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Crossed Products of C*-Algebras

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by

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Abstract

In this thesis, we give a detailed construction of the crossed product of a C*-algebra by a locally compact group. In Chapter 1, we review some preliminary results on locally compact groups and C*-algebras. In Chapter 2, Haar measures on locally compact groups are studied and a brief harmonic analysis is discussed. In Chapter 3, we study vector-valued integration on groups and prove a version of the Fubini theorem for vector-valued integrals. In Chapter 4, transformation groups are considered, and C*-dynamical systems and their covariant representations are investigated. Finally, we explore in Chapter 5 the construction of crossed products of C*-algebras and provide some examples of crossed products. Some representations associated with crossed products are also briefly discussed.

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Contents

Introduction

This work is based on materials contained in Folland [**7**], Hewitt and Ross [**8**], and Williams [**17**]. For many of the results, we give alternative proofs and add lots of details. We also extend some constructions of crossed products given in Williams [**17**], which we shall present in Section 5.2.

The crossed product $A \rtimes_\alpha G$ of a C*-algebra A by a locally compact group G is a C*-algebra built out of a continuous group action of G on A. In the special case where the C*-algebra A is trivial (that is, $A = \mathbb{C}$), the crossed product reduces to the group C*-algebra C$^*(G)$ of G. In the other special case where $G = \{e\}$, the crossed product is just the C*-algebra A itself. Hence, both locally compact groups and C*-algebras are considered before we look into their crossed products.

Chapter 1 provides a brief overview of the theories of locally compact groups and C*-algebras as well as some preliminary results on topology.

In Chapter 2, we define Haar measure on a locally compact group and consider some of its basic properties. In Section 2.2, we digress slightly into harmonic analysis on locally compact abelian groups. This is to bridge the gap between the constructions with abelian groups and the generalization to non-abelian groups.

In Chapter 3, we discuss vector-valued integration on groups, with particular emphasis on integration of compactly supported continuous functions, and prove a version of the Fubini theorem for vector-valued integrals.

In Chapter 4, we consider transformation groups which lead to the definition of C*-dynamical systems and their covariant representations. We show that all C*-dynamical systems (A, G, α) with A commutative arise from locally compact transformation groups.

In Chapter 5, we construct crossed products of C*-algebras and illustrate them with some examples. We conclude the thesis with Section 5.3 by describing some representations associated with crossed products.

CHAPTER 1

Preliminaries

Crossed products are built from locally compact group actions on C*-algebras as we shall see later. Therefore, we need to have sufficient information about C*-algebras and locally compact groups. In this chapter, the first section gives a brief introduction to C*-algebras while the last three sections cover topological considerations. In Sections 1.2 and 1.3, we give brief introductions to topological groups and locally compact groups. For detailed study of these topics, we refer to Folland [**7**] and Hewitt and Ross [**8**].

1.1. C*-Algebras

C*-algebras are special type of Banach algebras closely associated with the theory of operators on Hilbert spaces. For instance, if $\mathcal{H}$ is a Hilbert space, then the space $B(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ is a C*-algebra. On the other hand, every C*-algebra A is isomorphic to a subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

In this section, we give a sufficient information about C*-algebras following Conway [**4**]. Most results under this section will not be proved. A more thorough treatment of C*-algebras is available in Arveson [**2**].

Definition 1.1.1. If A is an algebra over $\mathbb{C}$, an *involution* on A is a map $a \mapsto a^*$ on A such that the following properties hold for all $a, b \in A$ and $\alpha \in \mathbb{C}$:

(1) $(a^*)^* = a$;

(2) $(ab)^* = b^*a^*$;

(3) $(\alpha a + b)^* = \bar{\alpha}a^* + b^*$,

where $\bar{\alpha}$ is the complex conjugate of α.

Note that if A is an involutive unital algebra, then $1^* = 1$. Indeed, we have $1^* \cdot a = (a^* \cdot 1)^* = (a^*)^* = a$, and, similarly, $a \cdot 1^* = a$. Since the identity is unique, $1^* = 1$. Also, when $A = \mathbb{C}$, we have $\alpha^* = \bar{\alpha}$.

DEFINITION 1.1.2. A C*-algebra is a Banach algebra A with an involution such that for every $a \in A$,

$$\|a^*a\| = \|a\|^2.$$

EXAMPLE 1.1.3. If $\mathcal{H}$ is a Hilbert space, $B(\mathcal{H})$ is a C*-algebra, where for each $T \in B(\mathcal{H})$, T^* is the adjoint of T.

EXAMPLE 1.1.4. If X is a compact topological space, then the space $C(X)$ of all continuous complex-valued functions on X is a unital C*-algebra, where $f^*(x) = \overline{f(x)}$ for all $f \in C(X)$ and $x \in X$.

EXAMPLE 1.1.5. If a topological space X is locally compact but not compact, then the space $C_0(X)$ of all continuous complex-valued functions on X varnishing at infinity is a C*-algebra without identity, where f^* is defined as in Example 1.1.4.

EXAMPLE 1.1.6. Let (X, Ω, μ) be a σ-finite measure space, and let $L^\infty(X, \Omega, \mu)$ be the space of equivalent classes of μ-essentially bounded measurable functions on X. Then $L^\infty(X, \Omega, \mu)$ is a C*-algebra, where the involution is defined as in Example 1.1.4.

EXAMPLE 1.1.7. Let $\mathbb{C}_{mm}$ denote the algebra of $m \times m$ matrices with entries from $\mathbb{C}$. By viewing elements of $\mathbb{C}_{mm}$ as operators on $\mathbb{C}^m$,

$$\|A\| := \sup\{\|Ax\| : x \in \mathbb{C}^m \text{ and } \|x\| = 1\}$$

is a norm on $\mathbb{C}_{mm}$ making it a C*-algebra.

PROPOSITION 1.1.8. *If A is a C*-algebra and $a \in A$, then $\|a^*\| = \|a\|$.*

PROOF. Note that $\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$; so, $\|a\| \leq \|a^*\|$. Since $a = a^{**}$, substituting a^* for a in the above inequality gives $\|a^*\| \leq \|a\|$. □

PROPOSITION 1.1.9. *If A is a C*-algebra and $a \in A$, then*

$$\|a\| = \sup\{\|ax\| : x \in A, \|x\| \leq 1\} = \sup\{\|xa\| : x \in A, \|x\| \leq 1\}.$$

PROOF. Let $\alpha = \sup\{\|ax\| : x \in A, \|x\| \leq 1\}$. Then $\|ax\| \leq \|a\|\|x\|$ for any $x \in A$, and hence $\alpha \leq \|a\|$. The equality is obvious if $a = 0$.

For $a \neq 0$, let $x = a^*/\|a\|$. Then $\|x\| = 1$ by the preceding proposition. For this x, $\|a\| = \|ax\| \leq \alpha$, and so $\alpha = \|a\|$. The proof of the other equality is similar. □

The last proposition has the following useful implication.

Corollary 1.1.10. *Let A be a C*-algebra. Then A is isometrically isomorphic to a subalgebra of the space $B(A)$ of all bounded linear operators on A.*

Proof. For $a \in A$, define $L_a : A \to A$ by $x \mapsto ax$. By Proposition 1.1.9, $L_a \in B(A)$ and $\|L_a\| = \|a\|$. If $\lambda : A \to B(A)$ is defined by $\lambda(a) = L_a$, then λ is an isometric homomorphism. □

The map λ in the above proof is called the *left regular representation* of A.

Definition 1.1.11. If A and C are involutive Banach algebras, then $v : A \to C$ is called a $*$-homomorphism when v is an algebraic homomorphism such that $v(a^*) = v(a)^*$ for all $a \in A$.

Now, we give a brief overview of the multiplier algebra of a C*-algebra A based on materials from [**3**], [**9**] and [**10**].

Definition 1.1.12. Let A be a C*-algebra. A *double centralizer* of A is a pair (L, R) of bounded linear maps on A such that for all $a, b \in A$,

$$L(ab) = L(a)b,\ R(ab) = aR(b), \text{ and } R(a)b = aL(b).$$

For example, if $c \in A$, then (L_c, R_c) is a double centralizer on A, where $R_c(a) = ac$. It follows from Proposition 1.1.9 that $\|L_c\| = \|R_c\| = \|c\|$.

Generally, for all double centralizers (L, R) of A, we have $\|L\| = \|R\|$ (cf. [**10**, Lemma 2.1.4]).

We denote the set of all double centralizers of A by $M(A)$. Let the norm of the double centralizer (L, R) be defined as $\|L\|$ $(= \|R\|)$, and let

$$(L_1, R_1)(L_2, R_2) = (L_1 L_2, R_2 R_1)$$

for all $(L_1, R_1), (L_2, R_2) \in M(A)$. It is easy to see that $(L_1 L_2, R_2 R_1)$ is again a double centralizer of A and that $M(A)$ is an algebra under this multiplication. It is also easy to check that $M(A)$ is a closed subalgebra of $B(A) \oplus_\infty B(A)^{op}$.

For $L \in B(A)$, define $L^* : A \to A$ by $a \mapsto (L(a^*))^*$. Then $L^* \in B(A)$, and the map $L \mapsto L^*$ is an isometric conjugate-linear map on $B(A)$ satisfying $L = L^{**}$ and $(L_1L_2)^* = L_1^*L_2^*$. If (L, R) is a double centralizer of A, so is $(L, R)^* = (R^*, L^*)$ and the map $(L, R) \mapsto (L, R)^*$ is an involution on $M(A)$.

In this way, as shown in [**10**, Theorem 2.1.5], $M(A)$ is a unital C*-algebra, called the *multiplier algebra* of A, with (id_A, id_A) as unit.

The map $A \to M(A)$, $a \mapsto (L_a, R_a)$ is obviously an isometric $*$-homomorphism. Therefore, we often identify A as a C*-subalgebra of $M(A)$.

Moreover, easy computations show that A is an ideal of $M(A)$ and thus $A = M(A)$ if and only if A is unital.

Definition 1.1.13. If A is a C*-algebra and $a \in A$, then

(1) a is called *hermitian* if $a = a^*$;
(2) a is called *normal* if $a^*a = aa^*$;
(3) a is called *unitary* if A is unital and $a^*a = aa^* = 1$.

The next theorem on automatic continuity is proved in [**4**, Proposition 8.1.11].

Theorem 1.1.14. *Let A be an involutive Banach algebra and let B be a C^*-algebra. Then every $*$-homomorphism $\pi : A \to B$ is contractive. That is, we have $\|\pi(a)\| \leq \|a\|$ for all $a \in A$.*

Definition 1.1.15. Let A be a Banach algebra. A *multiplicative functional* on A is a nonzero homomorphism from A to $\mathbb{C}$. The set of all multiplicative functionals on A is called the *spectrum* of A, and we denote it by $\sigma(A)$.

The proof of the following result can be found in [**7**, Proposition 1.10].

Proposition 1.1.16. *Let A be a Banach algebra. If $h \in \sigma(A)$, then $\|h\| \leq 1$. Therefore, $\sigma(A) \subset$ Ball (A^*), the unit ball of A^*.*

Definition 1.1.17. Let A be a Banach algebra with $\sigma(A) \neq \emptyset$. The map $\mathcal{G} : A \to C_b(\sigma(A))$, $a \mapsto \hat{a}$ is called the *Gelfand transform* on A, where $\sigma(A)$ is equipped with the relative weak*-topology from A^* and $\hat{a}(\varphi) = \varphi(a)$ $(\varphi \in \sigma(A))$.

The next proposition is from [**15**, Proposition I.3.10].

PROPOSITION 1.1.18. *The spectrum $\sigma(A)$ of a Banach algebra A is locally compact with respect to the weak*- topology of A^*. It is compact if A is unital.*

PROOF. Let $\sigma'(A) = \sigma(A) \cup \{0\}$. Then $\sigma'(A) \subset \mathrm{Ball}(A^*)$. Let $\{\omega_i\}$ be a net in $\sigma'(A)$ such that $\omega_i \to \omega_0$ in the weak*-topology. For all $x, y \in A$, we have

$$\omega_0(xy) = \lim \omega_i(xy) = \lim \omega_i(x)\omega_i(y) = \omega_0(x)\omega_0(y).$$

Hence, $\omega_0 \in \sigma'(A)$. Thus $\sigma'(A)$ is a weak*-closed subset of $\mathrm{Ball}(A^*)$, and so it is weak*-compact. Since $\{0\}$ is closed in $\sigma'(A)$, $\sigma'(A) \setminus \{0\} = \sigma(A)$ is open in the compact space $\sigma'(A)$. It follows that $\sigma(A)$ is locally compact.

Suppose that A is unital. Then $\omega(1_A) = 1$ for every $\omega \in \sigma(A)$. This implies that $(\hat{1}_A)^{-1}(\{1\}) = \sigma(A)$, and thus $\sigma(A)$ is weak*-closed in $\sigma'(A)$. Therefore, $\sigma(A)$ is compact. □

The next theorem (cf. [**7**, Theorems 1.20 and 1.31]) is the most fundamental result in the Gelfand theory.

THEOREM 1.1.19 (**Gelfand-Naimark Representation Theorem**). *Let A be a commutative C*-algebra. Then the Gelfand transform $\mathcal{G} : A \to C_0(\sigma(A))$ is an isometric $*$-isomorphism.*

The Gelfand-Naimark theorem says that every commutative C*-algebra can be identified with $C_0(X)$ for a suitable X. For general C*-algebras, which are not necessarily commutative, we have the following Gelfand-Naimark-Segal (GNS) representation theorem (cf. [**4**, Theorem 8.5.17]).

THEOREM 1.1.20 (**Gelfand-Naimark-Segal Representation Theorem**). *Let A be a C*-algebra. Then there exist a Hilbert space $\mathcal{H}$ and an isometric $*$-homomorphism $\pi : A \to B(\mathcal{H})$.*

1.2. Topological Groups

In this section, we study briefly the structure of topological groups.

DEFINITION 1.2.1. A *topological group* is a group G together with a topology τ such that

(a) points are closed in (G, τ);

(b) the map $G \times G \to G$, $(s, r) \mapsto sr^{-1}$ is continuous.

Condition (b) is equivalent to the condition below:

(c) the maps $(s, r) \mapsto sr$ and $s \mapsto s^{-1}$ are continuous.

Example 1.2.2. Any group G equipped with the discrete topology is a topological group.

Example 1.2.3. The groups $\mathbb{R}^n$, $\mathbb{T}^m$ and $\mathbb{Z}^d$ with their usual topologies are topological abelian groups.

Example 1.2.4. If G and H are topological groups, then $G \times H$ with the product topology is a topological group.

Example 1.2.5. Let $\mathcal{H}$ be a complex Hilbert space and let

$$U(\mathcal{H}) = \{U \in B(\mathcal{H}) : U^*U = UU^* = 1_{\mathcal{H}}\}.$$

With the relative strong operator topology (SOT), i.e., the topology of pointwise convergence, $U(\mathcal{H})$ is a topological group, which is non-abelian if $\dim(\mathcal{H}) \geq 2$.

Proof. Since $B(\mathcal{H})$ is Hausdorff in the SOT, $U(\mathcal{H})$ is Hausdorff and thus points in $U(\mathcal{H})$ are closed in the SOT.

Suppose that $U_\alpha \to U$ and $V_\alpha \to V$ in the SOT in $U(\mathcal{H})$. To show that $U_\alpha V_\alpha \to UV$ in the SOT, let $h \in \mathcal{H}$. Then

$$\begin{aligned} \|U_\alpha V_\alpha h - UVh\| &= \|U_\alpha V_\alpha h - U_\alpha Vh + U_\alpha Vh - UVh\| \\ &\leq \|(V_\alpha - V)h\| + \|(U_\alpha - U)(Vh)\| \to 0 \end{aligned}$$

and

$$\|U_\alpha^{-1}h - U^{-1}h\| = \|h - U_\alpha U^{-1}h\| = \|(U_\alpha - U)(U^{-1}h)\| \to 0.$$

Therefore, $(U(\mathcal{H}), SOT)$ is a topological group.

Now, suppose $\dim(\mathcal{H}) = 2$. Let $T, S \in U(\mathcal{H})$ be defined by

$$T(h_1, h_2) = (h_2, h_1) \quad \text{and} \quad S(h_1, h_2) = (h_2, -h_1).$$

We see easily that $ST \neq TS$. When $\dim(\mathcal{H}) > 2$, let $\mathcal{H}_0$ be a subspace of $\mathcal{H}$ with $\dim(\mathcal{H}_0) = 2$. In this case, we have $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$, and we define $\widetilde{T}, \widetilde{S} \in U(\mathcal{H})$ by

$$\widetilde{T}(x \oplus y) = Tx \oplus y \quad \text{and} \quad \widetilde{S}(x \oplus y) = Sx \oplus y,$$

where $x \in \mathcal{H}_0$ and $y \in \mathcal{H}_0^{\perp}$. Then $\widetilde{S}\widetilde{T} \neq \widetilde{T}\widetilde{S}$ since $ST \neq TS$. Therefore, $U(\mathcal{H})$ is non-abelian if $\dim(\mathcal{H}) \geq 2$. $\square$

EXAMPLE 1.2.6. Let A be a C*-algebra. Then the collection $\mathrm{Aut}(A)$ of all $*$-automorphisms of A is a group under composition. We equip $\mathrm{Aut}(A)$ with the so-called *point-norm topology*; that is, $\alpha_i \to \alpha$ if and only if $\alpha_i(a) \to \alpha(a)$ for all $a \in A$. Then $\mathrm{Aut}(A)$ is a topological group.

LEMMA 1.2.7. *If G is a topological group and $r \in G$, then the maps $s \mapsto s^{-1}$, $s \mapsto sr$, and $s \mapsto rs$ are homeomorphisms on G.*

PROOF. The assertion holds as the inverse of $t : s \mapsto s^{-1}$ is itself, the inverse of $R_r : s \mapsto sr$ is $s \mapsto sr^{-1}$, and the inverse of $s \mapsto rs$ is $s \mapsto r^{-1}s$, which is the map $t \circ R_r \circ t$ and hence is continuous. $\square$

One consequence of Lemma 1.2.7 is that the topology on a topological group G is *translation invariant*: a set V in G is open if and only if each of its translates rV is open. Thus the topology on G is completely determined by any neighborhood basis of e. More precisely, we have the following corollary of Lemma 1.2.7.

COROLLARY 1.2.8. *Let G be a topological group, and let $\mathcal{N}$ be a neighborhood basis of e. Then for each $r \in G$, $\{Nr\}_{N \in \mathcal{N}}$ and $\{rN\}_{N \in \mathcal{N}}$ are both neighborhood bases of r. In particular, if $\mathcal{N}$ consists of open neighborhoods of e, then*

$$\beta = \{Vr : V \in \mathcal{N}, r \in G\} \text{ and } \beta' = \{rV : V \in \mathcal{N}, r \in G\}$$

are bases for the topology on G.

LEMMA 1.2.9. *Let V be a neighborhood of e in G. Then $V \subset \overline{V} \subset V^2$.*

PROOF. It suffices to show that $\overline{V} \subset V^2$. Suppose that $s \in \overline{V}$. Then every neighborhood of s meets V. Since sV^{-1} is a neighborhood of s, it follows that

$sV^{-1} \cap V \neq \emptyset$. Let $t \in sV^{-1} \cap V$. Then $t = sr^{-1}$ for some $r \in V$. It follows that $s = tr \in V^2$. $\square$

LEMMA 1.2.10. *Let G be a topological group and $\mathcal{N}$ be an open neighborhood basis of e. Then for every $V \in \mathcal{N}$, there is $U \in \mathcal{N}$ such that $U^2 \subset V$ and hence $\overline{U} \subset V$.*

PROOF. By the continuity of the multiplication, there are open neighborhoods U_1 and U_2 of e such that $U_1U_2 \subset V$. Let $U = U_1 \cap U_2$. Then $U^2 \subset U_1U_2 \subset V$, and hence $\overline{U} \subset V$ by Lemma 1.2.9. $\square$

LEMMA 1.2.11. *If G is a topological group, then G is regular and Hausdorff.*

PROOF. The regularity of G follows from Lemma 1.2.10, and therefore G is Hausdorff as singletons in G are closed. $\square$

1.3. Locally Compact Groups

DEFINITION 1.3.1. A topological space is called *locally compact* if every point has a neighborhood basis consisting of compact sets.

LEMMA 1.3.2. *If X is a Hausdorff space, then X is locally compact whenever every point in X has a compact neighborhood.*

PROOF. Suppose that every point in X has a compact neighborhood. Let $x \in X$ and let U be a neighborhood of x. Let K be a compact neighborhood of $x \in X$ and let V be the interior of $U \cap K$. Then $\overline{V}$ is compact and Hausdorff, and therefore regular. Furthermore, $\overline{V} \setminus V$ is a closed subset of $\overline{V}$ not containing x. Thus there is an open set W in $\overline{V}$ such that $x \in W \subset \overline{W} \subset V$. Thus W is open in X and $\overline{W}$ is a compact neighborhood of x with $\overline{W} \subset U$. $\square$

DEFINITION 1.3.3. A *locally compact group* is a topological group for which the underlying topology is locally compact.

REMARK 1.3.4. Since topological groups are Hausdorff, Corollary 1.2.8 implies that a topological group G is locally compact if and only if there is a compact neighborhood of e, which holds if and only if there is a nonempty open set in G with compact closure.

Example 1.3.5. Any discrete group G is a locally compact group.

Example 1.3.6. The groups $\mathbb{R}^n$ and $\mathbb{Z}^d$ are locally compact, noncompact, abelian groups, and $\mathbb{T}^m$ is a compact abelian group.

Proposition 1.3.7. *Let X be a locally compact space, and let E be an open or closed subspace of X. Then E is locally compact.*

Proof. This assertion follows from Definition 1.3.1. $\square$

Remark 1.3.8. Since the intersection of two locally compact subsets of a space is also locally compact, Proposition 1.3.7 implies that the intersection of an open set and a closed set in a locally compact space X is also locally compact.

Definition 1.3.9. A subset Y of a space X is called *locally closed* if each point in Y has an open neighborhood P in X such that $P \cap Y$ is closed in P.

Lemma 1.3.10. *If X is a topological space and $Y \subset X$, then the following are equivalent:*

(a) *Y is locally closed in X;*
(b) *Y is open in $\overline{Y}$;*
(c) *$Y = C \cap O$, where C is closed in X and O is open in X.*

Proof. $(a) \Rightarrow (b)$. Suppose Y is locally closed in X. Let $y \in Y$ and let P be an open neighborhood of y in X such that $Y \cap P$ is closed in P. We only have to show that $\overline{Y} \cap P \subset Y$ since $\overline{Y} \cap P$ is open in $\overline{Y}$. To this end, let $x \in \overline{Y} \cap P$. Then there is a net $\{x_i\}$ in Y such that $x_i \to x$. Since $x \in P$ and P is open, we can assume that $x_i \in P \cap Y$ for all i. Since $P \cap Y$ is closed in P, $x \in P \cap Y \subset Y$ as required.

$(b) \Rightarrow (c)$. This is obvious.

$(c) \Rightarrow (a)$. Let $Y = C \cap O$ be as above. Fix $y \in Y$ and let $P = O$. Then P is an open neighborhood of y in X and $P \cap Y = C \cap P$ is closed in P. $\square$

Proposition 1.3.11. *Let Y be a subspace of a topological space X. If X is a locally compact space and Y is locally closed in X, then Y is locally compact. If Y is Hausdorff and locally compact, then Y is locally closed in X.*

PROOF. The first assertion follows from Remark 1.3.8 and Lemma 1.3.10. To prove the second assertion, fix $y \in Y$. Let U be an open neighborhood of y in X such that $U \cap Y$ has compact closure B in Y, which is possible as Y is locally compact and Hausdorff. It suffices to see that $U \cap Y$ is closed in U. So, we let $\{x_i\}$ be a net in $U \cap Y$ such that $x_i \to x \in U$. Since B is compact and hence closed, by passing to a subnet and by the uniqueness of limit, we have $x \in B \subset Y$. Therefore, $x \in U \cap Y$ and hence $U \cap Y$ is closed in Y. $\square$

We now digress slightly to describe the structure of $\mathrm{Aut}(A)$ for a commutative C*-algebra A, which is isomorphic to $C_0(X)$ for a suitable locally compact space X by Gelfand-Naimark Theorem. We will show in Theorem 1.3.16 that $\mathrm{Aut}(C_0(X))$ is homeomorphic to $\mathrm{Homeo}(X)$ when $\mathrm{Homeo}(X)$ is given the topology described in Definition 1.3.14.

DEFINITION 1.3.12. Let X and Y be topological spaces and let $C(X,Y)$ be the collection of continuous functions from X to Y. Then the *compact-open topology* on $C(X,Y)$ is the topology with a subbasis consisting of all the sets of the form

$$U(K,V) \quad = \quad \{f \in C(X,Y) : f(K) \subset V\}, \tag{1.3.1}$$

where $K \subset X$ is compact and $V \subset Y$ is open.

More information on compact-open topology can be seen in [**5**].

LEMMA 1.3.13. *Let X and Y be locally compact Hausdorff spaces and let $C(X,Y)$ be equipped with the compact-open topology. Then $f_i \to f$ in $C(X,Y)$ if and only if $f_i(x_i) \to f(x)$ in Y whenever $x_i \to x$ in X.*

PROOF. Suppose that $f_i \to f$ and $x_i \to x$. Let V be an open neighborhood of $f(x)$ in Y and let K be a compact neighborhood of x in X such that $f(K) \subset V$. Then $f \in U(K,V)$. Thus we eventually have both $f_i \in U(K,V)$ and $x_i \in K$. So, we eventually have $f_i(x_i) \in V$. Therefore, $f_i(x_i) \to f(x)$ in Y.

Conversely, suppose that $x_i \to x$ implies $f_i(x_i) \to f(x)$. To show that $f_i \to f$, we claim that f_i is eventually in any $U(K,V)$ that contains f. Otherwise, by passing to a subnet, we can assume that $f_i \notin U(K,V)$ for all i. Then for each i, there is $x_i \in K$ such that $f_i(x_i) \notin V$. Passing to another subnet, we can assume that $x_i \to x \in K$, which implies, by assumption, that $f_i(x_i) \to f(x)$. Since $Y \setminus V$ is closed, we must have $f(x) \notin V$, contradicting that $f \in U(K,V)$. $\square$

DEFINITION 1.3.14. Let X be a locally compact Hausdorff space. Let $\mathrm{Homeo}(X)$ be the set of all homeomorphisms on X. We give $\mathrm{Homeo}(X)$ the topology with a subbasis consisting of all the sets of the form

$$U(K,K',V,V') \quad = \quad \{h \in \mathrm{Homeo}(X) : h(K) \subset V \text{ and } h^{-1}(K') \subset V'\},$$

where K and K' are compact in X, and V and V' are open in X.

PROPOSITION 1.3.15. *Let X be a locally compact Hausdorff space. For each $h \in \mathrm{Homeo}(X)$ and $f \in C_0(X)$, let $\alpha(f)(x) = f(h(x))$ $(x \in X)$. Then we have $\alpha \in \mathrm{Aut}(C_0(X))$.*

PROOF. Let $f \in C_0(X)$. First note that $\alpha(f) = f \circ h$ is a continuous function on X, as h and f are both continuous.

To prove that $\alpha(f)$ is in $C_0(X)$, let $\epsilon > 0$, and we will show that the set $\{x \in X : |\alpha(f)(x)| \geq \epsilon\}$ is compact. This is indeed true since

$$\{x \in X : |\alpha(f)(x)| \geq \epsilon\} = \{x \in X : |f(h(x))| \geq \epsilon\} = h^{-1}(\{y \in X : |f(y)| \geq \epsilon\})$$

and $\{y \in X : |f(y)| \geq \epsilon\}$ is compact.

Clearly, $\alpha : C_0(X) \to C_0(X)$ is an injective $*$-homomorphism. It is also onto $C_0(X)$, since $\alpha(g \circ h^{-1}) = g$ for all $g \in C_0(X)$. Therefore, $\alpha \in \mathrm{Aut}(C_0(X))$. $\square$

THEOREM 1.3.16. *Let X be a locally compact Hausdorff space. Then for each $\alpha \in \mathrm{Aut}(C_0(X))$, there is $h \in \mathrm{Homeo}(X)$ such that $\alpha(f) = f \circ h^{-1}$ for all $f \in C_0(X)$.*

Moreover, the map $\alpha \mapsto h$ is a homeomorphic group isomorphism of $\mathrm{Aut}(C_0(X))$ *onto* $\mathrm{Homeo}(X)$, *where* $\mathrm{Aut}(C_0(X))$ *is equipped with the point-norm topology and* $\mathrm{Homeo}(X)$ *is equipped with the topology given in Definition* 1.3.14.

PROOF. Let $\triangle = \sigma(C_0(X))$ be the spectrum of $C_0(X)$. Then $\triangle \subset Ball(C_0(X)^*)$. Equipping $\triangle$ with the relative weak*-topology from $C_0(X)^*$, we have $X \cong \triangle$ via the homeomorphism $x \mapsto e(x)$, where $e(x)(f) = f(x)$ $(x \in X, f \in C_0(X))$.

Let $\alpha \in \mathrm{Aut}(C_0(X))$. Then $\alpha^* : C_0(X)^* \to C_0(X)^*$ is an isometric weak*-homeomorphism given by $\alpha^*(\varphi) = \varphi \circ \alpha$. It is easy to see that $\alpha^*(\triangle) = \triangle$. Let $h = e^{-1} \circ (\alpha^*)^{-1} \circ e : X \to X$. Then $h \in \mathrm{Homeo}(X)$ and $\alpha^* \circ e = e \circ h^{-1}$. In particular, we have

$$\alpha(f)(x) = e(x)(\alpha(f)) = \alpha^*(e(x))(f) = e(h^{-1}(x))(f) = f(h^{-1}(x)) = (f \circ h^{-1})(x)$$

for all $f \in C_0(X)$ and $x \in X$.

Combining with Proposition 1.3.15, we obtain that the above map

$$\Gamma : \mathrm{Aut}(C_0(X)) \to \mathrm{Homeo}(X),\ \alpha \mapsto h$$

is a surjection. It is also injective, since $\alpha(f) = f \circ h^{-1}$ for all $f \in C_0(X)$.

Clearly, $\Gamma : \mathrm{Aut}(C_0(X)) \to \mathrm{Homeo}(X)$ is a group isomorphism. Now we show that Γ is a homeomorphism. That is, $\alpha_i \to \alpha$ in $\mathrm{Aut}(C_0(X))$ if and only if $h_i \to h$ in $\mathrm{Homeo}(X)$, where $h_i = \Gamma(\alpha_i)$ and $h = \Gamma(\alpha)$.

Assume that $\alpha_i \to \alpha$ but $h_i \not\to h$ in $\mathrm{Homeo}(X)$. Note that $\alpha_i \to \alpha$ if and only if $\alpha_i^{-1} \to \alpha^{-1}$, since $\mathrm{Aut}(C_0(X))$ is a topological group. So, we may assume that $h_i^{-1} \not\to h^{-1}$ in the compact-open topology (otherwise, we consider the case where $\alpha_i^{-1} \to \alpha^{-1}$ but $h_i \not\to h$ in the compact-open topology). By Lemma 1.3.13, there exists a net $\{x_i\}$ in X such that $x_i \to x$ but $h_i^{-1}(x_i) \not\to h^{-1}(x)$. Thus, we obtain that $f(h_i^{-1}(x_i)) \not\to f(h^{-1}(x))$ for some $f \in C_0(X)$. That is, $\alpha_i(f)(x_i) \not\to \alpha(f)(x)$. However, we have

$$\|\alpha_i(f)(x_i) - \alpha(f)(x)\| \le \|\alpha_i(f)(x_i) - \alpha(f)(x_i)\| + \|\alpha(f)(x_i) - \alpha(f)(x)\| \to 0,$$

since $\|\alpha_i(f) - \alpha(f)\|_{C_0(X)} \to 0$, a contradiction.

Conversely, assume that $h_i \to h$ in $\mathrm{Homeo}(X)$ but $\alpha_i(f) \not\to \alpha(f)$ (i.e., we have $f \circ h_i^{-1} \not\to f \circ h^{-1}$) for some $f \in C_0(X)$. Passing to a subnet, we can assume that for some $\epsilon_0 > 0$ and a net (x_i) in X, we have

$$|f(h_i^{-1}(x_i)) - f(h^{-1}(x_i))| \ge \epsilon_0 \ \text{ for all } i. \tag{1.3.2}$$

By passing to a subnet again, we have that either $|f(h_i^{-1}(x_i))| \ge \frac{\epsilon_0}{2}$ for all i or $|f(h^{-1}(x_i))| \ge \frac{\epsilon_0}{2}$ for all i.

Let K be the compact set $\{x \in X : |f(x)| \ge \frac{\epsilon_0}{2}\}$ in X. Then we have either

$\{h_i^{-1}(x_i)\} \subset K$ or $\{h^{-1}(x_i)\} \subset K$. Suppose first that $\{h_i^{-1}(x_i)\} \subset K$. Then, passing to a subnet, we can assume that $h_i^{-1}(x_i) \to y \in K$. Since $h_i \to h$ in the compact-open topology, Lemma 1.3.13 implies that $x_i \to h(y)$; i.e., $h^{-1}(x_i) \to y$. Thus we obtain that $|f(h_i^{-1}(x_i)) - f(h^{-1}(x_i))| \to |f(y) - f(y)| = 0$, contradicting (1.3.2). Next, suppose that $\{h^{-1}(x_i)\} \subset K$. Again, passing to a subnet, we can assume that $h^{-1}(x_i) \to y \in K$; that is, $x_i \to h(y)$. Since $h_i^{-1} \to h^{-1}$ in the compact-open topology, we have $h_i^{-1}(x_i) \to y$. As discussed above, we obtain that $|f(h_i^{-1}(x_i)) - f(h^{-1}(x_i))| \to |f(y) - f(y)| = 0$, a contradiction.

Therefore, we conclude that the map $\Gamma : \mathrm{Aut}(C_0(X)) \to \mathrm{Homeo}(X)$, $\alpha \mapsto h$ is a homeomorphism. $\square$

1.3.1. Subgroups of Locally Compact Groups. If G is a topological group, then any subgroup H of G with the relative topology is also a topological group. However, by Proposition 1.3.11, if G is locally compact, then H is locally compact if and only if it is locally closed.

Remark 1.3.17. If H is a subgroup of a topological group G, so is $\overline{H}$, since the map $(x, y) \mapsto xy^{-1}$ is continuous.

Lemma 1.3.18. *A locally closed subgroup of a topological group is closed.*

Proof. Let H be a locally closed subgroup of a topological group G. Then there is an open neighborhood W of e such that $W \cap H$ is closed in W. Let U and V be neighborhoods of e in G such that

$$V^2 \subset U \subset U^2 \subset W.$$

Now let $x \in \overline{H}$ be fixed. Let $\{x_i\}$ be a net in H converging to x. Remark 1.3.17 implies that $x^{-1} \in \overline{H}$ as well. Since $x^{-1}V$ is a neighborhood of x^{-1}, it must meet H. Let $y \in x^{-1}V \cap H$. Since x_i is eventually in Vx and $x_i y$ is eventually in

$$V^2 \cap H \subset U \cap H \subset W \cap H,$$

it follows from Lemma 1.2.9 that $xy \in \overline{U} \subset U^2 \subset W$. Since $W \cap H$ is closed in W, we have $xy \in W \cap H$. Therefore, $x = (xy)y^{-1} \in H$. $\square$

Corollary 1.3.19. *Let H be a subgroup of a topological group G satisfying one of the following conditions:*

(a) *H is open;*

(b) *H is discrete;*

(c) *H is locally compact.*

Then H is closed.

PROOF. We only have to show that H is locally closed. This is trivial for case (a), and is true for case (c) by Proposition 1.3.11. For case (b), the discreteness of H implies that there is an open set W such that $W \cap H = \{e\}$, which is clearly closed in W, and hence H is locally closed. □

The fact that open subgroups are closed follows also from Lemma 1.2.9.

DEFINITION 1.3.20. A locally compact Hausdorff space X is called σ-compact if $X = \bigcup_{i=1}^{\infty} C_i$ for some sequence $\{C_i\}$ of compact subsets of X.

Every second countable locally compact Hausdorff space is σ-compact, since if $\{O_n\}$ is a countable basis for the topology, then the family consisting of those O_n with compact closure is still a basis. However, a topological disjoint union of uncountably many locally compact spaces fails to be σ-compact. In particular, uncountable discrete spaces are not σ-compact.

LEMMA 1.3.21. *Every locally compact group G has a σ-compact open subgroup. Hence, every locally compact group is a topological disjoint union of σ-compact spaces.*

PROOF. Let V be a symmetric open neighborhood of e in G with compact closure. Let $H = \bigcup_{n=1}^{\infty} V^n$. Then H is an open subgroup of G. By Lemma 1.2.9, $\overline{V}^n \subset V^{2n}$, which implies that $\bigcup_n V^n \subset \bigcup_n \overline{V}^n \subset \bigcup_n V^{2n} \subset \bigcup_n V^n$. Thus, $H = \bigcup \overline{V}^n$ is σ-compact since $\overline{V}^n$ is compact for each n. It then follows that for each $s \in G$, sH is a σ-compact subset of G, and thus G is the union of pairwise disjoint cosets sH. □

COROLLARY 1.3.22. *Every connected locally compact group is σ-compact.*

PROOF. Let G be a connected locally compact group. By Lemma 1.3.21, G has a σ-compact open subgroup H which is also closed by Corollary 1.3.19. The connectedness of G implies that $G = H$. □

1.4. Some Preliminary Results on Topology

Let X be a Hausdorff topological space.

DEFINITION 1.4.1. The space X is *paracompact* if every open cover of X has a locally finite refinement. Recall that a cover of X is *locally finite* if each point in X has a neighborhood that meets only finitely many elements of the cover.

Clearly, compact Hausdorff spaces are paracompact.

REMARK 1.4.2. We would like to have Urysohn's Lemma and the Tietze Extension Theorem for locally compact groups. This is only possible provided locally compact groups are normal topological spaces. But all we know is that locally compact groups are completely regular spaces (cf. [**14**, Corollary 4.1.5]), which are not necessarily normal.

However, σ-compact locally compact Hausdorff spaces are always paracompact (cf. [**11**, Proposition 1.7.11]). This will guarantee that all locally compact groups are paracompact (cf. Lemma 1.3.21) and hence normal (cf. [**8**, Theorem 8.13]). Without using this fact, we prove directly the following versions of Urysohn's Lemma and Tietze Extension Theorem.

LEMMA 1.4.3 (**Urysohn's Lemma**). *Suppose that X is a locally compact Hausdorff spaces and that V is an open set in X containing a compact set K. Then there is $f \in C_c(X)$ such that $0 \leq f(x) \leq 1$ for all x, $f(x) = 1$ for all $x \in K$, and $f(x) = 0$ if $x \notin V$.*

PROOF. Since X is locally compact, there is an open set W in X such that $\overline{W}$ is compact and $K \subset W \subset \overline{W} \subset V$. Since $\overline{W}$ is compact and therefore normal, by the usual Urysohn Lemma for normal spaces, we can find $h \in C(\overline{W})$ such that $0 \leq h(x) \leq 1$ for all $x \in \overline{W}$, $h(x) = 1$ for all $x \in K$, and $h(x) = 0$ if $x \in \overline{W} \setminus W$. Now define

$$f(x) = \begin{cases} h(x) & \text{if } x \in \overline{W}, \\ 0 & \text{if } x \in X \setminus \overline{W}. \end{cases}$$

Then $f \in C_c(X)$ is the required function. □

LEMMA 1.4.4 (**Tietze Extension Theorem**). *Suppose that X is a locally compact Hausdorff space and K is a compact subset of X. If $g \in C(K)$, then there is $f \in C_c(X)$ such that $f(x) = g(x)$ for all $x \in K$.*

Moreover, if $g(K) \subset [0,1]$, we can also have $f(X) \subset [0,1]$.

PROOF. Let W be an open set in X containing K with compact closure. By the usual Tietze Extension theorem for normal spaces, there is $k \in C(\overline{W})$ extending g. By Lemma 1.4.3, there is a $h \in C_c(X)$ such that $h(x) = 1$ for all $x \in K$ and $h(x) = 0$ if $x \notin W$. Now let

$$f(x) = \begin{cases} k(x)h(x) & \text{if } x \in \overline{W}, \\ 0 & \text{if } x \in X \setminus \overline{W}. \end{cases}$$

Then $f \in C_c(X)$ extends g. The second assertion also holds by the above proof. $\square$

DEFINITION 1.4.5. Let X be a Hausdorff space and $I = [0,1]$. A family of continuous maps $k_\alpha : X \to I$ is called a *partition of unity on X* if

(i) the family $\{\operatorname{supp}(k_\alpha)\}$ forms a locally finite closed covering of X;

(ii) $\sum_\alpha k_\alpha(x) = 1$ for each $x \in X$. (This sum is well-defined because each x lies in $\operatorname{supp}(k_\alpha)$ for finitely many α)

Given an open covering $\{U_\beta\}$ of X, we say that a partition $\{k_\beta\}$ of unity is *subordinated* to $\{U_\beta\}$ if $\operatorname{supp}(k_\beta) \subset U_\beta$ for each β.

The next result, cited from [**5**, Theorem 8.4.2], will be used in the proof of Proposition 1.4.7 below.

THEOREM 1.4.6. *Let X be a paracompact Hausdorff space. Then for each open covering $\{U_\alpha\}$ of X, there is a partition of unity subordinated to $\{U_\alpha\}$.*

PROPOSITION 1.4.7 (**Partitions of Unity**). *Suppose that X is a locally compact Hausdorff space and that $\{U_i\}_{i=1}^n$ is a cover of a compact set K in X by open sets with compact closures. Then for $i = 1, \cdots, n$, there are $\varphi_1, \cdots, \varphi_n \in C_c(X)$ such that*

(a) $0 \le \varphi_i(x) \le 1$ *for all $x \in X$;*

(b) *$\operatorname{supp}(\varphi_i) \subset U_i$ for all i;*

(c) $\sum_{i=1}^{n} \varphi_i(x) = 1$ *if* $x \in K$;

(d) $\sum_{i=1}^{n} \varphi_i(x) \leq 1$ *if* $x \notin K$.

PROOF. Let $C = \cup_{i=1}^{n} \overline{U_i}$. Then C is a compact neighborhood of K. Since C is compact, in particular, it is paracompact, Theorem 1.4.6 implies that there is a partition of unity $\{\psi_i\}_{i=0}^{n}$ of $C(C)$ subordinate to the cover $\{C\backslash K, U_1, \cdots, U_n\}$. By Lemmas 1.4.3 and 1.4.4, each ψ_i $(1 \leq i \leq n)$ can be extended to $\varphi_i \in C_c(X)$ satisfying (a) and (b). Clearly, now (c) and (d) are also satisfied. □

Suppose that H is a subgroup of a topological group G. Let G/H be the set of all left cosets of H. Equip G/H with the quotient topology τ, which is the strongest topology on G/H making the quotient map $q : G \to G/H$, $x \mapsto xH$ continuous. Then

$$\tau = \{U \subset G/H : q^{-1}(U) \text{ is open in } G\}.$$

LEMMA 1.4.8. *If H is a subgroup of a topological group G, then the quotient map $q : G \to G/H$ is open and continuous.*

PROOF. It suffices to show that q is open. In fact, if V is open in G, then

$$q^{-1}(q(V)) = \bigcup_{h \in H} Vh$$

is open in G as each Vh is open; that is, $q(V)$ is open in G/H. □

REMARK 1.4.9. Suppose that $\{V_\alpha\}$ is a basis for the topology on G. Let $sH \in G/H$ and W be an open set in G/H containing sH. Then $s \in q^{-1}(W)$, and $q^{-1}(W)$ is open in G. Thus, there exists α such that $s \in V_\alpha \subset q^{-1}(W)$; that is, $sH \in q(V_\alpha) \subset W$. Therefore, $\{q(V_\alpha)\}$ is a basis for the quotient topology on G/H. In particular, if G is second countable, then G/H is second countable.

The above argument shows that if $\{V_\alpha\}$ is a neighborhood basis at $s \in G$, then $\{q(V_\alpha)\}$ is a neighborhood basis at $sH \in G/H$. Therefore, if G is first countable, then so is G/H.

PROPOSITION 1.4.10. *Let H be a subgroup of a locally compact group G. Then G/H is locally compact.*

PROOF. This is obvious by Remark 1.4.9 and Lemma 1.4.8. □

CHAPTER 2

Haar Measures and a Brief Harmonic Analysis

The main references for this chapter are Folland [**7**] and William [**17**].

2.1. Haar Measures

Any locally compact group has a uniquely defined measure class which respects its group structure. We will study this measure class in this section. Let us begin with some definitions.

DEFINITION 2.1.1. Let G be a locally compact space. A measure μ on G is called a *Borel measure* if each open set in G is measurable. In this case,

(i) μ is called *compact inner regular* if for each open set V in G,

$$\mu(V) = sup\{\mu(C) : C \subset V \text{ and } C \text{ is compact}\};$$

(ii) μ is called *open outer regular* if for each measurable set A in G,

$$\mu(A) = inf\{\mu(V) : A \subset V \text{ and } V \text{ is open}\};$$

(iii) μ is called a *Radon measure* if it is both compact inner and open outer regular.

A Radon measure μ on G is called *left invariant* if for all $s \in G$ and measurable sets A in G, sA is measurable and $\mu(sA) = \mu(A)$. Right invariance can be defined similarly. If μ is both left and right invariant, we say that μ is *bi-invariant.*

DEFINITION 2.1.2. A nonzero left (right) invariant Radon measure on a locally compact group G is called a left (right) *Haar measure.*

REMARK 2.1.3. If μ is a left Haar measure, then $\nu(E) := \mu(E^{-1})$ is a right Haar measure. For convenience, in the rest of the thesis, the term Haar measure will be used for left Haar measures.

The following two results (cf. [**7**, Theorem 2.10]) and (cf. [**12**, Theorem 2.14], respectively) are fundamental in this section.

THEOREM 2.1.4 (**Riesz Representation Theorem**). *Let X be a locally compact Hausdorff space, and let I be a positive linear functional on $C_c(X)$. Then there exist a σ-algebra $\mathfrak{M}$ in X which contains all Borel sets in X and a unique positive measure μ on $\mathfrak{M}$ such that*

(a) $I(f) = \int_X f d\mu$ *for all* $f \in C_c(X)$*;*
(b) $\mu(K) < \infty$ *for every compact set* $K \subset X$*;*
(c) μ *is outer-open regular and compact-inner regular;*
(d) μ *is complete in the sense that if* $A \subset E \in \mathfrak{M}$ *and* $\mu(E) = 0$*, then* $A \in \mathfrak{M}$*.*

THEOREM 2.1.5. *Every locally compact group has a Haar measure which is unique up to a strictly positive scalar multiple.*

Thus, obtaining a Haar measure is equivalent to constructing a positive linear functional with left invariance. In fact, if

$$I : C_c(G) \longrightarrow \mathbb{C} \tag{2.1.1}$$

is a positive linear functional satisfying

$$I(\lambda(r)f) = I(f) \quad \text{for all } r \in G \text{ and } f \in C_c(G), \tag{2.1.2}$$

where

$$\lambda(r)f(s) = f(r^{-1}s) \ \ (s \in G),$$

then the Riesz Representation Theorem guarantees that (2.1.1) gives a Radon measure μ such that

$$I(f) = \int_G f(s) d\mu(s). \tag{2.1.3}$$

This Radon measure is left invariant by equation (2.1.2), and hence is a left Haar measure of G. Such positive linear functional I is called a *Haar functional* on G.

The following result is cited from [**7**, Proposition 2.19].

PROPOSITION 2.1.6. *If μ is a Haar measure on a locally compact group G, then $\mu(U) > 0$ for every nonempty open set U in G, and $\int f d\mu > 0$ for all $f \in C_c^+(G)$.*

PROOF. We prove the first assertion by contradiction. Suppose U is open and nonempty, and $\mu(U) = 0$. Then $\mu(sU) = 0$ for all $s \in G$ and, since any compact set K can be covered by finitely many translates of U, we have $\mu(K) = 0$ for every compact set K. But then $\mu(G) = 0$ by the compact inner regularity of μ, contradicting that $\mu \neq 0$. Thus $\mu(U) > 0$ for every nonempty open set U.

Now, for given $f \in C_c^+(G)$, let $U = \{s : f(s) > \frac{1}{2}\|f\|_{\sup}\}$. Then U is open, and hence $\int_G f d\mu \geq \int_U f d\mu \geq \frac{1}{2}\|f\|_{\sup}\mu(U) > 0$. □

The proposition above guarantees that

$$\|f\|_1 := \int_G |f(s)| d\mu(s)$$

defines a norm on $C_c(G)$. The completion of $(C_c(G), \|\cdot\|_1)$ is $L^1(G)$.

Let us illustrate Haar measure with some examples.

EXAMPLE 2.1.7. If G is a discrete group, then the counting measure μ is a Haar measure on G. In this case,

$$\int_G f d\mu = \sum_{x \in G} f(x)$$

for all functions f on G with finite supports, and $L^1(G) = \ell^1(G)$.

EXAMPLE 2.1.8. If G is $\mathbb{R}^n$ or $\mathbb{T}^m$, then the Lebesgue measure is a Haar measure on G.

Note that a Haar measure on a locally compact abelian group is automatically bi-invariant. This need not always be true for an arbitrary locally compact group as seen in the next example.

EXAMPLE 2.1.9. Let $G = \{(a, b) \in \mathbb{R}^2 : a > 0\}$ with binary operations $(a, b)(c, d) := (ac, ad + b)$ and $(a, b)^{-1} := (\frac{1}{a}, -\frac{b}{a})$. Then, with relative topology from $\mathbb{R}^2$, G is a locally compact group, called the $ax + b$ group, as it is identified with the group of affine transformations $x \mapsto ax + b$ of the real line. The functional $I : C_c(G) \to \mathbb{C}$ defined by

$$I(f) = \int_{-\infty}^{\infty} \int_0^{\infty} f(x, y) \frac{1}{x^2} dx dy \tag{2.1.4}$$

is a Haar functional on G, and the corresponding Haar measure on G is not right invariant.

PROOF. Clearly, I is a well-defined positive linear functional on $C_c(G)$. To verify the left invariance of I, let $(a,b) \in G$ and $f \in C_c(G)$. Then

$$\begin{aligned} I(\lambda(a,b)f) &= \int_{-\infty}^{\infty}\int_0^{\infty} f((a,b)^{-1}(x,y))\frac{1}{x^2}dxdy \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} f\left(\frac{x}{a},\frac{y}{a}-\frac{b}{a}\right)\left(\frac{a}{x}\right)^2\frac{dx}{a}\frac{dy}{a} \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} f\left(u,v-\frac{b}{a}\right)\frac{1}{u^2}dudv \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} f(u,t)\frac{1}{u^2}dudt \\ &= I(f). \end{aligned}$$

Thus I is a Haar functional on G.

Note that if $a,b>0$, $R=\{(x,y)\in G : a<x<b \text{ and } c<y<d\}$, and μ is the Haar measure determined by I, then

$$\mu(R) = \int_c^d\int_a^b \frac{1}{x^2}\,dx\,dy = \left(\frac{1}{a}-\frac{1}{b}\right)(d-c).$$

In particular, if $a=2$, $b=3$, $c=0$ and $d=1$, then $\mu(R)=\frac{1}{6}$. However, since $R(2,0)$ is the rectangle $(4,6)\times(0,1)$, we have $\mu(R(2,0))=\frac{1}{12}\neq\mu(R)$. Therefore, μ is not right invariant. $\square$

In the following, we consider a very important property of functions in $C_0(G)$, called *uniform continuity*.

PROPOSITION 2.1.10. *Let $f\in C_0(G)$ and $\epsilon>0$. Then there is a neighborhood V of e such that*

$$|f(s)-f(r)|<\epsilon \tag{2.1.5}$$

for all $s,r\in G$ satisfying $s^{-1}r\in V$ or $sr^{-1}\in V$.

PROOF. Let K be a compact subset of G such that $|f(s)|<\frac{\epsilon}{2}$ for all $s\in G\backslash K$. Choose a compact neighborhood U of e. Then $F=KU$ is a compact neighborhood of K. By Lemma 1.2.10 and the continuity of f, for each $s\in F$, there exists an open neighborhood V_s of e such that $|f(s) - f(r)| < \frac{\epsilon}{2}$ for all $r \in s(V_s)^2$. Since F is compact and $\{sV_s : s \in F\}$ is an open cover of F, we have $F \subset \cup_{i=1}^n s_i V_{s_i}$ for some $s_1, \cdots, s_n \in F$.

Let V be a symmetric neighborhood of e such that $V \subset U \cap (\cap_{i=1}^n V_{s_i})$. If $s \in F$ and $r \in sV$, then $s \in s_i V_{s_i}$ for some i and thus $r \in s_i(V_{s_i})^2$, which implies that

$$|f(s) - f(r)| \leq |f(s) - f(s_i)| + |f(s_i) - f(r)| < \epsilon.$$

If $s \notin F$ and $r \in sV$, then $s \notin K$ and $r \notin K$; in this case, we have

$$|f(s) - f(r)| \leq |f(s)| + |f(r)| < \epsilon.$$

Therefore, for all $s \in G$ and $r \in sV$, we have $|f(s) - f(r)| < \epsilon$.

The case of $sr^{-1} \in V$ can be proved similarly, and can also be obtained by replacing the above f by $\check{f}$, where $\check{f}(s) = f(s^{-1})$. $\square$

Remark 2.1.11. Let $\{f_i\}$ be a net in $C_0(G)$ such that $f_i \to f$ pointwise on G for some $f \in C_0(G)$ and $\mathrm{supp}(f_i) \subset K$ for all i, where K is a fixed compact subset of G. The proof of Proposition 2.1.10 shows that $f_i \to f$ uniformly on G if the family $\{f_i\}$ has an *equi uniform continuity.* That is, given $\epsilon > 0$, there exists a neighborhood U of e such that $|f_i(s) - f_i(r)| < \epsilon$ for all i and all $s, r \in G$ satisfying $sr^{-1} \in U$ or $r^{-1}s \in U$. Indeed, in this case, we have $f_i \to f$ uniformly on G if and only if $\{f_i\}$ has an equi uniform continuity.

In Example 2.1.9, we investigated a Haar measure that is not right invariant. We want to investigate the extent to which a Haar measure on a locally compact group G fails to be right invariant. A useful tool is a particular nonzero function on G, called the *modular function of* G.

Definition 2.1.12. Let $\{f_i\}$ be a net in $C_c(G)$ and $f \in C_c(G)$. We say that $\{f_i\}$ converges to f in the *inductive limit topology* if

(i) there exists a compact subset K of G such that $\mathrm{supp}(f_i) \subset K$ eventually;

(ii) $f_i \to f$ uniformly on G.

Remark 2.1.13. If $f_i \to f$ in the inductive limit topology on $C_c(G)$, then $\{f_i\}$ has an equi uniform continuity (cf. Remark 2.1.11) and $\|f_i - f\|_1 \to 0$.

For any $r \in G$ and any function f on G, we write $(\rho(r)f)(s) = f(sr)$ $(s \in G)$.

PROPOSITION 2.1.14. *Let μ be a Haar measure on a locally compact group G. Then there is a continuous homomorphism $\triangle : G \to \mathbb{R}^+$ such that*

$$\triangle(r)\int_G f(sr)d\mu(s) \quad = \quad \int_G f(s)d\mu(s) \tag{2.1.6}$$

for all $f \in C_c(G)$ and $r \in G$. That is,

$$I(\rho(r)f) = \frac{I(f)}{\triangle(r)} \quad \text{for all } r \in G \text{ and } f \in C_c(G).$$

This function $\triangle$ is independent of the choice of a Haar measure on G and is called the modular function *of G.*

PROOF. Let $r \in G$ and let $J_r : C_c(G) \to \mathbb{C}$ be defined by

$$J_r(f) = \int_G f(sr)d\mu(s) \quad (f \in C_c(G)).$$

Then

$$J_r(\lambda(t)f) \quad = \quad J_r(f) \quad \text{for all } f \in C_c(G) \text{ and } t \in G.$$

The uniqueness of Haar measure (or Haar functional) implies that there exists a positive scalar $\triangle(r)$ such that (2.1.6) holds. Furthermore, for all $r, s \in G$ and open sets E in G, we have

$$\triangle(rs)\mu(E) = \mu(Ers) = \triangle(s)\mu(Er) = \triangle(r)\triangle(s)\mu(E).$$

It follows that $\triangle(rs) = \triangle(r)\triangle(s)$; that is, $\triangle : G \to \mathbb{R}^+$ is a homomorphism.

Suppose that $r_i \to r$ in G. Then r_i is eventually in a compact neighborhood N of r. Choose $f \in C_c(G)$ such that $\int_G f d\mu \neq 0$. Let $g_i = \rho(r_i)f$ and $g = \rho(r)f$. By Proposition 2.1.10, we have $g_i \to g$ uniformly on G. Let K be the support of f. Then KN^{-1} is compact and $\mathrm{supp}(g_i) \subset KN^{-1}$ eventually. Thus, $g_i \to g$ in the inductive limit topology of $C_c(G)$. By Remark 2.1.13, we have

$$\int_G g_i d\mu \to \int_G g d\mu.$$

That is, $\triangle(r_i)^{-1}\int_G f d\mu \to \triangle(r)^{-1}\int_G f d\mu$. It follows that $\triangle(r_i) \to \triangle(r)$, since $\int_G f d\mu \neq 0$. Therefore, $\triangle : G \to \mathbb{R}^+$ is continuous.

It is clear that $\triangle$ is independent of the choice of a Haar measure on G. $\square$

EXAMPLE 2.1.15. The modular function of the $ax+b$ group is given by

$$\triangle((a,b)) = \frac{1}{a}.$$

In fact, for $(a,b) \in G$ and $f \in C_c(G)$, we have

$$\begin{aligned} I(\rho(a,b)f) &= \int_{-\infty}^{\infty}\int_0^{\infty} f(ax, bx+y)\frac{1}{x^2}\, dx\ dy \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} f(u, \frac{b}{a}u+y)\frac{1}{u^2}\, du\ ady \\ &= a\int_{-\infty}^{\infty}\int_0^{\infty} f(u, y+\frac{b}{a}u)\frac{1}{u^2}\, du\ dy \\ &= aI(f). \end{aligned}$$

Choose $f \in C_c(G)$ such that $I(f) > 0$. Then Proposition 2.1.14 implies that $\triangle((a,b)) = \frac{I(f)}{I(\rho(a,b)f)} = \frac{1}{a}$.

REMARK 2.1.16. It follows from Proposition 2.1.14 that

$$\mu(Er) = \triangle(r)\mu(E)$$

for all $r \in G$ and measurable sets E in G. Therefore, a Haar measure μ is bi-invariant if and only if $\triangle \equiv 1$; in this case, the group G is called *unimodular.*

Obviously, every abelian group is unimodular, and so is any discrete group (cf. Example 2.1.7). We show below that every compact group is also unimodular.

PROPOSITION 2.1.17. *If G is compact, then G is unimodular.*

PROOF. Since G is compact and $\triangle$ is a continuous homomorphism, $\triangle(G)$ is a compact subgroup of $\mathbb{R}^+ = (0,\infty)$. Assume that $\triangle$ is not identically one. Then $\triangle(s) \neq 1$ for some $s \in G$. Then we obtain that $\triangle(s^n) = \triangle(s)^n \to \infty$ if $\triangle(s) > 1$, and $\triangle(s^n) = \triangle(s)^n \to 0$ if $\triangle(s) < 1$, a contradiction. $\square$

PROPOSITION 2.1.18. *For all $f \in C_c(G)$, we have*

$$\int_G f(s^{-1})\triangle(s^{-1})d\mu(s) = \int_G f(s)d\mu(s). \tag{2.1.7}$$

Therefore, a left Haar measure and a right Haar measure are mutually absolutely continuous.

In particular, if ν is the right Haar measure defined by $\nu(E) = \mu(E^{-1})$, then

$$\frac{d\nu}{d\mu}(s) \quad = \quad \triangle(s^{-1}).$$

PROOF. Define $J : C_c(G) \to \mathbb{C}$ by

$$J(f) \quad = \quad \int_G f(s^{-1})\triangle(s^{-1})d\mu(s).$$

Then J is a positive linear functional on $C_c(G)$, and $J(\lambda(r)f) = J(f)$ for all $r \in G$ and $f \in C_c(G)$. Therefore, there exists $c > 0$ such that $J(f) = c\int_G f d\mu$ for all $f \in C_c(G)$. Choose $g \in C_c(G)^+$ with $\int_G g d\mu > 0$. Then

$$h(s) \quad := \quad g(s) + \triangle(s^{-1})g(s^{-1})$$

defines a function in $C_c(G)$ such that $h(s) = \triangle(s^{-1})h(s^{-1})$. Now, $\int_G h d\mu = J(h) = c\int_G h$ implies that $c = 1$, and thus (2.1.7) holds.

Let ν be a right Haar measure. By the uniqueness of Haar measure, we can assume that ν is given by $\nu(E) = \mu(E^{-1})$. Then we have

$$\int_G f d\mu = \int_G f(s^{-1})\triangle(s^{-1})d\mu(s) = \int_G f(s)\triangle(s)d\nu(s).$$

Suppose $\nu(E) = 0$. Then $\mu(E) = 0$, since for any compact $K \subset E$, we have

$$\mu(K) = \int_G 1_K(s)d\mu(s) = \int_G 1_K(s)\triangle(s)d\nu(s) = \int_K \triangle(s)d\nu(s) = 0.$$

Similarly, we have $\mu(E) = 0 \Rightarrow \nu(E) = 0$. Therefore, μ and ν are mutually absolutely continuous.

Finally, the equality

$$\int_G 1_E(s)d\mu(s) \quad = \quad \int_G 1_E(s)\triangle(s)d\nu(s)$$

holds for all Borel sets E in G, which implies that $d\mu(s) = \triangle(s)d\nu(s)$. $\square$

Note that the locally compact group G is not assumed to be σ-compact so that the Haar measure μ of G is not necessarily σ-finite. Thus the usual Radon-Nikodym Theorem (cf. [**12**, Theorem 6.10]) can not be applied. However, Proposition 2.1.18

shows that $\mu(A) = \int_A \triangle(s) d\nu(s)$ for all measurable subsets A of G, and $\triangle$ is indeed the Radon-Nikodym derivative of μ with respect to ν.

2.2. Abelian Harmonic Analysis

In this section, G will always be a locally compact *abelian* group with a fixed Haar measure μ, though many results in this section hold for all locally compact groups.

Suppose that f and g are L^1-functions on G. It follows that $(s, r) \mapsto f(r)g(r^{-1}s)$ is a measurable function with respect to the product measure $\mu \times \mu$, and $F(s, r) = f(r)g(r^{-1}s)$ defines a function in $L^1(G \times G)$. For this, it suffices to note that f and g have σ-finite supports, and thus F is supported on the σ-finite set $\mathrm{supp}(f) \times (\mathrm{supp}(f) \cdot \mathrm{supp}(g))$. Then we can apply the Fubini Theorem.

Proposition 2.2.1. *Let G be a locally compact abelian group, and let f and g be functions in $L^1(G)$.*

(1) *The function defined by*

$$f * g(s) \;=\; \int_G f(r)g(r^{-1}s)d\mu(r) \tag{2.2.1}$$

is in $L^1(G)$, and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

(2) *Proposition* 2.1.18 *guarantees that the function defined by*

$$f^*(s) \;=\; \overline{f(s^{-1})} \tag{2.2.2}$$

is in $L^1(G)$, and $\|f^*\|_1 = \|f\|_1$.

(3) *$f * g = g * f$ and $(f * g)^* = g^* * f^*$.*

Corollary 2.2.2. *$L^1(G)$ is an involutive commutative Banach algebra with respect to the operations* (2.2.1) *and* (2.2.2).

Proposition 2.2.3. *The involutive Banach algebra $L^1(G)$ has an approximate identity $\{u_i\}$ in $C_c(G)$ such that for each i, $u_i^* = u_i$ and $\|u_i\|_1 = 1$.*

Proof. Let $\mathcal{D}$ be the collection of all compact neighborhoods of e with the direction $V_1 \preceq V_2$ if and only if $V_2 \subset V_1$. For each $V \in \mathcal{D}$, let $u_V \in C_c(G)^+$ be such that $\mathrm{supp}(u_V) \subset V$, $u_V^* = u_V$, and $\|u_V\|_1 = 1$. Then $\{u_V\}$ is a net in $C_c(G)$, and we show below that $\{u_V\}$ is an approximate identity of $L^1(G)$.

Let $f \in C_c(G)$ and $\epsilon > 0$. Let $K = supp(f)$. By Proposition 2.1.10, there exists $V_1 \in \mathcal{D}$ such that $|f(s) - f(r)| < \frac{\epsilon}{(\mu(K)+1)}$ whenever $r^{-1}s \in V_1$. Choose $V_0 \in \mathcal{D}$ such that $V_0 \subset V_1$ and $\mu(V_0K) < \mu(K) + 1$. Then for all $V \in \mathcal{D}$ with $V \subset V_0$, we have

$$\begin{aligned}
\|u_V * f - f\| &= \int_G \left| \int_G u_V(r)(f(r^{-1}s) - f(s))d\mu(r) \right| d\mu(s) \\
&\leq \int_G \Big(\int_V u_V(r)|f(s) - f(r^{-1}s)|d\mu(r) \Big) d\mu(s) \\
&= \int_{VK} \int_V u_V(r)|f(s) - f(r^{-1}s)|d\mu(r)d\mu(s) \\
&\leq \frac{\epsilon\mu(VK)}{(\mu(K)+1)} < \epsilon.
\end{aligned}$$

Since $C_c(G)$ is dense in $L^1(G)$, we have $g * u_V = u_V * g \to g$ for all $g \in L^1(G)$. $\square$

Recall that the spectrum $\sigma(L^1(G))$ of $L^1(G)$ is a locally compact Hausdorff space with the relative weak*-topology from Ball$(L^1(G)^*)$. We will use $\triangle(G)$ to denote $\sigma(L^1(G))$.

Definition 2.2.4. Let $\widehat{G}$ denote the set of continuous homomorphisms from G to the circle group $\mathbb{T}$. Under pointwise multiplication, $\widehat{G}$ is a group, called the *character group* of G or the *Pontryagin dual* of G.

Lemma 2.2.5. *If $\omega \in \widehat{G}$ and $h_\omega : L^1(G) \to \mathbb{C}$ is defined by*

$$h_\omega(f) = \int_G f(s)\omega(s)d\mu(s), \tag{2.2.3}$$

then $h_\omega \in \triangle(G)$.

Proof. It is clear that the map $h_\omega : L^1(G) \to \mathbb{C}$ is a homomorphism. Choose $g \in L^1(G)$ such that $\int_G g d\mu \neq 0$. Let $f(s) = g(s)\omega(s^{-1})$. Then $f \in L^1(G)$ and $h_\omega(f) = \int_G g d\mu \neq 0$. Therefore, $h_\omega \in \triangle(G)$. $\square$

Proposition 2.2.6. *The map $\omega \mapsto h_\omega$ is a bijection of $\widehat{G}$ onto $\triangle(G)$.*

Proof. If $h_\omega = h_{\omega'}$, then

$$\int_G f(s)(\omega(s) - \omega'(s))d\mu(s) = 0 \quad \text{for all } f \in L^1(G).$$

In particular, we get $f\omega = f\omega'$ for all $f \in C_c(G)$. Thus, $\omega = \omega'$. This proves that $\omega \mapsto h_\omega$ is one-to-one.

Next, we claim that

$$\lambda(r)u * f \quad = \quad u * \lambda(r)f \tag{2.2.4}$$

for all $u, f \in L^1(G)$. Indeed, for any $s \in G$, we have

$$\begin{aligned}
(\lambda(r)u * f)(s) \quad &= \quad \int_G \lambda(r)u(t)f(t^{-1}s)d\mu(t) \quad = \quad \int_G u(r^{-1}t)f(t^{-1}s)d\mu(t) \\
&= \quad \int_G u(t)f(t^{-1}r^{-1}s)d\mu(t) \quad = \quad \int_G u(t)(\lambda(r)f)(t^{-1}s)d\mu(t) \\
&= \quad (u * \lambda(r)f)(s).
\end{aligned}$$

Now let $h \in \triangle(G)$ and let $\{u_i\} \subset C_c(G)$ be an approximate identity for $L^1(G)$. Then (2.2.4) implies that for all $f \in L^1(G)$, we have

$$h(\lambda(r)u_i * f) = h(u_i * \lambda(r)f) = h(u_i)h(\lambda(r)f) \to h(\lambda(r)f).$$

If $h(f) \neq 0$, then there is $\omega_h(r) \in \mathbb{C}$ such that

$$h(\lambda(r)u_i) \;\to\; \omega_h(r) \;:=\; \frac{h(\lambda(r)f)}{h(f)}. \tag{2.2.5}$$

Since the left hand side of (2.2.5) is independent of our choice of f, $\omega_h(r) = \frac{h(\lambda(r)g)}{h(g)}$ for all $g \in L^1(G)$ with $h(g) \neq 0$.

Notice that

$$|\omega_h(r)| \quad = \quad \left|\frac{h(\lambda(r)f)}{h(f)}\right| \;\leq\; \frac{\|f\|_1}{|h(f)|}, \tag{2.2.6}$$

and thus $\|\omega_h\|_\infty < \infty$. Replacing g by $\lambda(s)f$ in $\lambda(r)f * g = f * \lambda(r)g$, we get

$$(\lambda(r)f) * (\lambda(s)f) \quad = \quad f * \lambda(rs)f,$$

which, after applying h to both sides and dividing by $h(f)^2$, gives

$$\frac{h(\lambda(r)f)}{h(f)}\frac{h(\lambda(s)f)}{h(f)} \quad = \quad \frac{h(\lambda(rs)f)}{h(f)}.$$

Therefore, we have $\omega_h(r)\omega_h(s) = \omega_h(rs)$ for all $r, s \in G$. In particular, $\omega_h(s) \neq 0$ for all $s \in G$ as $\omega_h(e) = 1$. This shows that ω_h is a homomorphism of G into $\mathbb{C} \setminus \{0\}$.

Since translation is continuous in $L^1(G)$ (by Proposition 2.1.10) and

$$|\omega_h(s) - \omega_h(r)| = \tfrac{1}{|h(f)|}|h(\lambda(s)f - \lambda(r)f)| \leq \tfrac{1}{|h(f)|}\|\lambda(s)f - \lambda(r)f\|_1,$$

it follows that ω_h is continuous.

If $|\omega_h(s)| \neq 1$, we can assume that $|\omega_h(s)| > 1$ (otherwise replace s by s^{-1}), and obtain that $|\omega_h(s^n)| = |\omega_h(s)|^n \to \infty$, contradicting that $\|\omega_h\|_\infty < \infty$. Therefore, we must have $\omega(G) \subset \mathbb{T}$. Therefore, ω_h is a character of G.

All that remain to show is that $h_{\omega_h} = h$. As $h \in \triangle(G)$ is a bounded linear functional on $L^1(G)$, by [**1**, Theorem 31.16], there is an $\alpha \in L^\infty(G)$, such that

$$h(f) \;=\; \int_G f(s)\alpha(s)d\mu(s).$$

Fix $g \in C_c(G)$ such that $h(g) \neq 0$. The proof of Lemma 2.2.3 shows that

$$\|u_i * \lambda(s)g - \lambda(s)g\|_1 \to 0$$

uniformly for $s \in G$. Thus, $h(u_i * \lambda(s)g) \to h(\lambda(s)g)$ uniformly for $s \in G$. That is, $h(\lambda(s)u_i) \to \omega_h(s)$ uniformly for $s \in G$. Hence, for all $f \in L^1(G)$, we have

$$\begin{aligned}
h_{\omega_h}(f) = \int_G f(s)\omega_h(s)d\mu(s) &= \lim_i \int_G f(s)h(\lambda(s)u_i)d\mu(s) \\
&= \lim_i \int_G \int_G f(s)u_i(s^{-1}r)\alpha(r)\; d\mu(r)\; d\mu(s) \\
&= \lim_i \int_G (f * u_i)(r)\alpha(r)d\mu(r) \\
&= \lim_i h(f * u_i) = h(f).
\end{aligned}$$

It follows that $h_{\omega_h} = h$. $\square$

Proposition 2.2.6 shows that $\triangle(G) \;=\; \{h_\omega : \omega \in \widehat{G}\}$, where h_ω is as in (2.2.3).

LEMMA 2.2.7. *The map* $(s, h) \mapsto \omega_h(s)$ *is continuous from* $G \times \triangle(G)$ *to* $\mathbb{T}$.

PROOF. Let $(r_i, h_i) \to (r, h)$ in $G \times \triangle(G)$. Choose $f \in L^1(G)$ such that $h(f) \neq 0$. Then we have

$$\begin{aligned}
& |h(f)| \cdot |\omega_{h_i}(r_i) - \omega_h(r)| \\
\leq\ & |\omega_{h_i}(r_i)(h(f) - h_i(f))| + |\omega_{h_i}(r_i)h_i(f) - \omega_h(r)h(f)| \\
\leq\ & |h(f) - h_i(f)| + |h_i(\lambda(r_i)f) - h(\lambda(r)f)| \\
\leq\ & |h(f) - h_i(f)| + |h_i(\lambda(r_i)f) - h_i(\lambda(r)f)| + |h_i(\lambda(r)f) - h(\lambda(r)f)| \\
\leq\ & |h(f) - h_i(f)| + \|\lambda(r_i)f - \lambda(r)f\|_1 + |h_i(\lambda(r)f) - h(\lambda(r)f)|.
\end{aligned}$$

The assertion follows by the continuity of translation on $L^1(G)$ and the definition of the topology on $\triangle(G)$. $\square$

In view of Proposition 2.2.6, we can identify $\widehat{G}$ with $\triangle(G)$ and give $\widehat{G}$ the topology obtained from the weak*-topology on $\triangle(G)$.

PROPOSITION 2.2.8. *The weak*-topology on $\widehat{G}$ coincides with the topology of uniform convergence on compact sets in G.*

PROOF. Suppose $\omega_{h_i} \to \omega_h$ uniformly on compacta. Let $f \in C_c(G)$ and let $K = \mathrm{supp}(f)$, which is compact. Then

$$\begin{aligned}
|h_i(f) - h(f)| &= \Big| \int_G f(s)\omega_{h_i}(s)d\mu(s) - \int_G f(s)\omega_h(s)d\mu(s) \Big| \\
&\leq \int_K |f(s)||\omega_{h_i}(s) - \omega_h(s)|d\mu(s) \to 0.
\end{aligned}$$

Since $C_c(G)$ is dense in $L^1(G)$, it follows that $h_i \to h$ in the induced weak*-topology on $\widehat{G}$.

Conversely, assume that $h_i \to h$ in $\triangle(G)$ and there is a compact set C in G on which $\omega_{h_i} \not\to \omega_h$ uniformly. Passing to a subnet, we can assume that there exists $\epsilon > 0$ and a net $\{s_i\}$ in C such that

$$|\omega_{h_i}(s_i) - \omega_h(s_i)| \geq \epsilon$$

for all i. Passing to another subnet, we can assume that $s_i \to s$ in C. Then Lemma 2.2.7 implies that $|\omega_{h_i}(s_i) - \omega_h(s)| \to 0$. Thus

$$|\omega_{h_i}(s_i) - \omega_h(s_i)| \leq |\omega_{h_i}(s_i) - \omega_h(s)| + |\omega_h(s_i) - \omega_h(s)| \to 0,$$

a contradiction. □

Corollary 2.2.9. *Suppose that G is a locally compact abelian group. Then the character group $\widehat{G}$ of G is a locally compact abelian group in the topology of uniform convergence on compact sets in G.*

Proof. By Lemma 2.2.8, $\widehat{G} \cong \triangle(G)$ is a locally compact Hausdorff space. It is clear that $\widehat{G}$ is a group under pointwise multiplication. Let $k_i \to k$ and $t_i \to t$ in $\widehat{G}$, and let C be a compact set in G. Then for all $g \in C$, we have

$$\begin{aligned} |(k_i t_i)(g) - (kt)(g)| &= |k_i(g)t_i(g) - k(g)t(g)| \\ &\leq |k_i(g)t_i(g) - k_i(g)t(g)| + |k_i(g)t(g) - k(g)t(g)| \\ &\leq |t_i(g) - t(g)| + |k_i(g) - k(g)| \to 0 \end{aligned}$$

and

$$|t_i^{-1}(g) - t^{-1}(g)| = \left|\tfrac{1}{t_i(g)} - \tfrac{1}{t(g)}\right| = \left|\tfrac{t(g)-t_i(g)}{t_i(g)t(g)}\right| = |t(g) - t_i(g)| \to 0.$$

Thus $\widehat{G}$ is a topological group. □

Since $\widehat{G}$ is a locally compact abelian group, $\widehat{G}$ also has a dual. The next result, cited from [**7**], establishes the relationship between a locally compact abelian group G and the dual $\widehat{\widehat{G}}$ of its character group $\widehat{G}$.

Theorem 2.2.10 (**Pontryagin Duality Theorem**). *The map $\Phi : G \to \widehat{\widehat{G}}$, defined by $\Phi(x)(\omega) = \omega(x)$, is an isomorphism of topological groups.*

The Gelfand transform on a commutative Banach algebra A maps $a \in A$ to the function $\hat{a} \in C_0(\triangle(A))$ defined by $\hat{a}(h) = h(a)$. When $A = L^1(G)$, under the identification $\widehat{G} \cong \triangle(G) = \triangle(L^1(G))$, the Gelfand transform of $f \in L^1(G)$ is given by the function $\hat{f} \in C_0(\widehat{G})$ defined by

$$\hat{f}(\omega) = \int_G f(s)\omega(s)d\mu(s), \tag{2.2.7}$$

and $\hat{f}$ is known as the *Fourier transform of* f.

2.3. Non-abelian Harmonic Analysis

In the abelian harmonic analysis, the Pontryagin Duality Theorem identifies a locally compact abelian group G with its bidual $\widehat{\widehat{G}}$. It is natural to want to generalize this result to all locally compact groups.

However, for a general locally compact group G, there may not be enough characters to yield a substantial information on G. Therefore, we have to expand the notion of characters. The natural analogue of characters will be the unitary representations and irreducible representations which will be defined below.

DEFINITION 2.3.1. A *unitary representation* of a locally compact group G is a continuous homomorphism $U : G \to U(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $U(\mathcal{H})$ is equipped with the SOT. The *dimension* d_U of U is defined to be $dim(\mathcal{H})$.

In this case, the representation U is said to be *equivalent* to another unitary representation $V : G \to U(\mathcal{K})$ if there is a unitary $W : \mathcal{H} \to \mathcal{K}$ that satisfies $V_s W = W U_s$ for all $s \in G$.

EXAMPLE 2.3.2. Let G be a locally compact group and $\mathcal{H} = L^2(G)$. If $r \in G$, then $\lambda(r)$ and $\rho(r)$ are unitary operators on $L^2(G)$, where

$$\lambda(r)f(s) = f(r^{-1}s) \quad \text{and} \quad \rho(r)f(s) = \triangle(r)^{\frac{1}{2}} f(sr).$$

Since translation is continuous in $L^2(G)$, it follows that $\lambda : G \to U(L^2(G))$ and $\rho : G \to U(L^2(G))$ are unitary representations of G, called the *left regular representation* and the *right regular representation*, respectively.

DEFINITION 2.3.3. Let $U : G \to U(\mathcal{H})$ be a unitary representation. A closed linear subspace X of $\mathcal{H}$ is *invariant* for U if $U_s X \subset X$ for all $s \in G$. If $\{0\}$ and $\mathcal{H}$ are the only invariant subspaces for U, then U is *irreducible.*

DEFINITION 2.3.4. For a locally compact group G, the dual $\widehat{G}$ of G is defined to be the set of equivalent classes of irreducible representations of G.

If a locally compact group G is abelian, then all irreducible representations of G are one-dimensional and correspond to characters of G (cf. [**7**, Corollary 3.6]). For compact groups G, the Peter-Weyl Theorem (cf. [**7**, Theorems 5.2 and 5.12]) shows that every irreducible representation of G is finite dimensional. However, there are locally compact groups with infinite dimensional irreducible representations.

To generalize the Fourier transform to accommodate non-abelian groups, we have to replace ω in (2.2.7) with certain infinite-dimensional representation U on a Hilbert space $\mathcal{H}$. In this way, the integrand in (2.2.7) is taking values in $B(\mathcal{H})$. Vector-valued integrals will be discussed in the next chapter.

In Section 2.2, we see that if G is a locally compact abelian group, then the character group $\widehat{G}$ of G is isomorphically homeomorphic to the spectrum $\triangle(G)$ of $L^1(G)$. Now we consider general (non-abelian) locally compact groups. The material used below is based on Folland [**7**].

First, any unitary representation $U : G \to U(\mathcal{H})$ of a locally compact group G determines a representation of $L^1(G)$, still denoted by U, on the same Hilbert space. In fact, if $f \in L^1(G)$, we can define the bounded operator $U(f)$ on $\mathcal{H}$ by

$$U(f) = \int_G f(s)U_s d\mu(s).$$

This is the unique element of $B(\mathcal{H})$ satisfying

$$(U(f)u \mid v) = \int_G f(s)(U_s u \mid v)d\mu(s) \quad \text{for all } u, v \in \mathcal{H}.$$

See Chapter 3 for more information on vector-valued integration.

Next, for $f \in L^1(G)$, we define

$$\|f\|_* = \sup_{[U]\in\widehat{G}} \|U(f)\|.$$

By [**7**, Corollary 7.2], $\|\cdot\|_*$ is a norm on $L^1(G)$. We also have

$$\|f * g\|_* = \sup_{[U]\in\widehat{G}} \|U(f)U(g)\| \le \|f\|_*\|g\|_*,$$

$$\|f^*\|_* = \sup_{[U]\in\widehat{G}} \|U(f)^*\| = \|f\|_*,$$

and

$$\|f^* * f\|_* = \sup_{[U]\in\widehat{G}} \|U(f)^*U(f)\| = \sup_{[U]\in\widehat{G}} \|U(f)\|^2 = \|f\|_*^2.$$

Then the completion of $L^1(G)$ with respect to the norm $\|\cdot\|_*$ is a C*-algebra, which is called the *group C*-algebra of G* and is denoted by $C^*(G)$.

EXAMPLE 2.3.5. Suppose G is abelian. Then $(L^1(G), \|\cdot\|_*) \to C_0(\widehat{G})$, $f \mapsto \hat{f}$ is an isometry, since

$$\|f\|_* = \sup_{[U]\in\widehat{G}} \|\hat{f}(U)\| = \|\hat{f}\|_{\sup}.$$

This extends to a $*$-isomorphism $\mathrm{C}^*(G) \to C_0(\widehat{G})$, so we have $C^*(G) \cong C_0(\widehat{G})$.

CHAPTER 3

Vector-Valued Integration

Although some proofs in this chapter are constructed by the author, materials contained in this section are mainly based on William [**17**]. In order to make sense of integrals where the integrand is a function taking values in a normed space, we need a workable theory of what is referred to as *vector-valued* integration.

3.1. Vector-Valued Integration - the Norm Continuous Case

Let $\mathcal{D}$ be a Banach space. The idea is to assign to each function $f \in C_c(G, \mathcal{D})$ an element $I(f)$ of $\mathcal{D}$, which is meant to be the integral of f and denoted by

$$\int_G f(s)d\mu(s).$$

Naturally, we want I to be linear and bounded in some sense.

Note that if $f \in C_c(G, \mathcal{D})$, then $s \mapsto \|f(s)\|$ is in $C_c(G)$ and

$$\|f\|_1 \ := \ \int_G \|f(s)\|d\mu(s) \ \left(\leq \|f\|_\infty \cdot \mu(supp\ f) \ < \ \infty \right)$$

defines a norm on $C_c(G, \mathcal{D})$.

If $z \in C_c(G)$ and $a \in \mathcal{D}$, then the function $s \mapsto z(s)a$ in $C_c(G, \mathcal{D})$ is called an *elementary tensor* and will be denoted by $z \otimes a$. Clearly, $\|z \otimes a\|_1 = \|z\|_1 \|a\|$.

We shall need the uniform continuity property of functions in $C_c(G, \mathcal{D})$. This is stated in the lemma below, and can be proved following the arguments given in the proof of Proposition 2.1.10.

Proposition 3.1.1. *Let $f \in C_c(G, \mathcal{D})$ and $\epsilon > 0$. Then there is a neighborhood V of e in G such that*

$$\|f(s) - f(r)\| < \epsilon$$

for all $s, t \in G$ satisfying $s^{-1}r \in V$ or $sr^{-1} \in V$.

LEMMA 3.1.2. *Suppose that $\mathcal{D}_0$ is a dense subset of a Banach space $\mathcal{D}$. Then*

$$C_c(G) \odot \mathcal{D}_0 := span\{z \otimes a : z \in C_c(G) \text{ and } a \in \mathcal{D}_0\} \tag{3.1.1}$$

is dense in $C_c(G, \mathcal{D})$ in the inductive limit topology, and therefore in the L^1-norm on $C_c(G, \mathcal{D})$.

PROOF. Let $f \in C_c(G, \mathcal{D})$ with $K = \text{supp}(f)$. Fix a compact neighborhood W of e in G. Let $\epsilon > 0$. Using Lemma 3.1.1, choose an open neighborhood $V \subset W$ of e such that $sr^{-1} \in V$ implies that

$$\|f(s) - f(r)\| < \frac{\epsilon}{2}.$$

Then there are $r_i \in K$ such that $K \subset \bigcup_{i=1}^n V r_i$. By Proposition 1.4.7, there are continuous $z_i : G \to [0,1]$ such that $\text{supp}(z_i) \subset V r_i$ $(1 \le i \le n)$, $\sum_{i=1}^n z_i = 1$ on K and $\sum_{i=1}^n z_i \le 1$ on G. Let $x_i \in \mathcal{D}_0$ be such that $\|x_i - f(r_i)\| < \epsilon/2$ $(1 \le i \le n)$.

Let $g = \sum_{i=1}^n z_i \otimes x_i$ and let $s \in G$. Since for each i, $0 \le z_i(s) \le 1$ and $\text{supp}(z_i) \subset V r_i$, and noticing that $\sum_{i=1}^n z_i(s) \le 1$ and $K = \text{supp}(f)$, we have

$$\begin{aligned} \|f(s) - g(s)\| &= \Big\| \sum_{i=1}^n z_i(s)(f(s) - x_i) \Big\| \\ &= \Big\| \sum_{s \in V r_i} z_i(s)((f(s) - f(r_i)) + (f(r_i) - x_i)) \Big\| \\ &\le \sum_{s \in V r_i} z_i(s)(\epsilon/2 + \epsilon/2) \\ &\le \epsilon. \end{aligned}$$

For this $g_\epsilon = g$, we have $\text{supp}(g_\epsilon) \subset WK$, a fixed compact set depending on f only, and $\|f - g_\epsilon\|_1 \le \epsilon \mu(WK)$. Therefore, $C_c(G) \odot \mathcal{D}_0$ is dense in $C_c(G, \mathcal{D})$ in the inductive limit topology as well as in the L^1-norm. □

For $f \in C_c(G, \mathcal{D})$ and $\varphi \in \mathcal{D}^*$, we define

$$\hat{L}_f(\varphi) = \int_G \varphi(f(s)) d\mu(s).$$

Since $\varphi \mapsto \hat{L}_f(\varphi)$ is linear and

$$|\hat{L}_f(\varphi)| \le \|\varphi\| \|f\|_1, \tag{3.1.2}$$

$\hat{L}_f \in \mathcal{D}^{**}$ and $\|\hat{L}_f\| \le \|f\|_1$. Let $\widehat{\ } : \mathcal{D} \to \mathcal{D}^{**}$ be the canonical isometric embedding (that is, $\hat{a}(\varphi) = \varphi(a)$). It is straightforward to check that

$$\hat{L}_{z\otimes a} \;=\; c\hat{a}, \quad \text{where } c \;=\; \int_G z(s)d\mu(s).$$

Notice that if $f_i \to f$ in the inductive limit topology on $C_c(G,\mathcal{D})$, then for all $\varphi \in \mathcal{D}^*$, $\varphi \circ f_i \to \varphi \circ f$ in the inductive limit topology on $C_c(G)$, and hence we have $\hat{L}_{f_i}(\varphi) \to \hat{L}_f(\varphi)$; that is, $\hat{L}_{f_i} \to \hat{L}_f$ in the weak*-topology in $\mathcal{D}^{**}$.

We show below that $f \mapsto \hat{L}_f$ indeed maps $C_c(G,\mathcal{D})$ into the canonical image $\widehat{\mathcal{D}}$ of $\mathcal{D}$ in $\mathcal{D}^{**}$. The idea of the proof is from [**13**, §4.5].

Lemma 3.1.3. *If $f \in C_c(G,\mathcal{D})$, then $\hat{L}_f \in \widehat{\mathcal{D}}$.*

Proof. To show that $\hat{L}_f$ is weak*-continuous, it suffices to show that if $\{\varphi_i\}$ is a net in $\mathcal{D}^*$ such that $\|\varphi_i\| \le 1$ and $\varphi_i \to 0$ in the weak*-topology, then $\hat{L}_f(\varphi_i) \to 0$. To this end, let K be the compact support of f. We note that $\{\varphi_i \circ f\}$ is a net in $C_c(G)$, $\text{supp}(\varphi_i \circ f) \subset K$ for all i, and $\varphi_i \circ f \to 0$ pointwise. Also, $\{\varphi_i \circ f\}$ is equi uniformly continuous; that is, in Proposition 2.1.10, $V = V_\epsilon$ works for all $\varphi_i \circ f$. Thus, $\varphi_i \circ f \to 0$ uniformly on G. It easily follows that

$$|\hat{L}_f(\varphi_i)| = \Big| \int_K \varphi_i(f(s))d\mu(s)\Big| \le \int_K |\varphi_i(f(s))|d\mu(s) \to 0.$$

This shows that $\hat{L}_f$ is weak*-continuous and thus $\hat{L}_f \in \widehat{\mathcal{D}}$. $\square$

For $f \in C_c(G,\mathcal{D})$, due to Lemma 3.1.3, we will use $L_f \in \mathcal{D}$ to denote the preimage of $\hat{L}_f$ in $\mathcal{D}$, and define $\int_G f(s)d\mu(s) = L_f$.

Proposition 3.1.4. *Suppose that $\mathcal{D}$ is Banach space and G is a locally compact group with a fixed left Haar measure μ. Then*

$$C_c(G,\mathcal{D}) \to \mathcal{D}, \quad f \mapsto \int_G f(s)d\mu(s)$$

is a linear map satisfying

$$\varphi\Big(\int_G f(s)d\mu(s)\Big) \;=\; \int_G \varphi(f(s))d\mu(s), \tag{3.1.3}$$

$$\Big\|\int_G f(s)d\mu(s)\Big\| \;\le\; \|f\|_1, \; \textit{and} \; \int_G (z\otimes a)(s)d\mu(s) \;=\; a\int_G z(s)d\mu(s) \tag{3.1.4}$$

for all $f \in C_c(G, \mathcal{D})$, $\varphi \in \mathcal{D}^*$, $z \in C_c(G)$, *and* $a \in \mathcal{D}$.

Moreover, if $\mathcal{Y}$ *is a Banach space,* $L : \mathcal{D} \to \mathcal{Y}$ *is a bounded linear operator and* $f \in C_c(G, \mathcal{D})$, *then*

$$L\Big(\int_G f(s)d\mu(s)\Big) \;=\; \int_G L(f(s))d\mu(s). \tag{3.1.5}$$

PROOF. It is clear from the above discussions that the first assertion holds.

Now, let $\mathcal{Y}$ be a Banach space, $L : \mathcal{D} \to \mathcal{Y}$ be a bounded linear map, and $f \in C_c(G, \mathcal{D})$. Then $L \circ f \in C_c(G, \mathcal{Y})$ and $\Psi \circ L \in \mathcal{D}^*$ for all $\Psi \in \mathcal{Y}^*$. Therefore, by (3.1.3), for all $\Psi \in \mathcal{Y}^*$, we have

$$\Psi\Big(L\Big(\int_G f(s)d\mu(s)\Big)\Big) = \int_G \Psi(L(f(s)))d\mu(s) = \Psi\Big(\int_G L(f(s))d\mu(s)\Big),$$

and thus (3.1.5) follows. $\square$

REMARK 3.1.5. If $\mathcal{D}$ is Banach space with an involution $*$ satisfying $\|x^*\| = \|x\|$ (e.g., $\mathcal{D}$ is a C*-algebra), then it follows from (3.1.3) that

$$\Big(\int_G f(s)d\mu(s)\Big)^* \;=\; \int_G f(s)^* d\mu(s) \quad \text{for all } f \in C_c(G, \mathcal{D}). \tag{3.1.6}$$

For a C*-algebra A, let $M(A)$ denote the multiplier algebra of A.

PROPOSITION 3.1.6. *Let* G *be a locally compact group with a Haar measure* μ, *let* A *be a C*-algebra, and let* $\pi : A \to B(\mathcal{H})$ *be a representation. Then for all* $f \in C_c(G, A)$, $h, k \in \mathcal{H}$, *and* $a, b \in M(A)$, *we have*

$$\Big(\pi\Big(\int_G f(s)d\mu(s)\Big)h \mid k\Big) \;=\; \int_G (\pi(f(s))h \mid k)d\mu(s), \tag{3.1.7}$$

$$\pi\Big(\int_G f(s)d\mu(s)\Big) \;=\; \int_G \pi(f(s))d\mu(s), \tag{3.1.8}$$

and

$$a\int_G f(s)d\mu(s)\, b \;=\; \int_G af(s)b\; d\mu(s). \tag{3.1.9}$$

PROOF. Given $h, k \in \mathcal{H}$ and $a, b \in M(A)$, let

$$\varphi(x) = (\pi(x)h \mid k) \;\text{ and }\; L(x) = axb \;\; (x \in A).$$

Then $\varphi \in A^*$ and $L \in B(A)$. Therefore, (3.1.7) follows from (3.1.3), and (3.1.6) and (3.1.9) follow from (3.1.5). □

COROLLARY 3.1.7. *Let G be a locally compact group and let $\mathcal{H}$ be a Hilbert space. If $f \in C_c(G, B(\mathcal{H}))$, then*

$$\Big(\int_G f(s)d\mu(s)h \mid k\Big) \;=\; \int_G (f(s)h \mid k)d\mu(s)$$

for all $h, k \in \mathcal{H}$.

3.2. Vector-Valued Integration - the Strictly Continuous Case

In Section 2.2, we mentioned the Fourier transforms over locally compact abelian groups. In order to extend this notion to an arbitrary locally compact group G, we need to consider integrals of this form

$$\int_G f(s)U_s d\mu(s), \tag{3.2.1}$$

where $f \in C_c(G)$ or $f \in C_c(G, B(\mathcal{H}))$, and $U : G \to U(\mathcal{H})$ is a unitary representation of G. But the integrand in (3.2.1) is not necessarily a continuous function when $B(\mathcal{H})$ is equipped with the operator norm topology, and then Proposition 3.1.6 is not applicable. However, this integrand is indeed continuous when $B(\mathcal{H})$ is equipped with the strong operator topology. We will show that this is enough to define a well behaved integral.

To do this in sufficient generality, we need to discuss a bit more about the multiplier algebra $M(A)$ of a C*-algebra A.

DEFINITION 3.2.1. Let A be a C^*-algebra. If $a \in A$, let $\|b\|_a = \|ba\| + \|ab\|$ $(b \in M(A))$. Then $\|\cdot\|_a$ is a seminorm on $M(A)$. The *strict topology* on $M(A)$ is the topology generated by the seminorms $\{\|\cdot\|_a : a \in A\}$.

EXAMPLE 3.2.2. A net $\{b_i\}$ in $M(A)$ converges strictly to b if and only if $ab_i \to ab$ and $b_i a \to ba$ for all $a \in A$. Since $B(\mathcal{H}) = M(\mathcal{K}(\mathcal{H}))$, $B(\mathcal{H})$ has the strict topology, and $T_i \to T$ strictly in $B(\mathcal{H})$ if and only if $T_iK \to TK$ and $KT_i \to KT$ for all compact operators K on $\mathcal{H}$.

DEFINITION 3.2.3. Let $\mathcal{H}$ be a Hilbert space. The $*$-*strong operator topology* on $B(\mathcal{H})$ has subbasic open sets

$$N(T, h, \epsilon) := \{S \in B(\mathcal{H}) : \|Sh - Th\| + \|S^*h - T^*h\| < \epsilon\},$$

where $T \in B(\mathcal{H})$, $h \in \mathcal{H}$ and $\epsilon > 0$.

Therefore, $T_i \to T$ $*$-strongly in $B(\mathcal{H})$ if and only if both $T_i \to T$ strongly and $T_i^* \to T^*$ strongly.

More generally, let X be a Hilbert A-module and let $\mathcal{L}(X)$ be the algebra of all adjointable operators on X. Then, as $\mathcal{L}(X) \cong M(\mathcal{K}(X))$, $\mathcal{L}(X)$ has the strict topology. Since $\mathcal{L}(X) = M(\mathcal{K}(X))$, it also has the $*$-strong topology, in which a net $T_i \to T$ if and only if $T_i(x) \to T(x)$ and $T_i^*(x) \to T^*(x)$ for all $x \in X$. The link between the strict topology and the $*$-strong topology is given below.

PROPOSITION 3.2.4. *Let X be a Hilbert A-module. Then strict convergence implies $*$-strong convergence, and the strict and $*$-strong topologies coincide on norm bounded subsets of $\mathcal{L}(X)$.*

Since a Hilbert space $\mathcal{H}$ is a Hilbert $\mathbb{C}$-module, we have the following immediate corollary.

COROLLARY 3.2.5. *On norm bounded subsets of $B(\mathcal{H})$, the strict and $*$-strong topologies coincide.*

For a Hilbert A-module X, let $U(\mathcal{L}(X))$ be the unitary group of $\mathcal{L}(X)$.

COROLLARY 3.2.6. *Suppose that $u : G \to U(\mathcal{L}(X))$ is a homomorphism into $U(\mathcal{L}(X))$. Then u is strictly continuous if and only if it is strongly continuous.*

PROOF. Since $U(\mathcal{L}(X))$ is a norm bounded subset of $\mathcal{L}(X)$, Proposition 3.2.4 shows that the strict and $*$-strong topologies coincide on $U(\mathcal{L}(X))$. So, the strict continuity of u implies the strong continuity of u.

Conversely, suppose u is strongly continuous. It suffices to show that for all

$x \in X$, the map $G \to X$, $s \mapsto u_s^*(x)$ is continuous. In fact, we have

$$\begin{aligned}
\|u_s^*(x) - u_t^*(x)\|_X^2 &= \|((u_s - u_t)^*(x) \mid (u_s - u_t)^*(x))\|_A \\
&= \|(u_s(u_s - u_t)^* x \mid x)_A - (x \mid (u_s - u_t)u_t^*(x))\|_A \\
&= \|((u_t - u_s)u_t^* x \mid x)_A - (x \mid (u_s - u_t)u_t^*(x))\|_A \\
&\leq 2\|x\|\|(u_s - u_t)u_t^*(x)\|.
\end{aligned}$$

Therefore, $s \mapsto u_s^*(x)$ is continuous since $s \mapsto u_s(x)$ is continuous. □

The Principle of Uniform Boundedness implies that a strongly convergent *sequence* is bounded. Therefore, a $*$-strong convergent sequence is bounded and also strictly convergent by Proposition 3.2.4. Also, we note that any C*-algebra A is strictly dense in $M_s(A)$ (cf. [**3**, Proposition 3.5]).

We will use $M_s(A)$ to denote $M(A)$ with the strict topology. Notice that if $f \in C_c(G, M_s(A))$, then $s \mapsto f(s)a$ is in $C_c(G, A)$ for each $a \in A$. In particular, $\{f(s)a : s \in G\}$ is bounded in A, since any compact subset of A is bounded. Then the Uniform Boundedness Principle implies that

$$\{\|f(s)\| : s \in G\} \text{ is bounded for all } f \in C_c(G, M_s(A)).$$

Definition 3.2.7. A representation $\pi : A \to B(\mathcal{H})$ is *non-degenerate* if

$$\{\pi(a)h : a \in A \text{ and } h \in \mathcal{H}\}$$

spans a norm dense subset of $\mathcal{H}$. In this case, π can be uniquely extended to a representation $\bar{\pi} : M(A) \to B(\mathcal{H})$ (cf. [**3**, Propositions 3.7 and 3.8]).

More generally, suppose that B is a C*-algebra. We say that a $*$-homomorphism $L : A \to M(B)$ is *non-degenerate* if

$$\{L(a)b : a \in A \text{ and } b \in B\}$$

spans a norm dense subset of B. In this case, L can be uniquely extended to a $*$-homomorphism $\bar{L} : M(A) \to M(B)$, which is injective if L is injective.

Proposition 3.2.8. *Let A be a C^*-algebra. Then there is a unique linear map*

$$C_c(G, M_s(A)) \to M(A), \quad f \mapsto \int_G f(s)d\mu(s)$$

such that for any non-degenerate representation $\pi : A \to B(\mathcal{H}_\pi)$, $h, k \in \mathcal{H}_\pi$, and $f \in C_c(G, M_s(A))$, we have

$$\Big(\bar{\pi}\Big(\int_G f(s)d\mu(s)\Big)h \mid k\Big) \;=\; \int_G (\bar{\pi}(f(s))h \mid k)d\mu(s). \tag{3.2.2}$$

Furthermore, the map $f \mapsto \int_G f(s)d\mu(s)$ satisfies

$$\Big\|\int_G f(s)d\mu(s)\Big\| \;\leq\; \|f\|_\infty \cdot \mu(supp\ f),$$

and equations (3.1.6) *and* (3.1.9) *are valid in this context. In general, if B is a C^*-algebra and $L : A \to M(B)$ is a non-degenerate $*$-homomorphism, then*

$$\bar{L}\Big(\int_G f(s)d\mu(s)\Big) \;=\; \int_G \bar{L}(f(s))d\mu(s).$$

Proof. Let $f \in C_c(G, M_s(A))$. For each $a \in A$, the function $s \mapsto f(s)a$ is in $C_c(G, A)$ and thus $\int_G f(s)a\ d\mu(s) \in A$ is defined (cf. Lemma 3.1.4).

Define $L_f : A \to A$ by $a \mapsto \int_G f(s)a\ d\mu(s)$. Then for all $a, b \in A$, by (3.1.6) and (3.1.9), we have

$$(L_f(a) \mid b)_A = L_f(a)^*b = \int_G a^*f(s)^*bd\mu(s) = a^*L_{f^*}(b) = (a \mid L_{f^*}(b))_A,$$

noticing that $f^* \in C_c(G, M_s(A))$. Therefore, $L_f \in \mathcal{L}(A) = M(A)$.

We define $\int_G f(s)d\mu(s) = L_f$. Clearly, the map

$$C_c(G, M_s(A)) \to M(A),\ f \mapsto \int_G f(s)d\mu(s)$$

is linear. It is easy to see that (3.1.6) and (3.1.9) also hold in the present context.

Now, let $\pi : A \to B(\mathcal{H}_\pi)$ be a non-degenerate representation. By Proposition 3.1.6, for all $h, k \in \mathcal{H}_\pi$ and $a \in A$, we have

$$\begin{aligned}\Big(\bar{\pi}\Big(\int_G f(s)d\mu(s)\Big)\pi(a)h \mid k\Big) &= \Big(\pi\Big(\int_G f(s)ad\mu(s)\Big)h \mid k\Big)\\ &= \int_G (\pi(f(s)a)h \mid k)d\mu(s)\\ &= \int_G (\bar{\pi}(f(s))\pi(a)h \mid k)d\mu(s).\end{aligned}$$

Then (3.2.2) follows from the non-degeneracy of π.

Note that if a representation $\pi : A \to B(\mathcal{H}_\pi)$ is injective, so is its extension

$\bar{\pi} : M(A) \to B(\mathcal{H}_\pi)$. Therefore, the uniqueness follows from (3.2.2) by taking an injective representation π.

For $a \in A$ with $\|a\| \le 1$, since $f(\cdot)a \in C_c(G, A)$, by Proposition 3.1.6, we have

$$\begin{aligned} \Big\| \int_G f(s)d\mu(s)a \Big\| &= \Big\| \int_G f(s)ad\mu(s) \Big\| \le \|f(\cdot)a\|_1 \\ &= \int_{supp\ f} \|f(s)a\|d\mu(s) \le \|f\|_\infty \cdot \mu(supp\ f). \end{aligned}$$

That is, $\Big\| \int_G f(s)d\mu(s) \Big\| \le \|f\|_\infty \cdot \mu(supp\ f)$.

Finally, let B be a C*-algebra and $L : A \to M(B)$ be a non-degenerate $*$-homomorphism. Let $\bar{L} : M(A) \to M(B)$ be the unique homomorphic extension of L. Note that $\bar{L}$ is strictly continuous and hence $\bar{L} \circ f(\cdot) \in C_c(G, M_s(B))$. For all $a \in A$, by (3.1.5), we have

$$\begin{aligned} \bar{L}\Big(\int_G f(s)d\mu(s) \Big) L(a) &= L\Big(\int_G f(s)ad\mu(s) \Big) = \int_G L(f(s)a)d\mu(s) \\ &= \int_G \bar{L}(f(s))L(a)d\mu(s) = \int_G \bar{L}(f(s))d\mu(s)L(a). \end{aligned}$$

Therefore, $\bar{L}\Big(\int_G f(s)d\mu(s) \Big) = \int_G \bar{L}(f(s))d\mu(s)$, as L is non-degenerate. □

3.3. Fubini Theorem for Vector-Valued Integration

In this section, we consider a version of the Fubini Theorem for vector-valued integrals. Since our integrands are continuous with compact support, this can be achieved by applying certain bounded linear functionals and using the Fubini Theorem for scalar-valued integrals.

LEMMA 3.3.1. *Let G be a locally compact group, let X is a locally compact space, and let $F \in C_c(X \times G, \mathcal{D})$. Then the function*

$$x \mapsto \int_G F(x, s)d\mu(s)$$

is an element of $C_c(X, \mathcal{D})$.

PROOF. It will clearly suffice to prove the continuity of the function.

Suppose that $x_i \to x$. Let K and C be compact sets in X and G, respectively, such that $\text{supp}(F) \subset K \times C$. For each $x \in X$, let $\varphi(x)$ be the element of $C_c(G, \mathcal{D})$

defined by $\varphi(x)(s) = F(x,s)$. We claim that $\varphi(x_i) \to \varphi(x)$ uniformly on G. If this were not the case, then after passing to a subnet and relabeling, we can assume that there exists $\epsilon_0 > 0$ and $r_i \in G$ for each i such that

$$\|F(x_i, r_i) - F(x, r_i)\| \geq \epsilon_0 \text{ for all } i. \tag{3.3.1}$$

We certainly have $\{r_i\} \subset C$, and since C is compact, we can assume that $r_i \to r$. Taking the limit in (3.3.1), we obtain that $0 \geq \epsilon_0$, a contradiction.

Now we can assume that for large i, we have

$$\|\varphi(x_i) - \varphi(x)\|_\infty < \frac{\epsilon_0}{\mu(C)+1}.$$

Since $\operatorname{supp}(\varphi(x_i)) \subset C$ for all i, we have

$$\Big\| \int_G F(x_i, s) d\mu(s) - \int_G F(x,s) d\mu(s) \Big\| \;\leq\; \int_C \|\varphi(x_i)(s) - \varphi(x)(s)\| \, d\mu(s) \;\leq\; \epsilon_0.$$

This completes the proof. □

The following corollary will be very useful in the sequel.

COROLLARY 3.3.2. *Let G and G' be locally compact groups, and let F be a function in $C_c(G \times G', \mathcal{D})$. Then the functions*

$$s \mapsto \int_{G'} F(s,r) d\mu_{G'}(r) \quad \textit{and} \quad r \mapsto \int_G F(s,r) d\mu_G(s)$$

are in $C_c(G, \mathcal{D})$ and $C_c(G', \mathcal{D})$, respectively.

PROPOSITION 3.3.3. *Let G be a locally compact space, let A be a C^*-algebra, and let $F \in C_c(G \times G, M_s(A))$. Then*

$$s \mapsto \int_G F(s,r) d\mu(r) \quad \textit{and} \quad r \mapsto \int_G F(s,r) d\mu(s)$$

are in $C_c(G, M_s(A))$, and the iterated integrals

$$\int_G \int_G F(s,r) d\mu(s) d\mu(r) \quad \textit{and} \quad \int_G \int_G F(s,r) d\mu(r) d\mu(s)$$

are well defined in $M(A)$ and equal.

A similar statement holds for $F \in C_c(G \times G, \mathcal{D})$ with $\mathcal{D}$ a Banach space.

PROOF. For any fixed $s \in G$, since $F(s, \cdot) \in C_c(G, M_s(A))$, the integral $\int_G F(s,r)d\mu(r) \in M(A)$ is well-defined by Lemma 3.2.8. Now the map

$$G \to M(A), \ s \mapsto \int_G F(s,r)d\mu(r)$$

has a compact support. To show that it is in $C_c(G, M_s(A))$, let $a \in A$. Then $aF(\cdot,\cdot)$ and $F(\cdot,\cdot)a$ are in $C_c(G \times G, A)$. By Corollary 3.3.2, the maps

$$s \mapsto \int_G aF(s,r)d\mu(r) = a\int_G F(s,r)d\mu(r)$$

and

$$s \mapsto \int_G F(s,r)ad\mu(r) = \int_G F(s,r)d\mu(r)a$$

are both in $C_c(G, A)$. Therefore, the map $s \mapsto \int_G F(s,r)d\mu(r)$ is in $C_c(G, M_s(A))$, and hence so is the map $r \mapsto \int_G F(s,r)d\mu(s)$.

It is clear from Lemma 3.2.8 that both iterated integrals in the proposition are defined and take values in $M(A)$. To see that they have the same value, let π be an injective non-degenerate representation of A in $B(\mathcal{H})$. Then $\bar{\pi} : M(A) \to B(\mathcal{H})$ is also injective. By (3.2.2) and the usual scalar-valued Fubini Theorem, we have

$$\begin{aligned}\Big(\bar{\pi}\Big(\int_G\int_G F(s,r)d\mu(s)d\mu(r)\Big)h \mid k\Big) &= \int_G\int_G (\bar{\pi}(F(s,r))h \mid k)d\mu(s)d\mu(r) \\ &= \int_G\int_G (\bar{\pi}(F(s,r))h \mid k)d\mu(r)d\mu(s) \\ &= \Big(\bar{\pi}\Big(\int_G\int_G F(s,r)d\mu(r)d\mu(s)\Big)h \mid k\Big)\end{aligned}$$

for all $h, k \in \mathcal{H}$. Therefore, the first assertion follows since $\bar{\pi}$ is injective.

If $\mathcal{D}$ is a Banach space and $F \in C_c(G \times G, \mathcal{D})$, then the two iterated integrals are well-defined in A by Corollary 3.3.2 and Lemma 3.1.4. In this case, the equality

$$\int_G\int_G \varphi(F(s,r))d\mu(s)d\mu(r) \quad = \quad \int_G\int_G \varphi(F(s,r))d\mu(r)d\mu(s)$$

holds for all $\varphi \in \mathcal{D}^*$ due to (3.1.3) and the scalar-valued Fubini Theorem. Hence, we conclude that the two iterated integrals in the proposition are equal. $\square$

CHAPTER 4

Dynamical Systems and Their Covariant Representations

A C^*-dynamical system consists of a C^*-algebra A, a locally compact group G and an action of G on A. In this chapter, we study C*-dynamical systems and their covariant representations.

Section 4.1 contains materials from Dummit and Foote [**6**], and the rest of this chapter is mainly based on William [**17**].

4.1. Transformation Groups

Definition 4.1.1. A group G *acts* on the left on a set X if there is a map

$$G \times X \to X, \quad (s, x) \mapsto s \cdot x \tag{4.1.1}$$

such that for all $s, r \in G$ and $x \in X$,

$$e \cdot x = x \quad \text{and} \quad s \cdot (r \cdot x) \;=\; (sr) \cdot x.$$

Definition 4.1.2. Let G and X be a group and a set, respectively, and let $\tau : G \times X \to X$ be an action of G on X. For each $g \in G$, define $\tau_g : X \to X$ by $\tau_g(x) = g \cdot x$. Then each τ_g is a bijection on X, and the set $\{\tau_g : g \in G\}$ is a group under composition of functions. This group is called a *transformation group.*

If G is a topological group, X is a topological space, and there is a continuous action $\tau : G \times X \to X$, then X is called a *left G-space* and we denote the resulting transformation group by the pair (G, X).

In this case, each τ_g is a homeomorphism on X, and

$$\varphi : G \to \mathrm{Homeo}(X), \quad g \mapsto \tau_g$$

is a group homomorphism and hence

$$(G, X) \;=\; \varphi(G) \;\cong\; G/\,\mathrm{Ker}(\varphi)$$

as groups by the first isomorphism theorem.

We shall equip (G, X) with the topology from $G/\operatorname{Ker}(\varphi)$, which is a locally compact group when G is locally compact (cf. Proposition 1.4.10). Therefore, when both G and X are locally compact and Hausdorff, we call (G, X) a *locally compact transformation group,* and X is called a *left locally compact G-space.*

Right G-spaces can be defined analogously. Since we are most concerned with left G-spaces, we will omit the word "left" in our discussions.

EXAMPLE 4.1.3. Let X be a topological space and let $h \in \operatorname{Homeo}(X)$. Then $\mathbb{Z}$ acts on X by $n \cdot x = h^n(x)$, which is continuous. Thus $(\mathbb{Z}, X)$ is a transformation group, and X is a $\mathbb{Z}$-space.

EXAMPLE 4.1.4. Suppose that G is a locally compact group and H is a closed subgroup of G. Then H is a locally compact group and acts on G by left translation. Thus (H, G) is canonically a locally compact transformation group, and G is a locally compact H-space. In particular, when $H = G$, the action for the transformation group (G, G) is just the multiplication on G.

Let (G, X) be a locally compact transformation group with the action τ of G on X. Note that for each $s \in G$ and $f \in C_0(X)$, $\tau_s \in \operatorname{Homeo}(X)$ and hence $f \circ \tau_s \in C_0(X)$. If we let $\alpha_s : C_0(X) \to C_0(X)$ be the map $f \mapsto f \circ \tau_{s^{-1}}$, then it is easy to see that $\alpha_s \in \operatorname{Aut}(C_0(X))$ and

$$\alpha : G \to \operatorname{Aut}(C_0(X)), \ s \mapsto \alpha_s$$

is a homomorphism.

It is natural to ask whether α is continuous when $\operatorname{Aut}(C_0(X))$ is equipped with the point-norm topology. This is answered in the following result.

LEMMA 4.1.5. *Suppose that (G, X) is a locally compact transformation group and that $\operatorname{Aut}(C_0(X))$ is given the point-norm topology. Then the above associated homomorphism $\alpha : G \to \operatorname{Aut}(C_0(X))$ is continuous.*

PROOF. It suffices to show that for each $f \in C_0(X)$, $\|\alpha_s(f) - f\|_\infty \to 0$ as $s \to e$. If this were to fail, then there would be an $\epsilon > 0$ and nets $\{s_i\}$ and $\{x_i\}$ in

G and X, respectively, such that $s_i \to e$ and

$$|f(s_i^{-1} \cdot x_i) - f(x_i)| \geq \epsilon \quad \text{for all } i. \tag{4.1.2}$$

Choose a compact neighborhood V of e and we can assume that $s_i \in V$ for all i.

Since f varnishes at infinity, $K = \{x \in X : |f(x)| \geq \frac{\epsilon}{2}\}$ is compact. For each i, (4.1.2) shows that either $x_i \in K$ or $s_i^{-1} \cdot x_i \in K$ (i.e., $x_i \in s_i \cdot K \subset V \cdot K$). Thus $x_i \in V \cdot K$ for all i. By the compactness of $V \cdot K$, we can assume that $x_i \to x_0$ for some $x_0 \in V \cdot K$, and hence

$$s_i^{-1} \cdot x_i \to e \cdot x_0 = x_0.$$

Taking the limit in (4.1.2), we obtain that $0 \geq \epsilon$, a contradiction. □

4.2. Dynamical Systems

Definition 4.2.1. A C^**-dynamical system* is a triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G, and a homomorphism $\alpha : G \to \mathrm{Aut}(A)$, which is continuous when $\mathrm{Aut}(A)$ is equipped with the point-norm topology.

We will shorten C^*-dynamical system to just dynamical system. Notice that the continuity condition on α in Definition 4.2.1 is equivalent to $s \mapsto \alpha_s(a)$ being continuous for all $a \in A$. Lemma 4.1.5 shows that a locally compact transformation group (G, X) results in a dynamical system with $A = C_0(X)$ commutative.

It is natural to ask, when (A, G, α) is a dynamical system with A commutative and $X = \sigma(A)$, whether (G, X) is a locally compact transformation group so that (A, G, α) can be canonically recovered from (G, X). The answer is in the affirmative as the following proposition establishes.

Proposition 4.2.2. *Suppose that $(C_0(X), G, \alpha)$ is a dynamical system with X a locally compact Hausdorff space. Then there is a locally compact transformation group (G, X) such that*

$$\alpha_s(f)(x) \quad = \quad f(s^{-1} \cdot x) \tag{4.2.1}$$

for all $s \in G$, $f \in C_0(X)$ and $x \in X$.

PROOF. For each $s \in G$, by Theorem 1.3.16, there exists $h_s \in \text{Homeo}(X)$ such that

$$\alpha_s(f)(x) = f(h_s(x)) \tag{4.2.2}$$

for all $f \in C_0(X)$ and $x \in X$, and $s \mapsto h_s$ is a continuous map from G to $\text{Homeo}(X)$, which is equipped with the topology described in Definition 1.3.14.

From (4.2.2), we have,

$$f(h_{sr}(x)) = \alpha_{sr}(f)(x) = \alpha_s(\alpha_r(f))(x) = \alpha_r(f)(h_s(x)) = f(h_r(h_s(x))).$$

It follows that $h_{sr} = h_r \circ h_s$ and $h_s^{-1} = h_{s^{-1}}$. Therefore,

$$G \times X \to X,\ (s, x) \mapsto s \cdot x = h_s^{-1}(x) = h_{s^{-1}}(x)$$

is an action of G on X. This action is clearly continuous by the definition of the topology on $\text{Homeo}(X)$ and Lemma 1.3.13. Therefore, (G, X) is a locally compact transformation group such that (4.2.1) holds. □

EXAMPLE 4.2.3. Groups and C*-algebras give degenerate dynamical systems. Indeed, every locally compact group G gives rise to the dynamical system $(\mathbb{C}, G, 1)$, where 1 denotes the trivial homomorphism $G \to Aut(\mathbb{C}) = \{id_{\mathbb{C}}\}$, and every C*-algebra A is associated with the dynamical system $(A, \{e\}, id)$.

4.3. Covariant Representations of Dynamical Systems

DEFINITION 4.3.1. Let (A, G, α) be a dynamical system. A *covariant representation* of (A, G, α) is a pair (π, U) consisting of a representation $\pi : A \to B(\mathcal{H})$ and a unitary representation $U : G \to U(\mathcal{H})$ on the same Hilbert space $\mathcal{H}$ such that

$$\pi(\alpha_s(a)) \;=\; U_s\pi(a)U_s^* \quad (s \in G, a \in A). \tag{4.3.1}$$

A covariant representation (π, U) is called *non-degenerate* if the representation π is non-degenerate.

EXAMPLE 4.3.2. Let A be a C*-algebra and let G be a locally compact group. We consider covariant representations of the degenerate dynamical systems given in Example 4.2.3.

(1) Let $\pi : A \to B(\mathcal{H})$ be a representation, and let $U : \{e\} \to U(\mathcal{H})$, $e \mapsto id_{\mathcal{H}}$ be the trivial unitary representation. Then we have $U_e\pi(a)U_e^* = \pi(a) = \pi(id(a))$. Thus (π, U) is a covariant representation of $(A, \{e\}, id)$. Therefore, the covariant representations of $(A, \{e\}, id)$ correspond exactly to the representations of A.

(2) Let $U : G \to U(\mathcal{H})$ be any unitary representation of G on $\mathcal{H}$, and let $\pi : \mathbb{C} \to B(\mathcal{H})$, $c \mapsto c\ id_{\mathcal{H}}$ be the trivial representation. Then for any $s \in G$ and $c \in \mathbb{C}$, we have

$$U_s\pi(c)U_s^* = c\ id_{\mathcal{H}} = \pi(c) = \pi(1_s(c)).$$

Thus (π, U) is a covariant representation of $(\mathbb{C}, G, 1)$. Therefore, the covariant representations of $(\mathbb{C}, G, 1)$ correspond exactly to the unitary representations of G.

EXAMPLE 4.3.3. Let G be a locally compact group, and let $(C_0(G), G, \ell t)$ be the dynamical system associated with the transformation group (G, G) with G acting on itself by left translation. Let $M : C_0(G) \to B(L^2(G))$ be the representation of $C_0(G)$ given by pointwise multiplication; that is,

$$M(f) : h \mapsto fh,$$

and let $\lambda : G \to U(L^2(G))$ be the left regular representation of G.

We claim that (M, λ) is a covariant representation of $(C_0(G), G, \ell t)$. To see this, it suffices to show that $\lambda_s M(f) = M(\ell t_s(f))\lambda_s$ for all $s \in G$ and $f \in C_0(G)$. This is certainly true, as for all $h \in L^2(G)$, we have

$$\lambda_s M(f)(h) = \lambda_s(fh) = \ell t_s(f)\lambda_s(h) = M(\ell t_s(f))(\lambda_s(h)).$$

EXAMPLE 4.3.4. Let $h \in Homeo(\mathbb{T})$ be the "rotation by θ" map; that is,

$$h(z) \quad = \quad e^{2\pi i\theta}z.$$

Then $(\mathbb{Z}, \mathbb{T})$ is a transformation group (cf. Example 4.1.3). Let $(C(\mathbb{T}), \mathbb{Z}, \alpha)$ be the dynamical system associated with $(\mathbb{Z}, \mathbb{T})$. That is, for $n \in \mathbb{Z}$, $f \in C(\mathbb{T})$, and

$z \in \mathbb{T}$, we have

$$\alpha_n(f)(z) = f(h^{-n}(z)) = f(e^{-2\pi i n\theta} z).$$

(a) Let $M : C(\mathbb{T}) \to B(L^2(\mathbb{T}))$ be the representation given by pointwise multiplication, and let $U : \mathbb{Z} \to U(L^2(\mathbb{T}))$ be the unitary representation given by

$$U_n\xi(z) = (\xi \circ h^{-n})(z) = \xi(e^{-2\pi i n\theta} z).$$

Since $U_n M(f) = M(\alpha_n(f))U_n$ for all $n \in \mathbb{Z}$ and $f \in C(\mathbb{T})$, (M, U) is a covariant representation of $(C(\mathbb{T}), \mathbb{Z}, \alpha)$.

(b) Now fix $\omega \in \mathbb{T}$. Define $\pi_\omega : C(\mathbb{T}) \to B(L^2(\mathbb{Z}))$ to be the representation

$$\pi_\omega(f)\xi(n) = f(e^{2\pi i n\theta}\omega)\xi(n) = f(h^n(\omega))\xi(n).$$

Since $\lambda_n\pi_\omega(f) = \pi_\omega(\alpha_n(f))\lambda_n$ for all $n \in \mathbb{Z}$ and $f \in C(\mathbb{T})$, where λ is the left regular representation of $\mathbb{Z}$, (π_ω, λ) is a covariant representation of $(C(\mathbb{T}), \mathbb{Z}, \alpha)$.

Let G be a locally compact group and let $\mathcal{H}$ be a complex Hilbert space. We will take a slight detour to describe the structure of the Hilbert space $L^2(G, \mathcal{H})$, which will be needed in the subsequent examples. To begin with, we simply define $L^2(G, \mathcal{H})$ to be the completion of $C_c(G, \mathcal{H})$ with the $\|\cdot\|_2$-norm coming from the inner product

$$(\xi \mid \eta) = \int_G (\xi(s) \mid \eta(s)) d\mu(s) \qquad (\xi, \eta \in C_c(G, \mathcal{H})). \tag{4.3.2}$$

The above definition clearly gives us the usual space $L^2(G)$ when $\mathcal{H} = \mathbb{C}$.

On the other hand, it was shown in [**17**, §I.4] (see also Lemma 3.1.2) that $L^2(G, \mathcal{H})$ is naturally isomorphic to the Hilbert space tensor product $L^2(G) \otimes \mathcal{H}$, which is the completion of $C_c(G) \odot \mathcal{H}$ with the norm obtained from the inner product on $\mathcal{H}$ and the inner product on $C_c(G)$ given by

$$(\xi \mid \eta) = \int_G \xi(s)\overline{\eta(s)} d\mu(s).$$

In the next result, for any dynamical system, we define a very important class of covariant representations, called regular representations.

Lemma 4.3.5. *Let (A, G, α) be a dynamical system and let $\rho : A \to B(\mathcal{H}_\rho)$ be any representation of A on a Hilbert space $\mathcal{H}_\rho$. Let $\tilde{\rho} : A \to B(L^2(G, \mathcal{H}_\rho))$ and*

$U : G \to U(L^2(G, \mathcal{H}_\rho))$ be representations of A and G defined respectively by

$$(\tilde{\rho}(a)h)(r) \;=\; \rho(\alpha_r^{-1}(a))(h(r)) \quad \textit{and} \quad (U_s h)(r) \;=\; h(s^{-1}r), \tag{4.3.3}$$

where $a \in A$, $r \in G$ and $h \in L^2(G, \mathcal{H}_\rho)$. Then the pair $(\tilde{\rho}, U)$ of representations on the Hilbert space $L^2(G, \mathcal{H}_\rho)$ is a covariant representation of (A, G, α).

PROOF. Under the identification $L^2(G, \mathcal{H}_\rho) \cong L^2(G) \otimes \mathcal{H}_\rho$, it is clear that $U = \lambda \otimes id$, where λ is the regular representation of G on $L^2(G, \mathcal{H}_\rho)$. Now, we have

$$\begin{aligned} (U_s\tilde{\rho}(a)U_s^* h)(r) &= (\tilde{\rho}(a)U_s^* h)(s^{-1}r) \;=\; \rho(\alpha_{s^{-1}r}^{-1}(a))(U_s^* h(s^{-1}r)) \\ &= \rho(\alpha_r^{-1}(\alpha_s(a)))(h(r)) \;=\; \tilde{\rho}(\alpha_s(a))h(r). \end{aligned}$$

Then $(\tilde{\rho}, U)$ is a covariant representation of (A, G, α). □

The pair $(\tilde{\rho}, U)$ is called a *regular representation* of (A, G, α) and is denoted by $\mathrm{Ind}_e^G \rho$. Lemma 4.3.5 shows that for any given dynamical system, there always exists a covariant representation.

EXAMPLE 4.3.6. The covariant representation (π_ω, λ) of $(C(\mathbb{T}), \mathbb{Z}, \alpha)$ in Example 4.3.4(b) is exactly the regular representation $\mathrm{Ind}_e^G \mathrm{ev}_\omega$ of $(C(\mathbb{T}), \mathbb{Z}, \alpha)$, where $ev_\omega : C(\mathbb{T}) \to \mathbb{C} \cong B(\mathbb{C})$ and $L^2(\mathbb{Z}) \otimes \mathbb{C} \cong L^2(\mathbb{Z})$.

In fact, for $\xi \in L^2(\mathbb{Z})$, we have

$$(\pi_\omega(f)\xi)(n) = \alpha_n^{-1}(f)(\omega)\xi(n) = ev_\omega(\alpha_n^{-1}(f))\xi(n) = (\widetilde{ev}_\omega(f)\xi)(n).$$

This shows that $\pi_\omega = \widetilde{ev}_\omega$. Clearly, the unitary representation U in $\mathrm{Ind}_e^G ev_\omega$ is just the left regular representation λ of $\mathbb{Z}$. Therefore, $(\pi_\omega, \lambda) = \mathrm{Ind}_e^G ev_\omega$.

In the regular representation $\mathrm{Ind}_e^G \rho = (\tilde{\rho}, U)$ of a dynamical system (A, G, α), the representation $\tilde{\rho}$ depends on the representation ρ of A. We may want to know if $\tilde{\rho}$ inherits any property of ρ. The result below gives one important relationship between $\tilde{\rho}$ and ρ.

PROPOSITION 4.3.7. *Let (A, G, α) be a dynamical system and let $\mathrm{Ind}_e^G \rho = (\tilde{\rho}, U)$ be a regular representation of (A, G, α). Then the representation $\tilde{\rho}$ defined in Lemma 4.3.5 is non-degenerate if ρ is non-degenerate.*

Proof. Let $\{e_i\}$ be a bounded approximate identity of A. To show that $\tilde{\rho}$ is non-degenerate, it suffices to prove that

$$\tilde{\rho}(e_i)\xi \to \xi \quad \text{in the } \|\cdot\|_2\text{-norm} \tag{4.3.4}$$

for all ξ in the dense subset $C_c(G) \odot \mathcal{H}_\rho$ of $L^2(G, \mathcal{H}_\rho)$. Since ρ is non-degenerate, we can assume that $\xi = f \otimes (\rho(a)h)$ for some $f \in C_c(G)$, $a \in A$ and $h \in \mathcal{H}_\rho$.

Let $K = supp(f)$ and let $M = \sup\{|f(s)| : s \in K\}$. It follows that

$$\begin{aligned}
\|\tilde{\rho}(e_i)\xi - \xi\|_2^2 &= (\tilde{\rho}(e_i)\xi - \xi \mid \tilde{\rho}(e_i)\xi - \xi) \\
&= \int_G (\tilde{\rho}(e_i)\xi(s) - \xi(s) \mid \tilde{\rho}(e_i)\xi(s) - \xi(s))d\mu(s) \\
&= \int_G \|\tilde{\rho}(e_i)\xi(s) - \xi(s)\|^2 d\mu(s) \\
&= \int_K \|f(s)\rho(\alpha_s^{-1}(e_i)a)h - f(s)\rho(a)h\|^2 d\mu(s) \\
&\leq \int_K |f(s)|^2 \|\rho(\alpha_s^{-1}(e_i)a - a))\|^2 \|h\|^2 d\mu(s) \\
&\leq M^2\|h\|^2 \int_K \|\alpha_s^{-1}(e_i)a - a\|^2 d\mu(s) \\
&= M^2\|h\|^2 \int_K \|e_i\alpha_s(a) - \alpha_s(a)\|^2 d\mu(s).
\end{aligned}$$

For any fixed $s \in K$, we have $\|e_i\alpha_s(a) - \alpha_s(a)\| \to 0$. Since K is compact, the set $E = \{\alpha_s(a) : s \in K\}$ is a compact set in A. Note that $\{e_i\}$ is bounded. Thus

$$\|e_i x - x\| \to 0 \ \text{ uniformly for } x \in E.$$

Therefore, the integral $\int_K \|e_i\alpha_s(a) - \alpha_s(a)\|d\mu(s)$ is convergent to 0, and the proof is complete. □

Definition 4.3.8. Suppose that (A, G, α) is a dynamical system and that (π, U) and (ρ, V) are covariant representations of (A, G, α) on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. The *direct sum* $(\pi, U) \oplus (\rho, V)$ is the covariant representation $(\pi \oplus \rho, U \oplus V)$ of (A, G, α) on $\mathcal{H} \oplus \mathcal{K}$ given by

$$(\pi \oplus \rho)(a) = \pi(a) \oplus \rho(a) \ \text{ and } \ (U \oplus V)_s = U_s \oplus V_s.$$

A closed linear subspace $\mathcal{H}'$ of $\mathcal{H}$ is *invariant* for (π, U) if $\pi(a)(\mathcal{H}') \subset \mathcal{H}'$ and $U_s(\mathcal{H}') \subset \mathcal{H}'$ for all $a \in A$ and $s \in G$. If $\mathcal{H}'$ is invariant, π' is the restriction of π to $\mathcal{H}'$ and U' is the restriction of U to $\mathcal{H}'$, then (π', U') is a covariant representation of (A, G, α) on $\mathcal{H}'$, called a *subrepresentation* of (π, U).

We call (π, U) *irreducible* if $\{0\}$ and $\mathcal{H}$ are the only invariant subspaces of $\mathcal{H}$.

Finally, we say that (π, U) and (ρ, V) are *equivalent* if there is a unitary map $W : \mathcal{H} \to \mathcal{K}$ such that

$$\rho(a) = W\pi(a)W^* \quad \text{and} \quad V_s = WU_sW^* \quad \text{for all } a \in A \text{ and } s \in G.$$

Remark 4.3.9. If $\mathcal{V} \subset \mathcal{H}$ is invariant for (π, U), then it is clear that $\mathcal{V}^{\perp}$ is also invariant for (π, U). Thus if (π', U') and (π'', U'') are subrepresentations of (π, U) on $\mathcal{V}$ and $\mathcal{V}^{\perp}$, respectively, then $(\pi, U) = (\pi', U') \oplus (\pi'', U'')$. In particular, (π, U) is irreducible if and only if (π, U) is not equivalent to the direct sum of two nontrivial covariant representations of (A, G, α).

CHAPTER 5

Crossed Products of C*-Algebras and Examples

This chapter is mainly based on William [**17**].

5.1. Crossed Products

In this section, for a given dynamical system (A, G, α), we want to construct a $*$-algebra structure on $C_c(G, A)$, whose completion with respect to a norm defined later - the *universal norm* - will be a C*-algebra. This C*-algebra is called the *Crossed Product of A by G*, and is denoted by $A \rtimes_\alpha G$.

5.1.1. $*$-Algebraic Structure on $C_c(G, A)$. Let (A, G, α) be a dynamical system. If $f, g \in C_c(G, A)$, then $(s, r) \mapsto f(r)\alpha_r(g(r^{-1}s))$ is in $C_c(G \times G, A)$, and Corollary 3.3.2 guarantees that

$$s \mapsto \int_G f(r)\alpha_r(g(r^{-1}s))d\mu(r)$$

defines an element of $C_c(G, A)$. We define

$$f * g(s) \;=\; \int_G f(r)\alpha_r(g(r^{-1}s))d\mu(r). \tag{5.1.1}$$

For all $f, g, h \in C_c(G, A)$ and $s \in G$, using the left invariance of the Haar measure and equation (3.1.8) and applying the Fubini Theorem for vector-valued integrals (cf. Proposition 3.3.3), we have

$$\begin{aligned}
f * (g * h)(s) &= \int_G f(r)\alpha_r(g * h(r^{-1}s))d\mu(r) \\
&= \int_G f(r) \int_G \alpha_r(g(t))\alpha_{rt}(h(t^{-1}r^{-1}s))d\mu(t)d\mu(r) \\
&= \int_G f(r) \int_G \alpha_r(g(r^{-1}t))\alpha_t(h(t^{-1}s))d\mu(t)d\mu(r) \\
&= \int_G \Big(\int_G f(r)\alpha_r(g(r^{-1}t))d\mu(r) \Big) \alpha_t(h(t^{-1}s))d\mu(t) \\
&= (f * g) * h(s).
\end{aligned}$$

Thus the multiplication given by (5.1.1) is associative on $C_c(G,A)$, and we call it the *convolution* on $C_c(G,A)$.

Next, for $f \in C_c(G,A)$, we define

$$f^*(s) \quad = \quad \triangle(s^{-1})\alpha_s(f(s^{-1})^*). \tag{5.1.2}$$

Then $f^* \in C_c(G,A)$.

Now, for all $f, g \in C_c(G,A)$ and $s \in G$, we have

$$\begin{aligned}
f^* * g^*(s) &= \int_G f^*(r)\alpha_r(g^*(r^{-1}s))d\mu(r) \\
&= \int_G \triangle(r^{-1})\alpha_r(f(r^{-1})^*)\alpha_r(\triangle(s^{-1}r)\alpha_{r^{-1}s}(g(s^{-1}r)^*))d\mu(r) \\
&= \triangle(s^{-1})\int_G \alpha_r(\alpha_{r^{-1}s}(g(s^{-1}r))f(r^{-1}))^*d\mu(r) \\
&= \triangle(s^{-1})\int_G \alpha_{sr}(\alpha_{r^{-1}}(g(r))f(r^{-1}s^{-1}))^*d\mu(r) \\
&= \triangle(s^{-1})\int_G \alpha_s(g(r)\alpha_r(f(r^{-1}s^{-1})))^*d\mu(r) \\
&= \triangle(s^{-1})\alpha_s\Big(\int_G g(r)\alpha_r(f(r^{-1}s^{-1}))d\mu(r)\Big)^* \\
&= \triangle(s^{-1})\alpha_s(g * f(s^{-1}))^* \\
&= (g * f)^*(s).
\end{aligned}$$

Therefore, $f \mapsto f^*$ is an involution on $C_c(G,A)$.

Furthermore, if $\|\cdot\|_1$ is the norm on $C_c(G,A)$ given by

$$\|f\|_1 \quad = \quad \int_G \|f(s)\|d\mu(s), \tag{5.1.3}$$

then by Proposition 2.1.18, we have

$$\begin{aligned}
\|f^*\|_1 &= \int_G \|\triangle(s^{-1})\alpha_s(f(s^{-1})^*)\|d\mu(s) \\
&= \int_G \triangle(s^{-1})\|f(s^{-1})\|d\mu(s) \\
&= \int_G \|f(s)\|d\mu(s) \\
&= \|f\|_1.
\end{aligned}$$

Thus the involution $f \mapsto f^*$ on $(C_c(G,A), \|\cdot\|_1)$ is isometric.

Finally, for $f, g \in C_c(G,A)$, we have

$$\begin{aligned}
\|f * g\|_1 &= \int_G \|f * g(s)\| d\mu(s) \\
&= \int_G \Big\| \int_G f(r)\alpha_r(g(r^{-1}s)) d\mu(r) \Big\| d\mu(s) \\
&\leq \int_G \int_G \|f(r)\alpha_r(g(r^{-1}s))\| d\mu(r) d\mu(s) \\
&\leq \int_G \int_G \|f(r)\| \|g(r^{-1}s)\| d\mu(r) d\mu(s) \\
&= \int_G \|f(r)\| \Big(\int_G \|g(r^{-1}s)\| d\mu(s) \Big) d\mu(r) \\
&= \int_G \|f(r)\| \Big(\int_G \|g(s)\| d\mu(s) \Big) d\mu(r) \\
&= \|f\|_1 \|g\|_1.
\end{aligned}$$

Therefore, $(C_c(G,A), \|\cdot\|_1)$ is a normed associative algebra with the convolution (5.1.1) satisfying $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ and the isometric involution (5.1.2).

Example 5.1.1. (1) Let A be a C*-algebra. For the degenerate dynamical system $(A, \{e\}, id)$ (cf. Example 4.2.3), the normed $*$-algebra $(C_c(\{e\}, A), \|\cdot\|_1)$ is just the C*-algebra A.

(2) Let G be a locally compact group. For the degenerate dynamical system $(\mathbb{C}, G, 1)$ (cf. Example 4.2.3), the normed $*$-algebra $(C_c(G,\mathbb{C}), \|\cdot\|_1)$ is the usual space $(C_c(G), \|\cdot\|_1)$ with the convolution and the involution given by

$$f * g(s) = \int_G f(r) g(r^{-1}s) d\mu(r) \quad \text{and} \quad f^*(s) = \triangle(s^{-1}) \overline{f(s^{-1})}.$$

Definition 5.1.2. Let A be a C*-algebra and let $\mathcal{H}$ be a Hilbert space. A $*$-homomorphism $\pi : C_c(G,A) \to B(\mathcal{H})$ is called a $*$-representation of $C_c(G,A)$. If π satisfies $\|\pi(f)\| \leq \|f\|_1$, then π is called L^1-norm decreasing.

5.1.2. Integrated Form of a Covariant Representation. We will define a very useful form for a covariant representation, called the *integrated form*, which will be instrumental in defining the universal norm.

PROPOSITION 5.1.3. *Let (π, U) be a covariant representation of a dynamical system (A, G, α) on $\mathcal{H}$. Then the map $\pi \rtimes U : C_c(G, A) \to B(\mathcal{H})$ given by*

$$\pi \rtimes U(f) \;=\; \int_G \pi(f(s))U_s d\mu(s) \tag{5.1.4}$$

is a L^1-norm decreasing $*$-representation of the convolution algebra $C_c(G, A)$ on $\mathcal{H}$, called the integrated form *of (π, U).*

Furthermore, $\pi \rtimes U$ is non-degenerate if π is non-degenerate.

PROOF. Since $s \mapsto U_s$ is strongly continuous, Corollary 3.2.6 implies that it is strictly continuous. Consequently, the integrand in (5.1.4) is in $C_c(G, B_s(\mathcal{H}))$. By Proposition 3.2.8, the map $\pi \rtimes U : C_c(G, A) \to B(\mathcal{H})$ is well-defined, which is clearly linear. If h and k are unit vectors in $\mathcal{H}$, then by Proposition 3.2.8, we have

$$\begin{aligned} |(\pi \rtimes U(f)h \mid k)| &= \Big| \int_G (\pi(f(s))U_s h \mid k) d\mu(s) \Big| \\ &\leq \int_G \|\pi(f(s))\| \|U_s\| \|h\| \|k\| d\mu(s) \\ &\leq \int_G \|f(s)\| d\mu(s) \;=\; \|f\|_1. \end{aligned}$$

It follows that $\|\pi \rtimes U(f)\| \leq \|f\|_1$; that is, $\pi \rtimes U : C_c(G, A) \to B(\mathcal{H})$ is L^1-norm decreasing.

To see that $\pi \rtimes U$ is a $*$-homomorphism, we compute as follows by applying Proposition 3.2.8 and Proposition 2.1.18: for $f, g \in C_c(G, A)$, we have

$$\begin{aligned} \pi \rtimes U(f * g) &= \int_G \pi(f * g(s))U_s d\mu(s) \\ &= \int_G \pi\Big(\int_G f(r)\alpha_r(g(r^{-1}s)) d\mu(r) \Big) U_s d\mu(s) \\ &= \int_G \int_G \pi(f(r)\alpha_r(g(r^{-1}s)))U_s d\mu(r) d\mu(s) \\ &= \int_G \int_G \pi(f(r))U_r \pi(g(r^{-1}s))U_{r^{-1}s} d\mu(r) d\mu(s) \\ &= \int_G \int_G \pi(f(r))U_r \pi(g(s))U_s d\mu(r) d\mu(s) \\ &= \int_G \pi(f(r))U_r d\mu(r) \int_G \pi(g(s))U_s d\mu(s) \\ &= (\pi \rtimes U(f)) \circ (\pi \rtimes U(g)) \end{aligned}$$

and

$$\begin{aligned}
\pi \rtimes U(f)^* &= \int_G U_{s^{-1}}\pi(f(s)^*)d\mu(s) \\
&= \int_G U_{s^{-1}}\pi(\triangle(s^{-1})\alpha_s(f^*(s^{-1})))d\mu(s) \\
&= \int_G U_{s^{-1}}\triangle(s^{-1})\pi(\alpha_s(f^*(s^{-1})))d\mu(s) \\
&= \int_G U_{s^{-1}}\triangle(s^{-1})U_s\pi(f^*(s^{-1}))U_{s^{-1}}d\mu(s) \\
&= \int_G \triangle(s^{-1})\pi(f^*(s^{-1}))U_{s^{-1}}d\mu(s) \\
&= \int_G \pi(f^*(s))U_s d\mu(s) \;=\; \pi \rtimes U(f^*).
\end{aligned}$$

Now assume that π is non-degenerate. We want to show that the set

$$\{\pi \rtimes U(f)h : f \in C_c(G,A) \text{ and } h \in \mathcal{H}\}$$

is norm dense in $\mathcal{H}$. To this end, let $h \in \mathcal{H}$ and $\epsilon > 0$. Note that if $\{e_i\}$ is a bounded approximate identity of A, then, since π is non-degenerate, we have $\pi(e_i)h \to h$ in $\mathcal{H}$. Thus we can choose $u \in A$ of norm one such that $\|\pi(u)h - h\| < \epsilon/2$. Let V be a neighborhood of e in G such that $\|U_s h - h\| < \epsilon/2$ for all $s \in V$. Choose $\varphi \in C_c(G)^+$ such that $\mathrm{supp}(\varphi) \subset V$ and $\|\varphi\|_1 = 1$.

Let $f = \varphi \otimes u \in C_c(G,A)$. If k is an element of $\mathcal{H}$ with norm one, then, using Proposition 3.2.8 at some point, we have

$$\begin{aligned}
&|(\pi \rtimes U(f)h - h \mid k)| \\
=\;& \Big|\Big(\int_G \pi(f(s))U_s d\mu(s)h - h \mid k\Big)\Big| \\
=\;& \Big|\Big(\Big(\int_G \varphi(s)\pi(u)U_s d\mu(s) - \int_G \varphi(s)d\mu(s)\Big)h \mid k\Big)\Big| \\
=\;& \Big|\int_G (\varphi(s)(\pi(u)U_s h - h) \mid k)d\mu(s)\Big| \\
\leq\;& \int_G \varphi(s)|(\pi(u)(U_s h - h) \mid k)|d\mu(s) + \int_G \varphi(s)|(\pi(u)h - h \mid k)|d\mu(s) \\
\leq\;& \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.
\end{aligned}$$

Therefore, $\|\pi \rtimes U(f)h - h\| \leq \epsilon$. $\square$

EXAMPLE 5.1.4. Let A be a C*-algebra and let G be a locally compact group. Let $\pi : A \to B(\mathcal{H})$ be any representation of A and let $U : G \to U(\mathcal{H})$ be any unitary representation of G. It is known (cf. Example 4.3.2) that (π, id) and (id, U) are covariant representations of $(\{e\}, A, id)$ and $(\mathbb{C}, G, 1)$, respectively. In this case,

(i) $C_c(\{e\}, A) = A$ and $\pi \rtimes id = \pi$;

(ii) $C_c(G, \mathbb{C}) = C_c(G)$ and $id \rtimes U : C_c(G) \to B(\mathcal{H})$ is the restriction to $C_c(G)$ of the representation $L^1(G) \to B(\mathcal{H})$ associated with U (cf. Section 2.3). We will always shorten $id \rtimes U$ as just U.

Let (A, G, α) be a dynamical system. For each $r \in G$, let

$$i_G(r) : C_c(G, A) \to C_c(G, A)$$

be defined by

$$i_G(r)f(s) = \alpha_r(f(r^{-1}s)). \tag{5.1.5}$$

In particular, if $u \in C_c(G)$ and $a \in A$, then

$$i_G(r) : u \otimes a \mapsto (\lambda(r)u) \otimes \alpha_r(a) = (\lambda(r) \otimes \alpha_r)(u \otimes a).$$

So, we can write $i_G(r)$ as $\lambda(r) \otimes \alpha_r$.

This map has the following property: if (π, U) is a covariant representation of (A, G, α), then for all $f \in C_c(G, A)$, we have

$$\begin{aligned}
\pi \rtimes U(i_G(r)f) &= \int_G \pi(i_G(r)f(s))U_s d\mu(s) \\
&= \int_G \pi(\alpha_r(f(r^{-1}s))U_s d\mu(s) \\
&= \int_G U_r\pi(f(r^{-1}s))U_{r^{-1}}U_s d\mu(s) \\
&= \int_G U_r\pi(f(s))U_s d\mu(s) \\
&= (U_r \cdot \pi) \rtimes U(f),
\end{aligned}$$

where $(U_r \cdot \pi)(a) = U_r\pi(a)$ for $a \in A$. That is, we have

$$(\pi \rtimes U) \circ i_G(r) \;=\; (\pi \rtimes U) \circ (\lambda(r) \otimes \alpha_r) \;=\; (U_r \cdot \pi) \rtimes U. \tag{5.1.6}$$

Recall that, for a representation $\rho : A \to B(\mathcal{H})$, the regular representation $\mathrm{Ind}_e^G \rho = (\tilde{\rho}, U)$ of (A, G, α) on $L^2(G, \mathcal{H})$ was defined in (4.3.3). Then we can consider the integrated form $\tilde{\rho} \rtimes U$. We show below that $\tilde{\rho} \rtimes U$ is injective if ρ is injective.

PROPOSITION 5.1.5. *Let (A, G, α) be a dynamical system, let $\rho : A \to B(\mathcal{H})$ be an injective representation of A, and let $Ind_e^G \rho = (\tilde{\rho}, U)$ be the corresponding regular representation of (A, G, α). Then the integrated form*

$$\tilde{\rho} \rtimes U : C_c(G, A) \to B(L^2(G, \mathcal{H}))$$

is an injective representation of the convolution algebra $C_c(G, A)$.

PROOF. Let $f \in C_c(G, A)$ be such that $f(x) \neq 0$ for some $x \in G$. To get $\|\tilde{\rho} \rtimes U(f)\| \neq 0$, by (5.1.6), we only have to show that $\|\tilde{\rho} \rtimes U(i_G(x^{-1})f)\| \neq 0$. So, we let $g = i_G(x^{-1})f \in C_c(G, A)$ and show that $\|\tilde{\rho} \rtimes U(g)\| \neq 0$.

Since ρ is injective and $g(e) \neq 0$, there are vectors $h, k \in \mathcal{H}$ such that

$$(\rho(g(e))h \mid k) \neq 0.$$

We can find a neighborhood V of e such that for all $s, r \in V$, we have

$$|(\rho(\alpha_r^{-1}(g(s)))h \mid k) - (\rho(g(e))h \mid k)| \;<\; \frac{|(\rho(g(e))h \mid k)|}{3}.$$

Choose $\varphi \in C_c^+(G)$ with $\|\varphi\|_1 = 1$ and satisfying $\varphi(s^{-1}) = \varphi(s)$. Then

$$\int_G \int_G \varphi(s^{-1}r)\varphi(r) d\mu(s) d\mu(r) = 1.$$

Let $\xi, \eta \in L^2(G, \mathcal{H})$ be given by $\xi = \varphi \otimes h$ and $\eta = \varphi \otimes k$. Then

$$\begin{aligned}
&|(\tilde{\rho} \rtimes U(g)\xi \mid \eta) - (\rho(g(e))h \mid k)| \\
= \;& \Big| \int_G (\tilde{\rho} \rtimes U(g)\xi(r) \mid \eta(r)) d\mu(r) - (\rho(g(e))h \mid k) \Big| \\
= \;& \Big| \int_G \Big(\int_G \tilde{\rho}(g(s)) U_s d\mu(s) \xi(r) \mid \eta(r) \Big) d\mu(r) - (\rho(g(e))h \mid k) \Big| \\
= \;& \Big| \int_G \int_G (\tilde{\rho}(g(s)) U_s \xi(r) \mid \eta(r)) d\mu(s) d\mu(r) - (\rho(g(e))h \mid k) \Big|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \int_G \int_G (\varphi(s^{-1}r)\rho(\alpha_r^{-1}(g(s)))h \mid \varphi(r)k) d\mu(s) d\mu(r) - (\rho(g(e))h \mid k) \right| \\
&\leq \int_G \int_G \varphi(s^{-1}r)\varphi(r) \Big| (\rho(\alpha_r^{-1}(g(s)))h \mid k) - (\rho(g(e))h \mid k) \Big| d\mu(s) d\mu(r) \\
&< \frac{|(\rho(g(e))h \mid k)|}{3}.
\end{aligned}
$$

It follows that $(\tilde{\rho} \rtimes U(g)\xi \mid \eta) \neq 0$, and hence $\tilde{\rho} \rtimes U(g) \neq 0$. □

5.1.3. Crossed Products. We are ready now to define the crossed product $A \rtimes_\alpha G$ associated with a dynamical system (A, G, α).

PROPOSITION 5.1.6. *Let (A, G, α) be a dynamical system. For each f in $C_c(G, A)$, define*

$$
\|f\| = \sup \|\pi \rtimes U(f)\|, \tag{5.1.7}
$$

where the supremum is taking over all covariant representations (π, U) of (A, G, α). Then $\|\cdot\|$ is a norm on $C_c(G, A)$, called the universal norm, *and is dominated by the $\|\cdot\|_1$- norm.*

The completion of $(C_c(G, A), \|\cdot\|)$ is a C^*-algebra and is denoted by $A \rtimes_\alpha G$, called the crossed product *of A by G.*

PROOF. By Proposition 5.1.3, we have $\|\pi \rtimes U(f)\| \leq \|f\|_1$ for all $f \in C_c(G, A)$. Thus the function $\|\cdot\|$ given in (5.1.7) is well-defined on $C_c(G, A)$ and satisfies $\|\cdot\| \leq \|\cdot\|_1$.

Clearly, $\|f + g\| \leq \|f\| + \|g\|$ and $\|cf\| = |c|\|f\|$ for all $f, g \in C_c(G, A)$ and $c \in \mathbb{C}$. Suppose now that $\|f\| = 0$. Pick an injective representation ρ of A. Since $\|\tilde{\rho} \rtimes U(f)\| = 0$, by Proposition 5.1.5, we have $f = 0$. Therefore, $\|\cdot\|$ is a norm on $C_c(G, A)$ dominated by $\|\cdot\|_1$.

For any covariant representation (π, U) of (A, G, α) on a Hilbert space $\mathcal{H}$, $\pi \rtimes U(f) \in B(\mathcal{H})$ and hence we have

$$
\|\pi \rtimes U(f^* * f)\| = \|\pi \rtimes U(f)^* \circ \pi \rtimes U(f)\| = \|\pi \rtimes U(f)\|^2.
$$

Therefore, we have $\|f^* * f\| = \|f\|^2$ for all $f \in C_c(G, A)$, and hence the completion $A \rtimes_\alpha G$ of $(C_c(G, A), \|\cdot\|)$ is a C*-algebra. □

Due to Proposition 5.1.6, we often view $C_c(G, A)$ as a $*$-subalgebra of $A \rtimes_\alpha G$, and we will not distinguish between an element of $C_c(G, A)$ and its image in $A \rtimes_\alpha G$.

EXAMPLE 5.1.7. Let A and G be the same as in Example 5.1.1.

Note that $C_c(\{e\}, A) = A$. Therefore, for the degenerate dynamical system $(A, \{e\}, id)$, we have $A \rtimes_{id} \{e\} = A$.

For the degenerate dynamical system $(\mathbb{C}, G, 1)$, we have $C_c(G, \mathbb{C}) = C_c(G)$. Let $U : G \to U(\mathcal{H})$ be any unitary representation of G. Then

$$(id \rtimes U)(f) = U(f)$$

for all $f \in C_c(G)$, where U on the right side is the associated representation $L^1(G) \to B(\mathcal{H})$ given in Section 2.3. In this case, we have $\|f\|_* \leq \|f\|$ for all $f \in C_c(G)$, where $\|\cdot\|_*$ is the norm on $C_c(G)$ given in Section 2.3. It follows that $\|\cdot\|_* = \|\cdot\|$ on $C_c(G)$, and hence we have $\mathbb{C} \rtimes_1 G = C^*(G)$.

5.2. Non-degenerate Examples of Crossed Products

We have degenerate examples of crossed products in Example 5.1.7. In this section, we will construct a few illustrative examples other than the degenerate ones.

The following result, based on the discussions in [**17**, Section 2.3], will be fundamental in this section.

PROPOSITION 5.2.1. *Let G be a locally compact group, let X be a locally compact Hausdorff space, and let $(C_0(X), G, \alpha)$ be a dynamical system, which is associated with a locally compact transformation group (G, X) by Proposition* 4.2.2. *Then $C_c(G\times X)$ can be canonically identified with a $*$-subalgebra of the convolution algebra $C_c(G, C_0(X))$ such that $C_c(G \times X)$ is dense in $C_0(X) \rtimes_\alpha G$ with respect to the universal norm.*

PROOF. Clearly, we have the canonical embeddings

$$C_c(G) \odot C_c(X) \subset C_c(G \times X) \hookrightarrow C_c(G, C_c(X)) \subset C_c(G, C_0(X)), \qquad (5.2.1)$$

where $\hookrightarrow$ is the embedding $f \mapsto \tilde{f}$ given by $\tilde{f}(s)(x) = f(s, x)$. We will identify $C_c(G \times X)$ with its canonical image $\widetilde{C}_c(G \times X)$ in $C_c(G, C_0(X))$.

To show that $C_c(G\times X)$ is a $*$-subalgebra of the convolution algebra $C_c(G, C_0(X))$, we let $f, g \in C_c(G \times X)$, $s \in G$ and $x \in X$. Note that ev_x is a bounded linear functional on $C_0(X)$. Then, under the identifications $f \leftrightarrow \tilde{f}$ and $g \leftrightarrow \tilde{g}$, we have

$$\begin{aligned} f * g(s)(x) &= ev_x\Big(\int_G f(r)\alpha_r(g(r^{-1}s))d\mu(r)\Big) \\ &= \int_G f(r,x)g(r^{-1}s)(r^{-1}\cdot x)d\mu(r) \\ &= \int_G f(r,x)(g(r^{-1}s, r^{-1}\cdot x)d\mu(r) \end{aligned}$$

and

$$\begin{aligned} f^*(s)(x) &= \triangle(s^{-1})\alpha_s(f(s^{-1})^*)(x) \\ &= \triangle(s^{-1})\overline{f(s^{-1})(s^{-1}\cdot x)} \\ &= \triangle(s^{-1})\overline{f(s^{-1}, s^{-1}\cdot x)}. \end{aligned}$$

Let F and H be functions on $G \times X$ given by

$$F(s,x) = \int_G f(r,x)g(r^{-1}s, r^{-1}\cdot x)d\mu(r) \quad \text{and} \quad H(s,x) = \triangle(s^{-1})\overline{f(s^{-1}, s^{-1}\cdot x)}.$$

Clearly, $H \in C_c(G \times X)$, and we also have $F \in C_c(G \times X)$ by Lemma 3.3.1 (with X there replaced by $G \times X$). Therefore,

$$f * g = F \in C_c(G \times X) \quad \text{and} \quad f^* = H \in C_c(G \times X).$$

This shows that $C_c(G \times X)$ is a $*$-subalgebra of the involutive convolution algebra $C_c(G, C_0(X))$.

Since $C_c(X)$ is norm dense in $C_0(X)$, by (5.2.1) and Lemma 3.1.2, we have that $C_c(G) \odot C_c(X)$ and hence $C_c(G \times X)$ is L^1-norm dense in $C_c(G, C_0(X))$. It follows that

$$C_0(X) \rtimes_\alpha G \;=\; \overline{C_c(G \times X)}^{\|\cdot\|},$$

where $\|\cdot\|$ is the universal norm. $\square$

The next result (cf. [**17**, §2.5]) will be useful for giving more examples of crossed products.

LEMMA 5.2.2. *Let* (A, G, α) *be a dynamical system with the group* G *finite and let* $L : C(G, A) \to D$ *be a* $*$*-isomorphism of* $C(G, A)$ *onto a* C*-*algebra* D*. Then* $A \rtimes_\alpha G \cong D$*.*

PROOF. Since G is finite, we have $C_c(G, A) = C(G, A)$. By the assumption, $C(G, A)$ is also a C*-algebra. That is, $C(G, A)$ is complete with respect to the universal norm. Therefore, $A \rtimes_\alpha G = C(G, A) \cong D$. $\square$

When $G = \{e\}$ in Lemma 5.2.2, we just get the degenerate crossed product $A \rtimes_{id} \{e\} = A$ as given in Example 5.1.7.

In [**17**, Section 2.5], it was shown that if G is the group $\mathbb{Z}_2$ equipped with the discrete topology and (A, G, α) is a dynamical system, then $A \rtimes_\alpha G \cong \mathcal{D}_2$, where

$$\mathcal{D}_{\mathbf{2}} = \left\{ \begin{pmatrix} a & b \\ \alpha(b) & \alpha(a) \end{pmatrix} \in M_2(A) : a, b \in A \right\}.$$

Following the arguments given in [**17**] and using Lemma 5.2.2, we consider below the case $G = \mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$.

PROPOSITION 5.2.3. *Let* $(A, \mathbb{Z}_3, \alpha)$ *be a dynamical system, and let*

$$\mathcal{D}_{\mathbf{3}} = \left\{ \begin{pmatrix} a & b & c \\ \alpha_1(c) & \alpha_1(a) & \alpha_1(b) \\ \alpha_2(b) & \alpha_2(c) & \alpha_2(a) \end{pmatrix} \in M_3(A) : a, b, c \in A \right\}.$$

Then $A \rtimes_\alpha \mathbb{Z}_3 \cong \mathcal{D}_3$*.*

PROOF. It is clear that $\mathcal{D}_3$ is a C*-subalgebra of $M_3(A)$. Note that each f in $C(\mathbb{Z}_3, A)$ is a function $\{0, 1, 2\} \to A$, and thus the map $L : C(\mathbb{Z}_3, A) \to \mathcal{D}_3$ given by

$$L(f) = \begin{pmatrix} f(0) & f(1) & f(2) \\ \alpha_1(f(2)) & \alpha_1(f(0)) & \alpha_1(f(1)) \\ \alpha_2(f(1)) & \alpha_2(f(2)) & \alpha_2(f(0)) \end{pmatrix}$$

is a well-defined linear map.

Let $f, g \in C(\mathbb{Z}_3, A)$. By the definition of the convolution and the involution on

$C(\mathbb{Z}_3, A)$ and noticing that $\mathbb{Z}_3$ is modular, we have

$$(f * g)(0) = f(0)g(0) + f(1)\alpha_1(g(2)) + f(2)\alpha_2(g(1)),$$

$$(f * g)(1) = f(0)g(1) + f(1)\alpha_1(g(0)) + f(2)\alpha_2(g(2)),$$

$$(f * g)(2) = f(0)g(2) + f(1)\alpha_1(g(1)) + f(2)\alpha_2(g(0)),$$

$$f^*(0) = f(0)^*, \quad f^*(1) = \alpha_1(f(2)^*), \quad \text{and} \quad f^*(2) = \alpha_2(f(1)^*).$$

Since $\alpha_1 \circ \alpha_1 = \alpha_2$, $\alpha_2 \circ \alpha_2 = \alpha_1$, and $\alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1 = id$, we have

$$L(f * g) = L(f)L(g) \quad \text{and} \quad L(f^*) = L(f)^*.$$

That is, $L : C(\mathbb{Z}_3, A) \to \mathcal{D}_3$ is a $*$-homomorphism.

It is obvious that L is injective, and it is also onto $\mathcal{D}_3$ by the definition of $\mathcal{D}_3$. Therefore, L is a $*$-isomorphism from $C(\mathbb{Z}_3, A)$ onto $\mathcal{D}_3$. By Lemma 5.2.2, we obtain that $A \rtimes_\alpha \mathbb{Z}_3 \cong \mathcal{D}_3$. $\square$

The arguments given in the proof of Proposition 5.2.3 can be applied to the case $G = \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ $(n \geq 2)$, which yields the following proposition.

PROPOSITION 5.2.4. *Let $(A, \mathbb{Z}_n, \alpha)$ be a dynamical system with $n \geq 2$, and let*

$$\mathcal{D}_{\mathbf{n}} = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ \alpha_1(a_n) & \alpha_1(a_1) & \dots & \alpha_1(a_{n-1}) \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1}(a_2) & \alpha_{n-1}(a_3) & \dots & \alpha_{n-1}(a_1) \end{pmatrix} \in M_n(A) : a_1, \dots, a_n \in A \right\}.$$

Then $A \rtimes_\alpha \mathbb{Z}_n \cong \mathcal{D}_n$.

The following result, cited from [**17**], provides another important application of Lemma 5.2.2.

PROPOSITION 5.2.5. *Suppose that G is a finite group with $|G| = n$. Then for the canonical dynamical system $(C(G), G, \ell t)$, we have*

$$C(G) \rtimes_{\ell t} G \;\cong\; M_n,$$

where M_n denotes the C^*-algebra of $n \times n$ complex matrices.

PROOF. Let $G = \{s_i\}_{i=1}^n$. Then $\ell^2(G)$ is an n-dimensional Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^n$, where each e_i is the characteristic function of the singleton $\{s_i\}$. In this case, $\ell^2(G) \cong \mathbb{C}^n$ and $B(\ell^2(G)) \cong M_n$ canonically.

We take the counting measure as the Haar measure on G. Let (M, λ) be the natural covariant representation of $(C(G), G, \ell t)$. Since G is finite, we have that $C(G \times G) \cong C(G, C(G))$ canonically. Under this identification, we have the integrated form

$$M \rtimes_{\ell t} \lambda : C(G \times G) \to B(\ell^2(G)).$$

Then for all $f \in C(G \times G)$ and $h, k \in \ell^2(G)$, we have

$$\begin{aligned}
(M \rtimes_{\ell t} \lambda(f)h \mid k) &= \int_G (M(f(s))\lambda_s h \mid k) d\mu(s) \\
&= \int_G \int_G f(s,t) h(s^{-1}t)\overline{k(t)} d\mu(t) d\mu(s) \\
&= \int_G \int_G f(ts^{-1}, t) h(s)\overline{k(t)} d\mu(s) d\mu(t) \\
&= \sum_{i=1}^n \Big(\sum_{j=1}^n f(s_i s_j^{-1}, s_i) h(s_j) \Big) \overline{k(s_i)}.
\end{aligned}$$

Therefore, $\big((M \rtimes_{\ell t} \lambda)(f)h\big)(s) = \sum_{j=1}^n f(ss_j^{-1}, s)h(s_j)$.

With the identification $B(\ell^2(G)) \cong M_n$, we let

$$L = M \rtimes_{\ell t} \lambda : C(G \times G) \to M_n.$$

Then L is a $*$-homomorphism such that for each $f \in C(G \times G)$, we have

$$L(f) = (f(s_i s_j^{-1}, s_i)),$$

the $n \times n$ matrix whose (i, j)-th entry is $f(s_i s_j^{-1}, s_i)$.

Clearly, $L : C(G \times G) \to M_n$ is injective and surjective. Therefore, by Lemma 5.2.2, we obtain that $C(G) \rtimes_{\ell t} G \cong M_n$. $\square$

5.3. Representations Associated with Crossed Products

In the last two sections, we defined the crossed product of a dynamical system and illustrated it with some non-degenerate examples. In this section, we shall briefly study some representations associated with crossed products.

Notice that for any locally compact group G, the crossed product $C^*(G)$ of the trivial dynamical system $(\mathbb{C}, G, 1)$ contains a copy of $\mathbb{C}$. However, we can not convincingly affirm that it always contains a copy of G. Also, for any C*-algebra A, the crossed product $A \rtimes_\alpha \mathbb{Z}_n$ of A by $\mathbb{Z}_n$ contains a copy of $\mathbb{Z}_n$, whereas it may not contain a copy of A (cf. Section 5.2). On the other hand, the degenerate crossed product $A \rtimes_\alpha \{e\} = A$ (for any C*-algebra A) contains both a copy of A and a copy of $\{e\}$.

Generally, the crossed product $A \rtimes_\alpha G$ does not always contain a copy of A and a copy of G. However, we will show below that the multiplier algebra $M(A \rtimes_\alpha G)$ of the crossed product $A \rtimes_\alpha G$ does contain a copy of A and a copy of G.

In the following, we will view $C_c(G, A)$ as a $*$-subalgebra of $M(A \rtimes_\alpha G)$. Note that if T is a bounded linear map on $C_c(G, A)$ with respect to the universal norm, then T can be extended to a bounded linear map on $A \rtimes_\alpha G$, which is also denoted by T. In this case, $T \in M(A \rtimes_\alpha G)$ if the adjoint $T^* : A \rtimes_\alpha G \to A \rtimes_\alpha G$ of T exists. We will denote the unitary group of $M(A \rtimes_\alpha G)$ by $UM(A \rtimes_\alpha G)$.

PROPOSITION 5.3.1. *Let (A, G, α) be a dynamical system. Then there is an injective non-degenerate $*$-homomorphism*

$$i_A : A \to M(A \rtimes_\alpha G)$$

such that

$$(i_A(a)f)(s) = af(s) \quad (f \in C_c(G, A), a \in A),$$

and there is an injective strictly continuous homomorphism

$$i_G : G \to UM(A \rtimes_\alpha G)$$

such that

$$(i_G(r)f)(s) = \alpha_r(f(r^{-1}s)) \quad (r, s \in G, f \in C_c(G, A)).$$

Moreover, (i_A, i_G) is covariant in the sense that for all $a \in A$ and $r \in G$,

$$i_A(\alpha_r(a)) = i_G(r) i_A(a) i_G(r)^*.$$

Finally, for any non-degenerate covariant representation (π, U) of (A, G, α),

$$(\pi \rtimes U)^{-}(i_A(a)) = \pi(a) \quad \text{and} \quad (\pi \rtimes U)^{-}(i_G(s)) = U_s,$$

where $(\pi \rtimes U)^{-}$ is the unique strictly continuous $*$-homomorphic extension of $\pi \rtimes U$ to $M(A \rtimes_\alpha G)$.

PROOF. Let $a \in A$. Let $i_A(a) : C_c(G, A) \to C_c(G, A)$ be the map $f \mapsto af(\cdot)$. Then for any covariant representation (π, U) of (A, G, α), we have

$$\pi \rtimes U(i_A(a)f) \;=\; \pi(a)(\pi \rtimes U(f)) \;\text{ for all } f \in C_c(G, A). \tag{5.3.1}$$

Thus we have

$$\|i_A(a)f\| \leq \|a\|\|f\| \quad \text{for all } f \in C_c(G, A),$$

where $\|\cdot\|$ is the universal norm. So, $i_A(a)$ can be extended uniquely to a bounded linear map

$$i_A(a) : A \rtimes_\alpha G \to A \rtimes_\alpha G.$$

It is easy to see (from the definition of the convolution and the involution on $C_c(G, A)$) that

$$(i_A(a)f)^* * g \;=\; f^* * (i_A(a^*)g) \quad \text{for all } f, g \in C_c(G, A).$$

This shows that $i_A(a) : A \rtimes_\alpha G \to A \rtimes_\alpha G$ is adjointable with $i_A(a)^* = i_A(a^*)$. Therefore, $i_A(a) \in M(A \rtimes_\alpha G)$. Clearly, $i_A : A \to M(A \rtimes_\alpha G)$ is a $*$-homomorphism.

To show that i_A is injective, we suppose that $a \in A$ and $i_A(a) = 0$. Then for all $\varphi \in C_c(G)$, we have

$$0 = i_A(a)(\varphi \otimes a^*) = \varphi \otimes (aa^*),$$

which implies that $aa^* = 0$; that is, $a = 0$. Therefore, i_A is injective.

To get that $i_A : A \to M(A \rtimes_\alpha G)$ is non-degenerate, we observe that for all $\varphi \in C_c(G)$ and $a, b \in A$,

$$i_A(a)(\varphi \otimes a) = \varphi \otimes (ab).$$

It follows that $i_A(A)(C_c(G) \odot A) = C_c(G) \odot A$ is dense in $A \rtimes_\alpha G$. Therefore, $i_A : A \to M(A \rtimes_\alpha G)$ is an injective non-degenerate $*$-homomorphism.

Now we turn to consider the second map i_G. Let $r \in G$. Note that the map $i_G(r) : C_c(G, A) \to C_c(G, A)$, $f \mapsto \alpha_r(f(r^{-1}\cdot))$ was considered in Section 5.1. For any covariant representation (π, U) of (A, G, α), by (5.1.6), we have

$$\pi \rtimes U(i_G(r)f) \;=\; ((U_r \cdot \pi) \rtimes U)f \quad \text{for all } f \in C_c(G, A).$$

Thus $\|i_G(r)f\| \leq \|f\|$ for all $f \in C_c(G, A)$, and hence $i_G(r)$ can be extended to a contractive linear map $i_G(r) : A \rtimes_\alpha G \to A \rtimes_\alpha G$. It is also easy to see that for all $f, g \in C_c(G, A)$, we have

$$(i_G(r)f)^* * g \;=\; f^* * (i_G(r^{-1})g).$$

So, $i_G(r) : A \rtimes_\alpha G \to A \rtimes_\alpha G$ is adjointable with adjoint $i_G(r)^* = i_G(r^{-1}) = i_G(r)^{-1}$. Therefore, $i_G(r) \in UM(A \rtimes_\alpha G)$. It is clear that $i_G : G \to UM(A \rtimes_\alpha G)$ is a homomorphism.

To see that $i_G : G \to UM(A \rtimes_\alpha G)$ is injective, it suffices to show that if $t \in G \setminus \{e\}$, then there exists $f \in C_c(G, A)$ such that $\alpha_t(f(t^{-1})) \neq f(e)$. Indeed, this is true, since for $a \in A \setminus \{0\}$ and $\varphi \in C_c(G)$ with $\varphi(e) = 1$ and $\varphi(t^{-1}) = 0$, we have

$$\alpha_t((\varphi \otimes a)(t^{-1})) = \alpha_t(\varphi(t^{-1})a) = 0 \neq a = (\varphi \otimes a)(e).$$

To prove that $i_G : G \to UM(A \rtimes_\alpha G)$ is strictly continuous, which is equivalent to i_G being strongly continuous, we only have to show that if $r_i \to e$, then $\|i_G(r_i)(\varphi \otimes a) - \varphi \otimes a\|_1 \to 0$ for all $\varphi \in C_c(G)$ and $a \in A$. In fact, this is true, since

$$\|i_G(r_i)(\varphi \otimes a) - \varphi \otimes a\|_1 \leq \|\varphi(r^{-1}\cdot) - \varphi\|_1 \|a\| + \|\varphi\|_1 \|\alpha_{r_i}(a) - a\|,$$

Noticing that φ is uniformly continuous on G and $\alpha_{r_i} \to id_A$ in the point-norm topology. Consequently, $i_G : G \to UM(A \rtimes_\alpha G)$ is an injective strictly continuous homomorphism.

It is clear that (i_A, i_G) has the covariance with respect to (A, G, α).

Finally, let (π, U) be a non-degenerate covariant representation of (A, G, α). Then $\pi \rtimes U$ is a non-degenerate representation of $A \rtimes_\alpha G$ (cf. Proposition 5.1.3). For $a \in A$ and $f \in C_c(G, A)$, by (5.3.1), we have

$$(\pi \rtimes U)^-(i_A(a))(\pi \rtimes U)(f) = (\pi \rtimes U)(i_A(a)f) = \pi(a)((\pi \rtimes U)f).$$

By the density of $C_c(G, A)$ in $A \rtimes_\alpha G$ and the non-degeneracy of $\pi \rtimes U$, we have

$$(\pi \rtimes U)^-(i_A(a)) \;=\; \pi(a).$$

Similarly, for $s \in G$ and $f \in C_c(G, A)$, by (5.1.6), we have

$$(\pi \rtimes U)^-(i_G(s))(\pi \rtimes U(f)) = (\pi \rtimes U)(i_G(s)f) = U_s(\pi \rtimes U)(f).$$

For the same reason as given above, we can conclude that $(\pi \rtimes U)^-(i_G(s)) = U_s$. $\square$

As seen in Section 2.3, any unitary representation of G yields a representation of $L^1(G)$ and hence a representation of $\mathrm{C}^*(G)$. We show below that this can be extended to a unitary homomorphisms of G into the unitary group $UM(B)$ of the multiplier algebra of a C^*-algebra B.

PROPOSITION 5.3.2. *Let G be a locally compact group, let B be a C^*-algebra, and let $U : G \to M(B)$ be a strictly continuous homomorphism. Then there is a $*$-homomorphism $\widetilde{U} : C^*(G) \to M(B)$ such that*

$$\widetilde{U}(z) \;=\; \int_G z(s)U_s d\mu(s) \quad \textit{for all } z \in C_c(G).$$

PROOF. Let $z \in C_c(G)$. Then the map $G \to M(B)$, $s \mapsto z(s)U_s$ is strictly continuous with a compact support; that is, it is an element of $C_c(G, M_s(B))$. By Proposition 3.2.8, $\widetilde{U}(z) := \int_G z(s)U_s d\mu(s) \in M(B)$ is well defined, $\widetilde{U}(z^*) = \widetilde{U}(z)^*$, and $\widetilde{U}(z * \omega) = \widetilde{U}(z)\widetilde{U}(\omega)$ for $\omega \in C_c(G)$. Also, the $*$-homomorphism $\widetilde{U}$ on $C_c(G)$ is bounded with respect to the universal norm. Therefore, $\widetilde{U}$ can be extended to a $*$-homomorphism $\widetilde{U} : C^*(G) \to M(B)$. $\square$

COROLLARY 5.3.3. *Let (A, G, α) be a dynamical system. Let $a \in A$, $z \in C_c(G)$, and $g, h \in C_c(G, A)$. Then*

$$i_A(a)\widetilde{i_G}(z) \;=\; z \otimes a, \tag{5.3.2}$$

$$\int_G i_A(g(r))i_G(r)(h)d\mu(r) \;=\; g * h, \tag{5.3.3}$$

$$\int_G i_A(g(r))i_G(r)d\mu(r) \;=\; g. \tag{5.3.4}$$

Proof. Let (π, U) be a covariant representation of (A, G, α) with π injective and non-degenerate. Then, by Proposition 5.3.1, we have

$$\begin{aligned}
& (\pi \rtimes U)^-\Big(\int_G i_A(g(r))i_G(r)h d\mu(r)\Big) \\
= \ & (\pi \rtimes U)^-\Big(\int_G i_A(g(r))i_G(r)d\mu(r)\Big)(\pi \rtimes U)(h) \\
= \ & \int_G (\pi \rtimes U)^-(i_A(g(r)))(\pi \rtimes U)^-(i_G(r))d\mu(r)\,(\pi \rtimes U)(h) \\
= \ & \int_G \pi(g(r))U_r d\mu(r)\,(\pi \rtimes U)(h) \\
= \ & (\pi \rtimes U)(g)(\pi \rtimes U)(h) \\
= \ & (\pi \rtimes U)(g * h).
\end{aligned}$$

Since $\pi \rtimes U$ is injective (cf. Proposition 5.1.5), so is $(\pi \rtimes U)^-$. Thus (5.3.3) and (5.3.4) hold, noticing that $\pi \rtimes U$ is also non-degenerate.

Finally, taking $g = z \otimes a$ in (5.3.4), by Proposition 5.3.2, we have

$$\begin{aligned}
z \otimes a &= \int_G z(r)i_A(a)i_G(r)d\mu(r) \\
&= i_A(a)\int_G z(r)i_G(r)d\mu(r) \\
&= i_A(a)\widetilde{i_G}(z).
\end{aligned}$$

Therefore, we also have (5.3.4). $\square$

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