2017

Some applications of Erdos-Baum-Katz type theorems in high-dimensional data

Yueleng Wang
University of Windsor

Some applications of Erdös-Baum-Katz type theorems in High-Dimensional Data

by

Yueleng Wang

A Thesis
Submitted to the Faculty of Graduate Studies
through the Department of Mathematics and Statistics
in Partial Fulfillment of the Requirements for
the Degree of Master of Science at the
University of Windsor

Windsor, Ontario, Canada

Some applications of Erdös-Baum-Katz type theorems in High-Dimensional Data

by

Yueleng Wang

APPROVED BY:

A. Ngom
School of Computer Science

A. Hussein
Department of Mathematics and Statistics

T. Traynor
Department of Mathematics and Statistics

S. Nkurunziza, Advisor
Department of Mathematics and Statistics

May 12, 2017

Declaration of Co-Authorship / Previous Publication

I. Co-Authorship Declaration

I hereby certify that this master thesis incorporates the outcome of joint research undertaken in collaboration with my supervisor, Dr. Sévérien Nkurunziza. In all cases, the primary contributions and the provision of theoretical results were performed by the author and co-author.

I am aware of the University of Windsor Senate Policy on Authorship and I certify that I have properly acknowledge the contribution of other researchers to my master thesis, and have obtained written permission from each of the co-author(s) to include the above material(s) in my master thesis.

I certify that, with the above qualification, this master thesis, and the research to which it refers, is the product of my own work.

II. Declaration of Previous Publication

This major paper includes one original paper that has been previously submitted for publication in peer reviewed journals, as follows:

Thesis Chapter	Publication title / full citation	Publication Status
Chapter 3	S. Nkurunziza and Y. Wang, (2017). On convergence of the sample correlation matrices in high-dimensional data. *Bernoulli Journal*	Under review

I certify that I have obtained a written permission from the copyright owner(s) to include the above published material(s) in my master thesis. I certify that the above material describes work completed during my registration as graduate student at University of Windsor.

I certify that, to the best of my knowledge, this thesis does not infringe upon anyone's copyright nor violate any proprietary rights and that any ideas, techniques, quotations, or any other material from the work of other people included in my thesis, published or otherwise, are fully acknowledged in accordance with the standard referencing practices. Furthermore, to the extent that I have included copyrighted material that surpasses the bounds of fair dealing within the meaning of the Canada Copyright Act, I certify that I have obtained a written permission from the copyright owner(s) to include such material(s) in my thesis and have included copies of such copyright clearances to my appendix.

I declare that this is a true copy of my master thesis, including any final revisions, as approved by my thesis committee and the Graduate Studies office, and that this thesis has not been submitted for a higher degree to any other University or Institution.

Abstract

In this thesis, we consider an estimation problem concerning the matrix of correlation coefficients in context of high dimensional data settings. In particular, we generalise four main theorems in Li and Rosalsky [Li, D. and Rosalsky, A. (2006). *Some strong limit theorems for the largest entries of sample correlation matrices*, The Annals of Applied Probability, **16**, 1, 423–447]. In addition, by using Erdös-Baum-Katz type Theorems, we also simplify remarkably the proofs of some results of Li and Rosalsky (2006). Further, we generalize a theorem which is useful in deriving the existence of the p^{th} moment as well as in studying the convergence rates in Law of Large Numbers.

In loving memory of my grandpa

Bingsheng Wang

To my loving parents

Derong Wang and Yunfei Liang

Acknowledgments

I would first like to thank my thesis supervisor, Dr. Sévérien Nkurunziza for all I have learned from him and for his continuous support. The door to Professor Nkurunziza was always open whenever I ran into a trouble spot or had a question about my research or writing. He not only led me deep into the world of statistics, but also impressed me with his rigor in research as well as his sincerity and humor in communications.

I would also like to thank Dr. Abdulkadir Hussein and Dr. Tim Traynor for being my department readers and also I would like to thank Dr. Alioune Ngom who accepted to be the outside reader. They gave much useful advice which improved the quality of this thesis.

In addition, I would like to thank all faculty and staff members and graduate students in the Department of Mathematics and Statistics who helped me in many different ways during my study.

Finally, I must express my very profound gratitude to my parents for providing me with unfailing support and continuous encouragement throughout my years of study. This accomplishment would not have been possible without them. Thank you!

Contents

Chapter 1

Introduction

1.1 Motivation

The problem studied in this thesis has been inspired by the work of Li and Rosalsky (2006) who derived some properties of the largest entries of the sample matrices of correlation coefficients. Consider a p-variate population, $p \geq 2$, represented by a random vector $\boldsymbol{X} = (X_1, ..., X_p)$ with unknown mean $\boldsymbol{\mu} = (\mu_1, ..., \mu_p)$, unknown covariance matrix $\boldsymbol{\Sigma}$ and unknown correlation coefficient matrix $\boldsymbol{R}$. Let $\boldsymbol{\mathcal{M}}_{n,p} = (X_{k,i})_{1 \leq k \leq n, 1 \leq i \leq p}$ be an $n \times p$ matrix whose every row is an observed random sample from the population. That is, the rows of $\boldsymbol{\mathcal{M}}_{n,p}$ are independent copies of $\boldsymbol{X}$. In previous multivariate data analysis, the dimension of the random vector $\boldsymbol{X}$, p, is a fixed number. In this thesis, we consider the same context as in Jiang (2004) and Li and Rosalsky (2006), that p could be very large, possibly comparable with n. In fact, we consider the situation where p_n/n is bounded away from 0 and infinity, which means p_n could be greater than n.

When both n and p_n are large, Jiang (2004) considered the statistical test with the null hypothesis $H_0 : \boldsymbol{R} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the $p_n \times p_n$ identity matrix. This null hypothesis asserts that the components of $\boldsymbol{X} = (X_1, ..., X_{p_n})$ are uncorrelated. In particular, when $\boldsymbol{X}$ has a p_n-variate normal distribution, this null hypothesis asserts that these components are independent.

In the sequel, let $\boldsymbol{\mathcal{M}} = \{X_{k,i}; i \geq 1, k \geq 1\}$ be an array of i.i.d. r.v., Consider the $n \times p_n$ matrix of random variables selected from $\boldsymbol{\mathcal{M}}$, $\boldsymbol{\mathcal{M}}_{n,p_n} = (X_{k,i})_{1 \leq k \leq n, 1 \leq i \leq p_n}$. Set $\bar{X}_i^{(n)} = \sum_{k=1}^n X_{k,i}/n$ where $\boldsymbol{X}_i^{(n)}$ is the ith column of $\boldsymbol{\mathcal{M}}_{n,p_n}$. Further, define $\boldsymbol{e} = (1, ..., 1)' \in \mathbb{R}^n$ and $\| \cdot \|$ be the Euclidean norm in $\mathbb{R}^n$. Let

$$\begin{aligned}
\hat{\rho}_{i,j}^{(n)} &= \frac{\sum_{k=1}^n \left(X_{k,i} - \bar{X}_i^{(n)}\right)\left(X_{k,j} - \bar{X}_j^{(n)}\right)}{\left(\sum_{k=1}^n \left(X_{k,i} - \bar{X}_i^{(n)}\right)^2\right)^{1/2} \left(\sum_{k=1}^n \left(X_{k,j} - \bar{X}_j^{(n)}\right)^2\right)^{1/2}} \\
&= \frac{\left(\mathbf{X}_i^{(n)} - \bar{\mathbf{X}}_i^{(n)}\mathbf{e}\right)'\left(\mathbf{X}_j^{(n)} - \bar{\mathbf{X}}_j^{(n)}\mathbf{e}\right)}{\left\|\mathbf{X}_i^{(n)} - \bar{\mathbf{X}}_i^{(n)}\mathbf{e}\right\| \cdot \left\|\mathbf{X}_j^{(n)} - \bar{\mathbf{X}}_j^{(n)}\mathbf{e}\right\|},
\end{aligned}$$

which is the Pearson correlation coefficient between ith and jth columns of $\boldsymbol{\mathcal{M}}_{n,p_n}$, and let

$$L_n = \max_{1 \leq i < j \leq p_n} |\hat{\rho}_{i,j}^{(n)}|, \quad W_n = \max_{1 \leq i < j \leq p_n} \left| \sum_{k=1}^n X_{k,i} X_{k,j} \right|, n \geq 1.$$

The statistic L_n is defined in Jiang (2004) and used to derive an asympototic test for testing $H_0 : \boldsymbol{R} = \boldsymbol{I}$ vs $H_1 : \boldsymbol{R} \neq \boldsymbol{I}$. Further, Li and Rosalsky (2006) established the limit behaviors of W_n and L_n under certain conditions.

1.2 Organisation and highlight of contributions

The contribution of this thesis can be summarized as follows.

1. We first derive a theorem which generalises Theorem 3.2.1 in Chung (1974). The established results are useful in proving the existence of the p^{th} moment as well as in studying the convergence rates in law of large numbers.

2. We develop some techniques, which are useful in determining the behavior of tail probabilities. The results are used to simplify significantly the proof of Theorem 3.2 and 3.3 of Li and Rosalsky.

3. Finally, four main theorems of Li and Rosalsky are established in full generality. We heavily rely on the contributions in parts 1 and 2 in the process.

The remainder of this thesis is organised as follows. In Chapter 2, we present theorems from three papers, Erdös (1949), Katz (1963), Baum and Katz (1965), of which the results are used for deriving the main theorems of this thesis. Chapter 3 gives the main contributions of this thesis. In particular, Section 3.1 gives a theorem which is useful in deriving the the existence of the moments of a random variable. Further, this section also gives a theorem on the equivalence of certain tail events. Section 3.2 gives two simplified proofs of Theorems of Li and Rosalsky (2006) and four generalisations of theorems of Li and Rosalsky (2006). For the convenience of the reader, we present three appendices containing some technical results and proofs. Appendix A is a listing of classic probability theory results which are used in this thesis. Appendix B gives the proofs of the celebrated Erdös-Baum-Katz type theorems given in Chapter 2. Appendix C contains some proofs related to the main results.

Chapter 2

Erdös-Baum-Katz type theorems

In this chapter, we present some preliminary results which play a crucial role in deriving the main results of this thesis. In particular, we recall the celebrated Erdös-Baum-Katz type theorem. To this end, let $\{X_k, k \geq 1\}$ denote a sequence of random variables, let $\{a_n, n \geq 1\}$ be a sequence of real numbers, let $\{b_n, n \geq 1\}$ be a sequence of nondecreasing positive numbers, and let $S_n = \sum_{k=1}^{n} X_k$, i.e., the S_n are defined as partial sums. Many of the limit theorems of probability theory may then be formulated as theorems concerning the convergence of either the sequence

$$\mathbb{P}\left(|S_n - a_n|/b_n > \epsilon\right) \quad \text{or} \quad \mathbb{P}\left(\sup_{k \geq n} |S_k - a_k|/b_k > \epsilon\right),$$

$n = 1, 2, \ldots$, for some $\epsilon > 0$. The purpose of this section is to study the rates of convergence of such sequences.

Attention is restricted to sequences of independent and identically distributed (i.i.d.) random variables. In analogy with the law of large numbers, the normalizing constants b_n are chosen to be n^α, $\alpha > 1/2$, and the centering constants

$a_n = \mathbb{E}(S_n)$, provided that the expectation exists. Necessary and sufficient conditions are found, in terms of the order of magnitude of $\mathbb{P}(|X_k| > n)$, for the sequences $\mathbb{P}(|(S_n - ES_n)/n^\alpha| > \epsilon)$ and $\mathbb{P}(\sup_{k\geq n} |(S_n - ES_n)/n^\alpha| > \epsilon)$ to converge to zero at specified rates. In this chapter, we first present a distinguished result by Erdös (1949), namely Theorem 2.1, which provide a necessary and sufficient condition for the convergence of summation of such series, $\mathbb{P}(|S_n| > n)$. The result is profound and elegant. The method used in proving this theorem is widely applied in deriving other theorems given in this chapter. Second, we present some theorems established in Katz (1963) as well as some theorems due to Baum and Katz (1965). It should be noticed that the essence of these theorems rely heavily on the theorem in Erdös (1949). At the end of this chapter, we present a theorem established by Lai (1994), where b_n is considered as $\sqrt{n \log n}$.

Theorem 2.1 (P. Erdos, 1949)**.** *Let $X_n, n = 1, 2, ...,$ be a sequence of i.i.d. random variables. Put $M_n = \mathbb{P}(|S_n| > n)$. Then, a necessary and sufficient condition for the convergence of $\sum_{n=1}^{\infty} M_n$ is*

$$|\mathbb{E}(X_1)| < 1 \quad \textit{and} \quad \mathbb{E}(X_1^2) < \infty. \tag{2.1}$$

The proof of this theorem is given in Erdös (1949). For the convenience of the reader, we also give the proof with more details in Appendix B.

Theorem 2.2. *Let $X_i, i = 1, 2, ...$ be i.i.d random variables; let $t > 0$ and $r > 1$,*
(a) If $t > 1$ and $1/2 < r/t \leq 1$, then $\mathbb{E}|X_1|^t < \infty$ and $\mathbb{E}X_1 = \mu$ imply

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(|S_n - n\mu| > n^{r/t}\epsilon) < \infty, \quad \forall\epsilon > 0.$$

(b) If $t \geq 1$ *and* $r/t > 1$, *then* $\mathbb{E}|X_1|^t < \infty$ *implies*

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(|S_n| > n^{r/t}\epsilon) < \infty, \quad \forall \epsilon > 0.$$

(c) If $t < 1$ *and* $r/t > 1$, *then* $\mathbb{E}|X_1|^t < \infty$ *implies*

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(|S_n| > n^{r/t}\epsilon) < \infty, \quad \forall \epsilon > 0.$$

Remark 2.1. *For example, if* $\mathbb{E}X_1 = \mu$ *and* $\mathbb{E}|X_1|^{10} < \infty$, *then Theorem 2.2 implies that*

$$\mathbb{P}(|S_n - n\mu| > n\epsilon) = o(1/n^8) \quad \text{where } t = 10,\ r = 10,$$
$$\mathbb{P}(|S_n - n\mu| > n^{4/5}\epsilon) = o(1/n^6) \quad \text{where } t = 10,\ r = 8,$$
$$\mathbb{P}(|S_n| > n^2\epsilon) = o(1/n^{18}), \quad \text{where } t = 10,\ r = 20.$$

The proof is this theorem is given in Katz (1963). For the convenience of the reader, we also give the proof with more details in Appendix B. A partial converse to Theorem 2.2 is provided by Theorem 2.3.

Theorem 2.3. *Let* $X_i, i = 1, 2, ...$ *be i.i.d. random variables, let* $t > 0$ *and* $r \geq 2$.
(a) If $t > 1$, *and* $1/2 < r/t \leq 1$, *then*

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(|S_n - n\mu| > n^{r/t}\epsilon) < \infty, \quad \forall \epsilon > 0$$

implies that $\mathbb{E}|X_1|^t < \infty$ *and* $\mathbb{E}(X_1) = \mu$.

(b) If $t \geq 1$ *and* $r/t > 1$*, then*

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(|S_n| > n^{r/t}\epsilon) < \infty, \quad \forall \epsilon > 0$$

implies that $\mathbb{E}|X_1|^t < \infty$*.*

(c) If $t < 1$ *and* $r/t > 2$*, then*

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(|S_n| > n^{r/t}\epsilon) < \infty, \quad \forall \epsilon > 0$$

implies $\mathbb{E}|X_1|^t < \infty$*.*

The proof of this theorem is given in Katz (1963). For the convenience of the reader, we also give the proof with more details in Appendix B.

Theorem 2.4. *Let* $X_i, i = 1, 2, ...$ *be i.i.d random variables and let* $t \geq 1$*. Then* $\mathbb{E}|X_1|^t < \infty$ *and* $\mathbb{E}(X_1) = \mu$ *if and only if*

$$\sum_{n=1}^{\infty} n^{t-2}\mathbb{P}(|S_n - n\mu| > n\epsilon) < \infty, \quad \forall \epsilon > 0. \tag{2.2}$$

The proof of this theorem is given in Katz (1963). For this thesis to be self-contained, we also give the proof with more details in Appendix B.

Theorem 2.5. *Let* $X_i, i = 1, 2, ...$ *be i.i.d random variables.*

1. If $0 < t < 1$*, then the following are equivalent:*

$$(a) \quad \mathbb{E}|X_1|^t < \infty,$$

$$(b) \quad \sum_{n=1}^{\infty} n^{-1}\mathbb{P}(|S_n| > n^{1/t}\epsilon) < \infty, \quad \forall \epsilon > 0.$$

2. If $1 \le t < 2$*, then the following are equivalent:*

$$(c) \quad \mathbb{E}|X_1|^t < \infty \text{ and } \mathbb{E}X_1 = \mu,$$

$$(d) \quad \sum_{n=1}^{\infty} n^{-1}\mathbb{P}(|S_n - n\mu| > n^{1/t}\epsilon) < \infty, \quad \forall \epsilon > 0.$$

The proof of this theorem is given in Baum and Katz (1965). For the convenience of the reader, we also give the proof with more details in Appendix B.

Theorem 2.6. *Let* $\{X_i : i = 1, 2, ...\}$ *be a sequence of i.i.d. random variables.*

1. If $t > 1, r > 1$ *and* $1/2 < r/t \le 1$*, then the following are equivalent:*

$$(a) \quad \mathbb{E}|X_1|^t < \infty \text{ and } \mathbb{E}(X_1) = \mu,$$

$$(b) \quad \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}\big(|S_n - n\mu| > n^{r/t}\epsilon\big) < \infty \quad \forall \epsilon > 0.$$

2. If $t > 0$*,* $r > 1$ *and* $r/t > 1$*, then the following are equivalent:*

$$(c) \quad \mathbb{E}|X_1|^t < \infty,$$

$$(d) \quad \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(|S_n| > n^{r/t}\epsilon) < \infty \quad \forall \epsilon > 0.$$

The proof is this theorem is given in Baum and Katz (1965). For the convenience of the reader, we also give the proof with more details in Appendix B.

Theorem 2.7. *Let* $\{X_i : i = 1, 2, ...\}$ *be a sequence of i.i.d. random variables.*

1. If $t > 1, r > 1$ and $1/2 < r/t \leq 1$, then the following are equivalent:

$$(a) \quad \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}\big(|S_n - n\mu| > n^{r/t}\epsilon\big) < \infty \quad \forall\epsilon > 0,$$
$$(b) \quad \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}\left(\sup_{k\geq n}\frac{|S_k - k\mu|}{k^{r/t}} > \epsilon\right) < \infty \quad \forall\epsilon > 0.$$

2. If $t > 0$, $r > 1$ and $r/t > 1$, then the following are equivalent:

$$(c) \quad \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(|S_n| > n^{r/t}\epsilon) < \infty \quad \forall\epsilon > 0,$$
$$(d) \quad \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(\sup_{k\geq n}|S_k/k^{r/t}| > \epsilon) < \infty \quad \forall\epsilon > 0.$$

Proof. The proof that $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$ are established in Baum and Katz (1965), while $(b) \Rightarrow (a)$ and $(d) \Rightarrow (c)$ are trivial. □

To sum up Theorem 2.6 and Theorem 2.7, we present the following theorem.

Theorem 2.8. *Let $\{X_i : i = 1, 2, ...\}$ be a sequence of i.i.d. random variables.*

1. If $t > 1, r > 1$ and $1/2 < r/t \leq 1$, then the following are equivalent:

$$(a) \quad \mathbb{E}|X_1|^t < \infty \text{ and } \mathbb{E}(X_1) = \mu,$$
$$(b) \quad \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}\big(|S_n - n\mu| > n^{r/t}\epsilon\big) < \infty, \quad \forall\epsilon > 0,$$
$$(c) \quad \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}\left(\sup_{k\geq n}\frac{|S_k - k\mu|}{k^{r/t}} > \epsilon\right) < \infty, \quad \forall\epsilon > 0.$$

2. If $t > 0$, $r > 1$ and $r/t > 1$, then the following are equivalent:

$$
\begin{aligned}
&(d) \quad \mathbb{E}|X_1|^t < \infty, \\
&(e) \quad \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(|S_n| > n^{r/t}\epsilon) < \infty, \quad \forall \epsilon > 0, \\
&(f) \quad \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}\left(\sup_{k\geq n} |S_k/k^{r/t}| > \epsilon\right) < \infty, \quad \forall \epsilon > 0.
\end{aligned}
$$

This theorem follows directly from Theorem 2.6 and Theorem 2.7. In concluding of this chapter, we give a theorem established by Lai (1974), which will be used in Chapter 3 for simplifying the proof of theorem 3.3 of Li and Rosalsky (2006).

Theorem 2.9 (Lai, 1974)**.** *Suppose $X_1, X_2, ...$ are i.i.d. random variables and $p > 2$. If $E(X_1) = 0$, $\mathbb{E}(X_1)^2 = \sigma^2$ and*

$$\mathbb{E}\left(\frac{|X_1|^p}{(\log^+ |X_1| + 1))^{p/2}}\right) < \infty.$$

Then, for any $\epsilon > \sigma(p-2)^{1/2}$, we have

$$\sum_{n=2}^{\infty} n^{p/2-2}\mathbb{P}\left[|S_n| > \epsilon(n\log n)^{1/2}\right] < \infty$$

and

$$\sum_{n=2}^{\infty} n^{p/2-2}\mathbb{P}\left[\sup_{k\geq n} |S_n/(k\log k)^{1/2}| > \epsilon\right] < \infty.$$

The proof is established in Lai (1974). Note that the theorem we use in Chapter 3 is an immediate result of Theorem 2.9. By leting $\sigma = 1$, $p = 4\beta + 2$ and the fact that

$\log^+ |x| + 1 \sim \log |x| + e$ when $x \to \infty$, we have the following theorem.

Theorem 2.10. *Let $X_1, X_2, ..,$ be a sequence of i.i.d. random variables, and let $\beta > 0$. If*

$$\mathbb{E}(X_1) = 0,\ \mathbb{E}(X^2) = 1 \quad \textit{and} \quad \mathbb{E}\left(\frac{|X_1|^{4\beta+2}}{(\log(e + |X_1|))^{2\beta+1}}\right) < \infty,$$

then

$$\sum_{n=2}^{\infty} n^{2\beta-1}\mathbb{P}\left(\frac{|S_n|}{\sqrt{n\log n}} > \lambda\right) < \infty \quad \textit{for all } \lambda > 2\sqrt{\beta}.$$

Chapter 3

Main contributions

In this chapter, we present the main contribution of this thesis. The chapter is organised into two sections. Namely, in Section 3.1 we present results that are useful in studying the existence of the p^{th} moment of a r.v. We also establish some inequalities which are useful in studying the behaviours of tail probabilities. In Section 3.2, we present the generalisations of four theorems of Li and Rosalsky (2006). To introduce some notations, let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers. One writes $a_n = O(b_n)$ if there exists a positive real number M and an integer N, such that, $|a_n| \leq M|b_n|$ for all $n > N$. Also, one writes $a_n = o(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

3.1 Moment existence and tail equivalence

This section presents the first part of the main contribution of this thesis. First we generalise Theorem 3.2.1 in Chung (1974). The generalised version is useful in deriving the existence of the p^{th} moment. Further, an equivalence in the tail probability will be given. For the convenience of reader, we first recall Theorem 3.2.1 in

Chung (1974).

Theorem 3.1. *For any random variable X, we have*

$$\sum_{n=1}^{\infty} \mathbb{P}(|X| \geq n) \leq \mathbb{E}(|X|) \leq 1 + \sum_{n=1}^{\infty} \mathbb{P}(|X| \geq n)$$

so that $\mathbb{E}(|X|) < \infty$ if and only if the series above converges.

Proof. See Theorem 3.2.1 in Chung (1974). □

As far as the existence of $\mathbb{E}(X)$ is concerned, this theorem could be generalised as follows.

Theorem 3.2. *Let $\{\alpha_n, n \geq 1\}$ be nonnegative sequence of real numbers, $\{\beta_n, n \geq 1\}$ be nonnegative and nondecreasing sequences such that $\lim_{n\to\infty} \beta_n = \infty$, $\beta_{n+1} - \beta_n = \mathcal{O}(\beta_n)$ and $c^{-1}\alpha_{n+1} \leq \beta_{n+1} - \beta_n \leq c\alpha_{n+1}$, $n \geq 1$, for some $c \geq 1$. Then*

$$c^{-1} \sum_{n=1}^{\infty} \alpha_n \mathbb{P}(|X| \geq \beta_n) \leq \mathbb{E}(|X|) \leq Kc \sum_{n=1}^{\infty} \alpha_n \mathbb{P}(|X| \geq \beta_n) + \beta_1 \tag{3.1}$$

for some $K > 1$.

Proof. Let $\Lambda_n = \{\beta_n \leq |X| < \beta_{n+1}\}$, without loss of generality, let $\beta_0 = 0$, then

$$\mathbb{E}(|X|) = \int_{\bigcup_{n=0}^{\infty} \Lambda_n} |X| \, \mathrm{d}\mathbb{P} = \sum_{n=0}^{\infty} \int_{\Lambda_n} |X| \, \mathrm{d}\mathbb{P}.$$

We have,

$$\sum_{n=0}^{\infty} \beta_n \mathbb{P}(\Lambda_n) \leq \mathbb{E}(|X|) \leq \sum_{n=0}^{\infty} \beta_{n+1} \mathbb{P}(\Lambda_n) = \sum_{n=0}^{\infty} (\beta_{n+1} - \beta_n) \mathbb{P}(\Lambda_n) + \sum_{n=0}^{\infty} \beta_n \mathbb{P}(\Lambda_n). \tag{3.2}$$

Observe that,

$$\begin{aligned}\sum_{n=0}^{N}\beta_n\mathbb{P}(\Lambda_n) &= \sum_{n=0}^{N}\beta_n\big(\mathbb{P}(|X|\geq\beta_n)-\mathbb{P}(|X|\geq\beta_{n+1})\big)\\ &= \sum_{n=1}^{N}(\beta_n-\beta_{n-1})\mathbb{P}(|X|\geq\beta_n)-\beta_N\mathbb{P}(|X|\geq\beta_{N+1}) \qquad (3.3)\\ &\leq \sum_{n=1}^{N}(\beta_n-\beta_{n-1})\mathbb{P}(|X|\geq\beta_n), \quad N\geq 1. \qquad (3.4)\end{aligned}$$

First, suppose that $\mathbb{E}\left(|X|\right)<\infty$. In this case, we have

$$\beta_N\mathbb{P}(|X|\geq\beta_{N+1})\leq\beta_{N+1}\mathbb{P}(|X|\geq\beta_{N+1})\leq\mathbb{E}\left(|X|\mathbb{I}_{\{|X|\geq\beta_{N+1}\}}\right)\longrightarrow 0.$$

Then by combining (3.2) and (3.3), we have

$$\sum_{n=0}^{\infty}\beta_n\mathbb{P}(\Lambda_n)=\sum_{n=1}^{\infty}(\beta_n-\beta_{n-1})\mathbb{P}(|X|\geq\beta_n)\leq\mathbb{E}(|X|)<\infty,$$

which, by $\beta_n-\beta_{n-1}\geq c^{-1}\alpha_n$, implies $c^{-1}\sum_{n=1}^{\infty}\alpha_n\mathbb{P}\left(|X|\geq\beta_n\right)\leq\mathbb{E}(|X|)$. On the other hand by (3.2), since $\beta_{n+1}-\beta_n=\mathcal{O}(\beta_n)$ and $\beta_n-\beta_{n-1}\leq c\alpha_n$, $n\geq 1$, we have for some $B>0$,

$$\begin{aligned}\mathbb{E}(|X|) &\leq \sum_{n=0}^{\infty}(\beta_{n+1}-\beta_n)\mathbb{P}(\Lambda_n)+\sum_{n=0}^{\infty}\beta_n\mathbb{P}(\Lambda_n)\\ &\leq \sum_{n=1}^{\infty}B\beta_n\mathbb{P}(\Lambda_n)+\sum_{n=1}^{\infty}c\alpha_n\mathbb{P}(|X|\geq\beta_n)+\beta_1\mathbb{P}(\Lambda_0).\end{aligned}$$

Then, from (3.4), we get

$$\mathbb{E}(|X|) \le \sum_{n=1}^{\infty} B \cdot (\beta_n - \beta_{n-1})\mathbb{P}(|X| \ge \beta_n) + \sum_{n=1}^{\infty} c\alpha_n \mathbb{P}(|X| \ge \beta_n) + \beta_1,$$

and then

$$\mathbb{E}(|X|) \le (B+1)c \sum_{n=1}^{\infty} \alpha_n \mathbb{P}(|X| \ge \beta_n) + \beta_1. \tag{3.5}$$

By letting $K = B + 1$, we complete the proof of (3.1) in the case that $\mathbb{E}(|X|) < \infty$. Second, suppose that $\mathbb{E}(|X|) = \infty$. Note that the relation (3.5) does not require the condition $\mathbb{E}(|X|) < \infty$ to hold. Thus, by (3.5), we have$\sum_{n=1}^{\infty} \alpha_n \mathbb{P}(|X| \ge \beta_n) = \infty$. Then, the conclusion (3.1) also holds (all of them being infinity), this completes the proof. □

From Theorem 3.2, we establish the following theorem which is extensively used in this thesis.

Theorem 3.3. *Suppose that the conditions of Theorem 3.2 hold. Then* $\mathbb{E}(|X|) < \infty$ *if and only if* $\sum_{n=0}^{\infty} \alpha_n \mathbb{P}(|X| \ge \beta_n) < \infty$.

Proof. This is an immediate result of Theorem 3.2. □

Corollary 3.1. *Let* $\{\alpha_n, n \ge 1\}$ *be nonnegative sequence of real numbers,* $\{\beta_n, n \ge 1\}$ *be nonnegative and nondecreasing sequences such that* $\lim_{n\to\infty} \beta_n = \infty$, $\beta_{n+1} - \beta_n = \mathcal{O}(\beta_n)$ *and* $\beta_n - \beta_{n-1} = o(\alpha_n)$ *then*

$$\sum_{n=1}^{\infty} \alpha_n \mathbb{P}(|X| \ge \beta_n) < \infty \quad \textit{implies} \quad \mathbb{E}(|X|) < \infty.$$

Proof. Go through the proof of (3.5) in Theorem 3.2, we only used the condition that

$\beta_n - \beta_{n-1} \le c\alpha_n$, for some $c \ge 1$, which follows from $\lim_{n\to\infty} \dfrac{\beta_n - \beta_{n-1}}{\alpha_n} = 0$. The result is immediate. $\square$

Corollary 3.2. *Let $\alpha > 0$, let $\beta > 0$ and let X be a random variable. Then,* $\sum_{n=1}^{\infty} n^\alpha \mathbb{P}\Big(|X| > n^\beta\Big) < \infty$ *if and only if* $\mathbb{E}\Big(|X|^{\frac{\alpha+1}{\beta}}\Big) < \infty$.

Proof. The condition that $\sum_{n=1}^{\infty} n^\alpha \mathbb{P}\Big(|X| > n^\beta\Big) < \infty$ is equivalent with $\sum_{n=1}^{\infty} n^\alpha \mathbb{P}\Big(|X|^{\frac{\alpha+1}{\beta}} > n^{\alpha+1}\Big) < \infty$. Now let $\alpha_n := n^\alpha$ and $\beta_n := n^{\alpha+1}$. We have $\beta_{n+1} - \beta_n = (n+1)^{\alpha+1} - n^{\alpha+1} = n^{\alpha+1}\Big((\frac{n+1}{n})^{\alpha+1} - 1\Big) \le (2^{\alpha+1} - 1) \cdot n^{\alpha+1} = \mathcal{O}(\beta_n)$, and

$$\beta_n - \beta_{n-1} = n^{\alpha+1} - (n-1)^{\alpha+1} = n^\alpha \left[n - \left(\frac{n-1}{n}\right)^\alpha \cdot (n-1)\right] \ge n^\alpha = \alpha_n$$

Also, since $n - (\frac{n-1}{n})^\alpha (n-1) \to 1$, by the fact that every convergent sequence is bounded, there exists a constant C such that for all $n \ge 1$, $n - (\frac{n-1}{n})^\alpha (n-1) \le C$. Let $c = \max\{2, C\}$. One verifies that $c^{-1}\alpha_n \le \beta_n - \beta_{n-1} \le c\alpha_n \quad \forall n,$ for some $c \ge 1$. Then, the proof follows from Theorem 3.3. $\square$

Corollary 3.3. *Let $\alpha, \beta > 0$ and suppose that α_n/n^α and β_n/n^β are bounded away from 0 and ∞. Then, for a random variable X,*

$$\sum_{n=1}^{\infty} \alpha_n \mathbb{P}\Big(|X| > \beta_n\Big) < \infty$$

if and only if

$$\mathbb{E}\Big(|X|^{\frac{\alpha+1}{\beta}}\Big) < \infty.$$

Proof. The proof follows form Corollary 3.2. Since α_n/n^α and β_n/n^β are bounded away from 0 and ∞. Then exists $a > 1$ such that $a^{-1} < \frac{\alpha_n}{n^\alpha} < a$ and $a^{-1} < \frac{\beta_n}{n^\beta} < a$. Thus $\sum_{n=1}^{\infty} \alpha_n \mathbb{P}\Big(|X| > \beta_n\Big) < \infty$ implies $\sum_{n=1}^{\infty} a^{-1} n^\alpha \mathbb{P}\Big(|X| > an^\beta\Big) < \infty$, and this is equivalent to $\sum_{n=1}^{\infty} n^\alpha \mathbb{P}\Big(\frac{|X|}{a} > n^\beta\Big) < \infty$. Now by Corollary 3.2, $\mathbb{E}\Big((\frac{|X|}{a})^{\frac{\alpha+1}{\beta}}\Big) < \infty$, i.e. $\mathbb{E}\Big(|X|^{\frac{\alpha+1}{\beta}}\Big) < \infty$. Conversely, if $\mathbb{E}(|X|^{\frac{\alpha+1}{\beta}}) < \infty$, we have $\mathbb{E}\Big((a \cdot |X|)^{\frac{\alpha+1}{\beta}}\Big) < \infty$. Then, by Corollary 3.2, $\sum_{n=1}^{\infty} n^\alpha \mathbb{P}\Big(|X| > \frac{n^\beta}{a}\Big) < \infty$, which implies that $\sum_{n=1}^{\infty} \alpha_n \mathbb{P}\Big(|X| > \beta_n\Big) < \infty$, this completes the proof. □

We derived Theorem 3.2 and Theorem 3.3 which generalise Theorem 3.2.1 in Chung (1974). Now, we derive some lemmas which play a central role in generalising the main theorems of Li and Rosalsky (2006).

Conjecture 3.1. *Let $X_1, X_2, ...,$ be a sequence of i.i.d. random variables. Let m be a fixed positive integer and $\{u_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that $\lim_{n\to\infty} u_n = +\infty$ and if*

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{1\le i_1<i_2<\cdots<i_m\le n} \prod_{h=1}^{m} |X_{i_h}| \ge u_n\right) = 0.$$

Then,

$$\lim_{n\to\infty} \frac{\mathbb{P}\Big(\max_{1\le i_1<i_2<\cdots<i_m\le n} \prod_{h=1}^{m} |X_{i_h}| \ge u_n\Big)}{\frac{n!}{(n-m)!m!}\mathbb{P}(\prod_{h=1}^{m} |X_h| \ge u_n)} \ge 1,$$

given that for all large n, $\mathbb{P}(\prod_{h=1}^{m} |X_h| \ge u_n) > 0$

Lemma 3.1. *Let $X_1, X_2, ...,$ be a sequence of i.i.d. random variables. Let m be a* *fixed positive integer and* $\{u_n\}_{n=1}^{\infty}$ *be a sequence of nonnegative real numbers such that* $\lim_{n\to\infty} u_n = +\infty$. *Then*

$$\sum_{n=1}^{\infty} n^m \mathbb{P}\left(\prod_{h=1}^{m} |X_h| \geq u_n\right) < \infty$$

if and only if

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq i_1<i_2<\cdots<i_m\leq n} \prod_{h=1}^{m} |X_{i_h}| \geq u_n\right) < \infty.$$

Proof. The sufficient condition follows directly from the sub-additivity. To prove the necessary condition, let $\mathbb{A} = \{n : \mathbb{P}\Big(\max_{1\leq i_1<i_2<\cdots<i_m\leq n} \prod_{h=1}^{m} |X_{i_h}| \geq u_n\Big) = 0\}$. We have

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(\max_{1\leq i_1<i_2<\cdots<i_m\leq n} \prod_{h=1}^{m} |X_{i_h}| \geq u_n\Big) = \sum_{n\in\mathbb{N}\setminus\mathbb{A}} \mathbb{P}\Big(\max_{1\leq i_1<i_2<\cdots<i_m\leq n} \prod_{h=1}^{m} |X_{i_h}| \geq u_n\Big) \quad (3.6)$$

and note that if $n \in \mathbb{A}$, $\mathbb{P}\Big(\prod_{h=1}^{m}|X_h| \geq u_n\Big) = 0$ and then $n^m\mathbb{P}\Big(\prod_{h=1}^{m}|X_h| \geq u_n\Big) = 0$, $\forall n \in \mathbb{A}$. Then,

$$\sum_{n=1}^{\infty} n^m \mathbb{P}\Big(\prod_{h=1}^{m} |X_h| \geq u_n\Big) = \sum_{n\in\mathbb{N}\setminus\mathbb{A}} n^m \mathbb{P}\Big(\prod_{h=1}^{m} |X_h| \geq u_n\Big). \quad (3.7)$$

Hence, from (3.6) and (3.7), it suffices to prove that if $\sum_{n\in\mathbb{N}\setminus\mathbb{A}} \mathbb{P}\Big(\max_{1\leq i_1<i_2<\cdots<i_m\leq n} \prod_{h=1}^{m} |X_{i_h}| \geq u_n\Big) < \infty$ then $\sum_{n\in\mathbb{N}\setminus\mathbb{A}} n^m\mathbb{P}\Big(\prod_{h=1}^{m}|X_h| \geq u_n\Big) < \infty$. Hence, in the sequel, we suppose without loss of generality that $\mathbb{P}\Big(\prod_{h=1}^{m}|X_h| \geq u_n\Big) > 0$

for all $n = 1, 2, \ldots$ Thus, if $\sum_{n=1}^{\infty} \mathbb{P}\Big(\max_{1 \le i_1 < i_2 < \cdots < i_m \le n} \prod_{h=1}^{m} |X_{i_h}| \ge u_n \Big) < \infty$, we have

$$\lim_{n\to\infty} \mathbb{P}\Big(\max_{1 \le i_1 < i_2 < \cdots < i_m \le n} \prod_{h=1}^{m} |X_{i_h}| \ge u_n \Big) = 0. \tag{3.8}$$

Then, by Conjecture 3.1,

$$\lim_{n\to\infty} \frac{\mathbb{P}\Big(\max_{1 \le i_1 < i_2 < \cdots < i_m \le n} \prod_{h=1}^{m} |X_{i_h}| \ge u_n \Big)}{\frac{n!}{(n-m)!m!} \mathbb{P}(\prod_{h=1}^{m} |X_h| \ge u_n)} \ge 1. \tag{3.9}$$

Further from sub-additivity, we have

$$\limsup_{n\to\infty} \frac{\mathbb{P}\Big(\max_{1 \le i_1 < i_2 < \cdots < i_m \le n} \prod_{h=1}^{m} |X_{i_h}| \ge u_n \Big)}{\frac{n!}{(n-m)!m!} \mathbb{P}(\prod_{h=1}^{m} |X_h| \ge u_n)} \le 1. \tag{3.10}$$

Hence, combining (3.9) and (3.10), we get

$$\lim_{n\to\infty} \frac{\mathbb{P}\Big(\max_{1 \le i_1 < i_2 < \cdots < i_m \le n} \prod_{h=1}^{m} |X_{i_h}| \ge u_n \Big)}{\frac{n^m}{m!} \mathbb{P}(\prod_{h=1}^{m} |X_h| \ge u_n)} = 1.$$

Therefore, $\mathbb{P}\Big(\max_{1 \le i_1 < i_2 < \cdots < i_m \le n} \prod_{h=1}^{m} |X_{i_h}| \ge u_n \Big) \sim \frac{n^m}{m!} \mathbb{P}(\prod_{h=1}^{m} |X_h| \ge u_n)$, this completes the proof. □

We also prove the following lemma which is useful in generalising the main theorems of Li and Rosalsky (2006).

Lemma 3.2. *Let m and $\{u_n\}_{n=1}^{\infty}$ be as in Lemma 3.1 and $X_1, X_2, ...,$ be a sequence of i.i.d. random variables. Suppose that there exists a nonnegative, continuous and increasing function f such that $f(u_n)\big/n^{\beta}$ is bounded away from 0 and from infinity, for some $\beta > 0$. Then,*

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\max_{1\le i_1<i_2<\cdots<i_m\le n}\prod_{h=1}^{m}|X_{i_h}| > u_n\right) < \infty,$$

if and only if

$$\mathbb{E}\left(\left[f\left(\prod_{h=1}^{m}|X_h|\right)\right]^{\frac{m+1}{\beta}}\right) < \infty.$$

Proof. From Lemma 3.1, $\sum_{n=1}^{\infty}\mathbb{P}\left(\max_{1\le i_1<i_2<\cdots<i_m\le n}\prod_{h=1}^{m}|X_{i_h}| > u_n\right) < \infty$ if and only if $\sum_{n=1}^{\infty}n^m\mathbb{P}\left(\prod_{h=1}^{m}|X_h| > u_n\right) < \infty$. Since the function f is increasing and continuous, this last statement is equivalent to $\sum_{n=1}^{\infty}n^m\mathbb{P}\left[f\left(\prod_{h=1}^{m}|X_h|\right) > f(u_n)\right] < \infty$. Then, by using Corollary 3.3, this last statement is equivalent to $\mathbb{E}\left[\left(f\left(\prod_{h=1}^{m}|X_h|\right)\right)^{\frac{m+1}{\beta}}\right] < \infty$, this completes the proof. □

By using Lemma 3.2, we derive the following corollary which is useful in extending Theorem 3.3 (Theorem 3.6 of this thesis) of Li and Rosalsky (2006). The established corollary is also useful to simplify remarkably the statement in Remark 2.3 of Li and Rosalsky (2006).

Corollary 3.4. *We have*

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\max_{1\leq i_1<i_2<\cdots<i_m\leq n}\prod_{h=1}^{m}|X_{i_h}|\geq\sqrt{n\ln(n)}\right)<\infty$$

if and only if

$$\mathbb{E}\left[\prod_{h=1}^{m}|X_h|^{2(m+1)}\Bigg/\left(\ln\left(e+\prod_{h=1}^{m}|X_h|\right)\right)^{m+1}\right]<\infty,$$

with m a fixed positive integer.

Proof. By Lemma 3.1, $\sum_{n=1}^{\infty}\mathbb{P}\left(\max_{1\leq i_1<i_2<\cdots<i_m\leq n}\prod_{h=1}^{m}|X_{i_h}|\geq\sqrt{n\ln(n)}\right)<\infty$ if and only if $\sum_{n=3}^{\infty}n^m\mathbb{P}\left(\prod_{h=1}^{m}|X_h|\geq\sqrt{n\ln(n)}\right)<\infty$, and this is equivalent to $\sum_{n=3}^{\infty}n^m\mathbb{P}\left(\prod_{h=1}^{m}|X_h|^2\geq n\ln(n)\right)<\infty$. Then, by taking $f(x)=\frac{x}{\ln\left(e+\sqrt{x}\right)}$, $x\geqslant 3$, one can verify that $f(n\ln(n))/n\geqslant 1/2$ for all $n\geqslant 3$ and $\lim_{n\to\infty}f(n\ln(n))/n=2$ and this implies that $f(n\ln(n))/n$ is bounded away from 0 and from infinity. This completes the proof, by using Lemma 3.2 and setting $\beta=1$. □

Corollary 3.5. *Suppose that the conditions of Lemma 3.1 hold with u_n/n^{β} bounded away from 0 and from infinity, for some $\beta>0$. Then,*

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\max_{1\leq i_1<i_2<\cdots<i_m\leq n}\prod_{h=1}^{m}|X_{i_h}|\geq u_n\right)<\infty$$

if and only if

$$\mathbb{E}\left(|X_1|^{\frac{m+1}{\beta}}\right)<\infty.$$

Proof. By Lemma 3.2, with f being identity function, and $X_1,X_2,...,$ i.i.d., we have the result immediately. □

Corollary 3.6. *Let* $\{u_n\}_{n=1}^{\infty}$ *be a sequence of nonnegative real numbers and* $X_1, X_2, ...,$ *a sequence of i.i.d random variables. Then,*

$$\sum_{n=1}^{\infty} n^2 \mathbb{P}\left(|X_1 X_2| \geq u_n\right) < \infty$$

if and only if

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \leq i < j \leq n} |X_i X_j| \geq u_n\right) < \infty.$$

Proof. The proof follows directly from Lemma 3.1 by taking $m = 2$. □

3.2 Convergence of sample correlation matrices

In this section, we give the second part of the main contributions of this thesis. First, from Corollary 3.5, we generalise the statement in Remark 2.1 of Li and Rosalsky (2006). For the convenience of the reader, recall that in Remark 2.1, the authors concluded that if $\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \leq i < j \leq n} |X_i X_j| \geq n^{\alpha}\right) < \infty$, then $\mathbb{E}\left(|X_1|^{2/\alpha}\right) < \infty$. This statement becomes a special case of Corollary 3.5, with $m = 2$. Further, from Corollary 3.4, we establish the following result which improves the statement in Remark 2.3 of Li and Rosalsky (2006). For the convenience of the reader, we recall that in Remark 2.3 of Li and Rosalsky (2006), the authors conclude that $\sum_{n=1}^{\infty} \mathbb{P}\Big(\max_{1 \leq i < j \leq n} |X_1 X_2| \geq \sqrt{n \log n}\Big) < \infty$ implies $\mathbb{E}|X_1|^{\beta} < \infty$, for $0 \leqslant \beta < 6$. This becomes a special case of the following result.

Corollary 3.7. *Suppose that* $\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \prod_{h=1}^{m} |X_{i_h}| \geq \sqrt{n \ln(n)}\right) < \infty$, *for a fixed positive integer* m. *Then,* $\mathbb{E}\left[|X_1|^{\beta}\right] < \infty$, *for all* $0 \leqslant \beta < 2(m+1)$.

Proof. By Corollary 3.4, and by defining $Y := \prod_{h=1}^{m} |X_h|$ we have

$$\mathbb{E}\left[Y^{2(m+1)} \Big/ (\ln(e+Y))^{m+1}\right] < \infty.$$

Now, for all $0 \leq \beta < 2(m+1)$, we have

$$\mathbb{E}\left[Y^{2(m+1)} \Big/ (\ln(e+Y))^{m+1}\right] = \mathbb{E}\left[Y^{\beta} \cdot \frac{Y^{2(m+1)-\beta}}{(\ln(e+Y))^{m+1}}\right] < \infty.$$

Let M be a positive real number such that for all $x > M$,

$$\frac{x^{2(m+1)-\beta}}{(\ln(e+x))^{m+1}} > 1.$$

We have

$$\begin{aligned}\mathbb{E}\left[Y^{\beta} \cdot \frac{Y^{2(m+1)-\beta}}{(\ln(e+Y))^{m+1}}\right] &= \mathbb{E}\left[Y^{\beta} \cdot \frac{Y^{2(m+1)-\beta}}{(\ln(e+Y))^{m+1}} \mathbb{I}(Y > M)\right] \\ &\quad + \mathbb{E}\left[Y^{\beta} \cdot \frac{Y^{2(m+1)-\beta}}{(\ln(e+Y))^{m+1}} \mathbb{I}(Y \leq M)\right] < \infty.\end{aligned}$$

Since $\mathbb{E}\left[Y^{\beta} \cdot \frac{Y^{2(m+1)-\beta}}{(\ln(e+Y))^{m+1}} \mathbb{I}(Y \leq M)\right] < \infty$, then finally we get

$$\infty > \mathbb{E}\left[Y^{\beta} \cdot \frac{Y^{2(m+1)-\beta}}{(\ln(e+Y))^{m+1}} \mathbb{I}(Y > M)\right] \geq \mathbb{E}\left[Y^{\beta} \mathbb{I}(Y > M)\right],$$

which leads to $\mathbb{E}\left[Y^{\beta}\right] < \infty$, for all $0 \leq \beta < 2(m+1)$. □

Remark 3.1. *For $m = 2$, the result of Corollary 3.7 becomes the statement given in remark 2.3 of Li and Rosalsky (2006).*

Let $\{(U_{k,i}, V_{k,i}); i \geq 1, k \geq 1\}$ be an array of i.i.d 2-dimensional random vectors. Let $\{p_n, n \geq 1\}$ be a sequence of positive integers and consider the $n \times p_n$ matrices

$$A_n = (U_{k,i})_{1 \leq k \leq n, 1 \leq i \leq p_n}, \quad B_n = (V_{k,i})_{1 \leq k \leq n, 1 \leq i \leq p_n}, \quad n \geq 1.$$

Then $A_n^T B_n$ is a $p_n \times p_n$ matrix whose $(i,j)^{\text{th}}$ entry is $\sum_{k=1}^{n} U_{k,i} V_{k,j}$, $n \geq 1$. Let

$$T_n = \max_{1 \leq i \neq j \leq p_n} \left| \sum_{k=1}^{n} U_{k,i} V_{k,j} \right|$$

and let $Y_n = U_{n,1} V_{n,2}$ and set $S_n = \sum_{k=1}^{n} Y_k$. We first present Theorem 3.1 of Li and Rosalsky (2006), since this is used to derive some of the main results given in this section.

Theorem 3.4. *Let $\{a_n : n \geq 1\}$ be a sequence of positive nondecreasing real numbers such that $\lim_{n \to \infty} a_n = \infty$, and*

$$\lim_{c \downarrow 1} \limsup_{n \to \infty} \frac{a_{[cn]}}{a_n} = 1. \tag{3.11}$$

Suppose that the sequence $\{p_n; n \geq 1\}$ is nondecreasing. If

$$\frac{S_n}{a_n} \xrightarrow{P} 0 \tag{3.12}$$

and

$$\sum_{n=1}^{\infty} \frac{p_n^2}{n} \mathbb{P}\left(\frac{|S_n|}{a_n} > \lambda \right) < \infty, \text{ for some } 0 < \lambda < \infty \tag{3.13}$$

then

$$\limsup_{n \to \infty} \frac{T_n}{a_n} \leq \lambda \text{ a.s.}$$

The proof of this theorem is given in Li and Rosalsky (2006). Further, for this thesis to be self-contained, we also outline the proof in Appendix B. Below, we present two theorems which generalise Theorem 3.2 and Theorem 3.3 of Li and Rosalsky (2006). We also simplify significantly the proof of Theorem 3.2-3.3 of Li and Rosalsky (2006). To this end, we first recall a lemma of Li and Rosalsky (2006).

Lemma 3.3. *Suppose that n/p_n is bounded away from 0 and ∞. Let $\{a_n; n \geq 1\}$ be a nondecreasing sequence of positive constants such that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1, \; \liminf_{n\to\infty} \frac{a_{2n}}{a_n} = b \in (1, \infty] \textit{ and } \lim_{n\to\infty} \frac{a_n}{n} = 0.$$

Then, if $\limsup_{n\to\infty} \frac{T_n}{a_n} < \infty$ *a.s., we have* $\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq i\neq j\leq n} |U_{1,i}V_{1,j}| \geq a_n \right) < \infty$ *and* $\mathbb{E}(U_{1,1})\mathbb{E}(V_{1,1}) = 0$.

The proof of this theorem is given in Li and Rosalsky (2006). Further, for this thesis to be self-contained, we also outline the proof in Appendix B.

Theorem 3.5. *Suppose that n/p_n is bounded away from 0 and ∞. Let $1/2 < \alpha \leq 1$, then, the following statements are equivalent.*

1. $\frac{T_n}{n^\alpha} \xrightarrow[n\to\infty]{a.s.} 0,$
2. $\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq i\neq j\leq n} |U_{1,i}V_{1,j}| \geq n^\alpha \right) < \infty \quad \textit{and} \quad \mathbb{E}(U_{1,1})\mathbb{E}(V_{1,1}) = 0,$
3. $\mathbb{E}\left(|U_{1,1}|^{3/\alpha} |V_{1,2}|^{3/\alpha} \right) < \infty \quad \textit{and} \quad \mathbb{E}(U_{1,1})\mathbb{E}(V_{1,1}) = 0.$

Proof. The proof of (1) $\Rightarrow$ (2) is similar to that given by Li and Rosalsky (2006). For completeness, we present this here. To this end, define $Y_k = U_{k,1}V_{k,2}$, $k = 1, 2, ...$, and

$S_n = \sum_{k=1}^n Y_k$. We divided this into two cases: (i) $\alpha \in (1/2, 1)$, then applying Lemma 3.3, we get the result immediately. For the another case: (ii) $\alpha = 1$, we repeat the proof of Lemma 3.3, and get the first result, since in proving the first result, we did not use the condition that $\dfrac{a_n}{n} \to 0$. In proving the second result, since we already have $\lim_{n\to\infty} \dfrac{T_n}{n} = 0$, which implies

$$\lim_{n\to\infty} \frac{\left|\sum_{k=1}^n U_{k,1}V_{k,2}\right|}{n} = 0 \text{ a.s.}$$

Then by SLLN, we have the second result $\mathbb{E}(U_{1,1})\mathbb{E}(V_{1,1}) = 0$. Now, we give the proof of (2) $\Rightarrow$ (1). Note that by Corollary 3.6 and Corollary 3.2, the condition $\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\le i\neq j\le n} |U_{1,i}V_{1,j}| \ge n^{\alpha}\right) < \infty$ is equivalent to $\mathbb{E}|U_{1,1}V_{1,2}|^{\frac{3}{\alpha}} < \infty$. Next by Theorem 2.8 ($r = 3$, $t = 3/\alpha$), we have

$$\sum_{n=1}^{\infty} n\mathbb{P}\left(\frac{|S_n|}{n^{\alpha}} > \epsilon\right) < \infty \quad \text{for all } \epsilon > 0.$$

Since $c^{-1} \le p_n/n < c$, $n \ge 1$, which means $\frac{p_n^2}{n} < \frac{c^2n^2}{n} = c^2 n$, then

$$\sum_{n=1}^{\infty} \frac{p_n^2}{n}\mathbb{P}\left(\frac{|S_n|}{n^{\alpha}} > \epsilon\right) < \infty \quad \text{for all } \epsilon > 0. \tag{3.14}$$

One can verify that $\lim_{c\downarrow 1} \limsup_{n\to\infty} \dfrac{[cn]^{\alpha}}{n^{\alpha}} = 1$. Note that $p_n^2/n > c^{-2}n$, and by (3.14), it means $c^{-2}n\mathbb{P}\left(\frac{|S_n|}{n^{\alpha}} > \epsilon\right) \to 0$ and $\mathbb{P}\left(\frac{|S_n|}{n^{\alpha}} > \epsilon\right) \to 0$, i.e., $\dfrac{S_n}{n^{\alpha}} \xrightarrow[n\to\infty]{\mathbb{P}} 0$. Now by Theorem 3.1 of Li and Rosalsky (2006), $\limsup_{n\to\infty} \dfrac{T_n}{a_n} \le \epsilon$ a.s. for all $\epsilon > 0$. Letting $\epsilon \downarrow 0$, we get the desired result. The equivalence of (2) and (3) follows directly from Corollary 3.5, which completes the proof. $\square$

Theorem 3.6. *Suppose that n/p_n is bounded away from 0 and ∞. If*

$\mathbb{E}(U_{1,1})\mathbb{E}(V_{1,1}) = 0, \quad \mathbb{E}(U_{1,1}^2)\mathbb{E}(V_{1,1}^2) = 1$ *and*

$$\mathbb{E}\left(\frac{(U_{1,1}V_{1,2})^6}{\log^3(e+|U_{1,1}V_{1,2}|)}\right) < \infty \ \textit{ or } \ \sum_{n=1}^{\infty}\mathbb{P}\left(\max_{1\le i\ne j\le n}|U_{1,i}V_{1,j}| \ge \sqrt{n\log n}\right) < \infty, \tag{3.15}$$

then $\limsup_{n\to\infty}\frac{T_n}{\sqrt{n\log n}} \le 2$ *a.s. Conversely, if* $\limsup_{n\to\infty}\frac{T_n}{\sqrt{n\log n}} < \infty$ *a.s. then* $\mathbb{E}(U_{1,1})\mathbb{E}(V_{1,1}) = 0$, $\mathbb{E}(U_{1,1}^{\beta})\mathbb{E}(V_{1,1}^{\beta}) < \infty$, *for all* $0 \le \beta < 6$ *and* (3.15) *holds.*

Proof. The second part can be established by following the same steps as in Li and Rosalsky (2006). Specifically, let $a_n = \sqrt{n\log n}$, it is easy to verify that a_n is nondecreasing, and

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1, \quad \liminf_{n\to\infty}\frac{a_{2n}}{a_n} = \sqrt{2} \in (1,\infty], \quad \lim_{n\to\infty}\frac{a_n}{n} = 0.$$

As we have $\limsup_{n\to\infty}\frac{T_n}{\sqrt{n\log n}} \le \infty$ a.s. then by Lemma 3.3, we have $\mathbb{E}(U_{1,1})\mathbb{E}(V_{1,1}) = 0$ and (3.15). Left to prove that $\mathbb{E}\big(|Y_1|^{\beta}\big) < \infty$, for $\forall\, 0 \le \beta < 6$, which is, by Corollary 3.7, an immediate consequence of (3.15). We will prove the first part in a much simpler way than what is given in Li and Rosalsky. Let $m = 2$ and note that by Corollary 3.6, and Corollary 3.4, we have

$$\mathbb{E}\left(\frac{|U_{1,1}V_{1,2}|^6}{\left(\log(e+|U_{1,1}V_{1,2}|)\right)^3}\right) < \infty.$$

Next, by Theorem 2.10, we have

$$\sum_{n=2}^{\infty} n\mathbb{P}\left(\frac{|S_n|}{\sqrt{n\log n}} > \lambda\right) < \infty \quad \text{for all } \lambda > 2.$$

One can verify that $\lim_{c\downarrow 1}\limsup_{n\to\infty}\frac{\sqrt{[cn]\log[cn]}}{\sqrt{n\log n}}=1$, and by Chebyshev's inequality, for all $\epsilon>0$,

$$\mathbb{P}\left(\frac{|S_n|}{\sqrt{n\log n}}>\epsilon\right)\leq\frac{\mathrm{Var}(S_n)}{\epsilon^2 n\log n}=\frac{1}{\epsilon^2\log n}\to 0,$$

i.e., $\frac{S_n}{\sqrt{n\log n}}\xrightarrow[n\to\infty]{\mathbb{P}}0$. Now by Theorem 3.4, and letting $\lambda\downarrow 2$, we get the desired result. □

Below, we generalise Theorems 2.1 to 2.4 of Li and Rosalsky (2006).

Theorem 3.7. *Suppose that n/p_n is bounded away from 0 and ∞. Let $1/2<\alpha\leq 1$. Then, the following statements are equivalent.*

(1) $\sum_{n=1}^{\infty}n^2\mathbb{P}(|X_{1,1}X_{1,2}|\geq n^\alpha)<\infty$ and $\mathbb{E}(X_{1,1})=0$.

(2) $\sum_{n=1}^{\infty}\mathbb{P}(\max_{1\leq i<j\leq n}|X_{1,i}X_{1,j}|\geq n^\alpha)<\infty$ and $\mathbb{E}(X_{1,1})=0$.

(3) $\lim_{n\to\infty}\frac{W_n}{n^\alpha}=0$. *a.s.*

(4) $\mathbb{E}\Big(|X_{1,1}|^{3/\alpha}\Big)<\infty$ and $\mathbb{E}(X_{1,1})=0$.

Proof. The equivalence between (1) and (2) follows directly form Corollary 3.6 by taking $u_n=n^\alpha$. Further, the equivalence between the statements in (2) and (3) is established in Li and Rosalsky (2006). Finally, the equivalence between the statements (2) and (4) follows from Corollary 3.5 by taking $m=2$, this completes the proof. □

Note that the condition (4) is easy to verify than condition (2). For example, if $X_{1,i}\sim\mathcal{N}(\mu,\sigma^2)$, we know that directly $\mathbb{E}\left[|X_{1,1}|^{3/\alpha}\right]<\infty$, but it will be difficult to verify that condition (2) holds.

Theorem 3.8. *Suppose that n/p_n is bounded away from 0 and ∞. Let $\alpha \in (1/2, 1]$. Suppose $X_{1,1}$ is nondegenerate and suppose that one of the following three statements holds*

(1) $\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \le i < j \le n} |X_{1,i} X_{1,j}| \ge n^{\alpha} \right) < \infty,$

(2) $\sum_{n=1}^{\infty} n^2 \mathbb{P}\left(|X_{1,1} X_{1,2}| \ge n^{\alpha} \right) < \infty,$

(3) $\mathbb{E}\left(|X_{1,1}|^{3/\alpha} \right) < \infty.$

Then, we have

$$\lim_{n \to \infty} n^{1-\alpha} L_n = 0 \ a.s. \tag{3.16}$$

Proof. The proof of (1) $\Rightarrow$ (3.16) is given in Li and Rosalsky (2006). For the completeness of this thesis, the proof is also given in Appendix C with more details. The equivalence of (1) and (2) is by Lemma 3.1 with $m = 2$ and $u_n = n^{\alpha}$. The equivalence of (2) and (3) is given by Corollary 3.3, which completes the proof. □

Theorem 3.9. *Suppose that n/p_n is bounded away from 0 and ∞. Then the following statements are equivalent.*

(1) $\mathbb{E}(X_{1,1}) = 0, \quad \mathbb{E}(X_{1,1}^2) = 1$ and $\sum_{n=1}^{\infty} n^2 \mathbb{P}\left(|X_{1,1} X_{1,2}| \ge \sqrt{n \log n} \right) < \infty.$

(2) $\mathbb{E}(X_{1,1}) = 0, \quad \mathbb{E}(X_{1,1}^2) = 1$ and $\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \le i < j \le n} |X_{1,i} X_{1,j}| \ge \sqrt{n \log n} \right) < \infty.$

(3) $\lim_{n \to \infty} \frac{W_n}{\sqrt{n \log n}} = 2 \ a.s.$

(4) $\mathbb{E}(X_{1,1}) = 0, \quad \mathbb{E}(X_{1,1}^2) = 1$ and $\mathbb{E}\left(\frac{(X_{1,1} X_{1,2})^6}{\log^3(e + |X_{1,1} X_{1,2}|)} \right) < \infty.$

Proof. The equivalence between (1) and (2) follows directly from Lemma 3.1 by taking $m = 2$ and $u_n = \sqrt{n \ln(n)}$. The equivalence between (2) and (4) follows directly from Corollary 3.4 by taking $m = 2$. Further, the equivalence between (2) and (3) is established in Li and Rosalsky (2006). For completeness, we also give the proof in Appendix B. □

Note that the condition (4) is easy to verify than the condition (2). For example, if $X_{1.i} \sim \mathcal{N}(\mu, \sigma^2)$, one gets directly that the condition (4) holds since all moments exist. However, the verification of condition (2) is not straightforward.

Theorem 3.10. *Suppose that n/p_n is bounded away from 0 and ∞. If $X_{1,1}$ is non-degenerate and if one of the following three conditions hold,*

(1). $\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \le i < j \le n} |X_{1,i} X_{1,j}| \ge \sqrt{n \log n} \right) < \infty,$

(2). $\sum_{n=1}^{\infty} n^2 \mathbb{P}\left(|X_{1,1} X_{1,2}| \ge \sqrt{n \log n} \right) < \infty,$

(3). $\mathbb{E}\left(\frac{|X_{1,1} X_{1,2}|^6}{(\log(e + |X_{1,1} X_{1,2}|))^3} \right) < \infty,$

then,

$$\lim_{n \to \infty} \left(\frac{n}{\log n} \right)^{1/2} L_n = 2 \ a.s. \tag{3.17}$$

Proof. The proof of (1) to (3.17) is given in Li and Rosalsky (2006). For the completeness of this thesis, we also outline the proof in Appendix C. The equivalence of (1) and (2) is by Lemma 3.1 with $m = 2$ and $u_n = \sqrt{n \log n}$. The equivalence of (2) and (3) is given by Corollary 3.4 with $m = 2$, which completes the proof. □

Conclusion

In this thesis, we consider the estimation problem concerning the correlation coefficients in context of high dimensional data. In summary, we generalise some existing results in literature in four ways. First, we generalise Theorem 3.2.1 in Chung (1974), which is useful in studying the existence of the p^{th} moment. Second, we derive some inequalities which are useful in studying the behaviour of tail probabilities. Third, we simplify remarkably the proof of Theorems 3.2 and 3.3 as well as the proof of the statements of Remarks 2.1 and 2.3 of Li and Rosalsky (2006). Fourth, we generalise Theorem 2.1 to 2.4 of Li and Rosalsky (2006).

Appendix A

Background in probability theory

In this appendix, we present some probability results and concepts, which constitute the foundation of this thesis. Most of these results and concepts can be found in advanced textbooks of probability and analysis. To give some examples, we quote Chung (1974), Rudin (1976), Petrov (1995), Folland (1999), Durrett (2005), among others.

A.1 Modes of convergence

In this section, we present some basic definitions and theorems about modes of convergence of sequences of random variables. We recall the definitions of three modes of convergence: convergence almost surely, convergence in probability and convergence in distribution. All the definitions are cited from Chung (1974). We also present the famous Borel-Cantelli lemma which is used in establishing Theorem 3.4 of this thesis. To introduce some notations, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition A.1 (Convergence almost surely)**.** *A sequence of random variables* $\{X_n, n = 1, 2, ..., \}$ *is said to converge almost surely(a.s.) to the random variable* X *if and only if there exists a null set* N*,(*$\mathbb{P}(N) = 0$*), such that*

$$\forall \omega \in \Omega \backslash N : \lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \textit{is finite.} \tag{A.1}$$

To simplify the notation, we denote $X_n \xrightarrow[n\to\infty]{a.s.} X$.

Theorem A.1. $X_n \xrightarrow[n\to\infty]{a.s.} X$ *if and only if for all* $\epsilon > 0$*,*

$$\lim_{m\to\infty} \mathbb{P}(|X_n - X| \leq \epsilon \quad \textit{for all} \quad n \geq m) = 1; \tag{A.2}$$

or equivalently

$$\lim_{m\to\infty} \mathbb{P}(|X_n - X| > \epsilon \quad \textit{for some} \quad n \geq m) = 0.$$

Proof. The proof of this theorem is given in Chung (1974). □

Definition A.2 (Convergence in probability)**.** *The sequence* X_n *is said to converge in probability (pr.) to* X *if for every* $\epsilon > 0$*,*

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \epsilon) = 0.$$

We denote $X_n \xrightarrow[n\to\infty]{\mathbb{P}} X$.

Remark A.1. *By statement in Chung (1974), strictly speaking, the definition applies when all* X_n *and* X *are finite valued. But we may extend it to r.v.'s that are finite a.s. either by agreeing to ignore a null set or by the logical convention that a formula must first be defined in order to be valid or invalid. Thus for example, if* $X_n(\omega) = +\infty$ *and*

$X(\omega) = +\infty$ for some ω, then $X_n(\omega) - X(\omega)$ is not defined and therefore such an ω cannot belong to set $\{|X_n - X| > \epsilon\}$

Theorem A.2. *If $X_n \xrightarrow[n\to\infty]{a.s.} X$, then $X_n \xrightarrow[n\to\infty]{\mathbb{P}} X$.*

Proof. The proof follows directly from Theorem A.1. □

Theorem A.3 (Borel-Cantelli Lemma)**.** *We have events $E_n, n = 1, 2, ...,$ such that*

$$(1)\ \textit{If}\ \sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty,\ \textit{then}\ \mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = \mathbb{P}(E_n\, i.o.) = 0.$$

$$(2)\ \textit{If}\ \sum_{n=1}^{\infty} \mathbb{P}(E_n) = \infty, E_n\ \textit{are mutually independent},\ \textit{then}\ \mathbb{P}(E_n\, i.o.) = 1.$$

The proof of this theorem is given in Chung (1974). Below, we derive a proposition which is used in proof of Theorem 3.4 and other results. The proof is based on the classic probability techniques as given in Chung (1974).

Theorem A.4. *Let X_n be a sequence of random variables, then*

$$(1) \quad \left\{\limsup X_n \geq \epsilon\right\} \supseteq \limsup\{X_n \geq \epsilon\}$$

$$(2) \quad \left\{\limsup X_n > \epsilon\right\} \subseteq \limsup\{X_n > \epsilon\}$$

Proof. (1), $\forall \omega \in \limsup\{X_n \geq \epsilon\}$, i.e. $X_n(\omega) \geq \epsilon$ for infinitely many n. We collect those n and denote as $n_i, i = 1, 2, ...$ Then $X_{n_i}(\omega) \geq \epsilon$, for $i = 1, 2, ...$, thus

$$\sup_{i \geq k} X_{n_i}(\omega) \geq \epsilon, \quad \forall k = 1, 2, ...,$$

Since $\sup_{i\geq k} X_{n_i}$ is a decreasing sequence which has a lower bound ϵ, then

$$\lim_{k\to\infty} \sup_{i\geq k} X_{n_i}(\omega) \geq \epsilon$$

and

$$\limsup_{n\to\infty} X_n(\omega) \geq \limsup_{i\to\infty} X_{n_i}(\omega) \geq \epsilon.$$

(2), $\forall\omega \in \{\limsup_{n\to\infty} X_n > \epsilon\}$, $\exists$ infinitely many n such that $X_n(\omega) > \epsilon$, then

$$\omega \in \{X_n > \epsilon\} \text{ i.o.} = \limsup_{n\to\infty}\{X_n(\omega) > \epsilon\}$$

$\square$

Remark A.2. *We give a conter-example showing that the converse relationship of (1) does not hold. Let* $X_n = \epsilon - 1/n$. *Then* $\limsup_{n\to\infty} X_n = \epsilon$. *While* $\{X_n \geq \epsilon\} = \varnothing, \forall n \geq 1$ *and then* $\limsup \varnothing = \varnothing$. *We also give a conter-example showing that the converse relationship of (2) does not always hold either. Let* $X_n = \epsilon + 1/n$. *then*

$$\limsup_{n\to\infty} \{X_n > \epsilon\} = \limsup_{n\to\infty} \Omega = \Omega$$

while $\limsup_{n\to\infty} X_n = \epsilon$ *which gives*

$$\{\limsup_{n\to\infty} X_n > \epsilon\} = \varnothing$$

In proof of Theorem 3.4, we also use the following result.

Theorem A.5. *Let* $X_i, i = 1, 2, ...,$ *be independent and identically distributed (i.i.d.) random variables, and* $S_n = \sum_{i=1}^{n} X_i$, *defined as partial sums. If* $\frac{S_n}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$, *then,*

$\lim_{n\to\infty} \min_{1\le k\le n} \mathbb{P}\left(\frac{|S_n - S_k|}{n} < \epsilon\right) = 1$, *for all* $\epsilon > 0$.

Proof. It is equivalent to show that for all $\epsilon, \delta > 0$, $\exists$ N, such that, $\forall n > N$,

$$\min_{1\le k\le n} \mathbb{P}\left(\frac{|S_n - S_k|}{n} < \epsilon\right) > 1 - \delta.$$

Since X_i are i.i.d., we have

$$\min_{1\le k\le n} \mathbb{P}\Big(\frac{|S_n - S_k|}{n} < \epsilon\Big) = \min_{0\le k\le n-1} \mathbb{P}\Big(\frac{|S_k|}{n} < \epsilon\Big) \ge \min_{1\le k\le n} \mathbb{P}\Big(\frac{|S_k|}{n} < \epsilon\Big).$$

Thus, it suffices to show that

$$\min_{1\le k\le n} \mathbb{P}\left(\frac{|S_k|}{n} < \epsilon\right) > 1 - \delta.$$

By convergence in probability we have $N_{\epsilon,\delta}$, such that, for $\forall n > N_{\epsilon,\delta}$,

$$\mathbb{P}\left(\frac{|S_n|}{n} < \epsilon\right) > 1 - \delta.$$

Also $\exists N > N_{\epsilon,\delta}$, such that,

$$\mathbb{P}\left(\frac{|S_k|}{n} < \epsilon\right) > 1 - \delta,$$

for all $n > N, 1 \leq k \leq N_{\epsilon,\delta}$. Now we have $\forall n > N$,

$$\begin{aligned}
\min_{1\leq k\leq n} \mathbb{P}\Big(\frac{|S_k|}{n} < \epsilon\Big) &= \min\Big\{\min_{1\leq k\leq N_{\epsilon,\delta}} \mathbb{P}\big(\frac{|S_k|}{n} < \epsilon\big), \min_{N_{\epsilon,\delta}+1\leq k\leq n} \mathbb{P}\big(\frac{|S_k|}{n} < \epsilon\big)\Big\} \\
&\geq \min\Big\{\min_{1\leq k\leq N_{\epsilon,\delta}} \mathbb{P}\big(\frac{|S_k|}{n} < \epsilon\big), \min_{N_{\epsilon,\delta}+1\leq k\leq n} \mathbb{P}\big(\frac{|S_k|}{k} < \epsilon\big)\Big\} \\
&> 1-\delta,
\end{aligned}$$

and this completes the proof. □

Definition A.3 (Convergence in distribution)**.** *Suppose $X_n, n = 1, 2, ...,$ and X random variables with distribution function F_n and F respectively. We say that X_n converges in distribution to X, if*

$$F_n(x) \to F(x) \quad as \quad n \to \infty,$$

for all $x \in R$ at which F is continuous. We denote $X_n \xrightarrow[n\to\infty]{d} X$.

A.2 On maximum of i.i.d. random variables and probability inequalities

In this section, for a sequence of random variables $\{X_n, n \geq 1\}$, we study the property of $\max_{1\leq k\leq n} S_k$, $S_k = \sum_{i=1}^{k} X_i$ and we will derive some inequalities upon $\max_{1\leq k\leq n} S_k$. This is motivated by the fact that in proving the strong law of large numbers and other related results, we are always confronted with dealing the distribution of $\max_{1\leq k\leq n} S_k$. The inequalities provided ways to bound $\max_{1\leq k\leq n} S_k$ with end point S_n. Besides, we derived symmetrization inequalities and C_r inequality, which will be used in proving Kolmogorov-Marcinkiewicz-Zygmund SLLN, i.e. Theorem A.23.

Theorem A.6. *Let X_i i.i.d random variables and let $S_n = \sum_{i=1}^n X_i$. If* $\frac{S_n}{n^\alpha} \xrightarrow[n\to\infty]{\mathbb{P}} 0$, $\alpha > 0$, *then,* $n\mathbb{P}(|X_1| > n^\alpha) \to 0$.

Proof. The ideas of proof are from Erdös (1949). By the Lemma 3.1 ($m = 1$), it suffices to show that

$$\mathbb{P}\left(\max_{1\le i\le n} |X_i| > n^\alpha\right) \xrightarrow[n\to\infty]{} 0.$$

Note that

$$\Big\{|S_n| > n^\alpha/2\Big\} \supseteq \bigcup_{k=1}^n \left(\Big\{|X_k| > n^\alpha\Big\} \cap \Big\{\Big|\sum_{l=1,l\neq k}^n X_l\Big| < n^\alpha/2\Big\}\right).$$

Indeed, if $\exists k$ such that $|X_k| > n^\alpha$ and $|\sum_{l=1,l\neq k}^n X_k| < n^\alpha/2 \Rightarrow |S_n| > n^\alpha/2$. Let

$$A_k := \Big\{|X_k| > n^\alpha\Big\} \quad \text{and} \quad B_k := \Big\{\Big|\sum_{l=1,l\neq k}^n X_l\Big| < n^\alpha/2\Big\}.$$

Then

$$\mathbb{P}(|S_n| > n^\alpha/2) \ge \mathbb{P}\big(\cup_{k=1}^n (A_k \cap B_k)\big) = \mathbb{P}\big((\cup_{k=1}^n A_k) \cap (\cup_{k=1}^n B_k)\big).$$

Let

$$A'_n := \cup_{k=1}^n A_k \quad \text{and} \quad B'_n = \cup_{k=1}^n B_k,$$

then

$$\mathbb{P}(|S_n| > n^\alpha/2) \ge \mathbb{P}(A'_n \cap B'_n) \xrightarrow[n\to\infty]{} 0.$$

Now since

$$\mathbb{P}(B'_n) = \mathbb{P}\big(\cup_{k=1}^n B_k\big) \geq \mathbb{P}(B_1) = \mathbb{P}(|S_{n-1}| < n^\alpha/2) \xrightarrow[n\to\infty]{} 1,$$

by $S_n/n^\alpha \xrightarrow[n\to\infty]{\mathbb{P}} 0$, that

$$\lim_{n\to\infty} \mathbb{P}(B'_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} \mathbb{P}(B_n^{'c}) = 0.$$

Thus,

$$\mathbb{P}(A'_n \cap B'_n) + \mathbb{P}(B_n^{'c}) \geq \mathbb{P}(A'_n \cap B'_n) + \mathbb{P}(A'_n \cap B_n^{'c}) = \mathbb{P}(A'_n).$$

Then $\mathbb{P}(A'_n) \xrightarrow[n\to\infty]{} 0$ i.e.

$$\mathbb{P}\big(\cup_{k=1}^n \{|X_k| > n^\alpha\}\big) = \mathbb{P}\left(\max_{1\leq k\leq n} |X_k| > n^\alpha\right) \xrightarrow[n\to\infty]{} 0.$$

□

Definition A.4. *x is quantile of order q for a random variable X if and only if $\mathbb{P}(X < x) \leq q$ or equivalently $\mathbb{P}(X \geq x) \geq 1 - q$ and $\mathbb{P}(X \leq x) \geq q$ or equivalently $\mathbb{P}(X > x) \leq 1 - q$. x is often denoted by $\mathcal{K}_q(X)$. In case where $q = \frac{1}{2}$, we denote $K_{1/2}(X)$ as $m(X)$.*

Remark A.3. *For continuous random variable, there exists a unique x such that $F(x) = q$. But for discrete random variable, the situation is different. For example, we let $\mathbb{P}(X = 1/2) = \mathbb{P}(X = 1) = 1/2$. Then any $x \in [\frac{1}{2}, 1]$ is quantile of order $1/2$ for random variable X.*

Theorem A.7 (Symmetrization inequalities). *Let $X^s := X - X'$, where X' is an identically independent copy of X. For every ϵ and a*

$$\begin{aligned}
&(1) \quad \frac{1}{2}\mathbb{P}(X - m(X) \geq \epsilon) \leq \mathbb{P}(X^s \geq \epsilon),\\
&(2) \quad \frac{1}{2}\mathbb{P}(|X - m(X)| \geq \epsilon) \leq \mathbb{P}(|X^s| \geq \epsilon) \leq 2\mathbb{P}(|X - a| \geq \epsilon/2).
\end{aligned}$$

Proof. We reproduce the proof given in Loève (1977). Note that $\mu := m(X) = m(X')$, since X' and X are independent and identically distributed. Then

$$\begin{aligned}
&\mathbb{P}(X^s \geq \epsilon) = \mathbb{P}((X - \mu) - (X' - \mu) \geq \epsilon) \geq \mathbb{P}(X - \mu \geq \epsilon, X' - \mu \leq 0)\\
&= \mathbb{P}(X - \mu \geq \epsilon) \cdot \mathbb{P}(X' - \mu \leq 0) \geq 1/2\mathbb{P}(X - m(X) \geq \epsilon),
\end{aligned}$$

which finishes the proof of (1). Let X replaced by $-X$, we have

$$\mathbb{P}(-X \geq -m(X)) \geq 1/2 \leq \mathbb{P}(-X \leq -m(X)).$$

Then, by (1) and $m(-X) = -m(X)$,

$$\mathbb{P}(-X^s \geq \epsilon) \leq 1/2\mathbb{P}(-X + m(X) \geq \epsilon). \tag{A.3}$$

From part (1) and (A.3) follows first inequality of (2). Further,

$$\begin{aligned}
&\mathbb{P}(|X^s| \geq \epsilon) = \mathbb{P}(|(X - a) - (X' - a)| \geq \epsilon)\\
&\leq \mathbb{P}(|X - a| \geq \epsilon/2) + \mathbb{P}(|X' - a| \geq \epsilon/2) = 2\mathbb{P}(|X - a| \geq \epsilon/2)
\end{aligned}$$

which is the second inequality of part (2). □

Theorem A.8 (Classic C_r - inequality in L_r space)**.** *X, Y random variables, then*

$$\mathbb{E}(|X+Y|^r) \leq C_r\big(\mathbb{E}|X|^r + \mathbb{E}|Y|^r\big)$$

where $C_r = 1$ if $0 < r \leq 1$ and $C_r = 2^{r-1}$ if $r > 1$.

Proof. The proof of this result is given in Loève (1977). □

Theorem A.9. *For $r > 0$ and every a,*

$$1/2 \cdot \mathbb{E}|X - m(X)|^r \leq E|X^s|^r \leq 2C_r \cdot \mathbb{E}|X - a|^r$$

where $C_r = 1$ or 2^{r-1} according as $r < 1$ or $r \geq 1$.

Proof. This result is established in Loève (1977). For completeness, we reproduce the similar proof here. The second inequality follows from the C_r inequality,

$$\mathbb{E}(|X^s|^r) = \mathbb{E}|(X-a) - (X'-a)|^r \leq C_r\mathbb{E}|X-a|^r + C_r\mathbb{E}|X'-a|^r = 2C_r\mathbb{E}|X-a|^r.$$

As for the first inequality. It is trivial when $\mathbb{E}|X^s|^r = \infty$, as to the second inequality just proved (with $a = m(X)$), $E|X - m(X)|^r = \infty$. Thus we can assume that $\mathbb{E}|X^s|^r < \infty$. Let

$$q(t) = \mathbb{P}(|X - m(X)| > t) \text{ and } q^s(t) = \mathbb{P}(|X^s| > t).$$

So that by part (2) of Theorem A.7,

$$q(t) \leq 2q^s(t).$$

Note that

$$\int_0^\infty t^r \, \mathrm{d}q(t) = t^r q(t)\Big|_0^\infty - \int_0^\infty q(t) \, \mathrm{d}(t^r) = -\int_0^\infty q(t) \, \mathrm{d}(t^r)$$

since $t^r q(t) \leq 1/2 \cdot t^r q^s(t) \to 0$ as $t \to \infty$, by $\mathbb{E}|X^s|^r < \infty$. Then it follows by integration by parts, that

$$\begin{aligned}\mathbb{E}|X - m(X)|^r &= -\int_0^\infty t^r \, \mathrm{d}q(t) = \int_0^\infty q(t) \, \mathrm{d}(t^r) \leq 2\int_0^\infty q^s(t) \, \mathrm{d}(t^r) \\ &= -2\int_0^\infty t^r \, \mathrm{d}q^s(t) = 2\mathbb{E}|X^s|^r\end{aligned}$$

and the proof is complete. □

Theorem A.10. *For a random variable X, then for $r > 0$ and $a \in R$, then $\mathbb{E}|X - a|^r < \infty$ if and only if $\mathbb{E}|X|^r < \infty$.*

The proof is established in Loève (1977).

Theorem A.11. *If a sequence of random variables $\{X_n, n = 1, 2, ...,\}$, such that, $X_n \xrightarrow[n\to\infty]{\mathbb{P}} 0$. Then the median also converges to 0, i.e. $m(X_n) \xrightarrow[n\to\infty]{} 0$.*

The proof is established in Loève (1977). Below, we recall some inequalities established in Petrov (1995), which are useful in proving Theorem 3.4.

Theorem A.12. *For every q from interval $0 < q < 1$ and for every real x,*

$$\mathbb{P}\big(\max_{1\leq k\leq n}\{S_k - \mathcal{K}_q(S_k - S_n)\} \geq x\big) \leq \frac{1}{q}\mathbb{P}(S_n \geq x).$$

Proof. As in Petrov (1995), let

$$\bar{S}_k = \max_{1\le l\le k}\{S_l - \mathcal{K}_q(S_l - S_n)\} \quad (k = 1, ..., n)$$
$$D_1 = \{S_1 - \mathcal{K}_q(S_1 - S_n) \ge x\}$$
$$D_k = \{\bar{S}_{k-1} < x, S_k - \mathcal{K}_q(S_k - S_n) \ge x\} \quad (k = 2, ..., n)$$
$$E_k = \{S_n - S_k - \mathcal{K}_{1-q}(S_n - S_k) \ge 0\}.$$

Note that $\{\bar{S_n} \ge x\} = \bigcup_{k=1}^n D_k$ and D_k are pairwise disjoint. Thus, we have

$$\mathbb{P}(\bar{S_n} \ge x) = \sum_{k=1}^{n} \mathbb{P}(D_k). \tag{A.4}$$

Further,

$$\mathbb{P}\big(S_n - S_k \ge \mathcal{K}_{1-q}(S_n - S_k)\big) \ge 1 - (1-q) = q,$$

i.e.,

$$\mathbb{P}(E_k) \ge q. \tag{A.5}$$

Note that if $\mathcal{K}_q(X)$ is a quantile of order q for X, then $-\mathcal{K}_q$ is a quantile of order $1-q$ for random variable $-X$. Indeed, by definition

$$\mathbb{P}(X < \mathcal{K}_q(X)) \le q \quad \text{and} \quad \mathbb{P}(X \le \mathcal{K}_q(X)) \ge q.$$

Then, we have

$$\mathbb{P}(-X > -\mathcal{K}_q(X)) \le q \quad \text{and} \quad \mathbb{P}(-X \ge -\mathcal{K}_q(X)) \ge q,$$

this proves that $-\mathcal{K}_q$ is a quantile of order $1-q$. Using this property, we have

$$\mathcal{K}_{1-q}\big(S_n - S_k\big) = \mathcal{K}_{1-q}\big(-(S_k - S_n)\big) = \mathcal{K}_q(S_k - S_n).$$

By the definition of D_k and E_k, it implies

$$D_k \cap E_k \subseteq \{\bar{S}_{k-1} < x, S_n \geq x\},$$

then, we have

$$\bigcup_{k=1}^{n}(D_k \cap E_k) \subseteq \{S_n \geq x\},$$

and then, by (A.4) and (A.5), and the fact that D_k and E_k are independent,

$$\mathbb{P}(S_n \geq x) \geq \mathbb{P}(\bigcup_{k=1}^{n} D_k E_k) = \sum_{k=1}^{n}\mathbb{P}(D_k E_k) = \sum_{k=1}^{n}\mathbb{P}(D_k)\mathbb{P}(E_k) \geq q\sum_{k=1}^{n}\mathbb{P}(D_k).$$

Finally, by using (A.5), we have

$$\mathbb{P}(S_n \geq x) \geq q\sum_{k=1}^{n}\mathbb{P}(D_k) = q\cdot\mathbb{P}(\bar{S}_n \geq x) = q\cdot\mathbb{P}\big(\max_{1\leq k\leq n}\{S_k - \mathcal{K}_q(S_k - S_n)\} \geq x\big),$$

this completes the proof. □

From Theorem A.12, one establishes the inequality known as Lévy inequality. This is an inequality for the distribution of the maximum of sums of independent random variables, centred around the corresponding medians.

Theorem A.13 (Lévy inequality). *Let $X_1, ..., X_n$ be independent random variables, let $S_k = \sum_{i=1}^{k} X_i$ and let $m(X)$ be the median of random variable X; then for any x*

one has

$$(a) \quad \mathbb{P}\Big(\max_{1\le k\le n}\big(S_k - m(S_k - S_n)\big) \ge x\Big) \le 2\mathbb{P}(S_n \ge x),$$

$$(b) \quad \mathbb{P}\Big(\max_{1\le k\le n}\big|S_k - m(S_k - S_n)\big| \ge x\Big) \le 2\mathbb{P}(|S_n| \ge x).$$

Proof. This is an immediate result of Theorem A.12, by letting $q = 1/2$. □

Corollary A.1. *For symmetrically-distributed and independent random variables $X_1, ...X_n$, we have*

$$(a) \quad \mathbb{P}\left(\max_{1\le k\le n} S_k \ge x\right) \le 2\mathbb{P}(S_n \ge x),$$

$$(b) \quad \mathbb{P}\left(\max_{1\le k\le n} |S_k| \ge x\right) \le 2\mathbb{P}(|S_n| \ge x).$$

Proof. It is an immediate result of Theorem A.13. Indeed, for symmetrically-distributed random variable $m(X) = 0$. □

By using Theorem A.12, one establishes also the following theorem which is useful in deriving Theorem 3.4.

Theorem A.14. *Suppose that the conditions of Theorem A.12 hold and let $C \ge 0$, $\mathcal{K}_q(S_1 - S_n), ..., \mathcal{K}_q(S_{n-1} - S_n)$, $(0 < q < 1)$, such that, $\mathcal{K}_q(S_k - S_n) \le C$, for $k = 1, ..., n-1$. Then,*

$$\mathbb{P}(\max_{1\le k\le n} S_k \ge x) \le \frac{1}{q}\mathbb{P}(S_n \ge x - C).$$

Proof. As in Petrov (1995), the proof relies on Theorem A.12. We have

$$C \ge \mathcal{K}_q(S_k - S_n) \quad \forall k = 1, ..., n,$$

then

$$C \geq \max_{1\leq k\leq n} \mathcal{K}_q(S_k - S_n) \quad \text{and} -C \leq -\min_{1\leq k\leq n} \mathcal{K}_q(S_k - S_n),$$

and then

$$\max_{1\leq k\leq n} S_n - C \leq \max_{1\leq k\leq n} S_n - \min_{1\leq k\leq n} \mathcal{K}_q(S_k - S_n) = \max_{1\leq k\leq n} \big\{S_n - \mathcal{K}_q(S_k - S_n)\big\}.$$

Therefore

$$\left\{\max_{1\leq k\leq n} S_n \geq x + C\right\} \subseteq \left\{\max_{1\leq k\leq n} \big\{S_n - \mathcal{K}_q(S_k - S_n)\big\} \geq x\right\},$$

and then

$$\left\{\max_{1\leq k\leq n} S_n \geq x\right\} \subseteq \left\{\max_{1\leq k\leq n} \big\{S_n - \mathcal{K}_q(S_k - S_n)\big\} \geq x - C\right\}.$$

Hence, the proof follows by Theorem A.12. □

Theorem A.12 and Theorem A.14 are constructed to get Theorem A.15, which is used in the proof of Theorem 3.4.

Theorem A.15. *Let $X_1, X_2, ..., X_n$ be i.i.d. r.v. with $S_k = \sum_{i=1}^k X_i$, $\mathcal{K}_q(X)$ denote the quantile of order q, $q > 0$. If $\mathbb{P}(S_n - S_k \geq -C) \geq q$, $k = 1, ..., n-1$. for some constants $C \geq 0$,then $\mathbb{P}(\max_{1\leq k\leq n} S_k \geq x) \leq \frac{1}{q}\mathbb{P}(S_n \geq x - C)$, for all $x \in \mathbb{R}$.*

Proof. The proof of this theorem is given in Petrov (1995). □

A.3 Law of large numbers

In this section, we present those distinctive results and the derivation of L^1 version strong law of large numbers as well as Kolmogorov-Marcinkiewicz-Zygmund SLLN. Law of large numbers plays an important role in the development of probability theory as well as in mathematical statistics. Thus, this topic attracts many probabilists and statisticians. Now, let L^p, $p > 0$, space be a space containing all random variables, such that, $\mathbb{E}(|X|^p) < \infty$.

Theorem A.16 (Kolmogorov's inequality)**.** *Let $X_1, ..., X_n$ be independent r.v. with mean 0 and variance $\sigma_1^2, ..., \sigma_n^2$, and let $S_k = X_1 + \cdots + X_k$ as partial sums. Then $\forall \epsilon > 0$,*

$$\mathbb{P}\Big(\max_{1 \le k \le n} |S_k| \ge \epsilon \Big) \le \epsilon^{-2} \sum_{1}^{n} \sigma_k^2.$$

The proof of this theorem is given in Folland (1999). By using Kolmogorov's inequality, one establishes the following result.

Theorem A.17 (Kolmogorov's Strong Law of Large Numbers)**.** *If $X_n, n = 1, 2, ...,$ is a sequence of independent L^2 random variables with means $\{\mu_n\}_{n=1}^{\infty}$ and variance $\{\sigma_n^2\}_{n=1}^{\infty}$, such that, $\sum_{n=1}^{\infty} n^{-2}\sigma_n^2 < \infty$, then $n^{-1}\sum_{j=1}^{n}(X_j - \mu_j) \xrightarrow[n\to\infty]{a.s.} 0$.*

Proof. As in Folland (1999), let $S_n = \sum_{j=1}^{n}(X_j - \mu_j)$. Given $\epsilon > 0$ for $k \in \mathbb{N}$. Let

$$A_k = \bigcup_{2^{k-1} \le n < 2^k} \big\{ \omega : n^{-1}|S_n(\omega)| > \epsilon \big\}.$$

Then $\forall \omega \in A_k$, $\exists\, 2^{k-1} \le n_0 < 2^k$ such that

$$n_0^{-1}|S_{n_0}(\omega)| > \epsilon,$$

thus

$$|S_{n_0}(\omega)| > \epsilon 2^{k-1}.$$

We have

$$\begin{aligned}\mathbb{P}(A_k) &= \mathbb{P}\left(\bigcup_{2^{k-1}\leq n<2^k}\{\omega : n^{-1}|S_n(\omega)| > \epsilon\}\right)\\ &= \mathbb{P}\left(\max_{2^{k-1}\leq n<2^k} n^{-1}|S_n| > \epsilon\right) \leq \mathbb{P}\left(\max_{2^{k-1}\leq n<2^k}|S_n| > \epsilon 2^{k-1}\right)\end{aligned}$$

By Kolmogorov's inequality,

$$\mathbb{P}(A_k) \leq \mathbb{P}\left(\max_{1\leq n\leq 2^k}|S_n| > \epsilon 2^{k-1}\right) \leq (\epsilon\cdot 2^{k-1})^{-2}\sum_{k=1}^{2^k}\sigma_n^2.$$

Thus

$$\begin{aligned}\sum_{k=1}^{\infty}\mathbb{P}(A_k) &\leq \frac{4}{\epsilon}\sum_{k=1}^{\infty}\sum_{n=1}^{2^k}\sigma_n^2 2^{-2k} = \frac{4}{\epsilon}\sum_{n=1}^{\infty}\big(\sum_{k\geq[\log_2^n]}2^{-2k}\big)\sigma_n^2\\ &\leq \frac{4}{\epsilon}\sum_{n=1}^{\infty}n^{-2}\frac{4}{3}\sigma_n^2 \leq \frac{16}{3\epsilon}\sum_{n=1}^{\infty}n^{-2}\sigma_n^2 < \infty,\end{aligned}$$

and then, by Borel-Cantelli lemma, $\mathbb{P}(A_k \text{ i.o.}) = 0$. Thus,

$$\limsup_{k\to\infty}\max_{2^{k-1}\leq n<2^k} n^{-1}|S_n| \leq \epsilon \quad \text{a.s.}$$

i.e. $\limsup_{n\to\infty} n^{-1}|S_n| \leq \epsilon \quad$ a.s. Hence, letting $\epsilon \downarrow 0$, we have

$$\lim_{n\to\infty}\frac{S_n}{n} = 0 \quad \text{a.s.}$$

□

Theorem A.17 plays an important role in proof of the following result known as Kinchine's Strong Law of Large Numbers. The Law of Large Numbers has its deep root in history of probability theory and the rigorous strong law was finally been proved by Khinchine and Kolmogorov. The proof below is from Folland (1999), which is the shortest version among all kinds of references available.

Theorem A.18 (Khinchine's Strong Law of Large Numbers, Classic Strong Law of Large Numbers)**.** *If X_n is a sequence of independent identically distributed(i.i.d.) L^1 random variables with mean μ, then $n^{-1}\sum_{j=1}^{n} X_j \xrightarrow[n\to\infty]{a.s.} \mu$.*

Proof. Without loss of generality, let $\mu = 0$. Let λ be the probability measure induced by X_j's, such that, $\lambda((-\infty, a]) := F(a) = \mathbb{P}(X_1 \leq a)$, then

$$\int |t| \, \mathrm{d}F(t) < \infty, \quad \int t \, \mathrm{d}F(t) = 0.$$

Let $Y_j(\omega) = X_j(\omega) \cdot \mathbb{I}_{\{X_j(\omega) \leq j\}}$. Then, define by $\lambda((a, b]) = F(b) - F(a)$, and $\lambda(a, \infty) = 1 - F(a)$, we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} \mathbb{P}(Y_j \neq X_j) = \sum_{j=1}^{\infty} \mathbb{P}(|X_j| > j) = \sum_{j=1}^{\infty} \lambda(\{t : |t| > j\}) \\
&= \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \lambda(\{t : k < |t| \leq k+1\}) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} \lambda(\{t : k < |t| \leq k+1\}) \\
&= \sum_{k=1}^{\infty} k \cdot \lambda(\{t : k < |t| \leq k+1\}) \leq \int |t| \, \mathrm{d}F(t) < \infty.
\end{aligned}$$

By Borel-Cantelli Lemma,

$$\mathbb{P}(Y_j \neq X_j \text{ i.o.}) = \mathbb{P}\left(\bigcap_{j\geq 1}\bigcup_{k\geq j}\{Y_k \neq X_k\}\right) = 0,$$

i.e. $\mathbb{P}\left(\bigcup_{j\geq 1}\bigcap_{k\geq j}(Y_k = X_k)\right) = 1$. For all $\omega \in \cup_{j\geq 1}\cap_{k\geq j}(Y_k = X_k)$, $\exists j_\omega$, such that, $\omega \in \bigcap_{k\geq j_\omega}(Y_k = X_k)$, thus

$$\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} Y_j(\omega) = \lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} X_j(\omega).$$

Thus, we have

$$n^{-1}\sum_{j=1}^{n} Y_j - n^{-1}\sum_{j=1}^{n} X_j \xrightarrow[n\to\infty]{\text{a.s.}} 0.$$

From the above argument, it suffices to show

$$n^{-1}\sum_{j=1}^{n} Y_j \xrightarrow[n\to\infty]{\text{a.s.}} 0.$$

We have

$$\sigma^2(Y_n) \leq \mathbb{E}(Y_n^2) = \int_{|t|\leq n} t^2 \,\mathrm{d}F(t) < \infty \quad \forall n,$$

then,

$$\sum_{n=1}^{\infty} n^{-2}\sigma^2(Y_n) \leq \sum_{n=1}^{\infty}\sum_{j=1}^{n} n^{-2}\int_{j-1<|t|\leq j} t^2 \,\mathrm{d}\lambda(t) \leq \sum_{n=1}^{\infty}\sum_{j=1}^{n} jn^{-2}\int_{j-1<|t|\leq j} |t| \,\mathrm{d}\lambda(t),$$

and then,

$$\sum_{n=1}^{\infty} n^{-2}\sigma^2(Y_n) \leq \sum_{j=1}^{\infty}\sum_{n=j}^{\infty} jn^{-2}\int_{j-1<|t|\leq j} |t|\,\mathrm{d}\lambda(t) = \sum_{j=1}^{\infty}\left(j\int_{j-1<|t|\leq j}|t|\,\mathrm{d}\lambda(t)\cdot\sum_{n=j}^{\infty} n^{-2}\right)$$
$$\leq \sum_{j=1}^{\infty}\left(j\int_{j-1<|t|\leq j}|t|\,\mathrm{d}\lambda(t)\cdot 2j^{-1}\right) = 2\sum_{j=1}^{\infty}\left(\int_{j-1<|t|\leq j}|t|\,\mathrm{d}\lambda(t)\right) = 2\mathbb{E}(|X_1|) < \infty.$$

Let $\mu_j = \mathbb{E}(Y_j)$, then by Kolmogorov's Strong Law of Large Number, we have $n^{-1}\sum_{j=1}^{n}(Y_j - \mu_j) \xrightarrow[n\to\infty]{\text{a.s.}} 0$. Further, by Lebesgue Dominated Convergence theorem,

$$\mu_j = \mathbb{E}\left[X_1 \cdot \mathbb{I}_{\{|X_1|\leq j\}}\right] \xrightarrow[j\to\infty]{} \mathbb{E}(X_1) = 0 \Rightarrow n^{-1}\sum_{j=1}^{n}\mu_j \xrightarrow[n\to\infty]{} 0.$$

Finally, we conclude that

$$n^{-1}\sum_{j=1}^{n} Y_j \xrightarrow[n\to\infty]{a.s.} 0.$$

□

By Theorem A.18, we immediately have the following corollary.

Corollary A.2 (Weak Law of Large Number, L^2 version)**.** *Let $\{X_j\}_1^\infty$ be a sequence of independent L^2($\mathbb{E}(X_1)^2 < \infty$) random variables with means μ_j and variance σ_j^2. If*

$$\frac{\sum_{j=1}^{n}\sigma_j^2}{n^2} \to 0$$

then

$$\frac{1}{n}\sum_{j=1}^{n}(X_j - \mu_j) \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Below, we present the Khinchine-Kolmogorov Convergence theorem and Lemma A.1 which are useful in deriving the Kolmogorov Three Series theorem. This result plays an important role in proving Marcinkiewicz-Zygmund Convergence theorem.

Theorem A.19 (Khinchine-Kolmogorov Convergence theorem)**.** *Suppose* $X_1, X_2, \ldots$ *are independent with mean* 0*, such that,* $\sum_{n=1}^{\infty} Var(X_n) < \infty$*. Then* $\sum_{n=1}^{\infty} X_n < \infty$ *a.s.*

The proof of this theorem is given in Durrett (2005). Below, we recall a lemma from Durrett (2005), which shows that a sequence of uniformly bounded random variables could have a good property.

Lemma A.1. *If* $\sum_{n=1}^{\infty} Z_n$ *converges for a series of independent random variables* Z_n*, which have mean* 0*, variance* σ_n^2*, and are uniformly bounded by* $C > 0$*, i.e.*$|Z_n| < C, \forall n$*, then* $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$*.*

Proof. Let $S_n := \sum_{i=1}^{n} Z_i$. Fix $L > 0$ and let $\tau_L := \min\{i \geq 0 : |S_i| \geq L\}$. Since τ_L is the smallest number such that $|S_{\tau_L}| \geq L$, which means that $|S_{\tau_L - 1}| < L$, then we have $|S_{\tau_L}| < |S_{\tau_L - 1} + Z_{\tau_L}| \leq C + L$. By our assumptions, $\mathbb{P}(\{\tau_L \geq n, \forall n\}) \uparrow 1$ as $L \uparrow \infty$. Next, we observe that

$$\mathbb{E}[S_n^2] = \sum_{i=1}^{n} \mathbb{E}[Z_i^2] + 2 \sum_{1 \leq i < j \leq n} \mathbb{E}[Z_i Z_j] = \sum_{i=1}^{n} \sigma_i^2,$$

Then by defining $n \wedge \tau_L = \min\{n, \tau_L\}$

$$\begin{aligned}
\mathbb{E}[S_{n \wedge \tau_L}^2] &= \mathbb{E}\big[(\sum_{j=1}^{n} Z_j \mathbb{I}_{\{j \leq \tau_L\}})^2\big] \\
&= \sum_{j=1}^{n} \mathbb{E}[Z_j^2 \mathbb{I}_{\{j \leq \tau_L\}}] + 2 \sum_{1 \leq i < j \leq n} \mathbb{E}[Z_i Z_j \mathbb{I}_{\{j \leq \tau_L\}}].
\end{aligned}$$

Since Z_j is independent of $\mathbb{I}_{\{\tau_L \geq j\}} = \mathbb{I}_{\{\tau_L > j-1\}}$, we have

$$\begin{aligned}(L+C)^2 \geq \mathbb{E}[S_{n\wedge\tau_L}^2] &= \sum_{j=1}^n \sigma_j^2 \mathbb{P}(j \leq \tau_L) + 2 \sum_{i \leq i < j \leq n} \mathbb{E}[Z_i Z_j \mathbb{I}_{\{j \leq \tau_L\}}] \\ &\geq \mathbb{P}(\tau_L = \infty) \cdot \sum_{j=1}^n \sigma_j^2.\end{aligned}$$

Choose L sufficiently large that $\mathbb{P}(\tau_L = \infty) > 0$, we reach the result. □

Kolmogorov Three Series Theorem presented below is a very important theorem and has its wide applications in modern probability theory. It is used in proving Marcinkiewicz-Zygmund Convergence theorem.

Theorem A.20 (Kolmogorov Three Series Theorem)**.** *Suppose* $X_1, X_2, ...,$ *independent random variables, Let* $Y_n = X_n \mathbb{I}_{(|X_n| \leq 1)}$. *Then* $\sum_n X_n < \infty$ *a.s. if and only if*

$$\begin{aligned}&(i) \quad \sum_{n=1}^\infty \mathbb{P}(|X_n| > 1) < \infty; \\ &(ii) \quad \sum_{n=1}^\infty \mathbb{E}(Y_n) < \infty; \\ &(iii) \quad \sum_{n=1}^\infty Var(Y_n) < \infty.\end{aligned}$$

Proof. We first prove the if part of this theorem, $\sum_n \mathrm{Var}(Y_n) = \sum_n \mathrm{Var}(Y_n - \mathbb{E}Y_n) < \infty$. By TheoremA.19, $\sum_n (Y_n - \mathbb{E}(Y_n)) < \infty$ a.s. By (ii) we have $\sum_n Y_n < \infty$ a.s. On the other hand,

$$\sum_{n=1}^\infty \mathbb{P}(X_n \neq Y_n) = \sum_{n=1}^\infty \mathbb{P}(|X_n| > 1) < \infty,$$

the inequality is by (i). Then, we have by Borel-Cantelli lemma,

$$\mathbb{P}(X_n \neq Y_n \text{ i.o.}) = \mathbb{P}(\cap_{n\geq 1} \cup_{k\geq n} \{X_k \neq Y_k\}) = 0.$$

By the similar argument in proof of TheoremA.18, we have

$$\sum_n X_n < \infty \quad \text{a.s.}$$

The only if part is done as follows, (i) is verified by using the Borel-Cantelli lemma, i.e., if

$$\sum_n \mathbb{P}(|X_n| > 1) = \infty$$

then by Borel-Cantelli lemma, $\mathbb{P}(|X_n| > 1 \text{ i.o.}) = 1$, which contradicts the fact that $\sum_n X_n$ converges a.s. (ii) could be derived from (iii), i.e. if (iii) is true, then by Theorem A.19, that if $\sum_n \text{Var}(Y_n - \mathbb{E}(Y_n)) = \sum_n \text{Var}(Y_n) < \infty$, we have $\sum_n (Y_n - \mathbb{E}(Y_n)) < \infty$ a.s. By the convergence of $\sum_n Y_n$, (due to convergence of $\sum_n X_n$) we have $\sum_n \mathbb{E}(Y_n) < \infty$.

So left to prove (iii). Let $Z_n = Y_n - Y_n'$, where $\{Y_n'\}$ is an independent copy of $\{Y_n\}$, then Z_n i.i.d. with mean 0 and uniformly bounded by 2. By Lemma A.1, $\sum_n \text{Var}(Z_n) < \infty$, i.e. $2\sum_n \text{Var}(Y_n) < \infty$. Now the proof is complete. □

Remark A.4. *It should be noticed that if X_n is truncated at any constant $\epsilon > 0$ rather than 1, Theorem A.20 still holds.*

Below, we present a lemma and Marcinkiewicz-Zygmund convergence theorem, which are useful in deriving Kolmogorov-Marcinkiewicz-Zygmund strong law of large numbers. All these three results are given in Durrett (2005).

Lemma A.2. *If* $Y \geq 0$ *and* $p > 0$, *then*

$$\mathbb{E}(Y^p) = \int_0^\infty p y^{p-1} \mathbb{P}(Y > y)\, dy$$

The proof of this lemma is given in Durrett (2005). As in Durrett (2005), by using Kolmogorov Three Series theorem, one establishes the following result, known as Marcinkiewicz-Zygmund Convergence theorem.

Theorem A.21 (Marcinkiewicz-Zygmund Convergence theorem)**.** *Suppose that* $X_1, X_2, \ldots$ *are i.i.d. random variables with* $\mathbb{E}|X_1|^p < \infty$, *then*

$$\sum_{n=1}^{\infty} \frac{X_n}{n^{1/p}} < \infty \quad a.s. \quad \text{for } 0 < p < 1.$$

Proof. Let $Z_n = \frac{X_n}{n^{1/p}}$, then $Z_1, Z_2, \ldots$ are independent r.v. Let $Y_n = Z_n \mathbb{I}_{\{|Z_n| \leq 1\}}$. We have

$$\sum_{n=1}^{\infty} \mathbb{P}(|Z_n| > 1) = \sum_{n=1}^{\infty} \mathbb{P}\left(\frac{|X_n|}{n^{1/p}} > 1\right) = \sum_{n=1}^{\infty} \mathbb{P}(|X_n|^p > n)$$
$$= \sum_{n=1}^{\infty} \mathbb{P}(|X_1|^p > n) \leq \mathbb{E}(|X_1|^p) < \infty,$$

the condition (i) of Theorem A.20 is verified. Then let us verify the second condition of Theorem A.20,

$$\mathbb{E}(Z_n \mathbb{I}_{|Z_n| \leq 1}) = \mathbb{E}\left(\frac{X_n}{n^{1/p}} \mathbb{I}_{\{\frac{|X_n|}{n^{1/p}} \leq 1\}}\right) = n^{-1/p} \mathbb{E}(X_n \mathbb{I}_{\{|X_n|^p \leq n\}})$$
$$= n^{-1/p} \int_{|X_n|^p \leq n} X_n \, \mathrm{d}\mathbb{P} \leq n^{-1/p} \int_{|X_n|^p \leq n} |X_n| \, \mathrm{d}\mathbb{P} = n^{-1/p} \sum_{j=1}^{n} \int_{j-1 < |X_n|^p \leq j} |X_n| \, \mathrm{d}\mathbb{P}$$

in the set $\{j-1 < |X_n| \le j\}$, $|X_n| = |X_n|^p |X_n|^{1-p} \le |X_n|^p j^{1-p}$, $0 < p < 1$. Thus, we have

$$\mathbb{E}(Z_n \mathbb{I}_{\{|Z_n| \le 1\}}) \le n^{-1/p} \sum_{j=1}^{n} \int_{j-1<|X_n^p \le j} |X_n|^p j^{1-p} \, \mathrm{d}\mathbb{P},$$

then

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{E}(Z_n \mathbb{I}_{\{|Z_n| \le 1\}}) &\le \sum_{n=1}^{\infty} n^{-1/p} \sum_{j=1}^{n} \int_{j-1<|X_n|^p \le j} |X_n|^p j^{1-p} \, \mathrm{d}\mathbb{P} \\ &= \sum_{j=1}^{\infty} \left(\int_{j-1<|X_1|^p \le j} |X_1|^p j^{1-p} \, \mathrm{d}\mathbb{P} \cdot \sum_{n=j}^{\infty} n^{-1/p} \right)\end{aligned}$$

Notice that

$$\begin{aligned}&\sum_{n=j}^{\infty} n^{-1/p} \cdot j^{1-p} \le \int_{j-1}^{\infty} x^{-1/p} \, \mathrm{d}x \cdot j^{1-p} \\ &= \frac{1}{1/p-1}(j-1)^{1-1/p} j^{1-p} = \frac{1}{1/p-1} \left(\frac{j-1}{j} \right)^{1-1/p} \cdot j^{2-1/p-p},\end{aligned}$$

and this is uniformly bounded with $j \ge 2$. Let the above bounded by C. Then,

$$\sum_{n=1}^{\infty} \mathbb{E}\big(Z_n \mathbb{I}_{\{|Z_n| \le 1\}}\big) \le C \cdot \sum_{j=1}^{\infty} \int_{j-1<|X_1|^p \le j} |X_1|^p \, \mathrm{d}\mathbb{P} = C \cdot \mathbb{E}(|X_1|^p) < \infty.$$

Then, let us verify the condition (iii) of Theorem A.20. We have

$$\begin{aligned}&\mathrm{Var}(Y_n) = \mathrm{Var}(Z_n \mathbb{I}_{\{|Z_n| \le 1\}}) = \mathrm{Var}\left(\frac{X_n}{n^{1/p}} \mathbb{I}_{\{\frac{|X_n|}{n^{1/p}} \le 1\}} \right) = n^{-2/p} \mathrm{Var}(X_n \mathbb{I}_{\{|X_n|^p \le n\}}) \\ &\le n^{-2/p} \mathbb{E}(X_n^2 \mathbb{I}_{\{|X_n|^p \le n\}}) = n^{-2/p} \sum_{j=1}^{n} \int_{j-1<|X_1|^p \le j} |X_1|^2 \, \mathrm{d}\mathbb{P}\end{aligned}$$

then

$$\begin{aligned}\sum_{n=1}^{\infty}\operatorname{Var}(Y_n) &= \sum_{n=1}^{\infty} n^{-2/p}\sum_{j=1}^{n}\int_{j-1<|X_1|^p\le j} X_1^2 \,\mathrm{d}\mathbb{P}\\ &= \sum_{n=1}^{\infty} n^{-2/p}\sum_{j=1}^{n}\int_{j-1<|X_1|^p\le j} |X_1|^p|X_1|^{2-p} \,\mathrm{d}\mathbb{P}\\ &\le \sum_{n=1}^{\infty} n^{-2/p}\sum_{j=1}^{n}\int_{j-1<|X_1|^p\le j} |X_1|^p j^{2-p} \,\mathrm{d}\mathbb{P}\\ &= \sum_{j=1}^{\infty}\left(\int_{j-1<|X_1|^p\le j} |X_1|^p \,\mathrm{d}\mathbb{P}\sum_{n=j}^{\infty} n^{-2/p}\right)\cdot j^{2-p}\end{aligned}$$

Now consider

$$\sum_{n=j}^{\infty} n^{-2/p}\cdot j^{2-p} \le \frac{1}{2/p-1}(j-1)^{1-2/p} j^{2-p},$$

which is uniformly bounded for $j \ge 2$, if $0 < p < 1$. Let C be the upper bound. Thus,

$$\sum_{n=1}^{\infty}\operatorname{Var}(Y_n) \le C\sum_{j=1}^{\infty}\int_{j-1<|X_1|^p\le j} |X_1|^p \,\mathrm{d}\mathbb{P} \le C\cdot\mathbb{E}(|X_1|^p) < \infty.$$

this verifies the condition (iii) of Theorem A.20. Then, by Kolmogorov Three Series theorem,

$$\sum_{n=1}^{\infty} Z_n < \infty \quad \text{a.s., i.e.,} \quad \sum_{n=1}^{\infty}\frac{X_n}{n^{1/p}} < \infty \quad \text{a.s.}$$

□

To conclude this section, we present the crowning theorem on law of large numbers, also known as the Kolmogorov-Marcinkiewicz-Zygmund strong law of large numbers, which generalises the strong law of large numbers to a more profound essence. The proof of this theorem is given in Durrett (2005). For the convenience of the reader, we also present the proof have.

Theorem A.22 (Kronecker's lemma)**.** *If $x_n, n = 1, 2, ...$ is an infinite sequence of real numbers, such that $\sum_{m=1}^{\infty} x_m = s$ exists and is finite, then we have for all $0 < b_1 \leq b_2 \leq ...$ and $b_n \to \infty$, that*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} b_k x_k = 0.$$

Proof. The proof of Kronecker's lemma is given in Rudin (1976). □

Kronecker's lemma is particularly useful in deriving the well-known Kolmogorov-Marcinkiewicz-Zygmund SLLN as stated below.

Theorem A.23 (Kolmogorov-Marcinkiewicz-Zygmund SLLN)**.** *Suppose that $X_1, X_2, ...$ are i.i.d. r.v. and $0 < p < 2$. Let $S_n = \sum_{i=1}^{n} X_i$ as usual, then*

$$\frac{S_n - nc}{n^{1/p}} \xrightarrow[n\to\infty]{a.s.} 0, \text{ for some constant } c$$

if and only if $\mathbb{E}|X_1|^p < \infty$. Further, necessarily $\mathbb{E}(X_1) = c$ if $1 \leq p < 2$. Whereas c is arbitrary (and hence may be taken as 0) if $0 < p < 1$.

Proof. We first prove the necessity, i.e. given that $\frac{S_n - nc}{n^{1/p}} \xrightarrow[n\to\infty]{\text{a.s.}} 0$, we have

$$\begin{aligned}
\frac{X_n}{n^{1/p}} &= \frac{S_n - S_{n-1}}{n^{1/p}} = \frac{S_n - nc}{n^{1/p}} - \frac{S_{n-1} - nc}{n^{1/p}} \\
&= \frac{S_n - nc}{n^{1/p}} - (\frac{n-1}{n})^{1/p} \left(\frac{S_{n-1} - (n-1)c}{(n-1)^{1/p}} - \frac{c}{(n-1)^{1/p}} \right) \xrightarrow[n\to\infty]{\text{a.s.}} 0.
\end{aligned}$$

Thus, by the definition of almost sure convergence we have for all $\epsilon > 0$,

$$\mathbb{P}\left(\frac{|X_n|}{n^{1/p}} > \epsilon \text{ i.o.}\right) = 0.$$

Taking $\epsilon = 1$, we have

$$\mathbb{P}\left(\frac{|X_n|}{n^{1/p}} > 1 \text{ i.o.}\right) = 0.$$

Since X_n i.i.d. random variables, then by second Borel-Cantelli Lemma, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{|X_n|}{n^{1/p}} > 1\right) = \sum_{n=1}^{\infty} \mathbb{P}\big(|X_n|^p > n\big) < \infty,$$

which is, by Theorem 3.2, equivalent to $\mathbb{E}|X_1|^p < \infty$.

Then we deal with the sufficiency. Suppose $\mathbb{E}|X_1|^p < \infty$, $0 < p < 2$. First, consider the case where $0 < p < 1$. Then by Marcinkiewicz-Zygmund Convergence Theorem, we have $\sum_{n=1}^{\infty} \frac{X_n}{n^{1/p}}$ convergence a.s. Then, by Kronecker's lemma, $\frac{S_n}{n^{1/p}} = \frac{1}{n^{1/p}} \sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{\text{a.s.}} 0$, and from this it is clear that, if $0 < p < 1$,

$$\frac{S_n - nc}{n^{1/p}} \to 0 \quad \text{a.s.}$$

Second, if $p = 1$, this is Khinchine's strong law of large number, which is established in Theorem A.18. It left to prove the result for the case where $1 < p < 2$. Without loss of generality, let $\mathbb{E}(X_1) = 0$ and let $Y_k = X_k \mathbb{I}_{\{|X_k| \le k^{1/p}\}}$. We have

$$\sum_{k=1}^{\infty} \mathbb{P}(Y_k \neq X_k) = \sum_{k=1}^{\infty} \mathbb{P}(|X|^p > k) \le \mathbb{E}|X|^p < \infty.$$

Then, by Borel-Cantelli Lemma, we have

$$\mathbb{P}(Y_k \neq X_k \quad \text{i.o.}) = 0.$$

Thus, it suffices to show,

$$n^{-1/p} \sum_{m=1}^{n} Y_m \to 0 \quad \text{a.s.}$$

Note that $|Y_m| \leq |X_m|$. Then, $\mathbb{P}(|Y_m| > y) \leq \mathbb{P}(|X_m| > y)$, $\forall y \in \mathbb{R}$, and by Lemma A.2

$$\begin{aligned}
&\sum_{m=1}^{\infty} \mathrm{Var}(Y_m/m^{1/p}) \leq \sum_{m=1}^{\infty} \mathbb{E}(Y_m^2/m^{2/p}) = \sum_{m=1}^{\infty} \int_0^{\infty} \frac{2y}{m^{2/p}} \mathbb{P}(|Y_m| > y)\,\mathrm{d}y \\
&= \sum_{m=1}^{\infty} \int_0^{m^{1/p}} \frac{2y}{m^{2/p}} \mathbb{P}(|Y_m| > y)\,\mathrm{d}y = \sum_{m=1}^{\infty} \sum_{n=1}^{m} \int_{(n-1)^{1/p}}^{n^{1/p}} \frac{2y}{m^{2/p}} \mathbb{P}(|Y_m| > y)\,\mathrm{d}y \\
&\leq \sum_{m=1}^{\infty} \sum_{n=1}^{m} \int_{(n-1)^{1/p}}^{n^{1/p}} \frac{2y}{m^{2/p}} \mathbb{P}(|X_1| > y)\,\mathrm{d}y = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \int_{(n-1)^{1/p}}^{n^{1/p}} \frac{2y}{m^{2/p}} \mathbb{P}(|X_1| > y)\,\mathrm{d}y.
\end{aligned}$$

To bound the integral, we notice that for $n \geq 2$, comparing the sum with the integral of $x^{-2/p}$,

$$\sum_{m=n}^{\infty} m^{-2/p} \leq \frac{p}{2-p}(n-1)^{1-2/p} \leq C y^{p-2},$$

where $y \in [(n-1)^{1/p}, n^{1/p}]$. It follows that for some $C > 0$,

$$\sum_{m=1}^{\infty} \mathrm{Var}(Y_m/m^{1/p}) \leq \sum_{n=1}^{\infty} \int_{(n-1)^{1/p}}^{n^{1/p}} C y^{p-2} 2y \mathbb{P}(|X_1| > y)\,\mathrm{d}y = \frac{2C}{p} \mathbb{E}(|X_1|^p) < \infty.$$

Then, by Theorem A.19, we have

$$\sum_{n=1}^{\infty} \frac{Y_n - \mu_n}{n^{1/p}} < \infty \quad \text{a.s.},$$

where $\mu_n = \mathbb{E}(Y_n)$, $n = 1, 2, ...$ Now, by Kronecker's Lemma,

$$n^{-1/p} \sum_{m=1}^{n} (Y_m - \mu_m) \xrightarrow[n\to\infty]{\text{a.s.}} 0.$$

Since $\mathbb{E}(X_m) = 0$, we have

$$\mu_m = \mathbb{E}(Y_m - X_m) = \mathbb{E}(X_m \mathbb{I}_{\{|X_m| \le m^{1/p}\}} - X_m) = -\mathbb{E}(X_m \mathbb{I}_{\{|X_m| > m^{1/p}\}}).$$

So,

$$\begin{aligned} |\mu_m| &\le \mathbb{E}(|X_m| \mathbb{I}_{\{|X_m| > m^{1/p}\}}) = m^{1/p} \mathbb{E}\left(\frac{|X_m|}{m^{1/p}} \mathbb{I}_{\{|X_m| > m^{1/p}\}} \right) \\ &\le m^{1/p} \mathbb{E}(|X_m|^p / m \mathbb{I}_{\{|X_m| > m^{1/p}\}}) = m^{-1+1/p} \mathbb{E}(|X_1|^p \mathbb{I}_{\{|X_1| > m^{1/p}\}}). \end{aligned}$$

Notice that

$$\sum_{m=1}^{n} m^{-1+1/p} \le C_p n^{1/p}$$

and $\mathbb{E}(|X_1|^p \mathbb{I}_{\{|X_1| > m^{1/p}\}}) \xrightarrow[m\to\infty]{} 0$ by Lebesgue dominated convergence theorem. Now, For any $\epsilon > 0$, exists $N > 0$, such that, for all $n > N$, $\mathbb{E}(|X_1|^p \mathbb{I}_{\{|X_1| > n^{1/p}\}}) < \epsilon$. Thus, we have

$$\begin{aligned} & n^{-1/p} \sum_{m=1}^{n} \mu_m \le n^{-1/p} \sum_{m=1}^{n} |\mu_m| \le n^{-1/p} \sum_{m=1}^{n} m^{-1+1/p} \mathbb{E}\left(|X_1|^p \mathbb{I}_{\{|X_1| > m^{1/p}\}} \right) \\ & = n^{-1/p} \left(\sum_{m=1}^{N} m^{-1+1/p} \mathbb{E}\left(|X_1|^p \mathbb{I}_{\{|X_1| > m^{1/p}\}} \right) + \sum_{m=N+1}^{n} m^{-1+1/p} \mathbb{E}\left(|X_1|^p \mathbb{I}_{\{|X_1| > m^{1/p}\}} \right) \right) \\ & \le o(1) + \epsilon C_p. \end{aligned}$$

Then, we get

$$n^{-1/p}\sum_{m=1}^{n}\mu_m \xrightarrow[n\to\infty]{} 0,$$

which implies

$$n^{-1/p}\sum_{m=1}^{n}Y_m \xrightarrow[n\to\infty]{\text{a.s.}} 0,$$

this completes the proof □

Remark A.5. *It should be noted that Theorem A.23 is a generalisation of Khinchine's Strong Law of Large Number, as stated in Theorem A.18.*

Appendix B

Proofs of theorems from Chapter 2

Proposition B.1. *Let* $\{X_n, n = 1, 2, ...,\}$ *be a sequence of random variables. Let* $2^i \leq n < 2^{i+1}$, $i = 1, 2, ...,$ *and define*

$$S_n^{(1)} = \big\{|X_k| > 2^{i-2} \text{ for at least one } k \leq n\big\},$$
$$S_n^{(2)} = \big\{|X_{k_1}| > n^{4/5}, |X_{k_2}| > n^{4/5}, \text{ for at least two } k_1 \leq n, k_2 \leq n\big\},$$
$$S_n^{(3)} = \left\{\left|\sum_{k=1}^{n} X_k \mathbb{I}_{\{|X_k| \leq n^{4/5}\}}\right| > 2^{i-2}\right\}.$$

Then we have

$$\big\{\big|S_n\big| > n\big\} \subset S_n^{(1)} \cup S_n^{(2)} \cup S_n^{(3)}.$$

Proof. This is done by proving the opposite, i.e., showing that

$$\big\{\big|S_n\big| \leq n\big\} \supset S_n^{(1)c} \cap S_n^{(2)c} \cap S_n^{(3)c}.$$

By definition

$$S_n^{(1)c} = \big\{ \max_{1\le k\le n} |X_k| \le 2^{i-2} \big\},$$

$$S_n^{(2)c} = \big\{ |X_k| > n^{4/5}, \text{ for at most one } k \le n \big\},$$

$$S_n^{(3)c} = \big\{ \big| \sum_{k=1}^{n} X_k \mathbb{I}_{\{|X_k|\le n^{4/5}\}} \big| \le 2^{i-2} \big\}.$$

For if $\omega \in S_n^{(1)c} \cap S_n^{(2)c} \cap S_n^{(3)c}$, then clearly

$$\left| \sum_{k=1}^{n} X_k(\omega) \right| \le \left| \sum_{k=1}^{n} X_k \mathbb{I}_{\{|X_k|\le n^{4/5}\}}(\omega) \right| + |X_k(\omega)| \le 2^{i-2} + 2^{i-2} < n.$$

□

Lemma B.1. *$a_n \ge 0$, $\forall n = 1, 2, ...$ and suppose that a_n is non-increasing sequence with $\sum_{k=1}^{\infty} a_n < \infty$. Then, $\lim\limits_{n\to\infty} n a_n = 0$.*

Proof. Suppose that $\lim_{n\to\infty} n a_n \ne 0$. Then, there exists an $\epsilon > 0$ such that for any N, there is an $n \ge N$ so that $n a_n \ge \epsilon$. Since N is arbitrary, we can find infinitely many n_i, $i = 1, 2, ...$, such that $n_{i+1} > 2n_i$,

$$\sum_{k=n_i+1}^{2n_i} a_k \ge n_i \cdot \frac{\epsilon}{2n_i} = \frac{\epsilon}{2}$$

thus

$$\sum_{n=1}^{\infty} a_n \ge \sum_{i=1}^{\infty} \sum_{k=n_i+1}^{2n_i} a_k \ge \sum_{i=1}^{\infty} \frac{\epsilon}{2} = \infty,$$

contradiction. Thus

$$\lim_{n\to\infty} n \cdot a_n = 0.$$

□

Proof of Theorem 2.1. In proving the sufficiency of the theorem, we can firstly assume that $\mathbb{E}(X_1) = 0$. Put

$$a_i = \mathbb{P}(|X_1| > 2^i), \quad i = 1, 2, \ldots.$$

Then, for all $i = 1, 2, \ldots,$

$$a_i - a_{i+1} = \mathbb{P}(|X_1| > 2^i) - \mathbb{P}(|X_1| > 2^{i+1}) = \mathbb{P}(2^i < |X_1| \leq 2^{i+1}).$$

Since $\mathbb{E}(X_1^2) < \infty$, by Theorem 3.3, we have

$$\sum_{i=1}^{\infty} 2^{2i} a_i < \infty. \tag{B.1}$$

Now, let $2^i \leq n < 2^{i+1}$ and define $S_n^{(1)}, S_n^{(2)}, S_n^{(3)}$ as in Proposition B.1. Thus, by Proposition B.1, and for $2^i \leq n < 2^{i+1}$, $i = 1, 2, \ldots,$

$$\big\{ \big| S_n \big| > n \big\} \subset S_n^{(1)} \cup S_n^{(2)} \cup S_n^{(3)}.$$

To prove the convergence of $\sum_{n=1}^{\infty} M_n$, it suffices to show that

$$\sum_{n=1}^{\infty} \big(\mathbb{P}(S_n^{(1)}) + \mathbb{P}(S_n^{(2)}) + \mathbb{P}(S_n^{(3)}) \big) < \infty. \tag{B.2}$$

Observe that

$$\mathbb{P}(S_n^{(1)}) = \mathbb{P}\left(\max_{1 \leq k \leq n} |X_k| > 2^{i-2} \right) \leq n \mathbb{P}(|X_1| > 2^{i-2}) = n \cdot a_{i-2} \leq 2^{i+1} a_{i-2}.$$

Thus, by (B.1),

$$\sum_{n=2}^{\infty} \mathbb{P}\big(S_n^{(1)}\big) = \sum_{i=1}^{\infty} \sum_{2^i \leq n < 2^{i+1}} \mathbb{P}(S_n^{(1)}) \leq \sum_{i=1}^{\infty} \sum_{2^i \leq n < 2^{i+1}} 2^{i+1} a_{i-2} = \sum_{i=1}^{\infty} 2^{2i+1} a_{i-2} < \infty. \tag{B.3}$$

From (B.1), for large i, and for some integer u, such that, $2^i \leq u < 2^{i+1}$, we have

$$2^{2i} a_i < 1/4 \Rightarrow \mathbb{P}(|X_1| > 2^i) < \frac{1}{2^{2i}} \cdot \frac{1}{4},$$

then

$$\mathbb{P}(|X_1| > u) \leq \mathbb{P}(|X_1| > 2^i) < \frac{1}{2^{2i}} \frac{1}{4} \leq \frac{4}{u^2} \cdot \frac{1}{4} = \frac{1}{u^2}.$$

Next since the X_i are independent and have the same distribution function, it follows that for sufficiently large n,

$$\begin{aligned}
\mathbb{P}(S_n^{(2)}) &= \mathbb{P}\Big(\bigcup_{1 \leq k_1 < k_2 \leq n} \big\{ |X_{k_1}| > n^{4/5}, |X_{k_2}| > n^{4/5} \big\} \Big) \\
&\leq \sum_{1 \leq k_1 < k_2 \leq n} \mathbb{P}(|X_{k_1}| > n^{4/5}, |X_{k_2}| > n^{4/5}) \\
&= \binom{n}{2} \mathbb{P}(|X_{k_1}| > n^{4/5}) \cdot \mathbb{P}(|X_{k_2}| > n^{4/5}) \leq n^2 \cdot n^{-8/5} \cdot n^{-8/5} = n^{-6/5}.
\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \mathbb{P}(S_n^{(2)}) < \infty. \tag{B.4}$$

Next, put

$$X_k^{+} = X_k \mathbb{I}_{(|X_k| \leq n^{4/5})}.$$

Clearly X_k^+, $k \geq 1$ are i.i.d. Also, put

$$\mathbb{E}(X_1^+) = c \text{ and } Y_k = X_k^+ - \mathbb{E}(X_k^+). \tag{B.5}$$

We observe $\mathbb{E}(Y_k) = 0$ and $c \to 0$ as $n \to \infty$. We evidently have

$$\mathbb{E}\left(\sum_{k=1}^{n} Y_k\right)^4 = \mathbb{E}\left(\sum_{k=1}^{n} Y_k^4\right) + 6\mathbb{E}\left(\sum_{1 \leq k < l \leq n} Y_k^2 Y_l^2\right).$$

Now since $|Y_k| < n^{4/5} + c$,

$$\mathbb{E}(Y_k^4) < (n^{4/5} + c)^2 \cdot \mathbb{E}(Y_k^2) < c_1 n^{8/5}.$$

and

$$\mathbb{E}(Y_k^2 Y_l^2) = \mathbb{E}(Y_k^2) \cdot \mathbb{E}(Y_l^2) < c_2.$$

Thus

$$\mathbb{E}\left(\left|\sum_{k=1}^{n} Y_k\right|^4\right) < c_3 \cdot n \cdot n^{8/5} = c_3 n^{13/5}.$$

Hence, by Markov's inequality,

$$\mathbb{P}\left(\left|\sum_{k=1}^{n} Y_k\right| > n/16\right) < \frac{\mathbb{E}\left(\left|\sum_{k=1}^{n} Y_k\right|^4\right)}{(n/16)^4} < c_4 n^{-7/5}. \tag{B.6}$$

Then, from (B.5) and (B.6), and for sufficient large n, $|c| < 1/16$ and the fact that

$n/8 < 2^{i-2}$, we finally have

$$\begin{aligned}
&\mathbb{P}\Big(\Big|\sum_{k=1}^{n} X_k \mathbb{I}_{\{|X_k \le n^{4/5}|\}}\Big| > 2^{i-2}\Big) = \mathbb{P}\Big(\Big|\sum_{k=1}^{n} X_k^+\Big| > 2^{i-2}\Big) \\
&= \mathbb{P}\Big(\Big|\sum_{k=1}^{n} Y_k + nc\Big| > 2^{i-2}\Big) \le \mathbb{P}\Big(\Big|\sum_{k=1}^{n} Y_k\Big| + n|c| > 2^{i-2}\Big) \\
&\le \mathbb{P}\Big(\Big|\sum_{k=1}^{n} Y_k\Big| + n/16 > 2^{i-2}\Big) \le \mathbb{P}\Big(\Big|\sum_{k=1}^{n} Y_k\Big| > n/16\Big) < c_4 n^{-7/5}
\end{aligned}$$

or equivalently

$$\mathbb{P}\big(S_n^{(3)}\big) < c_4 n^{-7/5}. \tag{B.7}$$

Thus from (B.3), (B.4) and (B.7), we obtain (B.2). Note that we have shown given $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(X_1^2) < \infty$, that

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n| > n) < \infty.$$

Now, assume that $\mathbb{E}(X_1) = c$, $|c| < 1$. Then, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(\Big|\sum_{k=1}^{n} \frac{X_k - c}{1-|c|}\Big| > n\Big) = \sum_{n=1}^{\infty} \mathbb{P}\left(\left|\sum_{k=1}^{n} X_k - cn\right| > (1-|c|)n\right) < \infty.$$

And since

$$\begin{aligned}
&\mathbb{P}\left(\left|\sum_{k=1}^{n} X_k - nc\right| > (1-|c|)n\right) \\
&= \mathbb{P}\left(\sum_{k=1}^{n} X_k > nc + (1-|c|)n\right) + \mathbb{P}\left(\sum_{k=1}^{n} X_k < nc - (1-|c|)n\right) \\
&\ge \mathbb{P}\left(\sum_{n=1}^{n} X_k > n\right) + \mathbb{P}\left(\sum_{n=1}^{n} X_k < -n\right) = P\left(\left|\sum_{n=1}^{n} X_k\right| > n\right).
\end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\sum_{k=1}^{n} X_k\right| > n\right) < \infty,$$

this proves the sufficient part of the theorem. Next, we prove the necessary part of this theorem, In other words, we shall show that if $\sum_{n=1}^{\infty} M_n$ converges then (2.1) hold. First, we prove the second part of (2.1), by Theorem 3.3, it suffices to prove

$$\sum_{n=1}^{\infty} n\mathbb{P}(|X_1| > cn) = \sum_{n=1}^{\infty} n\mathbb{P}(|X_1|^2 > c^2n^2) < \infty, \tag{B.8}$$

for any $c > 0$. Now we have by triangle inequality,

$$\big\{|X_n| > 2n\big\} \subset \big\{\big|\sum_{k=1}^{n-1} X_k\big| > n\big\} \cup \big\{\big|\sum_{k=1}^{n} X_k\big| > n\big\}.$$

Hence

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_1| > 2n) \leq \sum_{n=1}^{\infty} \big(\mathbb{P}(|S_{n-1}| > n-1) + \mathbb{P}(|S_n| > n)\big) < \infty,$$

then, by Theorem 3.1, we obtain $\mathbb{E}|X_1| < \infty$. Since the terms of $\mathbb{P}(|X_1| > 2n)$ is non-increasing w.r.t. n, it follows from Lemma B.1 that

$$n\mathbb{P}(|X| > 2n) \to 0.$$

Our assumption being that $\sum_n M_n < \infty$, we have $M_n \to 0$ as $n \to \infty$ (i.e. $1 - M_n \to 1$). Then, there is a constant $\rho > 0$,(e.g. $\rho = 1/2$) independent of k and for sufficiently large n such that

$$\mathbb{P}\Big(\Big|\sum_{l=1,l\neq k}^{n} X_l\Big| < n\Big) = \mathbb{P}\big(|S_{n-1}| < n\big) \geq \mathbb{P}(|S_{n-1}| \leq n-1) \geq \rho.$$

We have

$$\bigcup_{k=1}^{n}\left(\{|X_k| > 2n\} \cap \left\{\Big|\sum_{l=1,l\neq k}^{n} X_l\Big| < n\right\}\right) \subset \{|S_n| > n\}. \tag{B.9}$$

We let $R_{k,n} = \{|X_k| > 2n\}$, and let

$$T_{k,n} = \left\{\Big|\sum_{l=1,l\neq k}^{n} X_l\Big| < n\right\}.$$

For conveniece, let $T_k = T_{k,n}$, $R_k = R_{k,n}$, also in the following, we denote events AB and $A \cdot B$ as $A \cap B$ for two events A, B, then we rewrite (B.9) as

$$\bigcup_{k=1}^{n}(R_k \cap T_k) \subset \{|S_n| > n\}.$$

In the sequel, we may write set intersections as products for the sake of simplicity, we have

$$\begin{aligned} M_n &= \mathbb{P}(|S_n| > n) \geq \mathbb{P}\Big(\cup_{k=1}^{n}(R_k \cdot T_k)\Big) \\ &= \mathbb{P}\Big(\cup_{k=1}^{n}(R_1T_1)^c \cdots (R_{k-1}T_{k-1})^c R_k T_k\Big). \end{aligned}$$

Then,

$$M_n \geq \sum_{k=1}^{n} \mathbb{P}\Big((R_1T_1)^c \cdots (R_{k-1}T_{k-1})^c R_k T_k\Big) \geq \sum_{k=1}^{n} \mathbb{P}(R_1^c \cdots R_{k-1}^c R_k T_k),$$

and then

$$
\begin{aligned}
M_n &\geq \sum_{k=1}^{n} \mathbb{P}\left((R_1^c \cdots R_{k-1}^c \cup R_k^c) \cdot R_k T_k\right) \\
&= \sum_{k=1}^{n} \mathbb{P}\big((R_1 \cup \cdots \cup R_{k-1} \cap R_k)^c \cap R_k T_k\big) \\
&\geq \sum_{k=1}^{n} \big[\mathbb{P}(R_k T_k) - \mathbb{P}((R_1 \cup \cdots \cup R_{k-1}) \cap R_k)\big].
\end{aligned}
$$

Note that T_k and R_k are independent and $R_k, R_{k-1}, ..., R_1$ are independent, then for all sufficiently large n, there exists $0 < \rho' < \rho$, such that,

$$
\begin{aligned}
&\mathbb{P}(R_k T_k) - \mathbb{P}((R_1 \cup \cdots \cup R_{k-1}) \cap R_k) = \mathbb{P}(R_k)\big[\mathbb{P}(T_k) - \mathbb{P}(R_1 \cup \cdots \cup R_{k-1})\big] \\
&\geq \mathbb{P}(R_k)\big[\mathbb{P}(T_k) - (k-1)\mathbb{P}(R_1)\big] \geq \mathbb{P}(R_k)(\rho - n\mathbb{P}(R_1)) = \mathbb{P}(R_k)(\rho - o(1)) \geq \rho' \mathbb{P}(R_k).
\end{aligned}
$$

Hence, for all sufficiently large n,

$$M_n \geq \rho' \sum_{k=1}^{n} \mathbb{P}(R_{k,n}) = \rho' n \mathbb{P}(R_{1,n}).$$

Thus

$$\sum_{n=1}^{\infty} n\mathbb{P}(R_{1,n}) = \sum_{n=1}^{n_0} n\mathbb{P}(R_{1,n}) + \sum_{n=n_0+1}^{\infty} n\mathbb{P}(R_{1,n}) \leq \sum_{n=1}^{n_0} n\mathbb{P}(R_{1,n}) + \frac{1}{\rho'} \sum_{n=n_0+1}^{\infty} M_n < \infty.$$

Hence by Theorem 3.3, $\mathbb{E}(X_1^2) < \infty$. Now we show the proof of the first part of (2.1), we can suppose without loss of generality that

$$\mathbb{E}(X_1) = c > 0.$$

If $c > 1$, by the second part of (2.1) and WLLN,

$$\mathbb{P}(|S_n| > n) = \mathbb{P}(S_n > n) + \mathbb{P}(S_n < -n) \geq \mathbb{P}(S_n > n) = \mathbb{P}\big(\frac{S_n}{n} - c > 1 - c\big)$$
$$\geq \mathbb{P}\Big(1 - c < \frac{S_n}{n} - c < c - 1\Big) = \mathbb{P}\Big(\Big|\frac{S_n}{n} - c\Big| < c - 1\Big) \xrightarrow[n\to\infty]{} 1,$$

contradicting $\sum_{n=1}^{\infty} M_n < \infty$. Thus $c \leq 1$. But if $c = 1$, from the second part of (2.1) and the central limit theorem,

$$\mathbb{P}(|S_n| > n) = \mathbb{P}(S_n > n) + \mathbb{P}(S_n < -n) \geq \mathbb{P}(S_n > n)$$
$$= \mathbb{P}\left(\sqrt{n}\left(\frac{S_n}{n} - 1\right) > 0\right) \xrightarrow[n\to\infty]{} 1/2,$$

which means that M_n does not tend to 0. Hence $C < 1$. The proof is complete. □

Proof of Theorem 2.2. To prove this theorem, we assume with no loss of generality that $\epsilon = 1$ and if $\mathbb{E}(X_k)$ exists, we assume that $\mathbb{E}(X_k) = 0$. Following the method of Theorem 2.1, we define

$$A_n := \left\{|S_n| > n^{r/t}\right\}$$

and

$$a_i := \mathbb{P}(|X_1| > 2^{ir/t}).$$

Then, by Theorem 3.3, we have

$$\sum_{i=0}^{\infty} 2^{ir} a_i < \infty \quad \Longleftrightarrow \quad \mathbb{E}|X_k|^t < \infty.$$

Let $2^i \le n < 2^{i+1}$ and define

$$A_n^{(1)} := \{|X_k| > 2^{(i-2)r/t} \text{ for at least one } k \le n\} = \{\max_{1\le k\le n} |X_k| > 2^{(i-2)r/t}\},$$

$$A_n^{(2)} := \{|X_{k_1}| > n^{\gamma r/t}, |X_{k_2}| > n^{\gamma r/t} \text{ for at least two } k's \le n\},$$

where γ is chosen so that $[(r+1)/2r] < \gamma < 1$ and $1 - 2\gamma r/t < 0$. Such a choice is possible by conditions. Further, let

$$A_n^{(3)} := \left\{\left|\sum_{k=1}^{n} X_k \mathbb{I}_{\{|X_k| \le n^{\gamma r/t}\}}\right| > 2^{(i-2)r/t}\right\}.$$

It follows by the similar arguments as in Proposition B.1, we have $A_n \subseteq A_n^{(1)} \cup A_n^{(2)} \cup A_n^{(3)}$. Since for all $\omega \in A_n^{(1)c} \cap A_n^{(2)c} \cap A_n^{(3)c}$ and $2^i \le n < 2^{i+1}$, and for some $1 \le k_0 \le n$, $|X_{k_0}(\omega)| > n^{\gamma r/t}$

$$\begin{aligned}
\left|\sum_{k=1}^{n} X_k(\omega)\right| &= \left|\sum_{k\in\{1,2,\dots,n\}\setminus k_0} X_k \mathbb{I}(|X_k| \le n^{\gamma r/t})\right| + |X_{k_0}| \\
&\le \left|\sum_{k=1}^{n} X_k \mathbb{I}(|X_k| \le n^{\gamma r/t})\right| + 2^{(i-2)r/t} \le 2^{(i-2)r/t} + 2^{(i-2)r/t} \\
&= 2^{(i-2)r/t+1} \le 2^{(i-2)r/t+2r/t} = 2^{ir/t} \le n^{r/t}
\end{aligned}$$

Therefore, similarly to Theorem 2.1, it suffices to prove that

$$\sum_{n=1}^{\infty} n^{r-2} \mathbb{P}\left(A_n^{(j)}\right) < \infty, \quad j = 1, 2, 3.$$

We begin by demonstrating that $\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(A_n^{(1)}) < \infty$. We have that

$$\mathbb{P}(A_n^{(1)}) = \mathbb{P}(|X_k| > 2^{(i-2)r/t} \quad \text{for some } k \leq n) \leq n \cdot \mathbb{P}(|X_1| > 2^{(i-2)r/t}) \leq 2^{i+1}a_{i-2}.$$

Thus, by $\mathbb{E}|X_1|^t < \infty$, we have

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(A_n^{(1)}) \leq \sum_{i=0}^{\infty} \sum_{2^i \leq n < 2^{i+1}} n^{r-2}2^{i+1}a_{i-2}$$
$$\leq \sum_{i=0}^{\infty} 2^{(2i+1)+(i+1)(r-2)}a_{i-2} = C + \sum_{i=0}^{\infty} 2^{ri+3r-1}a_i < \infty.$$

where $C = (r-1)a_{-2} + (2r-1)a_{-1}$. To show that $\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(A_n^{(2)}) < \infty$, we have

$$\mathbb{P}(A_n^{(2)}) = \mathbb{P}(|X_{k_1}| > n^{\gamma r/t}, |X_{k_2}| > n^{\gamma r/t} \quad \text{for at least two } k's \leq n)$$
$$\leq \sum_{1 \leq k_1 < k_2 \leq n} \mathbb{P}(|X_{k_1}| > n^{\gamma r/t}, |X_{k_2}| > n^{\gamma r/t}) \leq n^2\mathbb{P}^2(|X_1| > n^{\gamma r/t}).$$

Now, by Markov's inequality, for some $M > 0$,

$$\mathbb{P}(|X_1| > n^{\gamma r/t}) \leq \frac{\mathbb{E}|X_1|^t}{n^{\gamma r}} \leq \frac{M}{n^{\gamma r}},$$

and therefore $\mathbb{P}(A_n^{(2)}) \leq M^2n^2/n^{2\gamma r} = M^2n^{2(1-\gamma r)}$. Further, since $(r+1)/2r < \gamma < 1$, we have

$$r(1-2\gamma) < r\left(1 - 2\frac{1+r}{2r}\right) = -1.$$

Then,

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(A_n^{(2)}) \leq M^2 \sum_{n=1}^{\infty} n^{2(1-\gamma r)+r-2} = M^2 \sum_{n=1}^{\infty} n^{r(1-2\gamma)} < \infty.$$

It remains to prove the convergence of $\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(A_n^{(3)})$. We begin by proving convergence in the case $t < 1$, and $r/t > 1$. Let $\delta > 0$ be such that $t + 2\delta = 1$. For $2^i \leq n < 2^{i+1}$, we have $n/8 < 2^{(i-2)}$, and therefore, by Theorem A.8 with $C_r = 1$,

$$\mathbb{P}(A_n^{(3)}) \leq \mathbb{P}\left(\left|\sum_{k=1}^{n} X_k \mathbb{I}_{\{|X_k| \leq n^{\gamma r/t}\}}\right| > \left(\frac{n}{8}\right)^{r/t} \delta^{r/t}\right) \leq C\frac{\mathbb{E}|\sum_{k=1}^{n} X_k \mathbb{I}_{\{|X_k| \leq n^{\gamma r/t}\}}|^{t+\delta}}{n^{r+\delta r/t}}$$
$$\leq C\frac{\sum_{k=1}^{n} \mathbb{E}|X_k \mathbb{I}_{\{|X_k| \leq n^{\gamma r/t}\}}|^{t+\delta}}{n^{r+\delta r/t}} \leq Cn^{\delta\gamma r/t}\frac{\sum_{k=1}^{n} \mathbb{E}|X_k \mathbb{I}_{\{|X_k| \leq n^{\gamma r/t}\}}|^{t}}{n^{r+\delta r/t}},$$

for some $C > 0$. Then, for some $C_0 > 0$, such that,

$$n^{r-2}\mathbb{P}(A_n^{(3)}) \leq n^{r-2} \cdot C \cdot n^{\gamma r\delta/t} \sum_{k=1}^{n} \mathbb{E}|X_k \mathbb{I}_{\{|X_k| \leq n^{\gamma\delta/t}\}}|^t / n^{r+r\delta/t}$$
$$= C_0 n^{r-2+1-r-r\delta/t+\gamma r\delta/t} = C_0 n^{-1-(r/t)\delta(1-\gamma)}.$$

Since γ has been chosen so that $(1-\gamma) > 0$, it follows that if $t < 1$ and $r/t > 1$, and $1 + \frac{r}{t}\delta(1-\gamma) > 1$. Then $\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(A_n^{(3)}) < \infty$. This completes the proof of part (c). Now, let us prove that if $t > 1$ and $1/2 < r/t \leq 1$, then $\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(A_n^{(3)}) < \infty$, i.e. part (a) shall be proved. Recall that we have chosen $\mu = 0$. We now take j to be the smallest integer $\geq t$ and we let M to be a positive integer to be determined later. Define $\alpha_n = \mathbb{E}\left(X_1 \mathbb{I}_{\{|X_1| \leq n^{\gamma r/t}\}}\right)$ and note that $\alpha_n \to 0$ as $n \to \infty$. Define $Y_k = X_k \mathbb{I}_{\{|X_k| \leq n^{\gamma r/t}\}} - \alpha_n$ and note that $\mathbb{E}Y_k = 0$. We will find a bound for $\mathbb{P}(A_n^{(3)})$ by Markov's inequality and this requires that we find a bound for $\mathbb{E}\left|\sum_{k=1}^{n} Y_k\right|^{2Mj}$,

$$\mathbb{E}\left|\sum_{k=1}^{n} Y_k\right|^{2Mj} = \sum_{k=1}^{n} \mathbb{E}\left(Y_k^{2Mj}\right) + \cdots + c \sum_{k_1<k_2<\cdots<k_\tau} \mathbb{E}\left(Y_{k_1}^2\right)\mathbb{E}\left(Y_{k_2}^2\right)\cdots\mathbb{E}\left(Y_{k_\tau}^2\right) \quad \text{(B.10)}$$

where $\tau = Mj$. Thus, it suffices to bound each sum on the right-hand side of (B.10).

$$\mathbb{E}Y_k^{2Mj} = \mathbb{E}\big(|Y_k|^{2Mj-t}|Y_k|^t\big)$$
$$\leq (n^{\gamma r/t} + |\alpha|)^{2Mj-t} \cdot \mathbb{E}|Y_k|^t \leq (n^{\gamma r/t} + |\alpha|)^{2Mj-t} \cdot \mathbb{E}(|X_k| + |\alpha|)^t,$$

and then, for some $C_1 > 0$,

$$\sum_{k=1}^{n} \mathbb{E}Y_k^{2Mj} \leq C_1 n \cdot n^{\gamma r/t \cdot (2Mj-t)} = C_1 \cdot n^{2Mj\gamma r/t - \gamma r + 1}.$$

Now, consider any other sum on the right hand side of (B.10) where at least one of exponents of $Y_k's$ is greater than t. Suppose that for one of these sums exactly q of exponents in each summand exceed t and l of the exponents are less than or equal to t. Then, this sum is bounded by

$$C_2 n^{q+l} \cdot n^{(\gamma r/t)\{(d_1-t)+\cdots+(d_q-t)\}} \leq C_2 n^{q+l+(\gamma r/t)\{(2Mj-2l)-qt\}}$$
$$= C_2 n^{2Mj\gamma r/t + q(1-\gamma r) + l(1-2\gamma r/t)},$$

for some $C_2 > 0$ and $d_1, \cdots, d_q$ are exponents in each summand that exceed t. Now, γ has been chosen, such that, $1 - \gamma r < 0$ and $1 - 2\gamma r/t < 0$ and thus, this bound is maximized when $q = 1$ and $l = 0$. That is all sums when at least one exponents of Y_{k_i} is $> t$ are bounded by

$$C_3 n^{(2Mj\gamma r/t) - \gamma r + 1},$$

for some $C_3 > 0$. If all the exponents of the Y_{k_i} for a particular sum on the right-hand side of (B.10) are $\leq t$, then a bound for such a sum is given by cn^{Mj}. However if M

is sufficiently large,

$$(2Mj\gamma r/t) - \gamma r + 1 > Mj.$$

Thus, for M sufficiently large, we have

$$\mathbb{E}\left|\sum_{k=1}^{n} Y_k\right|^{2Mj} \leq C_4 n^{(2Mj\gamma r/t)-\gamma r+1}$$

for some $C_4 > 0$. Now that if $r/t = 1$, then

$$\mathbb{P}(A_n^{(3)}) = \mathbb{P}\left(\left|\sum_{k=1}^{n} X_k \mathbb{I}_{\{|X_k| \leq n^{\gamma r/t}\}}\right| > 2^{i-2}\right) = \mathbb{P}\left(\left|\sum_{k=1}^{n} Y_k + n\alpha\right| > 2^{i-2}\right).$$

Now, choose n so that $\alpha < 1/16$ and since $n/8 < 2^{i-2}$ it follows that

$$\mathbb{P}(A_n^{(3)}) = \mathbb{P}\left(\left|\sum_{k=1}^{n} Y_k + n\alpha\right| > 2^{i-2}\right) \leq \mathbb{P}\left(\left|\sum_{k=1}^{n} Y_k\right| + n/16 > n/8\right)$$
$$= \mathbb{P}\left(\left|\sum_{k=1}^{n} Y_k\right| > n/16\right) \leq C \cdot \mathbb{E}\left(\sum_{k=1}^{n} Y_k\right)^{2Mj} \Big/ n^{2Mj}.$$

Therefore, for n sufficiently large,

$$n^{r-2}\mathbb{P}(A_n^{(3)}) \leq C \cdot n^{2Mj\gamma r/t-\gamma r+1} \cdot n^{r-2} \cdot n^{-2Mj} = C \cdot n^{2Mj(\gamma-1)-r(\gamma-1)-1}.$$

However, $\gamma - 1 < 0$ and $2Mj > r$ $(j \geq t = r)$ and therefore,

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(A_n^{(3)}) < \infty.$$

If $1/2 < r/t < 1$, we first note that, by $t > 1$ in case (a), $(1-\gamma r) < 0$ and $(\gamma - 1) < 0$.

We have

$$\begin{aligned}
&n^{1-r/t}\mathbb{E}(X_1\mathbb{I}_{\{|X_1|>n^{\gamma r/t}\}}) \le n^{1-r/t}\mathbb{E}(|X_1|\mathbb{I}_{\{|X_1|>n^{\gamma r/t}\}})\\
&= n^{1-r/t}\cdot n^{\gamma r/t}\mathbb{E}\left(\frac{|X_1|}{n^{\gamma r/t}}\mathbb{I}_{\{|X_1|>n^{\gamma r/t}\}}\right) \le n^{1-r/t}\cdot n^{\gamma r/t}\mathbb{E}\left(\frac{|X_1|^t}{n^{\gamma r}}\mathbb{I}_{\{|X_1|>n^{\gamma r/t}\}}\right)\\
&= n^{1-r/t+\gamma r/t-\gamma r}\mathbb{E}(|X_1|^t\mathbb{I}_{|X_1|>n^{\gamma r/t}}) = n^{(1-\gamma r)+r/t(\gamma-1)}\mathbb{E}(|X_1|^t\mathbb{I}_{|X_1|>n^{\gamma r/t}}) \xrightarrow[n\to\infty]{} 0,
\end{aligned}$$

i.e., $n^{1-r/t}\mathbb{E}(X_1\mathbb{I}_{\{|X_1|>n^{\gamma r/t}\}}) \xrightarrow[n\to\infty]{} 0$. Thus,

$$\mathbb{P}(A_n^{(3)}) \le \mathbb{P}\left(\left|\sum_{k=1}^{n} Y_k + n\alpha\right| > cn^{r/t}\right) \le \mathbb{P}\left(\left|\sum_{k=1}^{n} Y_k\right| > cn^{r/t}(1-cn^{1-r/t}\alpha)\right).$$

However, since $\mathbb{E}X_k = 0$, $\alpha_n = \mathbb{E}(X_1\mathbb{I}_{\{|X_1|>n^{\gamma r/t}\}})$, then $n^{1-r/t}\alpha_n \xrightarrow[n\to\infty]{} 0$. Therefore, for sufficiently large n,

$$\begin{aligned}
&\mathbb{P}(A_n^{(3)}) \le \mathbb{P}(|\sum_{k=1}^{n} Y_k| \ge cn^{r/t}) \le c\mathbb{E}\big(|\sum_{k=1}^{n} Y_k|^{2Mj}\big/ n^{2Mjr/t}\big)\\
&\le C\cdot n^{2Mj\gamma r/t-\gamma r+1-2Mjr/t} = C\cdot n^{2Mjr/t(\gamma-1)+(1-\gamma r)}.
\end{aligned}$$

Then,

$$\begin{aligned}
&n^{r-2}\mathbb{P}(A_n^{(3)}) \le C\cdot n^{2Mjr/t(\gamma-1)+(1-\gamma r)+r-2} = C\cdot n^{2Mjr/t(\gamma-1)+r(1-\gamma)-1}\\
&= C\cdot n^{(2Mj/t-1)r(\gamma-1)-1}.
\end{aligned}$$

Choose M large enough that $2Mj/t-1>0$, and then, $(2Mj/t-1)r(\gamma-1)-1<-1$. Hence

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(A_n^{(3)}) < \infty,$$

which proves part (a). It remains to prove part (b). Recall that in part (b) of the theorem we assume that $t \geq 1$ and $r/t > 1$. Therefore, by Markov's inequality, for sufficiently large n and for some $0 < c' < c$, we have

$$\mathbb{P}(A_n^{(3)}) = \mathbb{P}\left(\left|\sum_{k=1}^{n} X_k \mathbb{I}_{\{|X_k| > n^{\gamma r/t}\}}\right| > cn^{r/t}\right) = \mathbb{P}\left(\left|\sum_{k=1}^{n} Y_k + n\alpha\right| > cn^{r/t}\right)$$
$$\leq \mathbb{P}\left(\left|\sum_{k=1}^{n} Y_k\right| > cn^{r/t} - n\alpha\right) \leq \mathbb{P}\left(\left|\sum_{k=1}^{n} Y_k\right| > c'n^{r/t}\right) \leq \frac{\mathbb{E}\left|\sum_{k=1}^{n} Y_k\right|^{2Mj}}{(c'n^{r/t})^{2Mj}}.$$

Finanlly, since

$$\mathbb{E}\left|\sum_{k=1}^{n} Y_k\right|^{2Mj} \leq c \cdot n^{2Mj\gamma r/t - \gamma r + 1},$$

we get $\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(A_n^{(3)}) < \infty$. □

Proof of Theorem 2.3. Without loss of generality, let $\mu = 0$. To check that $\mathbb{E}|X_k|^t < \infty$, in part (a), (b) and (c), we follow the method of Theorem 2.1. Note that

$$\left\{|X_n| > 2cn^{r/t}\right\} \subseteq \left\{\left|\sum_{k=1}^{n-1} X_k\right| > cn^{r/t}\right\} \cup \left\{\left|\sum_{k=1}^{n} X_k\right| > cn^{r/t}\right\}.$$

Thus, we have

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > 2cn^{r/t}) \leq \sum_{n=1}^{\infty} \mathbb{P}(|S_{n-1}| > cn^{r/t}) + \sum_{n=1}^{\infty} \mathbb{P}(|S_n| > cn^{r/t})$$
$$\leq \sum_{n=1}^{\infty} \mathbb{P}(|S_{n-1}| > c(n-1)^{r/t}) + \sum_{n=1}^{\infty} \mathbb{P}(|S_n| > cn^{r/t}).$$

Since $\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(|S_n| > cn^{r/t}) < \infty$, $r \geq 2$. Then, $\sum_{n=1}^{\infty} \mathbb{P}(|S_n| > cn^{r/t}) < \infty$. Thus,

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_1| > 2cn^{r/t}) < \infty.$$

Since $\mathbb{P}(|X_1| > 2cn^{r/t})$ is non-negative and non-increasing, by Lemma B.1, we have $n\mathbb{P}(|X_1| > 2cn^{r/t}) \to 0$, as $n \to \infty$. Further, similar to the argument in proof of Theorem 2.1, we observe that there exists $0 < \rho' < 1$, such that,

$$\begin{aligned}
&\mathbb{P}\left(|S_n| > cn^{r/t}\right) \\
&\geq \sum_{k=1}^{n} \left(\mathbb{P}\left(\left| \sum_{l=1, l\neq k}^{n} X_k \right| < cn^{r/t} \right) - (k-1)\mathbb{P}(|X_1| > 2cn^{r/t}) \right) \cdot \mathbb{P}(|X_k| > 2cn^{r/t})
\end{aligned}$$

This gives

$$\begin{aligned}
&\mathbb{P}\left(|S_n| > cn^{r/t}\right) \geq \sum_{k=1}^{n} \left(\mathbb{P}(|S_{n-1}| < cn^{r/t}) - n\mathbb{P}(|X_1| > 2cn^{r/t}) \right) \cdot \mathbb{P}(|X_k| > 2cn^{r/t}) \\
&\geq \rho' \sum_{k=1}^{n} \mathbb{P}(|X_k| > 2cn^{r/t}) = \rho' n \mathbb{P}(|X_1| > 2cn^{r/t}).
\end{aligned}$$

Then,

$$\sum_{n=1}^{\infty} n^{r-1}\mathbb{P}(|X_1| > 2cn^{r/t}) \leq \frac{1}{\rho'} \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(|S_n| > cn^{r/t}) < \infty,$$

and therefore, by using Corollary 3.2, we get $\mathbb{E}|X_1|^t < \infty$. Left to prove that in part

(a), $\mathbb{E}(X_1) = \mu$. By $r \geq 2$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\sum_{k=1}^{n} X_k - n\mu\right| > n^{r/t}\epsilon\right) < \infty,$$

which implies that

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\sum_{k=1}^{n} X_k - n\mu\right| > n^{r/t}\epsilon\right) = 0, \quad \forall\epsilon > 0,$$

i.e., $(S_n - n\mu)/n^{r/t} \xrightarrow[n\to\infty]{\mathbb{P}} 0$. This implies $(S_n - n\mu)/n \xrightarrow[n\to\infty]{\mathbb{P}} 0$, and then, $S_n/n \xrightarrow[n\to\infty]{\mathbb{P}} \mu$. With the fact we just proved that $\mathbb{E}|X_k|^t < \infty$, we can derive by WLLN, that $S_n/n \xrightarrow[n\to\infty]{\mathbb{P}} \mathbb{E}(X_1)$, then we get $\mathbb{E}(X_1) = \mu$, this completes the proof. □

Proof of Theorem 2.4. For necessity, for $t > 1$, we apply Theorem 2.2(a). Let $r = t$, then $\mathbb{E}(|X_k|^t) < \infty$ and $\mathbb{E}(X_k) = \mu$ imply (2.2). For $t = 1$, if $\mathbb{E}(X_k) = \mu$, by Theorem 4.2 of Spitzer (1956), we have

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\big(|S_n - n\mu| > n\epsilon\big) < \infty, \quad \forall\epsilon > 0, \tag{B.11}$$

This finishes the proof of necessity.

For sufficiency, in case $t \geq 2$, it follows from Theorem 2.3(a). Let $r = t$, then if (2.2) holds, we have $\mathbb{E}(X_k) = \mu$, and $\mathbb{E}|X_k|^t < \infty$. In case $t = 1$, the result follows from Theorem 4.2 of Spitzer (1956). Now, left to show the case where $1 < t < 2$. If (2.2) hold, then (B.11) holds, it follows that $\mathbb{E}(X_k) = \mu$ by Theorem 4.2 of Spitzer (1956). Left to show $\mathbb{E}|X_k|^t < \infty$. Without loss of generality, let $\mu = 0$, we have for some $0 < \rho' < 1$,

$$\mathbb{P}\left(|S_n| > \epsilon n\right) \geq \sum_{k=1}^{n} \left(\mathbb{P}\left(\left| \sum_{l=1.l\neq k}^{n} X_l \right| < \epsilon n \right) - (k-1)\mathbb{P}(|X_k| > 2\epsilon n) \right) \cdot \mathbb{P}(|X_k| > 2\epsilon n)$$

$$\geq \sum_{k=1}^{n} \left(\mathbb{P}\Big(|S_{n-1}| < \epsilon n\Big) - n\mathbb{P}(|X_k| > 2\epsilon n) \right) \cdot \mathbb{P}(|X_k| > 2\epsilon n) \geq \rho' \cdot n \cdot \mathbb{P}(|X_1| > 2\epsilon n).$$

Then,

$$\sum_{n=1}^{\infty} n^{t-1}\mathbb{P}(|X_1| > 2\epsilon n) \leq \frac{1}{\rho'} \sum_{n=1}^{\infty} n^{t-2}\mathbb{P}(|S_n| > n\epsilon) < \infty.$$

Thus, by Corollary 3.2, we have $\mathbb{E}(|X_1|^t) < \infty$, this completes the proof. □

Proof of Theorem 2.5. First, it will be shown that $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$. Assume with no loss of generality, that $\epsilon = 1$ and, if $\mathbb{E}(X_k)$ exists, we assume that $\mathbb{E}(X_k) = 0$. For $n = 1, 2, ...,$ and $k = 1, 2, ..., n$, define $X_{kn} = X_k \mathbb{I}_{(|X_k| < n^{1/t})}$ and observe that

$$\Big\{|S_n| > n^{1/t}\Big\} \subset \Big\{|X_k| \geq n^{1/t} \text{ for some } k \leq n\Big\}$$
$$\cup \Big\{|X_k| \leq n^{1/t} \text{ for all } k \leq n \cap \{|S_n| > n^{1/t}\}\Big\}$$

Then

$$\Big\{|S_n| > n^{1/t}\Big\} \subset \Big\{|X_k| \geq n^{1/t} \text{ for some } k \leq n\Big\} \cup \Big\{\Big|\sum_{k=1}^{n}(X_{kn} - \mathbb{E}(X_{kn}))\Big|$$
$$\geq n^{1/t}(1 - n^{1-1/t}|\mathbb{E}(X_{kn})|)\Big\}.$$

That is

$$\Big\{|S_n| > n^{1/t}\Big\} \subset \Big\{|X_k| \geq n^{1/t} \text{ for some } k \leq n\Big\} \\ \cup \left\{\left|\sum_{k=1}^{n}(X_{kn} - \mathbb{E}(X_{kn}))\right| \geq n^{1/t}(1 - n^{1-1/t}|\mathbb{E}(X_{kn})|)\right\}.$$

Then,

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}(|S_n| > n^{1/t}) \leq \sum_{n=1}^{\infty} n^{-1}\mathbb{P}(|X_k| \geq n^{1/t} \text{ for some } k \leq n) \\ + \sum_{n=1}^{\infty} n^{-1}\mathbb{P}\Big(\Big|\sum_{k=1}^{n}(X_{kn} - \mathbb{E}(X_{kn}))\Big| > n^{1/t}(1 - n^{1-1/t}|\mathbb{E}(X_{kn})|)\Big). \tag{B.12}$$

The first series on the right-hand side of (B.12) converges since

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\big(|X_k| \geq n^{1/t} \text{ for some } k \leq n\big) = \sum_{n=1}^{\infty} n^{-1}\mathbb{P}\big(\max_{1\leq k\leq n} |X_k| \geq n^{1/t}\big) \\ \leq \sum_{n=1}^{\infty} \mathbb{P}\big(|X_1| \geq n^{1/t}\big) = \sum_{n=1}^{\infty} \mathbb{P}\big(|X_1|^t \geq n\big).$$

Then by Theorem 3.1, the finiteness of this last series is equivalent to $\mathbb{E}|X_k|^t < \infty$. For $t \geq 1$, suppose that $\mathbb{E}(X_k) = 0$. We have $\mathbb{E}(X_{kn}) = \mathbb{E}(X_{kn} - X_k) = \mathbb{E}(-X_k\mathbb{I}_{(|X_k|\geq n^{1/t})})$, thus

$$|\mathbb{E}(X_{kn})| \leq \mathbb{E}(|X_k|\mathbb{I}_{(|X_k|\geq n^{1/t})}) = n^{1/t}\mathbb{E}\Big(\frac{|X_k|}{n^{1/t}}\mathbb{I}_{(|X_k|\geq n^{1/t})}\Big) \\ \leq n^{1/t}\mathbb{E}\Big(|X_k|^t/n\mathbb{I}_{(|X_k|\geq n^{1/t})}\Big) = n^{1/t-1}\mathbb{E}(|X_k|^t\mathbb{I}_{(|X_k|\geq n^{1/t})}),$$

then

$$n^{1-1/t}|\mathbb{E}X_{kn}| \leq \mathbb{E}(|X_k|^t\mathbb{I}_{(|X_k|\geq n^{1/t})}) \to 0.$$

For $0 < t < 1$, $|n^{1-1/t}X_{kn}| \leq |X_k|^t$, $\forall n \geq 1$. As

$$|n^{1-1/t}X_{kn}| = |X_{kn}|^t\Big(\frac{|X_{kn}|}{n^{1/t}}\Big)^{1-t} \leq |X_{kn}|^t \leq |X_k|^t.$$

Then, by Lebesgue dominated convergence theorem, $\lim\limits_{n\to\infty} n^{1-1/t}|\mathbb{E}(X_{kn})| = 0$. Thus, to show that the second series on the right-hand side of (B.12) is finite, it suffices to show that $\sum_{n=1}^{\infty} n^{-1}\mathbb{P}(|\sum_{k=1}^{n}(X_{kn} - \mathbb{E}X_{kn})| > cn^{1/t}) < \infty$ for some c, $0 < c < 1$. This is done as follows:

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\Big(\Big|\sum_{k=1}^{n}(X_{kn} - \mathbb{E}(X_{kn}))\Big| > cn^{1/t}\Big) \leq c^{-2}\sum_{n=1}^{\infty} n^{-1-2/t}\mathbb{E}\Big(\sum_{k=1}^{n}(X_{kn} - \mathbb{E}(X_{kn}))\Big)^2 \\
&= c^{-2}\sum_{n=1}^{\infty} n^{-1-2/t}\Big(\sum_{k=1}^{n}\mathbb{E}(X_{kn} - \mathbb{E}(X_{kn}))^2\Big) \leq c^{-2}\sum_{n=1}^{\infty} n^{-1-2/t}\Big(\sum_{k=1}^{n}\mathbb{E}(X_{kn}^2)\Big) \\
&= c^{-2}\sum_{n=1}^{\infty} n^{-2/t}\mathbb{E}(X_{1n}^2) \leq c^{-2}\sum_{n=1}^{\infty} n^{-2/t}\sum_{k=1}^{n} k^{2/t}\mathbb{P}(k-1 \leq |X_1|^t < k) \\
&= c^{-2}\sum_{k=1}^{\infty} k^{2/t}\mathbb{P}(k-1 \leq |X_1|^t < k)\cdot\sum_{n=k}^{\infty} n^{-2/t} \leq C\sum_{k=1}^{\infty} k\mathbb{P}(k-1 \leq |X_1|^t < k) \\
&\leq C(\mathbb{E}|X_1|^t + 1) < \infty.
\end{aligned}$$

To prove the converse assertions we may assume $\mu = 0$. The proof proceeds by showing first that $S_n^s/n^{1/t} \xrightarrow[n\to\infty]{\mathbb{P}} 0$, where $X^s = X - X'$. X' is an identical independent copy of X, X^s is denoted as the symmetrized random variable of X. Assume that $S_n^s/n^{1/t}$ does not converges in probability to 0. Then, there exists $\epsilon > 0$ such that either $\mathbb{P}(S_{n_i}^s/n_i^{1/t} > \epsilon) > \epsilon$ or $\mathbb{P}(S_{n_i}^s/n_i^{1/t} < -\epsilon) > \epsilon$ for infinitely many i. For argument's sake assume $\mathbb{P}(S_{n_i}^s/n_i^{1/t} > \epsilon) > \epsilon$ for infinitely many i. Without loss of generality choose $n_{i+1} > 2n_i$. Now, for each j such that $n_i < j \leq 2n_i$, it follows by symmetry that

$$\mathbb{P}\left(\sum_{k=n_i+1}^{j} X_j^s \geq 0\right) \geq 1/2,$$

and thus

$$\mathbb{P}\Big(S_j^s \geq j^{1/t}\epsilon/2^{1/t}\Big) = \mathbb{P}\left(\sum_{k=n_i+1}^{j} X_j^s \geq 0 \big| S_{n_i}^s \geq j^{1/t}\epsilon/2^{1/t}\right) \mathbb{P}\left(S_{n_i}^s \geq j^{1/t}\epsilon/2^{1/t}\right)$$
$$= \frac{1}{2} \cdot \mathbb{P}\left(S_{n_i}^s \geq \left(\frac{j}{2}\right)^{1/t} \epsilon\right) \geq \frac{1}{2}\mathbb{P}(S_{n_i}^s \geq n_i^{1/t}\epsilon) \geq \frac{\epsilon}{2},$$

for $n_i + 1 \leq j \leq 2n_i$. Therefore,

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}(S_n^s/n^{1/t} \geq \epsilon/2^{1/t}) \geq \sum_{i=1}^{\infty}\sum_{n=n_i}^{2n_i} n^{-1}\mathbb{P}(S_n^s/n^{1/t} \geq \epsilon/2^{1/t}) \geq \frac{\epsilon}{2}\sum_{i=1}^{\infty}\sum_{n=n_i}^{2n_i} n^{-1} = \infty. \tag{B.13}$$

However, by the fact that

$$\mathbb{P}(X^s > \epsilon) = \mathbb{P}(X - X' > \epsilon) \leq \mathbb{P}(X > \frac{\epsilon}{2}) + \mathbb{P}\left(-X' > \frac{\epsilon}{2}\right)$$
$$= \mathbb{P}\left(X > \frac{\epsilon}{2}\right) + \mathbb{P}\left(X < -\frac{\epsilon}{2}\right) = \mathbb{P}(|X| > \epsilon/2). \tag{B.14}$$

Thus, combining (B.14) and (B.13), we have

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n|/n^{1/t} \geq \epsilon/2^{1/t+1}) = \infty,$$

this is a contradiction. Thus $S_n^s/n^{1/t} \xrightarrow[n\to\infty]{\mathbb{P}} 0$. Also, note that for all $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}(X^s < -\epsilon) &= \mathbb{P}(X - X' < -\epsilon) \leq \mathbb{P}(X < -\epsilon/2) + \mathbb{P}(-X' < -\epsilon/2) \\ &= \mathbb{P}(X < -\epsilon/2) + \mathbb{P}(X > \epsilon/2) = \mathbb{P}(|X| > \epsilon/2). \end{aligned} \tag{B.15}$$

Combining (B.14) and (B.15), we conclude with the condition (b) or (d) that

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}(|S_n^s| > n^{1/t}\epsilon) < \infty. \tag{B.16}$$

In addition, we also conclude that by $S_n^s/n^{1/t} \xrightarrow[n\to\infty]{\mathbb{P}} 0$. By Theorem A.6, as if $\dfrac{S_n^s}{n^{1/t}} \xrightarrow[n\to\infty]{\mathbb{P}} 0$, then

$$n\mathbb{P}(|X_k^s| > n^{1/t}\epsilon) \xrightarrow[n\to\infty]{} 0.$$

Then, we proceed similarly as Theorem 2.1 and define

$$R_k = \{|X_k^s| > 2n^{1/t}\} \quad \text{and} \quad T_k = \left\{ \left| \sum_{l=1, l\neq k}^{n} X_k^s \right| < n^{1/t} \right\}.$$

Then

$$\{|S_n^s| > n^{1/t}\} \supseteq \bigcup_{k=1}^{n} (R_k \cap T_k),$$

and for sufficiently large n, with $0 < \rho' < \rho < 1$,

$$\begin{aligned}\mathbb{P}(|S_n^s| > n^{1/t}) &\geq \mathbb{P}\left(\bigcup_{k=1}^{n}(R_k \cap T_k)\right) \geq \sum_{k=1}^{n} \mathbb{P}(R_k)(\mathbb{P}(T_k) - (k-1)\mathbb{P}(R_k)) \\ &\geq \sum_{k=1}^{n} \mathbb{P}(R_k)(\rho - n\mathbb{P}(R_k)) = \sum_{k=1}^{n} \mathbb{P}(R_k)(\rho - o(1)) \geq \rho' n\mathbb{P}(R_k).\end{aligned}$$

Combining this last relation with (B.16), we have

$$\infty > \sum_{n=1}^{\infty} n^{-1}\mathbb{P}(|S_n^s| > n^{1/t}) \geq \sum_{n=1}^{\infty} \rho' \mathbb{P}(R_k) = \sum_{n=1}^{\infty} \rho' \mathbb{P}(|X_k^s| > 2n^{1/t})$$

which is, by Corollary 3.2, equivalent to $\mathbb{E}|X_1^s|^t < \infty$. Therefore, $\mathbb{E}|X_1^s|^t < \infty$ and consequently $\mathbb{E}|X_1|^t < \infty$, by Theorems A.9 and A.10. Finally, if $t \geq 1$, it follows from Theorem A.23,

$$\frac{S_n - \mathbb{E}(S_n)}{n^{1/t}} \xrightarrow[n\to\infty]{\text{a.s.}} 0$$

then

$$\frac{S_n - \mathbb{E}(S_n)}{n^{1/t}} \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

We assume that $\mathbb{E}(X_1) = c$, $c \neq 0$, therefore, $\mathbb{P}(|S_n - nc| > n^{1/t}\epsilon) \xrightarrow[n\to\infty]{} 0$, then $\lim_{n\to\infty} \mathbb{P}(|S_n| > n^{1/t}\epsilon) = 1$, which results to

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}(|S_n| > n^{1/t}\epsilon) = \infty,$$

this is a contradiction. Therefore, $\mathbb{E}(X_k) = 0$, this completes the proof. □

Proof of Theorem 2.6. The proof that $(a) \Rightarrow (b)$ has been given in Theorem 2.2 (a). (c) to (d) has been given in Theorem 2.2 (b),(c). It remains only to prove that $(b) \Rightarrow (a)$ and $(d) \Rightarrow (c)$. We need to consider only the case $1 < r < 2$, since for $r \geq 2$, this result has been proven in Theorem 2.3. Thus, we consider the case where $t > 1$, $1 < r < 2$, $1/2 < r/t \leq 1$ or the case where $t > 0$, $1 < r < 2$, $r/t > 1$.

We may assume that $\mu = 0$ if it exists. The proof proceeds by showing first that $S_n^s/n^{r/t} \xrightarrow[n\to\infty]{\mathbb{P}} 0$. Assume that $S_n^s/n^{1/t}$ does not converge in probability to zero. Then there exists $\epsilon > 0$ such that $\mathbb{P}(S_{n_i}^s/n_i > \epsilon) > \epsilon$ for infinitely many i. Without loss of generality, we choose $n_{i+1} > 2n_i$. Now, for each j such that $n_i < j \leq 2n_i$, it follows that $\mathbb{P}(\sum_{k=n_i+1}^{j} X_k^s \leq 0) \geq 1/2$ and then $\mathbb{P}(\sum_{k=1}^{j} X_k^s \geq j^{r/t}\epsilon/2^{r/t}) \geq \epsilon/2$. Since

$$\mathbb{P}\left(S_j^s \geq j^{r/t}/2^{r/t}\right) \geq \mathbb{P}(S_{n_i}^s \geq j^{r/t}/2^{r/t}) \cdot \mathbb{P}\left(\sum_{k=n_i+1}^{j} X_k^s \geq 0\right)$$
$$\geq \mathbb{P}(S_{n_i}^s \geq n_i^{r/t}) \cdot \mathbb{P}\left(\sum_{k=n_i+1}^{j} X_k^s \geq 0\right) \geq \epsilon/2.$$

Therefore,

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}(S_n^s/n^{r/t} > \epsilon/2) \geq \sum_{i=1}^{\infty}\sum_{n=n_i}^{2n_i} n^{-1}\mathbb{P}(S_n^s/n^{r/t}) \geq \frac{\epsilon}{2} \cdot \sum_{i=1}^{\infty}\sum_{n=n_i}^{2n_i} n^{-1} = \infty,$$

which leads to

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}(|S_n/n^{r/t}| > \epsilon/2^{r/t+1}) = \infty \quad \forall \epsilon > 0.$$

But, since $1 < r < 2$, $n^{r-2} > n^{-1}$, it implies that

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(|S_n/n^{r/t}| > \epsilon) = \infty \quad \forall \epsilon > 0.$$

This is a contradiction to the condition. Then $S_n^s/n^{r/t} \xrightarrow[n\to\infty]{\mathbb{P}} 0$. In addition, by Theorem A.6, we conclude that $n\mathbb{P}(|X_k^s| > n^{r/t}\epsilon) \xrightarrow[n\to\infty]{} 0$. Further, as in proof of

Theorem 2.3,

$$\mathbb{P}(|S_n^s| > n^{r/t}\epsilon) \geq \rho' n \mathbb{P}(|X_k^s| > 2\epsilon n^{r/t}).$$

Then,

$$\infty > \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}(|S_n^s| > n^{r/t}\epsilon) \geq \rho' \sum_{n=1}^{\infty} n^{r-1}\mathbb{P}(|X_k^s| > 2\epsilon n^{r/t}).$$

Therefore, we get, by Corollary 3.2, $\mathbb{E}|X_1^s|^t < \infty$, and thanks to Theorem A.9, this is equivalent to $\mathbb{E}|X_1|^t < \infty$. By Kolmogorov-Marcinkiewicz-Zygmund SLLN, and the fact that convergence a.s. implies convergence in probability, we have $(S_n - n\mathbb{E}(X_1))/n^{r/t} \xrightarrow[n\to\infty]{\mathbb{P}} 0$, then, since $\frac{r}{t} \leq 1$, $\mathbb{E}(X_1) = 0$. □

Appendix C

Proofs of theorems from Chapter 3

In this appendix, we complete the proofs of Theorems 3.7 to 3.10. In summary, the methods of proofs are similar to that used in Li and Rosalsky (2006). We also outline some other results given in Li and Rosalsky (2006), which are used in establishing the main results of this thesis.

Proof of Theorem 3.4. By (3.11) we have $\forall \epsilon_1 > 0$, $\exists \delta > 0$ such that $\forall\, 0 < c - 1 < \delta$, $\left| \limsup_{n\to\infty} \frac{a_{[cn]}}{a_n} - 1 \right| < \epsilon_1$, then $\forall \epsilon_2 > 0$, $\exists N_{\epsilon_2} > 0$, such that, $\forall n > N_{\epsilon_2}$,

$$\left| \sup_{k\geq n} \frac{a_{[ck]}}{a_k} - 1 \right| < \epsilon_1 + \epsilon_2,$$

then,

$$\frac{a_{[cn]}}{a_n} \leq \sup_{k\geq n} \frac{a_{[ck]}}{a_k} < 1 + \epsilon_1 + \epsilon_2.$$

Thus, it means that, for $\forall \delta > 0$, we can choose some $c_\delta > 1$ and big enough $N_{\delta,c}$, such that for all $n > N_{\delta,c}$, $a_{[cn]} \leq (1+\delta)a_n$. Also, note that for all large n,

$$\max_{c^{n-1}<m\leq c^n} T_m \leq \max_{c^{n-1}<m\leq c^n} \max_{1\leq i\neq j\leq p_{[c^n]}} \left|\sum_{k=1}^{m} U_{k,i}V_{k,j}\right|.$$

Then

$$\max_{c^{n-1}<m\leq c^n} T_m \leq \max_{1\leq i\neq j\leq p_{[c^n]}} \max_{c^{n-1}<m\leq c^n} \left|\sum_{k=1}^{m} U_{k,i}V_{k,j}\right| =: H_n,$$

and hence for all large n, we have

$$\mathbb{P}\left(\max_{c^{n-1}<m\leq c^n} \frac{T_m}{a_m} > (1+3\delta)^2\lambda\right) \leq \mathbb{P}\left(\frac{H_n}{a_{[c^{n-1}]}} > (1+3\delta)^2\lambda,\ \frac{a_{[c^{n-1}]}}{a_{[c^n]}} \geq \frac{1}{1+3\delta}\right)$$
$$\leq \mathbb{P}\left(\frac{H_n}{a_{[c^n]}} > (1+3\delta)\lambda\right) \leq \left(p_{[c^n]}\right)^2 \mathbb{P}\left(\max_{c^{n-1}<m\leq c^n} \frac{\left|\sum_{k=1}^{m} U_{k,1}V_{k,2}\right|}{a_{[c^n]}} > (1+3\delta)\lambda\right).$$

Therefore,

$$\mathbb{P}\left(\max_{c^{n-1}<m\leq c^n} \frac{T_m}{a_m} > (1+3\delta)^2\lambda\right) \leq \left(p_{[c^n]}\right)^2 \mathbb{P}\left(\max_{c^{n-1}<m\leq c^n} \frac{|S_m|}{a_{[c^n]}} > (1+3\delta)\lambda\right).$$

Note that, by Theorem A.5, (3.12) ensures that,

$$\lim_{n\to\infty} \min_{1\leq k\leq n} \mathbb{P}(S_n - S_k > -\delta\lambda a_n) = 1,$$

and

$$\lim_{n\to\infty} \min_{1\leq k\leq n} \mathbb{P}(S_n - S_k < \delta\lambda a_n) = 1.$$

We have for all large n, $\mathbb{P}\left(\frac{S_n - S_k}{a_n} \geq -\delta\lambda\right) \geq \frac{1}{2}, \quad k = 1, ..., n-1$. Let $S'_k = \frac{S_k}{a_n}$, we obtain

$$\mathbb{P}\left(S'_n - S'_k > -\delta\lambda\right) \geq \frac{1}{2}, k = 1, ..., n-1.$$

Further, we have

$$\mathbb{P}\bigg(\max_{c^{n-1}<m\leq c^n} \frac{T_m}{a_m} > (1+3\delta)^2\lambda\bigg) \leq \big(p_{[c^n]}\big)^2 \mathbb{P}\bigg(\max_{1\leq m\leq c^n} |S'_m| > (1+3\delta)\lambda\bigg).$$

Then, by applying Theorem A.15, we have (by letting $q = \frac{1}{2}$),

$$\mathbb{P}\bigg(\max_{c^{n-1}<m\leq c^n} \frac{T_m}{a_m} > (1+3\delta)^2\lambda\bigg) \leq 2\big(p_{[c^n]}\big)^2 \mathbb{P}\bigg(\frac{|S_{[c^n]}|}{a_{[c^n]}} > (1+2\delta)\lambda\bigg).$$

Next, for all large n and all $m \in \big[[c^n], [c^{n+1}] - 1\big]$,

$$\mathbb{P}\bigg(\frac{|S_{[c^n]}|}{a_{[c^n]}} > (1+2\delta)\lambda\bigg) \leq \mathbb{P}\bigg(\max_{1\leq j\leq m} \frac{|S_j|}{a_{[c^n]}} > (1+2\delta)\lambda\bigg),$$

and again, by applying Theorem A.15, we have

$$\begin{aligned}\mathbb{P}\bigg(\frac{|S_{[c^n]}|}{a_{[c^n]}} > (1+2\delta)\lambda\bigg) \leq\ & 2\mathbb{P}\bigg(\frac{|S_m|}{a_{[c^n]}} > (1+\delta)\lambda, \frac{a_{[c^n]}(1+\delta)}{a_{[c^{n+1}]}} \geq 1\bigg) \\ \leq 2\mathbb{P}\bigg(\frac{(1+\delta)|S_m|}{a_{[c^{n+1}]}} > (1+\delta)\lambda\bigg) =\ & 2\mathbb{P}\bigg(\frac{|S_m|}{a_{[c^{n+1}]}} > \lambda\bigg) \leq 2\mathbb{P}\bigg(\frac{|S_m|}{a_m} > \lambda\bigg).\end{aligned}$$

In summation, for all $m \in \big[[c^n], [c^{n+1}] - 1\big]$, and sufficiently large n,

$$\mathbb{P}\bigg(\max_{c^{n-1}<k\leq c^n} \frac{T_k}{a_k} > (1+3\delta)^2\lambda\bigg) \leq 4\big(p_{[c^n]}\big)^2 \mathbb{P}\bigg(\frac{|S_m|}{a_m} > \lambda\bigg).$$

Thus,

$$\mathbb{P}\left(\max_{c^{n-1}<k\leq c^n}\frac{T_k}{a_k}>(1+3\delta)^2\lambda\right)\leq\ 4\big(p_{[c^n]}\big)^2\frac{\sum_{m=[c^n]}^{[c^{n+1}]-1}\mathbb{P}\left(\frac{|S_m|}{a_m}>\lambda\right)}{[c^{n+1}]-[c^n]}.$$

Define α_n, such that, $c^{n+1}-\alpha_{n+1}=[c^{n+1}]$,

$$[c^{n+1}]-[c^n]=c^{n+1}-c^n-(\alpha_{n+1}-\alpha_n)=\frac{c-1}{c}[c^{n+1}]+\frac{c-1}{c}\alpha_{n+1}-(\alpha_{n+1}-\alpha_n).$$

We have

$$\lim_{n\to\infty}\frac{[c^{n+1}]-[c^n]}{\frac{c-1}{c}([c^{n+1}]-1)}=\lim_{n\to\infty}\frac{\frac{c-1}{c}[c^{n+1}]+\frac{c-1}{c}\alpha_{n+1}-(\alpha_{n+1}-\alpha_n)}{\frac{c-1}{c}[c^{n+1}]-\frac{c-1}{c}}=1,$$

which means that $[c^{n+1}]-[c^n]\sim\frac{c-1}{c}([c^{n+1}]-1)$. Then for sufficiently large n, and p_n being nondecreasing,

$$\begin{aligned}\mathbb{P}\left(\max_{c^{n-1}<k\leq c^n}\frac{T_k}{a_k}>(1+3\delta)^2\lambda\right)&\leq 4\big(p_{[c^n]}\big)^2\frac{\sum_{m=[c^n]}^{[c^{n+1}]-1}\mathbb{P}\left(\frac{|S_m|}{a_m}>\lambda\right)}{[c^{n+1}]-[c^n]}\\ \leq\ \frac{8c}{c-1}\big(p_{[c^n]}\big)^2\frac{\sum_{m=[c^n]}^{[c^{n+1}]-1}\mathbb{P}\left(\frac{|S_m|}{a_m}>\lambda\right)}{[c^{n+1}]-1}&\leq\frac{8c}{c-1}\sum_{m=[c^n]}^{[c^{n+1}]-1}\frac{p_m^2}{m}\mathbb{P}\left(\frac{|S_m|}{a_m}>\lambda\right).\end{aligned}$$

By (3.13), we have

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\max_{c^{n-1}<m\leq c^n}\frac{T_m}{a_m}>(1+3\delta)^2\lambda\right)<\infty.$$

Hence, by Borel-Cantelli Lemma,

$$\mathbb{P}\Big(\bigcap_{n\geq 1}\bigcup_{k\geq n}\Big\{\frac{T_k}{a_k} > (1+3\delta)^2\lambda\Big\}\Big) = 0 \Longleftrightarrow \mathbb{P}\Big(\frac{T_n}{a_n} > (1+3\delta)^2\lambda \quad \text{i.o.}(n)\Big) = 0.$$

In view of Theorem A.4, we have $\mathbb{P}\left(\limsup_{n\to\infty}\frac{T_n}{a_n} > (1+3\delta)^2\lambda\right) = 0, \quad \forall\delta > 0$. This is equivalent to

$$\mathbb{P}\Big(\limsup_{n\to\infty}\frac{T_n}{a_n} \leq (1+3\delta)^2\lambda\Big) = 1 \quad \forall\delta > 0.$$

Letting $\delta \downarrow 0$, we get $\limsup_{n\to\infty}\frac{T_n}{a_n} \leq \lambda$ a.s., as desired result. □

Corollary C.1. *Let $\beta > 0$ and $\alpha \in \left(\frac{1}{2}, 1\right]$. Suppose $\mathbb{E}(U_{1,1})\mathbb{E}(V_{1,1}) = 0$, $\mathbb{E}|U_{1,1}|^{\frac{2\beta+1}{\alpha}} < \infty$ and $\mathbb{E}|V_{1,1}|^{\frac{2\beta+1}{\alpha}} < \infty$, then*

$$\lim_{n\to\infty}\frac{\max_{1\leq i\neq j\leq n^\beta}\left|\sum_{k=1}^n U_{k,i}V_{k,j}\right|}{n^\alpha} = 0 \ a.s.$$

Proof. Let $a_n = n^\alpha$, easy to verify that $\lim_{c\downarrow 1}\limsup_{n\to\infty}\frac{a_{[cn]}}{a_n} = 1$. Let $p_n = [n^\beta]$, by the condition that $\mathbb{E}\left(|Y_1|^{\frac{2\beta+1}{\alpha}}\right) < \infty$ and by Theorem 2.8 $\left(t = \frac{2\beta+1}{\alpha}, r = 2\beta+1\right)$,

$$\sum_{n=1}^{\infty}\frac{p_n^2}{n}\mathbb{P}\Big(\frac{|S_n|}{a_n} > \lambda\Big) \leq \sum_{n=1}^{\infty}n^{2\beta-1}\mathbb{P}\Big(\frac{|S_n|}{n^\alpha} > \lambda\Big) < \infty, \quad \forall\lambda > 0.$$

Further, since $\mathbb{E}\left(|Y_1|^{\frac{2\beta+1}{\alpha}}\right) < \infty$, we have that $\mathbb{E}|Y_1|^{\frac{1}{\alpha}} < \infty$. Then by $\alpha \in (1/2, 1]$ and Kolmogrov-Marcinkiewicz-Zygmund SLLN (See Theorem A.23) $\frac{S_n}{n^\alpha} \xrightarrow{\text{a.s.}} 0$, then

$\frac{S_n}{n^\alpha} \xrightarrow{\mathbb{P}} 0$. Now applying Theorem 3.4, we have $\limsup_{n\to\infty} \frac{T_n}{a_n} \le \lambda$ a.s., i.e.,

$$\limsup_{n\to\infty} \frac{\max_{i\le i\ne j\le n^\beta} |\sum_{k=1}^n U_{k,i}V_{k,j}|}{n^\alpha} \le \lambda \text{ a.s. } \forall\lambda > 0.$$

Let $\lambda \downarrow 0$, we have $\lim_{n\to\infty} \frac{\max_{i\le i\ne j\le n^\beta} |\sum_{k=1}^n U_{k,i}V_{k,j}|}{n^\alpha} = 0$ a.s. □

Corollary C.2. *Let* $\beta > 0$, $\mathbb{E}(Y_1) = 0$, $E(Y_1^2) = 1$, *and* $\mathbb{E}\Big(\frac{|Y_1|^{4\beta+2}}{(\log(e+|Y_1|))^{2\beta+1}}\Big) < \infty$. *Then*

$$\limsup_{n\to\infty} \frac{\max_{1\le i\ne j\le n^\beta} \big|\sum_{k=1}^n U_{k,i}V_{k,j}\big|}{\sqrt{n\log n}} \le 2\sqrt{\beta} \ a.s.$$

Proof. Let $a_n = \sqrt{n\log n}$, one verifies that $\lim_{c\downarrow 1}\limsup_{n\to\infty} \frac{a_{[cn]}}{a_n} = 1$. By Chebyshev's inequality, we have

$$\mathbb{P}\Big(\frac{|S_n|}{a_n} \ge \epsilon\Big) \le \frac{\mathrm{Var}\big(\frac{S_n}{a_n}\big)}{\epsilon^2} = \frac{\mathrm{Var}(S_n)}{\epsilon^2 a_n^2} = \frac{n}{\epsilon^2 n\log n} = \frac{1}{\epsilon^2\log n} \longrightarrow 0, \quad \forall\epsilon > 0.$$

Then, we have $\frac{S_n}{a_n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$. Next by Theorem 2.10, and condition of this corollary and taking $p_n = [n^\beta]$, we have

$$\sum_{n=2}^\infty \frac{p_n^2}{n}\mathbb{P}\Big(\frac{|S_n|}{a_n} > \lambda\Big) \le \sum_{n=2}^\infty n^{2\beta-1}\mathbb{P}\Big(\frac{|S_n|}{\sqrt{n\log n}} > \lambda\Big) < \infty, \quad \forall\lambda > 2\sqrt{\beta}.$$

Then, by Theorem 3.4, we have $\limsup_{n\to\infty} \frac{T_n}{a_n} \le \lambda$ a.s. $\forall\lambda > 2\sqrt{\beta}$. By letting $\lambda \downarrow 2\sqrt{\beta}$, we get

$$\limsup_{n\to\infty} \frac{\max_{1\le i\ne j\le n^\beta} \big|\sum_{k=1}^n U_{k,i}V_{k,j}\big|}{\sqrt{n\log n}} \le 2\sqrt{\beta} \text{ a.s.}$$

□

Lemma C.1. *Let $\{a_n; n \geq 1\}$ be a nondecreasing sequence of positive constants such that* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ *and* $\liminf_{n\to\infty} \frac{a_{2n}}{a_n} = b \in (1, \infty]$. *Then, for every $c > 0$ and $q > 1$, the following three statements are equivalent,*

(i). $\sum_{n=1}^{\infty} \mathbb{P}\bigg(\max_{1\leq i\neq j\leq n} |U_{1,i}V_{1,j}| \geq a_n \bigg) < \infty$,

(ii). $\sum_{n=1}^{\infty} \mathbb{P}\bigg(\max_{1\leq i\neq j\leq cn} |U_{1,i}V_{1,j}| \geq \epsilon a_n \bigg) < \infty$ *for all* $\epsilon > 0$,

(iii). $\sum_{n=1}^{\infty} \mathbb{P}\bigg(\max_{1\leq m\leq q^n} \max_{1\leq i\neq j\leq cm} |U_{m,i}V_{m,j}| \geq \epsilon a_{[q^n]} \bigg) < \infty,$ *for all* $\epsilon > 0$.

The proof is given in Li and Rosalsky (2006).

Proof of Lemma 3.3. By n/p_n is bounded away from 0 to ∞, there exists a constant $c \geq 1$, such that, $c^{-1}n \leq p_n \leq cn, n \geq 1$, then

$$\limsup_{n\to\infty} \frac{\max_{1\leq i\neq j\leq c^{-1}n} \big| \sum_{k=1}^{n} U_{k,i}V_{k,j} \big|}{a_n} < \infty \text{ a.s.}$$

by $\frac{a_{n+1}}{a_n} \xrightarrow[n\to\infty]{} 1$, $\limsup_{n\to\infty} \frac{\max_{1\leq i\neq j\leq c^{-1}n} \big| \sum_{k=1}^{n+1} U_{k,i}V_{k,j} \big|}{a_n} < \infty$ a.s. then, by triangle inequality, we have

$$\begin{aligned}
&\max_{1\leq i\neq j\leq c^{n-1}n} |U_{n+1,i}V_{n+1,j}| \leq \max_{1\leq i\neq j\leq c^{-1}n} \bigg(\Big| \sum_{k=1}^{n} U_{k,i}V_{k,j} \Big| + \Big| \sum_{k=1}^{n+1} U_{k,i}V_{k,j} \Big| \bigg) \\
&\leq \max_{1\leq i\neq j\leq c^{-1}n} \Big| \sum_{k=1}^{n} U_{k,i}V_{k,j} \Big| + \max_{1\leq i\neq j\leq c^{-1}n} \Big| \sum_{k=1}^{n+1} U_{k,i}V_{k,j} \Big|, \quad n \geq 1
\end{aligned}$$

thus, we have

$$\limsup_{n\to\infty} \frac{\max_{1\leq i\neq j\leq c^{-1}n} \big| U_{n+1,i}V_{n+1,j} \big|}{a_n} < \infty \text{ a.s.}$$

By another application of Borel Cantelli lemma, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \le i \ne j \le c^{-1}n} |U_{1,i}V_{1,j}| \ge \lambda a_n \right) < \infty, \text{ for some } \lambda > 0$$

which is equivalent to, by Lemma C.1,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \le i \ne j \le n} |U_{1,i}V_{1,j}| \ge \lambda a_n \right) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \le i \ne j \le n} |U_{1,i}V_{1,j}| \ge a_n \right) < \infty.$$

Now the first conclusion has been proved. Next by

$$\limsup_{n\to\infty} \frac{\left|\sum_{k=1}^{n} U_{k,1}V_{k,2}\right|}{a_n} < \infty \text{ a.s.}$$

and the condition that $\frac{a_n}{n} \xrightarrow[n\to\infty]{} 0$ we have

$$\limsup_{n\to\infty} \frac{\left|\sum_{k=1}^{n} U_{k,1}V_{k,2}\right|}{n} = 0 \text{ a.s.} \Longleftrightarrow \lim_{n\to\infty} \frac{\left|\sum_{k=1}^{n} U_{k,1}V_{k,2}\right|}{n} = 0 \text{ a.s.}$$

by SLLN (Theorem A.18), we have $\mathbb{E}(Y_1) = 0$. This completes the proof. □

Corollary C.3. *Let* $\{X_{k,i}; k \ge 1, i \ge 1\}$ *be an array of i.i.d. random variables. Suppose that* n/p_n *is bounded away from* 0 *and* ∞.

(I) Let $\alpha \in \left(\frac{1}{2}, 1\right]$, *then,* $\lim_{n\to\infty} \frac{\max_{1\le i\le p_n} \left|\sum_{k=1}^{n} X_{k,i}\right|}{n^{\alpha}} = 0$ *a.s., if and only if* $\mathbb{E}|X_{1,1}|^{2/\alpha} < \infty, \quad \mathbb{E}(X_{1,1}) = 0.$

(II) If $\mathbb{E}(X_{1,1}) = 0, \quad \mathbb{E}(X_{1,1}^2) = 1$ *and* $\mathbb{E}\left(\frac{X_{1,1}^4}{\log^2(e + |X_{1,1}|)}\right) < \infty$, *then*

$$\limsup_{n\to\infty} \frac{\max_{1\le i\le p_n} \left|\sum_{k=1}^{n} X_{k,i}\right|}{\sqrt{n\log n}} \le 2 \text{ a.s.}$$

Conversely if

$$\limsup_{n\to\infty} \frac{\max_{1\le i\le p_n} \left|\sum_{k=1}^{n} X_{k,i}\right|}{\sqrt{n\log n}} < \infty \ a.s.$$

then,

$$\mathbb{E}(X_{1,1}) = 0, \quad \mathbb{E}(|X_{1,1}|^{\beta}) < \infty, \forall\, 0 \le \beta < 4 \text{ and } \mathbb{E}\left(\frac{X_{1,1}^4}{\log^2(e+|X_{1,1}|)}\right) < \infty.$$

Proof. It follows, respectively, from Theorem 3.5 and 3.6. □

Proof of part of Theorem 3.8. In view of Corollary 3.6 and Corollary 3.2, we have $\mathbb{E}|X_1|^{3/\alpha} < \infty$, with $\alpha \in (1/2, 1]$. Let $\mathbb{E}(X_{1,1}) = \mu$. Since $X_{1,1}$ is nondegenerate, $0 < \sigma^2 = \mathbb{E}(X_{1,1} - \mu)^2 < \infty$. Note that for $1 \le i \le p_n$,

$$\sum_{k=1}^{n}(X_{k,i} - \bar{X}_i^{(n)})^2 = \sum_{k=1}^{n}(X_{k,i} - \mu)^2 - n(\mu - \bar{X}_i^{(n)})^2 \tag{C.1}$$

and for $0 \le i, j \le p_n$,

$$\sum_{k=1}^{n}(X_{k,i} - \bar{X}_i^{(n)})(X_{k,j} - \bar{X}_j^{(n)}) = \sum_{k=1}^{n}(X_{k,i} - \mu)(X_{k,j} - \mu) - n(\bar{X}_i^{(n)} - \mu)(\bar{X}_j^{(n)} - \mu).$$

By Corollary C.3, with $\mathbb{E}|X_1|^{3/\alpha} < \infty$, $1/2 < \alpha \le 1$, we have

$$\lim_{n\to\infty} \max_{1\le i\le p_n} |\bar{X}_i^{(n)} - \mu| = 0 \quad \text{a.s.}$$

Thus,

$$\lim_{n\to\infty} n^{1-\alpha} \max_{1\le i,j\le p_n} |\bar{X}_i^{(n)} - \mu||\bar{X}_j^{(n)} - \mu| \le \lim_{n\to\infty} (n^{1-\alpha} \max_{1\le i\le p_n} |\bar{X}_i^{(n)} - \mu|)^2 = 0 \quad \text{a.s.}$$

This implies

$$\liminf_{n\to\infty} \min_{1\le i\le p_n} \frac{\sum_{k=1}^n (X_{k,i}-\bar{X}_i^{(n)})^2}{n} = \liminf_{n\to\infty} \min_{1\le i\le p_n} \frac{\sum_{k=1}^n (X_{k,i}-\mu)^2}{n}$$
$$\ge \liminf_{n\to\infty} \min_{1\le i\le p_n} \frac{\sum_{k=1}^n (X_{k,i}-\mu)^2 \mathbb{I}_{(|X_{k,i}-\mu|\le b)}}{n} = A.$$

Define $Y_{k,i}(b) := (X_{k,i}-\mu)\mathbb{I}_{(|X_{k,i}-\mu|\le b)}$. We have

$$A = \mathbb{E}Y_{1,1}^2(b) - \limsup_{n\to\infty} \max_{1\le i\le p_n} \frac{\sum_{k=1}^n \left[\mathbb{E}Y_{1,1}^2(b) - Y_{k,i}^2(b)\right]}{n}.$$

Next, by Corollary C.3(I) with $\mathbb{E}|Y_{1,1}(b)|^{4/\alpha} < \infty$, $\forall b > 0$, we have

$$A \ge \mathbb{E}Y_{1,1}^2(b) - \limsup_{n\to\infty} \max_{1\le i\le p_n} \frac{\sum_{k=1}^n \left|\mathbb{E}Y_{1,1}^2(b) - Y_{k,i}^2(b)\right|}{n} = \mathbb{E}Y_{1,1}^2(b) \quad \text{a.s.}$$

Let $b \uparrow \infty$, we get

$$\liminf_{n\to\infty} \min_{1\le i\le p_n} \frac{\sum_{k=1}^n \left(X_{k,i}-\bar{X}_i^{(n)}\right)^2}{n} \ge \sigma^2 \quad \text{a.s.}$$

Further, $\mathbb{E}\left(|X_1|^{3/\alpha}\right) < \infty$, which is, by Corollary 3.5, equivalent to

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\le i<j\le n} |(X_{1,i}-\mu)(X_{1,j}-\mu)| \ge n^\alpha\right) < \infty.$$

Hence, by applying Theorem 3.7, we get

$$\begin{aligned}
&\limsup_{n\to\infty} n^{1-\alpha} L_n \\
=&\limsup_{n\to\infty} \max_{1\le i<j\le p_n} n^{1-\alpha} \frac{\sum_{k=1}^{n}(X_{k,i}-\bar{X}_i^{(n)})(X_{k,j}-\bar{X}_j^{(n)})}{\Big(\sum_{k=1}^{n}(X_{k,i}-\bar{X}_i^{(n)})^2\Big)^{1/2}\Big(\sum_{k=1}^{n}(X_{k,i}-\bar{X}_i^{(n)})^2\Big)^{1/2}} \\
\le&\frac{1}{\sigma^2}\limsup_{n\to\infty} \frac{\max_{1\le i<j\le p_n}|\sum_{k=1}^{n}(X_{k,i}-\mu)(X_{k,j}-\mu)|}{n^{\alpha}} = 0 \quad \text{a.s.}
\end{aligned}$$

□

Theorem C.1. *Let $\{\xi, x_{i,j}; i,j = 1,2,...\}$ are i.i.d. with $\mathbb{E}(\xi) = 0$ and $Var(\xi) = 1$. Suppose that $\mathbb{E}(|\xi|^{30-\epsilon}) < \infty$ for $\epsilon > 0$. If $n/p_n \to \gamma \in (0,\infty)$, then*

$$\text{(i)} \limsup_{n\to\infty} \frac{W_n}{\sqrt{n\log n}} \le 2 \quad \text{a.s.}$$
$$\text{(ii)} \limsup_{n\to\infty} \frac{W_n}{\sqrt{n\log n}} \ge 2 \quad \text{a.s.}$$

The proof of this theorem is given in Jiang (2004).

Proof of the equivalence between parts (2) and (3) of Theorem 3.9. We suppose that part (2) holds. By Theorem 3.6, we have

$$\limsup_{n\to\infty} \frac{W_n}{\sqrt{n\log n}} \le 2 \text{ a.s.}$$

thus left to prove

$$\limsup_{n\to\infty} \frac{W_n}{\sqrt{n\log n}} \ge 2 \text{ a.s.}$$

to show this, for arbitrary $b > 0$ and $k \geq 1$, $i \geq 1$, set

$$U_{k,i}(b) = X_{k,i}\mathbb{I}_{(|X_{k,i}|\leq b)} - \mathbb{E}(X_{1,1}\mathbb{I}_{(|X_{1,1}|\leq b)})$$
$$V_{k,i}(b) = X_{k,i}\mathbb{I}_{(|X_{k,i}|> b)} - \mathbb{E}(X_{1,1}\mathbb{I}_{(|X_{1,1}|> b)})$$

Note that

$$\begin{aligned}
W_n &= \max_{1\leq i\neq j\leq p_n}\Big|\sum_{k=1}^{n} X_{k,i}X_{k,j}\Big| \geq \max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} X_{k,i}X_{k,j}\Big| \\
&= \max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n}\big[U_{k,i}(b)+V_{k,i}(b)\big]\big[U_{k,j}(b)+V_{k,j}(b)\big]\Big| \\
&= \max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} U_{k,i}(b)U_{k,j}(b) + \sum_{k=1}^{n} V_{k,i}(b)U_{k,j}(b) \\
&\qquad + \sum_{k=1}^{n} V_{k,j}(b)U_{k,i}(b) + \sum_{k=1}^{n} V_{k,i}(b)V_{k,j}(b)\Big|.
\end{aligned}$$

Then

$$\begin{aligned}
W_n \geq& \max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} U_{k,i}(b)U_{k,j}(b)\Big| - \max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} V_{k,i}(b)U_{k,j}(b)\Big| \\
&- \max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} U_{k,i}(b)V_{k,j}(b)\Big| - \max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} V_{k,i}(b)V_{k,j}(b)\Big|.
\end{aligned}$$

Since

$$\max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} V_{k,i}(b)U_{k,j}(b)\Big| = \max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} U_{k,i}(b)V_{k,j}(b)\Big|,$$

then,

$$W_n \geq \max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} U_{k,i}(b)U_{k,j}(b)\Big| - 2\cdot \max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} U_{k,i}(b)V_{k,j}(b)\Big|$$
$$- \max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} V_{k,i}(b)V_{k,j}(b)\Big|,$$

where $c \geq 1$ is a constant such that $n/c \leq p_n \leq cn$, $n \geq 1$. Applying Theorem C.1 and Theorem 3.6,

$$\liminf_{n\to\infty}\frac{W_n}{\sqrt{n\log n}} \geq \liminf_{n\to\infty}\frac{\max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} U_{k,i}(b)U_{k,j}(b)\Big|}{\sqrt{n\log n}}$$
$$- 2\cdot\limsup_{n\to\infty}\frac{\max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} U_{k,i}(b)V_{k,j}(b)\Big|}{\sqrt{n\log n}}$$
$$- \limsup_{n\to\infty}\frac{\max_{1\leq i\neq j\leq n/c}\Big|\sum_{k=1}^{n} U_{k,i}(b)V_{k,j}(b)\Big|}{\sqrt{n\log n}}.$$

This gives

$$\liminf_{n\to\infty}\frac{W_n}{\sqrt{n\log n}} \geq 2\mathbb{E}\left(U_{1,1}^2(b)\right) - 4\sqrt{E\left(U_{1,1}^2(b)\right)}\sqrt{\mathbb{E}\left(V_{1,1}^2(b)\right)} - 2\mathbb{E}\left(V_{1,1}^2(b)\right) \quad \text{a.s.}$$

Let $b\uparrow\infty$, we have

$$U_{1,1}^2(b) \xrightarrow[b\to\infty]{\text{a.s.}} X_{1,1}^2 \quad \text{and} \quad V_{1,1}^2(b) \xrightarrow[b\to\infty]{\text{a.s.}} 0.$$

Then, by Lebesgue dominated convergence theorem,

$$\lim_{b\to\infty}\mathbb{E}\left(U_{1,1}^2(b)\right) = 1 \quad \text{and} \quad \lim_{b\to\infty}\mathbb{E}\left(V_{1,1}^2(b)\right) = 0.$$

Thus

$$\liminf_{n\to\infty} \frac{W_n}{\sqrt{n\log n}} \geq 2.$$

We now show that part (3) of Theorem 3.9 implies part (2). In view of Lemma 3.3, $\lim_{n\to\infty} \frac{W_n}{\sqrt{n\log n}} = 2$ a.s. implies that $\mathbb{E}(X_{1,1}) = 0$ and $\mathbb{E}(X_{1,1}^2) = \sigma^2 < \infty$, also we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq i<j\leq n} |X_{1,i}X_{1,j}| \geq \sqrt{n\log n} \right) < \infty$$

which is, by Lemma C.1, equivalent to

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq i<j\leq n} |X_{1,i}X_{1,j}| \geq \sigma^2\sqrt{n\log n} \right) < \infty.$$

Then, we have

$$\lim_{n\to\infty} \frac{1}{\sigma^2} \cdot \frac{W_n}{\sqrt{n\log n}} = 2 = \lim_{n\to\infty} \frac{W_n}{\sqrt{n\log n}},$$

which implies that $\sigma = 1$ and the proof is complete. □

Proof of part (1) of Theorem 3.10 implying (3.17). In view of Corollary 3.4, it implies that $\mathbb{E}(X_{1.1}^{\beta}) < \infty$, for all $0 \leq \beta < 6$. Let $\mu = \mathbb{E}(X_{1,1})$, also, $\mathbb{E}(X_{1,1}-\mu)^2 < \infty$. By the assumption that $X_{1,1}$ is nondegenerate, we have $\mathbb{E}(X_{1,1}-\mu)^2 > 0$. By Corollary C.3, since $\mathbb{E}|X_{1,1}-\mu|^2 < \infty$ and $\mathbb{E}(X_{1,1}-\mu) = 0$, then

$$\lim_{n\to\infty} \max_{1\leq i\leq p_n} |\bar{X}_i^{(n)} - \mu| = 0, \quad \text{a.s.} \tag{C.2}$$

Further,

$$\mathbb{E}|(X_{1,1}-\mu)^2 - \sigma^2|^2 = \mathbb{E}(X_{1,1}-\mu)^4 - 2\sigma^2(X_{1,1}-\mu)^2 + \sigma^4 < \infty,$$

and

$$\mathbb{E}\big[(X_{1,1}-\mu)^2-\sigma^2\big]=0,$$

then, by Corollary C.3 again we have

$$\lim_{n\to\infty}\max_{1\le i\le p_n}\Big|\frac{\sum_{k=1}^{n}\big((X_{k,i}-\mu)^2-\sigma^2\big)}{n}\Big|=0,\quad \text{a.s.} \tag{C.3}$$

and by part (ii) of Corollary C.3, we have

$$\limsup_{n\to\infty}\Big(\frac{n}{\log n}\Big)^{1/2}\max_{1\le i\le p_n}|\bar{X}_i^{(n)}-\mu|\le 2\sigma\quad \text{a.s.} \tag{C.4}$$

By condition (1), it is easy to see

$$\sum_{n=1}^{\infty}\mathbb{P}\Big(\max_{1\le i<j\le n}|X_{1,i}-\mu||X_{1,j}-\mu|\ge \sigma^2\sqrt{n\log n}\Big)<\infty. \tag{C.5}$$

Also, by the fact that

$$\left|\sum_{k=1}^{n}(X_{k,i}-\bar{X}_i^{(n)})(X_{k,j}-\bar{X}_j^{(n)})-\sum_{k=1}^{n}(X_{k,i}-\mu)(X_{k,j}-\mu)\right|=n|\bar{X}_i-\mu||\bar{X}_j^{(n)}-\mu|, \tag{C.6}$$

we have

$$\begin{aligned}&\limsup_{n\to\infty}\max_{1\le i\le p_n}\left|\frac{\sum_{k=1}^{k}(X_{k,i}-\mu)^2}{n}-\sigma^2\right|+\limsup_{n\to\infty}\max_{1\le i\le p_n}|\bar{X}_i^{(n)}-\mu|^2\\ \ge&\limsup_{n\to\infty}\max_{1\le i\le p_n}\left|\frac{\sum_{k=1}^{k}(X_{k,i}-\bar{X}_i)^2}{n}-\sigma^2\right|.\end{aligned}$$

By (C.2) and (C.3), and the fact that

$$\limsup_{n\to\infty}\left(\max_{1\le i\le p_n}|\bar{X}_i^{(n)}-\mu|\right)^2=\limsup_{n\to\infty}\max_{1\le i\le p_n}|\bar{X}_i^{(n)}-\mu|^2$$

we have

$$\limsup_{n\to\infty}\max_{1\le i\le p_n}\left|\frac{\sum_{k=1}^{k}(X_{k,i}-\bar{X}_i)^2}{n}-\sigma^2\right|=0\quad\text{a.s.}\tag{C.7}$$

and by (C.2) and (C.4),

$$\begin{aligned}&\limsup_{n\to\infty}\frac{n\max\limits_{1\le i\le p_n,1\le j\le p_n}|\bar{X}_i^{(n)}-\mu||\bar{X}_j^{(n)}-\mu|}{\sqrt{n\log n}}\\&=\lim_{n\to\infty}\max_{1\le i\le p_n}|\bar{X}_i^{(n)}-\mu|\times\limsup_{n\to\infty}\left(\left(\frac{n}{\log n}\right)^{1/2}\max_{1\le i\le p_n}|\bar{X}_i^{(n)}-\mu|\right)=0\quad\text{a.s.}\end{aligned}\tag{C.8}$$

Then, by using (C.7), we get

$$\lim_{n\to\infty}\left(\frac{n}{\log n}\right)^{1/2}L_n=\frac{1}{\sigma^2}\lim_{n\to\infty}\frac{\max\limits_{1\le i<j\le p_n}\left|\sum_{k=1}^{n}(X_{k,i}-\bar{X}_i^{(n)})(X_{k,j}-\bar{X}_j^{(n)})\right|}{\sqrt{n\log n}},$$

and then, by combining (C.6) and (C.8), along with Theorem 3.9, we get

$$\lim_{n\to\infty}\left(\frac{n}{\log n}\right)^{1/2}L_n=\frac{1}{\sigma^2}\lim_{n\to\infty}\frac{\max\limits_{1\le i<j\le p_n}\left|\sum_{k=1}^{n}(X_{k,i}-\mu)(X_{k,j}-\mu)\right|}{\sqrt{n\log n}}=2.$$

□

Bibliography

[1] Leonard E. Baum and Melvin Katz. (1965). Convergence Rates in the Law of Large Numbers. *Transactions of the American Mathematical Society.* 120(1): 108-123.

[2] Kai Lai Chung. (1974). A Course in Probability Theory. Third Edition. Academic Press.

[3] Richard T. Durrett. (2005). Probability: Theory and Examples. Third Edition. Brooks/Cole Publishing Co..

[4] Paul Erdös. (1949). On a Theorem of Hsu and Robbins. *Annals of Mathematical Statistics*. 20(2): 286-291.

[5] Gerald B. Folland. (1999). Real Analysis: Modern Techniques and Their Applications, Second Edition. A Wiley-Interscience publicatin.

[6] Tiefeng Jiang. (2004). The Asymptotic Distributions of the Largest Entries of Sample Correlation Matrices. *Annals of Applied Probability.* 14(2): 865-880.

[7] Melvin Katz. (1963). The Probability in the Tail of a Distribution. *Annals of Mathematical Statistics*. 34(1): 312-318.

[8] Tze L. Lai. (1974). Limit Theorems for Delayed Sums. *Annals of Probability*. 2(3): 432-440.

[9] Deli Li and Andrew Rosalsky. (2006). Some Strong Limit theorems for the Largest Entries of Sample Correlation Matrices. *Annals of Applied Probability*. 16(1): 423-447.

[10] Michel Loève. (1977). Probability Theory I. Springer-Verlag, New York.

[11] Sévérien Nkurunziza and Yueleng Wang. (2017). On Convergence of the Sample Correlation Matrices in High-Dimensional Data. *Bernoulli*. Submitted.

[12] Valentin V. Petrov. (1995). Limit Theorems of Probability Theory: Sequence of Independent Random Variables. Clarendon Press. Oxford.

[13] Walter Rudin. (1976). Priciples of Mathematical Analysis. Third Edition. McGraw-Hill Education.

[14] Frank Spitzer. (1956). A Combinatorial Lemma and its Application to Probability Theory. *Transactions of the American Mathematical Society*. 82(2): 323-339.

Vita Auctoris

Mr. Yueleng Wang was born in 1990, Taizhou, Zhejiang, China. He graduated from Taizhou No.1 High School in 2008. From there he went to Chang'an University, where he obtained a Bachelor of Engineering in Automobile. After that he studied at Chalmers University of Technology and got a Master of Science degree in Engineering Mathematics. In 2015, he worked at Rational Stone Investment Inc. for half year as a trader. He is currently a candidate for the Master of Science degree in Statistics at the University of Windsor. He hopes to graduate in May 2017.