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Representations of banach algebras subordinate to topologically introverted subpaces

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Abstract

We show the existence of a natural bijection between continuous representations of a Banach algebra A on a reflexive Banach space Y subordinate to X, and normal representations of X^* on Y. We define the spaces $ap(A)$ and $wap(A)$ and study some of their properties. We show that if A has a bounded approximate identity, then a functional on A is in $wap(A)$ if and only if it is a coordinate function of a continuous representation of A on a reflexive Banach space. We prove that whenever A has a bounded right approximate identity, then a functional on A is in $luc(A)$ if and only if it is a coordinate function of some norm continuous representation of A on a dual Banach space.

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Contents

CHAPTER 1

Introduction and Preliminaries

1.1. Introduction

Suppose $\pi : A \longrightarrow \mathscr{L}(Y)$ is a continuous representation of a Banach algebra A on a Banach space Y (the precise definitions of the terms undefined below can be found later in this chapter). If $y \in Y$ and $\lambda \in Y^*$, then by the coordinate function of π corresponding to y and λ, we mean the continuous linear functional $\pi_{y,\lambda} \in A^*$, defined by $\pi_{y,\lambda}(a) = \langle \lambda, \pi(a)y \rangle$ $(a \in A)$. One of the main objectives of this thesis is to show that for certain types of representations, the coordinate functions $\pi_{y,\lambda}$ have specific analytic properties such as almost periodicity, weak almost periodicity, and left/right uniform continuity. Another major theme of this thesis is to study the converse problem. That is, if $f \in A^*$ has one of the properties of almost periodicity, weak almost periodicity, or left/right uniform continuity, whether we can represent f as a coordinate function of a representation of a specific type.

Below we shall give a brief description of the chapters in this thesis.

In the remainder of this chapter, we shall recall for the convenience of the reader some of the main definitions and results that are needed in the main part of the thesis. Unless otherwise mentioned, the proofs of the results in this chapter may be found in Conway [**4**], or Megginson [**37**].

Suppose A is a Banach algebra, X is a closed subspace of A^*, Y is a dual Banach space with predual Y_*, and π is a continuous representation of A on Y. We call π subordinate to X if $\pi_{y,\lambda} \in X$ for all $y \in Y$ and $\lambda \in Y_*$. In Chapter 2, we show the existence of a natural bijection between continuous representations of A on Y subordinate to X and normal representations of X^* on Y, whenever X is topologically left (right) introverted in A^* and Y is reflexive. This result can be regarded as a natural extension of the well known correspondence between the representations of a C^*-algebra and the representations of its enveloping von Neumann algebra (see Dixmier [**13**, (12.1.5)]).

We devote Chapter 3 to [weakly] almost periodic functionals, which have been studied by several authors in recent decades (see Dunkl–Ramirez [**17, 18**], Granirer [**25**], Ülger [**50**], Lau [**34**], Lau–Wong [**36**], Duncan–Ülger [**15**], and more recently, Hu [**30**], Mustafayev [**39**], Daws [**10, 11, 12**], Runde [**44, 45**] and Young [**51**]). We define the spaces $ap(A)$ and $wap(A)$ and study some of their properties. Our main interest in the space of weakly almost periodic functionals lies in the fact that it is a topologically introverted subspace of A^*.

It is known that for an involutive Banach algebra A with a bounded aproximate identity, every positive linear functional on A is a coordinate function of an involutive representation on some Hilbert space (see for example, Dixmier [**13**, Proposition 2.4.4]). Being a coordinate function of a representation on some Hilbert space, a positive linear functional is in fact weakly almost periodic by Young [**51**, page 102] or Filali-Monfared [**19**, Lemma 2.3]. Our aim in Chapter 4 is to extend this result on positive linear functionals to weakly almost periodic functionals. We show that if a Banach algebra A has a bounded approximate identity, then every weakly almost periodic functional on A is a coordinate function of a representation of A, on some reflexive Banach space Y, subordinate to $wap(A)$.

The spaces $luc(A)$ and $ruc(A)$ of left and right uniformly continuous functionals on A are studied in Chapter 5. We show that a function on a locally compact group G is left uniformly continuous if and only if it is a coordinate function of the conjugate representation of $L^1(G)$ associated to some unitary representation of G. We generalize the latter result to an arbitrary Banach algebra with bounded right approximate identity. We prove that the functionals in $luc(A)$ are all coordinate functions of some norm continuous representation of A on a dual Banach space.

Author's contribution. The results in this thesis are primarily from the paper by Filali–Neufang–Monfared [**20**]. Additionally, the results on [weakly] almost periodic functionals presented in Chapter 3 are mostly drawn from Duncan–Hosseiniun [**14**] and Duncan–Ülger [**15**]. Additional sources are mentioned in the relevant sections. The author's main contribution in this thesis has been to provide full details of the main results presented in the thesis when this can be done without diverging too much from the main subject of the research.

Among the results whose proofs have been substantially expanded, we can mention Theorems 2.18, Propositions 3.6 and 3.8, Theorem 3.12, Theorem 4.5, Lemma 4.7, Theorem 4.8, Lemmas 5.1 and 5.5, Theorems 5.7, 5.9, 5.10, 5.12 and Corollaries 5.11 and 5.13.

1.2. Banach spaces

DEFINITION 1.1. A *Banach space* X is a normed space which is complete with respect to the metric defined by the norm.

EXAMPLE 1.2. (1) The field $\mathbb{C}$ of complex numbers is a Banach space with the absolute value as the norm.

(2) Suppose X is a normed space and Y is a Banach space. Then the space $\mathscr{L}(X,Y)$ of all continuous linear operators from X into Y is a Banach space with the operator norm, defined by

$$\|T\| = \sup_{\substack{x\in X\\ \|x\|\leqslant 1}} \|T(x)\| \qquad (T \in \mathscr{L}(X,Y)).$$

DEFINITION 1.3. Let X be a normed space over field $\mathbb{F}$. The *dual space* of X, denoted by X^*, is the Banach space $\mathscr{L}(X,\mathbb{F})$ of all continuous linear functionals on X.

We define the second dual of X by $X^{**} = (X^*)^*$, and we embed X into X^{**} as follows.

PROPOSITION 1.4. *Let X be a normed space. The map*

$$\iota : X \longrightarrow X^{**}, \quad x \mapsto \hat{x},$$

where $\hat{x}(f) = f(x)$ for all $f \in X^*$, is an isometric isomorphism from X into X^{**}. Moreover, the subspace $\iota(X)$ of X^{**} is closed if and only if X is a Banach space.

DEFINITION 1.5. The map ι in the preceding proposition is called the *canonical embedding* of X into X^{**}.

The canonical embedding is not surjective in general. If $\iota(X) = X^{**}$, then we say that X is *reflexive*.

Every Banach space is endowed with the norm topology. Our next definition presents two additional well-known topologies which we will be used frequently in this thesis.

DEFINITION 1.6. Let X be a Banach space.

(i) The *weak topology* on X, denoted by $\sigma(X, X^*)$ (or simply w if there is no confusion), is the topology on X defined by the family of seminorms $\{p_f : f \in X^*\}$, where

$$p_f(x) = |f(x)| \qquad (x \in X).$$

Thus a net (x_α) in X converges to x in the w-topology if and only if $\lim\limits_{\alpha} f(x_\alpha) = f(x)$ for all $f \in X^*$.

(ii) The *weak* topology* on X^*, denoted by $\sigma(X^*, X)$ (or simply w^* if there is no confusion), is the topology on X^* defined by the family of seminorms $\{p_x : x \in X\}$, where

$$p_x(f) = |\hat{x}(f)| \qquad (f \in X^*).$$

Thus a net (f_α) in X^* converges to f in the w^*-topology if and only if $\lim\limits_{\alpha} f_\alpha(x) = f(x)$ for all $x \in X$.

The w-topology is weaker than the norm topology. On the dual space of any Banach space, the w^*-topology is the weaker than the w-topology.

We end this section by presenting Goldstine's theorem.

THEOREM 1.7. **(Goldstine's Theorem)** *Let X be a Banach space. Let $\iota : X \longrightarrow X^{**}$, $x \mapsto \hat{x}$ be the canonical embedding of X into X^{**}. If B_X and $B_{X^{**}}$ denote the closed unit balls of X and X^{**}, respectively, then $\iota(B_X)$ is w^*-dense in $B_{X^{**}}$, and $\iota(X)$ is w^*-dense in X^{**}.*

1.3. Convergence results

If W is a set and $f : W \longrightarrow [0, \infty)$ is a function, then we define $\sum\limits_{x \in W} f(x)$ to be the supremum of its finite partial sums, i.e.,

$$\sum_{x \in W} f(x) = \sup\{\sum_{x \in F} f(x) : F \subset W,\ F \text{ finite}\}.$$

Our next result can be found in Folland [**22**, Proposition 0.20].

PROPOSITION 1.8. *Given a function* $f : W \longrightarrow [0,\infty)$, *let* $S = \{x \in W : f(x) > 0\}$. *If* S *is uncountable, then* $\sum_{x\in W} f(x) = \infty$. *If* S *is countably infinite, then* $\sum_{x\in W} f(x) = \sum_{n=1}^{\infty} f(g(n))$, *where* $g : \mathbb{N} \longrightarrow S$ *is any bijection and the sum on the right is an ordinary infinite series.*

The frequently used consequence of the above proposition is that if X is a normed vector space and $\{x_\alpha\}_{\alpha\in I}$ is a subset of X such that $\sum_{\alpha\in I} \|x_\alpha\| < \infty$, then only countably many elements of $\{x_\alpha\}_{\alpha\in I}$ are different from zero. We will use this result in the work below, without explicitly referring to Proposition 1.8.

Recall that for a Banach space X, a series $\sum_n x_n$ is said to be absolutely convergent if $\sum_n \|x_n\|$ converges.

THEOREM 1.9. *A normed space* X *is a Banach space if and only if each absolutely convergent series in* X *converges.*

Many subsets of normed spaces that we work with are convex. For such sets the weak and norm closures are the same. This is justified by the following well-known result.

THEOREM 1.10. **(Mazur's Theorem)** *A convex subset of a Banach space is norm closed if and only if it is weakly closed.*

Our final result of this section is due to Grothendieck [**26**, Theorem 6].

THEOREM 1.11. **(Grothendieck's criterion)** *Let* E *be any topological space and let* $\mathscr{C}(E)$ *denote the Banach space of all bounded continuous functions on* E. *Then a set* $A \subset \mathscr{C}(E)$ *is weakly relatively compact if and only if it is bounded and it is impossible to find sequences* $\{f_i\}$ *in* A *and* $\{x_j\}$ *in* E *such that the limits* $\lim_i \lim_j f_i(x_j)$ *and* $\lim_j \lim_i f_i(x_j)$ *both exist and are distinct.*

1.4. Duals of subspaces

Let X be a normed space. Recall that for a subset $S \subset X$, the annihilator of S is defined by

$$S^\perp = \{f \in X^* : f|_S \equiv 0\}.$$

THEOREM 1.12. *Let M be a closed subspace of a normed space X. Then there is an isometric isomorphism between M^* and $X^*/M^\perp$ that identifies $f + M^\perp \in X^*/M^\perp$ with $f|_M \in M^*$.*

As a consequence of Theorem 1.12, we readily identify M^* with $X^*/M^\perp$ in our arguments. We can prove the following useful result as an easy corollary of Goldstine's theorem.

COROLLARY 1.13. *Suppose X is a Banach space, and M is a closed subspace of X^*. Then the image $\dot{X}$ of X, under the canonical mapping*

$$X \hookrightarrow X^{**} \longrightarrow X^{**}/M^\perp \cong M^*,$$

is w^*-dense in M^*:

$$\overline{\dot{X}}^{\sigma(M^*,M)} = M^*. \tag{1}$$

PROOF. Theorem 1.7 implies that for all $F \in X^{**}$, there exists a net $\{x_\alpha\}$ in X such that $\hat{x}_\alpha \xrightarrow{w^*} F$. Hence,

$$\lim_\alpha \langle \dot{x}_\alpha, m\rangle = \lim_\alpha \langle \hat{x}_\alpha + M^\perp, m\rangle = \lim_\alpha \langle \hat{x}_\alpha, m\rangle = \langle F, m\rangle = \langle F + M^\perp, m\rangle.$$

Therefore,

$$\dot{x}_\alpha \xrightarrow{w^*} F + M^\perp \in M^*.$$

$\square$

We end this section by introducing faithful subspaces of X^*.

DEFINITION 1.14. Let X be a Banach space and M be a closed subspace of X^*. Then M is said to be *faithful*, if $x = 0$ whenever $x \in X$ and $f(x) = 0$ for all $f \in M$.

LEMMA 1.15. *Let X be a Banach space and M be a closed faithful subspace of X^*. Then the canonical mapping*

$$X \hookrightarrow X^{**} \longrightarrow X^{**}/M^\perp \cong M^*$$

is injective, and hence $X \cong \dot{X}$.

PROOF. Suppose $\dot{x} = 0$ for some $x \in X$. Then $\hat{x} + M^{\perp} = 0$ and so $\hat{x} \in M^{\perp}$. Therefore, $\hat{x}(f) = 0$ for all $f \in M$. Thus $f(x) = 0$ for all $f \in M$ and hence $x = 0$, since M is faithful. □

1.5. Banach algebras

Our main objects of study in this research are Banach algebras and their duals. For the definition of an algebra see Hewitt–Ross [**27**, Definition C.1].

DEFINITION 1.16. Let A be an algebra over $\mathbb{C}$. Then A is a *Banach algebra* if A is a Banach space with a norm $\|\cdot\|$, which satisfies $\|ab\| \leqslant \|a\|\|b\|$ for all $a, b \in A$.

If A has an identity (an element e of A such that for all $a \in A$, $ae = ea = a$), then we say that A is *unital*.

EXAMPLES 1.17.
(1) Suppose Y is a Banach space. The space $\mathscr{L}(Y)$ of all continuous linear operators on Y is a Banach algebra with composition of operators as the product.
(2) Let G be a locally compact group and μ a left Haar measure on G. The space $L^1(G)$ of all (equivalence classes of) integrable functions on G (i.e., measurable functions f on G satisfying $\|f\|_1 = \int_G |f(s)| d\mu(s) < \infty$) is a Banach algebra, called the *group algebra* of G, with the convolution product $*$ defined by
$$(f * g)(t) = \int_G f(s) g(s^{-1}t) d\mu(s) \qquad (f, g \in L^1(G), t \in G).$$
The group algebra $L^1(G)$ is unital if and only if G is discrete, and is commutative if and only if G is commutative (see Rickart [**41**, Section A.3.1]).

We also work extensively with Banach A-bimodules.

DEFINITION 1.18. **(Banach module)** Let A be a Banach algebra. A *[right] left Banach A-module* is a Banach space E such that E is a [right] left A-module, and there exists $C > 0$ such that for all $a \in A$ and $x \in E$, we have
$$[\|x \cdot a\| \leqslant C\|a\|\,\|x\|]\ \|a \cdot x\| \leqslant C\|a\|\,\|x\|.$$

If E is a both left and right Banach A-module, then it is said to be a Banach A-bimodule, provided that $a \cdot (x \cdot b) = (a \cdot x) \cdot b$ for all $a, b \in A$ and $x \in E$.

EXAMPLE 1.19. The simplest example of a Banach A-bimodule is A itself with the multiplication as the module action. The dual of A has a canonical Banach A-bimodule structure which is defined by the operations (1) and (2) in Definition 1.20 below (see also Lemma 1.21).

The second dual A^{**} of a Banach algebra A can be turned into a Banach algebra by defining an appropriate product as we show below.

DEFINITION 1.20. **(Arens products)** Let A be a Banach Algebra. For all $a, b \in A$, $\lambda \in A^*$ and $\Psi, \Phi \in X^{**}$, we define $\Psi \square \Phi$ and $\Psi \Diamond \Phi$ in A^{**} in stages as follows:

(1) $\lambda \cdot a \in A^*$ is defined by $\langle \lambda \cdot a, b \rangle = \langle \lambda, ab \rangle$.
(2) $a \cdot \lambda \in A^*$ is defined by $\langle a \cdot \lambda, b \rangle = \langle \lambda, ba \rangle$.
(3) $\Phi \cdot \lambda \in A^*$ is defined by $\langle \Phi \cdot \lambda, a \rangle = \langle \Phi, \lambda \cdot a \rangle$.
(4) $\lambda \cdot \Psi \in A^*$ is defined by $\langle \lambda \cdot \Psi, a \rangle = \langle \Psi, a \cdot \lambda \rangle$.
(5) $\Psi \square \Phi \in A^{**}$ is defined by $\langle \Psi \square \Phi, \lambda \rangle = \langle \Psi, \Phi \cdot \lambda \rangle$.
(6) $\Psi \Diamond \Phi \in A^{**}$ is defined by $\langle \Psi \Diamond \Phi, \lambda \rangle = \langle \Phi, \lambda \cdot \Psi \rangle$.

The products $\square$ and $\Diamond$ are called the first and second Arens products on A^{**}, respectively. We note that $\Psi \square \Phi$ and $\Psi \Diamond \Phi$ are not equal in general. We say A is *Arens regular* if $\Psi \square \Phi = \Psi \Diamond \Phi$ for all $\Psi, \Phi \in A^{**}$.

We will use the following results extensively. The proofs can be found in Appendix B.

LEMMA 1.21. *With the operations defined in 1.20 above, we have*

(i) A^* *is a Banach A-bimodule with the canonical module actions defined in (1) and (2).*
(ii) $\Psi \cdot (\lambda \cdot a) = (\Psi \cdot \lambda) \cdot a$ *and* $(a \cdot \lambda) \cdot \Psi = a \cdot (\lambda \cdot \Psi)$.
(iii) *Each of the first and second Arens products turns A^{**} into a Banach algebra.*

(iv) *A^* is a left Banach $(A^{**}, \square)$-module with the canonical left module action defined in (3), and A^* is a right Banach $(A^{**}, \lozenge)$-module with the canonical right module action defined in (4).*

(v) *Both $\square$ and $\lozenge$ are extensions of the product of A to A^{**}, in the sense that if $a, b \in A$ and $\hat{a}, \hat{b}$ are the canonical images of a and b in A^{**}, then $\widehat{ab} = \hat{a}\square\hat{b} = \hat{a}\lozenge\hat{b}$.*

The following lemma follows easily from the definition.

LEMMA 1.22. *Let A be a Banach algebra, $a \in A$, $\lambda \in A^*$, and $\Psi, \Phi \in A^{**}$. Let $\hat{a}$ denote the canonical image of a in A^{**}. Then*

(1) $\lambda \cdot a = \lambda \cdot \hat{a}$ and $a \cdot \lambda = \hat{a} \cdot \lambda$;

(2) $\Psi\square\hat{a} = \Psi\lozenge\hat{a}$ and $\hat{a}\square\Psi = \hat{a}\lozenge\Psi$;

(3) the maps $\Psi \mapsto \Psi\square\Phi$ and $\Psi \mapsto \Phi\lozenge\Psi$ are w^*-continuous;

(4) the maps $\Psi \mapsto \hat{a}\square\Psi$ and $\Psi \mapsto \Psi\lozenge\hat{a}$ are w^*-continuous.

We end this section by introducing C^*-algebras.

DEFINITION 1.23. An *involution* on an algebra A is a map $a \mapsto a^*$ on A, that satisfies

$$(x+y)^* = x^* + y^*, \quad (\lambda x)^* = \bar{\lambda} x^*, \quad (xy)^* = y^* x^*, \quad x^{**} = x$$

for all $x, y \in A$ and $\lambda \in \mathbb{C}$.

DEFINITION 1.24. A *C^*-algebra* is a Banach algebra A equipped with an involution which satisfies

$$\|x^* x\| = \|x\|^2 \text{ for all } x \in A.$$

EXAMPLE 1.25. Suppose H is a Hilbert space. Then the Banach algebra $\mathscr{L}(H)$, defined in Example 1.17.1, is C^*-algebra. For each $T \in \mathscr{L}(H)$, the involution of T is the adjoint operator $T^* \in \mathscr{L}(H)$ defined by

$$\langle T^*\xi|\eta\rangle = \langle \xi|T\eta\rangle \quad \text{for all } \xi, \eta \in H.$$

1.6. Representations of Banach algebras

DEFINITION 1.26. **(Representation)** A *representation* of a Banach algebra A on a Banach space Y is an algebra homomorphism

$$\pi : A \longrightarrow \mathscr{L}(Y).$$

(1) π is said to be *faithful* if $a = 0$ whenever $\pi(a) = 0$ (i.e., $ker(\pi) = \{0\}$).
(2) π is *non-degenerate* if the linear span of the set $\{\pi(a)y : a \in A,\ y \in Y\}$ is dense in Y.
(3) A closed subspace $M \subset Y$ is said to be *invariant* under π, if $\pi(a)M \subset M$ for all $a \in A$. Clearly, $\{0\}$ and Y are trivially invariant for any representation of A on Y. If π does not have any non-trivial invariant subspace, then π is said to be *topologically irreducible*.
(4) Two representations, $\pi_1 : A \longrightarrow \mathscr{L}(Y_1)$ and $\pi_2 : A \longrightarrow \mathscr{L}(Y_2)$, are said to be *equivalent* if there exists a Banach space isomorphism $V : Y_1 \longrightarrow Y_2$ such that $V\pi_1(a)V^{-1} = \pi_2(a)$ for all $a \in A$.

DEFINITION 1.27. Let X be a Banach space. Then X is a *dual Banach space* if there exists a Banach space X_* such that $X = (X_*)^*$. The space X_* is called a predual of X.

DEFINITION 1.28. **(Continuity of representations)** Let A be a Banach algebra, Y a Banach space and $\pi : A \longrightarrow \mathscr{L}(Y)$ a representation of A on Y.

(1) We say π is *continuous* if it is continuous with respect to the norm topologies of A and $\mathscr{L}(Y)$.
(2) If both A and Y are dual spaces, then we say that π is *w^*-continuous*, if it is continuous with respect to the w^*-topologies of A and $\mathscr{L}(Y) \cong (Y\widehat{\otimes}Y_*)^*$, where $Y\widehat{\otimes}Y_*$ denotes the projective tensor product of Y and Y_* (see Ryan [**46**] for definition).
(3) If both A and Y are dual spaces and π is continuous, then we say π is *normal* if it is also w^*-continuous.

REMARK 1.29. Our definition of normality for representations is analogous to that in Takesaki [**49**, Definition III.2.15], where a continuous linear mapping between two von Neuman algebras is normal if it is also continuous with respect to the w^*-topologies.

DEFINITION 1.30. **(Coordinate function)** Let A be a Banach algebra, Y be a Banach space, and $\pi : A \longrightarrow \mathscr{L}(Y)$ be a representation of A on Y. The *coordinate function* of π corresponding to $y \in Y$ and $\lambda \in Y^*$ is the linear functional $\pi_{y,\lambda}$ on A defined by

$$\pi_{y,\lambda}(a) = \langle \pi(a)y, \lambda \rangle \qquad (a \in A).$$

LEMMA 1.31. *If π is continuous, then $\pi_{y,\lambda} \in A^*$ for all $y \in Y$ and $\lambda \in Y^*$. If both A and Y are dual Banach spaces and π is w^*-continuous, then $\pi_{y,\lambda} \in A^*$ is w^*-continuous for all $y \in Y$ and $\lambda \in Y_*$.*

PROOF. The linearity of $\pi_{y,\lambda}$ follows immediately from the linearity of π and the linearity of elements of Y^*. In fact, for fixed $y \in Y$ and $\lambda \in Y^*$, and for all $a, b \in A$ and scalars α and β, we have

$$\begin{aligned}
\pi_{y,\lambda}(\alpha a + \beta b) &= \langle \pi(\alpha a + \beta b)y, \lambda \rangle \\
&= \langle \alpha\pi(a)y + \beta\pi(b)y, \lambda \rangle \\
&= \alpha\langle \pi(a)y, \lambda \rangle + \beta\langle \pi(b)y, \lambda \rangle \\
&= \alpha\pi_{y,\lambda}(a) + \beta\pi_{y,\lambda}(b).
\end{aligned}$$

Next we verify that $\pi_{y,\lambda}$ is continuous. In fact, we have

$$\|\pi_{y,\lambda}(a)\| = \|\langle \pi(a)y, \lambda \rangle\| \leqslant \|\pi(a)y\|\|\lambda\| \leqslant \|\pi(a)\|\|y\|\|\lambda\| \leqslant (\|\pi\|\|y\|\|\lambda\|)\|a\|.$$

Thus $\pi_{y,\lambda}$ is a bounded linear functional, proving that $\pi_{y,\lambda}$ is continuous.

In the case when A and Y are dual spaces and π is w^*-continuous, if (a_α) is a net in A and $a_\alpha \to a$ in the w^*-topology of A, then $\pi(a_\alpha) \to \pi(a)$ in the w^*-topology of $\mathscr{L}(Y) = (Y \widehat{\otimes} Y_*)^*$. Hence if $y \in Y$ and $\lambda \in Y_*$, then

$$\pi_{y,\lambda}(a_\alpha) = \langle \pi(a_\alpha)y, \lambda \rangle = \langle \pi(a_\alpha), y \otimes \lambda \rangle \longrightarrow \langle \pi(a), y \otimes \lambda \rangle = \langle \pi(a)y, \lambda \rangle = \pi_{y,\lambda}(a),$$

proving that $\pi_{y,\lambda}$ is w^*-continuous. $\square$

We will sometimes work with *anti-representations* of A on Y, which are simply algebra anti-homomorphisms. Thus, if π is an anti-representation, then $\pi(ab) = \pi(b)\pi(a)$ for all $a, b \in A$. The terminology and results which are given above for representations apply to anti-representations without change.

1.7. Representations of groups

If X is a Banach space, we will let $Is(X) \subset \mathscr{L}(X)$ denote the group of all linear isometries of X:

$$Is(X) = \{T \in \mathscr{L}(X) : \|Tx\| = \|x\| \text{ for all } x \in X\}.$$

We will assume that $Is(X)$ carries the strong operator toplogy (SOT) (i.e., $T_\alpha \xrightarrow{SOT} T$ in $Is(X)$ if and only if $\|T_\alpha x - Tx\| \longrightarrow 0$ for all $x \in X$). See Takesaki [**49**, Section II.2] for detailed definitions of various topologies on $\mathscr{L}(X)$.

A unitary operator on a Hilbert space is defined to be a linear transformation $H \longrightarrow H$ that is a surjective isometry (see Conway [**4**, Page 20]). If H is a Hilbert space, then $U(H) \subset Is(H)$ denotes the group of unitary operators on H.

Analogously to our definition of Banach algebra representations, we now define representations of groups.

Definition 1.32. A *representation* of a topological group G on a Banach space X is a group homomorphism $V : G \longrightarrow \mathscr{L}(X)$. The representation V is *continuous* if it is continuous with respect to the SOT topology of $\mathscr{L}(X)$ and given topology of G.

We will often work with representations where the target Banach space is in fact a Hilbert space and the representation is unitary as defined below.

Definition 1.33. **(Unitary representation)** Let G be a locally compact group and H a Hilbert space. A representation $V : G \longrightarrow \mathscr{L}(H)$ is said to be *unitary* if $V(x)$ is a unitary operator on H for every $x \in G$ and V is continuous in the sense of Definition 1.32.

Thus a unitary representation $V : G \longrightarrow \mathscr{L}(H)$ satisfies $V(xy) = V(x)V(y)$ and $V(x^{-1}) = V(x)^{-1} = V(x)^*$, and the map $G \longrightarrow H, x \mapsto V(x)\xi$ is continuous for every $\xi \in H$.

EXAMPLE 1.34. For a locally compact group G, the *left regular representation* of G on $L^2(G)$ is the map

$$V : G \longrightarrow \mathscr{L}(L^2(G))$$

defined by

$$V(x)f = {}_{x^{-1}}f \qquad (f \in L^2(G)).$$

A verification that the left regular representation is unitary is presented in Folland [**21**, page 68].

The definitions 1.32 and 1.33 can be stated for anti-representations with the replacement of the homomorphism by an anti-homomorphism.

CHAPTER 2

Representations of the Dual of Introverted Subspaces

2.1. Dual Banach algebras

In this section, we introduce dual Banach algebras. We start by presenting some introductory materials regarding dual Banach spaces.

LEMMA 2.1. *Let X be a dual Banach space with a predual X_* and let $Y \subset X$ be a w^*-closed subspace of X. In this case, Y is a dual Banach space and $Y_* = X_*/{}^{\perp}Y$ is a predual of Y.*

PROOF. By definition,

$$^{\perp}Y = \{\lambda \in X_* : \langle y, \lambda \rangle = 0 \text{ for all } y \in Y\}.$$

This is a norm closed subspace of X_*, so we can form the Banach quotient space $X_*/{}^{\perp}Y$. By the standard results (see Megginson [**37**, Theorem 1.10.17 and Proposition 2.6.6(c)]), we have

$$(X_*/{}^{\perp}Y)^* \cong ({}^{\perp}Y)^{\perp} = \overline{Y}^{w^*} = Y.$$

In other words, $X_*/{}^{\perp}Y$ is a predual to Y. □

Before we present the next lemma, we recall the following proposition from Conway [**4**, Proposition A.1.4].

PROPOSITION 2.2. *Let X be a vector space and $f, f_1, \ldots, f_n$ be linear functionals on X. If $\bigcap_{k=1}^{n} ker f_k \subset ker f$, then there are scalars $\alpha_1, \ldots, \alpha_n$ such that $f = \sum_{k=1}^{n} \alpha_k f_k$.*

PROOF. It may be assumed without loss of generality that for $k = 1, 2, \ldots, n$, $\bigcap_{j \neq k} ker f_j \neq \bigcap_{j=1}^{n} ker f_j$. So for $k = 1, 2, \ldots, n$, there is a y_k in $\bigcap_{j \neq k} ker f_j$ such that $y_k \notin \bigcap_{j=1}^{n} ker f_j$. So $f_j(y_k) = 0$ for $j \neq k$ but $f_k(y_k) \neq 0$. Let $x_k = \frac{y_k}{f_k(y_k)}$. Then $f_k(x_k) = 1$ and $f_j(x_k) = 0$ for all $j \neq k$. Now, let f be as in the statement of the proposition and put $\alpha_k = f(x_k)$. If $x \in X$, let $y = x - \sum_{k=1}^{n} f_k(x)x_k$. Then $f_j(y) = f_j(x) - \sum_{k=1}^{n} f_k(x)f_j(x_k)$ for $j = 1, \ldots, n$, and hence $f(y) = 0$ by the hypothesis. Thus

$$0 = f(x) - \sum_{k=1}^{n} f_k(x)f(x_k) = f(x) - \sum_{k=1}^{n} \alpha_k f_k(x);$$

that is, $f = \sum_{k=1}^{n} \alpha_k f_k$. □

Lemma 2.3. *Let X be a Banach space with a predual X_*. Then X_* is isometrically isomorphic to the closed subspace of X^* consisting of all w^*-continuous linear functionals on X.*

Proof. Clearly we can identify X_* with its canonical image in X^*. This identification is isometric and by Goldstine's theorem, X_* is w^*-dense in X^*. If $\lambda \in X_*$, then λ is w^*-continuous on X by definition of the w^*-topology. Suppose that $\lambda \in X^*$ is continuous in the w^*-topology. Then from the continuity criterion of linear functionals on locally convex spaces (Conway [**4**, Theorem IV.3.1(f)]), it follows that there are $\mu_1, \ldots, \mu_n \in X_*$ such that $|\lambda(x)| \leqslant \sum_{k=1}^{n} |\mu_k(x)|$ for all $x \in X$. But then $\bigcap_{k=1}^{n} ker\mu_k \subset ker\lambda$, and hence for some scalars $\alpha_1, \ldots, \alpha_n$, $\lambda = \sum_{k=1}^{n} \alpha_k \mu_k$ (Proposition 2.2 above). It follows that $\lambda \in X_*$. □

We may now define dual Banach algebras and present some related results and examples.

Definition 2.4. A Banach algebra A is said to be a *dual* Banach algebra with a predual A_* if A_* is a Banach space such that $(A_*)^* = A$ and the multiplication on A is separately w^*-continuous (i.e., if (a_α) is a net in A and $a_\alpha \to a \in A$, then $a_\alpha b \xrightarrow{w^*} ab$ and $ba_\alpha \xrightarrow{w^*} ba$ for every $b \in A$).

Lemma 2.5. *A Banach algebra A is a dual Banach algebra if and only if there is a norm closed Banach A-bisubmodule A_* of A^* such that $(A_*)^* \cong A$ as Banach A-bimodules (i.e., there exists an isometric A-bimodule isomorphism between $(A_*)^*$ and A).*

PROOF. Suppose A_* is a norm closed Banach A-bisubmodule of A^* such that $(A_*)^* \cong A$ as Banach A-bimodules. Let $a_\alpha \xrightarrow{w^*} a \in A$, and let $a' \in A$. Then for every $f \in A_*$,

$$\lim_\alpha \langle a_\alpha a', f\rangle = \lim_\alpha \langle a_\alpha, a' \cdot f\rangle = \langle a, a' \cdot f\rangle = \langle aa', f\rangle.$$

Therefore, $a_\alpha a' \xrightarrow{w^*} aa'$. The proof that $a' a_\alpha \xrightarrow{w^*} a'a$ is similar.

Conversely, if A is a dual Banach algebra with a predual A_*, then we can take A_* to be the space of all w^*-continuous functionals on A (by Lemma 2.3). The above argument shows that A_* is a Banach A-bisubmodule of A^*. Clearly, we now have $(A_*)^* \cong A$ as Banach A-bimodules. □

LEMMA 2.6. *If A is a dual Banach algebra, then the unitization $A^\sharp$ of A is also a dual Banach algebra.*

PROOF. Let A be a dual Banach algebra with a predual A_*. We clearly have the Banach space identifications

$$A^\sharp = A \oplus_1 \mathbb{C} = (A_*)^* \oplus_1 \mathbb{C} = (A_* \oplus_\infty \mathbb{C})^*,$$

so it suffices to verify the continuity requirement.

Suppose $((a,\lambda)_\alpha)$ is a net in $A^\sharp$ such that $(a,\lambda)_\alpha \xrightarrow{w^*} (a,\lambda)$. By letting a_α and λ_α denote the projections of $(a,\lambda)_\alpha$ on its first and second components, respectively, we have

$$\langle (a_\alpha, \lambda_\alpha), (a_*, 0)\rangle = \langle a_\alpha, a_*\rangle \longrightarrow \langle (a,\lambda), (a_*, 0)\rangle = \langle a, a_*\rangle$$

for all $(a_*, 0) \in A_* \oplus \mathbb{C}$, and

$$\langle (a,\lambda)_\alpha, (0,1)\rangle = \lambda_\alpha \longrightarrow \langle (a,\lambda), (0,1)\rangle = \lambda.$$

Thus $a_\alpha \xrightarrow{w^*} a$ and $\lambda_\alpha \longrightarrow \lambda$. So $a_\alpha b \xrightarrow{w^*} b$ for every $b \in B$.

Let $(b,\beta) \in A^\sharp$ be arbitrary. Then, for all $(a_*, \phi) \in A_* \oplus \mathbb{C}$, we have

$$\begin{aligned}
\langle (a,\lambda)_\alpha (b,\beta), (a_*,\phi)\rangle &= \langle (a_\alpha, \lambda_\alpha)(b,\beta), (a_*,\phi)\rangle \\
&= \langle (a_\alpha b + \lambda_\alpha b + \beta a_\alpha, \lambda_\alpha \beta), (a_*,\phi)\rangle \\
&= \langle a_\alpha b, a_*\rangle + \lambda_\alpha \langle b, a_*\rangle + \beta \langle a_\alpha, a_*\rangle + \lambda_\alpha \beta \phi
\end{aligned}$$

$$\begin{aligned}
&\longrightarrow \langle ab, a_*\rangle + \lambda\langle b, a_*\rangle + \beta\langle a, a_*\rangle + \lambda\beta\phi \\
&= \langle ab + \lambda b + \beta a, a_*\rangle + \lambda\beta\phi \\
&= \langle (ab + \lambda b + \beta a, \lambda\beta), (a_*, \phi)\rangle \\
&= \langle (a, \lambda)(b, \beta), (a_*, \phi)\rangle.
\end{aligned}$$

Therefore, we have $(a, \lambda)_\alpha(b, \beta) \xrightarrow{w^*} (a, \lambda)(b, \beta)$. A similar calculation shows that $(b, \beta)(a, \lambda)_\alpha \xrightarrow{w^*} (b, \beta)(a, \lambda)$, proving that the multiplication on $A^\sharp$ is separately w^*-continuous. Hence $A^\sharp$ is a dual Banach algebra with respect to the predual $A^\sharp_* = A_* \oplus_\infty \mathbb{C}$. □

An analogue of Lemma 2.1 holds for dual Banach algebras.

LEMMA 2.7. *Let A be a dual Banach algebra with a predual A_*, and let $B \subset A$ be a w^*-closed subalgebra of A. Then B is a dual Banach algebra with a predual $B_* = A_*/{}^\perp B$.*

PROOF. In view of Lemma 2.1, it suffices to show that the multiplication on B is separately w^*-continuous with respect to $B_* = A_*/{}^\perp B$. Let $b_\alpha \xrightarrow{w^*} b \in B$. For each $f \in A_*$, let $\pi(f)$ denote the natural image of f in B_*. It follows from

$$\lim_\alpha \langle b_\alpha, f\rangle_{A,A_*} = \lim_\alpha \langle b_\alpha, \pi(f)\rangle_{B,B_*} = \langle b, \pi(f)\rangle_{B,B_*} = \langle b, f\rangle_{A,A_*}$$

that $b_\alpha \xrightarrow{w^*} b \in A$ as well. Therefore, for every $b' \in B$,

$$\lim_\alpha \langle b_\alpha b', \pi(f)\rangle_{B,B_*} = \lim_\alpha \langle b_\alpha b', f\rangle_{A,A_*} = \langle bb', f\rangle_{A,A_*} = \langle bb', \pi(f)\rangle_{B,B_*};$$

that is, $b_\alpha b' \xrightarrow{w^*} bb' \in B$. The assertion that $b' b_\alpha \xrightarrow{w^*} b'b \in B$ is proved similarly. □

EXAMPLES 2.8. (1) If Y is a reflexive Banach space, then $\mathscr{L}(Y)$ is a dual Banach algebra with a predual $\mathscr{L}(Y)_* = Y\widehat{\otimes}Y^*$. In fact, it is well-known (see Ryan [**46**, Page 24]) that $\mathscr{L}(Y) = (Y\widehat{\otimes}Y^*)^*$, so it suffices to verify the continuity requirement. Suppose (T_α) is a net in $\mathscr{L}(Y)$ and that $T_\alpha \xrightarrow{w^*} T \in \mathscr{L}(Y)$. Then for all $S \in \mathscr{L}(Y)$ and $u = \sum_{i=1}^{\infty} y_i \otimes \lambda_i \in Y \widehat{\otimes} Y^*$, we have

$$\begin{aligned}\langle T_\alpha S, u\rangle &= \langle T_\alpha S, \sum_{i=1}^{\infty} y_i \otimes \lambda_i\rangle = \sum_{i=1}^{\infty}\langle T_\alpha S, y_i \otimes \lambda_i\rangle = \sum_{i=1}^{\infty}\langle (T_\alpha S) y_i, \lambda_i\rangle \\ &= \sum_{i=1}^{\infty}\langle T_\alpha, S y_i \otimes \lambda_i\rangle = \langle T_\alpha, \sum_{i=1}^{\infty} S y_i \otimes \lambda_i\rangle \longrightarrow \langle T, \sum_{i=1}^{\infty} S y_i \otimes \lambda_i\rangle \\ &= \sum_{i=1}^{\infty}\langle T, S y_i \otimes \lambda_i\rangle = \sum_{i=1}^{\infty}\langle T S y_i, \lambda_i\rangle = \sum_{i=1}^{\infty}\langle TS, y_i \otimes \lambda_i\rangle = \langle TS, u\rangle.\end{aligned}$$

So $T_\alpha S \xrightarrow{w^*} TS$, and a similar argument shows that $ST_\alpha \xrightarrow{w^*} ST$.

(2) If A is an Arens regular Banach algebra, then $(A^{**}, \Box)$ is a dual Banach algebra with respect to the predual A^*. Suppose (Ψ_α) is a net in A^{**} and that $\Psi_\alpha \xrightarrow{w^*} \Psi \in A^{**}$. By Lemma 1.22, we immediately have $\Psi_\alpha \Box \Phi \xrightarrow{w^*} \Psi \Box \Phi$ and $\Phi \Diamond \Psi_\alpha \xrightarrow{w^*} \Phi \Diamond \Psi$ for every $\Phi \in A^{**}$. Since A is Arens regular, the products $\Box$ and $\Diamond$ coincide on A^{**}, and we conclude that the multiplication $\Box$ on A^{**} is separately w^*-continuous.

(3) Every von Neumann algebra (i.e., a w^*-closed, unital, $*$-subalgebra of $\mathscr{L}(H)$, where H is a Hilbert space) is a dual Banach algebra. This follows immediately from Lemma 2.7 and Example 2.8.1.

2.2. Introverted subspaces

Definition 2.9. Let A be a Banach algebra and $X \subset A^*$ a norm-closed Banach A-bisubmodule of A^*. For $\lambda \in X$ and $\Psi \in X^*$, we define $\Psi \cdot \lambda \in A^*$ and $\lambda \cdot \Psi \in A^*$ by

$$\langle \Psi \cdot \lambda, a\rangle = \langle \Psi, \lambda \cdot a\rangle \qquad (a \in A),$$

and

$$\langle \lambda \cdot \Psi, a\rangle = \langle \Psi, a \cdot \lambda\rangle \qquad (a \in A).$$

Definition 2.10. **(Introverted subspace)** Let A be a Banach algebra and $X \subset A^*$ a norm-closed Banach A-bisubmodule of A^*. X is said to be *topologically left introverted* if $\Psi \cdot \lambda \in X$ for all $\Psi \in X^*$ and $\lambda \in X$. X is called *topologically right introverted* if $\lambda \cdot \Psi \in X$ for all $\lambda \in X$ and $\Psi \in X^*$. X is called *topologically introverted* if it is both left and right introverted.

The following theorem relates topological introversion to dual Banach algebras. As the identification $(A_*)^* \cong A$ is given by a Banach A-bimodule isomorphism, the result is immediate.

THEOREM 2.11. *If A is a dual Banach algebra with respect to a predual A_*, then A_* is a topologically introverted subspace of A^*.*

The following proposition from Lau–Loy [**35**, Lemma 1.2] gives us a convenient method to check if a subspace of A^* is topologically introverted.

PROPOSITION 2.12. *Let A be a Banach algebra and X a norm closed Banach A-bisubmodule of A^*. Then X is topologically left introverted if and only if the $\sigma(A^*, A)$-closure of $B_A \cdot \lambda = \{a \cdot \lambda : a \in A,\ \|a\| \leqslant 1\}$ is a subset of X for all $\lambda \in X$. A similar statement holds for topologically right introverted subspaces.*

PROOF. Suppose that X is left introverted and let $\lambda \in X$. Given φ in the w^*-closure of $B_A \cdot \lambda$, there is a bounded net (a_α) in B_A with $a_\alpha \cdot \lambda \xrightarrow{w^*} \varphi$. Considering (a_α) as a bounded net in X^*, we let Φ_0 be a w^*-cluster point of (a_α) in X^*. By replacing (a_α) with a subnet if necessary, we may assume without loss of generality that $a_\alpha \xrightarrow{w^*} \Phi_0$. Then for all $b \in A$,

$$\langle \varphi, b \rangle = \lim_\alpha \langle a_\alpha \cdot \lambda, b \rangle = \lim_\alpha \langle a_\alpha, \lambda \cdot b \rangle = \langle \Phi_0, \lambda \cdot b \rangle = \langle \Phi_0 \cdot \lambda, b \rangle,$$

so that $\varphi = \Phi_0 \cdot \lambda \in X$.

Conversely, suppose that the w^*-closure of $B_A \cdot \lambda$ lies in X for all $\lambda \in X$. For $\Phi \in X^*$, let $\Psi \in A^{**}$ be a norm preserving extension of Φ (such an extension exists by the Hahn-Banach theorem). By Goldstine's theorem, there is a net (a_α) in A with $\|a_\alpha\| \leqslant \|\Phi\|$ and $a_\alpha \xrightarrow{w^*} \Psi \in A^{**}$. Then for $b \in A$,

$$\langle \Phi \cdot \lambda, b \rangle = \langle \Phi, \lambda \cdot b \rangle = \langle \Psi, \lambda \cdot b \rangle = \lim_\alpha \langle a_\alpha, \lambda \cdot b \rangle = \lim_\alpha \langle a_\alpha \cdot \lambda, b \rangle.$$

Therefore, $\Phi \cdot \lambda$ lies in the w^*-closure of $B_A \cdot \lambda$ and hence in X. Thus X is left introverted.

The proof for topologically right introverted subspaces of A^* is similar. $\square$

We will encounter several examples of topologically introverted subspaces of A^* in the following chapters.

2.3. Representations of the dual of introverted subspaces

DEFINITION 2.13. Let $\pi : A \longrightarrow \mathscr{L}(Y)$ be a continuous [anti-]representation of a Banach algebra A on a dual Banach space Y with a predual Y_*. Let X be a norm closed subspace of A^*. We call π *subordinate* to X if $\pi_{y,\lambda} \in X$ for all $y \in Y$ and $\lambda \in Y_*$

REMARK 2.14. In our definition of subordination, our choice of the coordinate functions $\pi_{y,\lambda}$, with λ in Y_* rather than in Y^*, is motivated by the important role played by the w^*-topology $\sigma(Y, Y_*)$ for dual Banach spaces Y in our work below. In fact, by Lemma 2.3, we have $(Y, w^*)^* = Y_*$, so the elements of Y_* are precisely the w^*-continuous functionals on Y. For this reason, our notion of subordination may also be called w^*-subordination.

The following lemma collects several useful properties of coordinate functions. First, we recall the following definition.

DEFINITION 2.15. Let E, F and G be normed spaces. A map $T : E \times F \longrightarrow G$ ia called a *bilinear* map if

$$T(x_1 + \alpha x_2, y) = T(x_1, y) + \alpha T(x_2, y)$$

and

$$T(x, y_1 + \alpha y_2) = T(x, y_1) + \alpha T(x, y_2)$$

for all $x_1, x_2 \in E$, $y_1, y_2 \in F$ and $\alpha \in \mathbb{C}$. A bilinear map is said to be continuous if there exists $C > 0$ such $\|T(x, y)\| \leqslant C\|x\|\|y\|$ for all $x \in E$ and $y \in F$.

LEMMA 2.16. *Let A be a Banach algebra, Y be a Banach space, and $\pi : A \longrightarrow \mathscr{L}(Y)$ be a continuous representation of A on Y. Then for all $a \in A$, $y \in Y$, and $\lambda \in Y^*$, we have*

(i) $a\cdot \pi_{y,\lambda} = \pi_{\pi(a)y,\lambda}$.
(ii) $\pi_{y,\lambda}\cdot a = \pi_{y,\pi(a)^*\lambda}$.
(iii) *The map $Y \times Y^* \longrightarrow A$, $(y, \lambda) \mapsto \pi_{y,\lambda}$ is bilinear.*
(iv) $\|\pi_{y,\lambda}\| \leqslant \|\pi\|\|y\|\|\lambda\|$, *and hence the bounded linear map $Y\widehat{\otimes}Y^* \longrightarrow A^*$ with $y \otimes \lambda \mapsto \pi_{y,\lambda}$ has a norm bounded by $\|\pi\|$.*

REMARK 2.17. Parts (i)-(iii) of the lemma hold for $(y, \lambda) \in Y \times Y_*$ if A and Y are dual spaces and π is w^*-continuous (see Definition 1.28).

PROOF. (i) For every $b \in A$,

$$\langle a \cdot \pi_{y,\lambda}, b \rangle = \langle \pi_{y,\lambda}, ba \rangle = \langle \pi(ba)y, \lambda \rangle = \langle \pi(b)(\pi(a)y), \lambda \rangle = \langle \pi_{\pi(a)y,\lambda}, b \rangle.$$

(ii) For every $b \in A$,

$$\begin{aligned} \langle \pi_{y,\lambda} \cdot a, b \rangle &= \langle \pi_{y,\lambda}, ab \rangle = \langle \pi(ab)y, \lambda \rangle = \langle \pi(a)(\pi(b)y), \lambda \rangle \\ &= \langle \pi(b)y, \pi(a)^* \lambda \rangle = \langle \pi_{y,\pi(a)^*\lambda}, b \rangle. \end{aligned}$$

(iii) For $a \in A$, $y_1, y_2 \in Y$, and scalars $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} \langle \pi_{\alpha y_1 + \beta y_2, \lambda}, a \rangle &= \langle \pi(a)(\alpha y_1 + \beta y_2), \lambda \rangle \\ &= \langle \alpha \pi(a) y_1 + \beta \pi(a) y_2, \lambda \rangle \\ &= \alpha \langle \pi(a) y_1, \lambda \rangle + \beta \langle \pi(a) y_2, \lambda \rangle \\ &= \alpha \langle \pi_{y_1,\lambda}, a \rangle + \beta \langle \pi_{y_2,\lambda}, a \rangle \\ &= \langle \alpha \pi_{y_1,\lambda} + \beta \pi_{y_2,\lambda}, a \rangle, \end{aligned}$$

so

$$\pi_{\alpha y_1 + \beta y_2, \lambda} = \alpha \pi_{y_1,\lambda} + \beta \pi_{y_2,\lambda}.$$

Similarly, for $a \in A$, $\lambda_1, \lambda_2 \in Y^*$, and scalars $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} \langle \pi_{y, \alpha\lambda_1 + \beta\lambda_2}, a \rangle &= \langle \pi(a)y, \alpha\lambda_1 + \beta\lambda_2 \rangle \\ &= \alpha \langle \pi(a)y, \lambda_1 \rangle + \beta \langle \pi(a)y, \lambda_2 \rangle \\ &= \alpha \langle \pi_{y,\lambda_1}, a \rangle + \beta \langle \pi_{y,\lambda_2}, a \rangle \\ &= \langle \alpha \pi_{y,\lambda_1} + \beta \pi_{y,\lambda_2}, a \rangle, \end{aligned}$$

so

$$\pi_{y, \alpha\lambda_1 + \beta\lambda_2} = \alpha \pi_{y,\lambda_1} + \beta \pi_{y,\lambda_2}.$$

(iv) This follows from the statements in the proof of Lemma 1.31. □

Next, we present the main result of this chapter.

THEOREM 2.18. *Let A be a Banach Algebra, X a topologically left (or right) introverted subspace of A^*, Y a reflexive Banach space, and $\pi : A \longrightarrow \mathscr{L}(Y)$ a continuous representation subordinate to X.*

(i) *The map*

$$\tilde{\pi} : X^* \longrightarrow \mathscr{L}(Y), \qquad \langle \tilde{\pi}(\Psi)y, \lambda \rangle = \langle \Psi, \pi_{y,\lambda} \rangle,$$

in which $\Psi \in X^*$, $y \in Y$, and $\lambda \in Y^*$, is a normal representation of X^* on Y. The image of $\tilde{\pi}$ is contained in the w^*-closure of the image of π, and for every $a \in A$, $\tilde{\pi}(\dot{a}) = \pi(a)$, where $\dot{a}$ is the canonical image of a in X^*.

(ii) *The map $\pi \mapsto \tilde{\pi}$ is a bijection between the set of all (equivalence classes of) norm continuous representations of A on Y which are subordinate to X and the set of all (equivalence classes of) normal representations of X^* on Y. Moreover, $\tilde{\pi}$ is topologically irreducible if and only if π is topologically irreducible.*

PROOF. (i) First, we show that $\tilde{\pi}(\Psi) \in \mathscr{L}(Y)$ for all $\Psi \in X^*$.
In fact, $\tilde{\pi}(\Psi)y$ has a well defined action on elements of Y^* given by

$$\langle \tilde{\pi}(\Psi)y, \lambda \rangle_{Y^{**},Y^*} = \langle \Psi, \pi_{y,\lambda} \rangle_{X^*,X}.$$

The linearity and continuity of this action follow directly from parts (iii) and (iv) of Lemma 2.16. Therefore, $\tilde{\pi}(\Psi)y \in Y^{**} = Y$. Moreover, for all $y \in Y$,

$$\begin{aligned}\langle \tilde{\pi}(\Psi)(\alpha y_1 + \beta y_2), \lambda \rangle &= \langle \Psi, \pi_{\alpha y_1 + \beta y_2, \lambda} \rangle = \langle \Psi, \alpha \pi_{y_1,\lambda} + \beta \pi_{y_2,\lambda} \rangle \\ &= \alpha \langle \Psi, \pi_{y_1,\lambda} \rangle + \beta \langle \Psi, \pi_{y_2,\lambda} \rangle = \alpha \langle \tilde{\pi}(\Psi)y_1, \lambda \rangle + \beta \langle \tilde{\pi}(\Psi)y_2, \lambda \rangle,\end{aligned}$$

where for the second equality, we used the linearity of $\pi_{y,\lambda}$ in y, proved in part (iii) of Lemma (2.16). Hence, $\tilde{\pi}(\Psi) : Y \longrightarrow Y$ is linear and $\|\tilde{\pi}(\Psi)\| \leqslant \|\pi\| \|\Psi\|$ (see Lemma 2.16(iv)). This completes the verification that $\tilde{\pi}(\Psi) \in \mathscr{L}(Y)$.

To prove that $\tilde{\pi}$ is a representation, we first observe that

$$\begin{aligned}\langle \tilde{\pi}(\alpha \Psi_1 + \beta \Psi_2)y, \lambda \rangle &= \langle \alpha \Psi_1 + \beta \Psi_2, \pi_{y,\lambda} \rangle = \alpha \langle \Psi_1, \pi_{y,\lambda} \rangle + \beta \langle \Psi_2, \pi_{y,\lambda} \rangle \\ &= \alpha \langle \tilde{\pi}(\Psi_1)y, \lambda \rangle + \beta \langle \tilde{\pi}(\Psi_2)y, \lambda \rangle,\end{aligned}$$

which proves that $\tilde{\pi}$ is linear. Thus $\tilde{\pi} : X^* \longrightarrow \mathscr{L}(Y)$ is linear and continuous satisfying $\|\tilde{\pi}\| \leqslant \|\pi\|$.

Next we must verify that $\tilde{\pi}$ is a homomorphism. Assuming X is topologically left introverted in A^*, we have

$$\Psi\cdot \pi_{y,\lambda} = \pi_{\tilde{\pi}(\Psi)y,\lambda} \qquad (y \in Y, \lambda \in Y^*, \Psi \in X^*). \tag{2}$$

In fact, for every $a \in A$,

$$\begin{aligned}\langle \Psi\cdot \pi_{y,\lambda}, a\rangle &= \langle \Psi, \pi_{y,\lambda}\cdot a\rangle = \langle \Psi, \pi_{y,\pi(a)^*\lambda}\rangle = \langle \tilde{\pi}(\Psi)y, \pi(a)^*\lambda\rangle \\ &= \langle \pi(a)(\tilde{\pi}(\Psi)y), \lambda\rangle = \langle \pi_{\tilde{\pi}(\Psi)y,\lambda}, a\rangle,\end{aligned}$$

where for the second equality we have used Lemma 2.16(ii). Therefore, for all $\Psi_1, \Psi_2 \in X^*, y \in Y$, and $\lambda \in Y^*$, we can write

$$\begin{aligned}\langle \tilde{\pi}(\Psi_1 \Box \Psi_2)y, \lambda\rangle &= \langle \Psi_1 \Box \Psi_2, \pi_{y,\lambda}\rangle = \langle \Psi_1, \Psi_2\cdot \pi_{y,\lambda}\rangle \\ &= \langle \Psi_1, \pi_{\tilde{\pi}(\Psi_2)y,\lambda}\rangle = \langle \tilde{\pi}(\Psi_1)\tilde{\pi}(\Psi_2)y, \lambda\rangle,\end{aligned}$$

which proves that $\tilde{\pi}(\Psi_1 \Box \Psi_2) = \tilde{\pi}(\Psi_1)\tilde{\pi}(\Psi_2)$.

Now, if we assume instead that X is topologically right introverted, then we have

$$\pi_{y,\lambda}\cdot \Psi = \pi_{y,\tilde{\pi}(\Psi)^*\lambda} \qquad (y \in Y, \lambda \in Y^*, \Psi \in X^*). \tag{3}$$

In fact, for every $a \in A$,

$$\begin{aligned}\langle \pi_{y,\lambda}\cdot \Psi, a\rangle &= \langle \Psi, a\cdot \pi_{y,\lambda}\rangle = \langle \Psi, \pi_{\pi(a)y,\lambda}\rangle = \langle \tilde{\pi}(\Psi)(\pi(a)y), \lambda\rangle \\ &= \langle \pi(a)y, \tilde{\pi}(\Psi)^*\lambda\rangle = \langle \pi_{y,\tilde{\pi}(\Psi)^*\lambda}, a\rangle,\end{aligned}$$

where for the second equality we have used Lemma 2.16(i). Therefore, for all $\Psi_1, \Psi_2 \in X^*, y \in Y$, and $\lambda \in Y^*$, we can write

$$\begin{aligned}\langle \tilde{\pi}(\Psi_1 \Diamond \Psi_2)y, \lambda\rangle &= \langle \Psi_1 \Diamond \Psi_2, \pi_{y,\lambda}\rangle = \langle \Psi_2, \pi_{y,\lambda} \cdot \Psi_1\rangle \\ &= \langle \Psi_2, \pi_{y,\tilde{\pi}(\Psi_1)^*\lambda}\rangle = \langle \tilde{\pi}(\Psi_2)y, \tilde{\pi}(\Psi_1)^* \cdot \lambda\rangle \\ &= \langle \tilde{\pi}(\Psi_1)\tilde{\pi}(\Psi_2)y, \lambda\rangle,\end{aligned}$$

which proves that $\tilde{\pi}(\Psi_1 \Diamond \Psi_2) = \tilde{\pi}(\Psi_1)\tilde{\pi}(\Psi_2)$. Thus we have shown that $\tilde{\pi}$ is a homomorphism whenever X is topologically left (or right) introverted in A^*.

To show that $\tilde{\pi}$ is normal, we first observe that by Lemma 2.16 (iii) and (iv), the bounded linear map $T : Y \widehat{\otimes} Y^* \longrightarrow X$ associated with the bilinear map

$$Y \times Y^* \longrightarrow X, \qquad (y, \lambda) \mapsto \pi_{y,\lambda}$$

satisfies $T(y \otimes \lambda) = \pi_{y,\lambda}$. The adjoint operator $T^* : X^* \longrightarrow (Y \widehat{\otimes} Y^*)^* \cong \mathscr{L}(Y)$ is given by

$$\langle T^*(\Psi), y \otimes \lambda \rangle = \langle \Psi, T(y \otimes \lambda) \rangle = \langle \Psi, \pi_{y,\lambda} \rangle \qquad (y \in Y,\ \lambda \in Y^*).$$

Thus $\tilde{\pi} = T^*$. This equality immediately implies the normality of $\tilde{\pi}$ (since T^* is both norm and w^*-continuous).

Next, we note that for $a \in A$, $\dot{a} \in X^*$ is given by $\dot{a} = \hat{a}|_X$. Thus for $y \in Y$ and $\lambda \in Y^*$, we have

$$\langle \tilde{\pi}(\dot{a})y, \lambda \rangle = \langle \hat{a}, \pi_{y,\lambda} \rangle = \langle \pi(a)y, \lambda \rangle,$$

proving that $\tilde{\pi}(\dot{a}) = \pi(a)$ for every $a \in A$.

We can now show that $\tilde{\pi}(X^*) \subset \overline{\pi(A)}^{w^*}$. Let $\Phi \in X^*$. By Corollary 1.13, there exists a net (a_α) in A such that $\dot{a}_\alpha \xrightarrow{w^*} \Phi$. The w^*-continuity of $\tilde{\pi}$ implies that $\pi(a_\alpha) = \tilde{\pi}(\dot{a}_\alpha) \xrightarrow{w^*} \tilde{\pi}(\Phi)$. Hence $\tilde{\pi}(\Phi) \in \overline{\pi(A)}^{w^*}$.

(ii) To prove injectivity of $\pi \mapsto \tilde{\pi}$, suppose that $\tilde{\pi}_1 = \tilde{\pi}_2$. This implies that $\tilde{\pi}_1(\Psi) = \tilde{\pi}_2(\Psi)$ for all $\Psi \in X^*$. In particular, $\tilde{\pi}_1(\dot{a}) = \tilde{\pi}_2(\dot{a})$, and thus $\pi_1(a) = \pi_2(a)$ for all $a \in A$. So $\pi_1 = \pi_2$, proving that $\pi \mapsto \tilde{\pi}$ is injective.

Turning to surjectivity, let $\sigma : X^* \longrightarrow \mathscr{L}(Y)$ be a normal representation. Let the map $\tau : A \longrightarrow X^*$ be defined by $\tau(a) = \dot{a} = \hat{a}|_X$, and let $\pi = \sigma \circ \tau$. Then $\pi : A \longrightarrow \mathscr{L}(Y)$ is a continuous homomorphism as both σ and τ are continuous homomorphisms. We show below that π is subordinate to X and that $\tilde{\pi} = \sigma$.

Since $\sigma : X^* \longrightarrow \mathscr{L}(Y) = (Y \widehat{\otimes} Y^*)^*$ is normal, $\sigma = T^*$ for some bounded linear map $T : Y \widehat{\otimes} Y^* \longrightarrow X$. Thus for $y \in Y$, $\lambda \in Y^*$ and $a \in A$, we have

$$\langle \pi_{y,\lambda}, a \rangle = \langle \pi(a), y \otimes \lambda \rangle = \langle \sigma(\dot{a}), y \otimes \lambda \rangle = \langle \dot{a}, T(y \otimes \lambda) \rangle = \langle T(y \otimes \lambda), a \rangle.$$

That is, $\pi_{y,\lambda} = T(y \otimes \lambda) \in X$ for all $y \in Y$ and $\lambda \in Y^*$. Therefore, π is subordinate to X.

Also, since $\dot{A}$ is w^*-dense in X^* (by Corollary 1.13) and $\tilde{\pi}(\dot{a}) = \pi(a) = \sigma(\dot{a})$ for all $a \in A$, we obtain from the w^*-continuity of $\tilde{\pi}$ and σ that $\tilde{\pi} = \sigma$. This completes the proof that $\pi \mapsto \tilde{\pi}$ is surjective.

Next, we will show that $\pi_1 \cong \pi_2 \iff \tilde{\pi}_1 \cong \tilde{\pi}_2$.

Suppose $\pi_1 \cong \pi_2$. Then there exists an isomorphism $V : Y_1 \longrightarrow Y_2$ such that $V\pi_1(a)V^{-1} = \pi_2(a)$ for all $a \in A$. Since $\tilde{\pi}(\dot{a}) = \pi(a)$ for every $a \in A$, we may substitute $\tilde{\pi}(\dot{a})$ for $\pi(a)$ in the above to get $V\tilde{\pi}_1(\dot{a})V^{-1} = \tilde{\pi}_2(\dot{a})$ for all $a \in A$. Therefore, $\tilde{\pi}_1|_{\dot{A}} \cong \tilde{\pi}_2|_{\dot{A}}$. To verify the equivalence on the whole of X^*, let $\Psi \in X^*$. By Corollary 1.13, there exists a net $(\dot{a}_\alpha)$ in X^* such that $\dot{a}_\alpha \xrightarrow{w^*} \Psi$. Since $\tilde{\pi}_j$ is w^*-continuous, we have $\tilde{\pi}_j(\dot{a}_\alpha) \xrightarrow{w^*} \tilde{\pi}_j(\Psi)$. Therefore, for all $y_j \in Y_j$ and $\lambda_j \in Y_j^*$, $j = 1, 2$, we have

$$\langle \tilde{\pi}_j(\dot{a}_\alpha)y_j, \lambda_j \rangle = \langle \tilde{\pi}_j(\dot{a}_\alpha), y_j \otimes \lambda_j \rangle \longrightarrow \langle \tilde{\pi}_j(\Psi), y_j \otimes \lambda_j \rangle = \langle \tilde{\pi}_j(\Psi)y_j, \lambda_j \rangle.$$

Now, for all $y_2 \in Y_2$ and $\lambda_2 \in Y_2^*$, we have

$$\begin{aligned}\langle V\tilde{\pi}_1(\Psi)V^{-1}y_2, \lambda_2 \rangle &= \langle \tilde{\pi}_1(\Psi)(V^{-1}y_2), (V^*\lambda_2) \rangle = \lim_\alpha \langle \tilde{\pi}_1(\dot{a}_\alpha)(V^{-1}y_2), (V^*\lambda_2) \rangle \\ &= \lim_\alpha \langle V\tilde{\pi}_1(\dot{a}_\alpha)V^{-1}y_2, \lambda_2 \rangle = \lim_\alpha \langle \tilde{\pi}_2(\dot{a}_\alpha)y_2, \lambda_2 \rangle \\ &= \lim_\alpha \langle \tilde{\pi}_2(\dot{a}_\alpha), y_2 \otimes \lambda_2 \rangle = \langle \tilde{\pi}_2(\Psi), y_2 \otimes \lambda_2 \rangle = \langle \tilde{\pi}_2(\Psi)y_2, \lambda_2 \rangle.\end{aligned}$$

So $V\tilde{\pi}_1(\Psi)V^{-1} = \tilde{\pi}_2(\Psi)$ for all $\Psi \in X^*$ and hence $\tilde{\pi}_1 \cong \tilde{\pi}_2$.

Conversely, if $\tilde{\pi}_1 \cong \tilde{\pi}_2$, then there exists an isomorphism $V : Y_1 \longrightarrow Y_2$ such that $V\tilde{\pi}_1(\Psi)V^{-1} = \tilde{\pi}_2(\Psi)$ for all $\Psi \in X^*$. In particular, $V\tilde{\pi}_1(\dot{a})V^{-1} = \tilde{\pi}_2(\dot{a})$ for all $\dot{a} \in X^*$. Since $\tilde{\pi}(\dot{a}) = \pi(a)$ for every $a \in A$, we have $V\pi_1(a)V^{-1} = \pi_2(a)$ for all $a \in A$, so $\pi_1 \cong \pi_2$.

Finally, we show that $\tilde{\pi}$ topologically irreducible if and only if π is topologically irreducible.

Suppose there exists $\{0\} \neq M \subsetneq Y$ such that $\pi(a)y \in M$ for all $a \in A$ and $y \in M$. Then since $\pi(a) = \tilde{\pi}(\dot{a})$, we immediately have $\tilde{\pi}(\dot{a})y \in M$ for all $y \in M$ and $a \in A$. If $\Psi \in X^*$ is an arbitrary element, then by Corollary 1.13, there is a net (a_α) in A such that $\dot{a}_\alpha \xrightarrow{w^*} \Psi$. Since $\tilde{\pi}$ is normal, this implies that

$\tilde{\pi}(\dot{a})y \xrightarrow{w} \tilde{\pi}(\Psi)y$ for all $y \in Y$, where the convergence is in the weak topology of Y. So $\tilde{\pi}(\Psi)y \in \overline{M}^{\sigma(Y,Y^*)} = M$ (since M is a norm, and hence weakly, closed subspace of Y). So M is a nontrivial invariant subspace for $\tilde{\pi}$.

Conversely, suppose there exists $\{0\} \neq M \subsetneq Y$ such that $\tilde{\pi}(\Psi)y \in M$ for all $\Psi \in X^*$ and $y \in M$. Then since $\tilde{\pi}(\dot{a}) = \pi(a)$ for all $a \in A$, we have $\pi(a)y \in M$ for all $y \in M$ and $a \in A$. Thus M is a nontrivial invariant subspace for π. $\square$

Remark 2.19. For the corresponding representations π and $\tilde{\pi}$, we have the following commutative diagram:

$$\begin{tikzcd}
A^{**} \arrow[r, "\pi^{**}"] \arrow[ddr, "Q"'] & (Y\widehat{\otimes}Y^*)^{***} \arrow[r, "P"] & (Y\widehat{\otimes}Y^*)^* \cong \mathscr{L}(Y) \\
& A \arrow[ul, "\iota"'] \arrow[ur, "\pi"] \arrow[d, "\sigma"] & \\
& X^* \arrow[uur, "\tilde{\pi}"'] &
\end{tikzcd}$$

where $\iota : A \longrightarrow A^{**}$ is the canonical embedding, P and Q are the canonical projections, and $\sigma : A \longrightarrow X^*$ is the map $a \mapsto \dot{a} = \hat{a}|_X = Q(\iota(a))$.

CHAPTER 3

[Weakly] Almost Periodic Functionals

3.1. Definition and examples

In this chapter we introduce [weakly] almost periodic functionals. For a given Banach algebra A, these functionals belong to A^*. The development of [weakly] almost periodic functionals was motivated by [weakly] almost periodic functions, which we define first.

Let G be a locally compact group and $\mathscr{C}^b(G)$ denote the space of all bounded continuous functions on G. Recall that for $f \in \mathscr{C}^b(G)$ and $s \in G$, the [right] left translation of f by s is defined by

$$[f_s(t) = f(ts)] \quad {}_sf(t) = f(st) \qquad (t \in G),$$

and the [right] left orbit of $f \in \mathscr{C}^b(G)$ is given by

$$[RO(f) = \{f_s : \ s \in G\}] \quad LO(f) = \{{}_sf : s \in G\}.$$

Definition 3.1. A function $f \in \mathscr{C}^b(G)$ is *almost periodic* on G if $LO(f)$ is relatively compact in the norm topology of $\mathscr{C}^b(G)$. A function $f \in \mathscr{C}^b(G)$ is *weakly almost periodic* on G if $LO(f)$ is relatively compact in the w-topology of $\mathscr{C}^b(G)$. The spaces of all almost periodic and weakly almost periodic functions on G are denoted $AP(G)$ and $WAP(G)$, respectively.

Both $AP(G)$ and $WAP(G)$ are closed subspaces of $\mathscr{C}^b(G)$. It is well known that the left orbit $LO(f)$ can be replaced with $RO(f)$ in the definition 3.1. We refer the reader to Burckel [**2**] and Dales–Lau [**6**] for details about these spaces and their properties.

Let A be a Banach algebra. The A-bimodule structure of A^* (see Lemma 1.21.i) gives rise to analogous subspaces of A^*.

DEFINITION 3.2. Let A be a Banach algebra. Then $\lambda \in A^*$ is [weakly] almost periodic if the set $\{\lambda \cdot a : a \in A,\ \|a\| \leqslant 1\}$ is relatively [weakly] compact in A^*.

REMARKS 3.3. (1) As will be shown below, the condition in the definition above is equivalent to $\{a \cdot \lambda : a \in A,\ \|a\| \leqslant 1\}$ being relatively [weakly] compact in A^*. Both versions of the definition are used in the literature.

(2) If $\lambda \in A^*$, we may define

$$L_\lambda : A \to A^* \text{ by } L_\lambda(a) = \lambda \cdot a \qquad (a \in A). \tag{4}$$

Since $L_\lambda \in \mathscr{L}(A, A^*)$, by Conway [**4**, Definitions VI.3.1 and VI.5.1], an equivalent condition for λ being [weakly] almost periodic is that L_λ is a [weakly] compact operator. If we similarly define

$$R_\lambda : A \to A^* \text{ by } R_\lambda(a) = a \cdot \lambda \qquad (a \in A), \tag{5}$$

then $R_\lambda \in \mathscr{L}(A, A^*)$, and λ is [weakly] almost periodic if and only if R_λ is a [weakly] compact operator. These are also frequently used in the literature to define [weakly] almost periodic functionals (for example, see Palmer [**40**, page 60]).

3.2. The spaces $wap(A)$ and $ap(A)$

The space of all [weakly] almost periodic functionals on A is denoted by [$wap(A)$] $ap(A)$. As a consequence of Megginson [**37**, Corollaries 3.4.9 and 3.5.10, and Propositions 3.4.10 and 3.5.11], both $ap(A)$ and $wap(A)$ are norm closed Banach A-bisubmodules of A^*. Since every compact operator in $\mathscr{L}(A, A^*)$ is weakly compact (see Megginson [**37**, Proposition 3.5.2]), it follows that $ap(A) \subset wap(A)$. In fact, as shown below, the spaces $ap(A)$ and $wap(A)$ are topologically introverted subspaces of A^*.

PROPOSITION 3.4. *Let A be a Banach algebra, and let X be a norm closed Banach A-bisubmodule of A^* with $X \subset wap(A)$. Then X is topologically introverted.*

PROOF. Let $\lambda \in X$ and let S denote the $\sigma(A^*, A^{**})$-closure of

$$B_A \cdot \lambda = \{a \cdot \lambda : a \in A,\ \|a\| \leqslant 1\}.$$

By Mazur's theorem, S is the norm closure of $B_A \cdot \lambda$. Since X is a norm closed Banach A-bisubmodule of A^*, we have $S \subset X$. Moreover, since S is compact in the w-topology, and since the w^*-topology is weaker than the w-topology, S is compact, and in particular closed, in the w^*-topology. Thus the $\sigma(A^*, A)$-closure of $B_A \cdot \lambda$ is a subset of S and hence a subset of X. Therefore, by Proposition 2.12, X is topologically left introverted. A similar argument proves that X is topologically right introverted. Therefore, X is topologically introverted. □

Corollary 3.5. *Let A be a Banach algebra. The spaces $ap(A)$ and $wap(A)$ are topologically introverted subspaces of A^*.*

We end this section by presenting an example of a Banach algebra A, in which the space $[wap(A)]\, ap(A)$ coincides with the space $[WAP(G)]\, AP(G)$ for a locally compact group G. Recall that if g is any function on a locally compact group G, then the functions $\check{g}$ and g^* on G are defined by $\check{g}(x) = g(x^{-1})$ and $g^*(x) = \frac{\overline{f(x^{-1})}}{\Delta(x)}$, where Δ is the modular function of G (see Folland [**21**, Section 2.4]). Moreover, if $\mathscr{F}$ is a collection of functions on G, then $\mathscr{F}^{\vee} = \{\check{f} : f \in \mathscr{F}\}$.

Proposition 3.6. *Let G be a locally compact group and $L^1(G)$ be the group algebra of G (see Example 1.17 for definition). Let $L^\infty(G)$ (as the dual of $L^1(G)$) be equipped with its canonical Banach $L^1(G)$-bimodule structure.*

(i) *For all $k \in L^1(G)$ and $f \in L^\infty(G)$, $f \cdot k = \overline{k^*} * f \in L^1(G) * L^\infty(G)$ and $k \cdot f = f * \check{k} \in L^\infty(G) * L^1(G)^{\vee}$.*
(ii) *$AP(G) = ap(L^1(G))$ and $WAP(G) = wap(L^1(G))$.*

Before presenting the proof, we recall the following result from Lau [**33**, Corollary 4.4].

Lemma 3.7. *Let G be any locally compact group. If $f \in L^\infty(G)$ is such that $\{{}_s f : s \in G\}$ is relatively compact in the [weak] norm topology of $L^\infty(G)$, then the maps $h \mapsto h * f$ and $h \mapsto f * \check{h}$ from $L^1(G)$ into $L^\infty(G)$ are [weakly] compact linear operators.*

PROOF OF PROPOSITION. (i) For $k, g \in L^1(G)$ and $f \in L^\infty(G)$, we have

$$\begin{aligned}
\langle f \cdot k, g\rangle = \langle f, k * g\rangle &= \int_G f(t)(k * g)(t)dt = \int_G \int_G f(t)\ k(s)\ g(s^{-1}t)\ ds\ dt \\
(s \to s^{-1}) &= \int_G \int_G f(t)\ \frac{k(s^{-1})}{\Delta(s)}\ g(st)\ ds\ dt \\
(\text{Fubini}) &= \int_G \int_G f(t)\ \overline{k^*}(s)\ g(st)\ dt\ ds \\
(t \to s^{-1}t) &= \int_G \int_G f(s^{-1}t)\ \overline{k^*}(s)\ g(t)\ dt\ ds \\
(\text{Fubini}) &= \int_G \int_G \overline{k^*}(s)\ f(s^{-1}t)\ g(t)\ ds\ dt \\
&= \int_G (\overline{k^*} * f)(t)\ g(t)\ dt = \langle \overline{k^*} * f, g\rangle.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\langle k \cdot f, g\rangle = \langle f, g * k\rangle &= \int_G f(t)(g * k)(t)dt = \int_G \int_G f(t)\ g(s)\ k(s^{-1}t)\ ds\ dt \\
(\text{Fubini}) &= \int_G \int_G f(t)\ \check{k}(t^{-1}s)\ g(s)\ dt\ ds \\
&= \int_G (f * \check{k})(s)\ g(s)\ ds \\
&= \langle f * \check{k}, g\rangle.
\end{aligned}$$

(ii) The full proof of this statement is given in Ülger [**50**]. We note that the inclusions $AP(G) \subset ap(L^1(G))$ and $WAP(G) \subset wap(L^1(G))$ follow directly from Lemma 3.7, using the identity $k \cdot f = f * \check{k}$ proved above and the respective definitions of $AP(G)$, $WAP(G)$, $ap(L^1(G))$, and $wap(L^1(G))$. □

3.3. Some classical results regarding $wap(A)$

Here we present some of the classical results involving weakly almost periodic functionals. Our first result is particularly interesting.

PROPOSITION 3.8. *Let A be a Banach algebra. Then $\lambda \in A^*$ is weakly almost periodic if and only if $\langle \Psi \Box \Phi, \lambda\rangle = \langle \Psi \Diamond \Phi, \lambda\rangle$ for all $\Psi, \Phi \in A^{**}$.*

Before proceeding to the proof of this proposition, we recall the following result on weakly compact operators.

THEOREM 3.9. *Let $T \in \mathscr{L}(X,Y)$. Then the following are equivalent:*

(1) T is weakly compact;

(2) $T^{**}(X^{**}) \subset \widehat{Y}$, where $\widehat{Y}$ denotes the canonical image of Y in Y^{**};

(3) T^* is continuous with respect to the w^* topology on Y^* and the w-topology on X^*.

For the proof, see Dunford-Schwartz [**16**, Theorem VI.4.2 and Lemma VI.4.7]).

We may now prove Proposition 3.8.

PROOF. Let $\lambda \in A^*$ and let $L_\lambda \in \mathscr{L}(A, A^*)$ be defined as in Remark 3.3. Note the following properties of the first and second adjoints of L_λ. For all $\Psi, \Phi \in A^{**}$, we have

$$L_\lambda^*(\Psi) = \Psi \cdot \lambda \tag{6}$$

and

$$\langle L_\lambda^{**}(\Psi), \Phi\rangle = \langle \Psi \Box \Phi, \lambda\rangle.$$

In fact, for $a \in A$,

$$\langle L_\lambda^*(\Psi), a\rangle = \langle \Psi, L_\lambda(a)\rangle = \langle \Psi, \lambda \cdot a\rangle = \langle \Psi \cdot \lambda, a\rangle,$$

and hence,

$$\langle L_\lambda^{**}(\Psi), \Phi\rangle = \langle \Psi, L_\lambda^*(\Phi)\rangle = \langle \Psi, \Phi \cdot \lambda\rangle = \langle \Psi \Box \Phi, \lambda\rangle.$$

Therefore, if $\langle \Psi \Box \Phi, \lambda\rangle = \langle \Psi \Diamond \Phi, \lambda\rangle$ for all $\Psi, \Phi \in A^{**}$, then

$$\langle L_\lambda^{**}(\Psi), \Phi\rangle = \langle \Psi \Box \Phi, \lambda\rangle = \langle \Psi \Diamond \Phi, \lambda\rangle = \langle \Phi, \lambda \cdot \Psi\rangle;$$

that is, $L_\lambda^{**}(\Psi) = \lambda \cdot \Psi \in \widehat{A^*}$, proving that $L_\lambda^{**}(A^{**}) \subset \widehat{A^*}$ and hence L_λ is weakly compact by Theorem 3.9, i.e., $\lambda \in wap(A)$.

Conversely, suppose that λ is weakly almost periodic. Then L_λ^* is w^*-w continuous by Theorem 3.9. So if (Ψ_α) is a net in A^{**} and $\Psi_\alpha \xrightarrow{w^*} \Psi \in A^{**}$, then $L_\lambda^*(\Psi_\alpha) \xrightarrow{w} L_\lambda^*(\Psi)$. Therefore, for all $\Phi \in A^{**}$, we have

$$\langle \Phi \Box \Psi_\alpha, \lambda\rangle = \langle \Phi, \Psi_\alpha \cdot \lambda\rangle = \langle \Phi, L_\lambda^*(\Psi_\alpha)\rangle \longrightarrow \langle \Phi, L_\lambda^*(\Psi)\rangle = \langle \Phi, \Psi \cdot \lambda\rangle = \langle \Phi \Box \Psi, \lambda\rangle.$$

Therefore, letting $\Phi \in A^{**}$ be arbitrary and $(a_\alpha) \subset A$ be such that $a_\alpha \xrightarrow{w^*} \Phi$, we have

$$\langle \Psi \Box \Phi, \lambda \rangle = \lim_\alpha \langle \Psi \Box a_\alpha, \lambda \rangle = \lim_\alpha \langle \Psi \Diamond a_\alpha, \lambda \rangle = \lim_\alpha \langle a_\alpha, \lambda \cdot \Psi \rangle = \langle \Phi, \lambda \cdot \Psi \rangle = \langle \Psi \Diamond \Phi, \lambda \rangle,$$

where we use Lemma 1.22 for the second equality. □

REMARK 3.10. An immediate consequence of the above proposition is that A is Arens regular if and only if $A^* = wap(A)$. In particular, since every C^*-algebra is Arens regular (see Dales [**5**, Corollary 3.2.37]), it follows that the duals of C^*-algebras consist entirely of weakly almost periodic functionals.

We will now show that when considering whether a functional λ is weakly almost periodic (i.e., L_λ is weakly compact), it is equivalent to considering the weak compactness of R_λ. A similar result will be presented for almost periodic functionals in Theorem 3.18 below.

THEOREM 3.11. *$\lambda \in A^*$ is weakly almost periodic if and only if $R_\lambda : A \longrightarrow A^*$ is weakly compact (equivalently, $\{a \cdot \lambda : a \in A, \|a\| \leqslant 1\}$ is relatively weakly compact).*

PROOF. As in the proof of the previous result, we will start by observing that the first and second adjoints of R_λ are given by

$$R_\lambda^*(\Psi) = \lambda \cdot \Psi \tag{7}$$

and

$$\langle R_\lambda^{**}(\Psi), \Phi \rangle = \langle \Phi \Diamond \Psi, \lambda \rangle.$$

In fact, for $a \in A$,

$$\langle R_\lambda^*(\Psi), a \rangle = \langle \Psi, R_\lambda(a) \rangle = \langle \Psi, a \cdot \lambda \rangle = \langle \lambda \cdot \Psi, a \rangle,$$

and for $\Phi \in A^{**}$,

$$\langle R_\lambda^{**}(\Psi), \Phi \rangle = \langle \Psi, R_\lambda^*(\Phi) \rangle = \langle \Psi, \lambda \cdot \Phi \rangle = \langle \Phi \Diamond \Psi, \lambda \rangle.$$

If $\lambda \in wap(A)$, then by Proposition 3.8, we have $\langle \Psi \Box \Phi, \lambda \rangle = \langle \Psi \Diamond \Phi, \lambda \rangle$ for all $\Psi, \Phi \in A^{**}$ and hence

$$\langle R_\lambda^{**}(\Psi), \Phi \rangle = \langle \Phi \Diamond \Psi, \lambda \rangle = \langle \Phi \Box \Psi, \lambda \rangle = \langle \Phi, \Psi \cdot \lambda \rangle,$$

implying that $R_\lambda^{**}(\Psi) = \Psi \cdot \lambda \in \widehat{A^*}$ for all $\Psi \in A^{**}$. Therefore, $R_\lambda^{**}(A^{**}) \subset \widehat{A^*}$ and R_λ is weakly compact by Theorem 3.9.

Conversely, suppose R_λ is weakly compact. Then R_λ^* is w^*-w continuous by Theorem 3.9. So if (Ψ_α) is a net in A^{**} and $\Psi_\alpha \xrightarrow{w^*} \Psi \in A^{**}$, then $R_\lambda^*(\Psi_\alpha) \xrightarrow{w} R_\lambda^*(\Psi)$. Hence, for all $\Phi \in A^{**}$, we have

$$\langle \Psi_\alpha \Diamond \Phi, \lambda \rangle = \langle \Phi, \lambda \cdot \Psi_\alpha \rangle = \langle \Phi, R_\lambda^*(\Psi_\alpha) \rangle \longrightarrow \langle \Phi, R_\lambda^*(\Psi) \rangle = \langle \Phi, \lambda \cdot \Psi \rangle = \langle \Psi \Diamond \Phi, \lambda \rangle.$$

Therefore, letting $\Phi \in A^{**}$ be arbitrary and (a_α) be a net in A such that $a_\alpha \xrightarrow{w^*} \Phi$, we have

$$\langle \Phi \Diamond \Psi, \lambda \rangle = \lim_\alpha \langle a_\alpha \Diamond \Psi, \lambda \rangle = \lim_\alpha \langle a_\alpha \Box \Psi, \lambda \rangle = \lim_\alpha \langle a_\alpha, \Psi \cdot \lambda \rangle = \langle \Phi, \Psi \cdot \lambda \rangle = \langle \Phi \Box \Psi, \lambda \rangle,$$

where we use Lemma 1.22 for the second equality. Hence, by Proposition 3.8, $\lambda \in wap(A)$. □

We may now consolidate the results proven thus far along with some additional equivalent conditions into the following theorem, which can be found in Palmer [**40**, Theorem 1.4.11] and Duncan–Hosseiniun [**14**].

THEOREM 3.12. *The following are equivalent for a Banach algebra A.*

(a) *A is Arens regular.*
(b) *The map $\Psi \mapsto \Phi \Box \Psi$ is w^*-continuous for all $\Phi \in A^{**}$.*
(c) *The map $\Phi \mapsto \Phi \Diamond \Psi$ is w^*-continuous for all $\Psi \in A^{**}$.*
(d) *The map L_λ is weakly compact for all $\lambda \in A^*$.*
(e) *The map R_λ is weakly compact for all $\lambda \in A^*$.*
(f) *Given bounded sequences $(a_n), (b_n) \in A$ and $\lambda \in A^*$, we have*

$$\lim_n \lim_m \lambda(a_n b_m) = \lim_m \lim_n \lambda(a_n b_m)$$

when both iterated limits exist.

PROOF. The equivalence between (a), (d), and (e) follows from Proposition 3.8 and Theorem 3.11. So, it suffices to prove the other equivalences.

To show that (a) implies (b), suppose A is Arens regular. Let $\Psi, \Phi \in A^{**}$ and suppose (Ψ_α) is a net in A^{**} such that $\Psi_\alpha \xrightarrow{w^*} \Psi$. Then for all $\lambda \in A^*$,

$$\langle \Phi \Box \Psi, \lambda \rangle = \langle \Phi \Diamond \Psi, \lambda \rangle = \lim_\alpha \langle \Phi \Diamond \Psi_\alpha, \lambda \rangle = \lim_\alpha \langle \Phi \Box \Psi_\alpha, \lambda \rangle.$$

Hence $\Psi \mapsto \Phi \Box \Psi$ is w^*-continuous for all $\Phi \in A^{**}$.

Conversely, suppose $\Psi \mapsto \Phi \Box \Psi$ is w^*-continuous for every $\Phi \in A^{**}$. Then, letting $\Psi \in A^{**}$ be arbitrary and (a_α) be a net in A such that $a_\alpha \xrightarrow{w^*} \Psi$, we have

$$\begin{aligned} \langle \Phi \Box \Psi, \lambda \rangle &= \lim_\alpha \langle \Phi \Box a_\alpha, \lambda \rangle = \lim_\alpha \langle \Phi \Diamond a_\alpha, \lambda \rangle = \lim_\alpha \langle a_\alpha, \lambda \cdot \Phi \rangle \\ &= \langle \Psi, \lambda \cdot \Phi \rangle = \langle \Phi \Diamond \Psi, \lambda \rangle, \end{aligned}$$

where we use Lemma 1.22 for the second equality. Since this holds for all $\Phi, \Psi \in A^{**}$ and $\lambda \in A^*$, we have shown that A is Arens regular. This completes the proof that (a) and (b) are equivalent.

Next, we show that (a) implies (c). Suppose A is Arens regular. Let $\Psi, \Phi \in A^{**}$ and suppose (Φ_α) is a net in A^{**} such that $\Phi_\alpha \xrightarrow{w^*} \Phi$. Then for all $\lambda \in A^*$,

$$\langle \Phi \Diamond \Psi, \lambda \rangle = \langle \Phi \Box \Psi, \lambda \rangle = \lim_\alpha \langle \Phi_\alpha \Box \Psi, \lambda \rangle = \lim_\alpha \langle \Phi_\alpha \Diamond \Psi, \lambda \rangle.$$

Hence $\Phi \mapsto \Phi \Diamond \Psi$ is w^*-continuous for all $\Psi \in A^{**}$.

Conversely, suppose $\Phi \mapsto \Phi \Diamond \Psi$ is w^*-continuous for every $\Psi \in A^{**}$. Letting $\Psi \in A^{**}$ be arbitrary and (a_α) be a net in A such that $a_\alpha \xrightarrow{w^*} \Phi$, we have

$$\begin{aligned} \langle \Phi \Diamond \Psi, \lambda \rangle &= \lim_\alpha \langle a_\alpha \Diamond \Psi, \lambda \rangle = \lim_\alpha \langle a_\alpha \Box \Psi, \lambda \rangle = \lim_\alpha \langle a_\alpha, \Psi \cdot \lambda \rangle \\ &= \langle \Phi, \Psi \cdot \lambda \rangle = \langle \Phi \Box \Psi, \lambda \rangle. \end{aligned}$$

Since this holds for all $\Phi, \Psi \in A^{**}$ and $\lambda \in A^*$, it follows that A is Arens regular. Thus we have shown that (a) and (c) are equivalent.

Finally, the equivalence of (d) and (f) is a direct application of Grothendieck's criterion (Proposition 1.11) to the set $\{\lambda \cdot a : a \in A,\ \|a\| \leqslant 1\}$. □

Next, we present here the classical theorem of Young [**51**, Theorem 1].

THEOREM 3.13. *A normed algebra A has a faithful continuous representation on some reflexive Banach space Y if and only if the weakly almost periodic functionals of unit norm on A separate the points of A.*

Before presenting the proof, we will state the following result from the same paper by Young [**51**, Corollary to Lemma 1].

LEMMA 3.14. *Let A and B be Banach algebras and let $\phi : A \longrightarrow B$ be a continuous homomorphism. If $\lambda \in wap(B)$, then $\phi^*(\lambda) \in wap(A)$.*

PROOF OF THEOREM 3.13. Let $\phi : A \longrightarrow \mathscr{L}(Y)$ be a continuous injective homomorphism, where Y is a reflexive Banach space. For $y \in Y$ and $\omega \in Y^*$, under the identification

$$Y\widehat{\otimes}Y^* \subseteq (Y\widehat{\otimes}Y^*)^{**} \cong \mathscr{L}(Y)^*,$$

$y \otimes \omega$ can be considered as a continuous linear functional on $\mathscr{L}(Y)$ given by

$$\langle y \otimes \omega, T\rangle = \langle Ty, \omega\rangle \qquad (T \in \mathscr{L}(Y))$$

satisfying $\|y \otimes \omega\| \leqslant \|y\|\|\omega\|$. We will show that $y \otimes \omega \in wap(\mathscr{L}(Y))$. First note that for all $y \in Y$, $\omega \in Y^*$ and $T \in \mathscr{L}(Y)$,

$$(y \otimes \omega) \cdot T = y \otimes T^*\omega.$$

In fact, for all $S \in \mathscr{L}(Y)$,

$$\begin{aligned}\langle (y \otimes \omega) \cdot T, S\rangle := \langle (y \otimes \omega), TS\rangle &= \langle \omega, (TS)y\rangle = \langle \omega, T(Sy)\rangle \\ &= \langle T^*\omega, Sy\rangle = \langle y \otimes T^*\omega, S\rangle.\end{aligned}$$

Hence, to prove that $y \otimes \omega \in wap(\mathscr{L}(Y))$, it suffices to show that $\{y \otimes \omega : \omega \in B^*\}$ is relatively weakly compact in $\mathscr{L}(Y)^*$ for all bounded subsets $B^* \subset Y^*$.

Note that the map $\omega \mapsto y \otimes \omega$, $Y^* \longrightarrow \mathscr{L}(Y)^*$ is w-w-continuous since it is norm continuous (see Megginson [**37**, Theorem 2.5.11]). Moreover, any bounded subset $B^* \subset Y^*$ is relatively w^*-compact in Y^* by Alaoglu's theorem. Since the w and w^* topologies agree on the reflexive space Y^*, the image $y \otimes B^*$ of B^* under the w-w-continuous operator $\omega \mapsto y \otimes \omega$ is also relatively w-compact in $\mathscr{L}(Y)^*$. We conclude that $y \otimes \omega \in wap(\mathscr{L}(Y))$ for all $y \in Y$ and $\omega \in Y^*$.

Now, it follows from Lemma 3.14 that $\phi^*(y \otimes \omega) \in wap(A)$ for all $y \in Y$ and $\omega \in Y^*$.

Finally, we verify that the functionals in the set

$$\{\phi^*(y \otimes \omega) : y \in Y,\ \omega \in Y^*\}$$

separate the points of A. Suppose that for some $a \in A$, we have $\langle \phi^*(y \otimes \omega), a\rangle = 0$ for all $y \in Y$ and $\omega \in Y^*$. Then

$$\langle \omega, \phi(a)y\rangle = \langle y \otimes \omega, \phi(a)\rangle = \langle \phi^*(y \otimes \omega), a\rangle = 0$$

for all $y \in Y$ and $\omega \in Y^*$. It follows from the Hahn-Banach theorem that $\phi(a)y = 0$ for all $y \in Y$, which implies that $\phi(a) = 0$. Since ϕ is injective, this implies that $a = 0$, completing the proof.

For the proof of the converse, we refer to Young [**51**, Page 118]. ☐

Our next result is implicit in the proof of Theorem 3.13. We give an explicit proof for completeness.

THEOREM 3.15. *Let A be a Banach algebra, Y a reflexive Banach space, and $\pi : A \longrightarrow \mathscr{L}(Y)$ a continuous representation of A. Then π is subordinate to $wap(A)$.*

PROOF. We must show that for all $y \in Y$ and $\lambda \in Y_* = Y^*$, the map

$$A \longrightarrow A^*, \qquad a \mapsto a \cdot \pi_{y,\lambda} = \pi_{\pi(a)y,\lambda}$$

is weakly compact. Since π is continuous with respect to the norm topology of $\mathscr{L}(Y)$, there is an $M > 0$ such that $\|\pi(a)\| \leqslant M$ if $\|a\| \leqslant 1$. If (a_α) is a net in the closed unit ball of A, then by the WOT-compactness of closed bounded sets in $\mathscr{L}(Y)$, there exists a subnet (a_β) of (a_α) and an operator $T \in \mathscr{L}(Y)$ such that $\pi(a_\beta) \xrightarrow{WOT} T$. To complete the proof, it remains to show that $a_\beta \cdot \pi_{y,\lambda} = \pi_{\pi(a_\beta)y,\lambda} \longrightarrow \pi_{Ty,\lambda}$ in the weak topology $\sigma(A^*, A^{**})$.

Indeed, if $\Phi \in A^{**}$, using the representation $\tilde{\pi} : A^{**} \longrightarrow \mathscr{L}(Y)$ given in Theorem 2.18, we have

$$\begin{aligned}
\lim_{\beta}\langle \Phi, \pi_{\pi(a_\beta)y,\lambda}\rangle_{A^{**},A^*} &= \lim_{\beta}\langle \widetilde{\pi}(\Phi)(\pi(a_\beta)y), \lambda\rangle_{Y,Y^*} \\
&= \lim_{\beta}\langle \widetilde{\pi}(\Phi)^*\lambda, \pi(a_\beta)y\rangle_{Y^*,Y} \\
(\pi(a_\beta) \xrightarrow{WOT} T) \qquad &= \langle \widetilde{\pi}(\Phi)^*\lambda, Ty\rangle_{Y^*,Y} \\
&= \langle \lambda, \widetilde{\pi}(\Phi)(Ty)\rangle_{Y^*,Y} \\
&= \langle \widetilde{\pi}(\Phi)(Ty), \lambda\rangle_{Y,Y^*} \\
&= \langle \Phi, \pi_{Ty,\lambda}\rangle_{A^{**},A^*}.
\end{aligned}$$

This shows that $a_\beta \cdot \pi_{y,\lambda} \xrightarrow{w} \pi_{Ty,\lambda}$. □

3.4. Some classical results regarding $ap(A)$

Our next theorem from Duncan–Hosseiniun [**14**] provides an $ap(A)$ analogue of Theorem 3.12. Before stating the theorem and its proof however, we must present some introductory materials regarding continuity and compactness of operators.

Definition 3.16. Let A be a Banach algebra. A map

$$\Upsilon : A^{**} \times A^{**} \longrightarrow A^{**}$$

is called *bounded jointly w^*-continuous* if whenever (Ψ_α) and (Φ_β) are two bounded nets in A^{**} such that $\Psi_\alpha \xrightarrow{w^*} \Psi$ and $\Phi_\beta \xrightarrow{w^*} \Phi$, then $\Upsilon(\Psi_\alpha, \Phi_\beta) \xrightarrow{w^*} \Upsilon(\Psi, \Phi)$.

Theorem 3.17. *Let X, Y be Banach spaces and $T \in \mathscr{L}(X, Y)$. Then the following are equivalent:*

(1) T is compact;
(2) T^* is compact;
(3) T^* maps bounded w^*-convergent nets in Y^* to norm-convergent nets in X^*.

The proof of Theorem 3.17 can be found in Dunford–Schwartz [**16**, Theorems VI.5.2 and VI.5.6]).

Theorem 3.18. *The following are equivalent for a Banach algebra A.*

(a) *The map L_λ is compact for all $\lambda \in A^*$.*

(b) *The map R_λ is compact for all $\lambda \in A^*$.*

(c) *The map $\Psi \mapsto \Psi \cdot \lambda$ is compact for all $\lambda \in A^*$.*

(d) *The map $\Phi \mapsto \lambda \cdot \Phi$ is compact for all $\lambda \in A^*$.*

(e) *The map $(\Psi, \Phi) \mapsto \Psi \Box \Phi$ is bounded jointly w^*-continuous.*

(f) *The map $(\Psi, \Phi) \mapsto \Psi \Diamond \Phi$ is bounded jointly w^*-continuous.*

Proof. The equivalences (a)$\Leftrightarrow$(c) and (b)$\Leftrightarrow$(d) follow from the properties of L_λ and R_λ given in (6) and (7) in Section 3.3 as well as the compactness equivalences in Theorem 3.17.

To show (a) implies (e), suppose (a) holds and let (Ψ_α) and (Φ_β) be bounded nets in A^{**} which are w^*-convergent to Ψ and Φ, respectively. Then for all $\lambda \in A^*$, we have $\|\Phi_\beta \cdot \lambda - \Phi \cdot \lambda\| \to 0$ by Theorem 3.17. Moreover, since

$$\begin{aligned} |\langle \Psi_\alpha \Box \Phi_\beta, \lambda\rangle - \langle \Psi \Box \Phi, \lambda\rangle| &\leqslant |\langle \Psi_\alpha \Box \Phi_\beta, \lambda\rangle - \langle \Psi_\alpha \Box \Phi, \lambda\rangle| + |\langle \Psi_\alpha \Box \Phi, \lambda\rangle - \langle \Psi \Box \Phi, \lambda\rangle| \\ &\leqslant \|\Psi_\alpha\|\|\Phi_\beta \cdot \lambda - \Phi \cdot \lambda\| + |\langle \Psi_\alpha, \Phi \cdot \lambda\rangle - \langle \Psi, \Phi \cdot \lambda\rangle| \to 0, \end{aligned}$$

it follows that $(\Psi, \Phi) \mapsto \Psi \Box \Phi$ is bounded jointly w^*-continuous.

A similar argument shows that (b) implies (f).

Now, we show that (e) implies (a). Suppose condition (e) holds and let (Ψ_α) be a bounded net in A^{**} with $\Psi_\alpha \xrightarrow{w^*} \Psi$. We claim that $\|\Psi_\alpha \cdot \lambda - \Psi \cdot \lambda\| \to 0$, proving that L_λ is compact for every $\lambda \in A^*$ by Theorem 3.17 above. In fact, if our claim does not hold, then there exist $\delta > 0$, a subnet (Ψ_β) of (Ψ_α), and a corresponding net (a_β) in B_A such that

$$|\langle \Psi_\beta \cdot \lambda, a_\beta\rangle - \langle \Psi \cdot \lambda, a_\beta\rangle| \geqslant \frac{\delta}{2},$$

that is,

$$|\langle \hat{a}_\beta \Box \Psi_\beta, \lambda\rangle - \langle \hat{a}_\beta \Box \Psi, \lambda\rangle| \geqslant \frac{\delta}{2}. \tag{8}$$

The w^*-compactness of the unit ball in A^{**} guarantees that (a_β) has a w^*-cluster point $\Phi \in A^{**}$, and condition (e) now implies that $\Phi \Box \Psi$ is a w^*-cluster point of $(\widehat{a}_\beta \Box \Psi_\beta)$, contradicting (8). The fact that (f) implies (b) can be similarly proved.

Finally, to show the equivalence of (a) and (b), note that for all $\lambda \in A^*$, $L_\lambda = R_\lambda^* \circ \iota$ and $R_\lambda = L_\lambda^* \circ \iota$, where $\iota : A \longrightarrow A^{**}$ is the canonical embedding.

Thus, L_λ is compact if and only if L_λ^* is compact, which implies that R_λ is compact. Similarly, we have that R_λ is compact implies that L_λ is compact. Therefore, we have (a) is equivalent to (b). $\square$

We proved in Theorem 3.15 that all continuous representations $\pi : A \to \mathscr{L}(Y)$, of a Banach algebra A on a reflexive Banach space Y, are subordinate to $wap(A)$. As the theorem below shows, this result can be enhanced if Y is finite dimensional. The proof, which can be found in Filali–Monfared [**19**, Lemma 2.3 (ii)], is similar to the proof of Theorem 3.15, noticing that $\|\pi_{y,\lambda}\| \leqslant \|\pi\|\|y\|\|\lambda\|$ and bounded closed sets in $\mathscr{L}(Y)$ are norm compact if Y is finite dimensional.

THEOREM 3.19. *Every continuous representation,*

$$\pi : A \longrightarrow M_n(C),$$

is subordinate to $ap(A)$.

It is well known (see Galindo [**23**, Proposition 1.6 (6)]) that for a group G and function f on G, $f \in AP(G)$ if and only if f is the uniform limit of coordinate functions of unitary representations $V_k : G \longrightarrow \mathscr{U}(H_k) = Is(H_k)$ of G on some finite dimensional Hilbert spaces H_k. The theorem above provides a Banach algebra analogue for one direction of this result. To date, it is not known if the converse is true.

Our next result gives us a class of linear functionals which are almost periodic. We start with a definition.

DEFINITION 3.20. Let A be a Banach algebra. By a *character* on A we mean a non-zero multiplicative linear functional $\varphi : A \longrightarrow \mathbb{C}$. The set of all characters of A is called the *spectrum* of A and is denoted by Φ_A.

It is well-known that if $\varphi \in \Phi_A$, then $\varphi \in A^*$ and $\|\varphi\| \leqslant 1$ (see Dales [**5**, Theorem 2.1.29 (i)]).

THEOREM 3.21. $\Phi_A \subset ap(A)$.

PROOF. Let $\varphi \in \Phi_A$. Then for all $a, b \in A$, we have:

$$\langle a \cdot \varphi, b\rangle = \langle \varphi, ba\rangle = \varphi(a)\langle \varphi, b\rangle.$$

Therefore, $a \cdot \varphi = \varphi(a)\varphi$. Similarly, $\varphi \cdot a = \varphi(a)\varphi$. Since $\|\varphi\| \leqslant 1$, we have $B_A \cdot \varphi \subset \varphi \cdot B_A = \mathbb{T}\varphi$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| \leqslant 1\}$ denotes the unit ball in $\mathbb{C}$. Now, $\mathbb{T}$ is compact and $\mathbb{T}\varphi$ is the image of $\mathbb{T}$ under the continuous operator

$$\Gamma : \mathbb{C} \to A^*, \qquad \alpha \mapsto \alpha\varphi.$$

Therefore, $\mathbb{T}\varphi$ is compact and hence L_φ and R_φ are compact. $\square$

3.5. Results regarding equalities for $ap(A)$ and $wap(A)$

Let A be a Banach algebra with a two sided approximate identity $(e_\alpha)_{\alpha\in I}$. Define

$$c_L(A^*) = \{\lambda \in A^* : \|\lambda \cdot e_\alpha - \lambda\| \to 0\}, \tag{9}$$

and

$$c_R(A^*) = \{\lambda \in A^* : \|e_\alpha \cdot \lambda - \lambda\| \to 0\}. \tag{10}$$

Let $A^* \cdot A$ and $A \cdot A^*$ denote the sets $\{\lambda \cdot a : \lambda \in A^*, a \in A\}$ and $\{a \cdot \lambda : \lambda \in A^*, a \in A\}$, respectively. If A has a two sided approximate identity, then $\overline{A^* \cdot A}$ and $\overline{A \cdot A^*}$ are closed linear subspaces of A^*.

For $\lambda \in A^*$ and $a \in A$, let $L_\lambda \in \mathscr{L}(A, A^*)$ be defined as in Remark 3.3 and $L_a \in \mathscr{L}(A)$ be the left multiplication operator by a. Similarly, let $R_\lambda \in \mathscr{L}(A, A^*)$ be defined as in Remark 3.3 and $R_a \in \mathscr{L}(A)$ be the right multiplication operator by a. We call $a \in A$ [weak] left compact if the operator L_a is [weakly] compact, and we call $a \in A$ [weak] right compact if the operator R_a is [weakly] compact. We recall the following result regarding the compactness of these operators. The proof can be found in Palmer [**40**, Proposition 1.4.13].

Proposition 3.22. *The following are equivalent for a Banach algebra A:*

(a) *$\iota(A)$ is a [left] right ideal in A^{**};*
(b) *[R_a] L_a is weakly compact for all $a \in A$;*
(c) *[$(R_a)^*$] $(L_a)^*$ is weakly compact for all $a \in A$.*

We further recall that if X and Y are Banach spaces, then the space of [weakly] compact operators from X into Y is a closed subspace of $\mathscr{L}(X, Y)$ by Megginson [**37**, Corollaries 3.4.9 and 3.5.10]. Moreover, if X, Y, and Z are Banach spaces, and either $S \in \mathscr{L}(X, Y)$ or $T \in \mathscr{L}(Y, Z)$ is [weakly] compact, then TS is [weakly] compact (see Megginson [**37**, Propositions 3.4.10 and 3.5.11]).

We can now present our next theorem, which compares the subsets/subspaces $c_L(A^*)$, $c_R(A^*)$, $A^* \cdot A$, $A \cdot A^*$, $\overline{A^* \cdot A}$, $\overline{A \cdot A^*}$, $ap(A)$ and $wap(A)$ of A^* for a Banach algebra A with a two-sided approximate identity.

THEOREM 3.23. *Let A be a Banach algebra with a two-sided approximate identity $(e_\alpha)_{\alpha \in I}$. We have*

(i) $A^* \cdot A \subset c_L(A^*) \subset \overline{A^* \cdot A}$, *and* $A \cdot A^* \subset c_R(A^*) \subset \overline{A \cdot A^*}$.

(ii) *If A is a right ideal in its second dual A^{**}, then $\overline{A^* \cdot A} \subset wap(A)$, and if A is a left ideal in its second dual A^{**}, then $\overline{A \cdot A^*} \subset wap(A)$.*

(iii) *If A is reflexive, then $\overline{A^* \cdot A} = \overline{A \cdot A^*} = A^*$.*

(iv) *If each $a \in A$ is left compact, then $\overline{A^* \cdot A} \subset ap(A)$, and if each $a \in A$ is right compact, then $\overline{A \cdot A^*} \subset ap(A)$.*

PROOF. (i) Let $a \in A$ and $\lambda \in A^*$. Since $(\lambda \cdot a) \cdot e_\alpha = \lambda \cdot (ae_\alpha)$ and

$$\|\lambda \cdot (ae_\alpha) - \lambda \cdot a\| \leqslant \|\lambda\| \|ae_\alpha - a\| \to 0,$$

we see that the inclusion $A^* \cdot A \subset c_L(A^*)$ holds. The inclusion $c_L(A^*) \subset \overline{A^* \cdot A}$ is obvious from the definition of $c_L(A^*)$. Similar arguments prove the other two inclusions.

(ii) Assume A is a right ideal in its second dual A^{**}. Then L_{e_α} is weakly compact by Proposition 3.22. For all $a \in A$ and $\lambda \in A^*$,

$$L_{\lambda \cdot e_\alpha}(a) = (\lambda \cdot e_\alpha) \cdot a = \lambda \cdot (e_\alpha a) = L_\lambda(e_\alpha a) = L_\lambda L_{e_\alpha}(a), \tag{11}$$

therefore, we have $L_{\lambda \cdot e_\alpha} = L_\lambda L_{e_\alpha}$, and hence $\lambda \cdot e_\alpha$ is weakly almost periodic. Moreover, for $\lambda \in c_L(A^*)$, we have

$$\|L_{\lambda \cdot e_\alpha} - L_\lambda\| \leqslant \|\lambda \cdot e_\alpha - \lambda\| \to 0.$$

It follows that $c_L(A^*) \subset wap(A)$. Since $A^* \cdot A \subset c_L(A^*)$ and $wap(A)$ is closed in A^*, we have $\overline{A^* \cdot A} \subset wap(A)$. A similar argument shows that $\overline{A \cdot A^*} \subset wap(A)$.

(iii) Assume that A is reflexive. Then for all $\lambda \in A^*$ and $\hat{a} \in A^{**}$,

$$\langle \lambda \cdot e_\alpha, \hat{a} \rangle = \langle \lambda, e_\alpha \hat{a} \rangle \longrightarrow \langle \lambda, \hat{a} \rangle.$$

So $\lambda \cdot e_\alpha \xrightarrow{w} \lambda$. Hence $\lambda \in \overline{\lambda \cdot A}^{w} = \overline{\lambda \cdot A}^{\|\cdot\|} \subset \overline{A^* \cdot A}^{\|\cdot\|}$. Therefore, $A^* \subset \overline{A^* \cdot A}$, proving that $A^* = \overline{A^* \cdot A}$. Similarly, we have $A^* = \overline{A \cdot A^*}$.

(iv) This assertion holds by a similar argument as given in the proof of (ii). □

Suppose the approximate identity of A is bounded. Then the sets $A^* \cdot A$ and $A \cdot A^*$ are norm closed linear subspaces of A^* by Hewitt–Ross [**28**, Theorem 32.22], and we have the following corollary of Theorem 3.23.

Corollary 3.24. *Let A be a Banach algebra with a bounded two-sided approximate identity $(e_\alpha)_{\alpha \in I}$.*

(i) *$A^* \cdot A = c_L(A^*) = \overline{A^* \cdot A}$, and $A \cdot A^* = c_R(A^*) = \overline{A \cdot A^*}$.*

(ii) *If A is a right ideal in its second dual A^{**}, then $\overline{A^* \cdot A} = A^* \cdot A = wap(A)$ and if A is a left ideal in its second dual, then $\overline{A \cdot A^*} = A^* \cdot A = wap(A)$.*

(iii) *If each $a \in A$ is left compact (i.e., L_a is compact), then $A^* \cdot A = c_L(A^*) = ap(A)$, and $A \cdot A^* = c_R(A^*) = ap(A)$.*

Proof. (i) This follows directly from Theorem 3.23(i) and Hewitt–Ross [**28**, Theorem 32.22].

(ii) By Theorem 3.23(ii), we have $\overline{A^* \cdot A} \subset wap(A)$ and $\overline{A \cdot A^*} \subset wap(A)$. To show the reverse inclusion, let $\lambda \in wap(A)$ and let Φ_0 be a w^* cluster point of (e_α) in A^{**} (which exists by Alaoglu's theorem). This is a mixed identity for A^{**} (i.e., a right identity for the first Arens product and a left identity for the second Arens product) by Palmer [**40**, Proposition 5.1.8]. Then for all $\Psi \in A^{**}$, we have

$$\begin{aligned} \lim_\alpha \langle \lambda \cdot e_\alpha, \Psi \rangle_{A^*, A^{**}} &= \lim_\alpha \langle e_\alpha, \Psi \cdot \lambda \rangle_{A^{**}, A^*} = \langle \Phi_0, \Psi \cdot \lambda \rangle \\ &= \langle \Phi_0 \Box \Psi, \lambda \rangle = \langle \Phi_0 \Diamond \Psi, \lambda \rangle \text{ (by Proposition 3.8)} \\ &= \langle \Psi, \lambda \rangle_{A^{**}, A^*}, \end{aligned}$$

so that $\lambda \cdot e_\alpha \xrightarrow{w} \lambda$. This proves that

$$\lambda \in \overline{\lambda \cdot A}^{w} = \overline{\lambda \cdot A}^{\|\cdot\|} \subset \overline{A^* \cdot A}^{\|\cdot\|} = A^* \cdot A.$$

Hence $wap(A) = A^* \cdot A = \overline{A^* \cdot A}$. A similar argument shows that $wap(A) = A \cdot A^* = \overline{A \cdot A^*}$.

(iii) This assertion holds by Theorem 3.23 (iv) and a similar argument as given in the proof of (ii) above. □

Recall that we have shown in Theorem 3.21 that the spectrum Φ_A of A is a subset of $ap(A)$.

LEMMA 3.25. *Assume that Φ_A separates the points of A. Let $\lambda \in A^*$ and let a be a weakly compact element of A. Then the functionals $\lambda \cdot a$ and $a \cdot \lambda$ are almost periodic.*

PROOF. We will show that $\lambda \cdot a$ is in the closed linear span of Φ_A (denoted by $\overline{\text{span}}\ \Phi_A$). Suppose it is not the case. Then by the Hahn-Banach theorem, there is $\Psi \in A^{**}$ such that $\langle \lambda \cdot a, \Psi \rangle$ is non-zero but $\langle \varphi, \Psi \rangle = 0$ for all φ in the span of Φ_A. In particular, for all $\varphi \in \Phi_A$, we have $\varphi \cdot a = \varphi(a)\varphi \in \text{span}\ \Phi_A$, and thus

$$\langle \varphi \cdot a, \Psi \rangle = \langle \varphi, a \Box \Psi \rangle = 0.$$

Since a is weakly compact, $a \Box \Psi \in A$ by Proposition 3.22. It follows that $a \Box \Psi = 0$ as Φ_0 separates the points of A, which contradicts the fact that $\langle \lambda \cdot a, \Psi \rangle \neq 0$. Thus we have shown by contradiction that $\lambda \cdot a \in \overline{\text{span}}\ \Phi_A$. A similar argument shows that $a \cdot \lambda \in \overline{\text{span}}\ \Phi_A$. □

The main result of this section is the following theorem.

THEOREM 3.26. *Let A be a Banach algebra with a bounded two-sided approximate identity $(e_\alpha)_{\alpha \in I}$. Assume that A is a two-sided ideal in its second dual. Then the equality $\overline{\text{span}}\ \Phi_A = wap(A)$ holds if and only if Φ_A separates the points of A. When this happens we have*

$$\overline{\text{span}}\ \Phi_A = wap(A) = A^* \cdot A = A \cdot A^* = c_L(A^*) = c_R(A^*) = ap(A). \qquad (12)$$

PROOF. The equalities

$$wap(A) = A^* \cdot A = A \cdot A^* = c_L(A^*) = c_R(A^*)$$

hold by Corollary 3.24. Moreover, since A is a two-sided ideal in its second dual, every $a \in A$ is weakly compact by Proposition 3.22. Suppose that Φ_A separates the points of A. Then by Lemma 3.25 and Theorem 3.21, we have

$$A^* \cdot A \subset \overline{\text{span}}\Phi_A \subset ap(A),$$

and (12) follows.

Conversely, suppose that (12) holds. Let $a \in A$ be such that $\langle \varphi, a\rangle = 0$ for all $\varphi \in \Phi_A$. Then $\langle \lambda, a\rangle = 0$ for all $\lambda \in wap(A)$. Since e_α is weakly compact, for all $\lambda \in A^*$, (11) implies that $\lambda \cdot e_\alpha$ is weakly almost periodic. Therefore, $\langle \lambda \cdot e_\alpha, a\rangle = \langle \lambda, e_\alpha a\rangle = 0$. This gives $e_\alpha a = 0$ for all $\alpha \in I$ and hence $a = 0$ (i.e., Φ_A separates the points of A). □

Before our next result, we must present some definitions.

Definition 3.27. Suppose that X and Y are Banach spaces. A linear operator T from X into Y is *completely continuous* if $T(K)$ is a compact subset of Y whenever K is a weakly compact subset of X.

Definition 3.28. A Banach space X has the *Dunford-Pettis property (DPP)* if, for every Banach space Y, each weakly compact operator from X into Y is completely continuous.

Examples 3.29. (1) Let Y be a Banach space. Then each member of $\mathscr{L}(\ell_1, Y)$ is completely continuous (see Megginson [**37**, Exercise 3.51]). Therefore, ℓ_1 has the DPP.

(2) The spaces $L^1(\Omega, \Sigma, \mu)$ and $\mathscr{C}(K)$ have the DPP, where (Ω, Σ, μ) is a finite measure space and K is a compact Hausdorff space (see Megginson [**37**, Page 345]).

The next result points out one class of Banach algebras for which the equality $wap(A) = ap(A)$ holds.

Theorem 3.30. *Let A be a Banach algebra with a bounded two-sided approximate identity $(e_\alpha)_{\alpha\in I}$. If A has the DPP and is a two-sided ideal in its second dual, then $wap(A) = ap(A)$.*

PROOF. By Corollary 3.24, the following equalities hold.

$$wap(A) = c_L(A^*) = c_R(A^*) = A^* \cdot A = A \cdot A^*.$$

Let $\lambda \in wap(A)$. Note that $L_{\lambda \cdot e_\alpha}(B_A) = L_\lambda(L_{e_\alpha}(B_A))$. Since L_{e_α} is weakly compact and L_λ is completely continuous (as A has the DPP), $L_{\lambda \cdot e_\alpha}$ is compact. Moreover, since $\|L_{\lambda \cdot e_\alpha} - L_\lambda\| \leqslant \|\lambda \cdot e_\alpha - \lambda\| \to 0$ (as $\lambda \in c_L(A^*)$), we conclude that λ is in $ap(A)$. Thus $wap(A) = ap(A)$. $\square$

In fact, this result holds under weaker conditions. The reader may consult Duncan–Ülger [**15**, Theorem 2.4].

3.6. The space $ap(A)$ for some classical Banach algebras

In this section we identify the space $ap(A)$ for some of the classical Banach algebras. We refer the reader to Duncan–Ülger [**15**] for the details and proofs.

We start by presenting the space $ap(A)$ for the Banach algebras c_0 and l^p $(1 \leqslant p < \infty)$. The multiplication in these spaces are defined coordinatewise.

PROPOSITION 3.31. *The following equalities hold.*

(i) $ap(c_0) = l^1$.
(ii) $ap(l^1) = c_0$.
(iii) $ap(l^p) = l^q$ $(1 < p < \infty,\ p^{-1} + q^{-1} = 1)$.

Next, we let G be a compact group, $1 < p < \infty$, and $A = L^p(G)$ with the usual convolution product.

PROPOSITION 3.32. $ap(L^p(G)) = L^q(G)$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

We will require the following definitions for our next result.

DEFINITION 3.33. A sequence (x_n) in a Banach space X is a *Schauder basis* for X if for any $x \in X$, there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n x_n$.

DEFINITION 3.34. A Banach space X has the *approximation property* if, for every Banach space Y, the set of finite-rank operators in $\mathscr{L}(Y, X)$ is dense in the set $K(Y, X)$ of compact operators from Y into X.

EXAMPLE 3.35. Every Banach space with a Schauder basis has the approximation property (see Ryan [**46**, Example 4.4]). In particular, c_0 and all l^p with $1 \leqslant p < \infty$ have the approximation property.

PROPOSITION 3.36. *Let X be an infinite dimensional Banach space with the approximation property. Then $ap(K(X)) = \{0\}$.*

Let K be a compact Hausdorff space. We have the well-known identification $\mathscr{C}(K)^* = M(K)$, where $M(K)$ is the space of regular complex Borel measures on K and the duality is given by

$$\langle a, \mu \rangle = \int_K a(t) d\mu(t) \qquad (a \in \mathscr{C}(K),\ \mu \in M(K)).$$

Recalling that $|\mu|$ denotes the total variation of the measure μ (see Hewitt–Stromberg [**29**, Definition 19.11]), we have the following definition.

DEFINITION 3.37. A measure $\mu \in M(K)$ is said to be *purely discontinuous* if there is a countable subset F of K such that $|\mu|(F^c) = 0$, where $F^c = K - F$. The set of all purely discontinuous measures in $M(K)$ is denoted by $M_d(K)$.

Hewitt–Ross [**27**, Theorem 19.15] show that the set $M_d(K)$ is a closed subalgebra of $M(K)$, and the equality $M_d(K) = M(K)$ holds if and only if K is discrete.

We can now state our final result from this section.

PROPOSITION 3.38. *Let K be a compact Hausdorff space. Then $ap(\mathscr{C}(K)) = M_d(K)$.*

CHAPTER 4

Representations of Banach Algebras Subordinate to $wap(A)$

4.1. Compactness and reflexivity

In this section we present some introductory materials which we require for our main results in the next section. Details can be found in Conway [**4**] and Megginson [**37**].

DEFINITION 4.1. Suppose that W is a subset of a vector space X. Then W is

(i) *Convex* if $tx + (1-t)y \in W$ whenever $x, y \in W$ and $0 < t < 1$;
(ii) *Balanced* if $\alpha W \subset W$ whenever $|\alpha| \leqslant 1$;
(iii) *Absorbing* if for all $x \in X$, there is a positive number ω_x such that $x \in tW$ whenever $t > \omega_x$.

EXAMPLE 4.2. The unit ball of B_X of a Banach space X is clearly convex, balanced and absorbing.

DEFINITION 4.3. Let W be an absorbing subset of a vector space X. For each $x \in X$, the *gauge functional* of W is defined by

$$p_W(x) = \inf\{t : t \geqslant 0 \text{ and } x \in tW\}.$$

It is well known that if W is both convex and balanced, then the gauge functional p_W is a seminorm on X (see Megginson [**37**, Proposition 1.9.14]). In the case when X is a Banach space, if W is also bounded, then for $n = 1, 2, \ldots$, we define the gauge $\|\cdot\|_n$ of the set $U_n = 2^n W + 2^{-n} B_X$ by

$$\|x\|_n = \inf\{t \geqslant 0 : x \in tU_n\}.$$

LEMMA 4.4. *Let $(X, \|\cdot\|)$ be a Banach space. For $n = 1, 2, \ldots$, the gauge $\|\cdot\|_n$ of U_n is a norm on X equivalent to $\|\cdot\|$.*

PROOF. Fix $n \in \mathbb{N}$. Clearly, the set $U_n = 2^n W + 2^{-n} B_X$ is convex, balanced and absorbing. Therefore, $\|x\|_n = \inf\{t \geqslant 0 : x \in tU_n\}$ is a semi-norm.

If $\|x\|_n = 0$, then for all $k > 0$, there exists a $t_k < \frac{1}{k}$ such that

$$x \in t_k U_n = t_k 2^n W + t_k 2^{-n} B_X.$$

Suppose M is a bound of elements in W, i.e., $\|w\| \leqslant M$ for all $w \in W$. Then

$$\|x\| \leqslant t_k 2^n M + t_k 2^{-n} = t_k(2^n M + 2^{-n}) \xrightarrow{k \longrightarrow \infty} 0.$$

Thus we have shown that $x = 0$ whenever $\|x\|_n = 0$. Therefore, $\|\cdot\|_n$ is a norm, and it remains to show that $\|\cdot\|_n$ is equivalent to $\|\cdot\|$.

If $x \in X$, then $x \in \|x\| B_X$. Therefore, $x \in 2^n \|x\| U_n$, and hence $\|x\|_n \leqslant 2^n \|x\|$. Moreover, if $x \in tU_n$, then $x = 2^n tw + 2^{-n} ty$, where $w \in W$ and $y \in B_X$. Therefore, $\|x\| \leqslant 2^n tM + 2^{-n} t = t(2^n M + 2^{-n})$. Hence $t \geqslant \frac{\|x\|}{2^n M + 2^{-n}}$ implying that

$$(2^n M + 2^{-n})^{-1} \|x\| \leqslant \|x\|_n \leqslant 2^n \|x\|.$$

□

With W and $\|\cdot\|_n$ as given above, we define

$$\|x\|' = (\sum_{n=1}^{\infty} \|x\|_n^2)^{1/2} \qquad (x \in X)$$

and

$$Y = \{x \in X : \|x\|' < \infty\}.$$

We let B_Y be the closed unit ball of Y and let $j : Y \longrightarrow X$ be the natural embedding. The following result, which shall be needed for Theorem 4.8, is due to Davis, Figiel, Johnson and Pelczynski [**8**, Lemma 1]. We provide a detailed proof below which is based on the proof given by Davis and his co-authors.

THEOREM 4.5. **(Davis et al)** *Under the above conditions,*

(i) $W \subset B_Y$;

(ii) $(Y, \|\cdot\|')$ *is a Banach space for which we have* $\|j\| \leqslant 1$;

(iii) $j^{**} : Y^{**} \longrightarrow X^{**}$ *is injective and* $(j^{**})^{-1}(X) = (Y)$;

(iv) Y *is reflexive if and only if* W *is relatively weakly compact.*

Before proving the theorem, we present the following lemma which we will require.

LEMMA 4.6. *Suppose that E is a Banach space and F is a closed subspace of E. Then $F^* = E^*/F^\perp$, $F^{**} = F^{\perp\perp} = \overline{F}^{w^*} \subset E^{**}$, and $F^{**} \bigcap E = F$.*

PROOF. The first two statements are standard results which can be found, for example, in Megginson [**37**, Theorems 1.10.16 and 2.6.26, and Proposition 1.11.14]. We prove that $F^{**} \bigcap E = F$.

If $x \in F^{**} \bigcap E = F^{\perp\perp} \bigcap E$, then $x|_{F^\perp} = 0$. Therefore, if $f \in F^\perp$, then $f(x) = 0$. Hence by the Hahn-Banach theorem, $x \in F$ and we get $F^{**} \bigcap E \subset F$. The reverse inclusion is immediate since $F \subset E$ and $F \subset F^{\perp\perp}$. □

We can now proceed to the proof of Theorem 4.5.

PROOF. (i) If $x \in W$, then $\|x\|_n \leqslant 2^{-n}$ since $x \in W + 2^{-2n} B_X = 2^{-n} U_n$. This holds for $n = 1, 2 \ldots$, so

$$\|x\|' \leqslant (\sum_{n=1}^{\infty} 2^{-2n})^{1/2} = (\frac{2^{-2}}{1 - 2^{-2}})^{1/2} = \sqrt{1/3} < 1.$$

Therefore, $x \in B_Y$.

(ii) Let $X_n = (X, \| \cdot \|_n)$, $Z = \ell_2\text{-}\bigoplus_n X_n$, and

$$\varphi : (Y, \| \cdot \|') \longrightarrow Z, \quad y \mapsto (j(y), j(y), \ldots).$$

Then φ is a linear isometric embedding onto its image. In fact, we have

$$\|\varphi(y)\|_Z = \|(j(y), j(y), \ldots)\|_Z = (\sum_{n=1}^{\infty} \|j(y)\|_n^2)^{1/2} = (\sum_{n=1}^{\infty} \|y\|_n^2)^{1/2} = \|y\|'.$$

Now,

$$\varphi(Y) = \{z = (x_n) \in Z : x_n = x \text{ for } n = 1, 2, \ldots\}.$$

We claim that $\varphi(Y)$ is a closed subspace of Z. To prove the claim, let $(\varphi(y_n))$ be a sequence in $\varphi(Y)$ such that

$$\varphi(y_n) = (y_n, y_n, \ldots) \xrightarrow{\|\cdot\|_{\ell_2}} z = (x_1, x_2, \ldots) \in Z.$$

Then $y_n \xrightarrow{\|\cdot\|_i} x_i$ as $n \longrightarrow \infty$ for $i = 1, 2, \ldots$. Since each $\| \cdot \|_i$ is equivalent to $\| \cdot \|$, it must be the case that $x_i = x_1$ for $i = 2, 3, \ldots$. Moreover, the requirement that

$\|z\|_{\ell_2} < \infty$ implies that

$$(\sum_{i=1}^{\infty} \|x_i\|_i^2)^{1/2} = (\sum_{i=1}^{\infty} \|x_1\|_i^2)^{1/2} = \|x_1\|' < \infty.$$

Therefore, $x_1 \in Y$ and $z = \lim_n \varphi(y_n) = \varphi(x_1) \in \varphi(Y)$, proving the claim.

Since $\varphi(Y)$ is a closed subspace of the Banach space Z, it is itself a Banach space. Moreover, since $(Y, \|\cdot\|')$ is isometrically isomorphic to $\varphi(Y)$ as shown above, we conclude that $(Y, \|\cdot\|')$ is also a Banach space.

Finally, let $p : Z \longrightarrow X$ be the projection $(x_1, x_2, \ldots) \mapsto x_1$. Then p is clearly continuous with $\|p\| \leqslant 1$. Hence $j = p \circ \varphi : (Y, \|\cdot\|') \longrightarrow X$ is the composition of two continuous functions of norm no greater than one, and as such is continuous with $\|j\| \leqslant 1$.

(iii) In the remainder of the proof, Y denotes the Banach space $(Y, \|\cdot\|')$. Before proving the claims in this part of the lemma, we would like to prove the following properties of j and φ which we will require below.

$$j^* : X^* \longrightarrow Y^*, \qquad j^*(\lambda) = \lambda|_Y \qquad (\lambda \in X^*), \tag{13}$$

$$\varphi^{**} : Y^{**} \longrightarrow Z^{**}, \qquad \varphi^{**}(\Psi) = (j^{**}(\Psi)) \qquad (\Psi \in Y^{**}). \tag{14}$$

In fact, for $\lambda \in X^*$ and $y \in Y$, $\langle j^*(\lambda), y\rangle = \langle \lambda, j(y)\rangle = \lambda(y)$, proving (13).

Turning to (14), we first note that for all $(\lambda_n) \in Z^* = \ell_2\text{-}\bigoplus X_n^*$ and $y \in Y$, we have

$$\begin{aligned} \langle \varphi^*((\lambda_n)), y\rangle &= \langle (\lambda_n), \varphi(y)\rangle = \langle (\lambda_n), (j(y))\rangle \\ &= \sum_{n=1}^{\infty} \langle \lambda_n, j(y)\rangle = \sum_{n=1}^{\infty} \langle j^*(\lambda_n), y\rangle = \langle \sum_{n=1}^{\infty} j^*(\lambda_n), y\rangle. \end{aligned}$$

Hence

$$\varphi^*((\lambda_n)) = \sum_{n=1}^{\infty} j^*(\lambda_n). \tag{15}$$

Therefore, for all $\Psi \in Y^{**}$ and $(\lambda_n) \in Z^* = \ell_2\text{-}\bigoplus X_n^*$, we have

$$\begin{aligned} \langle \varphi^{**}(\Psi), (\lambda_n)\rangle &= \langle \Psi, \varphi^*((\lambda_n))\rangle = \langle \Psi, \sum_{n=1}^{\infty} j^*(\lambda_n)\rangle \\ &= \sum_{n=1}^{\infty} \langle j^{**}(\Psi), \lambda_n\rangle = \langle (j^{**}(\Psi)), (\lambda_n)\rangle, \end{aligned}$$

proving (14).

To prove that j^{**} is injective, we first note that since φ is an isometry, φ^{**} is also an isometry and hence it is injective. In particular, we have

$$(\varphi^{**})^{-1}(\{0\}) = \{0\}, \tag{16}$$

$$(\varphi^{**})^{-1}(\varphi(Y)) = Y. \tag{17}$$

Hence if $j^{**}(\Psi) = 0$ for some $\Psi \in Y^{**}$, then $(j^{**}(\Psi)) = 0$ in Z^{**} which implies by (14) that $\varphi^{**}(\Psi) = 0$. Thus by (16), $\Psi = 0$, proving that j^{**} is injective.

Finally, we will show that $(j^{**})^{-1}(X) = Y$. Since $j^{**}(Y) = j(Y) \subset X$, we have $Y \subset (j^{**})^{-1}(X)$ and it remains to show that the inclusion is not strict.

Let $\Psi \in Y^{**}$ be such that $j^{**}(\Psi) \in X$. Then $\varphi^{**}(\Psi) = (j^{**}(\Psi)) \in Z$. Since $\varphi^{**} : Y^{**} \longrightarrow \varphi^{**}(Y^{**})$ is an isometric isomorphism, we have $\varphi^{**}(Y^{**}) = \varphi(Y)^{**}$. Therefore, $\varphi^{**}(\Psi) \in \varphi(Y)^{**}$ and we have

$$\varphi^{**}(\Psi) \in \varphi(Y)^{**} \bigcap Z.$$

Applying Lemma 4.6 to the Banach spaces $\varphi(Y)$ and Z, and recalling that $\varphi(Y)$ is a closed subspace of Z, we have $\varphi^{**}(\Psi) \in \varphi(Y)^{**} \bigcap Z = \varphi(Y)$, proving that $\Psi \in (\varphi^{**})^{-1}(\varphi(Y)) = Y$.

(iv) First, we show that

$$\overline{B_Y}^{\sigma(X^{**},X^*)} = j^{**}(B_{Y^{**}}), \tag{18}$$

where we are viewing B_Y as a subset of X. In fact, $B_{Y^{**}}$ is w^*-compact by Alaoglu's theorem (see Conway [**4**, Theorem V.3.1]), and $\overline{B_Y}^{\sigma(Y^{**},Y^*)} = B_{Y^{**}}$ by Theorem 1.7 (Goldstine's Theorem). Moreover, since j^{**} is w^*-continuous (see Megginson [**37**, Theorem 3.1.11]), we have $j^{**}(B_{Y^{**}})$ being $\sigma(X^{**}, X^*)$-compact in X^{**}, and thus

$$\overline{j^{**}(B_Y)}^{\sigma(X^{**},X^*)} = j^{**}(B_{Y^{**}}).$$

But $j^{**}(B_Y) = j(B_Y) = B_Y$. Therefore, $\overline{B_Y}^{\sigma(X^{**},X^*)} = j^{**}(B_{Y^{**}})$. We have shown that $j^{**}(B_{Y^{**}})$ is $\sigma(X^{**}, X^*)$-compact and B_Y is $\sigma(X^{**}, X^*)$-dense in it.

Now we prove that if W is weakly relatively compact, i.e., if $\overline{W}$ is compact in the $\sigma(X, X^*)$ topology, then the sets $2^n\overline{W} + 2^{-n}B_{X^{**}}$ contain B_Y for $n = 1, 2, \ldots$.

Let n be given. If $x \in B_Y$, then $\|x\|' \leqslant 1$ so $\|x\|_n \leqslant 1$. If $x \in 2^n W + 2^{-n} B_X$, then we have nothing to show. Otherwise, let $\epsilon > 0$ be arbitrary. By definition of the gauge, there exists some $t = t(n, \epsilon)$ such that $1 \leqslant t \leqslant 1 + \epsilon$ and

$$x \in 2^n tW + 2^{-n} tB_X \subset 2^n t\overline{W} + 2^{-n} tB_{X^{**}}.$$

So there must exist some $y_\epsilon \in 2^n \overline{W}$ and $z_\epsilon \in 2^{-n} B_{X^{**}}$ such that $x = t(n, \epsilon)(y_\epsilon + z_\epsilon)$. Using the fact that $2^n \overline{W}$ and $2^{-n} B_{X^{**}}$ are w^*-compact, we may assume without loss of generality that $y_\epsilon \xrightarrow{w^*} y \in 2^n \overline{W}$ and $z_\epsilon \xrightarrow{w^*} z \in 2^{-n} B_{X^{**}}$, where we replace each net by an appropriate subnet that guarantees simultaneous convergence. Since $t(n, \epsilon)$ decreases to one as ϵ decreases to zero, we have

$$t(n, \epsilon) y_\epsilon \xrightarrow{w^*} y \in 2^n \overline{W}, \text{ and } t(n, \epsilon) z_\epsilon \xrightarrow{w^*} z \in 2^{-n} B_{X^{**}},$$

hence

$$x = t(n, \epsilon)(y_\epsilon + z_\epsilon) \xrightarrow{w^*} (y + z) \in 2^n \overline{W} + 2^{-n} B_{X^{**}},$$

where the ϵ-net is constant and every element is always equal to x. Therefore, $x \in 2^n \overline{W} + 2^{-n} B_{X^{**}}$, proving that $B_Y \subset 2^n \overline{W} + 2^{-n} B_{X^{**}}$ for $n = 1, 2, \ldots$. Since each $2^n \overline{W} + 2^{-n} B_{X^{**}}$ is $\sigma(X^{**}, X^*)$-compact, it follows from (18) that

$$j^{**}(B_{Y^{**}}) \subset 2^n \overline{W} + 2^{-n} B_{X^{**}} \text{ for } n = 1, 2, \ldots. \tag{19}$$

Next we will show that

$$\bigcap_{n=1}^{\infty} (X + 2^{-n} B_{X^{**}}) = X. \tag{20}$$

To verify this, let $z \in \bigcap_{n=1}^{\infty} (X + 2^{-n} B_{X^{**}}) \subset X^{**}$. Then for $n = 1, 2, \ldots$, $z = x_n + 2^{-n} y_n$, for some $x_n \in X$ and $y_n \in B_{X^{**}}$. Since (y_n) is bounded, we have

$$x_n = z - 2^{-n} y_n \xrightarrow[n \to \infty]{\|\cdot\|} z \in X^{**}.$$

Since (x_n) is convergent in X^{**} and $x_n \in X$ for $n = 1, 2, \ldots$, (x_n) is Cauchy in X. It follows that (x_n) must be convergent in X. Since $X \subset X^{**}$, by uniqueness of limit, we conclude that $x_n \xrightarrow[n \to \infty]{\|\cdot\|} z \in X$, proving our claim.

Finally, using the fact that X is a Banach space and $W \subset X$ is convex, we have $2^n\overline{W} \subset X$ for each n, and hence by (19) and (20),

$$j^{**}(B_{Y^{**}}) \subset \bigcap_{n=1}^{\infty} 2^n\overline{W} + 2^{-n}B_{X^{**}} \subset \bigcap_{n=1}^{\infty}(X + 2^{-n}B_{X^{**}}) = X.$$

Therefore, $j^{**}(B_{Y^{**}}) \subset X$ and hence by part (iii) above, $B_{Y^{**}} \subset Y$, proving that $Y^{**} \subset Y$ and Y is reflexive.

Conversely, if $Y = Y^{**}$, then $B_Y = B_{Y^{**}}$ is weakly compact and hence $W \subset B_Y$ is relatively weakly compact. $\square$

4.2. A WAP-representation theorem and its applications

It is well-known that for an involutive Banach algebra A with a bounded aproximate identity, every positive linear functional on A is a coordinate function of an involutive representation of A on some Hilbert space (see for example, Dixmier [**13**, Proposition 2.4.4]). In this section, we will extend this result on positive linear functionals to weakly almost periodic functionals. We will show that if a Banach algebra A has a bounded approximate identity, then every weakly almost periodic functional on A is a coordinate function of a representation of A, on some reflexive Banach space Y.

Recall that if A is a Banach algebra and X is a closed subspace of A^*, we say X is faithful if $a = 0$ whenever $\lambda(a) = 0$ for all $\lambda \in X$ (see Definition 1.14). It was shown in Lemma 1.15 that whenever X is faithful, then the natural map of A into X^* is an embedding, and we will regard A as a subalgebra of $(X^*, \square)$ when X is topologically left introverted in A^*. The space X^* can be equipped with the w^*-topology $\sigma(X^*, X)$. It is well known and easy to verify that if X is topologically [right] left introverted in A^*, then for all $\Phi \in X^*$, the map

$$X^* \longrightarrow X^*, [\Psi \mapsto \Phi\Diamond\Psi]\ \Psi \mapsto \Psi\square\Phi$$

is w^*-continuous.

LEMMA 4.7. *Let A be a Banach algebra and $X \subset A^*$ be a faithful subspace of A^*.*

(i) *If A has a bounded left approximate identity and if X is topologically left introverted in A^*, then every $\lambda \in X$ is a coordinate function of the continuous representation*

$$L : A \longrightarrow \mathscr{L}(X), \qquad L(a)\mu = a \cdot \mu \qquad (a \in A, \mu \in X).$$

(ii) *If A has a bounded right approximate identity and if X is topologically right introverted in A^*, then every $\lambda \in X$ is a coordinate function of the continuous anti-representation*

$$R : A \longrightarrow \mathscr{L}(X), \qquad R(a)\mu = \mu \cdot a \qquad (a \in A, \mu \in X).$$

Proof. (i) Since X is a closed Banach A-bisubmodule of A^*, $L(a) \in \mathscr{L}(X)$ for every $a \in A$. Moreover,

$$L(ab)\mu = (ab) \cdot \mu = a \cdot (b \cdot \mu) = L(a)L(b)\mu,$$

where for the second equality, we used the associative property of the module action. Hence L is a representation.

Furthermore, L is continuous, since

$$\|L(a)\| = \sup_{\mu \in B_X} \|L(a)\mu\| = \sup_{\mu \in B_X} \|a \cdot \mu\| \leqslant \sup_{\mu \in B_X} \|a\|\|\mu\| \leqslant \|a\|.$$

Let (e_α) be a bounded left approximate identity of A, and let Φ_0 be any w^*-cluster point of this net in X^*. Then $\Phi_0 \cdot a = a$, since

$$\Phi_0 \cdot a = \Phi_0 \Box a = \lim_\alpha e_\alpha \Box a = \lim_\alpha e_\alpha a = a,$$

where for the second equality, we used the w^*-continuity of the map $\Psi \mapsto \Psi \Box \Phi$. Suppose $\lambda \in X$. Given any $a \in A$, we have

$$L_{\lambda,\Phi_0}(a) = \langle L(a)\lambda, \Phi_0 \rangle = \langle a \cdot \lambda, \Phi_0 \rangle = \langle \lambda, \Phi_0 \cdot a \rangle = \langle \lambda, a \rangle.$$

So $L_{\lambda,\Phi_0} = \lambda$.

(ii) Since X is a closed Banach A-bisubmodule of A^*, $R(a) \in \mathscr{L}(X)$ for every $a \in A$. Moreover,

$$R(ab)\mu = \mu \cdot (ab) = (\mu \cdot a) \cdot b = R(b)R(a)\mu,$$

where for the second equality, we used the associative property of the module action. Hence R is an anti-representation.

Furthermore, R is continuous, since

$$\|R(a)\| = \sup_{\mu \in B_X} \|R(a)\mu\| = \sup_{\mu \in B_X} \|\mu \cdot a\| \leqslant \sup_{\mu \in B_X} \|\mu\|\|a\| \leqslant \|a\|.$$

Let (e_α) be a bounded right approximate identity of A, and let Ψ_0 be any w^*-cluster point of this net in X^*. Then $a \cdot \Psi_0 = a$, since

$$a \cdot \Phi_0 = a \Diamond \Phi_0 = \lim_\alpha a \Diamond e_\alpha = \lim_\alpha a e_\alpha = a,$$

where for the second equality, we used the w^*-continuity of the map $\Psi \mapsto \Phi \Diamond \Psi$. Suppose $\lambda \in X$. Given any $a \in A$, we have

$$R_{\lambda,\Psi_0}(a) = \langle R(a)\lambda, \Psi_0 \rangle = \langle \lambda \cdot a, \Psi_0 \rangle = \langle \lambda, a \cdot \Psi_0 \rangle = \langle \lambda, a \rangle.$$

So $R_{\lambda,\Psi_0} = \lambda$. $\square$

Let $\lambda \in wap(A)$. Using the notation of Section 4.1, let $X = A^*$, and W be the norm closure of $B_A \cdot \lambda = \{a \cdot \lambda : a \in A, \|a\| \leqslant 1\}$ in A^*. Then W is a convex, bounded, balanced, and weakly compact subset of A^*. Hence if Y is as in Theorem 4.5, then Y is a reflexive Banach space, $Y \subset A^*$, the natural inclusion $j : Y \longrightarrow A^*$ is continuous and $\|j\| \leqslant 1$, and moreover, $W \subset B_Y$. Recall that $\Psi_0 \in A^{**}$ is a mixed identity for A^{**} if it is a right identity for the first Arens product and a left identity for the second Arens product. We can now state the main result of this chapter.

Theorem 4.8. *Let A be a Banach algebra with a bounded two-sided approximate identity (BAI) and let $0 \neq \lambda \in wap(A)$. Let $L : A \longrightarrow \mathscr{L}(A^*)$ be defined by $L(a)\mu = a \cdot \mu$ ($a \in A, \mu \in A^*$). Then the map*

$$\pi : A \longrightarrow \mathscr{L}(Y), \qquad \pi(a) = L(a) \circ j = L(a)|_Y,$$

is a continuous representation of A on the reflexive Banach space Y, and λ is a coordinate function of π. Moreover, π is faithful if Y separates the points of A^*.

Before we present the proof of Theorem 4.8, we will prove the following Lemma which is needed in the proof of the theorem.

LEMMA 4.9. *Let A be a Banach algebra with a BAI and let $L : A \longrightarrow \mathscr{L}(A^*)$ be defined as above. Then for all $a \in A$,*

(i) $L(a)^* : A^{**} \longrightarrow A^{**}$ *is given by* $L(a)^*\Psi = \Psi \cdot a$ $(\Psi \in A^{**})$.

(ii) $L(a)^{**} : A^{***} \longrightarrow A^{***}$ *is given by* $L(a)^{**}\Gamma = a \cdot \Gamma$ $(\Gamma \in A^{***})$. *In particular, for all* $\lambda \in A^*$, $L(a)^{**}\lambda = a \cdot \lambda$.

PROOF. Let $a \in A$, $\Psi \in A^{**}$, and $\Gamma \in A^{***}$. Then

(i) $\langle L(a)^*\Psi, \lambda\rangle = \langle \Psi, L(a)\lambda\rangle = \langle \Psi, a \cdot \lambda\rangle = \langle \Psi \cdot a, \lambda\rangle$.

(ii) $\langle L(a)^{**}\Gamma, \Psi\rangle = \langle \Gamma, L(a)^*\Psi\rangle = \langle \Gamma, \Psi \cdot a\rangle = \langle a \cdot \Gamma, \Psi\rangle$. □

PROOF OF THEOREM 4.8. Note first that the continuity of L and the fact that every $\mu \in A^*$ is a coordinate function of L follow immediately from Lemma 4.7, since A^* is clearly a faithful subspace of itself and is also topologically introverted in itself. Moreover, Theorem 4.5 guarantees that Y is a reflexive Banach space.

Now we show that $\pi(a) \in \mathscr{L}(Y)$ for every $a \in A$.

Let $a \in A$ and $\mu \in Y$. If $t \geqslant 0$ is such that $\mu \in tU_n$, then

$$a \cdot \mu \in ta \cdot U_n = 2^n ta \cdot W + 2^{-n} ta \cdot B_{A^*} \subset 2^n t\|a\| W + 2^{-n} t\|a\| B_{A^*}. \qquad (21)$$

The inclusion is derived as follows: $a \cdot W = \|a\| \frac{a}{\|a\|} \cdot W$, where $\frac{a}{\|a\|} \cdot W \subset W$. Similarly, $a \cdot B_{A^*} = \|a\| \frac{a}{\|a\|} \cdot B_{A^*}$, and $\frac{a}{\|a\|} \cdot B_{A^*} \subset B_{A^*}$. It follows from (21) that $\|a \cdot \mu\|_n \leqslant t\|a\|$ $(n = 1, 2, \ldots)$. Since this holds for all $t \geqslant 0$ such that $\mu \in tU_n$, we have $\|a \cdot \mu\|_n \leqslant \|a\|\|\mu\|_n$ $(n = 1, 2, \ldots)$. Consequently, we have

$$\|a \cdot \mu\|' = (\sum_{n=1}^{\infty} \|a \cdot \mu\|_n^2)^{1/2} \leqslant \|a\|(\sum_{n=1}^{\infty} \|\mu\|_n^2)^{1/2} \leqslant \|a\|\|\mu\|' < \infty.$$

This proves that $L(a)\mu = a \cdot \mu \in Y$ for all $a \in A$ and $\mu \in Y$. Moreover, π is continuous since

$$\|\pi(a)\mu\|' = \|a \cdot \mu\|' \leqslant \|a\|\|\mu\|'$$

for all $a \in A$ and $\mu \in Y$, and hence $\|\pi(a)\| \leqslant \|a\|$, i.e., $\|\pi\| \leqslant 1$.

Next we will show that λ is a coordinate function of π. Let (e_α) be a BAI for A with bound $r > 0$. Then for all $a \in A$, $\langle e_\alpha \cdot \lambda, a\rangle = \langle \lambda, ae_\alpha\rangle \longrightarrow \langle \lambda, a\rangle$. So

$e_\alpha \cdot \lambda \xrightarrow{w^*} \lambda$. Moreover,

$$e_\alpha \cdot \lambda \in C = \{a \cdot \lambda : a \in A, \|a\| \leqslant r\} = r\{a \cdot \lambda : a \in A, \|a\| \leqslant 1\}.$$

Since $\lambda \in wap(A)$, C is relatively weakly compact. Therefore, there exists a subnet (e_{α_β}) of (e_α) such that

$$e_{\alpha_\beta} \cdot \lambda \xrightarrow{w} \lambda' \in \overline{C}^w.$$

Since the w^*-topology is weaker than the w-topology on A^*, $e_{\alpha_\beta} \cdot \lambda \xrightarrow{w^*} \lambda'$. By the uniqueness of w^*-limits, we have

$$\lambda = \lambda' \in \overline{C}^w = \overline{C}^{\|\cdot\|} = rW \subset Y,$$

where for the last inclusion, we used Theorem 4.5(i) and the definition of Y.

Let Φ_0 be a w^*-cluster point of (e_α) in A^{**}. Then Φ_0 is a mixed identity for A^{**} by Palmer [**40**, Proposition 5.1.8]. So $\Phi_0 \cdot a = a$ for all $a \in A$. Define $\Phi_{00} := \Phi_0|_Y$. Then $\Phi_{00} = j^*(\Phi_0) \in Y^*$.

Now, since we have shown that $\lambda \in Y$ and $\Phi_{00} \in Y^*$, the coordinate function $\pi_{\lambda,\Phi_{00}}$ is well defined. In fact, if $a \in A$, we have

$$\pi_{\lambda,\Phi_{00}}(a) = \langle \pi(a)\lambda, \Phi_{00}\rangle_{Y,Y^*} = \langle a \cdot \lambda, \Phi_0\rangle_{A^*,A^{**}} = \langle \lambda, \Phi_0 \cdot a\rangle_{A^*,A^{**}} = \lambda(a).$$

So $\lambda = \pi_{\lambda,\Phi_{00}}$.

Finally we show that π is faithful whenever Y separates the points of A^*; that is, π is injective whenever Y is a faithful subspace of A^*.

Suppose Y is faithful, and let $a_0 \in A$ be such that $\pi(a_0) = 0$. Then $L(a_0) \circ j = 0$ and hence $a_0 \cdot y = 0$ for every $y \in Y$. Therefore, $\langle a_0 \cdot y, a\rangle = 0$ for all $y \in Y$ and $a \in A$ which implies that $\langle y, aa_0\rangle = 0$ for all $y \in Y$ and $a \in A$. In particular, $\langle y, e_\alpha a_0\rangle = 0$ for all $y \in Y$ and for all α, and hence $\langle y, a_0\rangle = 0$ for all $y \in Y$, which implies that $a_0 = 0$. Therefore, π is faithful. □

Let G be a topological group and f be a function on G. Megrelishvili showed in [**38**] that $f \in WAP(G)$ if and only if f is a coordinate function of some representation $V : G \longrightarrow Is(Y)$ of G on some reflexive Banach space Y. Theorems 4.8 and 3.15 provide a Banach algebra analogue of this result.

The following lemma will be useful in proving our next theorem.

LEMMA 4.10. *Let A be a Banach algebra with a BAI. For each $\lambda \in wap(A)$, let $\pi^\lambda : A \longrightarrow \mathscr{L}(Y_\lambda)$ be a continuous representation of A on a reflexive Banach space Y_λ such that $\|\pi^\lambda\| \leqslant 1$ and λ is a coordinate function of π^λ (see Theorem 4.8).*

(i) *The map*

$$\widetilde{T}_\lambda : Y_\lambda \times Y_\lambda^* \longrightarrow wap(A), \qquad (y_\lambda, \phi_\lambda) \mapsto \pi^\lambda_{y_\lambda,\phi_\lambda}$$

is bilinear and continuous and $\|\widetilde{T}_\lambda\| \leqslant 1$.

(ii) *Let $Y = \ell^2\text{-}\bigoplus_\lambda Y_\lambda$, be the ℓ^2-direct sum of the family $\{Y_\lambda\}_{\lambda \in wap(A)}$. Then the map*

$$\widetilde{T} : Y \times Y^* \longrightarrow wap(A),$$
$$((y_\lambda), (\phi_\lambda)) \mapsto \sum_\lambda \widetilde{T}_\lambda(y_\lambda, \phi_\lambda) = \sum_\lambda \pi_{y^\lambda_\lambda,\phi_\lambda}$$

is bilinear and continuous and $\|\widetilde{T}\| \leqslant 1$.

REMARKS 4.11. (1) The fact that the images of $\widetilde{T}_\lambda$ and $\widetilde{T}$ above are subsets of $wap(A)$ is a consequence of Theorem 3.15.

(2) Note that $Y^* = \ell^2\text{-}\bigoplus_\lambda Y_\lambda^*$ and hence Y is reflexive by Megginson [**37**, Theorem 1.10.13].

(3) Although the sums in the definition of $\widetilde{T}$ are over an uncountably infinite set, the fact that Y and Y^* are ℓ^2-direct sums implies by Proposition 1.8 that only countable many of the y_λ and ϕ_λ are non-zero. Consequently, only countably many $\pi_{y_\lambda,\phi_\lambda}$ terms are nonzero. Thus the sums are in fact countable.

PROOF. (i) The bilinearity of $\widetilde{T}_\lambda$ follows from Lemma 2.16(iii). Moreover, a direct application of part (iv) of the same lemma yields

$$\|\widetilde{T}_\lambda(y_\lambda, \phi_\lambda)\| = \|\pi_{y_\lambda,\phi_\lambda}\| \leqslant \|y_\lambda\|\|\phi_\lambda\|,$$

since $\|\pi\| \leqslant 1$. Therefore, $\widetilde{T}_\lambda$ is bilinear and continuous and $\|\widetilde{T}_\lambda\| \leqslant 1$.

(ii) Since $\|\widetilde{T}_\lambda\| \leqslant 1$ and Y and Y^* are ℓ^2 direct sums, by part (3) of the above remark, $\tilde{T} : Y \times Y^* \longrightarrow wap(A)$ is well defined. The bilinearity of $\widetilde{T}$ follows from its definition and the bilinearity of $\widetilde{T}_\lambda$. In fact, suppose (y^1_λ) and (y^2_λ) are two elements of Y, $(\phi_\lambda) \in Y^*$ and $\alpha, \beta \in \mathbb{C}$ are scalars, then

$$\begin{aligned}
\widetilde{T}(\alpha(y_\lambda^1) + \beta(y_\lambda^2), (\phi_\lambda)) &= \sum_\lambda \widetilde{T}_\lambda(\alpha y_\lambda^1 + \beta y_\lambda^2, \phi_\lambda) \\
&= \sum_\lambda (\alpha \widetilde{T}_\lambda(y_\lambda^1, \phi_\lambda) + \beta \widetilde{T}_\lambda(y_\lambda^2, \phi_\lambda)) \\
&= \alpha \sum_\lambda \widetilde{T}_\lambda(y_\lambda^1, \phi_\lambda) + \beta \sum_\lambda \widetilde{T}_\lambda(y_\lambda^2, \phi_\lambda) \\
&= \alpha \widetilde{T}((y_\lambda^1), (\phi_\lambda)) + \beta \widetilde{T}((y_\lambda^2), (\phi_\lambda)),
\end{aligned}$$

showing that $\widetilde{T}$ is linear in its first component. A similar argument, which we omit for briefness, shows that $\widetilde{T}$ is linear in its second component.

To show continuity of $\widetilde{T}$, we write

$$\begin{aligned}
\|\widetilde{T}((y_\lambda), (\phi_\lambda))\| &= \| \sum_\lambda \widetilde{T}_\lambda(y_\lambda, \phi_\lambda)\| \\
&\leqslant \sum_\lambda \|\widetilde{T}_\lambda(y_\lambda, \phi_\lambda)\| \\
&\leqslant \sum_\lambda \|y_\lambda\| \|\phi_\lambda\| \\
\text{(Cauchy-Schwarz)} &\leqslant (\sum_\lambda \|y_\lambda\|^2)^{1/2} (\sum_\lambda \|\phi_\lambda\|^2)^{1/2} \\
&= \|(y_\lambda)\| \|(\phi_\lambda)\|.
\end{aligned}$$

Hence $\widetilde{T}$ is continuous and $\|\widetilde{T}\| \leqslant 1$. □

The well-known GNS theorem (see Folland [**21**]) states that every C^*-algebra is isometrically isomorphic to a closed C^*-subalgebra of $\mathscr{L}(H)$ for some Hilbert space H. The following theorem of Daws [**10**, Theorem 3.6 and Corollary 3.8] provides an analogue result for dual Banach algebras. In the following, we give an alternative proof which utilizes Theorem 4.8.

THEOREM 4.12. **(Daws)** *Every dual Banach Algebra is isometrically isomorphic to a closed subalgebra of $\mathscr{L}(Y)$ for some reflexive Banach space Y.*

PROOF. Let A be a dual Banach Algebra with predual A_*. Since the unitization of a dual Banach algebra remains a dual Banach algebra (see Lemma 2.6), we may assume without any loss of generality that A is unital.

For each $\lambda \in wap(A)$, let π^λ, Y_λ and Y be given as in Lemma 4.10 and let T_λ and T be the bounded linear maps corresponding to $\widetilde{T}_\lambda$ and $\widetilde{T}$, respectively (Theorem A.19). Then

$$T_\lambda : Y_\lambda \widehat{\otimes} Y_\lambda^* \longrightarrow wap(A), \qquad y_\lambda \otimes \phi_\lambda \mapsto \pi^\lambda_{y_\lambda,\phi_\lambda},$$

and

$$T : Y \widehat{\otimes} Y^* \longrightarrow wap(A), \quad (y_\lambda) \otimes (\phi_\lambda) \mapsto \sum_\lambda T_\lambda(y_\lambda \otimes \phi_\lambda) = \sum_\lambda \pi^\lambda_{y_\lambda,\phi_\lambda},$$

satisfying $\|T_\lambda\| \leqslant 1$ and $\|T\| \leqslant 1$.

We claim that the adjoint map

$$T^* : wap(A)^* \longrightarrow (Y \widehat{\otimes} Y^*)^* \cong \mathscr{L}(Y)$$

is an isometric algebra homomorphism. To prove this claim, we first show that given $\nu \in wap(A)^*$, $y = (y_\lambda) \in Y$ and $\phi = (\phi_\lambda) \in Y^*$, we have

$$(T^*\nu)y = ((T_\lambda^*\nu)y_\lambda)_\lambda \in Y, \tag{22}$$

$$\sum_\lambda \nu \cdot \pi^\lambda_{y_\lambda,\phi_\lambda} = T((T^*\nu)y \otimes \phi) \in wap(A). \tag{23}$$

In fact,

$$\begin{aligned}
\langle (T^*\nu)y, \phi\rangle_{Y,Y^*} &= \langle T^*\nu, y \otimes \phi\rangle_{\mathscr{L}(Y),\mathscr{L}(Y)_*} \\
&= \langle \nu, T(y \otimes \phi)\rangle_{wap(A)^*,wap(A)} \\
&= \langle \nu, \sum_\lambda T_\lambda(y_\lambda \otimes \phi_\lambda)\rangle \\
&= \sum_\lambda \langle \nu, T_\lambda(y_\lambda \otimes \phi_\lambda)\rangle \\
&= \sum_\lambda \langle T_\lambda^*\nu, y_\lambda \otimes \phi_\lambda\rangle_{\mathscr{L}(Y_\lambda),\mathscr{L}(Y_\lambda)_*} \\
&= \sum_\lambda \langle (T_\lambda^*\nu)y_\lambda, \phi_\lambda\rangle_{Y_\lambda,Y_\lambda^*} \\
&= \langle ((T_\lambda^*\nu)y_\lambda), (\phi_\lambda)\rangle_{Y,Y^*} \\
&= \langle ((T_\lambda^*\nu)y_\lambda), \phi\rangle,
\end{aligned}$$

which proves (22). To prove (23), we note that for every $a \in A$, we have

$$
\begin{aligned}
\langle \sum_\lambda \nu \cdot \pi^\lambda_{y_\lambda,\phi_\lambda}, a\rangle_{A^*,A} &= \sum_\lambda \langle \nu, \pi^\lambda_{y_\lambda,\phi_\lambda} \cdot a\rangle_{wap(A)^*,wap(A)} \\
\text{(Lemma 2.16(ii))} &= \sum_\lambda \langle \nu, \pi^\lambda_{y_\lambda,\pi^\lambda(a)^*\phi_\lambda}\rangle_{wap(A)^*,wap(A)} \\
&= \langle \nu, T(y \otimes (\pi^\lambda(a)^*\phi_\lambda))\rangle \\
&= \langle (T^*\nu)y, (\pi^\lambda(a)^*\phi_\lambda)\rangle_{Y,Y^*} \\
\text{(by (22))} &= \langle ((T^*_\lambda\nu)y_\lambda), (\pi^\lambda(a)^*\phi_\lambda)\rangle \\
&= \sum_\lambda \langle \pi^\lambda(a)(T^*_\lambda\nu)y_\lambda, \phi_\lambda\rangle_{Y_\lambda,Y^*_\lambda} \\
&= \langle \sum_\lambda \pi^\lambda_{(T^*_\lambda\nu)y_\lambda,\phi_\lambda}, a\rangle_{A^*,A} \\
&= \langle T(((T^*_\lambda\nu)y_\lambda) \otimes (\phi_\lambda)), a\rangle \\
\text{(by (22))} &= \langle T((T^*\nu)y \otimes \phi), a\rangle,
\end{aligned}
$$

which completes the proof of (23).

Next we verify that

$$T^*(\mu\Box\nu) = (T^*\mu)(T^*\nu) \text{ for all } \mu,\, \nu \in wap(A)^*.$$

In fact, for all $y \in Y$ and $\phi \in Y^*$, we have

$$
\begin{aligned}
\langle T^*(\mu\Box\nu)y, \phi\rangle_{Y,Y^*} &= \langle T^*(\mu\Box\nu), y \otimes \phi\rangle_{\mathscr{L}(Y),\mathscr{L}(Y)_*} \\
&= \langle \mu\Box\nu, T(y \otimes \phi)\rangle_{wap(A)^*,wap(A)} \\
&= \langle \mu\Box\nu, \sum_\lambda \pi^\lambda_{y_\lambda,\phi_\lambda}\rangle \\
&= \langle \mu, \sum_\lambda \nu \cdot \pi^\lambda_{y_\lambda,\phi_\lambda}\rangle \\
\text{(by (23))} &= \langle \mu, T((T^*\nu)y \otimes \phi)\rangle \\
&= \langle T^*\mu, (T^*\nu)y \otimes \phi\rangle_{\mathscr{L}(Y),\mathscr{L}(Y)_*} \\
&= \langle (T^*\mu)(T^*\nu)y, \phi\rangle_{Y,Y^*},
\end{aligned}
$$

which is what we wanted to show.

Now, we verify that T^* is an isometry. We immediately have $\|T^*\| = \|T\| \leqslant 1$, so $\|T^*\mu\| \leqslant \|\mu\|$ for every $\mu \in wap(A)^*$.

Let e be the identity of A, $\mu \in wap(A)^*$ be given, and $j_\lambda : Y_\lambda \longrightarrow A^*$ be the natural inclusion map. For $y \in Y_\lambda$ and $\Psi \in A^{**}$, we have $\langle j_\lambda^*(\Psi), y\rangle = \langle \Psi, j_\lambda(y)\rangle$. Hence $j_\lambda^*(\Psi) = \Psi|_{Y_\lambda}$.

Let $e_\lambda = j_\lambda^*(e) = e|_{Y_\lambda}$. Then since $\|j_\lambda\| \leqslant 1$ and $\|e\| = 1$, we have

$$\|e_\lambda\| = \|j_\lambda^*(e)\| \leqslant \|j_\lambda\|\|e\| \leqslant 1.$$

Let $\epsilon > 0$ be given and pick $\lambda \in wap(A)$ such that $\|\lambda\| \leqslant 1$ and $|\mu(\lambda)| \geqslant \|\mu\| - \epsilon$. Since $\lambda \in Y_\lambda$ and $e_\lambda \in Y_\lambda^*$, $T_\lambda(\lambda \otimes e_\lambda) = \pi^\lambda_{\lambda, e_\lambda} \in wap(A)$ is well defined. In fact, for every $a \in A$,

$$\pi^\lambda_{\lambda,e_\lambda}(a) = \langle \pi^\lambda(a)\lambda, e_\lambda\rangle = \langle a \cdot \lambda, j_\lambda^*(e)\rangle = \langle a \cdot \lambda, e\rangle = \lambda(a),$$

so that

$$T_\lambda(\lambda \otimes e_\lambda) = \pi^\lambda_{\lambda,e_\lambda} = \lambda.$$

Thus we have

$$\|T^*\mu\| \geqslant \|T_\lambda^*\mu\| \geqslant |\langle T_\lambda^*\mu, \lambda \otimes e_\lambda\rangle| = |\langle \mu, T_\lambda(\lambda \otimes e_\lambda)\rangle| = |\mu(\lambda)| \geqslant \|\mu\| - \epsilon. \quad (24)$$

The justification for the first inequality is as follows:

$$\begin{aligned} \|T^*\mu\| &= \sup_{y \in B_Y} \|(T^*\mu)y\| \overset{(22)}{=} \sup_{y \in B_Y} \|((T_\lambda^*\mu)y_\lambda)\| = \sup_{y \in B_Y} \Big(\sum_\lambda \|(T_\lambda^*\mu)y_\lambda\|^2\Big)^{1/2} \\ &\geqslant \sup_{y_\lambda \in B_{Y_\lambda}} \|(T_\lambda^*\mu)y_\lambda\| = \|T_\lambda^*\mu\|. \end{aligned}$$

The second inequality in (24) follows by definition of the norm and the fact that $\|\lambda \otimes e_\lambda\| = \|\lambda\|\|e_\lambda\| \leqslant 1$.

Since (24) holds for every $\epsilon > 0$, we conclude that $\|T^*\mu\| \geqslant \|\mu\|$, completing the proof that T^* is an isometry.

We have shown that $wap(A)^*$ embeds isometrically into $\mathscr{L}(Y)$, which combined with the fact that the dual Banach algebra A can be isometrically embedded in

$wap(A)^*$ (Runde [**43**, Corollary 4.6]) proves that A can be isometrically embedded in $\mathscr{L}(Y)$, completing the proof. $\square$

CHAPTER 5

LUC and RUC-Representation Theorems

5.1. Preliminaries

We begin this chapter with some basic definitions and results. For more details the reader may consult Folland [**21**] or Burckel [**2**].

The following lemma will prove quite useful for later purposes. Recall the definition of unitary representation given in Section 1.7. As in Folland [**21**, Section 3.2], every unitary representation $V : G \longrightarrow \mathscr{L}(H)$ of a locally compact group G on a Hilbert space H determines a continuous representation $\widetilde{V} : L^1(G) \longrightarrow \mathscr{L}(H)$ of the group algebra $L^1(G)$ on H in the following way: if $f \in L^1(G)$, then

$$\widetilde{V}(f) = \int_G f(t)V(t)dt, \tag{25}$$

where the integration is with respect to a left Haar measure on G.

Since $\widetilde{V}$ is continuous with respect to the SOT on $\mathscr{L}(H)$, its coordinate functions belong to $\mathscr{C}^b(G)$.

LEMMA 5.1. *Let V be a unitary representation of a locally compact group G on some Hilbert space H, and suppose $\widetilde{V}$ is the associated representation of $L^1(G)$ on H defined in* (25)*. Then for all $\xi, \eta \in H$, we have $\widetilde{V}_{\xi,\eta} = V_{\xi,\eta}$.*

PROOF. For all $\xi, \eta \in H$ and $f \in L^1(G)$, we have

$$\begin{aligned}
\langle \widetilde{V}_{\xi,\eta}, f\rangle_{L^\infty(G),L^1(G)} &= \langle \widetilde{V}(f)\xi|\eta\rangle \\
&= \int_G \langle V(t)\xi|\eta\rangle f(t)dt \\
&= \int_G V_{\xi,\eta}(t)f(t)dt \\
&= \langle V_{\xi,\eta}, f\rangle_{L^\infty(G),L^1(G)}.
\end{aligned}$$

□

Let G be a locally compact group, and $\mathscr{C}^b(G)$ be the space of all bounded continuous functions on G. Recall that for a function $f \in \mathscr{C}^b(G)$ we define the [right] left translations of f by $[f_s(t) = f(ts)]$ ${}_sf(t) = f(st)$ $(t \in G)$.

Definition 5.2. A function f in $\mathscr{C}^b(G)$ is said to be *[left] right uniformly continuous* if

$$G \longrightarrow \mathscr{C}^b(G), \quad [s \mapsto {}_sf]\; s \mapsto f_s$$

is continuous. The space of all [left] right uniformly continuous functions is denoted by $[LUC(G)]\, RUC(G)$.

The following result is excerpted from Galindo [**23**, Proposition 1.6] and we state it here without proof for comparison with our later results.

Proposition 5.3. *Let G be a topological group and $f : G \longrightarrow \mathbb{C}$ be a function. Then $f \in [LUC(G)]\, RUC(G)$ if and only if f is a coordinate function of some [anti-]representation $V : G \longrightarrow Is(X)$ of G on some Banach space X.*

Suppose A is a Banach algebra. The spaces of left uniformly continuous and right uniformly continuous functionals on A are defined respectively by

$$luc(A) = \overline{\text{span}}(A^* \cdot A)$$

and

$$ruc(A) = \overline{\text{span}}(A \cdot A^*),$$

where $\overline{\text{span}}$ denotes the closed linear span. Since $a \cdot (\lambda \cdot b) = (a \cdot \lambda) \cdot b$ for all $a, b \in A$ and $\lambda \in A^*$ (see part (ii) of Lemma 1.21), Proposition 2.12 implies that $luc(A)$ is topologically left-introverted in A^* and $ruc(A)$ is topologically right-introverted in A^*.

As in the case of [weakly] almost periodic functionals, our next lemma provides a situation in which the space $[LUC(G)]\, RUC(G)$ for a locally compact group G coincides with the space $[luc(A)]\, ruc(A)$ for a Banach algebra A.

Lemma 5.4. *Let G be a locally compact group and $L^1(G)$ be the group algebra. Then $LUC(G) = luc(L^1(G))$ and $RUC(G) = ruc(L^1(G))$.*

PROOF. This follows from Proposition 3.6(i) and the identities

$$LUC(G) = L^1(G) * L^\infty(G), \; RUC(G) = L^\infty(G) * L^1(G)^\vee$$

(see Hewitt–Ross [**28**, (32.45)]). □

5.2. Results for the group algebra

Throughout this section, G denotes a locally compact group with a fixed left Haar measure. We shall consider representations that are subordinate to the space $LUC(G)$. In Theorem 5.7 we show that to every continuous unitary representation of G on a Hilbert space H, one can associate a conjugate [anti-]representation $[\pi']$ π of $L^1(G)$ on $\mathscr{L}(H)$, which is subordinate to $[RUC(G)]$ $LUC(G)$. The converse of this result will be proved in Theorem 5.9.

LEMMA 5.5. *Let $V : G \longrightarrow \mathscr{L}(H)$ be a continuous unitary representation of G.*

(i) *The map $\pi : M(G) \longrightarrow \mathscr{L}(\mathscr{L}(H))$, defined by*

$$\pi(\mu)T = \int_G V(t)TV(t)^* \, d\mu(t) \qquad (\mu \in M(G), \; T \in \mathscr{L}(H)), \tag{26}$$

is a continuous representation with $\|\pi\| \leqslant 1$.

(ii) *The map $\pi' : M(G) \longrightarrow \mathscr{L}(\mathscr{L}(H))$, defined by*

$$\pi'(\mu)T = \int_G V(t)^*TV(t) \, d\mu(t) \qquad (\mu \in M(G), \; T \in \mathscr{L}(H)), \tag{27}$$

is a continuous anti-representation with $\|\pi'\| \leqslant 1$.

The integrals appearing in (26) and (27) are operator-valued, and the identity (26) (and those similar to it) is to be interpreted as

$$\langle \pi(\mu)Tx|y\rangle = \int_G \langle V(t)TV(t)^*x|y\rangle \, d\mu(t) \qquad (x, y \in H),$$

where $\langle V(\cdot)TV(\cdot)^*x|y\rangle \in \mathscr{C}^b(G)$. For a brief review of these integrals, see Folland [**21**, Appendix 3], and for a more detailed treatment, see Bourbaki [**1**, Section VI.1.3].

PROOF. (i) Let μ, $\nu \in M(G)$ and $T \in \mathscr{L}(H)$. Then for all $x, y \in H$,

$$\begin{aligned}
\langle \pi(\mu * \nu)Tx|y\rangle &= \int_G \langle V(t)TV(t)^*x|y\rangle\, d(\mu * \nu)(t) \\
&= \int_G \int_G \langle V(gs)TV(gs)^*x|y\rangle d\nu(s)d\mu(g) \\
&= \int_G \int_G \langle V(g)V(s)TV(s)^*V(g)^*x|y\rangle d\nu(s)d\mu(g) \\
&= \int_G \int_G \langle V(s)TV(s)^*V(g)^*x|V(g)^*y\rangle d\nu(s)d\mu(g) \\
&= \int_G \langle \pi(\nu)V(g)^*x|V(g)^*y\rangle d\mu(g) \\
&= \int_G \langle V(g)\,(\pi(\nu)T)\,V(g)^*x|y\rangle d\mu(g) \\
&= \langle \pi(\mu)\pi(\nu)Tx|y\rangle,
\end{aligned}$$

where the second equality is justified by the definition of $\mu * \nu$ (see Folland [**21**, Equation (2.34), p.49]).

Moreover, using the fact that $V(t)$ is a unitary operator for all $t \in G$ and hence $\|V(t)\| = \|V(t)^*\| = 1$, we have

$$\begin{aligned}
\|\pi(\mu)T\| &= \sup_{\substack{x,y\in H\\ \|x\|,\|y\|\leqslant 1}} |\langle \pi(\mu)Tx|y\rangle| \\
&= \sup_{\substack{x,y\in H\\ \|x\|,\|y\|\leqslant 1}} \left|\int_G \langle V(t)TV(t)^*x|y\rangle d\mu(t)\right| \\
&\leqslant \sup_{\substack{x,y\in H\\ \|x\|,\|y\|\leqslant 1}} \int_G |\langle V(t)TV(t)^*x|y\rangle| d|\mu|(t) \\
&\leqslant \sup_{\substack{x,y\in H\\ \|x\|,\|y\|\leqslant 1}} \|T\||\mu|(G) = \|\mu\|\|T\|.
\end{aligned}$$

Therefore, $\|\pi(\mu)\| \leqslant \|\mu\|$.

(ii) Let μ, $\nu \in M(G)$ and $T \in \mathscr{L}(H)$. Then for all $x, y \in H$, we have

$$\begin{aligned}
\langle \pi'(\mu * \nu)Tx|y\rangle &= \int_G \langle V(t)^*TV(t)x|y\rangle d(\mu * \nu)(t) \\
&= \int_G \int_G \langle V(gs)^*TV(gs)x|y\rangle d\mu(g)d\nu(s) \\
&= \int_G \int_G \langle V(s)^*V(g)^*TV(g)V(s)x|y\rangle d\mu(g)d\nu(s)
\end{aligned}$$

$$
\begin{aligned}
&= \int_G \int_G \langle V(g)^* T V(g) V(s)x | V(s)y \rangle d\mu(g) d\nu(s) \\
&= \int_G \langle \pi'(\mu) T V(s)x | V(s)y \rangle d\nu(s) \\
&= \int_G \langle V(s)^* \pi'(\mu) T V(s)x | y \rangle d\nu(s) \\
&= \langle \pi'(\nu)\pi'(\mu) T x | y \rangle.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\|\pi'(\mu)T\| &= \sup_{\substack{x,y\in H \\ \|x\|,\|y\|\leqslant 1}} |\langle \pi'(\mu)Tx|y\rangle| \\
&= \sup_{\substack{x,y\in H \\ \|x\|,\|y\|\leqslant 1}} \left| \int_G \langle V(t)^* T V(t)x | y \rangle d\mu(t) \right| \\
&\leqslant \sup_{\substack{x,y\in H \\ \|x\|,\|y\|\leqslant 1}} \int_G |\langle V(t)^* T V(t)x | y \rangle| d|\mu|(t) \\
&\leqslant \sup_{\substack{x,y\in H \\ \|x\|,\|y\|\leqslant 1}} \|T\| |\mu|(G) = \|\mu\| \|T\|.
\end{aligned}
$$

Therefore, $\|\pi'(\mu)\| \leqslant \|\mu\|$. $\square$

Remark 5.6. The preceding result was proved for the left regular representation of G on $L^2(G)$ by Størmer [**48**] when G is abelian, and by Ghahramani [**24**] for general locally compact groups. Both Størmer and Ghahramani showed that the representation associated to the left regular representation is an isometry. This of course is not true in general. For example, suppose V is the identity representation of G on $L^2(G)$, that is,

$$V : G \longrightarrow \mathscr{L}(L^2(G)), \qquad t \mapsto V(t) = I,$$

and let π be the representation of $M(G)$ associated to V. Then π is not even injective, since if f is a function in $L^1(G)$ such that $\int_G f(t)dt = 0$, then $\pi(f) = 0$.

In the following, we shall call π and π' given in (26) and (27), as well as their restrictions, $\tilde{\pi}$ and $\tilde{\pi}'$, to $L^1(G)$, the conjugate representation and anti-representation associated with V, respectively. We recall that the predual of $\mathscr{L}(H)$ is the space

$\mathscr{L}(H)_* \cong H\widehat{\otimes}H$, where $\widehat{\otimes}$ denotes the projective tensor product (see Definition A.16).

THEOREM 5.7. *Let $V : G \longrightarrow \mathscr{L}(H)$ be a continuous unitary representation, and $[\tilde{\pi}']$ $\tilde{\pi}$ be the associated conjugate [anti-]representation of $L^1(G)$.*

(i) *If $T \in \mathscr{L}(H)$ and $T_* = \sum_{i=1}^{\infty} x_i \otimes y_i \in \mathscr{L}(H)_*$, then in $L^\infty(G)$,*

$$\tilde{\pi}_{T,T_*} = \sum_{i=1}^{\infty} \langle TV(\cdot)^* x_i | V(\cdot)^* y_i \rangle, \tag{28}$$

$$\tilde{\pi}'_{T,T_*} = \sum_{i=1}^{\infty} \langle TV(\cdot) x_i | V(\cdot) y_i \rangle. \tag{29}$$

(ii) *$\tilde{\pi}$ and $\tilde{\pi}'$ are subordinate to $LUC(G)$ and $RUC(G)$, respectively.*

Before presenting the proof, we make a remark which we will require later on.

REMARK 5.8. Suppose f is a continuous function on a locally compact group G, and (s_α) is a net in G such that $s_\alpha \to s \in G$. Then $s_\alpha s^{-1} \to e$, and

$$\begin{aligned} \|{}_{s_\alpha}f - {}_s f\|_\infty &= \sup_{t\in G} |({}_{s_\alpha}f - {}_s f)(t)| \\ &= \sup_{t\in G} |f(s_\alpha t) - f(st)| \\ (\text{letting } y = st,) &= \sup_{y\in G} |f(s_\alpha s^{-1} y) - f(y)| \\ &= \|{}_{s_\alpha s^{-1}}f - f\|_\infty. \end{aligned}$$

A similar argument shows that $\|f_{s_\alpha} - f_s\|_\infty = \|f_{s^{-1}s_\alpha} - f\|_\infty$.

Hence, when verifying the left/right uniform continuity of f, it suffices to consider the continuity of the translation operator at the identity of G.

PROOF. (i) Since $\tilde{\pi}$ is continuous, it follows that $\tilde{\pi}_{T,T_*} \in L^1(G)^* = L^\infty(G)$. Let $f_i, f_i' \in L^\infty(G)$ be defined by $f_i(t) = \langle TV(t)^* x_i | V(t)^* y_i \rangle$ and $f_i'(t) = \langle TV(t) x_i | V(t) y_i \rangle$, respectively. As a consequence of the continuity of V in the strong operator topology, we have if $t_\alpha \to t \in G$, then for every $x \in H$, $V(t_\alpha)x \to V(t)x \in H$, and

$$V(t_\alpha)^* x = V(t_\alpha^{-1})x \to V(t^{-1})x = V(t)^* x.$$

Hence f_i and f_i' are continuous. Moreover, by the Cauchy-Schwarz inequality, since V is unitary, we have

$$\begin{aligned}\|f_i\|_\infty &= \sup_{t\in G}|f_i(t)| = \sup_{t\in G}|\langle TV(t)^*x_i|V(t)^*y_i\rangle| \\ &\leqslant \sup_{t\in G}\|T\|\|V(t)^*\|\|x_i\|\|V(t)^*\|\|y_i\| \\ &= \|T\|\|x_i\|\|y_i\|,\end{aligned}$$

and similarly, $\|f_i'\|_\infty \leqslant \|T\|\|x_i\|\|y_i\|$. Thus $f_i,\ f_i' \in \mathscr{C}^b(G)$.

We note moreover that

$$\sum_{i=1}^\infty \|f_i\|_\infty \leqslant \|T\|\sum_{i=1}^\infty \|x_i\|\|y_i\| < \infty,$$

and hence $\sum\limits_{i=1}^\infty f_i$ is uniformly and absolutely convergent in $\mathscr{C}^b(G)$. A similar argument shows that $\sum\limits_{i=1}^\infty f_i'$ is also uniformly and absolutely convergent in $\mathscr{C}^b(G)$.

Let $h \in L^1(G)$. Then we have

$$\begin{aligned}\langle \tilde{\pi}_{T,T_*}, h\rangle_{L^\infty(G),L^1(G)} &= \langle \tilde{\pi}(h)T, T_*\rangle_{\mathscr{L}(H),\mathscr{L}(H)_*} \\ &= \sum_{i=1}^\infty \langle \tilde{\pi}(h)T, x_i\otimes y_i\rangle \\ &= \sum_{i=1}^\infty \langle \tilde{\pi}(h)Tx_i|y_i\rangle \\ &= \int_G \sum_{i=1}^\infty \langle V(t)TV(t)^*x_i|y_i\rangle h(t)dt \\ &= \int_G \sum_{i=1}^\infty \langle TV(t)^*x_i|V(t)^*y_i\rangle h(t)dt \\ &= \int_G \sum_{i=1}^\infty f_i(t)h(t)dt \\ &= \langle \sum_{i=1}^\infty f_i, h\rangle_{\mathscr{C}^b(G),L^1(G)}.\end{aligned}$$

Therefore, $\tilde{\pi}_{T,T_*} = \sum\limits_{i=1}^\infty f_i$ in $L^\infty(G)$. Similarly,

$$\langle \tilde{\pi}'_{T,T_*}, h\rangle_{L^\infty(G),L^1(G)} = \langle \tilde{\pi}'(h)T, T_*\rangle_{\mathscr{L}(H),\mathscr{L}(H)_*}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \langle \tilde{\pi}'(h)T, x_i \otimes y_i \rangle \\
&= \sum_{i=1}^{\infty} \langle \tilde{\pi}'(h)Tx_i | y_i \rangle \\
&= \int_G \sum_{i=1}^{\infty} \langle V(t)^* T V(t) x_i | y_i \rangle h(t) dt \\
&= \int_G \sum_{i=1}^{\infty} \langle T V(t) x_i | V(t) y_i \rangle h(t) dt \\
&= \int_G \sum_{i=1}^{\infty} f_i'(t) h(t) dt \\
&= \langle \sum_{i=1}^{\infty} f_i', h \rangle_{\mathscr{C}^b(G), L^1(G)}.
\end{aligned}
$$

Hence, $\tilde{\pi}'_{T,T_*} = \sum\limits_{i=1}^{\infty} f_i'$ in $L^\infty(G)$.

(ii) Continuing to work with f_i and f_i' as defined above, we show that $f_i \in LUC(G)$ and $f_i' \in RUC(G)$, and thus it will follow that $\tilde{\pi}_{T,T_*} \in LUC(G)$ and $\tilde{\pi}'_{T,T_*} \in RUC(G)$.

Let $x, y \in H$ and define

$$f(t) = \langle TV(t)^* x | V(t)^* y \rangle, \; f'(t) = \langle TV(t)x | V(t)y \rangle.$$

As shown above, $f, f' \in \mathscr{C}^b(G)$. Let (s_α) be a net in G such that $s_\alpha \to e$. We need to show that $\|_{s_\alpha} f - f\|_\infty \to 0$ and $\|f'_{s_\alpha} - f'\|_\infty \to 0$. In fact, we have

$$
\begin{aligned}
\|_{s_\alpha} f - f\|_\infty &= \sup_{t \in G} |\langle TV(s_\alpha t)^* x | V(s_\alpha t)^* y \rangle - \langle TV(t)^* x | V(t)^* y \rangle| \\
&= \sup_{t \in G} |\langle TV(s_\alpha t)^* x | V(s_\alpha t)^* y \rangle - \langle TV(t)^* x | V(s_\alpha t)^* y \rangle \\
&+ \langle TV(t)^* x | V(s_\alpha t)^* y \rangle - \langle TV(t)^* x | V(t)^* y \rangle| \\
&\leqslant \sup_{t \in G} \|T(V(s_\alpha t)^* - V(t)^*)x\| \|V(s_\alpha t)^* y\| \\
&+ \sup_{t \in G} \|TV(t)^* x\| \|V(s_\alpha t)^* y - V(t)^* y\| \\
&\leqslant \|T\| \sup_{t \in G} \|V(t)^* (V(s_\alpha)^* x - x)\| \|y\| \\
&+ \|T\| \sup_{t \in G} \|V(t)^* x\| \|V(t)^* (V(s_\alpha)^* y - y)\|
\end{aligned}
$$

$$= \|T\|\|V(s_\alpha)^*x - x\|\|y\| + \|T\|\|x\|\|V(s_\alpha)^*y - y\|$$
$$= \|T\|\|y\|\|V(s_\alpha)x - x\| + \|T\|\|x\|\|V(s_\alpha)y - y\|.$$

And we also have

$$\begin{aligned}
\|f'_{s_\alpha} - f'\|_\infty &= \sup_{t\in G} |\langle TV(ts_\alpha)x|V(ts_\alpha)y\rangle - \langle TV(t)x|V(t)y\rangle| \\
&= \sup_{t\in G} |\langle TV(ts_\alpha)x|V(ts_\alpha)y\rangle - \langle TV(t)x|V(ts_\alpha)y\rangle \\
&+ \langle TV(t)x|V(ts_\alpha)y\rangle - \langle TV(t)x|V(t)y\rangle| \\
&\leqslant \sup_{t\in G} \|T(V(ts_\alpha) - V(t))x\|\|V(ts_\alpha)y\| \\
&+ \sup_{t\in G} \|TV(t)x\|\|V(ts_\alpha)y - V(t)y\| \\
&\leqslant \|T\| \sup_{t\in G} \|V(t)(V(s_\alpha)x - x)\|\|y\| \\
&+ \|T\| \sup_{t\in G} \|V(t)x\|\|V(t)(V(s_\alpha)y - y)\| \\
&= \|T\|\|V(s_\alpha)x - x\|\|y\| + \|T\|\|x\|\|V(s_\alpha)y - y\|.
\end{aligned}$$

Since V is continuous in the SOT, it follows that

$$\|V(s_\alpha)x - x\| \longrightarrow 0 \text{ and } \|V(s_\alpha)y - y\| \longrightarrow 0,$$

and hence $\|_{s_\alpha}f - f\|_\infty \to 0$ and $\|f'_{s_\alpha} - f'\|_\infty \to 0$. □

Next, we use the left regular representation of G on $L^2(G)$ (see Example 1.34) to prove the converse of Theorem 5.7. Since the left regular representation of G on $L^2(G)$ is unitary, it is associated with the left regular representation

$$\widetilde{V} : L^1(G) \longrightarrow \mathscr{L}(L^2(G)),$$

of $L^1(G)$ on $L^2(G)$.

THEOREM 5.9. *Let G be a locally compact group. Then every function in $[RUC(G)]$ $LUC(G)$ is a coordinate function of the conjugate [anti-]representation of $L^1(G)$ on $\mathscr{L}(L^2(G))$ associated with the left regular representation of G.*

PROOF. First let $f \in LUC(G)$. Using the identity $LUC(G) = L^1(G) * L^\infty(G)$ (Hewitt–Ross [**28**, (32.45)]), we can set $f = h * g$, where $h \in L^1(G)$ and $g \in L^\infty(G)$.

We define $\phi, \psi \in L^2(G)$ by $\psi(x) = |h(x)|^{1/2}$ ($x \in G$), and

$$\phi(x) = \begin{cases} 0 & h(x) = 0 \\ h(x)/|h(x)|^{1/2} & h(x) \neq 0 \end{cases}$$

Then $h = \phi\psi$ and $\phi, \psi \in L^2(G)$, since

$$\int_G |\psi(x)|^2 dx = \int_G |h(x)| dx < \infty \text{ and } \int_G |\phi(x)|^2 dx \leqslant \int_G |h(x)| dx < \infty.$$

Let

$$T_* = \phi \otimes \psi \in L^2(G) \widehat{\otimes} L^2(G) = \mathscr{L}(L^2(G))_*,$$

and let $M_{\check{g}} \in \mathscr{L}(L^2(G))$ be the multiplication operator by $\check{g}$. Then $M_{\check{g}} \in \mathscr{L}(L^2(G))$ since it is clearly linear, and for $k \in L^2(G)$,

$$\|M_{\check{g}} k\|_2^2 = \int_G |g(x^{-1})k(x)|^2 dx \leqslant \int_G \|g\|_\infty^2 |k(x)|^2 dx = \|g\|_\infty^2 \|k\|_2 < \infty.$$

Let V be the left regular representation of G on $L^2(G)$, and let

$$\tilde{\pi} : L^1(G) \longrightarrow \mathscr{L}\left(\mathscr{L}(L^2(G))\right), \qquad \tilde{\pi}(k)T = \int_G V(t)TV(t)^* k(t) dt,$$

$k \in L^1(G)$, $T \in \mathscr{L}(L^2(G))$, be the conjugate representation associated to V. We claim that f is a coordinate function of $\tilde{\pi}$ corresponding to $T = M_{\check{g}} \in \mathscr{L}(L^2(G))$ and $T_* = \phi \otimes \psi \in \mathscr{L}(L^2(G))_*$; in other words,

$$\tilde{\pi}_{M_{\check{g}}, \phi \otimes \psi} = f. \tag{30}$$

In fact, for every $k \in L^1(G)$, we have

$$\langle \tilde{\pi}_{M_{\check{g}}, \phi\otimes\psi}, k \rangle = \langle \tilde{\pi}(k) M_{\check{g}}, \phi \otimes \psi \rangle = \langle \tilde{\pi}(k) M_{\check{g}} \phi | \psi \rangle = \int_G \langle V(t) M_{\check{g}} V(t)^* \phi | \psi \rangle k(t) dt.$$

Therefore, to prove (30), it suffices to show that for all $t \in G$,

$$\langle V(t) M_{\check{g}} V(t)^* \phi | \psi \rangle = f(t). \tag{31}$$

Note that $\bar{\psi} = \psi$. Given $t \in G$, we can write

$$\begin{aligned} \langle V(t) M_{\check{g}} V(t)^* \phi | \psi \rangle &= \int_G \left(V(t) M_{\check{g}} V(t)^* \phi\right)(s) \overline{\psi(s)} ds \\ &= \int_G \left(M_{\check{g}} V(t)^* \phi\right)(t^{-1}s) \psi(s) ds \end{aligned}$$

$$
\begin{aligned}
&= \int_G \check{g}(t^{-1}s)\left(V(t)^*\phi\right)(t^{-1}s)\psi(s)ds \\
(V(t)^* = V(t)^{-1} = V(t^{-1})) &= \int_G g(s^{-1}t)\phi(s)\psi(s)ds \\
&= \int_G g(s^{-1}t)h(s)ds \\
&= (h * g)(t) = f(t),
\end{aligned}
$$

which proves (31) and hence (30).

Next, we let $f = g * \check{h} \in RUC(G) = L^\infty(G) * L^1(G)^\vee$ with $h \in L^1(G)$ and $g \in L^\infty(G)$. We chose ϕ, ψ and T_* as before, and let $M_g \in \mathscr{L}(L^2(G))$ be the multiplication operator by g. Let

$$\tilde{\pi}' : L^1(G) \longrightarrow \mathscr{L}\left(\mathscr{L}(L^2(G))\right), \qquad \tilde{\pi}'(k)T = \int_G V(t)^*TV(t)k(t)dt,$$

$k \in L^1(G), T \in \mathscr{L}(L^2(G))$, be the conjugate anti-representation associated to V, where V is again the left regular representation of G on $L^2(G)$. We claim that f is a coordinate function of $\tilde{\pi}'$ corresponding to $T = M_g \in \mathscr{L}(L^2(G))$ and $T_* = \phi \otimes \psi \in \mathscr{L}(L^2(G))_*$; in other words,

$$\tilde{\pi}'_{M_g,\phi\otimes\psi} = f. \tag{32}$$

In fact, for every $k \in L^1(G)$, we have

$$\langle \tilde{\pi}'_{M_g,\phi\otimes\psi}, k\rangle = \langle \tilde{\pi}'(k)M_g, \phi \otimes \psi\rangle = \langle \tilde{\pi}'(k)M_g\phi|\psi\rangle = \int_G \langle V(t)^*M_gV(t)\phi|\psi\rangle k(t)dt.$$

Therefore, to prove (32), it suffices to show that for all $t \in G$,

$$\langle V(t)^*M_gV(t)\phi|\psi\rangle = f(t). \tag{33}$$

Given $t \in G$, we can write

$$
\begin{aligned}
\langle V(t)^*M_gV(t)\phi|\psi\rangle &= \int_G \left(V(t)^*M_gV(t)\phi\right)(s)\overline{\psi(s)}ds \\
&= \int_G \left(M_gV(t)\phi\right)(ts)\psi(s)ds \\
&= \int_G g(ts)\left(V(t)\phi\right)(ts)\psi(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&= \int_G g(ts)\phi(s)\psi(s)ds \\
&= \int_G g(ts)h(s)ds \\
&= \int_G g(ts)\check{h}(s^{-1})ds \\
(s \to t^{-1}s) &= \int_G g(s)\check{h}(s^{-1}t)ds \\
&= (g * \check{h})(t) = f(t),
\end{aligned}
$$

which proves (33) and hence (32). □

5.3. Results for arbitrary Banach algebras

Next we turn our attention to the space of left uniformly continuous functionals $luc(A)$ on a Banach algebra A. We shall present a class of representations of A that are subordinate to $luc(A)$, and moreover, we shall show that every element in $luc(A)$ is a coordinate function of one such representation.

THEOREM 5.10. *Let A be a Banach algebra with a bounded right approximate identity. Let $\theta : A \longrightarrow \mathscr{L}(A)$ be a continuous, non-degenerate anti-representation. Then each of the two maps*

$$
\begin{aligned}
&\text{(i) } \pi^\theta : A \longrightarrow \mathscr{L}(\mathscr{L}(A, A^*)), \qquad \pi^\theta(a)T = T \circ \theta(a), \\
&\text{(ii) } \check{\pi}^\theta : A \longrightarrow \mathscr{L}(\mathscr{L}(A, A^*)), \qquad \check{\pi}^\theta(a)T = \theta(a)^* \circ T,
\end{aligned}
$$

where $a \in A$ and $T \in \mathscr{L}(A, A^*)$, is a continuous representation of A subordinate to $luc(A)$.

PROOF. By the definition, $\pi^\theta(a)T, \check{\pi}^\theta(a)T \in \mathscr{L}(A, A^*)$ for all $a \in A$ and $T \in \mathscr{L}(A, A^*)$, since $\theta(a) \in \mathscr{L}(A)$ and $\theta(a)^* \in \mathscr{L}(A^*)$.

Now we show that $\pi^\theta(a), \check{\pi}^\theta(a) \in \mathscr{L}(\mathscr{L}(A, A^*))$ for all $a \in A$. Let $a \in A$, $T, S \in \mathscr{L}(A, A^*)$, and $\alpha, \beta \in \mathbb{C}$. Then

$$\pi^\theta(a)(\alpha T + \beta S) = (\alpha T + \beta S) \circ \theta(a) = \alpha T \circ \theta(a) + \beta S \circ \theta(a) = \alpha \pi^\theta(a)T + \beta \pi^\theta(a)S,$$

and

$$\check{\pi}^\theta(a)(\alpha T + \beta S) = \theta(a)^* \circ (\alpha T + \beta S) = \alpha \theta(a)^* \circ T + \beta \theta(a)^* \circ S = \alpha \check{\pi}^\theta(a)T + \beta \check{\pi}^\theta(a)S.$$

Furthermore,

$$\|\pi^{\theta}(a)T\| = \|T \circ \theta(a)\| \leqslant \|\theta\|\|a\|\|T\|,$$

$$\|\check{\pi}^{\theta}(a)T\| = \|\theta(a)^* \circ T\| \leqslant \|\theta\|\|a\|\|T\|.$$

Clearly, π^{θ} and $\check{\pi}^{\theta}$ are linear. It follows that $\|\pi^{\theta}\| \leqslant \|\theta\|$ and $\|\check{\pi}^{\theta}\| \leqslant \|\theta\|$.

Next we check that π^{θ} and $\check{\pi}^{\theta}$ are homomorphisms. For all $a, b \in A$ and $T \in \mathscr{L}(A, A^*)$,

$$\pi^{\theta}(ab)T = T \circ \theta(ab) = T \circ (\theta(b) \circ \theta(a)) = (T \circ \theta(b)) \circ \theta(a) = \pi^{\theta}(a)(\pi^{\theta}(b)T),$$

and

$$\check{\pi}^{\theta}(ab)T = \theta(ab)^* \circ T = (\theta(a)^* \circ \theta(b)^*) \circ T = \theta(a)^* \circ (\theta(b)^* \circ T) = \check{\pi}^{\theta}(a)(\check{\pi}^{\theta}(b)T).$$

Therefore, π^{θ} and $\check{\pi}^{\theta}$ are continuous representations of A on $\mathscr{L}(A, A^*)$. It remains to show that they are subordinate to $luc(A)$. In other words, it remains to show that for all $T \in \mathscr{L}(A, A^*)$ and $T_* \in \mathscr{L}(A, A^*)_*$,

$$\pi^{\theta}_{T,T_*}, \check{\pi}^{\theta}_{T,T_*} \in luc(A) = A^* \cdot A.$$

We observe that A is a right Banach A-module with the module action $b \bullet a = \theta(a)b$. In fact, for all $a, b, c \in A$, we have

$$(a \bullet b) \bullet c = \theta(c)\theta(b)a = \theta(bc)a = a \bullet (bc),$$

and

$$\|a \bullet b\| = \|\theta(b)a\| \leqslant \|\theta\|\|b\|\|a\|.$$

Moreover, since θ is non-degenerate, the linear span of the set $\{\theta(a)b : a, b \in A\} = A \bullet A$ is norm dense in A. Therefore, by Cohen–Hewitt's factorization theorem (Hewitt–Ross [**28**, Theorem 32.23]), for all $a \in A$, we can find elements $a', a'' \in A$ such that $a = a'' \bullet a' = \theta(a')a''$.

Let $T_* = a_1 \otimes a_2 \in A\widehat{\otimes}A = \mathscr{L}(A, A^*)_*$ and let $T \in \mathscr{L}(A, A^*)$. Write $a_1 = \theta(a_1')a_1''$ and $a_2 = \theta(a_2')a_2''$. Then for all $c \in A$, we have

$$\begin{aligned}\pi^{\theta}_{T,T_*}(c) &= \langle \pi^{\theta}(c)T, a_1 \otimes a_2 \rangle_{\mathscr{L}(A,A^*),\mathscr{L}(A,A^*)_*} \\ &= \langle (T \circ \theta(c))a_1, a_2 \rangle_{A^*,A}\end{aligned}$$

$$
\begin{aligned}
&= \langle \theta(c)a_1, T^* a_2\rangle_{A,A^*} \\
&= \langle \theta(c)(\theta(a_1')a_1''), T^* a_2\rangle_{A,A^*} \\
&= \langle \theta(a_1'c)a_1'', T^* a_2\rangle_{A,A^*} \\
(A\widehat{\otimes}A^* \subset \mathscr{L}(A)^* \text{ by Lemma } A.21) \quad &= \langle \theta(a_1'c), a_1'' \otimes T^* a_2\rangle_{\mathscr{L}(A),\mathscr{L}(A)^*} \\
&= \langle c, \theta^*(a_1'' \otimes T^* a_2)\cdot a_1'\rangle_{A,A^*}.
\end{aligned}
$$

Hence,

$$\pi^\theta_{T,T_*} = \theta^*(a_1'' \otimes T^* a_2)\cdot a_1' \in A^*\cdot A = luc(A).$$

Similarly,

$$
\begin{aligned}
\check{\pi}^\theta_{T,T_*}(c) &= \langle \check{\pi}^\theta(c)T, a_1 \otimes a_2\rangle_{\mathscr{L}(A,A^*),\mathscr{L}(A,A^*)_*} \\
&= \langle (\theta(c)^* \circ T)a_1, a_2\rangle_{A^*,A} \\
&= \langle Ta_1, \theta(c)a_2\rangle_{A^*,A} \\
&= \langle Ta_1, \theta(c)(\theta(a_2')a_2'')\rangle_{A^*,A} \\
&= \langle Ta_1, \theta(a_2'c)a_2''\rangle_{A^*,A} \\
&= \langle a_2'' \otimes Ta_1, \theta(a_2'c)\rangle_{\mathscr{L}(A)^*,\mathscr{L}(A)} \\
&= \langle \theta^*(a_2'' \otimes Ta_1), a_2'c\rangle_{A^*,A} \\
&= \langle \theta^*(a_2'' \otimes Ta_1)\cdot a_2', c\rangle_{A^*,A}.
\end{aligned}
$$

Hence, $\check{\pi}^\theta_{T,T_*} = \theta^*(a_2'' \otimes Ta_1)\cdot a_2' \in A^*\cdot A = luc(A)$.

To complete the proof for general $T_* \in A\widehat{\otimes}A$, we let $\eta = \sum_{i=1}^{\infty} a_i \otimes b_i \in A\widehat{\otimes}A$ and we let $\eta_n = \sum_{i=1}^n a_i \otimes b_i$. Then $\pi^\theta_{T,\eta_n}, \check{\pi}^\theta_{T,\eta_n} \in luc(A)$ and $\|\eta - \eta_n\| \longrightarrow 0$ as $n \to \infty$. For all $T \in \mathscr{L}(A, A^*)$, we have

$$
\begin{aligned}
\|\pi^\theta_{T,\eta} - \pi^\theta_{T,\eta_n}\| &= \sup_{\substack{c\in A\\ \|c\|\leqslant 1}} |\pi^\theta_{T,\eta}(c) - \pi^\theta_{T,\eta_n}(c)| = \sup_{\substack{c\in A\\ \|c\|\leqslant 1}} |\langle \pi^\theta(c)T, \eta-\eta_n\rangle| \\
&\leqslant \|\eta-\eta_n\| \sup_{\substack{c\in A\\ \|c\|\leqslant 1}} \|T\circ\theta(c)\| \leqslant \|\eta-\eta_n\|\|T\|\|\theta\| \longrightarrow 0 \text{ as } n\to\infty.
\end{aligned}
$$

So $\pi^\theta_{T,\eta} = \lim_n \pi^\theta_{T,\eta_n} \in luc(A)$.

Similarly,

$$\|\breve{\pi}^{\theta}_{T,\eta} - \breve{\pi}^{\theta}_{T,\eta_n}\| = \sup_{\substack{c\in A\\ \|c\|\leqslant 1}} |\breve{\pi}^{\theta}_{T,\eta}(c) - \breve{\pi}^{\theta}_{T,\eta_n}(c)| = \sup_{\substack{c\in A\\ \|c\|\leqslant 1}} |\langle \breve{\pi}^{\theta}(c)T, \eta - \eta_n\rangle|$$
$$\leqslant \|\eta - \eta_n\| \sup_{\substack{c\in A\\ \|c\|\leqslant 1}} \|\theta(c)^* \circ T\| \leqslant \|\eta - \eta_n\|\|\theta\|\|T\| \longrightarrow 0 \text{ as } n \to \infty.$$

So $\breve{\pi}^{\theta}_{T,\eta} = \lim_n \breve{\pi}^{\theta}_{T,\eta_n} \in luc(A)$. □

COROLLARY 5.11. *Let A be a Banach algebra with a bounded left approximate identity. Let $\theta : A \longrightarrow \mathscr{L}(A)$ be a continuous, non-degenerate representation. Then each of the two maps*

(i) $\pi^{\theta} : A \longrightarrow \mathscr{L}(\mathscr{L}(A, A^*))$, $\qquad \pi^{\theta}(a)T = T \circ \theta(a)$,

(ii) $\breve{\pi}^{\theta} : A \longrightarrow \mathscr{L}(\mathscr{L}(A, A^*))$, $\qquad \breve{\pi}^{\theta}(a)T = \theta(a)^* \circ T$,

where $a \in A$ and $T \in \mathscr{L}(A, A^*)$, is a continuous anti-representation of A subordinate to $ruc(A)$.

PROOF. By Lemma C.24 and the given assumptions above, A^{op} has a bounded right approximate identity and θ is a continuous non-degenerate anti-representation of A^{op} on $\mathscr{L}(A^{op})$. And hence, by Theorem 5.10, the maps defined in (i) and (ii) are both continuous representations of A^{op} subordinate to $luc(A^{op})$. Applying Lemma C.24 again implies that the same maps are continuous anti-representations of A subordinate to $(A^{op})^* \cdot A^{op} = A \cdot A^* = ruc(A)$. □

THEOREM 5.12. *Let A be a Banach algebra with a bounded right approximate identity, and let $\theta : A \longrightarrow \mathscr{L}(A)$ be the continuous anti-representation defined by $\theta(a) = R_a$, with R_a being the right multiplication operator by a. Then every $f \in luc(A) = A^* \cdot A$ is a coordinate function of the continuous representations π^{θ} and $\breve{\pi}^{\theta}$ of A on $\mathscr{L}(A, A^*)$, where $\pi^{\theta}(a)T = T \circ \theta(a)$ and $\breve{\pi}^{\theta}(a)T = \theta(a)^* \circ T$.*

PROOF. Let $f = \lambda \cdot a \in luc(A)$, where $\lambda \in A^*$ and $a \in A$. Using Cohen's factorization theorem (see Hewitt–Ross [**28**, (32.26)]), we can write $a = a_1 a_2$ with $a_1, a_2 \in A$. Define $L_\lambda, R_\lambda \in \mathscr{L}(A, A^*)$ as in (4) and (5) in Section 3.1, and let

$$T_* = a_2 \otimes a_1 \in A \widehat{\otimes} A = \mathscr{L}(A, A^*)_*.$$

We claim that $f = \pi^\theta_{R_{\lambda,T_*}} = \breve{\pi}^\theta_{L_{\lambda,T_*}}$. In fact, for every $c \in A$,

$$\begin{aligned}\pi^\theta_{R_{\lambda,T_*}}(c) &= \langle \pi^\theta(c)R_\lambda, a_2 \otimes a_1\rangle = \langle R_\lambda \circ \theta(c), a_2 \otimes a_1\rangle = \langle R_\lambda(a_2c), a_1\rangle \\ &= \langle (a_2c)\cdot\lambda, a_1\rangle = \langle \lambda, a_1(a_2c)\rangle = \langle \lambda \cdot a, c\rangle = f(c).\end{aligned}$$

Moreover, since for all $x \in A$ and $\omega \in A^*$, we have

$$\langle R_a^*\omega, x\rangle = \langle \omega, R_a(x)\rangle = \langle \omega, xa\rangle = \langle a\cdot\omega, x\rangle,$$

$R_a^* \in \mathscr{L}(A^*)$ is given by $R_a^*(\omega) = a\cdot\omega$ $(\omega \in A^*)$. Hence we can write

$$\begin{aligned}\breve{\pi}^\theta_{L_{\lambda,T_*}}(c) &= \langle \breve{\pi}^\theta(c)L_\lambda, a_2 \otimes a_1\rangle = \langle \theta(c)^* \circ L_\lambda, a_2 \otimes a_1\rangle = \langle R_c^* \circ L_\lambda, a_2 \otimes a_1\rangle \\ &= \langle c\cdot(\lambda\cdot a_2), a_1\rangle = \langle c\cdot\lambda, a_2a_1\rangle = \langle \lambda, ac\rangle = \langle \lambda\cdot a, c\rangle = f(c).\end{aligned}$$

□

Corollary 5.13. *Let A be a Banach algebra with a bounded left approximate identity, and let $\theta : A \longrightarrow \mathscr{L}(A)$ be the continuous representation defined by $\theta(a) = L_a$, with L_a being the left multiplication operator by a. Then every $f \in ruc(A) = A\cdot A^*$ is a coordinate function of the continuous anti-representations π^θ and $\breve{\pi}^\theta$ of A on $\mathscr{L}(A, A^*)$, where $\pi^\theta(a)T = T\circ\theta(a)$ and $\breve{\pi}^\theta(a)T = \theta(a)^*\circ T$.*

Proof. By Lemma C.24 and the given assumptions above, A^{op} has a bounded right approximate identity and θ is the continuous anti-representation of A^{op} on $\mathscr{L}(A^{op})$ defined by

$$\theta(a) = R_a \qquad (a \in A^{op}).$$

Hence, by Theorem 5.12, every $f \in luc(A^{op}) = ruc(A)$ is a coordinate function of the continuous anti-representations π^θ and $\breve{\pi}^\theta$ of A^{op} on $\mathscr{L}(A^{op}, (A^{op})^*)$, where $\pi^\theta(a)T = T\circ\theta(a)$ and $\breve{\pi}^\theta(a)T = \theta(a)^*\circ T$. These maps, with the same definitions, are continuous representations of A on $\mathscr{L}(A, A^*)$ by Lemma C.24. □

Conclusion and Future Work

This research has highlighted the significance of topological introversion as a property of subspaces of the dual space of a Banach algebra A. In Chapter 2, we proved the existence of a natural bijection between continuous representations of A on Y subordinate to X, and normal representations of X^* on Y whenever X is topologically left (right) introverted in A^* and Y is reflexive. This result can be regarded as a natural extension of the well known correspondence between the representations of a C^*-algebra and the representations of its enveloping von Neumann algebra.

We then moved on to study specific examples of topologically introverted subspaces of A^* and examine representations of A which are subordinate to such spaces. We devoted special attention to [weakly] almost periodic functionals, which have been studied by several authors in recent decades. In Chapter 3, we defined these spaces and studied some of their properties.

In Chapter 4, we extended the well-known result on positive linear functionals being coordinate functions of involutive representations on Hilbert spaces to weakly almost periodic functionals. We showed that if A has a bounded approximate identity, then every weakly almost periodic functional of A is a coordinate function of a representation of A, on some reflexive Banach space Y, subordinate to $wap(A)$.

Finally, in Chapter 5 we studied the topologically introverted subspaces $luc(A)$ and $ruc(A)$ of A^*. We showed that a function f on a locally compact group G is left uniformly continuous if and only if it is the coordinate function of the conjugate representation of $L^1(G)$, associated to some unitary representation of G. We generalized the latter result to an arbitrary Banach algebra with bounded right approximate identity. We proved that the functionals in $luc(A)$ are all coordinate functions of some norm continuous representation of A on a dual Banach space Y.

There are many interesting questions which arise from this research and offer directions for future work. We mention a few of them below.

(1) The dual B^* of an arbitrary Banach A-bimodule B can be made into a Banach A-bimodule via the module actions $\odot$, where

$$\langle \varphi \odot a, b\rangle = \langle \varphi, a \cdot b\rangle, \quad \langle a \odot \varphi, b\rangle = \langle \varphi, b \cdot a\rangle \qquad (a \in A,\ b \in B,\ \varphi \in B^*). \quad (34)$$

In particular, A^* is canonically a Banach A-bimodule. Our definition of [weakly] almost periodic functionals uses this structure. It would be interesting to generalize our results to spaces of [weakly] almost periodic functionals defined on an arbitrary Banach A-module B, i.e., [weakly] almost periodic functionals that belong to B^* (see Kaijser [**32**] and Runde [**43**]).

(2) Suppose the Banach space X has a Schauder basis, then every compact operator $T \in \mathscr{L}(X)$ is the limit of finite rank operators (see Conway [**4**, Page 175]). Therefore, as a means of better understanding almost periodic functionals, it would be interesting to study functionals $\lambda \in A^*$ such that L_λ (defined in Section 3.1) is a finite rank operator.

(3) As noted above, the converse of Theorem 3.19 is not known. In particular, we don't have a Banach algebra analogue of the well know result for groups which states that a function f on G is in $AP(G)$ if and only if f is the uniform limit of coordinate functions of unitary representations $V_k : G \longrightarrow \mathscr{L}(H_k)$ of G on some finite dimensional Hilbert spaces H_k. It would be interesting to know if this result holds for particular classes of Banach algebras (such as, for example, C^*-algebras).

Appendix A: Tensor Products

Following are some of the well-known results regarding tensor products which we require in this thesis. For the definition of tensor products and a more detailed treatment of this subject, the reader may refer to Ryan [**46**].

We start with the universal property of tensor products.

THEOREM A.14 (Universal Property of Tensor Products). *Let E and F be two vector spaces, and*

$$\iota: \ E \times F \longrightarrow E \otimes F, \qquad (x, y) \mapsto x \otimes y$$

be the canonical bilinear map.

(a) *Given any vector space X and any bilinear map $\widetilde{\varphi}: E \times F \longrightarrow X$, there exists a unique linear map $\varphi : E \otimes F \longrightarrow X$ such that $\widetilde{\varphi} = \varphi \circ \iota$.*

(b) *The map φ is surjective if and only if the span of the image of $\widetilde{\varphi}$ is the entire space X.*

(c) *The map φ is injective if and only if $\widetilde{\varphi}$ satisfies the following "disjointness property": (DP) Suppose that $\{x_1, \ldots, x_n\} \subset E$, $\{y_1, \ldots, y_n\} \subset F$, and suppose that $\sum_{i=1}^{n} \widetilde{\varphi}(x_i, y_i) = 0$. Then if $x_1, \ldots, x_n$ are linearly independent, $y_1 = y_2 = \cdots = y_n = 0$; and if $y_1, \ldots, y_n$ are linearly independent, $x_1 = x_2 = \cdots = x_n = 0$.*

PROOF. (a) Let $(x_\alpha)_{\alpha \in I}$ and $(y_\beta)_{\beta \in J}$ be bases for E and F, respectively. Then $(x_\alpha \otimes y_\beta)_{(\alpha,\beta) \in I \times J}$ form a basis for $E \otimes F$ by Ryan [**46**, Proposition 1.1]. We define

$$\varphi(x_\alpha \otimes y_\beta) := \widetilde{\varphi}(x_\alpha, y_\beta) \quad \text{for } \alpha \in I \text{ and } \beta \in J.$$

We extend φ to the entire space $E \otimes F$ by linearity; clearly φ with such properties is uniquely determined.

(b) Since $\widetilde{\varphi}(x_\alpha, y_\beta) = \varphi \circ \iota(x_\alpha, y_\beta) = \varphi(x_\alpha \otimes y_\beta)$, and since $\widetilde{\varphi}$ is bilinear and φ is linear, we can write

$$\begin{aligned}\operatorname{Span}\{\operatorname{Im}\widetilde{\varphi}\} &= \operatorname{Span}\{\widetilde{\varphi}(x_\alpha, y_\beta) : \quad \alpha \in I,\ \beta \in J\}\\ &= \operatorname{Span}\{\varphi(x_\alpha \otimes y_\beta) : \quad \alpha \in I,\ \beta \in J\}\\ &= \operatorname{Im}\varphi.\end{aligned}$$

Here, Im denotes the image. It follows that φ is surjective if and only if the span of the image of $\widetilde{\varphi}$ is the entire space X.

(c) First suppose that φ is injective. If $x_1, \ldots, x_n$ are linearly independent and

$$\sum_{i=1}^{n} \widetilde{\varphi}(x_i, y_i) = 0, \text{ then } \sum_{i=1}^{n} \varphi(x_i \otimes y_i) = 0 \text{ which implies that } \varphi\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = 0.$$

Since φ is injective, we have $\sum_{i=1}^{n} x_i \otimes y_i = 0$. It follows that $y_1 = y_2 = \ldots = y_n = 0$.

A similar argument shows that if $y_1, \ldots, y_n$ were linearly independent, then we must have $x_1 = \ldots = x_n = 0$. Thus $\widetilde{\varphi}$ satisfies (DP).

Conversely, suppose that $\widetilde{\varphi}$ satisfies (DP) and let $\varphi(u) = 0$ for some

$$u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F,$$

where $x_1, \ldots, x_n$ are linearly independent. Then

$$\sum_{i=1}^{n} \varphi(x_i \otimes y_i) = 0, \text{ which implies that } \sum_{i=1}^{n} \widetilde{\varphi}(x_i, y_i) = 0.$$

It follows by the (DP) that $y_1 = \ldots = y_n = 0$, and hence $u = 0$, since $x_i \otimes y_i = 0$ for every i. Thus φ is injective. □

Corollary A.15. *Let E, F and X be vector spaces. If $\widetilde{\varphi} : E \times F \longrightarrow X$ is a bilinear map satisfying (DP), and the image of $\widetilde{\varphi}$ spans X, then*

$$\varphi : E \otimes F \longrightarrow X, \qquad x \otimes y \mapsto \widetilde{\varphi}(x, y)$$

is an isomorphism between $E \otimes F$ and X.

Next we will define the projective tensor product.

Definition A.16 (The greatest cross norm). Given $u \in E \otimes F$, we define the *projective tensor norm* of u as follows:

$$\|u\| = \inf\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i\}.$$

The *projective tensor product* $E\widehat{\otimes}F$ is the completion of $E \otimes F$ with respect to the norm $\|\cdot\|$.

Theorem A.17. *(1) $\|\cdot\|$ is a norm on $E \otimes F$.*

(2) $\|x \otimes y\| = \|x\|\|y\|$ for all $x \in X$ and $y \in Y$.

Remark A.18. Part (2) of Theorem A.17 is refered to as the 'cross' property of $\|\cdot\|$. It follows that if $\|\cdot\|_\beta$ is another cross norm on $E \otimes F$, then $\|u\|_\beta \leqslant \sum_{i=1}^{n} \|x_i \otimes y_i\|_\beta = \sum_{i=1}^{n} \|x_i\|\|y_i\|$ so $\|u\|_\beta \leqslant \|u\|$, proving that $\|\cdot\|$ is the greatest cross norm on $E \otimes F$.

For the proof, see Ryan [**46**, Proposition 2.1].

The power of the tensor product construction is that it can be used to linearize bilinear maps. We presented this universal property in Theorem A.14 above for algebraic tensor products. In fact, this result extends to the projective tensor product as shown below.

Theorem A.19. *Let X, Y and Z be Banach spaces and $\widetilde{T} : X \times Y \longrightarrow Z$ be a bounded bilinear mapping. Then there exists a unique bounded linear map $T : X\widehat{\otimes}Y \longrightarrow Z$ satisfying $\widetilde{T} = T \circ \iota$. More specifically, $T(x \otimes y) = \widetilde{T}(x, y)$ for all $x \in X$ and $y \in Y$. The correspondence $\widetilde{T} \longleftrightarrow T$ is an isometric isomorphism between the Banach space $\mathscr{B}(X \times Y, Z)$ of all bounded bilinear maps from $X \times Y$ to Z and $\mathscr{L}(X\widehat{\otimes}Y, Z)$.*

Proof. Theorem A.14 immediately implies that there exists a unique linear map $T : X \otimes Y \longrightarrow Z$ satisfying $\widetilde{T} = T \circ \iota$. We first show that T is bounded under the projective norm on $X \otimes Y$. For $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, we have

$$\|T(u)\| = \|\sum_{i=1}^{n} \widetilde{T}(x_i, y_i)\| \leqslant \|\widetilde{T}\| \sum_{i=1}^{n} \|x_i\|\|y_i\|.$$

Since this holds for every representation of u, it follows that $\|T(u)\| \leqslant \|\widetilde{T}\|\|u\|$. Therefore, T is bounded and satisfies $\|T\| \leqslant \|\widetilde{T}\|$. On the other hand, $\|\widetilde{T}(x,y)\| = \|T(x \otimes y)\| \leqslant \|T\|\|x\|\|y\|$ implies that $\|\widetilde{T}\| \leqslant \|T\|$. Therefore, $\|T\| = \|\widetilde{T}\|$. Now, let $X \otimes Y$ be equipped with the projective tensor norm and let $X\widehat{\otimes}Y$ denote the completion of $X\otimes Y$ with respect to this norm. Then the operator $T : X\otimes Y \longrightarrow Z$ has a unique extension to an operator $T : X\widehat{\otimes}Y \longrightarrow Z$ with the same norm. The mapping $\widetilde{T} \mapsto T$ is clearly a linear isometry and it remains to show that this mapping is surjective. Let $S \in \mathscr{L}(X\widehat{\otimes}Y, Z)$. The bounded bilinear mapping $\widetilde{T} \in \mathscr{B}(X \times Y, Z)$, defined by $\widetilde{T}(x,y) = S(x \otimes y)$, satisfies $\widetilde{T} = S \circ \iota$. $\square$

With the canonical identification $\mathscr{B}(X \times Y, Z) = \mathscr{L}(X\widehat{\otimes}Y, Z)$ established above, we can let Z be the scalar field to get $\mathscr{B}(X \times Y, \mathbb{C}) = (X\widehat{\otimes}Y)^*$, where the action of a bounded bilinear form B as a bounded linear functional on $X\widehat{\otimes}Y$ is given by

$$\langle B, \sum_{i=1}^{\infty} x_i \otimes y_i \rangle = \sum_{i=1}^{\infty} B(x_i, y_i),$$

which yields a new formula for the projective norm

$$\|u\| = \sup\{|\langle B, u\rangle| : B \in \mathscr{B}(X \times Y, \mathbb{C}),\ \|B\| \leqslant 1\}.$$

This new formula is sometimes more convenient for calculations than the original formula.

With each bounded bilinear form $B \in \mathscr{B}(X \times Y, \mathbb{C})$, there is an associated operator $L_B \in \mathscr{L}(X, Y^*)$, defined by $\langle L_B(x), y\rangle = B(x,y)$ for all $x \in X$ and $y \in Y$. The map $B \mapsto L_B$ is an isometric isomorphism between the spaces $\mathscr{B}(X \times Y, \mathbb{C})$ and $\mathscr{L}(X, Y^*)$. Thus we have the identification:

$$(X\widehat{\otimes}Y)^* = \mathscr{L}(X, Y^*),$$

where the action of an operator $S \in \mathscr{L}(X, Y^*)$ as a bounded linear functional on $X\widehat{\otimes}Y$ is given by

$$\langle S, \sum_{i=1}^{\infty} x_i \otimes y_i \rangle = \sum_{i=1}^{\infty} \langle Sx_i, y_i\rangle.$$

Similarly, we have another identification:

$$(X\widehat{\otimes}Y)^* = \mathscr{L}(Y, X^*).$$

REMARK A.20. An extensively used special case of the identifications above occurs when Y is a reflexive Banach space. In this case, we identify $\mathscr{L}(Y) = \mathscr{L}(Y, Y^{**})$ with $(Y \widehat{\otimes} Y^*)^*$.

LEMMA A.21. *Let X be a Banach space. Then there is an injective bounded linear map $\phi : X \widehat{\otimes} X^* \longrightarrow \mathscr{L}(X)^*$.*

PROOF. Let

$$\tau : \mathscr{L}(X) \hookrightarrow \mathscr{L}(X, X^{**}) \cong (X \widehat{\otimes} X^*)^*$$

be the canonical embedding. Let $\phi : X \widehat{\otimes} X^* \longrightarrow \mathscr{L}(X)^*$ be the map $u \mapsto \tau^*(\hat{u})$, where $\hat{u}$ is the canonical image of u in $(X \widehat{\otimes} X^*)^{**}$. Then ϕ is a bounded linear map. To prove that ϕ is injective, we may use part (c) of Theorem A.14 and show that the map $\widetilde{\phi} : X \times X^* \longrightarrow \mathscr{L}(X)^*$ given by $\widetilde{\phi} = \phi \circ \iota$ satisfies the disjointness property (DP).

Let $x_1, x_2, \ldots, x_n \in X$ and $f_1, f_2, \ldots, f_n \in X^*$ be such that $\sum\limits_{i=1}^{n} \widetilde{\phi}(x_i, f_i) = 0$. This is equivalent to the requirement that

$$0 = \langle T, \sum_{i=1}^{n} \widetilde{\phi}(x_i, f_i) \rangle = \sum_{i=1}^{n} \langle T, \widetilde{\phi}(x_i, f_i) \rangle = \sum_{i=1}^{n} \langle f_i, T x_i \rangle \text{ for all } T \in \mathscr{L}(X).$$

Suppose that $x_1, x_2, \ldots, x_n$ are linearly independent. Then by the Hahn-Banach theorem (see Conway [**4**, Corollary III.6.6]), there exists a bounded linear functional $g \in X^*$ such that $g(x_1) = 1$ and $g(x_j) = 0$ for all $j \geqslant 2$. Let $y \in X$ and define

$$T \in \mathscr{L}(X), \qquad T(x) = g(x) y.$$

Then

$$\sum_{i=1}^{n} \langle f_i, T x_i \rangle = f_1(y) = 0.$$

Since y was arbitrary, we have $f_1(y) = 0$ for all $y \in X$ and hence $f_1 = 0$. A similar argument shows that $f_j = 0$ for all $j \geqslant 2$, hence $f_1 = f_2 = \ldots = f_n = 0$.

Conversely, suppose that $f_1, f_2, \ldots, f_n$ are linearly independent. Then by Proposition 2.2, there exists some $y \in X$ such that $y \notin \ker(f_1)$ and $y \in \bigcap\limits_{i=2}^{n} \ker(f_i)$. Let $g \in X^*$ be such that $g(x_1) = 1$. Such a functional g exists by the Hahn-Banach theorem. Define

$$S \in \mathscr{L}(X), \qquad S(x) = g(x) y.$$

Then

$$0 = \sum_{i=1}^{n} \langle f_i, Sx_i \rangle = \sum_{i=1}^{n} \langle f_i, g(x_i)y \rangle = \sum_{i=1}^{n} g(x_i) \langle f_i, y \rangle = f_1(y),$$

contradicting our assumption that $f_1(y) \neq 0$. Therefore, $x_1 = 0$ and $g(x_1) = 0$ for all $g \in X^*$. Repeating the argument for all $i \geqslant 2$ implies that $x_1 = x_2 = \cdots = x_n = 0$. Therefore, $\widetilde{\phi}$ satisfies the disjointness property and hence ϕ is injective. $\square$

As a consequence of the above lemma, we can consider $x \otimes f \in X \widehat{\otimes} X^*$ (and clearly, arbitrary $u = \sum_{i=1}^{\infty} x_i \otimes f_i \in X \widehat{\otimes} X^*$) as an element of $\mathscr{L}(X)^*$ with duality action given by $\langle x \otimes f, T \rangle = \langle f, Tx \rangle$.

Appendix B: Proof of Lemmas 1.21 and 1.22

Proof of Lemma 1.21. Throughout the proof below, we have left out the verification of linearity of the canonical module actions for briefness, since it follows directly from the linear operations in the spaces involved as well as the linearity of the functionals in A^* and A^{**}.

(i) For all $\lambda \in A^*$ and $a, b, x \in A$,

$$\langle \lambda \cdot (ab), x \rangle = \langle \lambda, (ab)x \rangle = \langle \lambda, a(bx) \rangle = \langle \lambda \cdot a, bx \rangle = \langle (\lambda \cdot a) \cdot b, x \rangle.$$

So $\lambda \cdot (ab) = (\lambda \cdot a) \cdot b$. Similarly, $(ab) \cdot \lambda = a \cdot (b \cdot \lambda)$.

Moreover, we have

$$\langle (a \cdot \lambda) \cdot b, x \rangle = \langle a \cdot \lambda, bx \rangle = \langle \lambda, (bx)a \rangle = \langle \lambda, b(xa) \rangle = \langle \lambda \cdot b, xa \rangle = \langle a \cdot (\lambda \cdot b), x \rangle.$$

Therefore, $(a \cdot \lambda) \cdot b = a \cdot (\lambda \cdot b)$.

Furthermore, we have

$$\|\lambda \cdot a\| = \sup_{x \in B_A} |\langle \lambda \cdot a, x \rangle| = \sup_{x \in B_A} |\langle \lambda, ax \rangle| \leqslant \sup_{x \in B_A} \|\lambda\| \, \|a\| \, \|x\| \leqslant \|\lambda\| \, \|a\|.$$

Similarly, $\|a \cdot \lambda\| \leqslant \|a\| \|\lambda\|$. Therefore, A^* is a Banach A-bimodule.

(ii) For all $\Psi \in A^{**}$, $\lambda \in A^*$, and $a, x \in A$, we have

$$\langle \Psi \cdot (\lambda \cdot a), x \rangle = \langle \Psi, (\lambda \cdot a) \cdot x \rangle = \langle \Psi, \lambda \cdot (ax) \rangle = \langle \Psi \cdot \lambda, ax \rangle = \langle (\Psi \cdot \lambda) \cdot a, x \rangle$$

Therefore, $\Psi \cdot (\lambda \cdot a) = (\Psi \cdot \lambda) \cdot a$. Similarly, $(a \cdot \lambda) \cdot \Psi = a \cdot (\lambda \cdot \Psi)$.

(iii) The verification that $\Box$ and $\Diamond$ are algebra products is straightforward and has been left out for briefness. Here, we verify that these products satisfy the Banach algebra condition. For $\Psi, \Phi \in A^{**}$, we have

$$\|\Psi \Box \Phi\| = \sup_{\lambda \in B_{A^*}} |\langle \Psi \Box \Phi, \lambda \rangle| = \sup_{\lambda \in B_{A^*}} |\langle \Psi, \Phi \cdot \lambda \rangle|$$

$$\leqslant \sup_{\lambda \in B_{A^*}} \|\Psi\| \, \|\Phi\| \, \|\lambda\| \leqslant \|\Psi\| \, \|\Phi\|.$$

Similarly, $\|\Psi \Diamond \Phi\| \leqslant \|\Psi\| \, \|\Phi\|$.

(iv) For all $\Psi, \Phi \in A^{**}$, $\lambda \in A^*$, and $a \in A$,

$$\langle (\Psi \Box \Phi) \cdot \lambda, a \rangle = \langle \Psi \Box \Phi, \lambda \cdot a \rangle = \langle \Psi, \Phi \cdot (\lambda \cdot a) \rangle = \langle \Psi, (\Phi \cdot \lambda) \cdot a \rangle = \langle \Psi \cdot (\Phi \cdot \lambda), a \rangle.$$

So $(\Psi \Box \Phi) \cdot \lambda = \Psi \cdot (\Phi \cdot \lambda)$. Moreover,

$$\|\Psi \cdot \lambda\| = \sup_{a \in B_A} |\langle \Psi \cdot \lambda, a \rangle| = \sup_{a \in B_A} |\langle \Psi, \lambda \cdot a \rangle| \leqslant \sup_{a \in B_A} \|\Psi\| \, \|\lambda\| \, \|a\| \leqslant \|\Psi\| \, \|\lambda\|.$$

Therefore, A^* is a left Banach $(A^{**}, \Box)$-module. Similarly, A^* is a right Banach $(A^{**}, \Diamond)$-module.

(v) For each $\lambda \in A^*$, we have

$$\langle \widehat{ab}, \lambda \rangle = \langle \lambda, ab \rangle = \langle \lambda \cdot a, b \rangle = \langle \hat{b}, \lambda \cdot \hat{a} \rangle = \langle \hat{a} \Diamond \hat{b}, \lambda \rangle.$$

Therefore, $\widehat{ab} = \hat{a} \Diamond \hat{b}$. The proof of the second equality is similar.

$\square$

Proof of Lemma 1.22. (1) Let $a \in A$ and $\lambda \in A^*$. Then for all $x \in A$,

$$\langle \lambda \cdot a, x \rangle = \langle \lambda, ax \rangle = \langle x \cdot \lambda, a \rangle = \langle \hat{a}, x \cdot \lambda \rangle = \langle \lambda \cdot \hat{a}, x \rangle.$$

Similarly, $a \cdot \lambda = \hat{a} \cdot \lambda$.

(2) Let $a \in A$ and $\Psi \in A^{**}$. Then for all $\lambda \in A^*$,

$$\langle \Psi \Box \hat{a}, \lambda \rangle = \langle \Psi, \hat{a} \cdot \lambda \rangle = \langle \Psi, a \cdot \lambda \rangle = \langle \lambda \cdot \Psi, a \rangle = \langle \hat{a}, \lambda \cdot \Psi \rangle = \langle \Psi \Diamond \hat{a}, \lambda \rangle.$$

Similarly, $\hat{a} \Box \Psi = \hat{a} \Diamond \Psi$.

(3) Suppose $\{\Psi_\alpha\}$ is a net in A^{**} such that $\Psi_\alpha \xrightarrow{w^*} \Psi \in A^{**}$. Then for all λ in A^*,

$$\langle \Psi \Box \Phi, \lambda \rangle = \langle \Psi, \Phi \cdot \lambda \rangle = \lim_\alpha \langle \Psi_\alpha, \Phi \cdot \lambda \rangle = \lim_\alpha \langle \Psi_\alpha \Box \Phi, \lambda \rangle.$$

So $\Psi_\alpha \Box \Phi \xrightarrow{w^*} \Psi \Box \Phi$. Similarly, $\Phi \Diamond \Psi_\alpha \xrightarrow{w^*} \Phi \Diamond \Psi$.

(4) Suppose $\{\Psi_\alpha\}$ is a net in A^{**} such that $\Psi_\alpha \xrightarrow{w^*} \Psi \in A^{**}$. Then for all λ in A^*,

$$\begin{aligned}\langle \hat{a}\Box\Psi, \lambda\rangle &= \langle \hat{a}, \Psi\cdot\lambda\rangle = \langle \Psi\cdot\lambda, a\rangle = \langle \Psi, \lambda\cdot a\rangle \\ &= \lim_\alpha \langle \Psi_\alpha, \lambda\cdot a\rangle = \lim_\alpha \langle \Psi_\alpha\cdot\lambda, a\rangle = \lim_\alpha \langle \hat{a}, \Psi_\alpha\cdot\lambda\rangle \\ &= \lim_\alpha \langle \hat{a}\Box\Psi_\alpha, \lambda\rangle.\end{aligned}$$

Therefore, $\hat{a}\Box\Psi_\alpha \xrightarrow{w^*} \hat{a}\Box\Psi$. Similarly, $\Psi_\alpha\Diamond\hat{a} \xrightarrow{w^*} \Psi\Diamond\hat{a}$.

$\Box$

Appendix C: Opposite Algebra

Let A be a Banach algebra. Define

$$\star : A \times A \longrightarrow A, \qquad a \star b = ba \qquad (a, b \in A). \tag{35}$$

DEFINITION C.22. The *opposite algebra* of A, denoted A^{op}, is $(A, \star)$.

THEOREM C.23. *Let A be a Banach algebra. Then A^{op} is a Banach algebra.*

PROOF. First note that the normed space A^{op} is identical to A and hence is a Banach space.

Let $a, b, c \in A^{op}$ and $\alpha \in \mathbb{C}$. Then

$$a \star (b + c) = (b + c)a = ba + ca = a \star b + a \star c,$$

$$(a + b) \star c = c(a + b) = ca + cb = a \star c + b \star c,$$

$$(a \star b) \star c = c(ba) = (cb)a = a \star (b \star c),$$

$$\alpha(a \star b) = \alpha(ba) = (\alpha b)a = b(\alpha a) = a \star (\alpha b) = (\alpha a) \star b.$$

Therefore, A^{op} is an algebra. Finally, A^{op} is a Banach algebra since

$$\|a \star b\| = \|ba\| \leqslant \|b\|\|a\| = \|a\|\|b\|.$$

$\square$

LEMMA C.24. *Let A be a Banach algebra and A^{op} its opposite algebra. Then*

(i) *A is unital if and only if A^{op} is unital.*
(ii) *If A has a [bounded][two-sided][right] left approximate identity, then A^{op} has a [bounded][two-sided][left] right approximate identity.*
(iii) *If θ is any algebra homomorphism acting on A, then θ is an algebra anti-homomorphism acting on A^{op} and vice versa.*
(iv) $(A^{op})^* = A^*$.

Proof. (i) Let e be the identity element of A. Then $ae = ea = a$ for every $a \in A$. This happens if and only if $e \star a = a \star e = a$ for every $a \in A^{op}$, implying that e is the identity element of A^{op}.

(ii) Suppose $(e_\alpha)_{\alpha\in I}$ is a net in A such that $\lim_\alpha \|e_\alpha a - a\| = 0$. Then $(e_\alpha)_{\alpha\in I}$ is a net in A^{op} and $\lim_\alpha \|a \star e_\alpha - a\| = 0$. The proof of the other statements is similar.

(iii) If θ is such that $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in A$, then $\theta(a \star b) = \theta(ba) = \theta(b)\theta(a)$ for all $a, b \in A^{op}$. The converse is also true.

(iv) This is obvious since A^{op} and A are identical as normed spaces and only differ in their respective algebraic multiplication operations. □

Lemma C.25. *Let A be a Banach algebra. If X is a [left] right Banach A-module, then X is a [right] left Banach A^{op}-module. In particular, every Banach A-bimodule is a Banach A^{op}-bimodule.*

Proof. Suppose X is a left Banach A-module with module action

$$A \times X \longrightarrow X, \quad (a, x) \mapsto a \cdot x.$$

Define

$$X \times A^{op} \longrightarrow X, \quad (x, a) \mapsto x \odot a = a \cdot x.$$

Then for all $a, b \in A^{op}$ and $x, y \in X$, we have

$$x \odot (a + b) = (a + b) \cdot x = a \cdot x + b \cdot x = x \odot a + x \odot b,$$

$$(x + y) \odot a = a \cdot (x + y) = a \cdot x + a \cdot y = x \odot a + y \odot a,$$

$$x \odot (a \star b) = (ba) \cdot x = b \cdot (a \cdot x) = (x \odot a) \odot b.$$

Therefore, X is a right A^{op}-module. Moreover, for all $a \in A^{op}$ and $x \in X$,

$$\|x \odot a\| = \|a \cdot x\| \leqslant C\|a\|\|x\| = C\|x\|\|a\|.$$

Hence X is a right Banach A^{op}-module. The proofs of the remaining statements are similar. □

Corollary C.26. $A \cdot A^* = (A^{op})^* \odot A^{op}$

Proof. This follows immediately from Lemmas C.24(iv) and C.25. □

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