken along the \( \omega, \omega_z \), and \( \omega = \omega \), axes which are shown in Figures 4.150, 4.151, and 4.152, the actual 3-D plot in Figure 4.153, and the contour plot in Figure 4.154.

This lowpass filter is then varied to \( \omega_c = 1.5 \text{ rad/s} \). The various cross sections for this filter are shown in Figures 4.155, 4.156, and 4.157, the actual 3-D plot in Figure 4.158, and the contour plot in Figure 4.159.

The design of a 2-D highpass IIR filter has a cutoff frequency \( \omega_c = 1.1 \text{ rad/s} \). The various cross sections are taken along the \( \omega, \omega_z \), and \( \omega = \omega \), axes which are shown in Figures 4.160, 4.161, and 4.162, the actual 3-D plot in Figure
4.163, and the contour plot in Figure 4.164.

This highpass filter is then varied to $\omega_c = 1.8 \text{ rad/s}$. The various cross sections for this filter are shown in Figures 4.165, 4.166, and 4.167, the actual 3-D plot in Figure 4.168, and the contour plot in Figure 4.169.

The design of a 2-D IIR bandpass filter is designed with $\omega_{c1} = 1.1 \text{ rad/s}$ and $\omega_{c2} = 2.5 \text{ rad/s}$. Thus a second filter is required. This second prototype 2-D IIR filter has a cutoff frequency $\omega_c = 1.9 \text{ rad/s}$. The specifications for this second prototype filter are as follows:

$$H(j\omega, T, j\omega_1 T) = 1 \text{ for } 0 \leq (\omega_1^2 + \omega_1^2)^{1/2} \leq 1.5 \text{ rad/s}$$

$$H(j\omega, T, j\omega_1 T) = 0 \text{ for } 2.3 \leq (\omega_1^2 + \omega_1^2)^{1/2} \leq \pi \text{ rad/s}$$

Sampling frequency, $\omega_s = 2\pi (6.28) \text{ rad/s}$

The values of the coefficients for this second designed prototype filter are given in Table 4.11. To obtain this bandpass filter, this second designed prototype 2-D IIR lowpass filter is added with a 2-D IIR highpass filter. This highpass filter is the transformed filter of the first designed prototype 2-D IIR lowpass filter designed above. Thus the various cross sections for this bandpass filter are shown in Figures 4.170, 4.171, and 4.172, the actual 3-D plot in Figure 4.173, and the contour plot in Figure 4.174.

This bandpass filter is then varied to $\omega_{c1} = 1.5 \text{ rad/s}$ and $\omega_{c2} = 2.1 \text{ rad/s}$. The various cross sections for this filter are shown in Figures 4.175, 4.176, and 4.177, the actual 3-D plot in Figure 4.178, and the contour plot in Figure 4.179.
The design of a 2-D IIR bandstop filter is designed with $\omega_{c1} = 1.3 \text{ rad/s}$ and $\omega_{c2} = 2.3 \text{ rad/s}$. This filter is designed using the first prototype designed 2-D IIR lowpass filter added with a 2-D IIR highpass filter. This highpass filter is the transformed filter of the second designed prototype 2-D IIR lowpass filter designed previously. Thus the various cross sections for this bandpass filter are shown in Figures 4.180, 4.181, and 4.182, the actual 3-D plot in Figure 4.183, and the contour plot in Figure 4.184.

This bandstop filter is then varied to $\omega_{c1} = 1.2 \text{ rad/s}$ and $\omega_{c2} = 2.7 \text{ rad/s}$. The various cross sections for this filter are shown in Figures 4.185, 4.186, and 4.187, the actual 3-D plot in Figure 4.188, and the contour plot in Figure 4.189.
### Values of the numerator coefficients

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### Values of the denominator coefficients

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Table 4.10: The values of the coefficients for the first of two designed Type 4 Prototype Lowpass 2-D IIR Filters.

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Figure 4.150: Type 4 Lowpass 2-D IIR Filter with
\( \omega_{c1} = 1.1 \text{ rad/s}, \ \omega_{c2} = 1.1 \text{ rad/s} \)
along \( \omega_1 \) axis.
Figure 4.151: Type 4 Lowpass 2-D IIR Filter with
\( \omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s} \)
along \( \omega_2 \) axis.
Figure 4.152: Type 4 Lowpass 2-D IIR Filter with
$\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$
along $\omega_1 = \omega_2$ axis.
Figure 4.153: Three Dimensional Plot of Type 4 Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$.
Figure 4.154: Contour Plot of Type 4 Lowpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$. 
Figure 4.155: Type 4 Lowpass 2-D IIR Filter transformed to \( \omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 1.5 \text{ rad/s} \) along \( \omega_1 \) axis.
Figure 4.156: Type 4 Lowpass 2-D IIR Filter transformed to 
\[ \omega_{c_1} = 1.5 \text{ rad/s}, \ \omega_{c_2} = 1.5 \text{ rad/s} \]
along \( \omega_2 \) axis.
Figure 4.157: Type 4 Lowpass 2-D IIR Filter transformed to 
\( \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s} \)
along \( \omega_1 = \omega_2 \) axis.
Figure 4.158: Three Dimensional Plot of Type 4 Lowpass 2-D IIR Filter transformed to
\[ \omega_{c1} = 1.5 \text{ rad/s}, \ \omega_{c2} = 1.5 \text{ rad/s}. \]
Figure 4.159: Contour Plot of Type 4 Lowpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 1.5 \text{ rad/s}$. 
Figure 4.160: Type 4 Highpass 2-D IIR Filter transformed to $\omega_c = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.161: Type 4 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.162: Type 4 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}, \omega_{c2} = 1.1 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.163: Three Dimensional Plot of Type 4 Highpass 2-D IIR Filter with $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 1.1$ rad/s.
Figure 4.164: Contour Plot of Type 4 Highpass 2-D IIR Filter with $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 1.1 \text{ rad/s}$. 
Figure 4.165: Type 4 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8 \text{ rad/s}$, $\omega_{c2} = 1.8 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.166: Type 4 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s along $\omega_2$ axis.
Figure 4.167: Type 4 Highpass 2-D IIR Filter transformed to $\omega_{c1} = 1.8 \text{ rad/s}$, $\omega_{c2} = 1.8 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.168: Three Dimensional Plot of Type 4 Highpass 2-D IIR Filter transformed to
\[ \omega_{c1} = 1.8 \, \text{rad/s}, \quad \omega_{c2} = 1.8 \, \text{rad/s}. \]
Figure 4.169: Contour Plot of Type 4 Highpass 2-D IIR Filter
transformed to $\omega_{c1} = 1.8$ rad/s, $\omega_{c2} = 1.8$ rad/s.
### Values of the numerator coefficients

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### Values of the denominator coefficients

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</tr>
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</tr>
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</tr>
<tr>
<td>q₂</td>
<td>3.8999993324280E-01</td>
</tr>
</tbody>
</table>

**Table 4.11:** The values of the coefficients for the second of two designed Type 4 Prototype Lowpass 2-D IIR Filters.
Figure 4.170: Type 4 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1 \text{ rad/s}$, $\omega_{c2} = 2.5 \text{ rad/s}$ along $\omega_1$ axis.
Figure 4.171: Type 4 Bandpass 2-D IIR Filter transformed to $\omega_c 1 = 1.1 \text{ rad/s}, \omega_c 2 = 2.5 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.172: Type 4 Bandpass 2-D IIR Filter transformed to $\omega_c1 = 1.1 \text{ rad/s}$, $\omega_c2 = 2.5 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.173: Three Dimensional Plot of Type 4 Bandpass 2-D IIR Filter transformed to
\[ \omega_{c1} = 1.1 \text{ rad/s}, \quad \omega_{c2} = 2.5 \text{ rad/s}. \]
Figure 4.174: Contour Plot of Type 4 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.1$ rad/s, $\omega_{c2} = 2.5$ rad/s.
Figure 4.175: Type 4 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 2.1 \text{ rad/s}$ along $\omega_\perp$ axis.
Figure 4.176: Type 4 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 2.1 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.177: Type 4 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5$ rad/s, $\omega_{c2} = 2.1$ rad/s along $\omega_1 = \omega_2$ axis.
Figure 4.178: Three Dimensional Plot of Type 4 Bandpass 2-D IIR Filter transformed to
\( \omega_{c1} = 1.5 \text{ rad/s}, \omega_{c2} = 2.1 \text{ rad/s}. \)
Figure 4.179: Contour Plot of Type 4 Bandpass 2-D IIR Filter transformed to $\omega_{c1} = 1.5 \text{ rad/s}$, $\omega_{c2} = 2.1 \text{ rad/s}$. 

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Figure 4.180: Type 4 Bandstop 2-D IIR Filter transformed to $\omega_c1 = 1.3$ rad/s, $\omega_c2 = 2.3$ rad/s along $\omega_1$ axis.
Figure 4.181: Type 4 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.3 \text{ rad/s}, \omega_{c2} = 2.3 \text{ rad/s}$ along $\omega_2$ axis.
Figure 4.182: Type 4 Bandstop 2-D IIR Filter transformed to $\omega_c = 1.3 \text{ rad/s}, \omega_c = 2.3 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.183: Three Dimensional Plot of Type 4 Bandstop 2-D IIR Filter transformed to
\[ \omega_{c1} = 1.3 \text{ rad/s}, \ \omega_{c2} = 2.3 \text{ rad/s}. \]
Figure 4.184: Contour Plot of Type 4 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.3 \text{ rad/s}, \omega_{c2} = 2.3 \text{ rad/s}$. 

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Figure 4.185: Type 4 Bandstop 2-D IIR Filter transformed to $\omega_c = 1.2$ rad/s, $\omega_c = 2.7$ rad/s along $\omega_1$ axis.
Figure 4.186: Type 4 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2$ rad/s, $\omega_{c2} = 2.7$ rad/s along $\omega_2$ axis.
Figure 4.187: Type 4 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2 \text{ rad/s}$, $\omega_{c2} = 2.7 \text{ rad/s}$ along $\omega_1 = \omega_2$ axis.
Figure 4.188: Three Dimensional Plot of Type 4 Bandstop 2-D IIR Filter transformed to
\[ \omega_{c1} = 1.2 \text{ rad/s}, \ \omega_{c2} = 2.7 \text{ rad/s}. \]
Figure 4.189: Contour Plot of Type 4 Bandstop 2-D IIR Filter transformed to $\omega_{c1} = 1.2$ rad/s, $\omega_{c2} = 2.7$ rad/s.
V. IMPLEMENTATION OF FIR AND IIR 2-D FILTERS

This is a brief introduction of various implementation structures available to design the variable cutoff frequency filters. Some of the common designs for both the FIR and IIR filter structures are considered.

5.1 2-D FIR IMPLEMENTATION

The various network structures of linear phase variable cutoff filters are considered.

5.1.1. Network Structures

From Equation 2.3a and Equation 2.6, the transfer function of the prototype filter is given by [7,18]

\[ H(z) = z^{-N} \sum_{n=0}^{N} a(n) \left( \frac{z + z^{-1}}{2} \right)^{n} \]  \hspace{1cm} (5.1a)

\[ = \sum_{n=0}^{N} a(n) z^{-N+n} \left( \frac{1 + z^{-2}}{2} \right)^{n} \]  \hspace{1cm} (5.1b)

A direct implementation of \( H(z) \) as expressed in Equation 5.1b is shown in Figure 5.1. Each of the branches with transmittance \((1 + z^{-2})/2\) is implemented by a subnetwork. And for Equation 5.1a, the implementation is seen in Figure 5.2, where \( b(n) = a(n) \) for \( n = 0,1,2,\ldots,N \) and \( H_f = (z^{-1} + z)/2 \). The second form, the transpose direct transformed structure, is shown in Figure 5.3. The cascade transformed structure is another implementation. The transfer function, \( H(z) \), as given in Equation 5.1a, can be factored into the form
\[ H(z) = \pi \frac{N/2}{k=1} \left[ \sum_{n=0}^{2} b_k(n) \left[ \frac{z + z^{-1}}{2} \right]^n \right] \] (5.2a)

\[ = \pi \frac{N/2}{k=1} \left[ \sum_{n=0}^{2} b_k(n) z^{-2+n} \left[ \frac{1 + z^{-2}}{2} \right]^n \right] \] (5.2b)

By observing Equation 5.2b, this corresponds to a cascade of the fourth-order Taylor sections which can be seen in Figure 5.4.

For Equation 5.2a, the implementation is seen in Figure 5.5 where \( d_k(n) = b_k(n) \) for \( n=0,1,2 \), and \( k=1,2,...,N/2 \) and \( H_f = \frac{(z + z^{-1})}{2} \). A new structure derivation is the Chebyshev structure [6]. Combining Equations 2.3b and 2.5, the following is obtained

\[ H_0(z) = h_0(z) + \sum_{n=1}^{N} 2h_0(n)T_{n}\begin{bmatrix} p(z) \end{bmatrix} \] (5.3a)

where

\[ p(z) = H_f(z, z_s) = \frac{z + z^{-1}}{2} \] (5.3b)

and \( T_n(x) \) is a Chebyshev polynomial of \( n \)th order. This structure is shown in Figure 5.6. It has been shown that this structure is superior over the other structures [6].

5.1.2. **Variable Cutoff Structures**

Denoting the transfer function, \( H_0(z) \) of Equation 2.10, and \( H_0(z) \) from Equation 2.6, the transfer function of the zero phase prototype filter, from Equation 2.8, \( \hat{H}_0(z) \) and \( H_0(z) \) are related through the substitution

\[ \frac{z + z^{-1}}{2} = \sum_{k=0}^{p} A_k \left[ \frac{z + z^{-1}}{2} \right]^k \] (5.4)
Figure 5.1: Implementation of 1-D zero-phase filter in terms of zero-phase operators \((z + z^{-1})/2\).

Figure 5.2: Direct transformed structure.

Figure 5.3: Transpose direct transformed structure.

Figure 5.4: Cascade form implementation of linear phase FIR filter using fourth-order Taylor sections.
and by combining Equations 2.6, 2.10, and 5.3

\[ \hat{H}(z) = z^{-NP} \sum_{n=0}^{N} a(n) \left[ \sum_{k=0}^{P} A_k \left( \frac{z + z^{-1}}{2} \right)^k \right]^n \]  

(5.5)

which can be rewritten as

\[ \hat{H}(Z) = \sum_{n=0}^{N} a(n) z^{-P(N-n)} \left[ \sum_{k=0}^{P} A_k z^{k-P} \left( \frac{1 + z^{-2}}{2} \right)^k \right]^n \]  

(5.6)

or

\[ \hat{H}(Z) = \sum_{n=0}^{N} a(n) z^{-P(N-n)} \left[ \left( \frac{\hat{H}_p(Z)}{z^P} \right) \right]^n \]  

(5.7a)

where

\[ \frac{\hat{H}_p(Z)}{z^P} = \sum_{k=0}^{P} A_k z^{k-P} \left( \frac{1 + z^{-2}}{2} \right)^k \]  

(5.7b)

Figure 5.7 shows the implementation of Equation 5.7. Here it can be seen that it has the same general structure as Figure 5.8 where \((1 + z^{-2})/2\) is replaced by \(\hat{H}_p(Z)\) and \(z^{-1}\) by \(z^{-P}\). By varying \(A_0, A_1, ..., A_p\), the transformed filter can be obtained. The structure of Figure 5.7 is not in canonic form in terms of the number of delay elements required.

If the first-order transformation is used, \(P=1\), then Equation 5.7 becomes

\[ \hat{H}(Z) = \sum_{n=0}^{N} a(n) z^{-(N-n)} \left[ A_0 z^{-1} + A_1 \left( \frac{1 + z^{-2}}{2} \right) \right]^n \]  

(5.8)

and the configuration of Figure 5.7 reduces to that shown in Figure 5.9. For the case of the cascade implementation,
Figure 5.5: Cascade transformed structure.

Figure 5.6: Chebyshev structure.

Figure 5.7: Implementation of linear phase variable cutoff filter with Pth-order transformation.
Equation 5.4 can be substituted into Equation 5.2a to obtain the variable cutoff linear phase filter which is seen in Figure 5.10. Again by using Equation 5.4 and combining it for the Chebyshev structure, Equation 5.3, the variable cutoff filter can be implemented. The network structures shown are those which allow transformations as Equation 5.4 to be used, hence the convenience of modularity can be incorporated into the filter structure.

5.2 2-D IIR IMPLEMENTATION

5.2.1. Predistortion Realization

A method of realizing digital filters from their analog counterparts is the predistortion method. This gives no delay-free loops since delay-free loops are unrealizable [1]. This method shifts all the roots one unit to the left, therefore stability is preserved.

The general mapping \( S_i = G_i(z_i) \), \( i = 1,2 \) in the \( z_i \)-\( z_i \) plane is given by

\[
G_i(z_i) = \frac{1 - F_{i1}(z_i)}{1 + F_{i2}(z_i)} , \quad i = 1,2 \tag{5.9}
\]

where \( F_{i1}(\cdot) \) and \( F_{i2}(\cdot) \) contain no constant terms.

The predistorted variables \( (S_i - 1) \) are replaced by a network function implementing the analog-to-digital transformation. In general, this network function according to Equation 5.9 is

\[
(S_i - 1) \rightarrow \left[ G_i(z_i) - 1 \right] = - \frac{F_{i1}(z_i) + F_{i2}(z_i)}{1 + F_{i2}(z_i)} , \quad i = 1,2 \tag{5.10}
\]

In the case of the double bilinear transformation where
Figure 5.8: Implementation of linear phase FIR filter.

Figure 5.9: Implementation of linear phase variable cutoff filter with first-order transformation.

Figure 5.10: Canonic implementation of linear phase variable cutoff filter with first-order transformation.
\[ G_i(z_i) = \frac{1 - z_i^{-1}}{1 + z_i^{-1}} \; , \; i = 1, 2 \], the digital networks of Equation 5.10 reduce to the simple network

\[ z_i = -\frac{2z_i^{-1}}{1 + z_i^{-1}} \; ; \; i = 1, 2 \tag{5.11} \]

This network is shown in Figure 5.11. This network contains no delay-free forward paths, thus it can be placed within delay-free structures without creating delay-free loops.

One arbitrary all-pole analog filter can be characterized by its predistorted transfer function as

\[ H(s_i, s_z) = \frac{1}{\sum_{j=0}^{n} \sum_{l=0}^{j} b_{l,j-1}(s_z - 1)^{l}(s_i - 1)^{j-l}} \] \tag{5.12a}

\[ = \frac{1}{\sum_{j=0}^{n} \sum_{l=0}^{j} b_{l,j-1} s_z^{l} s_i^{j-l}} = \frac{1}{D(., .)} \tag{5.12b} \]

where \( D(., .) \) is the denominator polynomial in \( z_i = s_i - 1 \) and \( z_z = s_z - 1 \). A direct form realization is achieved by direct implementation of Equation 5.12 which is converted to a digital network by using Figure 5.11. This final realization can be seen in Figure 5.12 for \( n = 3 \), where \( b_{00} = 1 \). This structure is also a canonic realization.

The separable denominator, which is considered in the filter designs in this report, can be expressed as

\[ H(s_i, s_z) = \frac{1}{\sum_{i=0}^{n} a_{i1}s_i^{i} \sum_{j=0}^{m} a_{2j}s_z^{j}} \tag{5.13} \]

The predistorted transfer function for this filter would then be
Figure 5.11: The digital network implementing Equation (5.11).

Figure 5.12: A direct form realization of the third-order all-pole filter characterized by Equation (5.12).
the following

\[ H(Z_i, Z_j) = \frac{1}{\sum_{k=0}^{n} b_{1k}(s_i - 1)^k \sum_{l=0}^{m} b_{2l}(s_j - 1)^l} \quad (5.14a) \]

\[ = \frac{1}{D_i(Z_i) D_j(Z_j)} \quad (5.14b) \]

where \( D_i \) and \( D_j \) are denominator polynomials in \( Z_i = s_i - 1 \) and \( Z_j = s_j - 1 \), respectively. According to the predistortion technique in [1], the polynomials coefficients are related by

\[ b_{1k} = \sum_{i=0}^{n} C_k a_{1i} , \quad i \neq k \quad (5.15a) \]

and

\[ b_{2k} = \sum_{j=0}^{m} C_j a_{2j} , \quad j \neq l \quad (5.15b) \]

A ladder realization of Equation 5.14 is achieved by expressing it as

\[ H(. . .) = \frac{1}{\text{even } D_i + \text{odd } D_i} \cdot \frac{1}{\text{even } D_j + \text{odd } D_j} \quad (5.16) \]

and expanding \( t_{zz} = \text{even } D_i/\text{odd } D_i, \quad i = 1, 2 \) for odd \( n \), or \( t_{zz} = \text{odd } D_i/\text{even } D_i \) for even \( n \), in a continued fraction expansion around \( Z_i \) [19]. This expansion is given by

\[ t_{zz} = \frac{1}{\alpha_{11} Z_i + \frac{1}{\alpha_{12} Z_i + \frac{1}{\ddots \frac{1}{\alpha_{ln} Z_i}}} \quad (5.17) \]
If the denominator is a Hurwitz, polynomial \( D_i(Z_i) \), the continued fraction expansion, Equation 5.17 always exists. The realization of \( 1/D_i(Z_i) \), \( i = 1,2 \) from Equation 5.17 can be seen in which \( Z_i \) represents the network of Figure 5.13.

To obtain the tunability characteristics of the above structures, the modular network structures of the Pendergrass transformations are then incorporated into the realizations.
Figure 5.13: A ladder form realization of the third-order all-pole filter $1/D_i(z_i)$. 
VI. TUNABILITY CHARACTERISTICS AND MAXIMUM ERROR

6.1 TUNABILITY CHARACTERISTICS

The purpose of the tunability characteristics is to test the correspondence of the desired cutoff frequency selected to the actual cutoff frequency that is obtained. In order to see if the actual cutoff frequency, \( \omega_c \), is obtained, the test is performed along the \( \omega_i = \omega_i \) axis for various selected values of \( \omega_c \). This can be seen in Figure 6.1.

In Figure 6.2, the magnitude for \( \omega_c \) or \( \omega_p \) is determined in the prototype lowpass filter. This magnitude is then found on the transformed filter along the \( \omega_i = \omega_i \) axis and hence the corresponding \( \omega_c \) or \( \omega_p \) is then determined. These values are then plotted for the lowpass filters designed previously. The plots show the relationship between the actual \( \omega_c \) (or \( \omega_p \)) versus the desired \( \omega_c \) (or \( \omega_p \)). Here the comparison of the actual cutoff frequency, \( \omega_{ca} \), to the desired cutoff frequency, \( \omega_{cd} \), can be seen. These are shown for the various 2-D designs given in the previous chapters. The tunability characteristics for the FIR filter using Kaiser window from Figures 3.4 to 3.13 are given in Figure 6.3 and Figure 6.5. The tunability characteristics for the FIR filter using the optimization technique from Figures 3.4-4 to 3.53 is given in Figure 6.7. The tunability characteristics for the Type 1 IIR filter from Figures 4.29 to 4.38 is given in Figure 6.9, for the Type 2 IIR filter from Figure 4.69 to 4.78 is given in Figure 6.11, for the Type 3 IIR filter from Figures 4.110
to 4.119 is given in Figure 6.13, and for the Type 4 IIR filter from Figures 4.150 to 4.159 is given in Figure 6.15.

6.2 MAXIMUM ERROR

During this tuning process, the maximum error is searched for in the passband and stopband regions of the entire filter for various selected values of $\omega_c$. This is done for the various lowpass filters that are used for the tunability characteristic testing given above, after its corresponding tunability characteristic graph. The maximum error characteristics for the FIR filter using Kaiser window from Figures 3.4 to 3.13 are given in Figure 6.4 and Figure 6.6. The maximum error characteristics for the FIR filter using the optimization technique from Figures 3.44 to 3.53 is given in Figure 6.8. The maximum error characteristics for the Type 1 IIR filter from Figures 4.29 to 4.38 is given in Figure 6.10, for the Type 2 IIR filter from Figures 4.69 to 4.78 is given in Figure 6.12, for the Type 3 IIR filter from Figures 4.110 to 4.119 is given in Figure 6.14, and for the Type 4 IIR filter from Figures 4.150 to 4.159 is given in Figure 6.16.
Prototype Filter

Transformed Frequency Filter

$H(e^{j\omega})$

Test Axis

$H(e^{j\Omega})$

Test Axis

Figure 6.1: Testing of tunability characteristics along the positive $\omega_i = \omega$, axis for various values of $\omega_c$.

Prototype Filter

Higher Cutoff Frequency Filter

$H(e^{j\omega})$

$H(e^{j\Omega})$

$\omega_c < \Omega_c$

$0 \leq \omega_c \leq \omega_c$

$\Omega_c \leq \Omega_c$

Figure 6.2: The comparison of the actual cutoff or passband frequency, $\Omega_c$, to the desired cutoff or passband frequency, $\Omega_d$. 431
Figure 6.3: Lowpass 2-D FIR Filter (Kaiser window) Tunability Characteristics from $\omega_c = 1.0$ to 2.0 rad/s.
Figure 6.4: Lowpass 2-D FIR Filter (Kaiser window) Maximum Error Characteristics from $\omega_c = 1.0$ to 2.0 rad/s.
Figure 6.5: Lowpass 2-D FIR Filter (Kaiser window) Tunability Characteristics from $\omega_c = 1.0$ to 2.2 rad/s.
Figure 6.6: Lowpass 2-D FIR Filter (Kaiser window) Maximum Error Characteristics from $\omega_C = 1.0$ to $2.2$ rad/s.
Figure 6.7: Lowpass 2-D FIR Filter (Optimized) Tunability Characteristics from $\omega_p = 0.7$ to 1.8 rad/s.
Figure 6.8: Lowpass 2-D FIR Filter (Optimized) Maximum Error Characteristics from $\omega_p = 0.7$ to $1.8$ rad/s.
Figure 6.9: Type 1 Lowpass 2-D IIR Filter Tunability Characteristics.
Figure 6.10: Type 1 Lowpass 2-D IIR Filter Maximum Error Characteristics.
Figure 6.11: Type 2 Lowpass 2-D IIR Filter Tunability Characteristics.
Figure 6.12: Type 2 Lowpass 2-D IIR Filter Maximum Error Characteristics.
Figure 6.13: Type 3 Lowpass 2-D IIR Filter Tunability Characteristics.
Figure 6.14: Type 3 Lowpass 2-D IIR Filter Maximum Error Characteristics.
Figure 6.15: Type 4 Lowpass 2-D IIR Filter Tunability Characteristics.
Figure 6.16: Type 4 Lowpass 2-D IIR Filter Maximum Error Characteristics.
VII. CONCLUSIONS AND SUMMARY

In this thesis several approaches for the design of 1-D FIR and IIR digital filters have been examined. The various types of transfer functions for these filters have been considered. The feasibility of design of various 1-D cutoff frequency FIR and IIR filters have been studied. The use of different spectral transformations such as the Oppenheim and the Constantinides transformations have been utilized in the design of these filters.

Several methods of designing 2-D FIR and IIR filters have been discussed. The 2-D FIR filter is designed using the McClellan transformation which transforms a 1-D zero-phase FIR filter to a 2-D FIR filter. Another powerful design technique for the 2-D FIR filter involves the use of optimization. The following transfer functions for 2-D IIR filters have been examined:

(1) separable products
(2) separable denominator and nonseparable numerator filter
   i) class 1: zero-phase numerator
   ii) class 2: non zero-phase numerator
(3) a general class of 2-D filters
   i) derived by Hurwitz polynomial with non zero-phase numerator
   ii) derived by Hurwitz polynomial with zero-phase numerator

and (4) transformation technique.
   i) derived by Reactance function Transformation
Various variable cutoff boundary 2-D FIR and IIR filters have been examined using the above transfer functions along with the spectral transformations. These include the Oppenheim, the Constantinides, and the Pendergrass transformations, and the two-variable Reactance function.

In this thesis a new transformation technique has been proposed for the design of a class of 2-D variable cutoff boundary filters. This technique has been presented along with various examples to show the usefulness of the proposed transformation. In the variable 2-D FIR filter in conjunction with the McClellan transformation followed by the Oppenheim spectral transformation using the Kaiser window, the circular contours become increasingly square as the frequency increases. This is unlike the second design where the optimization technique is used which results in circular contours under the Oppenheim spectral transformation. It should be noted that the first design using the Kaiser window is a suboptimal design since the McClellan transformation is itself a suboptimal transformation.

The design of variable cutoff frequency 2-D IIR filters using the separable product transfer function, separable numerator separable denominator and finally the general case of non separable numerator and denominator transfer function is treated in detail in this thesis. The first class of such filters i.e. separable product are the easiest to handle. The spectral transformation of Constantinides can easily be applied to the transfer function in each variable therefore obtaining the desired magnitude squared frequency specification. This is
simply an extension of the 1-D IIR filters where two 1-D IIR filters are cascaded together. In the second class of 2-D IIR filters which have separable denominator and non-separable numerators, two different types of transfer functions are identified. These include the zero-phase and non-zero-phase numerator. The zero-phase numerator makes use of the Oppenheim spectral transformations while the denominator uses the Pendergrass or Constantinides transformations. In the non-zero-phase numerator, the numerator along with the denominator both utilize the Pendergrass or Constantinides transformations. The third class of 2-D IIR filters include the general case of non-separable numerator and denominator transfer function. In this case the Pendergrass or Constantinides transformations are applied directly. The other type of 2-D IIR filter in this third class is one with a zero-phase numerator makes use of the Oppenheim spectral transformations while the denominator uses the Pendergrass or Constantinides transformations as is done in the second class of 2-D IIR filters. Lastly there is the transformation technique which used the two-variable Reactance function. A 1-D IIR filter is transformed directly to a variable 2-D IIR filter via the Reactance function.

The optimized 2-D FIR and the first class of 2-D IIR filters have improved circular contours over the 2-D FIR filter using Kaiser window and the other classed of 2-D IIR filters. This can be seen in the figures of the contour responses: circularity is kept at all frequencies. This applies for all the filters designed: lowpass, highpass, bandpass, and bandstop. Since these
filters yields better circular contour response at all frequencies, then the various other filter designs (highpass, bandpass, and bandstop) can be designed properly resulting in a non-distorted magnitude response. The circular contour response is very important since the bandpass and bandstop filters are made up of two filters: both a lowpass and a highpass are required. Thus it is important to have circular response in the lowpass and the highpass filters.

The transformation technique using the two-variable Reactance function has little resemblance of circular contour response. In fact, this cannot be obtained since this would make the filter unstable.

Various examples using the optimized 2-D FIR filter have been used to illustrate the design of elliptical filters derived from the circular prototype filter using the variable cutoff boundary filter transformations.

A discussion on the realization problem for such filters has been examined. Various structures have been presented for both the FIR and IIR filters.

The tunability characteristics of these filters are linear for a wide range of values. The maximum error can also be seen to be of the same proportion for all the cutoff frequency values as the prototype filter cutoff frequency.

The advantages of the proposed design are:
- optimization is not required after the initial prototype filter is designed
- the design is simple to use
- attenuation and gain in the passband and stopband is likely to remain constant
- hardware realization for the transformed filter are obtainable by keeping the original structure of the prototype filter and simply adding the structure for the spectral transformation
- mathematical calculations are satisfied, and the number of calculations are kept to a minimum

The disadvantages of the proposed design are:
- the transformation is non-linear and the phase will be distorted for the IIR filters
- one cannot obtain bandpass or bandstop from the lowpass filter easily
- multiband filters are not possible through this transformation
- the transition region will generally be distorted

In conclusion, the proposed design has improved circular contours whereby the loss of circularity can be seen in most other designs. With this design, circular to elliptical contours are also possible.
APPENDIX

The Parameters for the Various Designed Filters
THE PARAMETERS FOR THE VARIOUS DESIGNED FILTERS

The parameters for the various filters designed in this thesis report are listed here in the order they appear in the thesis. The lists indicate the prototype and the transformed filters' designed passband and/or cutoff frequency (or frequencies) and the corresponding parameter(s) that transform the prototype filter to the transformed filters as given in the various examples of the report.
1-D FIR FILTER USING KAISER WINDOW

PROTOTYPE FILTER WITH $\omega_c = 1.10000000E+00$ rad/s

LOWPASS FILTER

$\omega_c = 1.10000000E+00$ rad/s
$A_o = 0.00000000E+00$

LOWPASS FILTER

$\omega_c = 1.50000000E+00$ rad/s
$A_o = 4.12002848E-01$

LOWPASS FILTER

$\omega_c = 7.00000000E-01$ rad/s
$A_o = -1.76359149E-01$
1-D IIR FILTER

PROTOTYPE FILTER WITH $\omega_p = 7.00000000E-01$ rad/s

LOWPASS FILTER

$\omega_p = 7.00000000E-01$ rad/s
$\alpha = 0.00000000E+00$

LOWPASS FILTER

$\omega_p = 1.10000000E+00$ rad/s
$\alpha = -2.53622502E-01$

LOWPASS FILTER

$\omega_p = 3.00000000E-01$ rad/s
$\alpha = 4.14390379E-01$

HIGHPASS FILTER

$\omega_p = 1.50000000E+00$ rad/s
$\alpha = 2.03057511E+00$

HIGHPASS FILTER

$\omega_p = 1.90000000E+00$ rad/s
$\alpha = 3.08538030E+00$

HIGHPASS FILTER

$\omega_p = 1.10000000E+00$ rad/s
$\alpha = 1.57665840E+00$
BANDPASS FILTER
\[ \omega_{p1} = 1.10000000E+00 \text{ rad/s} \quad \omega_{p2} = 2.10000000E+00 \text{ rad/s} \]
\[ \alpha = -3.32726783E-02 \quad k = 6.68180178E-01 \]

BANDPASS FILTER
\[ \omega_{p1} = 1.50000000E+00 \text{ rad/s} \quad \omega_{p2} = 2.50000000E+00 \text{ rad/s} \]
\[ \alpha = -4.74196793E-01 \quad k = 6.68180178E-01 \]

BANDPASS FILTER
\[ \omega_{p1} = 7.00000000E-01 \text{ rad/s} \quad \omega_{p2} = 1.70000000E+00 \text{ rad/s} \]
\[ \alpha = 4.12904471E-01 \quad k = 6.68180178E-01 \]

BANDSTOP FILTER
\[ \omega_{p1} = 1.10000000E+00 \text{ rad/s} \quad \omega_{p2} = 2.10000000E+00 \text{ rad/s} \]
\[ \alpha = -3.32726783E-02 \quad k = 1.99415976E-01 \]

BANDSTOP FILTER
\[ \omega_{p1} = 1.50000000E+00 \text{ rad/s} \quad \omega_{p2} = 2.50000000E+00 \text{ rad/s} \]
\[ \alpha = -4.74196793E-01 \quad k = 1.99415976E-01 \]

BANDSTOP FILTER
\[ \omega_{p1} = 7.00000000E-01 \text{ rad/s} \quad \omega_{p2} = 1.70000000E+00 \text{ rad/s} \]
\[ \alpha = 4.12904471E-01 \quad k = 1.99415976E-01 \]
2-D FIR FILTER USING KAISER WINDOW

PROTOTYPE FILTER WITH $\omega_c = 1.10000000E+00$ rad/s

LOWPASS FILTER

$\omega_c1 = 1.10000000E+00$ rad/s  $\omega_c2 = 1.10000000E+00$ rad/s
$A_0 = 0.00000000E+00$  $B_0 = 0.00000000E+00$

LOWPASS FILTER

$\omega_c1 = 1.50000000E+00$ rad/s  $\omega_c2 = 1.50000000E+00$ rad/s
$A_0 = 4.12002848E-01$  $B_0 = 4.12002848E-01$

HIGHPASS FILTER

$\omega_c1 = 1.90000000E+00$ rad/s  $\omega_c2 = 1.90000000E+00$ rad/s
$A_0 = 0.00000000E+00$  $B_0 = 0.00000000E+00$

HIGHPASS FILTER

$\omega_c1 = 2.30000000E+00$ rad/s  $\omega_c2 = 2.30000000E+00$ rad/s
$A_0 = 2.05840119E-01$  $B_0 = 2.05840119E-01$
BANDPASS FILTER

\[ \omega_{c11} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.10000000E+00 \text{ rad/s} \]
\[ A_{01} = 3.55577691E-16 \quad B_{01} = 3.55577691E-16 \]
\[ \omega_{c21} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.10000000E+00 \text{ rad/s} \]
\[ A_{02} = 1.20647910E-01 \quad B_{02} = 1.20647910E-01 \]

BANDPASS FILTER

\[ \omega_{c11} = 1.50000000E+00 \text{ rad/s} \quad \omega_{c12} = 1.90000000E+00 \text{ rad/s} \]
\[ A_{01} = 4.12002848E-01 \quad B_{01} = 4.12002848E-01 \]
\[ \omega_{c21} = 1.50000000E+00 \text{ rad/s} \quad \omega_{c22} = 1.90000000E+00 \text{ rad/s} \]
\[ A_{02} = 0.00000000E+00 \quad B_{02} = 0.00000000E+00 \]

BANDSTOP FILTER

\[ \omega_{c11} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.40000000E+00 \text{ rad/s} \]
\[ A_{01} = 3.55577691E-16 \quad B_{01} = 3.55577691E-16 \]
\[ \omega_{c21} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.40000000E+00 \text{ rad/s} \]
\[ A_{02} = 2.38347903E-01 \quad B_{02} = 2.38347903E-01 \]

BANDSTOP FILTER

\[ \omega_{c11} = 1.20000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.90000000E+00 \text{ rad/s} \]
\[ A_{01} = 1.43087080E-01 \quad B_{01} = 1.43087080E-01 \]
\[ \omega_{c21} = 1.20000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.90000000E+00 \text{ rad/s} \]
\[ A_{02} = 3.28605959E-01 \quad B_{02} = 3.28605959E-01 \]
OPTIMIZED 2-D FIR FILTER: CIRCULAR

PROTOTYPE FILTER WITH $\omega_c = 1.10000000E+00$ rad/s

LOWPASS FILTER

$\omega_{c1} = 1.10000000E+00$ rad/s  $\omega_{c2} = 1.10000000E+00$ rad/s
$A_0 = 0.00000000E+00$  $B_0 = 0.00000000E+00$  

LOWPASS FILTER

$\omega_{c1} = 1.50000000E+00$ rad/s  $\omega_{c2} = 1.50000000E+00$ rad/s
$A_0 = 4.12002848E-01$  $B_0 = 4.12002848E-01$

HIGHPASS FILTER

$\omega_{c1} = 1.10000000E+00$ rad/s  $\omega_{c2} = 1.10000000E+00$ rad/s
$A_0 = 0.00000000E+00$  $B_0 = 0.00000000E+00$

HIGHPASS FILTER

$\omega_{c1} = 1.80000000E+00$ rad/s  $\omega_{c2} = 1.80000000E+00$ rad/s
$A_0 = 5.54756400E-01$  $B_0 = 5.54756400E-01$
BANDPASS FILTER

\[ \omega_{c11} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.50000000E+00 \text{ rad/s} \]

\[ A_{01} = 3.55577691E-16 \quad B_{01} = 3.55577691E-16 \]

\[ \omega_{c21} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.50000000E+00 \text{ rad/s} \]

\[ A_{02} = 2.65305912E-01 \quad B_{02} = 2.65305912E-01 \]

BANDPASS FILTER

\[ \omega_{c11} = 1.50000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.10000000E+00 \text{ rad/s} \]

\[ A_{01} = 4.12002848E-01 \quad B_{01} = 4.12002848E-01 \]

\[ \omega_{c21} = 1.50000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.10000000E+00 \text{ rad/s} \]

\[ A_{02} = 1.20647910E-01 \quad B_{02} = 1.20647910E-01 \]

BANDSTOP FILTER

\[ \omega_{c11} = 1.30000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.30000000E+00 \text{ rad/s} \]

\[ A_{01} = 2.54057331E-01 \quad B_{01} = 2.54057331E-01 \]

\[ \omega_{c21} = 1.30000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.30000000E+00 \text{ rad/s} \]

\[ A_{02} = 2.05840119E-01 \quad B_{02} = 2.05840119E-01 \]

BANDSTOP FILTER

\[ \omega_{c11} = 1.20000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.70000000E+00 \text{ rad/s} \]

\[ A_{01} = 1.43087080E-01 \quad B_{01} = 1.43087080E-01 \]

\[ \omega_{c21} = 1.20000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.70000000E+00 \text{ rad/s} \]

\[ A_{02} = 3.05021308E-01 \quad B_{02} = 3.05021308E-01 \]
OPTIMIZED 2-D FIR FILTER: ELLIPTICAL

PROTOTYPE FILTER WITH $\omega_c = 1.10000000E+00$ rad/s

LOWPASS FILTER

$\omega_{c1} = 1.10000000E+00$ rad/s  $\omega_{c2} = 1.70000000E+00$ rad/s  
$A_o = 3.55577691E-16$  $B_o = 5.15961781E-01$

LOWPASS FILTER

$\omega_{c1} = 2.50000000E+00$ rad/s  $\omega_{c2} = 1.50000000E+00$ rad/s  
$A_o = 6.96635030E-01$  $B_o = 4.12002848E-01$

HIGHPASS FILTER

$\omega_{c1} = 1.10000000E+00$ rad/s  $\omega_{c2} = 1.40000000E+00$ rad/s  
$A_o = 3.55577691E-16$  $B_o = 3.41708134E-01$

HIGHPASS FILTER

$\omega_{c1} = 2.00000000E+00$ rad/s  $\omega_{c2} = 1.30000000E+00$ rad/s  
$A_o = 6.14161565E-01$  $B_o = 2.54057331E-01$
BANDPASS FILTER

\[ \omega_{c11} = 1.20000000 \times 10^0 \text{ rad/s} \quad \omega_{c12} = 1.90000000 \times 10^0 \text{ rad/s} \]

\[ A_{01} = 1.43087080 \times 10^{-01} \quad B_{01} = 4.69098200 \times 10^{-01} \]

\[ \omega_{c21} = 1.60000000 \times 10^0 \text{ rad/s} \quad \omega_{c22} = 2.30000000 \times 10^0 \text{ rad/s} \]

\[ A_{02} = 0.00000000 \times 10^0 \quad B_{02} = 2.05840119 \times 10^{-01} \]

BANDPASS FILTER

\[ \omega_{c11} = 1.40000000 \times 10^0 \text{ rad/s} \quad \omega_{c12} = 2.00000000 \times 10^0 \text{ rad/s} \]

\[ A_{01} = 3.41708134 \times 10^{-01} \quad B_{01} = 3.55576916 \times 10^{-16} \]

\[ \omega_{c21} = 1.10000000 \times 10^0 \text{ rad/s} \quad \omega_{c22} = 2.40000000 \times 10^0 \text{ rad/s} \]

\[ A_{02} = 6.55703683 \times 10^{-02} \quad B_{02} = 2.38347903 \times 10^{-01} \]

BANDSTOP FILTER

\[ \omega_{c11} = 1.10000000 \times 10^0 \text{ rad/s} \quad \omega_{c12} = 2.00000000 \times 10^0 \text{ rad/s} \]

\[ A_{01} = 3.55576916 \times 10^{-16} \quad B_{01} = 2.54057331 \times 10^{-01} \]

\[ \omega_{c21} = 1.30000000 \times 10^0 \text{ rad/s} \quad \omega_{c22} = 2.00000000 \times 10^0 \text{ rad/s} \]

\[ A_{02} = 6.55703683 \times 10^{-02} \quad B_{02} = 6.55703683 \times 10^{-02} \]

BANDSTOP FILTER

\[ \omega_{c11} = 1.30000000 \times 10^0 \text{ rad/s} \quad \omega_{c12} = 2.20000000 \times 10^0 \text{ rad/s} \]

\[ A_{01} = 2.54057331 \times 10^{-01} \quad B_{01} = 5.15961781 \times 10^{-01} \]

\[ \omega_{c21} = 1.70000000 \times 10^0 \text{ rad/s} \quad \omega_{c22} = 2.60000000 \times 10^0 \text{ rad/s} \]

\[ A_{02} = 1.66957107 \times 10^{-01} \quad B_{02} = 2.87361957 \times 10^{-01} \]

BANDSTOP FILTER

\[ \omega_{c11} = 1.30000000 \times 10^0 \text{ rad/s} \quad \omega_{c12} = 2.50000000 \times 10^0 \text{ rad/s} \]

\[ A_{01} = 2.54057331 \times 10^{-01} \quad B_{01} = 5.15961781 \times 10^{-01} \]

\[ \omega_{c21} = 1.70000000 \times 10^0 \text{ rad/s} \quad \omega_{c22} = 2.90000000 \times 10^0 \text{ rad/s} \]

\[ A_{02} = 2.65305912 \times 10^{-01} \quad B_{02} = 3.28605959 \times 10^{-01} \]
2-D IIR FILTER: SPECIAL CASE (SQUARE TYPE)

PROTOTYPE FILTER WITH $\omega_p = 7.00000000E-01$ rad/s

LOWPASS FILTER

$\omega_{p1} = 7.00000000E-01$ rad/s $\omega_{p2} = 7.00000000E-01$ rad/s

$\alpha_1 = 0.00000000E+00$ $\alpha_2 = 0.00000000E+00$

LOWPASS FILTER

$\omega_{p1} = 1.10000000E+00$ rad/s $\omega_{p2} = 1.10000000E+00$ rad/s

$\alpha_1 = -2.53622502E-01$ $\alpha_2 = -2.53622502E-01$

LOWPASS FILTER

$\omega_{p1} = 3.00000000E-01$ rad/s $\omega_{p2} = 3.00000000E-01$ rad/s

$\alpha_1 = 4.14390379E-01$ $\alpha_2 = 4.14390379E-01$
TYPE 1: 2-D IIR FILTER

PROTOTYPE FILTER WITH $\omega_c = 1.10000000E+00$ rad/s

LOWPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

$\omega_{c1} = 1.10000000E+00$ rad/s  $\omega_{c2} = 1.10000000E+00$ rad/s
$A_o = 0.00000000E+00$  $B_o = 0.00000000E+00$

VALUES FOR THE DENOMINATOR SECTION:

$\omega_{p1} = 7.00000000E-01$ rad/s  $\omega_{p2} = 7.00000000E-01$ rad/s
$\alpha_1 = 0.00000000E+00$  $\alpha_2 = 0.00000000E+00$

LOWPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

$\omega_{c1} = 1.50000000E+00$ rad/s  $\omega_{c2} = 1.50000000E+00$ rad/s
$A_o = 4.12002848E-01$  $B_o = 4.12002848E-01$

VALUES FOR THE DENOMINATOR SECTION:

$\omega_{p1} = 1.10000000E+00$ rad/s  $\omega_{p2} = 1.10000000E+00$ rad/s
$\alpha_1 = -2.53622502E-01$  $\alpha_2 = -2.53622502E-01$

HIGHPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

$\omega_{c1} = 1.10000000E+00$ rad/s  $\omega_{c2} = 1.10000000E+00$ rad/s
$A_o = 0.00000000E+00$  $B_o = 0.00000000E+00$

VALUES FOR THE DENOMINATOR SECTION:

$\omega_{p1} = 7.00000000E-01$ rad/s  $\omega_{p2} = 7.00000000E-01$ rad/s
$\alpha_1 = 0.00000000E+00$  $\alpha_2 = 0.00000000E+00$
HIGHPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c1} = 1.80000000E+00 \text{ rad/s} \quad \omega_{c2} = 1.80000000E+00 \text{ rad/s} \]
\[ A_o = 5.54756400E-01 \quad B_o = 5.54756400E-01 \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p1} = 1.40000000E+00 \text{ rad/s} \quad \omega_{p2} = 1.40000000E+00 \text{ rad/s} \]
\[ \alpha_1 = -3.95306233E-01 \quad \alpha_2 = -3.95306233E-01 \]

BANDPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c11} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.50000000E+00 \text{ rad/s} \]
\[ A_{01} = 3.55577691E-16 \quad B_{01} = 3.55577691E-16 \]
\[ \omega_{c21} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.50000000E+00 \text{ rad/s} \]
\[ A_{02} = 2.65305912E-01 \quad B_{02} = 2.65305912E-01 \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p11} = 7.00000000E-01 \text{ rad/s} \quad \omega_{p12} = 2.10000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = -1.72336626E-16 \quad \alpha_{12} = -3.03456308E-01 \]
\[ \omega_{p21} = 7.00000000E-01 \text{ rad/s} \quad \omega_{p22} = 2.10000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = -1.72336626E-16 \quad \alpha_{22} = -3.03456308E-01 \]
BANDPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c11} = 1.50000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.10000000E+00 \text{ rad/s} \]
\[ A_{01} = 4.12002848E-01 \quad B_{01} = 4.12002848E-01 \]
\[ \omega_{c21} = 1.50000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.10000000E+00 \text{ rad/s} \]
\[ A_{02} = 1.20647910E-01 \quad B_{02} = 1.20647910E-01 \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p11} = 1.10000000E+00 \text{ rad/s} \quad \omega_{p12} = 1.70000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = -2.53622502E-01 \quad \alpha_{12} = -9.98760035E-02 \]
\[ \omega_{p21} = 1.10000000E+00 \text{ rad/s} \quad \omega_{p22} = 1.70000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = -2.53622502E-01 \quad \alpha_{22} = -9.98760035E-02 \]

BANDSTOP FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c11} = 1.30000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.30000000E+00 \text{ rad/s} \]
\[ A_{01} = 2.54057331E-01 \quad B_{01} = 2.54057331E-01 \]
\[ \omega_{c21} = 1.30000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.30000000E+00 \text{ rad/s} \]
\[ A_{02} = 2.05840119E-01 \quad B_{02} = 2.05840119E-01 \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p11} = 9.00000000E-01 \text{ rad/s} \quad \omega_{p12} = 1.90000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = -1.39168563E-01 \quad \alpha_{12} = -2.00339196E-01 \]
\[ \omega_{p21} = 9.00000000E-01 \text{ rad/s} \quad \omega_{p22} = 1.90000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = -1.39168563E-01 \quad \alpha_{22} = -2.00339196E-01 \]
BANDSTOP FILTER

VALUES FOR THE NUMERATOR SECTION:

$\omega_{c11} = 1.20000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.70000000E+00 \text{ rad/s}$

$A_{01} = 1.43087080E-01 \quad B_{01} = 1.43087080E-01$

$\omega_{c21} = 1.20000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.70000000E+00 \text{ rad/s}$

$A_{02} = 3.05021308E-01 \quad B_{02} = 3.05021308E-01$

VALUES FOR THE DENOMINATOR SECTION:

$\omega_{p11} = 8.00000000E-01 \text{ rad/s} \quad \omega_{p12} = 2.30000000E+00 \text{ rad/s}$

$\alpha_{11} = -7.33220764E-02 \quad \alpha_{12} = -4.11516756E-01$

$\omega_{p21} = 8.00000000E-01 \text{ rad/s} \quad \omega_{p22} = 2.30000000E+00 \text{ rad/s}$

$\alpha_{21} = -7.33220764E-02 \quad \alpha_{22} = -4.11516756E-01$
TYPE 2: 2-D IIR FILTER

PROTOTYPE FILTER WITH $\omega_c = 1.10000000E+00$ rad/s

LOWPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

$\omega_{c1} = 1.10000000E+00$ rad/s  $\omega_{c2} = 1.10000000E+00$ rad/s
$\alpha_1 = 0.00000000E+00$  $\alpha_2 = 0.00000000E+00$

VALUES FOR THE DENOMINATOR SECTION:

$\omega_{p1} = 7.00000000E-01$ rad/s  $\omega_{p2} = 7.00000000E-01$ rad/s
$\alpha_1 = 0.00000000E+00$  $\alpha_2 = 0.00000000E+00$

LOWPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

$\omega_{c1} = 1.50000000E+00$ rad/s  $\omega_{c2} = 1.50000000E+00$ rad/s
$\alpha_1 = -2.53622502E-01$  $\alpha_2 = -2.53622502E-01$

VALUES FOR THE DENOMINATOR SECTION:

$\omega_{p1} = 1.10000000E+00$ rad/s  $\omega_{p2} = 1.10000000E+00$ rad/s
$\alpha_1 = -2.53622502E-01$  $\alpha_2 = -2.53622502E-01$

HIGHPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

$\omega_{c1} = 1.10000000E+00$ rad/s  $\omega_{c2} = 1.10000000E+00$ rad/s
$\alpha_1 = 0.00000000E+00$  $\alpha_2 = 0.00000000E+00$

VALUES FOR THE DENOMINATOR SECTION:

$\omega_{p1} = 7.00000000E-01$ rad/s  $\omega_{p2} = 7.00000000E-01$ rad/s
$\alpha_1 = 0.00000000E+00$  $\alpha_2 = 0.00000000E+00$
HIGHPASS FILTER

VALUES FOR THE NUMERATOR SECTION:
\[ \omega_{c1} = 1.80000000E+00 \text{ rad/s} \quad \omega_{c2} = 1.80000000E+00 \text{ rad/s} \]
\[ \alpha_1 = -3.95306233E-01 \quad \alpha_2 = -3.95306233E-01 \]

VALUES FOR THE DENOMINATOR SECTION:
\[ \omega_{p1} = 1.40000000E+00 \text{ rad/s} \quad \omega_{p2} = 1.40000000E+00 \text{ rad/s} \]
\[ \alpha_1 = -3.95306233E-01 \quad \alpha_2 = -3.95306233E-01 \]

BANDPASS FILTER

VALUES FOR THE NUMERATOR SECTION:
\[ \omega_{c11} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.50000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = -1.72336626E-16 \quad \alpha_{12} = -6.53729630E-01 \]
\[ \omega_{c21} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.50000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = -1.72336626E-16 \quad \alpha_{22} = -6.53729630E-01 \]

VALUES FOR THE DENOMINATOR SECTION:
\[ \omega_{p11} = 7.00000000E-01 \text{ rad/s} \quad \omega_{p12} = 2.10000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = -1.72336626E-16 \quad \alpha_{12} = -6.53729630E-01 \]
\[ \omega_{p21} = 7.00000000E-01 \text{ rad/s} \quad \omega_{p22} = 2.10000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = -1.72336626E-16 \quad \alpha_{22} = -6.53729630E-01 \]
BANDPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c11} = 1.50000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.60000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = -2.53622502E-01 \quad \alpha_{12} = -6.86642329E-01 \]
\[ \omega_{c21} = 1.50000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.60000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = -2.53622502E-01 \quad \alpha_{22} = -6.86642329E-01 \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p11} = 1.10000000E+00 \text{ rad/s} \quad \omega_{p12} = 2.20000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = -2.53622502E-01 \quad \alpha_{12} = -6.86642329E-01 \]
\[ \omega_{p21} = 1.10000000E+00 \text{ rad/s} \quad \omega_{p22} = 2.20000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = -2.53622502E-01 \quad \alpha_{22} = -6.86642329E-01 \]

BANDSTOP FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c11} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.40000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = -1.72336626E-16 \quad \alpha_{12} = -6.20243844E-01 \]
\[ \omega_{c21} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.40000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = -1.72336626E-16 \quad \alpha_{22} = -6.20243844E-01 \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p11} = 7.00000000E-01 \text{ rad/s} \quad \omega_{p12} = 2.00000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = -1.72336626E-16 \quad \alpha_{12} = -6.20243844E-01 \]
\[ \omega_{p21} = 7.00000000E-01 \text{ rad/s} \quad \omega_{p22} = 2.00000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = -1.72336626E-16 \quad \alpha_{22} = -6.20243844E-01 \]
BANDSTOP FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c11} = 1.20000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.70000000E+00 \text{ rad/s} \]

\[ \alpha_{11} = -7.33220764E-02 \quad \alpha_{12} = -7.19157590E-01 \]

\[ \omega_{c21} = 1.20000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.70000000E+00 \text{ rad/s} \]

\[ \alpha_{21} = -7.33220764E-02 \quad \alpha_{22} = -7.19157590E-01 \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p11} = 8.00000000E-01 \text{ rad/s} \quad \omega_{p12} = 2.30000000E+00 \text{ rad/s} \]

\[ \alpha_{11} = -7.33220764E-02 \quad \alpha_{12} = -7.19157590E-01 \]

\[ \omega_{p21} = 8.00000000E-01 \text{ rad/s} \quad \omega_{p22} = 2.30000000E+00 \text{ rad/s} \]

\[ \alpha_{21} = -7.33220764E-02 \quad \alpha_{22} = -7.19157590E-01 \]
TYPE 3: 2-D IIR FILTER

PROTOTYPE FILTER WITH $\omega_c = 2.50000000E+00 \text{ rad/s}$

LOWPASS FILTER

VALUES FOR THE NUMERATOR SECTION:
$\omega_{c1} = 2.50000000E+00 \text{ rad/s} \quad \omega_{c2} = 2.50000000E+00 \text{ rad/s}$
$\alpha_1 = 0.00000000E+00 \quad \alpha_2 = 0.00000000E+00$

VALUES FOR THE DENOMINATOR SECTION:
$\omega_{p1} = 2.00000000E+00 \text{ rad/s} \quad \omega_{p2} = 2.00000000E+00 \text{ rad/s}$
$\alpha_1 = 0.00000000E+00 \quad \alpha_2 = 0.00000000E+00$

LOWPASS FILTER

VALUES FOR THE NUMERATOR SECTION:
$\omega_{c1} = 3.50000000E+00 \text{ rad/s} \quad \omega_{c2} = 3.50000000E+00 \text{ rad/s}$
$\alpha_1 = -3.09016994E-01 \quad \alpha_2 = -3.09016994E-01$

VALUES FOR THE DENOMINATOR SECTION:
$\omega_{p1} = 3.00000000E+00 \text{ rad/s} \quad \omega_{p2} = 3.00000000E+00 \text{ rad/s}$
$\alpha_1 = -3.09016994E-01 \quad \alpha_2 = -3.09016994E-01$

HIGHPASS FILTER

VALUES FOR THE NUMERATOR SECTION:
$\omega_{c1} = 2.50000000E+00 \text{ rad/s} \quad \omega_{c2} = 2.50000000E+00 \text{ rad/s}$
$\alpha_1 = 0.00000000E+00 \quad \alpha_2 = 0.00000000E+00$

VALUES FOR THE DENOMINATOR SECTION:
$\omega_{p1} = 2.00000000E+00 \text{ rad/s} \quad \omega_{p2} = 2.00000000E+00 \text{ rad/s}$
$\alpha_1 = 0.00000000E+00 \quad \alpha_2 = 0.00000000E+00$
HIGHPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

\( \omega_{c1} = 3.80000000E+00 \text{ rad/s} \quad \omega_{c2} = 3.80000000E+00 \text{ rad/s} \)
\( \alpha_1 = -3.98918304E-01 \quad \alpha_2 = -3.98918304E-01 \)

VALUES FOR THE DENOMINATOR SECTION:

\( \omega_{p1} = 3.30000000E+00 \text{ rad/s} \quad \omega_{p2} = 3.30000000E+00 \text{ rad/s} \)
\( \alpha_1 = -3.98918304E-01 \quad \alpha_2 = -3.98918304E-01 \)

BANDPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

\( \omega_{c11} = 2.50000000E+00 \text{ rad/s} \quad \omega_{c12} = 3.70000000E+00 \text{ rad/s} \)
\( \alpha_{11} = 0.00000000E+00 \quad \alpha_{12} = -3.68852399E-01 \)
\( \omega_{c21} = 2.50000000E+00 \text{ rad/s} \quad \omega_{c22} = 3.70000000E+00 \text{ rad/s} \)
\( \alpha_{21} = 0.00000000E+00 \quad \alpha_{22} = -3.68852399E-01 \)

VALUES FOR THE DENOMINATOR SECTION:

\( \omega_{p11} = 2.00000000E+00 \text{ rad/s} \quad \omega_{p12} = 3.20000000E+00 \text{ rad/s} \)
\( \alpha_{11} = 0.00000000E+00 \quad \alpha_{12} = -3.68852399E-01 \)
\( \omega_{p21} = 2.00000000E+00 \text{ rad/s} \quad \omega_{p22} = 3.20000000E+00 \text{ rad/s} \)
\( \alpha_{21} = 0.00000000E+00 \quad \alpha_{22} = -3.68852399E-01 \)
BANDPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c11} = 3.00000000E+00 \text{ rad/s} \quad \omega_{c12} = 4.20000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = -1.58384440E-01 \quad \alpha_{12} = -5.21603312E-01 \]
\[ \omega_{c21} = 3.00000000E+00 \text{ rad/s} \quad \omega_{c22} = 4.20000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = -1.58384440E-01 \quad \alpha_{22} = -5.21603312E-01 \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p11} = 2.50000000E+00 \text{ rad/s} \quad \omega_{p12} = 3.70000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = -1.58384440E-01 \quad \alpha_{12} = -5.21603312E-01 \]
\[ \omega_{p21} = 2.50000000E+00 \text{ rad/s} \quad \omega_{p22} = 3.70000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = -1.58384440E-01 \quad \alpha_{22} = -5.21603312E-01 \]

BANDSTOP FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c11} = 2.50000000E+00 \text{ rad/s} \quad \omega_{c12} = 4.10000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = 0.00000000E+00 \quad \alpha_{12} = -4.90440728E-01 \]
\[ \omega_{c21} = 2.50000000E+00 \text{ rad/s} \quad \omega_{c22} = 4.10000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = 0.00000000E+00 \quad \alpha_{22} = -4.90440728E-01 \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p11} = 2.00000000E+00 \text{ rad/s} \quad \omega_{p12} = 3.60000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = 0.00000000E+00 \quad \alpha_{12} = -4.90440728E-01 \]
\[ \omega_{p21} = 2.00000000E+00 \text{ rad/s} \quad \omega_{p22} = 3.60000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = 0.00000000E+00 \quad \alpha_{22} = -4.90440728E-01 \]
BANDSTOP FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c11} = 2.70000000E+00 \text{ rad/s} \quad \omega_{c12} = 3.90000000E+00 \text{ rad/s} \]

\[ \alpha_{11} = -6.48271847E-02 \quad \alpha_{12} = -4.29163373E-01 \]

\[ \omega_{c21} = 2.70000000E+00 \text{ rad/s} \quad \omega_{c22} = 3.90000000E+00 \text{ rad/s} \]

\[ \alpha_{21} = -6.48271847E-02 \quad \alpha_{22} = -4.29163373E-01 \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p11} = 2.20000000E+00 \text{ rad/s} \quad \omega_{p12} = 3.40000000E+00 \text{ rad/s} \]

\[ \alpha_{11} = -6.48271847E-02 \quad \alpha_{12} = -4.29163373E-01 \]

\[ \omega_{p21} = 2.20000000E+00 \text{ rad/s} \quad \omega_{p22} = 3.40000000E+00 \text{ rad/s} \]

\[ \alpha_{21} = -6.48271847E-02 \quad \alpha_{22} = -4.29163373E-01 \]
TYPE 4: 2-D IIR FILTER

PROTOTYPE FILTER WITH $\omega_c = 1.10000000E+00$ rad/s

LOWPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

$\omega_{c1} = 1.10000000E+00$ rad/s  $\omega_{c2} = 1.10000000E+00$ rad/s
$A_0 = 0.00000000E+00$  $B_0 = 0.00000000E+00$

VALUES FOR THE DENOMINATOR SECTION:

$\omega_{p1} = 7.00000000E-01$ rad/s  $\omega_{p2} = 7.00000000E-01$ rad/s
$\alpha_1 = 0.00000000E+00$  $\alpha_2 = 0.00000000E+00$

LOWPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

$\omega_{c1} = 1.50000000E+00$ rad/s  $\omega_{c2} = 1.50000000E+00$ rad/s
$A_0 = 4.12002848E-01$  $B_0 = 4.12002848E-01$

VALUES FOR THE DENOMINATOR SECTION:

$\omega_{p1} = 1.10000000E+00$ rad/s  $\omega_{p2} = 1.10000000E+00$ rad/s
$\alpha_1 = -2.53622502E-01$  $\alpha_2 = -2.53622502E-01$

HIGHPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

$\omega_{c1} = 1.10000000E+00$ rad/s  $\omega_{c2} = 1.10000000E+00$ rad/s
$A_0 = 0.00000000E+00$  $B_0 = 0.00000000E+00$

VALUES FOR THE DENOMINATOR SECTION:

$\omega_{p1} = 7.00000000E-01$ rad/s  $\omega_{p2} = 7.00000000E-01$ rad/s
$\alpha_1 = 0.00000000E+00$  $\alpha_2 = 0.00000000E+00$
HIGHPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c1} = 1.80000000E+00 \text{ rad/s} \quad \omega_{c2} = 1.80000000E+00 \text{ rad/s} \]

\[ A_o = 5.54756400E-01 \quad B_o = 5.54756400E-01 \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p1} = 1.40000000E+00 \text{ rad/s} \quad \omega_{p2} = 1.40000000E+00 \text{ rad/s} \]

\[ \alpha_1 = -3.95306233E-01 \quad \alpha_2 = -3.95306233E-01 \]

BANDPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c11} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.50000000E+00 \text{ rad/s} \]

\[ A_{01} = 3.55577691E-16 \quad B_{01} = 3.55577691E-16 \]

\[ \omega_{c21} = 1.10000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.50000000E+00 \text{ rad/s} \]

\[ A_{02} = 2.65305912E-01 \quad B_{02} = 2.65305912E-01 \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p11} = 7.00000000E-01 \text{ rad/s} \quad \omega_{p12} = 2.10000000E+00 \text{ rad/s} \]

\[ \alpha_{11} = -1.72336626E-16 \quad \alpha_{12} = -3.03456308E-01 \]

\[ \omega_{p21} = 7.00000000E-01 \text{ rad/s} \quad \omega_{p22} = 2.10000000E+00 \text{ rad/s} \]

\[ \alpha_{21} = -1.72336626E-16 \quad \alpha_{22} = -3.03456308E-01 \]
BANDPASS FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c11} = 1.50000000 \times 10^0 \text{ rad/s} \quad \omega_{c12} = 2.10000000 \times 10^0 \text{ rad/s} \]
\[ A_{01} = 4.12002848 \times 10^{-1} \quad B_{01} = 4.12002848 \times 10^{-1} \]
\[ \omega_{c21} = 1.50000000 \times 10^0 \text{ rad/s} \quad \omega_{c22} = 2.10000000 \times 10^0 \text{ rad/s} \]
\[ A_{02} = 1.20647910 \times 10^{-1} \quad B_{02} = 1.20647910 \times 10^{-1} \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p11} = 1.10000000 \times 10^0 \text{ rad/s} \quad \omega_{p12} = 1.70000000 \times 10^0 \text{ rad/s} \]
\[ \alpha_{11} = -2.53622502 \times 10^{-1} \quad \alpha_{12} = -9.98760035 \times 10^{-2} \]
\[ \omega_{p21} = 1.10000000 \times 10^0 \text{ rad/s} \quad \omega_{p22} = 1.70000000 \times 10^0 \text{ rad/s} \]
\[ \alpha_{21} = -2.53622502 \times 10^{-1} \quad \alpha_{22} = -9.98760035 \times 10^{-2} \]

BANDSTOP FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c11} = 1.30000000 \times 10^0 \text{ rad/s} \quad \omega_{c12} = 2.30000000 \times 10^0 \text{ rad/s} \]
\[ A_{01} = 2.54057331 \times 10^{-1} \quad B_{01} = 2.54057331 \times 10^{-1} \]
\[ \omega_{c21} = 1.30000000 \times 10^0 \text{ rad/s} \quad \omega_{c22} = 2.30000000 \times 10^0 \text{ rad/s} \]
\[ A_{02} = 2.05840119 \times 10^{-1} \quad B_{02} = 2.05840119 \times 10^{-1} \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p11} = 9.00000000 \times 10^{-1} \text{ rad/s} \quad \omega_{p12} = 1.90000000 \times 10^0 \text{ rad/s} \]
\[ \alpha_{11} = -1.39168563 \times 10^{-1} \quad \alpha_{12} = -2.00339196 \times 10^{-1} \]
\[ \omega_{p21} = 9.00000000 \times 10^{-1} \text{ rad/s} \quad \omega_{p22} = 1.90000000 \times 10^0 \text{ rad/s} \]
\[ \alpha_{21} = -1.39168563 \times 10^{-1} \quad \alpha_{22} = -2.00339196 \times 10^{-1} \]
BANDSTOP FILTER

VALUES FOR THE NUMERATOR SECTION:

\[ \omega_{c11} = 1.20000000E+00 \text{ rad/s} \quad \omega_{c12} = 2.70000000E+00 \text{ rad/s} \]
\[ A_{01} = 1.43087080E-01 \quad B_{01} = 1.43087080E-01 \]
\[ \omega_{c21} = 1.20000000E+00 \text{ rad/s} \quad \omega_{c22} = 2.70000000E+00 \text{ rad/s} \]
\[ A_{02} = 3.05021308E-01 \quad B_{02} = 3.05021308E-01 \]

VALUES FOR THE DENOMINATOR SECTION:

\[ \omega_{p11} = 8.00000000E-01 \text{ rad/s} \quad \omega_{p12} = 2.30000000E+00 \text{ rad/s} \]
\[ \alpha_{11} = -7.33220764E-02 \quad \alpha_{12} = -4.11516756E-01 \]
\[ \omega_{p21} = 8.00000000E-01 \text{ rad/s} \quad \omega_{p22} = 2.30000000E+00 \text{ rad/s} \]
\[ \alpha_{21} = -7.33220764E-02 \quad \alpha_{22} = -4.11516756E-01 \]
REFERENCES


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VITA AUCTORIS

Bruno Romanzin was born in Windsor, Ontario, Canada, on March 16, 1963. He received his B.A.Sc. degree in Electrical Engineering from the University of Windsor, Windsor, Ontario in 1986, and is currently a candidate for the Master's degree also in Electrical Engineering. He has been the class representative in the second, third, and fourth years of the undergraduate program. His research interests include digital filtering and computer optimization.