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Further contributions to the theory of steady rotational flow of gases.

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FURTHER CONTRIBUTIONS TO THE THEORY

OF STEADY ROTATIONAL FLOW

OF GASES

by

Om Parkash Chandna

A Thesis
Submitted to the Faculty of Graduate Studies through the
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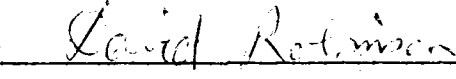
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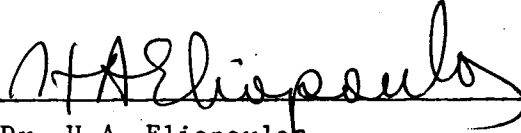
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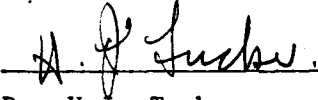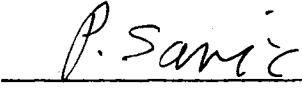
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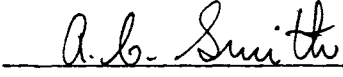
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ABSTRACT

Steady rotational flow of gases is studied in the plane and in three dimensions. Starting from the flow equations in orthogonal curvilinear coordinate system, sets of restrictions on the geometries of flow corresponding to different flow conditions are deduced.

For the planar flow, density, entropy and velocity are eliminated from the flow equations to obtain a pressure equation in natural coordinates. The resulting equation is a linear hyperbolic partial differential equation of the second order. An exceptional case is encountered when the pressure equation is not obtainable from the equations of flow. In this case the stream lines are demonstrated to be straight.

The pressure equation for planar flow is solved explicitly for vortex flow, flow through a parabolic channel and the flow through a hyperbolic channel. Substitution of the solution into the flow equations yields properties of flow, namely velocity, entropy and density.

For three dimensional flow the general flow equations are seen to reduce to two independent pressure equations. Three categories of three dimensional flow are identified. Typical examples of flow for each category are investigated in detail. In the first category flow of gases emanating from a spherical ball and from a cylindrical bar is studied. Flow of gases swirling about the axis of cylinder and through a hyperboloidal tunnel are investigated under the second category. Flow of gases through a tunnel with elliptical cross sections is studied to illustrate the nature of solution encountered in the third category of flow.

iii

Finally, it is proved that the only flow possible, with the stream lines as taken in the example of the third category of flow, is incompressible and irrotational.

iv

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v

TABLE OF CONTENTS

vii

Chapter 1

INTRODUCTION

(a) Historical Sketch

Advances in the theory of compressible fluid flow have been rather slow because of the basic non-linearity of the fundamental flow equations. In particular, rotational problems have provided quite a challenge due to the presence of thermodynamic variables like density and pressure in the flow equations.

Friedmann [1924] investigated rotational and irrotational flow of gases and was the first to deduce from the equations of state, motion and energy some relationships that he termed the 'compatibility equations'. The establishment of these relations, achieved by the elimination of the pressure and the density from the equations satisfied by the motion, was done by Friedmann for the most general case of unsteady flows.

Crocco [1937] deduced a pressure theorem and obtained a single differential equation governing the stream function for the plane, rotational flow of a perfect gas. However, the treatment was limited to isoenergetic flows.

Little progress was made in the next decade as far as the solution of the rotational problem was concerned. Prim ([1947], [1948], [1949]) carried out an extensive investigation into the nature of rotational flow. His work has the status of a landmark. Munk and Prim [1948] discovered the canonical equations of motion and the substitution principle that made possible a simplified formulation of the flow problem.

1

Munk and Prim showed that for steady flow of gases, with equation of state in the product form, if reduced velocity vector w defined as $\frac{q}{a}$, where a is the ultimate velocity magnitude, is used in place of velocity vector q, then the density can be eliminated from the general equations. The field properties of the flow are then completely defined by the dynamic equation and the continuity equation. These constitute a set of four equations in four dependent variables, namely the three components of w and pressure. Given a solution of this set of equations, one can solve the system by further preassigning entropy. This canonical system of four equations was obtained independently by Hicks, Wasserman and Guenther [1947].

Prim utilized inverse methods to obtain the flow solutions. The solutions sought were those defined by the differential equations and were prescribed having certain geometrical or kinematical properties, rather than prescribed boundary conditions. The problems that Prim considered belong to plane flow fields, axially symmetric flow fields and truly spatial flow fields. Some of these problems are as follows:

(i)   Three parameter generalization of the Prandtl-Meyer Corner flow.

(ii)   Investigation of conditions under which the stream lines of flow are isometric curves for the perfect gases.

(iii)   Investigation of the conditions under which the stream lines coincide with lines of constant **speed** for gases with product equation of state. This work was carried out by Nemenyi and Prim in a concurrent manner.

(iv)   Investigation of generalized Beltrami flows.

(v)   Generalization of Crocco's stream function equation for plane and axially symmetric flows.

(vi)   Generalization of Crocco's pressure theorem for plane and axially symmetric flows.

(vii)   Generalization of Poritsky's superposition principle. Berker [1956] rederived the compatibility equations which were discovered by Friedmann. Berker, however, considered the specific case of steady motion. This results in a considerable simplification of the compatibility conditions.

Ozoklav [(1959), (1964)] applied the inverse method technique to Berker's compatibility equations. Ozoklav considered the flow through a hyperbolic channel, and later, extended the work for plane and axially symmetric flows.

(b) <u>Scope of the present work</u>.

It is apparent from the survey of the literature that not much attention has so far been directed to three dimensional flow. Very few exact solutions are available. Furthermore the method of solution, so popular with the investigators thus far, suffers from a serious drawback in that it assumes a velocity form a priori. For a fixed family of stream lines an infinite family of velocity fields is possible. The velocity field, therefore, is not unique for a given family of stream lines and pressure distribution. [Prim (1952)]

The present work is intended to be a comprehensive treatment of the rotational flow problem. The problem is discussed from the twin points of views of dimensionality and exactness.

In section 1 of chapter 2 the non-linear partial differential equations governing the flow of inviscid and thermally non-conducting gases subject to no extraneous force field are formulated in general orthogonal curvilinear coordinate system. In section 2 of this chapter the complex variable technique for finding the geometry of the flow (in plane) is explained. This technique was first employed by W. Tollmien (1937) and later used by P. Nemenyi and R. Prim (1948).

Chapter 3 deals with plane, steady, rotational flow of gases.

In first part of this chapter (section 2) we investigate the following:

(i)   The conditions under which the stream lines coincide with lines of sonic velocity magnitude for any gas.

(ii)   The conditions under which the stream lines coincide with lines of constant velocity magnitude for any gas. These conditions were investigated by Prim and Nemenyi (1952) for only those gases which obeyed the product equation of state. This, therefore, represents a generalization of   Prim and Nemenyi's result.

(iii) The conditions under which the stream lines are concentric circles or parallel straight lines.

(iv)   The conditions on pressure, density, velocity, speed of sound and vorticity if any of the first four variables is constant along the stream lines.

(v)   The conditions under which the orthogonal trajectories are isobaric curves in an orthogonal isometric net.

(vi)   The conditions on straight stream lines in orthogonal isometric net.

In the second part of chapter 3 (sections 3 and 4) we develop a linear hyperbolic partial differential equation of the second order in pressure for the general natural coordinate system formed by the family of stream lines and their orthogonal trajectories. We find that this pressure equation, which fails only for the case of straight stream lines, gives us the unique pressure distribution for a given set of boundary conditions. We find the velocity, the density, the entropy and Mach number in terms of the pressure distribution in section 4. We find that whereas the Mach number field is unique for a given pressure distribution and the family of stream lines, the other variables velocity, density and entropy can be chosen in an infinite number of ways when the equation of state for the gas is in the product form. These results, therefore, satisfy Prim's substitution principle.

In section 5 we solve the flow problem when the stream lines are straight. This is the case when pressure equation is not obtainable.

In section 6 we study the different forms of isometric nets and find the pressure equations which hold for these forms. These pressure equations are solvable for the correspondingly prescribed boundary value problems.

In sections 7, 8 and 9 we study in details the vortex flow, the flow through a parabolic channel and the flow through a hyperbolic channel. The study of flow through a hyperbolic channel was done by Ozoklav (1959) by using the inverse method.

Chapter 4 deals with the solutions of flow problems in three dimensional space without any restricting assumptions of symmetry which make a reduction to two independent variables possible.

In the first part of this chapter (sections 1 and 2) we develop the three pressure equations, of which any two are independent. We obtain these pressure equations from the equations of flow in natural coordinates (ξ, η, ψ) with metric coefficients g_1, g_2 and g_3. Here g_1 is the scale factor along the stream line. We divide the three dimensional problems in three categories

I. When $\dfrac{\partial g_1}{\partial \eta} = \dfrac{\partial g_1}{\partial \psi} = 0$

II. Either $\dfrac{\partial g_1}{\partial \eta}$ or $\dfrac{\partial g_1}{\partial \psi}$ is zero

III. Neither $\dfrac{\partial g_1}{\partial \eta}$ nor $\dfrac{\partial g_1}{\partial \psi}$ is zero

In all these three categories pressure distribution is uniquely defined by the properly posed boundary value problem.

In section 3 and 4 we obtain the expressions for velocity, the density, the entropy and the Mach number as functions of pressure; and study the choice of coordinate systems.

In sections 5 and 6 we solve for the flow of gases emanating from a spherical ball and a cylindrical bar. These cases belong to the first category.

In sections 7 and 8 we study the flow of gases swirling about an axis of cylinder and the flow of gases through a hyperboloidal tunnel with circular cross sections. These are the examples of flow that belong to second category.

Finally, we study an example of the third category problems and solve the flow of gases through a tunnel with elliptic cross sections. This example is studied to illustrate the nature of solution encountered

in this category of flow.

Finally, we prove that the only flow possible when the stream lines are the curves of intersection of the hyperboloids of one sheet and the hyperboloids of two sheets is incompressible and irrotational flow.

Chapter 2

PRELIMINARIES

Section 1. Equations of fluid motion in general orthogonal curvilinear coordinates.

The differential equations governing the three dimensional unsteady motion of a compressible fluid, in the absence of external forces and heat conduction, are:

$$\rho \, \underset{\sim}{a} = - \text{grad } p \qquad\qquad (21.01)$$

(Conservation of momentum)

$$\frac{\partial \rho}{\partial t} + \text{div } (\rho \underset{\sim}{q}) = 0 \qquad\qquad (21.02)$$

(Conservation of mass)

$$\frac{\partial s}{\partial t} + \underset{\sim}{q} \cdot \text{grad} s = 0 \qquad\qquad (21.03)$$

(Changes of state are adiabatic)

$$\rho = \rho(p,s) \qquad\qquad (21.04)$$

(Caloric Equation of State)

wherein $\underset{\sim}{a}$ denotes the acceleration vector, $\underset{\sim}{q}$ the velocity vector, ρ the density, p the pressure and s the specific entropy. In this section we express these five non-linear partial differential equations in orthogonal curvilinear coordinates [Lamb (1932)]. We consider the orthogonal curvilinear coordinate system (ξ, η, ψ) obtained from the three families of surfaces $\xi = c_1$, $\eta = c_2$ and $\psi = c_3$. Let

$$x = x \, (\xi, \eta, \psi)$$

$$y = y \, (\xi, \eta, \psi) \qquad\qquad (21.05)$$

$$z = z \, (\xi, \eta, \psi)$$

be the equations of the three families of surfaces. The squared element of arc length in this coordinate system is of the form

$$ds^2 = g_1^2(\xi,\eta,\psi) \, d\xi^2 + g_2^2(\xi,\eta,\psi) d\eta^2 + g_3^2(\xi,\eta,\psi) d\psi^2 \qquad (21.06)$$

where g_1, g_2, and g_3 are the metric coefficients. Now through any point P pass the three curves which are mutually orthogonal. Along the curve of intersection of $\eta = $ const and $\psi = $ const varies ξ-only. We call it a ξ-curve. Likewise η-varies along η-curve and ψ varies along ψ-curve.
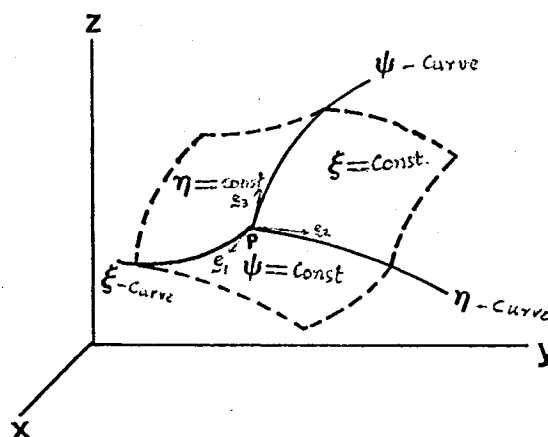
Let $\underset{\sim}{e}_1$ be the unit tangent vector to the curve along which ξ increases in the direction of ξ increasing; $\underset{\sim}{e}_2$ the unit tangent vector to the curve along which η increases in the direction of η increasing and $\underset{\sim}{e}_3$ the unit tangent vector to the curve along which ψ increases in the direction of ψ increasing.

In these coordinates, let

$$\underset{\sim}{q} = \underset{\sim}{e}_1 \, u(\xi, \eta, \psi, t) + \underset{\sim}{e}_2 \, v(\xi, \eta, \psi, t) + \underset{\sim}{e}_3 \, w(\xi, \eta, \psi, t)$$

$$(21.07)$$

Using (21.07), we get

$$\underset{\sim}{a} = \frac{\partial \underset{\sim}{q}}{\partial t} + (\underset{\sim}{q} \cdot \underset{\sim}{\nabla}) \, \underset{\sim}{q} = \underset{\sim}{e}_1 \, a_1 + \underset{\sim}{e}_2 \, a_2 + \underset{\sim}{e}_3 \, a_3$$

where

$$a_1 = \frac{\partial u}{\partial t} + \frac{u}{g_1}\frac{\partial u}{\partial \xi} + \frac{v}{g_2}\frac{\partial u}{\partial \eta} + \frac{w}{g_3}\frac{\partial u}{\partial \psi} + \frac{1}{g_1 g_2}\left\{ uv\,\frac{\partial g_1}{\partial \eta} - v^2\,\frac{\partial g_2}{\partial \xi}\right\} + \frac{1}{g_1 g_3}$$

$$\left\{ uw\,\frac{\partial g_1}{\partial \psi} - w^2\,\frac{\partial g_3}{\partial \xi}\right\},$$

$$a_2 = \frac{\partial v}{\partial t} + \frac{u}{g_1}\frac{\partial v}{\partial \xi} + \frac{v}{g_2}\frac{\partial v}{\partial \eta} + \frac{w}{g_3}\frac{\partial v}{\partial \psi} + \frac{1}{g_1 g_2}\left\{ uv\,\frac{\partial g_2}{\partial \xi} - u^2\,\frac{\partial g_1}{\partial \eta}\right\} + \frac{1}{g_2 g_3}$$

$$\left\{ vw\,\frac{\partial g_2}{\partial \psi} - w^2\,\frac{\partial g_3}{\partial \eta}\right\}, \tag{21.08}$$

$$a_3 = \frac{\partial w}{\partial t} + \frac{u}{g_1}\frac{\partial w}{\partial \xi} + \frac{v}{g_2}\frac{\partial w}{\partial \eta} + \frac{w}{g_3}\frac{\partial w}{\partial \psi} + \frac{1}{g_1 g_3}\left\{ uw\,\frac{\partial g_3}{\partial \xi} - u^2\,\frac{\partial g_1}{\partial \psi}\right\} + \frac{1}{g_2 g_3}$$

$$\left\{ vw\,\frac{\partial g_3}{\partial \eta} - v^2\,\frac{\partial g_2}{\partial \psi}\right\},$$

and

$$\text{div}\,(\rho\underset{\sim}{q}) = \frac{1}{g_1 g_2 g_3}\left\{ \frac{\partial}{\partial \xi}(g_2 g_3 \rho u) + \frac{\partial}{\partial \eta}(g_1 g_3 \rho v) + \frac{\partial}{\partial \psi}(g_1 g_2 \rho w)\right\} \tag{21.09}$$

Also

$$\text{grad}\,\phi = \underset{\sim 1}{e}\,\frac{1}{g_1}\frac{\partial \phi}{\partial \xi} + \underset{\sim 2}{e}\,\frac{1}{g_2}\frac{\partial \phi}{\partial \eta} + \underset{\sim 3}{e}\,\frac{1}{g_3}\frac{\partial \phi}{\partial \psi} \tag{21.10}$$

where ϕ is any scalar point function.

Putting the results (21.07) — (21.10) in the vector equations of motion (21.01) — (21.03) we get the five non-linear partial differential equations in general orthogonal curvilinear $(\xi,\ \eta,\ \psi)$ net as:

$$\frac{\partial u}{\partial t} + \frac{u}{g_1}\frac{\partial u}{\partial \xi} + \frac{v}{g_2}\frac{\partial u}{\partial \eta} + \frac{w}{g_3}\frac{\partial u}{\partial \psi} + \frac{1}{g_1 g_2}\left\{ uv\,\frac{\partial g_1}{\partial \eta} - v^2\,\frac{\partial g_2}{\partial \xi}\right\} + \frac{1}{g_1 g_3}$$

$$\left\{ uw\,\frac{\partial g_1}{\partial \psi} - w^2\,\frac{\partial g_3}{\partial \xi}\right\} + \frac{1}{g_1 \rho}\frac{\partial p}{\partial \xi} = 0 \tag{21.11}$$

$$\frac{\partial v}{\partial t} + \frac{u}{g_1}\frac{\partial v}{\partial \xi} + \frac{v}{g_2}\frac{\partial v}{\partial \eta} + \frac{w}{g_3}\frac{\partial v}{\partial \psi} + \frac{1}{g_2 g_3}\left\{ vw\frac{\partial g_2}{\partial \psi} - w^2\frac{\partial g_3}{\partial \eta}\right\} + \frac{1}{g_1 g_2}$$

$$\left\{ uv\frac{\partial g_2}{\partial \xi} - u^2\frac{\partial g_1}{\partial \eta}\right\} + \frac{1}{g_2 \rho}\frac{\partial p}{\partial \eta} = 0 \qquad (21.12)$$

$$\frac{\partial w}{\partial t} + \frac{u}{g_1}\frac{\partial w}{\partial \xi} + \frac{v}{g_2}\frac{\partial w}{\partial \eta} + \frac{w}{g_3}\frac{\partial w}{\partial \psi} + \frac{1}{g_3 g_1}\left\{ uw\frac{\partial g_3}{\partial \xi} - u^2\frac{\partial g_1}{\partial \psi}\right\} + \frac{1}{g_2 g_3}$$

$$\left\{ vw\frac{\partial g_3}{\partial \eta} - v^2\frac{\partial g_2}{\partial \psi}\right\} + \frac{1}{g_3 \rho}\frac{\partial p}{\partial \psi} = 0 \qquad (21.13)$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{g_1 g_2 g_3}\left\{ \frac{\partial}{\partial \xi}(g_2 g_3 \rho u) + \frac{\partial}{\partial \eta}(g_1 g_3 \rho v) + \frac{\partial}{\partial \psi}(g_1 g_2 \rho w)\right\} = 0 \qquad (21.14)$$

$$\frac{\partial s}{\partial t} + \frac{u}{g_1}\frac{\partial s}{\partial \xi} + \frac{v}{g_2}\frac{\partial s}{\partial \eta} + \frac{w}{g_3}\frac{\partial s}{\partial \psi} = 0 \qquad (21.15)$$

These five equations and the state equation

$$\rho = \rho\,(p,s) \qquad (21.16)$$

form a system of six equations involving six unknown functions.

Since g_1, g_2 and g_3 correspond to a triply orthogonal set of surfaces in Euclidean 3-space, these metric coefficients satisfy a set of six partial differential equations of Gauss [Moon and Spencer (1960)]:

$$\frac{\partial}{\partial u^i}\left(\frac{1}{g_i}\cdot\frac{\partial g_i}{\partial u^i}\right) + \frac{\partial}{\partial u^j}\left(\frac{1}{g_j}\frac{\partial g_i}{\partial u^j}\right) + \frac{1}{g_k^2}\frac{\partial g_i}{\partial u^k}\frac{\partial g_i}{\partial u^k} = 0 \qquad (21.17)$$

and

$$\frac{\partial^2 g_i}{\partial u^j \partial u^k} = \frac{1}{g_j}\frac{\partial g_i}{\partial u^k}\frac{\partial g_j}{\partial u^j} + \frac{1}{g_k}\frac{\partial g_k}{\partial u^j}\frac{\partial g_i}{\partial u^k} \qquad (21.18)$$

where i,j,k are $1,2,3$ respectively or any cyclic permutation and u^1,u^2,u^3 are ξ, η, ψ or any corresponding cyclic permutation.

Section 2.   The geometry of orthogonal curvilinear net in plane.

In this section we give an account of a method which was first employed by W. Tollmien (1937) and later used by P. Nemenyi and R. Prim (1948).

We consider the plane orthogonal curvilinear net (ξ, η) which is composed of two families of curves $\xi = $ constant and $\eta = $ constant. The squared element of arc length in this coordinate system is given by

$$ds^2 = g_1^2(\xi, \eta)\, d\xi^2 + g_2^2(\xi, \eta)\, d\eta^2 \qquad (22.01)$$

Here $g_1(\xi, \eta)$ and $g_2(\xi, \eta)$ are the metric coefficients such that the necessary and sufficient restriction on g_1 and g_2 to correspond to an orthogonal set of curves in Euclidean plane [Moon and Spencer (1960)] is given by Gauss's equation

$$\frac{\partial}{\partial \xi}\left[\frac{1}{g_1}\frac{\partial g_2}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[\frac{1}{g_2}\frac{\partial g_1}{\partial \eta}\right] = 0 \qquad (22.02)$$

By definition

$$g_1(\xi, \eta) = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2} \qquad (22.03)$$

$$g_2(\xi, \eta) = \sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2} \qquad (22.04)$$

Also since the curvilinear net (ξ, η) is an orthogonal net, we have

$$\frac{\partial x}{\partial \xi} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} = 0 \qquad (22.05)$$

Now introducing the complex variable $z = x + iy$, we get from (22.03) and (22.04)

$$\frac{\partial z}{\partial \xi} = g_1 e^{i\alpha} \qquad (22.06)$$

and

$$\frac{\partial z}{\partial \eta} = g_2 e^{i\beta} \qquad (22.07)$$

wherein α and β are real functions of ξ and η.

Putting (22.06) and (22.07) in (22.05), we get

$$g_1 g_2 \left\{ \cos\alpha \, \cos\beta + \sin\alpha \, \sin\beta \right\} = 0$$

or

$$\cos(\beta - \alpha) = 0$$

which implies that

$$\sin(\beta - \alpha) = \underline{+} 1$$

Whence

$$e^{i\beta} = e^{i\alpha} \, e^{i(\beta - \alpha)} = \underline{+} \, i e^{i\alpha} \tag{22.08}$$

In the sequel, we take $e^{i\beta} = i e^{i\alpha}$ (22.09)

Eliminating z between (22.06) and (22.07) and using (22.09),

we get $$\frac{\partial}{\partial \eta} (g_1 e^{i\alpha}) - \frac{\partial}{\partial \xi} (i \, g_2 \, e^{i\alpha}) = 0$$

or

$$\frac{\partial g_1}{\partial \eta} + i g_1 \frac{\partial \alpha}{\partial \eta} - i \frac{\partial g_2}{\partial \xi} + g_2 \frac{\partial \alpha}{\partial \xi} = 0 \tag{22.10}$$

By separating (22.10) into real and imaginary parts, we get

$$\frac{\partial \alpha}{\partial \xi} = -\frac{1}{g_2} \frac{\partial g_1}{\partial \eta} \tag{22.11}$$

and

$$\frac{\partial \alpha}{\partial \eta} = \frac{1}{g_1} \frac{\partial g_2}{\partial \xi} \tag{22.12}$$

Having thus obtained the set of relations (22.06), (22.07), (22.09),

(22.11) and (22.12) the geometry of the net can be obtained by the know-

ledge of two limitations on the metric coefficients $g_1(\xi, \eta)$ and $g_2(\xi, \eta)$.

Let us assume that we are given one of the limitations on

metric coefficient $g_2(\xi, \eta)$ as:

$$\frac{\partial g_2}{\partial \xi} = 0 \qquad (22.13)$$

From (22.12) and (22.13), we get

$$g_2 = g_2 (\eta) \qquad (22.14)$$

and

$$\alpha = \alpha (\xi) \qquad (22.15)$$

Using (22.15) in (22.11), we get

$$-g_2 \, \alpha'(\xi) = \frac{\partial g_1}{\partial \eta}$$

or

$$g_1(\xi,\eta) = -\alpha'(\xi) \int g_2(\eta)d\eta + \phi(\xi) \qquad (22.16)$$

where ϕ is a real arbitrary function of ξ.

Using (22.14) and (22.16) in (12.06) and (12.07), we get

$$\frac{\partial z}{\partial \xi} = \left\{ - \alpha'(\xi) \int g_2(\eta) \, d\eta + \phi(\xi) \right\} e^{i\alpha(\xi)} \qquad (22.17)$$

and

$$\frac{\partial z}{\partial \eta} = i \, g_2 \, e^{i\alpha(\xi)} \qquad (22.18)$$

Using (22.18) and integrating (22.17), we get

$$z = i \, e^{i\alpha(\xi)} \int g_2(\eta)d\eta + \int \phi(\xi) \, e^{i\alpha(\xi)} \, d\xi + C_1 + iC_2$$

where $C_1 + iC_2$ is a complex constant of integration.

or

$$x = \int \phi(\xi) \, \cos\alpha(\xi)d\xi - \sin\alpha(\xi) \int g_2(\eta)d\eta + C_1 \qquad (22.19)$$

and

$$y = \int \phi(\xi) \, \sin\alpha(\xi)d\xi + \cos\alpha(\xi) \int g_2(\eta)d\eta + C_2 \qquad (22.20)$$

Equations (22.19) and (22.20) describe the geometry of orthogonal curvilinear net. Since $\phi(\xi)$ is an unknown function, the geometry of the net depends upon one more condition on the metric coefficient g_1. In the

particular case when the condition on g_1 is such that $\phi(\xi) = 0$ <u>or</u> $\phi(\xi) = .(\text{const}).(\alpha'(\xi))$, we get $\eta = \text{const.}$ as the family of concentric circles which in the limiting case is a family of parallel straight lines.

CHAPTER 3

STEADY PLANE ROTATIONAL GAS FLOW

Section 1. The flow equations in natural coordinates.

In this section we set the equations governing the steady, plane and rotational gas flow in natural coordinates consisting of the stream lines $\eta = $ const. and their orthogonal trajectories $\xi = $ const. i.e. we take these two systems of curves as our coordinate curves. Let

$$\left. \begin{array}{l} x = x\ (\xi,\eta) \\ \\ y = y\ (\xi,\eta) \end{array} \right\} \tag{31.01}$$

be the equations of these two families of curves.

In this way we use the cylindrical coordinate system (ξ,η,z) based on the plane orthogonal net (ξ,η) so chosen that the velocity vector $\underset{\sim}{q}$ has the components (u,o) at any point P or if we take $\underset{\sim 1}{e}$ as the unit vector tangential to the curve $\eta = $ const. in the direction of ξ- increasing and $\underset{\sim 2}{e}$ as the unit vector tangential to the curve $\xi = $ const. in the direction of η-increasing, then $\underset{\sim}{q} = u(\xi,\eta)\ \underset{\sim 1}{e}$ \hfill (31.02)

The squared element of arc length is of the form

$$ds^2 = g_1^2\ (\xi,\eta)d\xi^2 + g_2^2(\xi,\eta)\ d\eta^2 \tag{31.02'}$$

Since the plane flow is steady, equations (21.11) to (21.16) for such a flow are:

$$u\ \frac{\partial u}{\partial \xi} + \frac{1}{\rho}\ \frac{\partial p}{\partial \xi}\ = 0 \tag{31.03}$$

$$\frac{u^2}{g_1}\ \frac{\partial g_1}{\partial \eta}\ = \frac{1}{\rho}\ \frac{\partial p}{\partial \eta} \tag{31.04}$$

$$\frac{\partial}{\partial \xi}\ (g_2 \rho u)\ = 0 \tag{31.05}$$

$$\frac{\partial s}{\partial \xi} = 0 \tag{31.06}$$

16

$$\rho = \rho(p,s) \tag{31.07}$$

where the metric coefficients g_1 and g_2 satisfy

$$\frac{\partial}{\partial \xi}\left[\frac{1}{g_1}\frac{\partial g_2}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[\frac{1}{g_2}\frac{\partial g_1}{\partial \eta}\right] = 0 \tag{31.08}$$

In the case when the motion is rotational our knowledge is limited and does not extend much beyond the important results of R. Prim (1952). We shall study such a flow in chapters 3 and 4.

Section 2. General Theorems

Theorem I. If the velocity is sonic throughout the two dimensional steady compressible fluid flow, then the stream lines are a system of concentric circles or in the limiting case a family of parallel straight lines.

Proof: Since the velocity is sonic throughout the flow, we have by (31.06)

$$u\ (\xi,\eta) = c\ (\rho(\xi,\eta),s(\eta))$$

Here c is the local speed of sound, given by $c^2 = \dfrac{\partial p}{\partial \rho}$

or

$$\frac{\partial u}{\partial \xi} = \frac{\partial c}{\partial \rho} \frac{\partial \rho}{\partial \xi} \tag{32.01}$$

Putting (32.01) in (31.03), we get

$$c\frac{\partial c}{\partial \rho} \frac{\partial \rho}{\partial \xi} + \frac{c^2}{\rho} \frac{\partial \rho}{\partial \xi} = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial \xi}\left[c\ \frac{\partial c}{\partial \rho} + \frac{c^2}{\rho}\right] = 0 \tag{32.02}$$

Now $c\ \dfrac{\partial c}{\partial \rho} + \dfrac{c^2}{\rho} > 0$, therefore, from (32.02) we get

$$\frac{\partial \rho}{\partial \xi} = 0 \tag{32.03}$$

Putting (32.03) in (31.03), we get

$$\frac{\partial u}{\partial \xi} = 0 \ \text{and} \ \frac{\partial b}{\partial \xi} = 0$$

Therefore, if $u(\xi,\eta) = c(\xi,\eta)$, then

$$\frac{\partial u}{\partial \xi} = 0 \qquad , \tag{32.04}$$

$$\frac{\partial b}{\partial \xi} = 0 \tag{32.05}$$

and

$$\frac{\partial \rho}{\partial \xi} = 0 \tag{32.03}$$

Substituting (32.03) and (32.04) in (31.05), we get

$$\frac{\partial g_2}{\partial \xi} = 0 \qquad \text{or} \qquad g_2 = g_2(\eta) \tag{32.06}$$

which is one of the two limitations needed to find the geometry of flow. From (31.04), we get

$$\frac{\partial}{\partial \eta} (\log g_1) = \frac{1}{\rho u^2} \frac{\partial p}{\partial \eta} .$$

Now the right hand side of this equation is a function of η only, therefore, differentiating partially w.r.t. ξ, we get the second limitation as:

$$\frac{\partial^2}{\partial \xi \partial \eta} (\log g_1) = 0 \qquad\qquad (32.07)$$

Equation (32.07) is satisfied by setting

$$g_1 (\xi, \eta) = b(\xi) \, c(\eta) \qquad\qquad (32.08)$$

Putting (32.08) in (31.08), we get

$$\left[\frac{c'(\eta)}{g_2(\eta)} \right]' b(\xi) = 0$$

Whence $b(\xi) \neq 0$, $c'(\eta) = A \, g_2(\eta)$ where A is an arbitrary constant.

or $\qquad\qquad c(\eta) = A \int g_2(\eta) \, d\eta + B$ where B is an arbitrary constant.

Therefore,

$$g_1(\xi, \eta) = b(\xi) \left[A \int g_2(\eta) \, d\eta + B \right] \qquad\qquad (32.09)$$

and

$$g_2 (\xi, \eta) = g_2(\eta) \qquad\qquad (32.06)$$

Putting (32.06) in (22.12), we get

$$\frac{\partial \alpha}{\partial \eta} = 0 \qquad \text{i.e. } \alpha = \alpha(\xi) \qquad\qquad (32.10)$$

Putting (32.09) in (22.11) and using (32.10), we get

$$b(\xi) = -\frac{\alpha'(\xi)}{A}$$

Therefore,

$$g_1(\xi, \eta) = -\alpha'(\xi) \int g_2(\eta) \, d\eta - \frac{B}{A} \alpha'(\xi) \qquad\qquad (32.11)$$

Comparing (22.16) and (32.11) we find that the unknown function $\phi(\xi)$ in

(22.16) is given in terms of $\alpha'(\xi)$ i.e.

$$\phi(\xi) = -\frac{B}{A} \alpha'(\xi) \tag{32.12}$$

So from (22.19) and (22.20), we get

$$x = c_1 - \sin\alpha(\xi) \left[\int g_2(\eta) d\eta + \frac{B}{A} \right] \tag{32.13}$$

and

$$y = c_2 - \cos\alpha(\xi) \left[\int g_2(\eta) d\eta + \frac{B}{A} \right] \tag{32.14}$$

which are the equations of two families of curves.

Eliminating ξ and η from (32.13) and (32.14), we get

$$(x - c_1)^2 + (y - c_2)^2 = \left[\int g_2(\eta) d\eta + \frac{B}{A} \right]^2 \tag{32.15}$$

and

$$\frac{y - c_2}{x - c_1} = - \cot\alpha(\xi) \tag{32.16}$$

From (32.15) we conclude that the stream lines $\eta =$ constant are concen-
tric circles which have $A = 0$ for the limiting case. From equation

(32.12) when $A = 0$, $\alpha'(\xi) = 0$ i.e. α is a constant and, therefore,

equation (32.16) gives a system of parallel straight lines. So in the

limiting case when $A = 0$, the orthogonal trajectories and, therefore,

the stream lines are a family of parallel straight lines.

In the case when $A \neq 0$, from (32.09), we get $\frac{\partial g_1}{\partial \xi} \neq 0$

and $\frac{\partial g_1}{\partial \eta} \neq 0$. Therefore, from (31.04) $\frac{\partial p}{\partial \eta} \neq 0$ i.e. $p = p(\eta)$. So when

$A \neq 0$ (i.e. in the case of concentric circles as stream lines), $p = p(\eta)$.

However, in the case when $A = 0$, we get from (32.09) that

$\frac{\partial g_1}{\partial \eta} = 0$ and, therefore, from (31.04) $\frac{\partial p}{\partial \eta} = 0$. Since $\frac{\partial p}{\partial \xi} = 0$ also from

(32.05), we find that $p =$ constant in the case of straight parallel

stream lines.

Theorem II. <u>If the velocity magnitude is constant along each</u> <u>stream line in two dimensional compressible fluid motion, then the only</u> <u>possible flow fields are the general vortex flow or in the limiting case</u> <u>flow in parallel straight lines.</u>

Proof: This theorem was first proved for the special case of a perfect gas (Nemenyi and Prim (1948)) and was later generalized for only those gases which had a product equation of state i.e. $\rho = P(p)\, S(s)$ [Prim (1952)]. In both the proofs the method employed was the same which we have mentioned in section 2 of Chapter 2.

Here, we have this theorem for any gas with no restriction on the state equation and use the same method.

We are given that $u = u(\eta)$ i.e. $\dfrac{\partial u}{\partial \xi} = 0$. By substituting $\dfrac{\partial u}{\partial \xi} = 0$ in (31.03) and using (31.06), we get

$$\frac{\partial p}{\partial \xi} = 0 \quad \text{and} \quad \frac{\partial \rho}{\partial \xi} = 0$$

So the assumption of this Theorem gives us (32.03), (32.04) and (32.05) by which we established in Theorem I that the only possible flow fields are the general vortex flow or in the limiting case flow in parallel straight lines.

<u>Corollary 1</u>

If pressure or density is constant along each individual stream line in the flow, then the result of Theorem I (or Theorem II) also holds.

Theorem III. <u>If one of the four unknown scalar functions u,</u> <u>p, ρ and c (the local velocity of sound) is constant along each stream</u> <u>line of the flow, then the other three unknown scalar functions and the</u>

vorticity vector ω are also constant along each stream line of the flow.

Proof: From equations (31.03) and (31.06), we get

$$\frac{\partial u}{\partial \xi} = 0 \Rightarrow \frac{\partial \rho}{\partial \xi} = \frac{\partial p}{\partial \xi} = 0$$

$$\frac{\partial p}{\partial \xi} = 0 \Rightarrow \frac{\partial u}{\partial \xi} = \frac{\partial \rho}{\partial \xi} = 0$$

and

$$\frac{\partial \rho}{\partial \xi} = 0 \Rightarrow \frac{\partial p}{\partial \xi} = \frac{\partial u}{\partial \xi} = 0$$

Therefore, we are required to prove

(i) $\quad \dfrac{\partial \rho}{\partial \xi} = 0 \Longrightarrow \dfrac{\partial c}{\partial \xi} = 0$

and $\quad \dfrac{\partial c}{\partial \xi} = 0 \Longrightarrow \dfrac{\partial \rho}{\partial \xi} = 0$

(ii) $\quad \dfrac{\partial c}{\partial \xi} = 0 \Longrightarrow \dfrac{\partial \omega}{\partial \xi} = 0$

To prove (i) we know

$$c = c(\rho, s)$$

or

$$\frac{\partial c}{\partial \xi} = \frac{\partial c}{\partial \rho}\frac{\partial \rho}{\partial \xi} + \frac{\partial c}{\partial s}\frac{\partial s}{\partial \xi} = \frac{\partial c}{\partial \rho}\frac{\partial \rho}{\partial \xi} \quad \left(\because \frac{\partial s}{\partial \xi} = 0 \text{ by (31.06)}\right)$$

Since $\dfrac{\partial c}{\partial \rho} > 0$, therefore,

$$\frac{\partial c}{\partial \xi} = 0 \Longleftrightarrow \frac{\partial \rho}{\partial \xi} = 0$$

(ii) Finally, since the vorticity vector $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{q}$, therefore,

$$\omega = -\frac{1}{g_1 g_2}\frac{\partial}{\partial \eta}(g_1 u) \text{ along the direction perpendicular to}$$

the plane.

or

$$\omega = -\frac{1}{g_2}\left[u\frac{\partial}{\partial \eta}(\log g_1) + \frac{\partial u}{\partial \eta}\right]$$

Differentiating partially w.r.t. ξ and partially w.r.t. η, we get

$$\frac{\partial \omega}{\partial \xi} = \frac{1}{g_2^2} \frac{\partial g_2}{\partial \xi} \left[u \frac{\partial}{\partial \eta}(\log g_1) + \frac{\partial u}{\partial \eta} \right] - \frac{1}{g_2} \left[\frac{\partial u}{\partial \xi} \frac{\partial}{\partial \eta}(\log g_1) + u \frac{\partial^2}{\partial \xi \partial \eta}(\log g_1) \right.$$
$$\left. + \frac{\partial^2 u}{\partial \xi \partial \eta} \right] \qquad (32.17)$$

and

$$\frac{\partial \omega}{\partial \eta} = \frac{1}{g_2^2} \frac{\partial g_2}{\partial \eta} \left[u \frac{\partial}{\partial \eta}(\log g_1) + \frac{\partial u}{\partial \eta} \right] - \frac{1}{g_2} \left[\frac{\partial u}{\partial \eta} \frac{\partial}{\partial \eta}(\log g_1) + u \frac{\partial^2}{\partial \eta^2}(\log g_1) \right.$$
$$\left. + \frac{\partial^2 u}{\partial \eta^2} \right] \qquad (32.18)$$

Using (32.06), (32.07) and (32.04) in (32.17) and (32.18), we get

$$\frac{\partial \omega}{\partial \xi} = 0 \qquad \text{but } \frac{\partial \omega}{\partial \eta} \neq 0$$

Therefore the vorticity vector ω is also constant along each stream line of the flow.

Theorem IV (Converse of theorems I and II). If the stream lines are concentric circles or parallel straight lines, then either $\frac{\partial u}{\partial \xi} = 0$ or $u = c(\rho, s)$.

Proof: We consider the two cases separately.

Case I. (Stream lines form a family of concentric circles).

In this case $\eta = $ constant are the concentric circles and $\xi = $ constant are the radial lines.

The squared element of arc length for this system is
$$ds^2 = \eta^2 \, d\xi^2 + d\eta^2 \qquad (32.19)$$

Using $g_1 = \eta$ and $g_2 = 1$ (since g_1 and g_2 are non-zero and positive) in (31.03) - (31.06), we get:
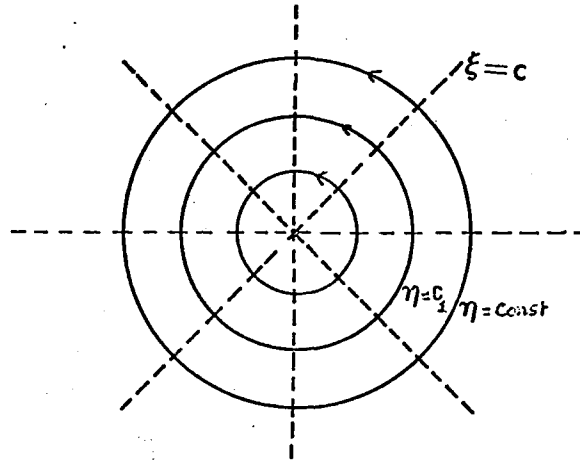
$$u \frac{\partial u}{\partial \xi} + \frac{1}{\rho} \frac{\partial p}{\partial \xi} = 0 \qquad (32.20)$$

$$\frac{u^2}{\eta} = \frac{1}{\rho} \frac{\partial p}{\partial \eta} \qquad (32.21)$$

$$\frac{\partial}{\partial \xi} (\rho u) = 0 \qquad (32.22)$$

$$\frac{\partial s}{\partial \xi} = 0 \qquad (32.23)$$

From (32.22), we get

$$u \frac{\partial \rho}{\partial \xi} + \rho \frac{\partial u}{\partial \xi} = 0$$

or

$$\frac{1}{\rho} \frac{\partial \rho}{\partial \xi} = -\frac{1}{u} \frac{\partial u}{\partial \xi} \qquad (32.24)$$

From (32.20) and (32.23), we get

$$u \frac{\partial u}{\partial \xi} + \frac{c^2}{\rho} \frac{\partial \rho}{\partial \xi} = 0 \qquad (32.25)$$

Putting (32.24) in (32.25), we get

$$u \frac{\partial u}{\partial \xi} - \frac{c^2}{u} \frac{\partial u}{\partial \xi} = 0$$

or

$$\frac{\partial u}{\partial \xi} [u - \frac{c^2}{u}] = 0 \qquad (32.26)$$

From (32.26) we conclude that either $\frac{\partial u}{\partial \xi} = 0$ or $u - \frac{c^2}{u} = 0$

i.e. if the stream lines are concentric circles, then either $\frac{\partial u}{\partial \xi} = 0$ or $u = c$.

Case II. (Stream lines are parallel straight lines).

Let $\eta =$ constant be the stream lines $||$ to ξ-axis and $\xi =$ constant be the orthogonal trajectories $||$ to η-axis. Squared element of arc length is:

$$ds^2 = d\xi^2 + d\eta^2 \qquad (32.27)$$

so that $g_1 = g_2 = 1$.
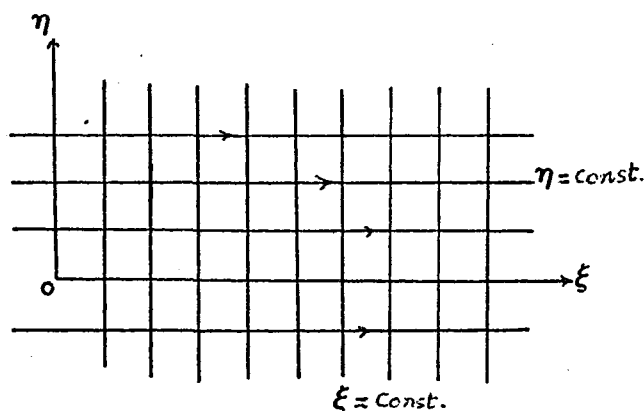
Using $g_1 = g_2 = 1$ in (31.03) - (31.06), we get

$$u \frac{\partial u}{\partial \xi} + \frac{1}{\rho} \frac{\partial p}{\partial \xi} = 0 \qquad (32.28)$$

$$\frac{\partial p}{\partial \eta} = 0 \qquad (32.29)$$

$$\frac{\partial}{\partial \xi}(\rho u) = 0 \qquad (32.30)$$

$$\frac{\partial s}{\partial \xi} = 0 \qquad (32.31)$$

Using $\frac{1}{\rho} \frac{\partial p}{\partial \xi} = -\frac{1}{u} \frac{\partial u}{\partial \xi}$ as obtained from (32.30) in (32.28) taking in account that $\frac{\partial s}{\partial \xi} = 0$ (32.31), we get

$$\frac{\partial u}{\partial \xi} \left(u - \frac{c^2(\rho,s)}{u} \right) = 0$$

i.e. Either $\frac{\partial u}{\partial \xi} = 0$ or $u = c$.

So if the stream lines are parallel straight lines, then either

$$\frac{\partial u}{\partial \xi} = 0 \qquad \text{or} \quad u = c.$$

Theorem V. If the orthogonal trajectories of stream lines are isobaric curves and the orthogonal curvilinear net is isometric, then the stream lines are simple source flows or flow in parallel straight lines.

Proof: Since the orthogonal trajectories given by $\xi = $ constant are the isobaric curves, therefore, $\frac{\partial p}{\partial \eta} = 0$ 　　　(32.32)

Also by assumption $g_1(\xi,\eta) = g_2(\xi,\eta) = g(\xi,\eta)$ (say) 　(32.33)

Using (32.32) and (32.33) in (31.04), we get

$$\frac{\partial g}{\partial \eta} = 0 \qquad\qquad (32.34)$$

Putting (32.34) in (31.08) and using (32.33), we get

$$\frac{\partial^2}{\partial \xi^2}(\log g) = 0$$

which is satisfied by setting

$$g(\xi,\eta) = e^{A\xi+B} \qquad\qquad (32.35)$$

where A and B are the constants.

Putting (32.35) in (22.11) and (22.12), we get

$$\frac{\partial \alpha}{\partial \xi} = 0 \qquad\qquad (32.36)$$

and

$$\frac{\partial \alpha}{\partial \eta} = A \qquad\qquad (32.37)$$

From (32.36) and (32.37), we get

$$\alpha = A\eta + c \qquad\qquad (32.38)$$

where c is a constant.

Putting (32.38) and (32.35) in (22.06) and (22.07), we get

$$\frac{\partial z}{\partial \xi} = e^{A\xi + B} \ e^{i(A\eta + C)} \tag{32.39}$$

and

$$\frac{\partial z}{\partial \eta} = ie^{A\xi + B} \ e^{i(A\eta + c)} \qquad \text{(using 12.09)} \tag{32.40}$$

Integrating (32.40) and using (32.39), we get

$$Z = \frac{e^{A\xi + B} \ e^{i(A\eta + c)}}{A} + (k_1 + ik_2) \quad \text{where } k_1 + ik_2 \text{ is a constant}$$

By separating into real and imaginary parts, we get

$$x = \frac{e^{A\xi + B}}{A} \ \cos (A\eta + C) + k_1 \tag{32.41}$$

and

$$y = \frac{e^{A\xi + B}}{A} \ \sin (A\eta + C) + k_2 \tag{32.42}$$

which are the equations of the two families of curves.

Eliminating $\xi \& \eta$ from (32.41) and (32.42), we get

$$(x - k_1)^2 + (y - k_2)^2 = \frac{1}{A^2} \cdot e^{2A\xi + 2B} \tag{32.43}$$

and

$$\frac{y - k_2}{x - k_1} = \tan (A\eta + C) \tag{32.44}$$

From (32.43) we conclude that the orthogonal trajectories are concentric circles which in the limiting case when A = 0 become a family of parallel straight lines. From (32.43) we conclude that the stream lines are lines through (k_1, k_2) and in the limiting case when A = 0 become a family of parallel straight lines.

So the two cases are when A = 0 and when A ≠ 0. In the case when A ≠ 0, from (32.35), we get $\frac{\partial g}{\partial \xi} \neq 0$ and hence $\frac{\partial p}{\partial \xi} \neq 0$ though $\frac{\partial p}{\partial \eta} = 0$. Therefore,

when the stream lines are radial lines, $p=p(\xi)$. In the case when $A = 0$, from (32.35), we get

$$\frac{\partial g}{\partial \xi} = 0 \qquad\qquad (32.45)$$

Putting (32.44) in (31.05), we get:

$$\frac{1}{\rho}\frac{\partial \rho}{\partial \xi} + \frac{1}{u}\frac{\partial u}{\partial \xi} = 0$$

Putting from this in (31.03) and using (31.06), we get

$$\frac{\partial u}{\partial \xi} = \frac{\partial p}{\partial \xi} = \frac{\partial \rho}{\partial \xi} = 0 \qquad \text{or} \qquad u = c$$

Therefore, again in the limiting case $p = $ constant.

Theorem VI. When the stream lines are straight lines and orthogonal curvilinear net is isometric, the stream lines must be a family of radial lines or a family of parallel lines.

Proof: As the stream lines are straight, the stream line inclination angle α given by (22.11) and (22.12) must remain constant along each stream line i.e.

$$\frac{\partial \alpha}{\partial \xi} = 0 \qquad\qquad (32.46)$$

Putting (32.46) in (12.11), we get

$$\frac{\partial g_1}{\partial \eta} = 0 \qquad\qquad (32.47)$$

Putting (32.47) in (31.04), we get

$$\frac{\partial p}{\partial \eta} = 0 \qquad \text{or} \quad p = p(\xi)$$

i.e. if stream lines are straight, then the orthogonal trajectories are isobaric.

Therefore by Theorem V, the result of this Theorem VI is established.

Section 3 Pressure Equation

To obtain the solutions of problems in the case of plane, steady and rotational gas flow Ozoklav[(1959), (1964)]used the compatibility equations for steady gas flows which were obtained by Berker (1956). These equations contain only the velocity field $\underset{\sim}{q}$ and are obtained by eliminating the pressure, the density and the specific entropy from the equations (21.01) - (21.04).

However, we eliminate the velocity, the density and the specific entropy from the non-linear partial differential equations (31.03) — (31.06) which govern the plane, steady and rotational flow of gas in natural coordinates and thus obtain a linear partial differential equation of second order in pressure.

From equation (31.05), we get

$$(g_2 \rho u) = \phi (\eta)$$

where ϕ is an arbitrary function of η

or

$$\rho = \frac{\phi (\eta)}{g_2 u} \tag{33.01}$$

Substituting for ρ from (33.01) in equation (31.04), we get

$$\frac{u^2}{g_1} \frac{\partial g_1}{\partial \eta} = \frac{g_2 u}{\phi(\eta)} \frac{\partial p}{\partial \eta}$$

Assuming that $u \neq 0$, we get

$$u = \frac{g_1 g_2}{\dfrac{\partial g_1}{\partial \eta}} \cdot \frac{1}{\phi(\eta)} \cdot \frac{\partial p}{\partial \eta} \tag{33.02}$$

Equation (33.02) expresses u in terms of p.

Substituting for u from (33.02) in (33.01), we get:

$$\rho = \frac{\phi^2(\eta)}{g_1 g_2^2} \frac{\partial g_1}{\partial \eta} \frac{1}{\frac{\partial p}{\partial \eta}} \tag{33.03}$$

Equation (33.03) expresses ρ in terms of p.

We eliminate ρ and u from (31.03), (33.02) and (33.03) to

obtain

$$\left(\frac{1}{\phi(\eta)} \frac{g_1 g_2}{\frac{\partial g_1}{\partial \eta}} \cdot \frac{\partial p}{\partial \eta} \right) \frac{\partial}{\partial \xi} \left(\frac{1}{\phi(\eta)} \frac{g_1 g_2}{\frac{\partial g_1}{\partial \eta}} \frac{\partial p}{\partial \eta} \right) + \frac{g_1 g_2^2}{\frac{\partial g_1}{\partial \eta}} \frac{1}{\phi^2(\eta)} \cdot \frac{\partial p}{\partial \xi} \frac{\partial p}{\partial \eta} = 0$$

or

$$\left(\frac{1}{\rho} \frac{g_1}{\frac{\partial g_1}{\partial \eta}} \right) \left\{ \frac{\partial^2 p}{\partial \xi \partial \eta} + \frac{\partial}{\partial \eta} (\log g_1) \frac{\partial p}{\partial \xi} + \frac{\partial}{\partial \xi} (\log G) \frac{\partial p}{\partial \eta} \right\} = 0 \tag{33.04}$$

where $G = \dfrac{g_1 g_2}{\dfrac{\partial g_1}{\partial \eta}} = \dfrac{g_2}{\partial/\partial \eta (\log g_1)}$

If $1 \Big/ \dfrac{\partial g_1}{\partial \eta} \neq 0$ or ∞ (since ρ and g_1 are non-zero and positive), then the

pressure function $p(\xi,\eta)$ in steady, plane and rotational gas flow is

given by:

$$\frac{\partial^2 p}{\partial \xi \partial \eta} + \frac{\partial}{\partial \eta} (\log g_1) \frac{\partial p}{\partial \xi} + \frac{\partial}{\partial \xi} (\log G) \frac{\partial p}{\partial \eta} = 0 \tag{33.05}$$

Equation (33.05) is a linear hyperbolic partial differential equation

of the second order in canonical form. In this equation pressure p is

the dependent variable and the natural coordinates ξ, η are the indepen-

dent variables. For a particular flow problem this equation can be

solved to get the pressure as a function of position.

However, the exceptional cases when the pressure field is

not given by (33.05) are:

(i) $\dfrac{1}{\dfrac{\partial g_1}{\partial \eta}} = 0$ or $\dfrac{g_1}{\dfrac{\partial g_1}{\partial \eta}} = 0$ ($\because g_1$ is non-zero and positive)

i.e. when the local radius of curvature of the stream line is zero.

This case is of no physical importance.

(ii) $\dfrac{1}{\dfrac{\partial g_1}{\partial \eta}} = \infty$ or $\dfrac{g_1}{\dfrac{\partial g_1}{\partial \eta}} = \infty$

This is the case when the local radius of curvature of the stream line is infinity i.e. a case of straight stream lines. In this case pressure is the solution of the equation.

$$\frac{\partial p}{\partial \eta} = 0 \qquad\qquad\qquad (33.06)$$

Section 4. Velocity, density, entropy and Mach number for the flows whose pressure is the solution of equation (33.05).

From the previous section we know that if the radius of curvature of the stream lines of flow is not infinity, then the pressure function is given as a solution of the pressure equation (33.05). In this section we consider only such flows. Let

$$p = P\ (\xi,\eta) \tag{34.01}$$

be the solution of equation (33.05) for a given problem of plane, steady and rotational gas flow. Substituting (34.01) in (31.03) and (31.04), we get:

$$\rho u\ \frac{\partial u}{\partial \xi} = -\ \frac{\partial P}{\partial \xi} \tag{34.02}$$

and

$$\rho u^2 = \frac{1}{\dfrac{\partial}{\partial \eta}(\log g_1)} \cdot \frac{\partial P}{\partial \eta} \tag{34.03}$$

Dividing (34.02) by (34.03), we eliminate ρ and get

$$\frac{1}{u}\ \frac{\partial u}{\partial \xi} = -\frac{\partial}{\partial \eta}\ (\log g_1)\ \frac{\partial P}{\partial \xi} \cdot \frac{1}{\dfrac{\partial P}{\partial \eta}}$$

Integrating w.r.t. ξ, we get

$$u(\xi,\eta) = f(\eta)\ F(\xi,\eta) \tag{34.04}$$

where $f(\eta)$ is an arbitrary function of η and $F(\xi,\eta)$ is a known function of ξ and η given by

$$F(\xi,\eta) = e^{-\int [\frac{\partial}{\partial \eta}\ (\log g_1)] \cdot [\frac{\partial P}{\partial \xi} \cdot \frac{1}{\frac{\partial P}{\partial \eta}}]\ d\xi} \tag{34.05}$$

Equation (34.04) gives us the velocity distribution for the flow problem.

Substituting (34.04) in (34.02) or (34.03), we get

$$\rho(\xi,\eta) = \frac{G(\xi,\eta)}{f^2(\eta)} \qquad (34.06)$$

wherein $f(\eta)$ is the same unknown function used in (34.04) and $G(\xi,\eta)$ is a known function given by

$$G(\xi,\eta) = \frac{\partial P}{\partial \eta} \cdot \frac{1}{\frac{\partial}{\partial \eta}(\log g_1)} \cdot e^{2\int [\frac{\partial}{\partial \eta}(\log g_1)] \cdot [\frac{\partial P}{\partial \xi} \cdot \frac{1}{\frac{\partial P}{\partial \eta}}]d\xi} \qquad (34.07)$$

Equation (34.06) gives us the density function for the flow. Now we find the Mach number at any point in the flow region. By definition Mach number at any point is given by

$$M = \frac{u}{c} \qquad (34.08)$$

Where u is the local speed of gas there at that point and c is the speed of sound.

From (31.05), we get:

$$\frac{\partial}{\partial \xi}(\log \rho) = - \frac{\partial}{\partial \xi}[\log (g_2 u)]$$

Using (34.04) for u in this equation, we get

$$\frac{\partial}{\partial \xi}(\log \rho) = -\frac{\partial}{\partial \xi}[\log (g_2 F)] \qquad (34.09)$$

Putting for u from (34.04) in (31.03) and for $\frac{1}{\rho}\frac{\partial \rho}{\partial \xi}$ from (34.09) in (31.03), equation (31.03) is

$$f^2(\eta) \ F(\xi,\eta) \ \frac{\partial}{\partial \xi} \ F(\xi,\eta) = c^2 \cdot \frac{1}{g_2 F(\xi,\eta)} \cdot \left\{ g_2 \frac{\partial F}{\partial \xi} + F \frac{\partial g_2}{\partial \xi} \right\}$$

or

$$M^2 = \frac{u^2}{c^2} = 1 + \frac{\partial}{\partial \xi}(\log g_2) \cdot \frac{1}{\frac{\partial}{\partial \xi}(\log F)}$$

or

$$M = \sqrt{1 + \frac{\partial}{\partial \xi} (\log g_2) \cdot \frac{1}{\frac{\partial}{\partial \xi}(\log F)}} \qquad (34.10)$$

Substituting for $\dot{F}(\xi, \eta)$ from (34.05) in (34.10), we get

$$M = \sqrt{1 - \frac{\frac{\partial}{\partial \xi}(\log g_2) \frac{\partial P}{\partial \eta}}{\frac{\partial}{\partial \eta}(\log g_1) \frac{\partial P}{\partial \xi}}} \qquad (34.11)$$

Equation (34.10) or (34.11) gives us the Mach number at any point in the flow in terms of the pressure there.

Now since the flow at any point is subsonic, sonic or super-sonic according as $M \lessgtr 1$, from (34.11), we get:

$$\frac{[\frac{\partial}{\partial \xi}(\log g_2)] \cdot \frac{\partial P}{\partial \eta}}{[\frac{\partial}{\partial \eta}(\log g_1)] \cdot \frac{\partial P}{\partial \xi}} \gtrless 0 \qquad (34.12)$$

according as the flow at any point is subsonic, sonic or supersonic.

Finally from equation (31.06) we find that the entropy is constant on each stream line i.e.

$$s = \psi(\eta) \qquad (34.13)$$

wherein ψ is an unknown function of η. We have in both the velocity function and the density function given by (34.04) and (34.06) an unknown function $f(\eta)$ involved. Also, we have in entropy function given by (34.13) another unknown function $\psi(\eta)$ involved. These two functions are non-vanishing differentiable functions and remain constant on each stream line since $\underset{\sim}{q} \cdot \text{grad } \psi = 0$, $\underset{\sim}{q} \cdot \text{grad } f = 0$.

We cannot determine these two functions uniquely from the knowledge of state equation of the gas in general. For a given gas

when the pressure function is given by equation (34.01), we have many

flow fields possible and the number of these flow fields depends upon

the state equation of the gas. There are two types of gases.

Type I: Gases obeying the product equation of state.

Type II: Gases having the equation of state not in product

form.

For gases having the product equation of state the unknown

functions $f(\eta)$ and $\psi(\eta)$ can be chosen in an infinite number of ways

so that the state equation of the gas holds. Let

$$\rho = R(p)S(s) \tag{34.14}$$

be the equation of state for a gas of type I.

Substituting for p, ρ and s from (34.01), (34.06) and

(34.13), we get

$$\frac{G(\xi,\eta)}{R[P(\xi,\eta)]} = f^2(\eta) \; S[\psi(\eta)] \tag{34.15}$$

As the left hand side of this equation is a known function

of η say $a(\eta)$ but the right hand side is the product of two unknown

functions of η, therefore, we can have an infinite number of choices of

$f(\eta)$ and $\psi(\eta)$ under the restriction that (34.15) holds.

Therefore, corresponding to a unique pressure distribution

we get an infinite family of flow fields sharing the same stream lines

if the equation of state is of the product form. From this we find that

our results satisfy the Prim's substitution principle [Prim (1952)] .

For the gases of type II the unknown functions $f(\eta)$ and

$\psi(\eta)$ related by the state equation

$$p = p(\rho,s)$$

may be uniquely determined as illustrated by the following example. Let

$$p = (a\rho^{-\frac{1}{2}} + bs)^n \tag{34.16}$$

be the equation of state not in the product form. Here a and b are constants and n is any +ve or -ve integer or a fraction.

Differentiating (34.16) partially w.r.t. ξ and using equation (31.06), we get:

$$\frac{\partial p}{\partial \xi} = \frac{na}{2} \ p^{\frac{n-1}{n}} \cdot \frac{1}{\sqrt{\rho}} \cdot \frac{\partial \rho}{\partial \xi} \tag{34.17}$$

Substituting for p and ρ from equations (34.01) and (34.06), we obtain

$$\frac{2}{na} \frac{\partial P}{\partial \xi} \cdot \frac{1}{(P)^{\frac{n-1}{n}}} \ \sqrt{G} \cdot \frac{1}{\frac{\partial G}{\partial \xi}} = \frac{1}{f(\eta)} \tag{34.18}$$

Since left hand side is known, the function $f(\eta)$ is uniquely determined from this equation. Substituting this value of $f(\eta)$ in (34.16), we obtain the function $\psi(\eta)$ in a unique way.

Therefore, when the equation of state is of the form (34.16), then the velocity field, the density field and the entropy field are uniquely determined for flow fields sharing the same stream lines, the same pressure field and the same Mach number field.

Section 5. Plane steady flows when pressure is not given as a solution of the pressure equation (33.05)

We know that pressure of a given problem is not the solution of equation (33.05) when the radius of curvature of stream lines of flow is either zero or infinity (section 3, page 29). Therefore, in this section we study the case when the radius of curvature of the stream lines is infinity i.e. the stream lines are straight lines (the other case of zero radius of curvature being of no physical importance).

For such a case

$$\frac{\partial \alpha}{\partial \xi} = 0 \qquad (35.01)$$

where α is the local angle between the straight stream lines and the x-axis and $\eta =$ constant are straight stream lines.

Putting (35.01) in (22.11), we get

$$\frac{\partial g_1}{\partial \eta} = 0 \qquad (35.02)$$

Using (35.02) in (31.04), we get

$$\frac{\partial p}{\partial \eta} = 0 \qquad (35.03)$$

Now we divide the flow patterns with straight stream lines into two groups as:

      (i) The straight parallel flows.

      (ii) The straight non-parallel flows.

and study these one by one.

(i)   The straight parallel flows: By Theorem IV (Section 2, page 23) and Theorem III (Section 2, page 21), we have for such flows

$$\frac{\partial u}{\partial \xi} = 0 \qquad (35.04)$$

$$\frac{\partial \rho}{\partial \xi} = 0 \tag{35.05}$$

$$\frac{\partial p}{\partial \xi} = 0 \tag{35.06}$$

From (35.03) and (35.06), we get

$$p(\xi,\eta) = A \tag{35.07}$$

where A is an arbitrary constant and can be fixed for a given problem.

Having thus obtained the pressure, the other flow variables (i.e. u, ρ and s) are obtained by solving the linear partial differential equations (35.04) and (35.05) by prescribing appropriate boundary conditions and using the caloric equation of state (31.07) for the regions of continuous motion.

(ii)   The straight non-parallel flows.

Let

$$p(\xi,\eta) = P(\xi) \tag{35.08}$$

be the solution of equation (35.03) for a given problem with non-parallel straight lines.

Putting (35.08) in (31.03), we get

$$\rho u \frac{\partial u}{\partial \xi} + P'(\xi) = 0 \tag{35.09}$$

Putting for ρu from (35.09) in (31.05), we get

$$g_2 \, P'(\xi) \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial}{\partial \xi} [g_2 P'(\xi)] \frac{\partial u}{\partial \xi} = 0$$

or

$$\frac{\partial}{\partial \xi} \left[\log \left(\frac{\frac{\partial u}{\partial \xi}}{g_2 P'(\xi)} \right) \right] = 0$$

or

$$u(\xi,\eta) = f(\eta) \left\{ \int g_2 P'(\xi) \, d\xi + \frac{\phi(\eta)}{f(\eta)} \right\} \tag{35.10}$$

where $f(\eta)$ and $\phi(\eta)$ are the arbitrary functions of η.

Putting (35.10) in (35.09), we get

$$\rho(\xi,\eta) = \frac{-1}{g_2 f^2(\eta)\left\{\int g_2 P'(\xi)d\xi + \frac{\phi(\eta)}{f(\eta)}\right\}}$$

(35.11)

Using (35.11) and (35.08) in (31.07), we get the specific entropy s so that $\frac{\partial s}{\partial \xi} = 0$ also holds.

Finally, from (31.05) and (31.03), we get

$$\frac{1}{g_2 u}\frac{\partial}{\partial \xi}(g_2 u) = -\frac{1}{\rho}\frac{\partial \rho}{\partial \xi}$$

and

$$u\frac{\partial u}{\partial \xi} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial \xi} \qquad (\because \text{ of } (31.06))$$

Eliminating $\frac{1}{\rho}\frac{\partial \rho}{\partial \xi}$ between these two equations, we get

$$u\frac{\partial u}{\partial \xi} = \frac{c^2}{g_2 u}\frac{\partial}{\partial \xi}(g_2 u)$$

or

$$M^2 = \frac{u^2}{c^2} = 1 + \left(u\frac{\partial g_2}{\partial \xi} \Big/ g_2\frac{\partial u}{\partial \xi}\right)$$

or

$$M = \sqrt{1 + \frac{\frac{\partial}{\partial \xi}(\log g_2)}{\frac{\partial}{\partial \xi}(\log u)}}$$

(35.12)

where u is given by (31.10)

Section 5 b. Source Flows

A familiar flow pattern, which belongs to group (ii) of non-parallel straight stream lines, is of radial flow due to some source. Choosing the radial lines through the source as our stream lines and the concentric circles with centre at the source as our orthogonal trajectories, we get

$$g_1(\xi,\eta) = g_2(\xi,\eta) = e^{\xi} \tag{35.13}$$

Putting for $g_2(\xi,\eta)$ from (35.13) in (35.10), (35.11) and (35.12), we get:

$$u(\xi,\eta) = f(\eta)\left\{\int e^{\xi}P'(\xi)d\xi + \frac{\phi(\eta)}{f(\eta)}\right\} \tag{35.14}$$

$$\rho(\xi,\eta) = \frac{-1}{e^{\xi}f^2(\eta)\left\{\int e^{\xi}P'(\xi)d\xi + \frac{\phi(\eta)}{f(\eta)}\right\}} \tag{35.15}$$

and

$$M = \frac{e^{\xi}P'(\xi)}{\left\{\int e^{\xi}P'(\xi)d\xi + \frac{\phi(\eta)}{f(\eta)}\right\}} \tag{35.16}$$

Section 6. Some special forms of isometric nets and pressure equation.

In this section we take some special forms of isometric nets and find the restrictions on these forms from Gauss's equation obtaining, thereby, some well known coordinate systems (Moon and Spencer (1961)).

We then obtain the pressure equation (33.05) for these forms and give the Riemann-Green function, associated with each of these linear hyperbolic equations in pressure, for the coordinate systems of form I considered.

A plane orthogonal curvilinear net (ξ, η) is an isometric net if the metric coefficients $g_1(\xi, \eta)$ and $g_2(\xi, \eta)$ are equal i.e.

$$g_1(\xi, \eta) = g_2(\xi, \eta) = g\ (\xi, \eta)\ \text{(say)}$$

For such a net Gauss's equation (31.08) is

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) (\log g) = 0 \tag{36.01}$$

__Form I.__

$$g_1^2(\xi, \eta) = g_2^2(\xi, \eta) = \{f(\xi) + h(\eta)\}^n = g^2(\xi, \eta) \tag{36.02}$$

where the different cases for this form under study are (i) when $n = 1$ (ii) when $n = -1$ (iii) when $n = -3$ (iv) when $n = -\frac{1}{2}$ and (v) when $n = -\frac{3}{2}$.

From (36.02), we get

$$\frac{\partial^2}{\partial \xi^2} (\log g) = \frac{n}{2} \left\{ \frac{(f(\xi) + h(\eta))\ f''(\xi) - f'^2(\xi)}{(f(\xi) + h(\eta))^2} \right\} \tag{36.03}$$

and

$$\frac{\partial^2}{\partial \eta^2} (\log g) = \frac{n}{2} \left\{ \frac{(f(\xi) + h(\eta))\ h''(\eta) - h'^2(\eta)}{(f(\xi) + h(\eta))^2} \right\} \tag{36.04}$$

Putting (36.03) and (36.04) in (36.01), we get:

$$f(\xi) \, f''(\xi) + h(\eta) \, h''(\eta) - f'^{2}(\xi) - h'^{2}(\eta) + f(\xi)h''(\eta) + h(\eta)f''(\xi) = 0$$

Taking $\dfrac{\partial^2}{\partial\xi\partial\eta}$ of this equation, we get

$$f'(\xi)h'''(\eta) + h'(\eta)f'''(\xi) = 0$$

Whence $h'(\eta) \neq 0, \; f'(\xi) \neq 0$

$$\frac{h'''(\eta)}{h'(\eta)} + \frac{f'''(\xi)}{f'(\xi)} = 0 \tag{36.05}$$

is the restriction on the form I from Gauss's equation.

If (36.05) holds, then

$$\frac{h'''(\eta)}{h'(\eta)} = - \frac{f'''(\xi)}{f'(\xi)}$$

Since the left hand side is independent of ξ and the right hand side is

independent of η and the two sides are equal, each side is equal to k^2

(some constant).

Therefore:

$$f'''(\xi) + k^2 \, f'(\xi) = 0 \tag{36.06}$$

and

$$h'''(\eta) - k^2 \, h'(\eta) = 0 \tag{36.07}$$

The eigenfunctions of the ordinary differential equation (36.06) are

$\sin k\xi$, $\cos k\xi$ and c (constant) when $k^2 \neq 0$ or 0 and the eigenfunctions

of (36.07) are $e^{+k\eta}$, $e^{-k\eta}$ and c (constant).

Now we substitute (36.02) in pressure equation (33.05) and

find that the pressure equation for the form I is

$$L(p) = \frac{\partial^2 p}{\partial\xi\partial\eta} + \frac{n \, h'(\eta)}{2 \, \{f(\xi)+h(\eta)\}} \frac{\partial p}{\partial\xi} + \frac{n+2}{2} \frac{f'(\xi)}{\{f(\xi)+h(\eta)\}} \frac{\partial p}{\partial\eta} = 0 \tag{36.08}$$

This is a linear partial hyperbolic differential equation of second order and the general solution of Cauchy's problem for this equation can be obtained by Riemann method. This method depends on finding a certain subsidiary function, often called the Riemann-Green function, which is the solution of a characteristic boundary value problem for the adjoint equation. We, therefore, find the Riemann-Green function for the equation we wish to solve by Riemann method. For our form I, we first reduce (36.08) to a type of hyperbolic equation whose Riemann-Green function is already found by Mackie (1956).

We let

$$f(\xi) = r,$$
$$h(\eta) = s$$

(36.09)

and get the equation (36.08) reduced to

$$\frac{\partial^2 p}{\partial r \partial s} + \frac{1}{(r+s)} \left[\frac{n}{2} \frac{\partial p}{\partial r} + \frac{(n+2)}{2} \frac{\partial p}{\partial s} \right] = 0$$

(36.10)

for the region where $f'(\xi) \neq 0$ and $h'(\eta) \neq 0$.

Mackie constructed the complex integral solution of this type of equation and such a complex integral gives the Riemann-Green function for an appropriate choice of contour. [Copson (1957-58)]. For equation (36.10) the Riemann-Green function is

$$R(r,s;R,S) = \frac{(r+s)^{\frac{n}{2}}(r+s)}{(R+S)^{\frac{n+2}{2}}} \quad F[1-\frac{n}{2}, 1+\frac{n}{2}, 1; \ -\frac{(R-r)(S-s)}{(R+S)(r+s)}]$$

(36.11)

where $F[a,b,c;z]$ is a hypergeometric function which is convergent for $|z| < 1$ only since $c-a-b \not\Rightarrow 0$.

Now we consider some important well known isometric nets which belong to this form I for different values of n. These different

important nets are used for finding the solutions of different problems.

In the sequel, $z = x + iy$ and $w = \xi + i\eta$.

(i) when $n = 1$,

$$g_1^2\,(\xi,\eta) = g_2^2\,(\xi,\eta) = f(\xi) + h(\eta) = g^2\,(\xi,\eta)$$

and the coordinate nets are

(a) Parabolic cylinder coordinates $z = \dfrac{1}{2}\,w^2$

and   (b) Elliptic cylinder coordinates $z = a \cos h\ w$.

For both these curvilinear nets the pressure equation (33.05) is given

by (36.08) wherein $n = 1$ is substituted and the functions $f(\xi)$ and $h(\eta)$

are put in for the two systems.

(ii) when $n = -1$,

$$g_1^2 = g_2^2 = \frac{1}{f(\xi)+h(\eta)} = g^2$$

and the two nets are

(a) logarithmic curves $z = \dfrac{2a}{\pi}\,\log w$

(b) logarithmic tangent curves $z = \dfrac{2a}{\pi}\,\log \tan w$.

For these nets $n = -1$ for the pressure equation (36.08) and $f(\xi)$ and

$h(\eta)$ are given by the two nets respectively.

(iii) when $n = -2$

$$g_1^2 = g_2^2 = \frac{1}{\{f(\xi)+h(\eta)\}^2} = g^2$$

and the coordinate nets are

(a) tangent cylinder coordinates $\bar{z} = \dfrac{1}{w}$

(b) bicylinder coordinates $\bar{z} = \dfrac{a(e^{w}+1)}{e^{w}-1}$

For these systems $n = -2$ and $f(\xi)$, $h(\eta)$ are the respective

values of the systems in (36.08).

(iv) when n = -3,

$$g_1^2 = g_2^2 = \frac{1}{\{f(\xi)+h(\eta)\}^3} = g^2$$

The coordinate net for this case is cardioid cylinder coordinates $\bar{z} = \dfrac{1}{2w^2}$ and in pressure equation (36.08) we have n = -3.

(v) when n = -3/2,

$$g_1^2 = g_2^2 = \frac{1}{\sqrt{\{f(\xi)+h(\eta)\}^3}} = g^2$$

For this case coordinate system is rose-coordinates $\bar{z} = \sqrt{2}\ w^{-\frac{1}{2}}$ and n = -3/2 in (36.08) for pressure equation.

(vi) when n = -1/2,

$$g_1^2 = g_2^2 = \frac{1}{\sqrt{f(\xi)+h(\eta)}} = g^2$$

The coordinate net for this case is hyperbolic cylinder coordinates $z = \sqrt{2}\ w^{1/2}$ and the pressure equation is obtained from (36.08) by taking n = -1/2 and the functions $f(\xi)$, $h(\eta)$ of the system.

Form II.

$$g_1^2(\xi,\eta) = g_2^2(\xi,\eta) = g^2(\eta) = h^2(\eta) \tag{36.12}$$

Putting (36.12) in (36.01), we get

$$h(\eta)h''(\eta) - h'^2(\eta) = 0 \tag{36.13}$$

The general solution of equation (36.13) is

$$h(\eta) = e^{c\eta+d} \tag{36.14}$$

where c and d are arbitrary constants. An important curvilinear net belonging to this form is given by (36.14) when c = 1 and d = 0 is substituted in it. For this case the coordinate system is circular-

cylinder coordinates $z = e^{i\overline{w}}$ such that $\eta = $ constant are circles and $\xi = $ constant are radial lines. Now as $\frac{\partial g}{\partial \eta} \neq 0$ for this case, we have the pressure equation (33.05) for this case as:

$$\frac{\partial^2 p}{\partial \xi \partial \eta} + \frac{\partial p}{\partial \xi} = 0 \qquad (36.15)$$

This equation can be solved for Cauchy's problem without using the Riemann's method.

Form III

$$g_1^2(\xi,\eta) = g_2^2(\xi,\eta) = f^2(\xi) = g^2(\xi) \qquad (36.16)$$

Putting (36.16) in (36.01), we get

$$f(\xi) \, f''(\xi) - f'^2(\xi) = 0 \qquad (36.17)$$

The general solution for this equation is

$$f(\xi) = e^{c\xi + d} \qquad (36.18)$$

wherein c and d are the arbitrary constants.

(i) when $c = 0$, $d = 0$, we get the rectangular net $z = w$. Taking the stream lines as $\eta = $ constant and the orthogonal trajectories as $\xi = $ constant, we get the flow in parallel straight lines parallel to x-axis in (x,y) plane. This case has already been discussed in section 5 of this chapter.

(ii) when $c = 1$, $d = 0$, we get the circular-cylinder coordinate net $z = e^w$ so that the stream lines $\eta = $ constant are the radial lines. This case has also been discussed in section 5 and for these two cases of this form III we cannot use the pressure equation (33.05) as proved in section 3 of the present chapter.

Section 7. Vortex Flow.

When η = constant, the stream lines, are concentric circles ($\eta>0$) and ξ = constant, the orthogonal trajectories, are radial lines, we get a vortex flow.
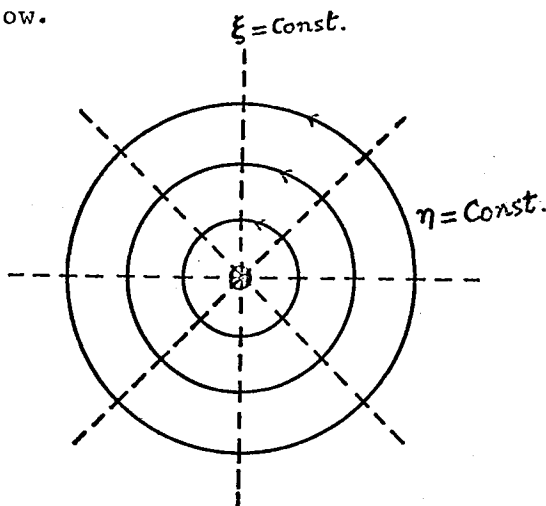
For this flow our net is

$$x = \eta \cos \xi,$$

$$y = \eta \sin \xi,$$

and

$$ds^2 = \eta^2 d\xi^2 + d\eta^2 \qquad (37.01)$$

Pressure equation for this flow is given by

$$\eta \frac{\partial^2 p}{\partial \xi \partial \eta} + \frac{\partial p}{\partial \xi} = 0 \qquad (37.02)$$

or

$$\frac{\partial}{\partial \eta} [\eta \frac{\partial p}{\partial \xi}] = 0$$

i.e.

$$p(\xi,\eta) = \frac{\phi(\xi)}{\eta} + \psi(\eta) \qquad (37.03)$$

where $\phi(\xi)$ and $\psi(\eta)$ are the two arbitrary functions of ξ and η respectively.

Now there are two possibilities

(i) $\phi'(\xi) \neq 0$ i.e. $\frac{\partial p}{\partial \xi} \neq 0$

or

$$(ii) \quad \phi'(\xi) = 0 \qquad i.e. \quad \frac{\partial p}{\partial \xi} = 0$$

In case (i)

Let $p = P(\xi,\eta)$ be given by (37.03). We put for p in

(34.04) and (34.06) to get the velocity and density distributions.

However, substituting (37.03) in (34.11), we get

$M = \frac{u}{c} = 1$ i.e.   the flow is sonic at every point in the

flow region.   Since in this case $\frac{\partial p}{\partial \xi} \neq 0$, we get $\frac{\partial \rho}{\partial \xi} \neq 0$ and $\frac{\partial u}{\partial \xi} \neq 0$.

Putting $u = c$ in (31.03), we get

$$\frac{c}{\rho} + \frac{\partial c}{\partial \rho} = 0$$

i.e. $c\rho = e^{f(s)}$ where $f(s)$ is an arbitrary function of entropy.

or using $\frac{\partial p}{\partial \rho} = c^2$, we get

$$p = -\frac{e^{2f(s)}}{\rho} + h(s) \tag{37.04}$$

wherein $h(s)$ is another arbitrary function of s.

Now here the fundamental property of all actual media that,

entropy remaining constant, the pressure increases with density and,

density remaining constant, pressure increases with entropy holds.   There-

fore, for these gases which obey form (37.04) for the state equation,

our results hold.

In case (ii)

$$p(\xi,\eta) = \psi(\eta) \tag{37.05}$$

Putting (37.05) in (34.04) and (34.05), we get

$$u = f(\eta) \tag{37.06}$$

and

$$\rho = \frac{\eta}{f^2(\eta)} \, \psi'(\eta) \tag{37.07}$$

Wherein $f(\eta)$ and $\psi(\eta)$ are arbitrary functions. Putting (37.05) and (37.07) in state equation, we get $s = s(\eta)$.

Section 8.   Flow in a parabolic channel
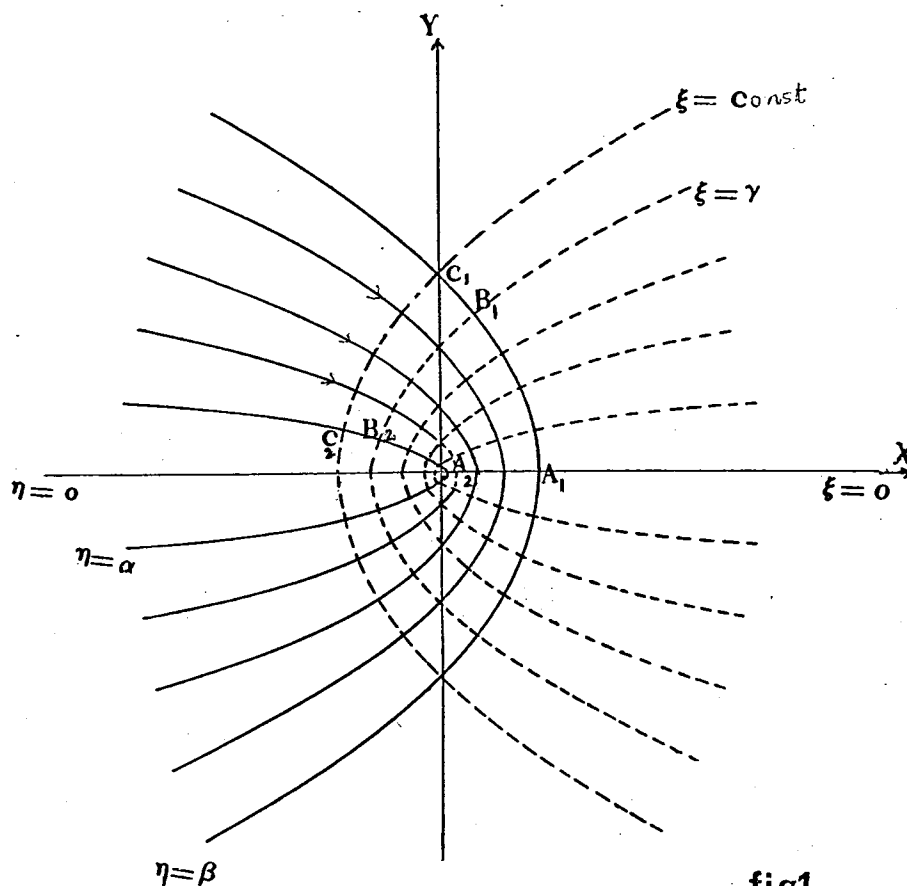
fig1.

In this section we study the flow of gases in an infinite channel whose walls are two confocal parabolas. We take the family of confocal parabolas confocal with the parabolic walls as our stream lines in order to obtain the flow in this channel. We denote this family of parabolas by η = constant such that when the walls are given by $\eta = \alpha$ and $\eta = \beta$, the stream lines of flow are $\alpha \leq \eta \leq \beta$.

With this choice of stream lines made for the parabolic channel, we take the other family of confocal parabolas ξ = constant, which are also confocal with the family η = constant, as our orthogonal trajectories where $-\infty < \xi < +\infty$.

So our infinite channel is given by $\alpha \leq \eta \leq \beta$; $-\infty < \xi < +\infty$. The equations for the two families of curves for this choice of natural coordinates are:

$$x = \frac{1}{2}(\eta^2 - \xi^2), \quad y = \xi\eta \qquad (38.01)$$

The squared element of arc length is

$$ds^2 = (\xi^2 + \eta^2)[d\xi^2 + d\eta^2] \qquad (38.02)$$

wherein the metric coefficients for this coordinate system are

$$g_1(\xi,\eta) = g_2(\xi,\eta) = (\xi^2 + \eta^2)^{1/2} \qquad (38.03)$$

Since $g_1 = g_2$ in this system of curvilinear coordinates, by equation (38.03) our system corresponds to $n = 1$ in form I of section 6 of this chapter and is an isometric net.

We solve this problem of flow of gases, in this section, by using the pressure equation (36.08) established for coordinate systems of form I. However, this problem could also be solved by the method of using compatibility equations [Berker (1956)] and making the applications of the inverse method [Nemenyi (1951)] as carried out by Ozoklav (1959) for finding the flow of gases in a hyperbolic channel.

Pressure equation (36.08) for the present flow in the channel takes the form

$$\frac{\partial^2 p}{\partial\xi\partial\eta} + \frac{\eta}{\xi^2+\eta^2}\frac{\partial p}{\partial\xi} + \frac{3\xi}{\xi^2+\eta^2}\frac{\partial p}{\partial\eta} = 0 \qquad (38.04)$$

as $n = 1$, $f(\xi) = \xi^2$ and $h(\eta) = \eta^2$ in this flow. Equation (38.04) is a second order linear partial differential equation of hyperbolic type in canonical form. In this equation pressure p is the dependent variable and ξ,η are the independent variables. Our aim is to represent a solution

p by properly prescribing the boundary value problem for it. For solving (38.04), to get the pressure at any point, we must prescribe the values of p and that of the 'outgoing' derivative of p on a curve C which is a free curve (i.e. C is nowhere tangent to a characteristic direction). If, however, the intitial curve degenerates into a right angle formed by the characteristics $\xi = c_1$, $\eta = c_2$, then we pose the boundary value problem called the characteristic boundary value problem in which merely the values of one quantity p on $\xi = c_1$ and $\eta = c_2$ are prescribed.

Therefore, to get p we prescribe the pressure along the wall (i.e. characteristic curve of (38.04)) $\eta = \alpha$ (or $\eta = \beta$) and along the orthogonal trajectory $\xi = \gamma$ (i.e. second characteristic curve of (38.04)) since these two are at right angle.

As the domain of dependence of a point P is enclosed by the two characteristic curves through P to the boundary, therefore, if pressure is prescribed along an orthogonal trajectory $\xi = \gamma$ and the arc $B_2 C_2$ of $\eta = \alpha$ (or the arc $B_1 C_1$ of $\eta = \beta$), we can then solve for the pressure at any point inside the region $C_2 B_2 B_1 C_1$ (fig. 1).

Let

$$p(\xi, \alpha) = f(\xi) \qquad 0 \leq \xi \leq \gamma,$$
$$p(\gamma, \eta) = g(\eta) \qquad \alpha \leq \eta \leq \beta \qquad\qquad (38.05)$$
$$\text{with the restriction } f(\gamma) = g(\alpha)$$

be the values of pressure prescribed along the arc $A_2 B_2$ and the arc $B_2 B_1$ so that it attains the unique value at $B_2(\gamma, \alpha)$.

So our problem is to solve for pressure in the region $A_2 A_1 B_1 B_2$ when it is governed by (38.04) and is subjected to the boundary conditions (38.05). To achieve this we first transform the equations (38.04) and (38.05), by changing the independent variables, to obtain a more

convenient form.

We let

$$\xi^2 = r, \quad \Big\}$$
$$\eta^2 = s \quad \Big\}$$
(38.06)

and get

$$L(p) \equiv \frac{\partial^2 p}{\partial r \partial s} + \frac{1}{r+s} \left(\frac{1}{2} \frac{\partial p}{\partial r} + \frac{3}{2} \frac{\partial p}{\partial s}\right) = 0$$
(38.07)

with the boundary conditions

$$p(r, s_1) = a(r) \qquad 0 \le r \le r_1 \quad \Big\}$$
$$p(r_1, s) = b(s) \qquad s_1 \le s \le s_2 \quad \Big\}$$
(38.08)

with the restriction $a(r_1) = b(s_1)$

Here $r_1 = \gamma^2$, $s_1 = \alpha^2$ and $s_2 = \beta^2$. By this transformation the original flow in parabolic channel (infinite) is transformed into the infinite rectangular channel bounded by $r = 0$, $r = \infty$ and $s = s_1$, $s = s_2$. This transformation is from Oxy plane to the first quadrant of $\overline{O}rs$ plane such that the stream lines $\eta = $ constant (bounded by $\eta = \alpha$, $\eta = \beta$) and their orthogonal trajectories $\xi = $ constant (bounded by $\xi = 0$, $\xi = \infty$ for upper half) are mapped into straight lines parallel to r-axis (bounded by $s = s_1$, $s = s_2$) and parallel to s-axis (bounded by $r = 0$, $r = \infty$). The region $A_2 A_1 B_1 B_2$ of channel, under study, is mapped into the region $\overline{A}_2 \overline{A}_1 \overline{B}_1 \overline{B}_2$ as shown in fig. 2.

By this transformation, our problem is to solve for pressure at any point in $\overline{A}_1 \overline{A}_2 \overline{B}_2 \overline{B}_1$ when pressure is governed by (38.07) inside this region and the boundary conditions are prescribed by (38.08) on the sides $r = r_1$ and $s = s_1$.
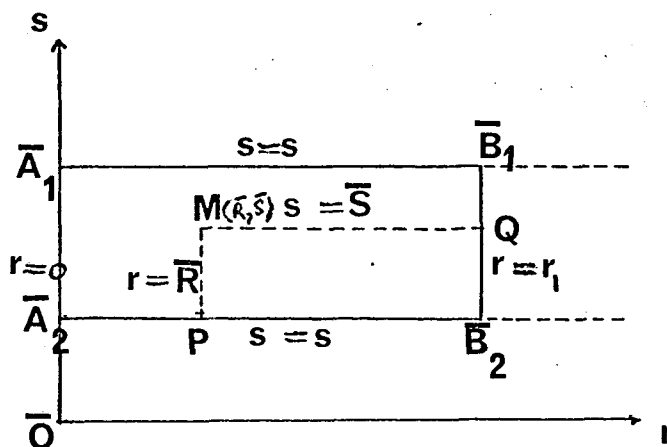
fig 2.

We shall solve this problem by Riemann's method. Let

$M(\overline{R},\overline{S})$ by any point inside the rectangle so that MQ and MS are the

two characteristic curves through M, given by $s = \overline{S}$ and $r = \overline{R}$, meeting

the boundary curves in Q,P. Riemann's method of solving the problem

depends ultimately on finding a certain subsidiary function, associated

with the operator L of (38.07) & often called the Riemann-Green function,

which is the solution of a characteristic boundary value problem for the

adjoint equation. Therefore, we first find this function $R(r,s;\overline{R},\overline{S})$

which is the solution of the adjoint equation of (38.07) i.e.

$$\overset{*}{L}(R) \equiv \frac{\partial^2 R}{\partial r \partial s} - \frac{1}{2(r+s)} \frac{\partial R}{\partial r} - \frac{3}{2(r+s)} \frac{\partial R}{\partial s} + \frac{2}{(r+s)^2} R = 0 \quad (38.09)$$

Such that

$$R(r,\overline{S};\overline{R},\overline{S}) = \exp\left\{\int_{\overline{R}}^{r} \frac{3}{2(\lambda+\overline{S})} \, d\lambda\right\} \qquad \text{on } s = \overline{S},$$

$$R(\overline{R},s;\overline{R},\overline{S}) = \exp\left\{\int_{\overline{S}}^{s} \frac{1}{2(\overline{R}+\lambda)} \, d\lambda\right\} \qquad \text{on } r = \overline{R}$$

$$\left.\right\} \quad (38.10)$$

and $R(\overline{R},\overline{S};\overline{R},\overline{S}) = 1$ at $(\overline{R},\overline{S})$.

From (38.10) $R(r,s;\bar{R},\bar{S})$ is such that

$$R = \frac{(r+\bar{S})^{3/2}}{(\bar{R}+\bar{S})^{3/2}} \qquad \text{on } s = \bar{S},$$

$$R = \frac{(\bar{R}+s)^{1/2}}{(\bar{R}+\bar{S})^{1/2}} \qquad \text{on } r = \bar{R} \qquad\qquad\qquad (38.11)$$

and $\qquad R = 1 \qquad\qquad \text{at } r = \bar{R}, \ s = \bar{S}$

From (38.11) we guess that $R(r,s;\bar{R},\bar{S})$ is of the form

$$R(r,s;\bar{R},\bar{S}) = \frac{(r+s)^{1/2}(r+\bar{S})}{(\bar{R}+\bar{S})^{3/2}} F(\alpha,\beta,\gamma;z) \qquad\qquad (38.12)$$

where

$$z = - \frac{(r-\bar{R})(s-\bar{S})}{(\bar{R}+\bar{S})(r+s)} \qquad\qquad\qquad (38.13)$$

so that conditions (38.11) hold. Here $F(\alpha,\beta,\gamma;z)$ is a hypergeometric function in which α,β and γ are unknown constants to be determined and

$$F(\alpha,\beta,\gamma;z) = 1 + \frac{\alpha\beta}{\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} + \text{-----} \qquad (38.14)$$

This series is convergent for $|z| < 1$ and is convergent for $|z| = 1$ also, if $\gamma - \alpha - \beta > 0$. [Friedrichs (1965)] .

Substituting (38.12) in (38.09), we get

$$F''(z) \frac{\partial z}{\partial r} \frac{\partial z}{\partial s} + \left(\frac{\partial^2 z}{\partial r \partial s} - \frac{1}{r+s} \frac{\partial z}{\partial s} + \frac{1}{r+\bar{S}} \frac{\partial z}{\partial s}\right) F'(z) + \frac{3}{4(r+s)^2} F(z) = 0$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (38.15)$$

From (38.13), we get

$$\frac{\partial z}{\partial r} \frac{\partial z}{\partial s} = \frac{1}{(r+s)^2} (z^2 - z),$$

$$\frac{\partial^2 z}{\partial r \partial s} = \frac{2z - 1}{(r+s)^2},$$

and $\left(\frac{1}{(r+\bar{s})} - \frac{1}{(r+s)} \right) \frac{\partial z}{\partial s} = \frac{z}{(r+s)^2}$ (38.16)

Substituting (38.16) in (38.15), we get

$$z(1-z)F''(z) + (1-3z)F'(z) - \frac{3}{4} F(z) = 0 \tag{38.17}$$

This equation is the Gaussian differential equation with

$\alpha = 1/2$, $\beta = 3/2$ and $\gamma = 1$. Therefore, equation (38.17) possesses a

unique solution $F(\frac{1}{2}, \frac{3}{2}, 1; z)$. Hence the Riemann-Green function is

$$R(r,s;\bar{R},\bar{S}) = \frac{(r+s)^{1/2}(r+\bar{s})}{(\bar{R}+\bar{S})^{3/2}} F(\frac{1}{2}, \frac{3}{2}, 1; z) \tag{38.18}$$

wherein z is given by (38.13). This hypergeometric function $F(\frac{1}{2}, \frac{3}{2}, 1; z)$

is only convergent for $|z| < 1$ as $\gamma - \alpha - \beta < 0$ here & $M(\bar{R}, \bar{S})$ is the point

where representation for pressure (i.e. $p(M) = p(\bar{R}, \bar{S})$) is to be found

and (r, s) is any point. Mackie [(1954),(1955),(1956)] obtained a

Contour integral formula for the Riemann-Green function of the partial

differential equation of the type we have.

Now having found Riemann-Green function, given by (38.18),

for the operator L of (38.07), we get the solution of pressure at M for

(38.07) with the boundary conditions (38.08) as: [Courant and Hilbert

(1965) pages 454-55].

$$p(M) = p(\bar{R}, \bar{S}) = p(r_1, s_1) \frac{(r_1 + s_1)^{1/2}(r_1 + \bar{s})}{(\bar{R} + \bar{S})^{3/2}} F\left[\begin{matrix} \frac{1}{2}, \frac{3}{2}, 1; \\ \end{matrix} \frac{(r_1 - \bar{R})(\bar{S} - s_1)}{(\bar{R} + \bar{S})(r_1 + s_1)}\right]$$

$$- \int_{\bar{R}}^{r_1} \frac{(x + s_1)^{1/2}(x + \bar{s})}{(\bar{R} + \bar{S})^{3/2}} F\left[\begin{matrix} \frac{1}{2}, \frac{3}{2}, 1; \\ \end{matrix} \frac{(x - \bar{R})(\bar{S} - s_1)}{(\bar{R} + \bar{S})(x + s_1)}\right] \left\{ a'(x) + \frac{3a(x)}{2(x + s_1)} \right\} dx$$

$$-\int_{\overline{S}}^{s_1} \frac{(r_1+y)^{1/2}(r_1+\overline{S})}{(\overline{R}+\overline{S})^{3/2}} \; F\left[\frac{1}{2},\frac{3}{2},1,\frac{(r_1-\overline{R})(\overline{S}-y)}{(\overline{R}+\overline{S})(r_1+y)}\right] \left\{b'(y)+\frac{1}{2}\frac{b(y)}{(r_1+y)}\right\} \; dy$$

$$(38.19)$$

where in the integrals on the right hand side (x,y) are the variables and (r_1,s_1) will be considered as constant. Equation (38.19) gives us the solution of the pressure equation (38.07) with the boundary conditions (38.08). We now express this solution (38.19) in a form which will be more helpful in numerical calculations by finding a relationship between the hypergeometric function entering in our solution and the complete elliptic integral of the second type.

Now the complete elliptic integral of the second type, denoted by $E(k)$, is

$$E(k) = \int_o^{\pi/2} \sqrt{1-k^2\sin^2\phi} \;\; d\phi$$

or

$$E(k) = \frac{\pi}{2}\left[1-(1/2)^2 k^2 -\left(\frac{1\cdot3}{2\cdot4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 \frac{k^6}{5} - - - - \right]$$

or

$$E(k) = \frac{\pi}{2} F\left[-\frac{1}{2},\frac{1}{2};1;k^2\right] \qquad\qquad (38.20)$$

However, the hypergeometric function in our problem is $= F\left[\frac{1}{2},\frac{3}{2},1;k^2\right]$

where

$$k^2 = \frac{(r-\overline{R})(\overline{S}-s)}{(\overline{R}+\overline{S})(r+s)} \qquad\qquad (38.21)$$

In order to find a relationship between this hypergeometric function in

(38.21) and the complete elliptic integral of second type expressed in terms of hypergeometric function in (38.20), we make use of a relation of contiguity between the hypergeometric functions given as:

$$(\gamma-\alpha-\beta)F[\alpha,\beta,\gamma;z] \equiv (\gamma-\alpha)\ F[\alpha-1,\beta,\gamma;z] -\beta(1-z)\ F[\alpha,\beta+1,\gamma;z] \qquad (38.22)$$

Putting $\alpha = 1/2$, $\beta = 1/2$, $\gamma = 1$ and $z = k^2$ in (38.22), we get

$$F\left[\frac{1}{2}\ \frac{3}{2},1;k^2\right] = \frac{2}{\pi}\ \frac{E(k)}{1-k^2} \qquad (38.23)$$

where we assumed that $k^2 = \dfrac{(r-\bar{R})(\bar{S}-s)}{(\bar{R}+\bar{S})(r+s)}$ is +ve. If however,

$\dfrac{(r-\bar{R})(\bar{S}-s)}{(\bar{R}+\bar{S})(r+s)} = -k^2$ where k^2 is +ve, then we replace k^2 by $-k^2$ in (38.15) to get

$$F\left[\frac{1}{2},\frac{3}{2},1;-k^2\right] = \frac{2}{\pi}\ \frac{E_1(k)}{(1+k^2)}$$

wherein $E_1(k) = \displaystyle\int_0^{\pi/2}\sqrt{1+k^2\sin^2\phi}\ \ d\phi$

$$= \sqrt{1 + k^2}\ E\left(\frac{k}{\sqrt{1+k^2}}\right)$$

Therefore, if $\dfrac{(r-\bar{R})(\bar{S}-s)}{(\bar{R}+\bar{S})(r+s)} = -k^2$, then

$$F\left[\frac{1}{2},\frac{3}{2},1;-k^2\right] = \frac{2}{\pi}\ \sqrt{1+k^2}\ \ E\left(\frac{k}{\sqrt{1+k^2}}\right). \qquad (38.24)$$

By using (38.23) or (38.24), as the case may be, we solve for pressure inside the rectangle (thus the parabolic channel) which is given by (38.11) and do the numerical calculations.

Section 8b.

Let $p = P(\xi, \eta)$ be the pressure distribution in the region $A_2 A_1 B_2 B_1$ as obtained by solving the properly prescribed boundary value problem given by equations (38.04) and (38.05). This distribution of pressure is unique for the chosen system of stream lines for our problem.

However, when we substitute for pressure in equations (34.04) and (34.06) for finding the density distribution and the velocity distribution in the same region $A_2 A_1 B_2 B_1$, we find that the arbitrary function $f(\eta)$, remaining constant on each stream line, enters into the two expressions for velocity and density. Again, if we use the caloric equation of state for the flowing gas, then we shall have for the expression of specific entropy the unknown function $f(\eta)$ involved. The number of ways for which this unknown function of η can be determined further depends upon the form of the state equation of the flowing gas.

As proved earlier in section 4 this number is not unique, in general, whether the equation of state is in product form or the equation of state is in nonproduct form. For product equation of state for any function of η (for $f(\eta)$) which is non-vanishing differentiable function the family of velocity fields, the density fields and the entropy fields obtained satisfy the Euler's equations of motion for the defined unique pressure field i.e. to have a unique flow field we have to have a uniquely defined function $f(\eta)$ which will have the given pressure field $P(\xi, \eta)$.

Likewise, for gases with non-product equation of state we have to define uniquely the arbitrary function $f(\eta)$ in order to get a unique flow for our problem.

We can achieve this in two ways. By solving (31.06), when we properly prescribe the boundary value problem for the entropy, we get the entropy distribution, say

$$s(\eta) = a(\eta) \qquad (38.25)$$

wherein $a(\eta)$ is a known function.

Putting (38.17) in the state equation $\rho = R(p)S(s)$ or $\rho = f(p,s)$ and using the expression for ρ and p also, we get

$$\frac{G[\xi,\eta]}{R[P(\xi,\eta)]S[a(\eta)]} = f^2(\eta) \text{ which defines } f(\eta) \text{ and hence the}$$

flow uniquely. $\left\{ \text{For } \rho = \rho(p,s), \text{ we get } \frac{G(\xi,\eta)}{\rho[P(\xi,\eta)],a(\eta)]} = f^2(\eta) \right\}$

The second way of getting $f(\eta)$ and, therfore, the flow is to find $f(\eta)$ by prescribing $u(\xi,\eta)$ or $\rho(\xi,\eta)$ on an orthogonal trajectory $\xi = $ constant, lying in our region $A_2A_1B_2B_1$.

We now study the Mach number field. By putting for g_1 and g_2 in (34.11), we get

$$M = \sqrt{1 - \frac{\xi}{\eta} \cdot \frac{\partial P}{\partial \eta} \cdot \frac{1}{\frac{\partial P}{\partial \xi}}} \qquad (38.26)$$

Since this expression for M does not depend upon ρ, u, s or $f(\eta)$, there-fore, the Mach number field is unique like the pressure field for the same stream lines.

In the region under consideration, $A_2A_1B_1B_2$, we have

$$\xi \geq 0,$$

and $\qquad \eta > 0$

Therefore, for the region above A_2A_1 $\xi > 0$ and $\eta > 0$ and

on the line A_2A_1 $\xi = 0$ and $\eta > 0$. We take these two cases separately.

For the region above A_2A_1 the flow is subsonic, sonic or

supersonic according as

$$\frac{\partial P}{\partial \eta} \cdot \frac{1}{\frac{\partial P}{\partial \xi}} \gtrless 0 \qquad\qquad (38.27)$$

On the line $\xi = 0$ (assuming that $\frac{\partial P}{\partial \xi} \neq 0$) the flow is sonic at every

point.

So we conclude from these that the flow is sonic on line

A_2A_1 but the flow at any other point in the region is supersonic if the

rates of change of pressure w.r.t. ξ and η at that point have opposite

signs, subsonic if these rates of change have the same sign and is sonic

if the rate of change of pressure w.r.t. η is zero there.

Finally, by using equation (34.10) we study the flow when a

particle of gas moves down along a stream line from the point P on the

orthogonal trajectory $\xi = \gamma$ to the point Q on the orthogonal trajectory

$\xi = 0.$
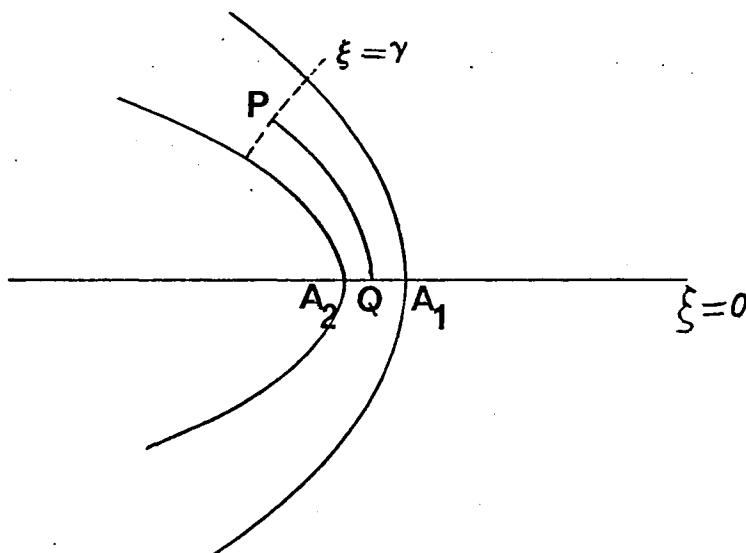
Case I Let the flow be subsonic in the region under consideration.

Since
$$M = \sqrt{1 + \frac{\xi}{\xi^2 + \eta^2} \frac{1}{\frac{\partial}{\partial \xi}(\log F)}} \quad ,$$

therefore, $\frac{\partial}{\partial \xi}(\log F) < 0$ i.e. log F increases as ξ decreases, or

$F(\xi, \eta)$ and therefore the velocity increases when the flow is subsonic

and the particle of gas moves down from P to Q.

Case II Let the flow be supersonic. In this case, we get

$$\frac{\partial}{\partial \xi}(\log F) > 0$$

i.e. as ξ decreases log F decreases, or $F(\xi, \eta)$ and therefore the velo-

city magnitude in this case of supersonic flow decreases when the gas

particle moves from P to Q.

Section 9. Flow in a hyperbolic channel
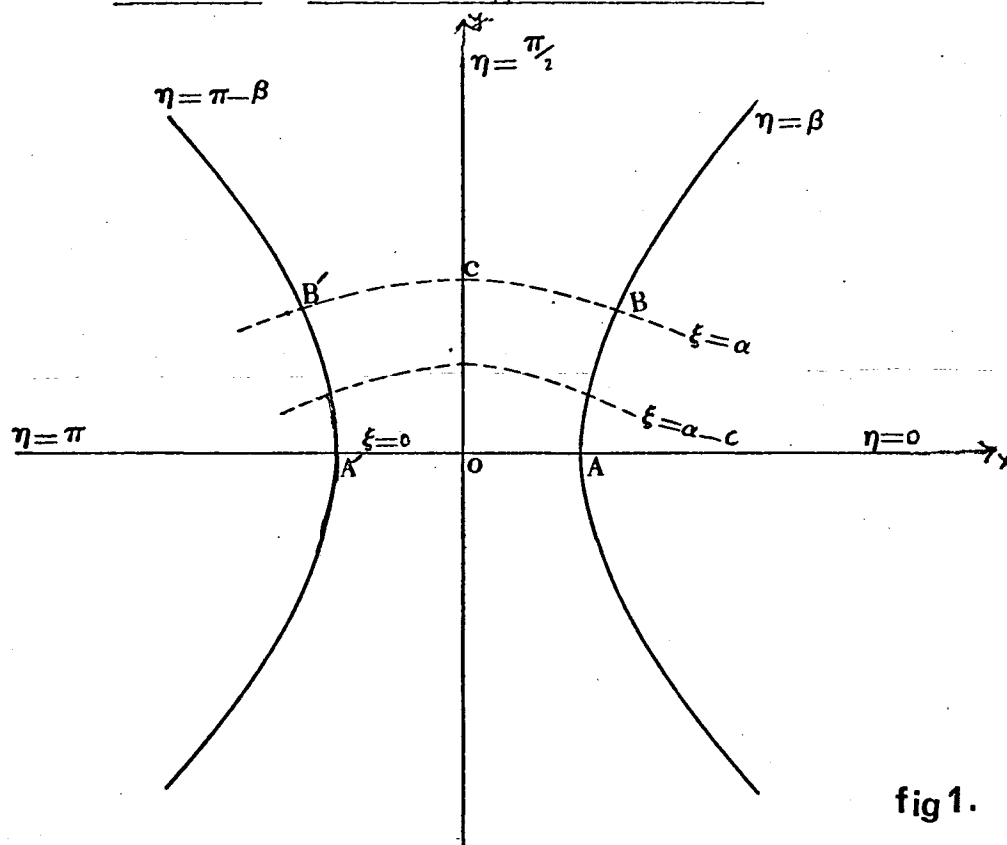
fig 1.

In this section we study the flow in a channel whose walls

are the two branches of a hyperbola. We take the family of confocal

hyperbolas which are also confocal with the hyperbola of our channel as

the stream lines. This choice gives us the flow in the hyperbolic chan-

nel. We take the family of confocal ellipses also confocal with the

stream lines as our orthogonal trajectories given by ξ = constant. Let

$\eta = \beta$ and $\eta = \pi - \beta$ be the walls of this infinite channel. The stream

lines and orthogonal trajectories defining different points of this

channel are given by η = constant and ξ = constant where

$$\pi - \beta \le \eta \le \beta \quad \text{and} \quad 0 \le \xi < + \infty$$

The equations of these two families of curves are

$$x = a \cosh \xi \cos\eta , \quad y = a \sinh\xi \sin\eta \qquad (39.01)$$

The squared element of arc length is

$$ds^2 = a^2(\sinh^2\xi + \sin^2\eta) \left[d\xi^2 + d\eta^2 \right] \qquad (39.02)$$

where the metric coefficients for this coordinate system are

$$g_1(\xi,\eta) = g_2(\xi,\eta) = a(\sinh^2\xi + \sin^2\eta)^{1/2} \qquad (39.03)$$

Since $g_1(\xi,\eta) = g_2(\xi,\eta)$ in this system of curvilinear coordinates

(by equation (39.03)), our system corresponds to $n = 1$ in form I of

section 6 of this chapter and is an isometric net.

In this section, like section 8, we use the pressure equation (36.08), established for coordinate systems of form I, to study

the present flow problem. This problem has been solved by Ozoklav using

the compatibility equations of gas flow and the inverse method [Ozoklav

(1959)].

Substituting $n = 1$, $f(\xi) = \sqrt{a} \sinh^2\xi$, $h(\eta) = \sqrt{a} \sin^2\eta$ in

equation (36.08), we get the pressure equation for the flow in a hyper-

bolic channel as:

$$\frac{\partial^2 P}{\partial\xi\partial\eta} + \frac{1}{2} \frac{\sin 2\eta}{(\sinh^2\xi + \sin^2\eta)} \frac{\partial p}{\partial\xi} + \frac{3}{2} \frac{\sinh 2\xi}{(\sinh^2\xi + \sin^2\eta)} \frac{\partial p}{\partial\eta} = 0$$

$$(39.04)$$

Equation (39.04) is a second order linear hyperbolic partial differential equation in canonical form with p as the dependent variable and ξ, η as the independent variables.

Here in this problem as the pressure function, given by (39.04), is symmetrical about $\eta = \frac{\pi}{2}$ (i.e. y-axis), therefore, we study the distribution of pressure in the region $0 \leq \xi < \infty$, $\alpha \leq \eta \leq \beta$ by only considering the region $0 \leq \xi < \infty$, $\pi/2 \leq \eta \leq \beta$. For the lower part of the channel we shall have the corresponding distribution of pressure.

As in this problem, also, the characteristic curves of equation (38.04) intersect at a right angle, we can prescribe the characteristic boundary value problem in which the boundary curve (i.e. the initial curve) is considered to degenerate into two characteristic arcs and we merely prescribe the quantity p on these two arcs $\xi = $ constant and $\eta = $ constant.

Therefore, let

$$
\left.
\begin{aligned}
p(\xi,\beta) &= f(\xi) \quad \text{on } \eta = \beta \quad \text{for } 0 \leq \xi \leq \alpha, \\
p(\alpha,\eta) &= g(\eta) \quad \text{on } \xi = \alpha \quad \text{for } \beta \leq \eta \leq \pi/2 \\
&\text{with the restriction } g(\beta) = f(\alpha)
\end{aligned}
\right\} \quad (39.05)
$$

be the values of pressure prescribed along the arc AB and the arc BC so that it attains the unique value at $B(\alpha,\beta)$.

So our problem is to solve for pressure in the region ABCO when it is given by (39.04) subject to the boundary conditions (39.05).

We first transform the equations (39.04) and (39.05) to a more convenient form from the (ξ, η) net to $\bar{o}rs$ plane by letting

$$\left. \begin{array}{l} \sinh^2 \xi = r, \\[2mm] \sin^2 \eta = s \end{array} \right\} \tag{39.06}$$

Our transformed problem is to solve

$$L(p) \equiv \frac{\partial^2 p}{\partial r \partial s} + \frac{1}{r+s} \left[\frac{1}{2} \frac{\partial p}{\partial r} + \frac{3}{2} \frac{\partial p}{\partial s} \right] = 0 \tag{39.07}$$

with the boundary conditions

$$\left. \begin{array}{ll} p(r, s_1) = a(r) & 0 \leq r \leq r_1 \\[3mm] p(r_1, s) = b(s) & s_1 \leq s \leq 1 \end{array} \right\} \tag{39.08}$$

with the restriction $a(r_1) = b(s_1)$.

Here $r_1 = \sinh^2 \alpha$ and $s_1 = \sin^2 \beta$.

By this transformation the flow in the infinite hyperbolic channel is transformed into the infinite rectangular channel bounded by $r = 0$, $r = \infty$ and $s = s_1$, $s = 1$. The region ABCO in which we wish to find the flow in oxy plane is mapped into the region $\overline{ABCO}_1$, in $\bar{o}rs$ plane as shown in fig 2.

The stream lines and the orthogonal trajectories are mapped into mutually orthogonal families of parallel straight lines given by $s = $ constant $(s_1 \leq s \leq 1)$ and $r = $ constant $(0 \leq r < \infty)$. Our problem is to solve for pressure, inside $\overline{ABCO}_1$, given by (39.07) and (39.08).
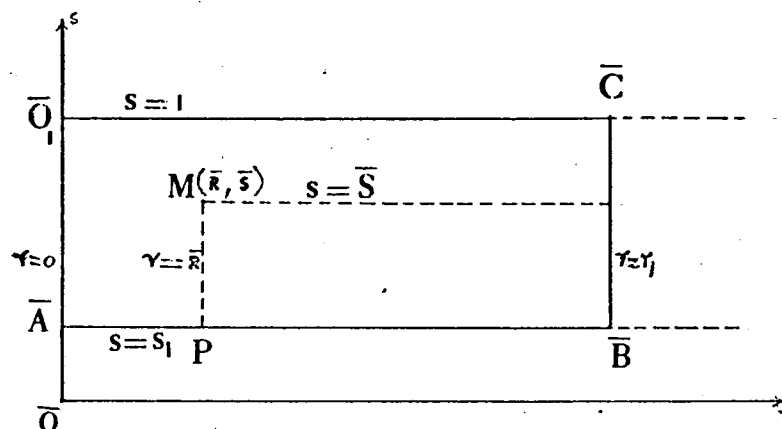
fig 2.

Since these equations (39.07) and (39.08) giving us the pressure distribution are exactly the same as (38.07) and (38.08) of previous section 8, we use the Riemannian method of solving the problem for finding the pressure at any point $M(\bar{R},\bar{S})$ in the region as done there.

We get the pressure distribution as given in (38.19) wherein for this problem r and s are given by (39.06). Hence, we solve for the flow in hyperbolic channel as was done for the case of parabolic channel. However, only in that part of the problem where we discuss the Mach number at any point in the region we do not use (34.26) for Mach numbers at different points of the stream line $\eta = \pi/2$. We know from (34.10) and (34.11), giving the expression for Mach number, that

$$- \frac{\frac{\partial P}{\partial \eta}}{\frac{\partial P}{\partial \xi}} = \frac{a(\sin 2\eta)}{(\sinh^2 \xi + \sin^2 \eta)^{1/2}} \cdot \frac{1}{\frac{\partial}{\partial \xi}(\log F)}$$

i.e. $\frac{\partial P}{\partial \eta} = 0$ on the stream line $\eta = \pi/2$ and therefore, equation (34.11) for the Mach number i.e.

$$M = \sqrt{1 - \frac{\sinh 2\xi}{\sin 2\eta} \frac{\frac{\partial P}{\partial \eta}}{\frac{\partial P}{\partial \xi}}}$$

is an indeterminate form on $\eta = \pi/2$. Therefore, only for the line

$\eta = \pi/2$ we cannot determine M from pressure only.

CHAPTER 4

STEADY ROTATIONAL GAS FLOW IN THREE DIMENSIONS

Section 1. The flow equations in natural coordinates

In this section we set the differential equations for the three dimensional steady and rotational flow of gases in natural coordinates when the assumptions are that there is no heat conduction and that the gases are subjected to no extraneous forces.

We consider the orthogonal curvilinear coordinate system (ξ,η,ψ) obtained from the three families of surfaces $\xi = c_1$, $\eta = c_2$ and $\psi = c_3$ where c_1, c_2 and c_3 are the parameters of three families.

Let

$$\left. \begin{array}{l} x = x(\xi,\eta,\psi) \\ y = y(\xi,\eta,\psi) \\ z = z(\xi,\eta,\psi) \end{array} \right\} \qquad (41.01)$$

be the equations of three families of surfaces. The squared element of arc length in this coordinate system is of the form

$$ds^2 = g_1^2(\xi,\eta,\psi)\, d\xi^2 + g_2^2(\xi,\eta,\psi)d\eta^2 + g_3^2(\xi,\eta,\psi)d\psi^2 \qquad (41.02)$$

where g_1, g_2 and g_3, the metric coefficients, satisfy the six Gauss's equations (21.17) and (21.18).

Now taking the curve, through any point, along which ξ-increases as our stream lines, we get the orthogonal curvilinear net in natural coordinates. We now let $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$ be the unit tangential vectors to the three orthogonal curves at a point in the directions of ξ-increasing, η-increasing and ψ-increasing respectively so that the velocity vector $\underset{\sim}{q}$ is given by

69

$$\underset{\sim}{q} = u(\xi, \eta, \psi) \; \underset{\sim}{e}_1 \tag{41.03}$$

Since our flow is steady, we obtain the flow equations in natural coordinates by putting (41.03) in (21.11)— (21.16) and get

$$u\frac{\partial u}{\partial \xi} + \frac{1}{\rho}\frac{\partial p}{\partial \xi} = 0 \tag{41.04}$$

$$u^2 \frac{\partial}{\partial \eta} (\log g_1) - \frac{1}{\rho}\frac{\partial p}{\partial \eta} = 0 \tag{41.05}$$

$$u^2 \frac{\partial}{\partial \psi} (\log g_1) - \frac{1}{\rho}\frac{\partial p}{\partial \psi} = 0 \tag{41.06}$$

$$\frac{\partial}{\partial \xi} (g_2 g_3 \rho u) = 0 \tag{41.07}$$

$$\frac{\partial s}{\partial \xi} = 0 \tag{41.08}$$

$$\rho = \rho(p, s) \tag{41.09}$$

We shall study the symmetric flows and the flows without the restricting assumptions of symmetry in this chapter when the flow of gases is three dimensional, rotational and steady.

Section 2. Pressure Equations

Three dimensional rotational steady gas flow is governed by the equations (41.04) - (41.09). To obtain the solutions of problems for such flows we eliminate the velocity, the density and the specific entropy from these six non-linear partial differential equations and obtain three linear partial differential equations satisfied by the pressure function.

From equation (41.07), we get

$$g_2 g_3 \, \rho u = \phi \, (\eta, \psi)$$

where ϕ is an arbitrary function of η and ψ.

or

$$\rho = \frac{\phi(\eta, \psi)}{g_2 g_3 u} \tag{42.01}$$

Substituting for ρ from equation (42.01) in equations (41.05) and (41.06), we get

$$u^2 \frac{\partial}{\partial \eta}(\log g_1) = \frac{g_2 g_3 u}{\phi(\eta, \psi)} \frac{\partial p}{\partial \eta}$$

and

$$u^2 \frac{\partial}{\partial \psi}(\log g_1) = \frac{g_2 g_3 u}{\phi(\eta, \psi)} \frac{\partial p}{\partial \psi}$$

Assuming that $u \neq 0$, we get

$$u = \frac{g_2 g_3}{\frac{\partial}{\partial \eta}(\log g_1)} \cdot \frac{1}{\phi(\eta, \psi)} \frac{\partial p}{\partial \eta} \tag{42.02}$$

and

$$u = \frac{g_2 g_3}{\frac{\partial}{\partial \psi}(\log g_1)} \cdot \frac{1}{\phi(\eta, \psi)} \frac{\partial p}{\partial \psi} \tag{42.03}$$

Either of the two equations (42.02) and (42.03) expresses u in terms of p. Substituting for u from equation (42.02) in (42.01) and from equation

(42.03) in (42.01), we get

$$\rho = \frac{\phi^2(\eta,\psi)}{g_2^2 \, g_3^2} \, \frac{\partial}{\partial\eta}(\log g_1) \cdot \frac{1}{\frac{\partial p}{\partial\eta}} \tag{42.04}$$

and

$$\rho = \frac{\phi^2(\eta,\psi)}{g_2^2 \, g_3^2} \, \frac{\partial}{\partial\psi}(\log g_1) \cdot \frac{1}{\frac{\partial p}{\partial\psi}} \tag{42.05}$$

respectively.

Equation (42.04) or (42.05) gives ρ in terms of p. We eliminate ρ from equations (42.04) and (42.05) (or eliminate u from equations (42.02) and (42.03)) to obtain

$$\frac{\partial g_1}{\partial\psi} \frac{\partial p}{\partial\eta} - \frac{\partial g_1}{\partial\eta} \frac{\partial p}{\partial\psi} = 0 \tag{42.06}$$

We call equation (42.06) the first pressure equation. This equation is a first order linear partial differential equation in which p is the dependent variable and ξ,η,ψ are the independent variables. From this equation if p is constant along a curve on any one of the given surfaces $\xi =$ constant, then g_1 is also constant and vice versa   i.e.

$$p = p(g_1) \tag{42.07}$$

on a surface $\xi =$ constant.

By eliminating p from (41.05) and (41.06), we get

$$\frac{\partial}{\partial\psi}(\log g_1) \frac{\partial}{\partial\eta}(\rho u^2) - \frac{\partial}{\partial\eta}(\log g_1) \frac{\partial}{\partial\psi}(\rho u^2) = 0$$

i.e. on a surface $\xi =$ constant, we have

$$\rho u^2 = f(g_1) \tag{42.08}$$

where f is an unknown function.

From (42.07) and (42.08), we get

$$p = p(g_1) = p(\rho u^2) \qquad (42.09)$$

We eliminate ρ and u from equations (41.04), (42.02) and (42.04) to

obtain

$$\left(\frac{\frac{g_1 g_2 g_3}{\partial g_1}}{\frac{\partial g_1}{\partial \eta}} \cdot \frac{1}{\phi(\eta,\psi)} \frac{\partial p}{\partial \eta} \right) \frac{\partial}{\partial \xi} \left[\frac{\frac{g_1 g_2 g_3}{\partial g_1}}{\frac{\partial g_1}{\partial \eta}} \frac{1}{\phi(\eta,\psi)} \frac{\partial p}{\partial \eta} \right] + \frac{\frac{g_1 g_2^2 g_3^2}{\partial g_1}}{\frac{\partial g_1}{\partial \eta}} \cdot \frac{1}{\phi^2(\eta,\psi)} \frac{\partial p}{\partial \eta} \frac{\partial p}{\partial \xi} = 0$$

or

$$\left(\frac{g_1}{\rho} \cdot \frac{1}{\frac{\partial g_1}{\partial \eta}} \right) \left[\frac{\partial^2 p}{\partial \xi \partial \eta} + \frac{\partial}{\partial \xi}(\log H) \frac{\partial p}{\partial \eta} + \frac{\partial}{\partial \eta}(\log g_1) \frac{\partial p}{\partial \xi} \right] = 0 \qquad (42.10)$$

where

$$H = \frac{g_1 g_2 g_3}{\frac{\partial g_1}{\partial \eta}} \qquad (42.11)$$

From equation (42.10) the pressure function satisfies the equation

$$\frac{\partial^2 p}{\partial \xi \partial \eta} + \frac{\partial}{\partial \xi}(\log H) \frac{\partial p}{\partial \eta} + \frac{\partial}{\partial \eta}(\log g_1) \frac{\partial p}{\partial \xi} = 0 \qquad (42.12)$$

if $\frac{\partial g_1}{\partial \eta} \neq 0$ or ∞ ___ (since ρ and g_1 are positive)

We shall call equation (42.12) the second pressure equation.

Likewise, by eliminating ρ and u from (41.04), (42.03) and

(42.05), we get

$$\frac{\partial^2 p}{\partial \xi \partial \psi} + \frac{\partial}{\partial \xi}(\log J) \frac{\partial p}{\partial \psi} + \frac{\partial}{\partial \psi}(\log g_1) \frac{\partial p}{\partial \xi} = 0 \qquad (42.13)$$

as the equation satisfied by pressure if $\frac{\partial g_1}{\partial \psi} \neq 0$ or ∞.

Here

$$J = \frac{g_1 g_2 g_3}{\frac{\partial g_1}{\partial \psi}} \qquad (42.14)$$

We shall call (42.13) the third pressure equation wherein J is given by equation (42.14). Since equations (42.06) and (42.12) together give (42.13) (or (42.06) and (42.13) together give (42.12)), the three pressure equations are not independent but two of these three form an independent set.

Thus the pressure function in a three dimensional flow problem satisfies (42.06), (42.12) and (42.13). These three equations are linear partial differential equations of first order, second order and second order respectively.

For a particular flow problem when the flow equations are written in natural coordinates, there are three possible cases.

Case I When $\dfrac{\partial g_1}{\partial \eta} \neq 0$ and $\dfrac{\partial g_1}{\partial \psi} \neq 0$

Case II Either $\dfrac{\partial g_1}{\partial \eta} = 0$ or $\dfrac{\partial g_1}{\partial \psi} = 0$

Case III $\dfrac{\partial g_1}{\partial \eta} = \dfrac{\partial g_1}{\partial \psi} = 0$

Case I For this case pressure is given as a solution of any two of the three pressure equations (42.06), (42.12) and (42.13). Here $p = p(\xi, \eta, \psi)$.

Case II (a) When $\dfrac{\partial g_1}{\partial \eta} = 0$ but $\dfrac{\partial g_1}{\partial \psi} \neq 0$, pressure is given as a solution of the third pressure equation (42.13) and $\dfrac{\partial p}{\partial \eta} = 0$. Here $p = p(\xi, \psi)$.

(b) When $\dfrac{\partial g_1}{\partial \psi} = 0$ but $\dfrac{\partial g_1}{\partial \eta} \neq 0$, pressure is given as a solution of the second pressure equation (42.12) and $\dfrac{\partial p}{\partial \psi} = 0$. Here $p = p(\xi, \eta)$.

Case III When $\dfrac{\partial g_1}{\partial \eta} = \dfrac{\partial g_1}{\partial \psi} = 0$, pressure is given as a solution of $\dfrac{\partial p}{\partial \eta} = 0$ and $\dfrac{\partial p}{\partial \psi} = 0$. Here $p = p(\xi)$. We study now the geometric implication of this case. In this case $g_1 = g_1(\xi)$ and, therefore, the differential arc

length along the stream line is given by

$$ds_1 = g_1(\xi)d\xi \tag{42.15}$$

Integrating (42.15) from $\xi = d_1$ to $\xi = d_2$, we get

$$s_1 = \int_{d_1}^{d_2} g_1(\xi)\, d\xi \tag{42.16}$$

as the arc length between the two surfaces $\xi = d_1$ and $\xi = d_2$.

From (42.16) the arc length of a stream line between the two isobaric surfaces $\xi = d_1$ and $\xi = d_2$ does not depend upon η and ψ i.e. in this case when pressure is a function of ξ only the arc length measured along a stream line, from one isobaric surface to the second isobaric surface, does not change from stream line to stream line and is the same s_1.

We shall discuss the flow problems belonging to these three cases in the following sections of this chapter.

Section 3. Velocity, density, entropy and Mach number

In this section we consider only those problems for which pressure is not constant on the surfaces $\xi = $ constant i.e. we consider the problems for which either pressure is a solution of any two of the three pressure equations (i.e. case of $\dfrac{\partial g_1}{\partial \eta} \neq 0$ and $\dfrac{\partial g_1}{\partial \psi} \neq 0$) or pressure is a solution of the second or third pressure equation only (i.e. $\dfrac{\partial g_1}{\partial \eta} = 0$ or $\dfrac{\partial g_1}{\partial \psi} = 0$). We consider these two cases separately.

Case I. Let

$$p = P(\xi, \eta, \psi) \qquad (43.01)$$

be a solution of the pressure equations (42.06) and (42.12) for a given problem of three dimensional rotational steady flow when $\dfrac{\partial g_1}{\partial \eta} \neq 0$ and $\dfrac{\partial g_1}{\partial \psi} \neq 0$. Substituting (43.01) in (41.04) and (41.05), we get

$$\rho u \frac{\partial u}{\partial \xi} = -\frac{\partial P}{\partial \xi} \qquad (43.02)$$

and

$$\rho u^2 = \frac{1}{\frac{\partial}{\partial \eta}(\log g_1)} \frac{\partial P}{\partial \eta} \qquad (43.03)$$

Dividing (43.02) by (43.03), we get

$$\frac{1}{u} \frac{\partial u}{\partial \xi} = -\frac{\partial}{\partial \eta}(\log g_1) \frac{\partial P}{\partial \xi} \bigg/ \frac{\partial P}{\partial \eta}$$

Integrating w.r.t. ξ, we get

$$u(\xi, \eta, \psi) = f(\eta, \psi) F(\xi, \eta, \psi) \qquad (43.04)$$

wherein $f(\eta, \psi)$ is an arbitrary function of η and ψ and the function $F(\xi, \eta, \psi)$ is a known function given by

$$F(\xi,\eta,\psi) = e^{-\int\left[\frac{\partial}{\partial\eta}(\log\, g_1)\right]\left[\frac{\frac{\partial P}{\partial\xi}}{\frac{\partial P}{\partial\eta}}\right]\,d\xi} \tag{43.05}$$

Equation (43.04) gives us the velocity function.

Substituting (43.04) in (43.03), we get

$$\rho(\xi,\eta,\psi) = \frac{G(\xi,\eta,\psi)}{f^2(\eta,\psi)} \tag{43.06}$$

where $f(\eta,\psi)$ is the same arbitrary function used in (43.04) and $G(\xi,\eta,\psi)$ is a known function given by

$$G(\xi,\eta,\psi) = \frac{1}{\frac{\partial}{\partial\eta}(\log\, g_1)}\,\frac{\partial P}{\partial\eta}\, e^{2\int\frac{\partial}{\partial\eta}(\log\, g_1)\cdot\frac{\frac{\partial P}{\partial\xi}}{\frac{\partial P}{\partial\eta}}\,d\xi} \tag{43.07}$$

Equation (43.06) gives us the density distribution. Now we find the Mach number at any point in the flow region. By definition, Mach number at any point is given by

$$M = \frac{u}{c} \tag{43.08}$$

where u is the local speed of gas at the point and c is the speed of sound there. By using (43.04) for u in (41.07), we get

$$\frac{\partial}{\partial\xi}(\log\,\rho) = -\frac{\partial}{\partial\xi}\left[\log\{g_2 g_3 f(\eta,\psi)\, F(\xi,\eta,\psi)\}\right] \tag{43.09}$$

Putting for u from (43.04) and for $\frac{1}{\rho}\frac{\partial\rho}{\partial\xi}$ from (43.09) in equation (41.04), we get

$$f^2(\eta,\psi)F(\xi,\eta,\psi)\,\frac{\partial F}{\partial\xi} = c^2\,\frac{\partial}{\partial\xi}\left[\log\{g_2 g_3 f(\eta,\psi)F(\xi,\eta,\psi)\}\right]$$

or

$$M = \frac{u}{c} = \sqrt{1 + \frac{\frac{\partial}{\partial\xi}(\log\, g_2 g_3)}{\frac{\partial}{\partial\xi}(\log\, F)}} \tag{43.10}$$

Substituting for $F(\xi,\eta,\psi)$ from (43.05) in (43.10), we get

$$M = \sqrt{1 - \frac{\frac{\partial}{\partial \xi}(\log g_2 g_3) \cdot \frac{\partial P}{\partial \eta}}{\frac{\partial}{\partial \eta}(\log g_1) \cdot \frac{\partial P}{\partial \xi}}} \qquad (43.11)$$

Equation (43.10) or equation (43.11) gives the Mach number at any point, in the flow region, in terms of the known pressure function.

The flow at any point is subsonic, sonic or supersonic according as $M = \frac{u}{c} \lessgtr 1$, therefore, from (43.10) and (43.11) we get

$$\frac{\frac{\partial}{\partial \xi}(\log g_2 g_3)}{\frac{\partial}{\partial \eta}(\log g_1)} \cdot \frac{\frac{\partial P}{\partial \eta}}{\frac{\partial P}{\partial \xi}} \lessgtr 0 \qquad (43.12)$$

or

$$\frac{\frac{\partial}{\partial \xi}(\log g_2)}{\frac{\partial}{\partial \xi}(\log F)} \gtrless 0 \qquad (43.13)$$

according as the flow at any point is subsonic, sonic or supersonic.

We have in both the velocity and the density functions, given by (43.04) and (43.06) respectively, an unknown function $f(\eta, \psi)$ involved. This function is a non-vanishing differentiable function which is constant upon each stream line $(\underset{\sim}{q} \cdot \text{grad } f = 0)$. Determining the velocity and density is equivalent to determining this unknown function $f(\eta, \psi)$.

For a gas flow with the pressure function given by (43.01) this function $f(\eta, \psi)$ and therefore u, ρ and s may or may not be unique depending upon the state equation of gas. For gases there are two forms of state equations.

Form I: Gases obeying the equation of state in product form.

Form II: Gases with the equation of state not in product form.

For gases of Form I the function $f(\eta,\psi)$ and therefore u, ρ and s can

be obtained in an infinite number of ways.

Let

$$\rho = R[p]S[s] \tag{43.14}$$

be the equation of state.

Substituting for ρ and p from (43.06) and (43.01) respectively, we get

$$\frac{G(\xi,\eta,\psi)}{f^2(\eta,\psi)} = R[P(\xi,\eta,\psi)]S[s(\eta,\psi)]$$

or

$$\frac{G(\xi,\eta,\psi)}{R(\xi,\eta,\psi)} = S[s(\eta,\psi)] \cdot f^2(\eta,\psi) \tag{43.15}$$

Since left hand side is a known function, we take it

$= a(\eta,\psi)$ (say)

$$\therefore \ S[s(\eta,\psi)]f^2(\eta,\psi) = a(\eta,\psi) \quad (\text{ a known function}) \tag{43.16}$$

Obviously this equation can remain true by an infinite number of choices

of $f(\eta,\psi)$ as long as $s(\eta,\psi)$ is correspondingly chosen each time satisfy-

ing the state equation (43.14).

Therefore, the only possible method of obtaining a unique

flow field is to prescribe one well defined boundary condition for the

velocity or the density or the specific entropy.

However, if we do not prescribe such a boundary condition,

then we get an infinite family of flow fields corresponding to a unique

pressure field sharing the same stream lines. This is known as Prim's

substitution principle (Prim [1952]) for gases obeying the product

equation of state and this principle is obviously satisfied by our flow results.

Likewise, for gases obeying the equation of state in Form II we can show that the unknown functions $f(\eta,\psi)$ and $s(\eta,\psi)$ cannot be determined uniquely from the knowledge of state equation of the gas.

Therefore, for both forms of state equations of gases we get a unique solution if one of the three unknown functions u, ρ and s is prescribed.

Case II.   Let $p = P(\xi,\eta)$ (43.17)

be a solution of the pressure equation (42.12) for a given problem of three dimensional rotational steady flow when $\dfrac{\partial g_1}{\partial \psi} = 0$. For such a problem the velocity u, the density ρ and the Mach number are given by

$$u(\xi,\eta,\psi) = f(\eta,\psi)\, F(\xi,\eta) \qquad\qquad (43.18)$$

$$\rho(\xi,\eta,\psi) = G(\xi,\eta)/f^2(\eta,\psi) \qquad\qquad (43.19)$$

and equation (43.10) or (43.11) respectively.

Here $f(\eta,\psi)$ is an arbitrary function of (η,ψ) and $F(\xi,\eta)$ and $G(\xi,\eta)$ are given by (43.05) and (43.07).

The rest of the problem is solved as done in case I.

Section 4. Choice of coordinate systems

We used the general orthogonal curvilinear coordinate system with the metric coefficients g_1, g_2 and g_3 in section 2 and obtained the three pressure equations (42.06), (42.12) and (42.13). These three pressure equations hold when these are in natural coordinates i.e. when one of the curves, along which ξ-increases, of the orthogonal curvilinear net (ξ, η, ψ) is the stream line of flow of gas.

Now to solve these pressure equations i.e. to obtain the pressure distribution of a particular flow problem we take that orthogonal coordinate system which fits in with the physical boundaries of our problem and thereby assists us to substitute (or insert) the boundary conditions in a reasonably simple way.

Rectangular coordinate system which is a coordinate system composed of the three families of orthogonal surfaces of the first degree (i.e. planes) does not make a good choice, in many of the actual flow problems, for their solutions. For these problems, therefore, we use the coordinate systems formed by the family of orthogonal surfaces of the second degree (and degenerate cases) or we use the coordinate systems formed by the orthogonal surfaces of the fourth degree. However, in most of the physical problems, we study in the following sections, we use the families of orthogonal surfaces of the second degree (and degenerate cases) to build up the coordinate systems useful for the problem. We have ten important coordinate systems which include three cylindrical systems, four rotational systems and three general systems formed by using the surfaces of the second degree (and degenerate cases) and these are most

commonly used.

These coordinate systems are as follows:

<u>Cylindrical Coordinate Systems</u>

(i)   Parabolic-cylinder coordinates.

(ii)   Elliptic-cylinder coordinates.

(iii) Circular-cylinder coordinates.

<u>Rotational Coordinate Systems</u> (where the coordinate surfaces are symmetrical about an axis).

(i)   Spherical coordinates

(ii)   Parabolic coordinates

(iii) Prolate spheroidal coordinates

(iv)   Oblate spheroidal coordinates

<u>General Coordinate Systems</u>

(i)   Conical coordinates

(ii)   Ellipsoidal coordinates

(iii) Paraboloidal coordinates

In three dimensional problems often restricting assumptions of symmetry are made. Under such assumptions of symmetry a reduction to two independent variables in place of three is done to make a mathematical problem easier when, however, it is more complicated. By these restricting assumptions a condition is put on all the pertinent quantities of flow to depend only on two independent variables.

In the following sections, however, we shall not put any restricting assumptions of symmetry.

In sections 5 and 6 we study the three dimensional problems when the chosen natural coordinate systems have the property that

$$\frac{\partial g_1}{\partial \eta} = \frac{\partial g_1}{\partial \psi} = 0 \quad \text{for the metric coefficient } g_1(\xi, \eta, \psi) \text{ i.e. pressure function}$$

is not the solution of the pressure equations and $p = p(\xi)$ only.

Section 5. Radial flow emanating from the surface of a small spherical ball.
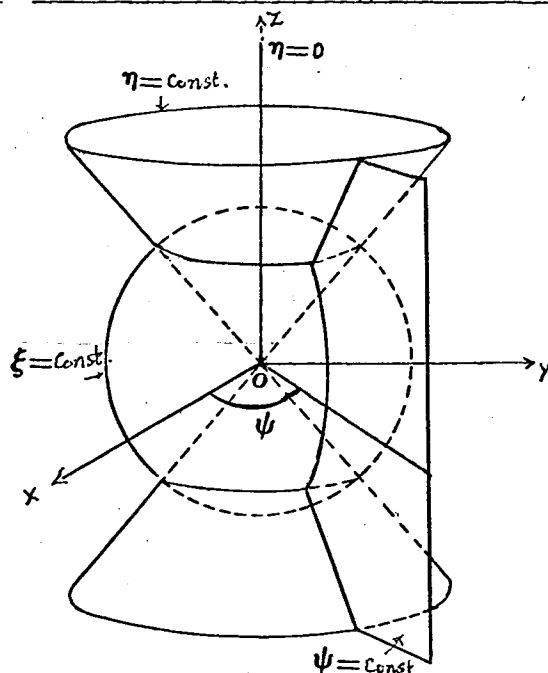
We study the flow of gases emanating from a spherical surface of a small radius $r_1 > 0$ and moving outwards along the rays that pass through O, the centre of spherical ball (with coordinates $(0,0,0)$).

For finding the solution of flow of gases in this problem we choose the spherical coordinates consisting of the coordinate surfaces

$$x^2 + y^2 + z^2 = \xi^2 \qquad r_1 \leq \xi < \infty,$$

$$(x^2 + y^2)^{1/2} \Big/ z = \tan\eta \qquad 0 \leq \eta \leq \pi,$$

and $\qquad \dfrac{y}{x} = \tan\psi \qquad\qquad 0 \leq \psi < 2\pi$

i.e. $\qquad x = \xi \sin\eta\cos\psi,$

$\qquad\qquad y = \xi \sin\eta \sin\psi,$

and $\qquad z = \xi \cos\eta$ $\qquad\qquad\qquad\qquad$ (45.01)

Here ξ = constant are concentric spheres, η = constant the circular cones and ψ = constant the half planes.

The squared element of arc length is

$$ds^2 = d\xi^2 + \xi^2 \, d\eta^2 + \xi^2 \sin^2\eta \, d\psi^2 \tag{45.02}$$

wherein the metric coefficients are

$$\left.\begin{aligned} g_1(\xi,\eta,\psi) &= 1 \\ g_2(\xi,\eta,\psi) &= \xi \\ g_3(\xi,\eta,\psi) &= \xi\sin\eta \end{aligned}\right\} \tag{45.03}$$

From (45.03) $\dfrac{\partial g_1}{\partial \eta} = \dfrac{\partial g_1}{\partial \psi} = 0$, therefore

$$\frac{\partial p}{\partial \eta} = 0 \tag{45.04}$$

and

$$\frac{\partial p}{\partial \psi} = 0 \tag{45.05}$$

By equations (45.04) and (45.05) the pressure function of the flow and, therefore, velocity, density are not obtained as the solutions from the pressure equations and equations (43.04), (43.06) respectively.

Substituting the equations (45.03) in (41.04) and (41.07), we get

$$u \frac{\partial u}{\partial \xi} + \frac{1}{\rho} \frac{\partial p}{\partial \xi} = 0 \tag{45.06}$$

and

$$\frac{\partial}{\partial \xi}[\xi^2 \sin\eta \, \rho \, u] = 0 \tag{45.07}$$

From equations (45.04) and (45.05), we get

$$p(\xi,\eta,\psi) = \phi(\xi) \quad \text{where } \phi(\xi) \text{ is an arbitrary function of } \xi.$$

Now by prescribing suitably the pressure function (e.g. on a ray of a circular cone $\eta = $ constant, $\psi = $ constant) on a curve we get the pressure distribution.

Let the solution of prescribed problem for pressure be

$$p(\xi,\eta,\psi) = P(\xi) \tag{45.08}$$

From equation (45.07), we get

$$\sin^2\eta \frac{\partial}{\partial\xi}[\xi^2\rho\, u] = 0$$

or for the region of flow leaving the only vertical ray along z-axis $(\eta = 0)$, we have $\sin^2\eta \neq 0$ and, therefore, for this region

$$\rho = \frac{a(\eta,\psi)}{\xi^2 u} \tag{45.09}$$

where $a(\eta,\psi)$ is an unknown function of η and ψ.

Substituting (45.09) and (45.08) in equation (45.06), we get

$$u \frac{\partial u}{\partial\xi} + \frac{\xi^2 u}{a(\eta,\psi)}\; P'(\xi) = 0 \;.$$

Taking $u \neq 0$ in this equation, we get

$$u(\xi,\eta,\psi) = \frac{-1}{a(\eta,\psi)}\; \int \xi^2 P'(\xi)d\xi + b(\eta,\psi) \tag{45.10}$$

which gives us the velocity distribution of the flow where $b(\eta,\psi)$ is also an arbitrary function of η,ψ. From (45.10) and (45.09), we get

$$\rho(\xi,\eta,\psi) = \frac{-a^2(\eta,\psi)}{\xi^2\left[\int \xi^2 P'(\xi)d\xi + a(\eta,\psi)b(\eta,\psi)\right]} \tag{45.11}$$

Equation (45.11) gives us the density function. Now we find the Mach number at any point in the flow region.

From equation (45.08), we get

$$c^2 \frac{\partial\rho}{\partial\xi} = P'(\xi)$$

Putting in this equation the derivative of ρ w.r.t. ξ from (45.11), we get

$$c^2 = \frac{\xi^3 P'(\xi)}{a^2(\eta,\psi)} \; \frac{[\int \xi^2 P'(\xi)d\xi + a(\eta,\psi)b(\eta,\psi)]^2}{[2\{\int\xi^2 P'(\xi)d\xi + a(\eta,\psi)b(\eta,\psi)\} + \xi^3 P'(\xi)]} \tag{45.12}$$

From (45.10) and (45.11), we get

$$M = \frac{u}{c} = \sqrt{1 + \frac{2}{\xi^3 P'(\xi)} \cdot [\int \xi^2 P'(\xi)d\xi + a(\eta,\psi)b(\eta,\psi)]}$$

$$(45.13)$$

which gives us the Mach number at any point.

Equations (45.10) and (45.11) give the general solution of velocity and density of the flow. Substituting (45.08) and (45.11) in the caloric equation of state for the gas we find the entropy distribution.

Section 6. Flow emanating from the surface of an infinite cylindrical bar of small radius.

We study, in this section, the flow of gases emanating from the cylindrical surface of a bar of small radius $r_1 > 0$ having as its axis the coordinate axis $Z'OZ$ and moving along the rays perpendicular to this axis.
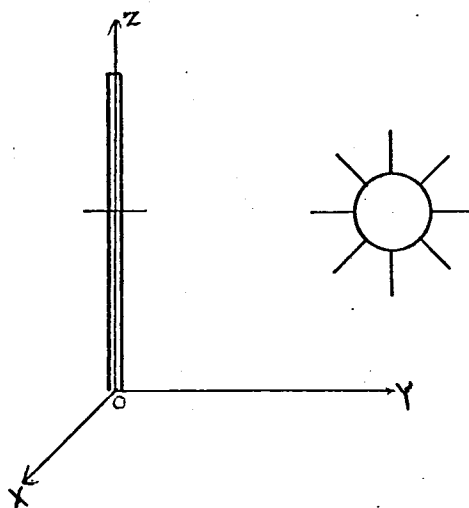
For solving this problem we choose the circular-cylinder coordinates consisting of the coordinate surfaces

$$x^2 + y^2 = \xi^2 \qquad\qquad r_1 \leq \xi < \infty,$$

$$\frac{y}{x} = \tan \psi \qquad\qquad 0 \leq \psi < 2\pi,$$

and $\qquad z = \eta \qquad\qquad -\infty < \eta < +\infty$

i.e.

$$\left. \begin{array}{l} x = \xi\cos\psi, \\[4pt] y = \xi\sin\psi, \\[4pt] z = \eta \end{array} \right\} \qquad\qquad (46.01)$$

and

Here $\xi = $ constant are the circular cylinders, $\eta = $ constant the half-planes perpendicular to the axis of cylinders and $\psi = $ constant the half-

planes through axis of cylinders.

The squared element of arc length is

$$ds^2 = d\xi^2 + d\eta^2 + \xi^2 d\psi^2 \qquad (46.02)$$

wherein

$$\left. \begin{array}{l} g_1(\xi,\eta,\psi) = g_2(\xi,\eta,\psi) = 1 \\[2ex] g_3(\xi,\eta,\psi) = \xi \end{array} \right\} \qquad (46.03)$$

From (46.03)

$$\frac{\partial g_1}{\partial \eta} = \frac{\partial g_1}{\partial \psi} = 0, \quad \text{therefore, we get}$$

$$\frac{\partial p}{\partial \eta} = 0 \qquad (46.04)$$

and

$$\frac{\partial p}{\partial \psi} = 0 \qquad (46.05)$$

By equations (46.04) and (46.05), we get

$$p(\xi,\eta,\psi) = \phi(\xi) \quad \text{where } \phi(\xi) \text{ is an arbitrary function of } \xi.$$

By suitably prescribing the distribution on a line given by $\eta = \text{constant}$

and $\psi = \text{constant}$, we remove the arbitrariness of the function ϕ.

Let

$$p = P(\xi) \qquad (46.06)$$

be the solution of the problem prescribed.

Substituting (46.06) in (41.04) and (46.03) in (41.07), we get

$$u \frac{\partial u}{\partial \xi} + \frac{P'(\xi)}{\rho} = 0 \qquad (46.07)$$

and

$$\frac{\partial}{\partial \xi}(\xi \rho u) = 0 \qquad (46.08)$$

From (46.08), we get

$$\xi \rho u = a(\eta,\psi)$$

or

$$\rho = \frac{a(\eta,\psi)}{\xi u} \qquad (46.09)$$

wherein $a(\eta,\psi)$ is an arbitrary function of η,ψ.

Substituting (46.09) in (46.08), we get

$$\frac{\partial u}{\partial \xi} + \frac{\xi P'(\xi)}{a(\eta,\psi)} = 0 \qquad\qquad \text{(since } u \neq 0)$$

or

$$u(\xi,\eta,\psi) = -\frac{1}{a(\eta,\psi)} \int \xi P'(\xi) d\xi + b(\eta,\psi) \qquad (46.10)$$

which is velocity function for the flow with $a(\eta,\psi)$, $b(\eta,\psi)$ as two

arbitrary functions of η,ψ.

From (46.09) and (46.10), we get

$$\rho(\xi,\eta,\psi) = \frac{-a^2(\eta,\psi)}{\xi\left[\int \xi P'(\xi) d\xi + a(\eta,\psi) b(\eta,\psi)\right]} \qquad (46.11)$$

Equation (46.11) gives us the density field.

Finally, the Mach number at any point in the flow region is

$$M = \frac{u}{c}$$

From (46.06), we get

$$c^2 = \frac{\xi^2 P'(\xi) \left[\int \xi P'(\xi) d\xi + a(\eta,\psi) b(\eta,\psi) \right]^2}{a^2(\eta,\psi) \left[\left(\int \xi P'(\xi) d\xi + a(\eta,\psi) b(\eta,\psi)\right) + \xi^2 P'(\xi)\right]} \qquad (46.12)$$

Therefore,

$$M = \sqrt{1 + \frac{\int \xi P'(\xi) d\xi + a(\eta,\psi) b(\eta,\psi)}{\xi^2 P'(\xi)}} \qquad (46.13)$$

is the Mach number at any point.

Equations (46.10) and (46.11) give us the velocity and density of the

flow. Substituting (46.06) and (46.11) in the equation (41.09) we find

the entropy for the flow.

In sections 7 and 8 we study the problems of three dimensional, steady and rotational gas flow when the chosen natural coordinate systems have the property that either $\dfrac{\partial g_1}{\partial \eta} = 0$ or $\dfrac{\partial g_1}{\partial \psi} = 0$ for the metric coefficient $g_1(\xi, \eta, \psi)$.

<u>Section 7</u>   <u>Flow of gases in a circular tunnel when the</u>

<u>gases are swirling about its axis.</u>
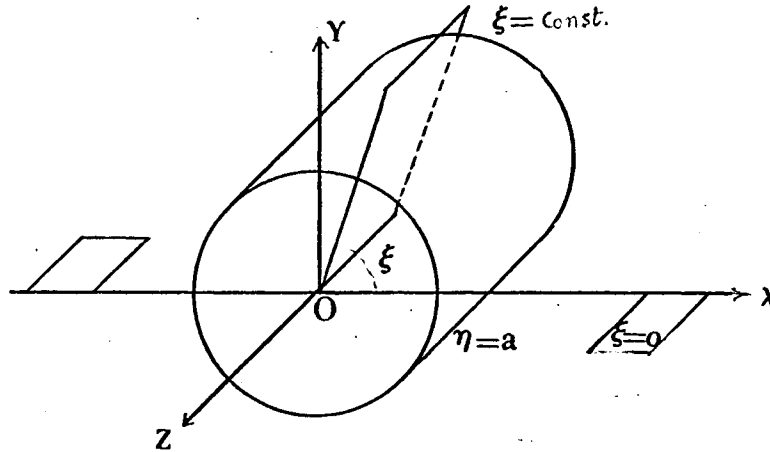
We study the flow of gases, inside the circular tunnel of

radius a, circulating about the axis of tunnel when there are no external

forces. For finding the solution of flow of gases in this problem we

choose the circular-cylinder coordinates consisting of the coordinate

surfaces

$$x^2 + y^2 = \eta^2 \qquad\qquad 0 < \eta \le a$$

$$\frac{y}{x} = \tan \xi \qquad\qquad 0 \le \xi < 2\pi$$

$$z = \psi \qquad\qquad -\infty < \psi < +\infty$$

i.e.

$$\left.\begin{array}{l} x = \eta \cos\xi \\[2mm] y = \eta \sin\xi \\[2mm] z = \psi \end{array}\right\} \qquad\qquad (47.01)$$

Here η = constant are the circular cylinders, ξ = constant the half planes

through the axis of the tunnel and ψ = constant the planes perpendicular

to the axis of the cylinder.

For this system of coordinates the squared element of arc

length is given by

$$ds^2 = \eta^2 \, d\xi^2 + d\eta^2 + d\psi^2 \tag{47.02}$$

so that the metric coefficients are given by

$$\left.\begin{aligned} g_1(\xi,\eta,\psi) &= \eta \\ g_2(\xi,\eta,\psi) &= 1 \\ g_3(\xi,\eta,\psi) &= 1 \end{aligned}\right\} \tag{47.03}$$

From (47.03), we get

$$\frac{\partial g_1}{\partial \eta} = 1 \neq 0 \quad \text{but} \quad \frac{\partial g_1}{\partial \psi} = 0.$$

Therefore,

$$\frac{\partial p}{\partial \psi} = 0 \tag{47.04}$$

i.e. pressure function is independent of ψ and is the solution of second pressure equation (42.12).

So the solution of pressure is given by

$$\frac{\partial p}{\partial \psi} = 0 \tag{47.04}$$

and

$$L(p) \equiv \eta \frac{\partial^2 p}{\partial \xi \partial \eta} + \frac{\partial p}{\partial \xi} = 0 \tag{47.05}$$

From (47.05), we get

$$\frac{\partial}{\partial \eta} \left(\eta \frac{\partial p}{\partial \xi} \right) = 0$$

i.e. $\eta \dfrac{\partial p}{\partial \xi} = a(\xi)$ where $a(\xi)$ is an arbitrary function of ξ.

or $\quad p(\xi,\eta,\psi) = \dfrac{1}{\eta} \displaystyle\int a(\xi)d\xi + b(\eta)$

or $\quad p(\xi,\eta) = \dfrac{d(\xi)}{\eta} + \mathbf{b}(\eta) \tag{47.06}$

where $b(\eta)$ is an arbitrary function of η and $d'(\xi) = a(\xi)$. Therefore, in pressure distribution of the flow given by (47.06) we have two arbitrary functions $a(\xi)$ and $b(\eta)$ which can be easily found by prescribing

the properly posed boundary value problem.

As ξ = constant and η = constant are the characteristic surfaces of equation (47.05) and are mutually orthogonal families of lines in ξ,η plane, we can prescribe the pressure along these. Obviously, from equation (47.04) curves of intersection of the surfaces ξ = constant and η = constant are isobaric curves and these curves are points in (ξ,η) plane.
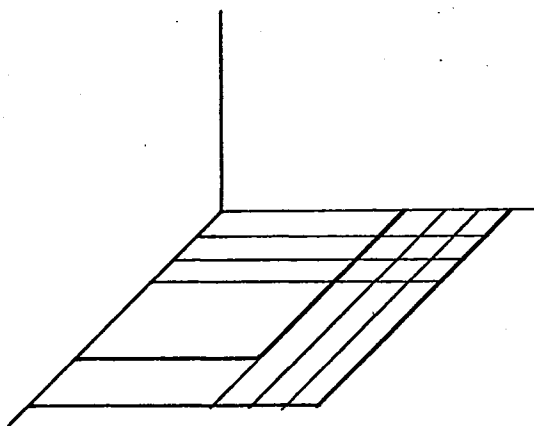
Therefore, we shall solve the characteristic boundary value problem for our flow. Let

$$\left.\begin{array}{ll} p(\xi,a) = \phi(\xi) & 0 \leq \xi < 2\pi \\[2mm] p(2\pi,\eta) = \psi(\eta) & 0 < \eta \leq a \\[2mm] \text{with the restriction } \phi(2\pi) = \psi(a). \end{array}\right\} \qquad (47.07)$$

The Riemann-Green function associated with the operator L in equation (47.05) is

$$R = (\xi,\eta;x,y) = \frac{\eta}{y} \qquad\qquad (47.08)$$

Therefore, the pressure at any point $M(x,y)$ is

$$p(M) = p(2\pi,a)\,\frac{a}{y} + \int_{2\pi}^{x} \phi'(\lambda)d\lambda + \int_{a}^{y} \frac{\lambda}{y}\,[\psi'(\lambda) + \frac{\psi(\lambda)}{\lambda}]d\lambda$$

$$(47.09)$$

Equation (47.09) gives us the solution of (47.05), (47.07) at any point M when the boundary value problem is prescribed. Therefore, equation (47.09) defines the two arbitrary functions of (47.06) by (47.05) and (47.07).

Substituting (47.06) in (43.04), (43.06) and (43.11), we get the velocity, the density and the Mach number at any point in the region as

$$u(\xi,\eta,\psi) = f(\eta,\psi) [\eta^2 b'(\eta) - d(\xi)] \tag{47.11}$$

$$\rho(\xi,\eta,\psi) = \frac{1}{\eta \, f^2(\eta,\psi) \{ \eta^2 b'(\eta) - d(\xi) \}} \tag{47.12}$$

and

$$M = 1 \tag{47.13}$$

Thus our flow is given by (47.06), (47.11) and (47.12). The entropy function can be obtained from the state equation (41.09), (47.06) and (47.12).

Mach number at any point in the region is 1 if $\frac{\partial p}{\partial \xi} \neq 0$. Therefore, we consider this exceptional case when $\frac{\partial p}{\partial \xi} = 0$.

In this case, from (47.06), we get

$$d'(\xi) = a(\xi) = 0 \quad \text{i.e.}$$

$$d(\xi) = \text{constant} = k \text{ (say)} \tag{47.14}$$

Substituting (47.14) in (47.11) and (47.12), we get

$$u(\eta,\psi) = f(\eta,\psi)[\eta^2 b'(\eta) - k]$$

and

$$\rho(\eta,\psi) = \frac{1}{\eta \, f^2(\eta,\psi) \{ \eta^2 b'(\eta) - k \}}$$

In the case when $\frac{\partial p}{\partial \xi} \neq 0$, $u = c$ from $M = 1$ in (47.13). Putting $u = c(\rho,s)$ in (41.04), we get

$$c \, \frac{\partial \rho}{\partial \xi} \left[\frac{\partial c}{\partial \rho} + \frac{c}{\rho} \right] = 0$$

or

As $\frac{\partial \rho}{\partial \xi} \neq 0$, by assumption that $\frac{\partial p}{\partial \xi} \neq 0$, we get

$$\frac{\partial c}{\partial \rho} + \frac{c}{\rho} = 0$$

or

$$c(\rho, s) = \frac{f(s)}{\rho} \tag{47.15}$$

where $f(s)$ is an arbitrary function of s.

From (47.15), we get

$$\frac{\partial p}{\partial \rho} = \frac{f^2(s)}{\rho^2}$$

or

$$p = -\frac{f^2(s)}{\rho} + h(s) \quad \text{is the form of state equation in the}$$

case if $\frac{\partial p}{\partial \xi} \neq 0$. Here $f(s)$ and $h(s)$ are the arbitrary functions of s.

Section 8. Flow through the hyperboloidal tunnel.

fig1.
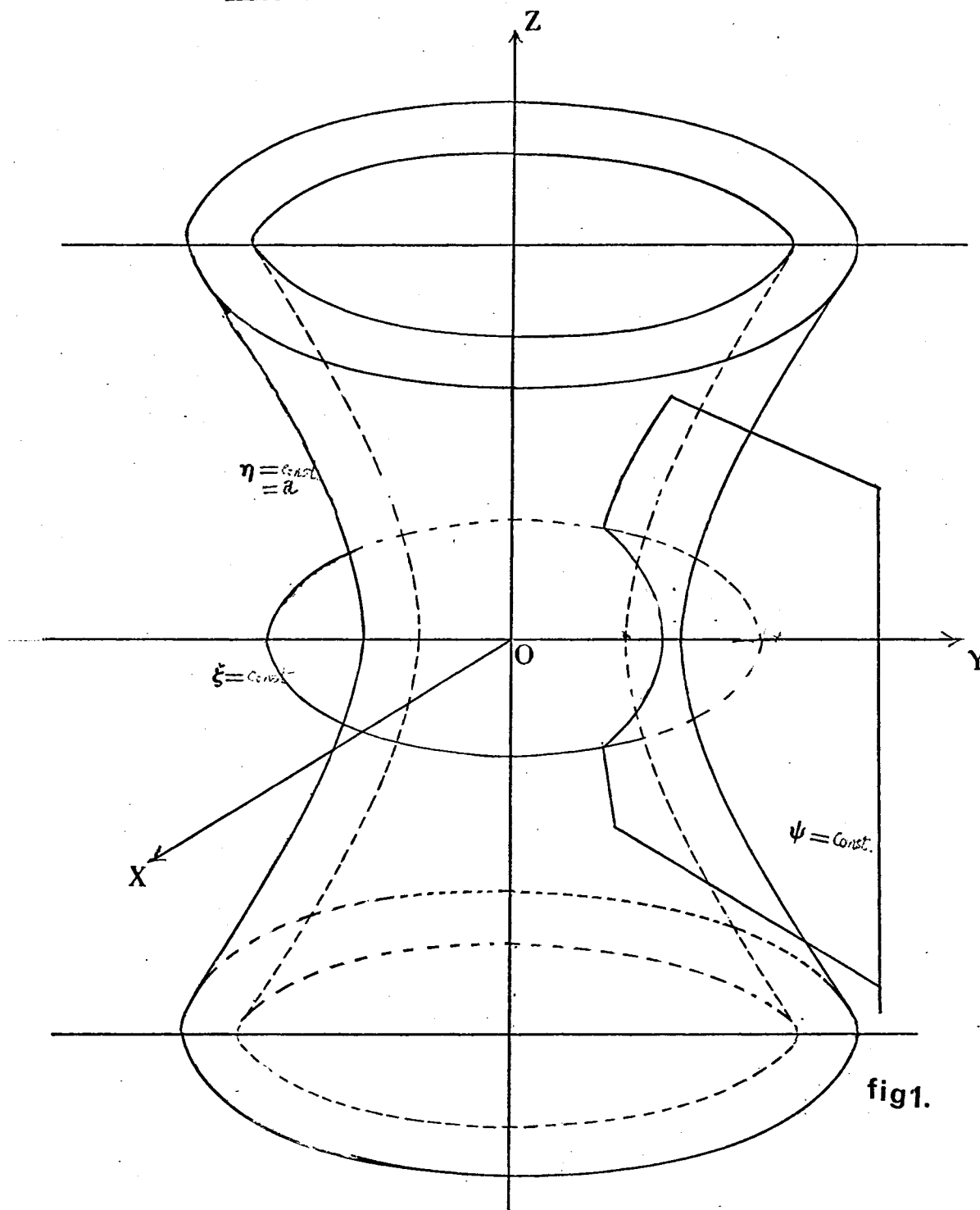
To study the steady flow of gases through a tunnel whose wall

is a hyperboloid of one sheet, we take the oblate spheroidal coordinates

(ξ,η,ψ) consisting of the three orthogonal surfaces

$$\frac{x^2}{a^2\cos^2\eta} + \frac{y^2}{a^2\cos^2\eta} - \frac{z^2}{a^2\sin^2\eta} = 1 \qquad ; \; 0 \leq \eta \leq \pi,$$

$$\frac{x^2}{a^2\cosh^2\xi} + \frac{y^2}{a^2\cosh^2\xi} + \frac{z^2}{a^2\sinh^2\xi} = 1 \qquad ; \; 0 \leq \xi < \infty,$$

and $\dfrac{y}{x} = \tan\psi$ $\qquad\qquad\qquad ; \; 0 \leq \psi < 2\pi.$

i.e.

$$x = a\cosh\xi \, \cos\eta \, \cos\psi,$$
$$y = a\cosh\xi \, \cos\eta \, \sin\psi, \qquad\qquad\qquad (48.01)$$
and $$z = a\sinh\xi \, \sin\eta.$$

Here $\xi =$ constant are oblate spheroids, $\eta =$ constant the hyperboloids of one sheet and $\psi =$ constant the half planes.

The squared element of arc length is

$$ds^2 = a^2(\sinh^2\xi + \sin^2\eta)\left[d\xi^2 + d\eta^2\right] + a^2\cosh^2\xi \, \cos^2\eta \, d\psi^2$$

$$(48.02)$$

wherein

$$g_1(\xi,\eta,\psi) = g_2(\xi,\eta,\psi) = a(\sinh^2\xi + \sin^2\eta)^{1/2}$$
$$g_3(\xi,\eta,\psi) = a\cosh\xi \, \cos\eta \qquad\qquad\qquad\Bigg] \quad (48.03)$$

From (48.03), we get

$$\frac{\partial g_1}{\partial \psi} = 0 \quad \text{but} \quad \frac{\partial g_1}{\partial \eta} \neq 0.$$

Therefore, pressure function is the solution of

$$\frac{\partial p}{\partial \psi} = 0 \qquad\qquad\qquad (48.04)$$

and

$$\frac{\partial^2 p}{\partial \xi \partial \eta} + \left\{ \frac{3}{2}\frac{\sinh 2\xi}{(\sinh^2\xi + \sin^2\eta)} + \frac{\sinh \xi}{\cosh \xi} \right\}\frac{\partial p}{\partial \eta} + \frac{1}{2}\frac{\sin 2\eta}{(\sinh^2\xi + \sin^2\eta)}\frac{\partial p}{\partial \xi} = 0$$

$$(48.05)$$

Solution of (48.04) and (48.05) gives us the pressure distribution in the hyperboloidal tunnel with circular cross sections. Hence by equation (48.04) we conclude that the pressure function is independent of ψ and, therefore, $p = p(\xi,\eta)$ i.e. the curves of intersection of oblate spheroids and the hyperboloids of one sheet are isobaric curves.

We can solve for pressure, thereby, from equation (48.05) which is a linear hyperbolic partial differential equation of the second order in canonical form. In this equation p is the dependent variable and ξ,η are the independent variables. Our objective is to obtain the solution of this equation by properly posing the boundary value problem for it.

Since the curves of intersection of the surfaces $\xi = $ constant and $\eta = $ constant are isobaric curves, we make a transformation from Oxyz space to $\overline{0}\alpha\beta$ plane so that these curves are points of the plane where-in pressure will be the point function.

Let

$$\sinh^2\xi = \alpha,$$

and $\qquad \sin^2 \eta = \beta \qquad \Bigg\} \qquad (48.06)$

be the transformation.

Here in this problem of flow through the tunnel $\eta = a$ the region inside the tunnel given by $0 < a \leq \eta \leq \pi/2$, $0 \leq \xi < \infty$ is transformed into an infinite rectangular channel ($0 \leq \alpha < \infty$, $\beta_1 = \sin^2 a \leq \beta \leq 1$) in $\overline{0}\alpha\beta$ plane as shown in the figure 2.

Further, the problem of posing a well defined boundary value problem is also made easier by the transformation (48.06). By this transformation the families of surfaces $\xi = $ constant and $\eta = $ constant

are mapped into the families of orthogonal straight lines $\alpha = $ constant

and $\beta = $ constant respectively in $\overline{0}\alpha\beta$ plane.

Therefore, if pressure is prescribed along the surface $\eta = a$

and some oblate spheroid $\xi = $ constant, then these boundary values are

well prescribed for their region of influence. This region of influence

can be made larger and larger by taking ξ greater in value.

Let us suppose that we want to solve for pressure inside the

region of tunnel bounded by $\xi = \gamma$. For finding the pressure inside this

we prescribe the pressure on the surface $\xi = \gamma$ as a function of η and on

the walls of this region belonging to the tunnel $\eta = a$ as a function of

ξ. This way we prescribe pressure on the part of a line $\beta = \beta_1$ and on

the line $\alpha = \alpha_1$ (where $\alpha_1 = \sinh^2 \gamma$) in $\overline{0}\alpha\beta$ plane when the pressure has

a unique value at (α_1, β_1). This problem is thus the characteristic

boundary value problem in which the boundary curve degenerates into a right

angle formed by the characteristics $\alpha = \alpha_1$ and $\beta = \beta_1$ in $\overline{0}\alpha\beta$ plane.

Let

$$\left.\begin{array}{ll} p(\xi,a) = f(\xi) & \text{on } \eta = a \; ; \quad 0 \leq \xi \leq \gamma, \\ p(\gamma,\eta) = g(\eta) & \text{on } \xi = \gamma \; ; \quad a \leq \eta \leq \pi/2 \end{array}\right\} . \tag{48.07}$$

with the restriction $f(\gamma) = g(a)$ be the prescribed boundary conditions

of flow.

By transformation (48.06) the boundary value problem of

(48.05) and (48.07) in xyz space transforms in $\overline{0}\alpha\beta$ plane as

$$L(p) \equiv \frac{\partial^2 p}{\partial\alpha\partial\beta} + \left[\frac{3}{2(\alpha+\beta)} + \frac{1}{2(1+\alpha)}\right]\frac{\partial p}{\partial\beta} + \frac{1}{2(\alpha+\beta)}\frac{\partial p}{\partial\alpha} = 0 \tag{48.08}$$

and

$$\left.\begin{array}{ll} p(\alpha,\beta_1) = a(\alpha) & 0 \leq \alpha \leq \alpha_1, \\ p(\alpha_1,\beta) = b(\beta) & \beta_1 \leq \beta \leq 1 \end{array}\right\} \tag{48.09}$$

such that $a(\alpha_1) = b(\beta_1)$ where $\sinh^2 \gamma = \alpha_1$ and $\sin^2 a = \beta_1$.
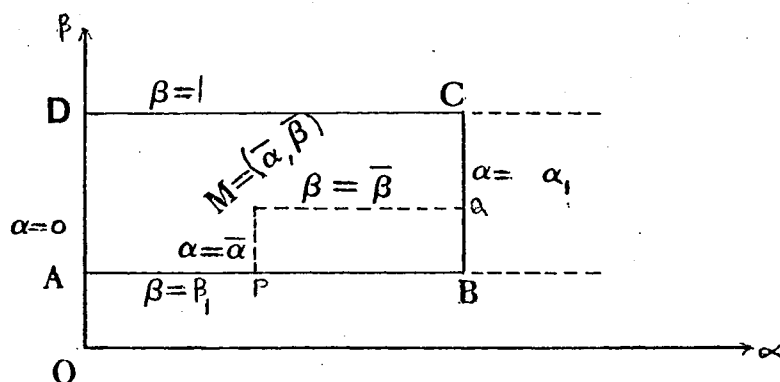

fig 2.

To solve the problem defined by (48.08) and (48.09) we use Riemann's method. Let $M(\bar{\alpha}, \bar{\beta})$ by any point inside the rectangle so that MQ and MP are the characteristic lines through M. We find the pressure at M.

Now adjoint equation of (48.08) is

$$L^*(R) = \frac{\partial^2 R}{\partial \alpha \partial \beta} - \left[\frac{3}{2(\alpha+\beta)} + \frac{1}{2(1+\alpha)} \right] \frac{\partial R}{\partial \beta} - \frac{1}{2(\alpha+\beta)} \frac{\partial R}{\partial \alpha} + \frac{2}{(\alpha+\beta)^2} R = 0 \qquad (48.10)$$

The Riemann-Green function, which is the solution of a characteristic boundary value problem for this adjoint equation (48.10), is found in the next section **9** and is given by

$$R(\alpha,\beta;\bar{\alpha},\bar{\beta}) = \frac{(\alpha + \beta)^{1/2} (\alpha+\bar{\beta}) (\alpha+1)^{1/2}}{(\bar{\alpha}+\bar{\beta})^{3/2}(\bar{\alpha}+1)^{1/2}} F(\tfrac{1}{2},\tfrac{3}{2},1,z) \qquad (48.11)$$

(where N = 1 is put in)

Here $F(\tfrac{1}{2},\tfrac{3}{2},1,z)$ is the same hypergeometric function as used in (38.18) with z given by $-\frac{(\beta-\bar{\beta})(\alpha-\bar{\alpha})}{(\alpha+\beta)(\bar{\alpha}+\bar{\beta})}$.

Having found the Riemann-Green function associated with the operator L of (48.08), we have the solution for the pressure at M given by

$$p(M) = p(\bar{\alpha},\bar{\beta}) = p(\alpha_1,\beta_1) \frac{(\alpha_1+\beta_1)^{1/2}(\alpha_1+\bar{\beta})(\alpha_1+1)^{1/2}}{(\bar{\alpha}+\bar{\beta})^{3/2}(\bar{\alpha}+1)^{1/2}} F[\tfrac{1}{2},\tfrac{3}{2},1;\frac{(\alpha_1-\bar{\alpha})(\bar{\beta}-\beta_1)}{(\bar{\alpha}+\bar{\beta})(\alpha_1+\beta_1)}]$$

$$-\int_{\bar{\alpha}}^{\alpha_1} \frac{(\lambda+\beta_1)^{1/2}(\lambda+\bar{\beta})(\lambda+1)^{1/2}}{(\bar{\alpha}+\bar{\beta})^{3/2}(\bar{\alpha}+1)^{1/2}} F[\tfrac{1}{2},\tfrac{3}{2},1;\frac{(\lambda-\bar{\alpha})(\bar{\beta}-\beta_1)}{(\bar{\alpha}+\bar{\beta})(\lambda+\beta_1)}]\left\{\alpha'(\lambda) + \frac{a(\lambda)}{2(\lambda+\beta_1)}\right\} d\lambda$$

$$-\int_{\bar{\beta}}^{\beta_1} \frac{(\alpha_1+\lambda)^{1/2}(\alpha_1+\bar{\beta})(\alpha_1+1)^{1/2}}{(\bar{\alpha}+\bar{\beta})^{3/2}(\bar{\alpha}+1)^{1/2}} F[\tfrac{1}{2},\tfrac{3}{2},1;\frac{(\alpha_1-\bar{\alpha})(\bar{\beta}-\lambda)}{(\bar{\alpha}+\bar{\beta})(\alpha_1+\lambda)}]\{b'(\lambda) + \frac{3b(\lambda)}{2(\alpha_1+\lambda)} + \frac{b(\lambda)}{2(1+\alpha_1)}\}\, d\lambda$$

$$(48.12)$$

where in the integrals on the right hand side λ is the variable and α_1, β_1 are constants. Equation (48.12) gives us the pressure distribution in the region ABCD and, therefore, in the part of the tunnel bounded by the surface $\xi = \gamma$ and $\eta = a$ (i.e. the walls).

Let

$$p = P(\xi,\eta) \qquad\qquad (48.13)$$

be the solution of our boundary problem (48.05) and (48.07).

Substituting (48.13) in (43.04) and (43.06), we get the velocity and the density as

$$u(\xi,\eta,\psi) = e^{-\int \frac{\sin\eta\,\cos\eta}{\sinh^2\xi+\sin^2\eta}\cdot\begin{bmatrix}\frac{\partial P}{\partial \xi}\\\frac{\partial P}{\partial \eta}\end{bmatrix} d\xi} \cdot f(\eta,\psi) \qquad (48.14)$$

and

$$\rho(\xi,\eta,\psi) = \frac{2(\sinh^2\xi+\sin^2\eta)}{\sin 2\eta} \cdot \frac{\frac{\partial P}{\partial \eta}}{f^2(\eta,\psi)} \cdot e^{2\int \frac{\sin\eta\cos\eta}{\sinh^2\xi+\sin^2\eta}\cdot\frac{\frac{\partial P}{\partial \xi}}{\frac{\partial P}{\partial \eta}} d\xi} \qquad (48.15)$$

Substituting (48.13) in the expressions (43.10), (43.11) for Mach number, we get

$$M = \sqrt{1 + \tanh\xi\left[1 + \frac{\cosh^2\xi}{\sinh^2\xi+\sin^2\eta}\right] \frac{1}{\frac{\partial}{\partial\xi}(\log F)}} \qquad (48.16)$$

or

$$M = \sqrt{1 - \tanh\xi \left[\frac{\sinh^2\xi + \sin^2\eta}{\sin\eta \, \cos\eta}\right]\left[1 + \frac{\cosh^2\xi}{\sinh^2\xi + \sin^2\eta}\right] \cdot \frac{\frac{\partial P}{\partial \eta}}{\frac{\partial P}{\partial \xi}}}$$

wherein

$$F = e^{-\int \frac{\sin\eta\cos\eta}{(\sinh 2\xi + \sin 2\eta)}\left(\frac{\frac{\partial P}{\partial \xi}}{\frac{\partial P}{\partial \eta}}\right) \cdot d\xi} \tag{48.17}$$

From (48.16) and (48.17), we get $M = 1$ when $\xi = 0$ provided $\frac{\partial P}{\partial \xi} \neq 0$ on $\xi = 0$ i.e. the flow is sonic on the surface $\xi = 0$. In the region bounded by $\xi = \gamma$, $\eta = a$, we have

$$\tanh \xi \left[1 + \frac{\cosh^2\xi}{\sinh^2\xi + \sin^2\eta}\right] > 0$$

and

$$\tanh\xi \left\{\frac{\sinh^2\xi + \sin^2\eta}{\sin\eta \, \cos\eta}\right\}\left\{1 + \frac{\cosh^2\xi}{\sinh^2\xi + \sin^2\eta}\right\} > 0$$

when $\xi > 0$.

Therefore, from (48.16) and (48.17) we conclude that the flow is subsonic, sonic or supersonic according as

$$\left.\begin{aligned}\frac{\partial}{\partial\xi}(\log F) &\lessgtr 0 \\[2ex] \text{or} \\[2ex] \frac{\partial P}{\partial \eta} \Big/ \frac{\partial P}{\partial \xi} &\gtrless 0\end{aligned}\right\} \tag{48.18}$$

So we find that the flow of gases on the surface $\xi = 0$ is sonic but the flow at any other point in the region is supersonic (subsonic) if the rates of change of pressure w.r.t. ξ and η at that point have opposite (same) signs.

Finally, we study the flow when a particle of gas moves down along a stream line from the surface $\xi = \gamma$ to the surface $\xi = 0$.

<u>Case I</u> (When the flow is subsonic in the region.)

In this case M < 1. By this assumption we get from (48.16) that $\frac{\partial}{\partial \xi}(\log F) < 0$ i.e. log F and ξ are such that if one increases, the other decreases. Now along the motion considered ξ decreases and, therefore, log F increases i.e. F increases and thus the velocity increases when the flow is subsonic and the particle moves down the stream line.

<u>Case II</u> Likewise, we prove that the velocity magnitude decreases when a particle of gas flows from $\xi = \gamma$ to $\xi = 0$ in this case of supersonic flow in the region.

In section 9, we study the problem of three-dimensional, steady and   rotational gas flow when the chosen natural coordinate system has the property that $\frac{\partial g_1}{\partial \eta} \neq 0$ and $\frac{\partial g_1}{\partial \psi} \neq 0$ for the metric coefficient g_1.

Section 9. Flow of gases through a tunnel with elliptic cross sections
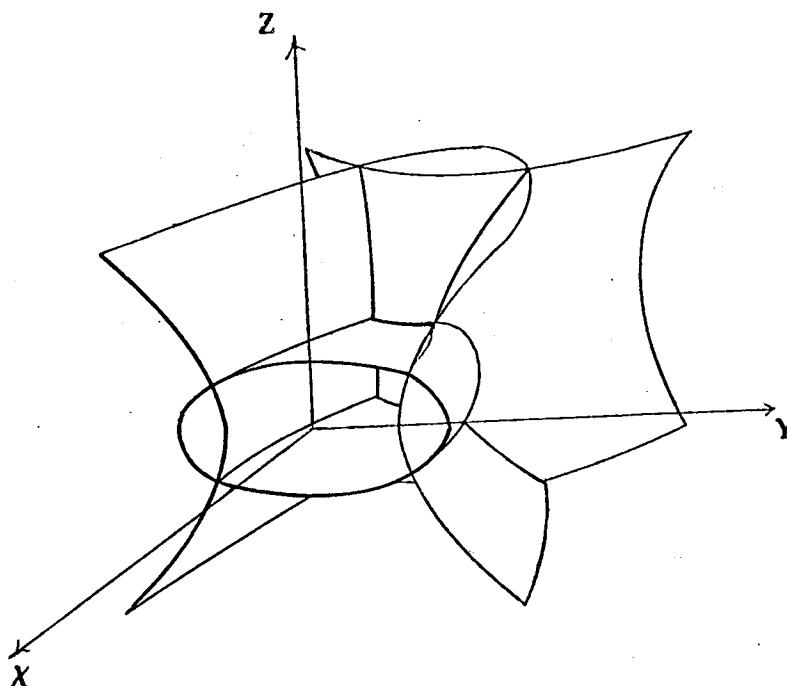
We study the flow of gases through a tunnel with elliptic cross sections and, therefore, choose the ellipsoidal coordinates for the natural coordinates here. In this coordinate system the coordinate surfaces are

$$\frac{x^2}{\xi^2} + \frac{y^2}{\xi^2 - b^2} + \frac{z^2}{\xi^2 - c^2} = 1 \qquad c^2 < \xi^2 < \infty^2,$$

$$\frac{x^2}{\eta^2} + \frac{y^2}{\eta^2 - b^2} - \frac{z^2}{c^2 - \eta^2} = 1 \qquad b^2 < \eta^2 < c^2,$$

and

$$\frac{x^2}{\psi^2} - \frac{y^2}{b^2-\psi^2} - \frac{z^2}{c^2-\psi^2} = 1 \qquad 0 \leq \psi^2 < b^2$$

with $\xi^2 > c^2 > \eta^2 > b^2 > \psi^2 \geq 0$

i.e.

$$(x)^2 = \left(\frac{\xi\eta\psi}{bc}\right)^2 , \quad (y)^2 = \frac{(\xi^2-b^2)(\eta^2-b^2)(b^2-\psi^2)}{b^2(c^2-b^2)} \qquad (49.01)$$

and

$$(z)^2 = \frac{(\xi^2-c^2)(c^2-\eta^2)(c^2-\psi^2)}{c^2(c^2-b^2)}$$

Here $\xi^2 = $ constant are the ellipsoids, $\eta^2 = $ constant the hyperboloids of one sheet and $\psi^2 = $ constant the hyperboloids of two sheets.

The squared element of arc length is

$$ds^2 = \frac{(\xi^2-\eta^2)(\xi^2-\psi^2)}{(\xi^2-b^2)(\xi^2-c^2)} d\xi^2 + \frac{(\eta^2-\psi^2)(\xi^2-\eta^2)}{(\eta^2-b^2)(c^2-\eta^2)} d\eta^2 + \frac{(\xi^2-\psi^2)(\eta^2-\psi^2)}{(b^2-\psi^2)(c^2-\psi^2)} d\psi^2 \qquad (49.02)$$

From (49.02), the metric coefficients of this system are:

$$\left. \begin{array}{c} g_1(\xi,\eta,\psi) = \sqrt{\dfrac{(\xi^2-\eta^2)(\xi^2-\psi^2)}{(\xi^2-b^2)(\xi^2-c^2)}} , \\[3mm] g_2(\xi,\eta,\psi) = \sqrt{\dfrac{(\eta^2-\psi^2)(\xi^2-\eta^2)}{(\eta^2-b^2)(c^2-\eta^2)}} , \\[3mm] g_3(\xi,\eta,\psi) = \sqrt{\dfrac{(\xi^2-\psi^2)(\eta^2-\psi^2)}{(b^2-\psi^2)(c^2-\psi^2)}} \end{array} \right\} \qquad (49.03)$$

From (49.03)

$$\frac{\partial g_1}{\partial \eta} \neq 0 \quad \text{and} \quad \frac{\partial g_1}{\partial \psi} \neq 0$$

Therefore, in this case pressure at any point in the region

of flow is given as a solution of any two of the three pressure equations (42.06), (42.12) and (42.13). Here $p = p(\xi, \eta, \psi)$.

Substituting (49.03) in (42.06), we get

$$\psi(\xi^2 - \eta^2) \frac{\partial p}{\partial \eta} = \eta(\xi^2 - \psi^2) \frac{\partial p}{\partial \psi} \qquad (49.04)$$

Substituting (49.03) in (42.12), we get

$$\frac{\partial^2 p}{\partial \xi \partial \eta} + [\frac{3\xi}{\xi^2 - \eta^2} + \frac{\xi}{\xi^2 - \psi^2}] \frac{\partial p}{\partial \eta} - \frac{\eta}{\xi^2 - \eta^2} \frac{\partial p}{\partial \xi} = 0 \qquad (49.05)$$

The third pressure equation (42.13) can be obtained by eliminating $\frac{\partial p}{\partial \eta}$ between (49.04) and (49.05) and, therefore, is an equation depending upon (49.04) and (49.05). So the solution obtained from (49.04) and (49.05) will also be the solution of the third pressure equation.

Now equation (49.04) is a partial differential equation of order one and equation (49.05) is a partial differential equation of order two. In both these equations p is the dependent variable which is a function of three independent variables ξ, η, ψ.

In solving these equations for pressure we shall have three arbitrary functions involved which we have to prescribe by a well posed boundary value problem.

We shall solve this problem by first solving the boundary value problem for equation (49.05) in which we shall consider ψ as a constant parameter for the equation. This equation then is a second order linear hyperbolic partial differential equation, for the function p, in canonical form with ξ, η as the independent variables.

Let $\eta^2 = a^2$, where $b^2 < a^2 < c^2$, be the wall of this tunnel. We now transform the equation (49.05) which gives us the pressure in the

tunnel to another equation which holds in plane.

Let

$$\xi^2 = \alpha,$$
$$\eta^2 = -\beta$$

(49.06)

be the transformation.

By this transformation the wall of the tunnel $\eta^2 = a^2$ is mapped into the line $\beta = -a^2$, the surface $\eta^2 = b^2 + \epsilon^2$ ($\epsilon^2 > 0$) into the line $\beta = -(b^2 + \epsilon^2)$ and the surfaces $\eta^2 = d^2$ where $b^2 + \epsilon^2 < d^2 < a^2$ into the lines parallel to α-axis and bounded by $\beta = -a^2$ and $\beta = -(b^2 + \epsilon)$. Likewise, the surfaces $\xi^2 = f^2$ where $c^2 < f^2 < \infty^2$ are mapped into the lines parallel to β-axis and are bounded by $\alpha = c^2 + \epsilon^2$ and $\alpha = \infty^2$.

Therefore, the region inside the tunnel in xyz space is mapped into the $\overline{0}\alpha\beta$ plane. The curves of intersection of the surfaces $\xi = $ constant and $\eta = $ constant are points in $\overline{0}\alpha\beta$ plane and the infinite tunnel is mapped into the rectangular region bounded by $\alpha = c^2 + \epsilon^2$, $\alpha = \infty^2$ and $\beta = -a^2$, $\beta = -(b^2 + \epsilon^2)$.
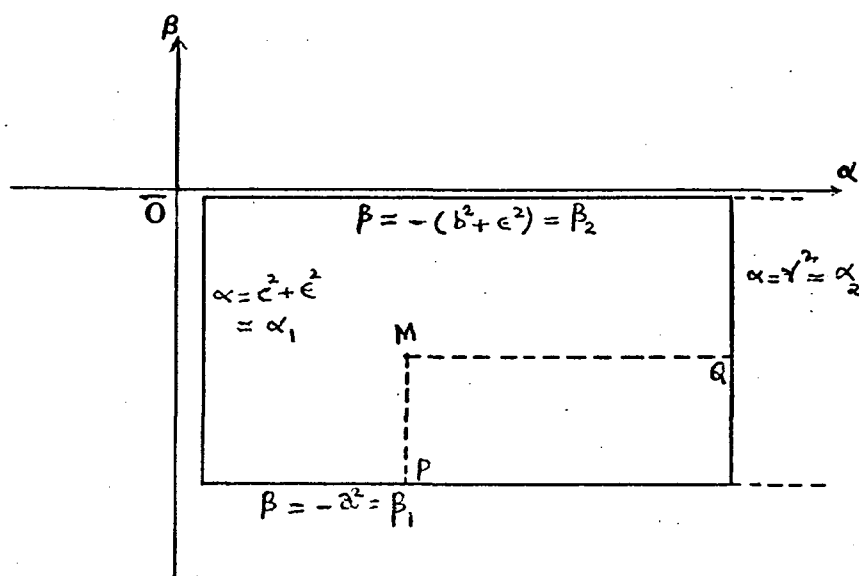
Now from equation (49.05) the characteristic hypersurfaces of this equation are ξ = constant and η = constant i.e. the ellipsoids and the hyperboloids of one sheet. Since these surfaces are mutually orthogonal, we can prescribe the boundary value problem for (49.05) by prescribing pressure along one of the surfaces ξ = constant and one of the surfaces η = constant. We, therefore, regard ψ as a constant parameter in this equation (say, $\psi^2 = -N^2$)

Let

$$
\left.
\begin{aligned}
&p(\xi^2, a^2) = f(\xi^2) \qquad c^2 + e^2 \le \xi^2 \le \gamma^2 < \infty^2, \\
&p(\gamma^2, \eta^2) = g(\eta^2) \qquad b^2 + e^2 \le \eta^2 \le a^2 \\
&\text{with the restriction } g(a^2) = f(\gamma^2)
\end{aligned}
\right\} \tag{49.07}
$$

be the boundary values on the wall $\eta^2 = a^2$ and an ellipsoid $\xi^2 = \gamma^2$ for the pressure function so that on their curve of intersection the two prescribed functions have the same value.

By transformation (49.06) the equations (49.05) and (49.07) are transformed into

$$
\frac{\partial^2 p}{\partial \alpha \partial \beta} + \left[\frac{3}{2(\alpha+\beta)} + \frac{1}{2(\alpha+N^2)} \right] \frac{\partial p}{\partial \beta} + \frac{1}{2(\alpha+\beta)} \frac{\partial p}{\partial \alpha} = 0 \tag{49.08}
$$

subject to the boundary conditions,

$$
\left.
\begin{aligned}
&p(\alpha, \beta_1) = a(\alpha) \qquad \alpha_1 \le \alpha \le \alpha_2 \\
&p(\alpha_2, \beta) = b(\beta) \qquad \beta_1 \le \beta \le \beta_2 \\
&\text{such that } a(\alpha_2) = b(\beta_1)
\end{aligned}
\right\} \tag{49.09}
$$

Here $\alpha = c^2 + e^2 = \alpha_1$, $\alpha = \gamma^2 = \alpha_2$ and $\beta = -a^2 = \beta_1$, $\beta = -(b^2 + e^2) = \beta_2$.

By transformation our problem is to solve (49.08) for pressure at any point inside the rectangle when on $\alpha = \alpha_2$, $\beta = \beta_1$ pressure

is prescribed. We solve this problem given by (49.08) and (49.09) by using Riemann's method. Let $M = (\bar{\alpha}, \bar{\beta})$ be any point inside the rectangular region through which we pass the two characteristics MP and MQ.

As in Riemann's method we first find the Riemann-Green function which is the solution of a characteristic boundary value problem for the adjoint equation,

$$L^*(R) = \frac{\partial^2 R}{\partial\alpha\partial\beta} - \left[\frac{3}{2(\alpha+\beta)} + \frac{1}{2(\alpha+N^2)}\right]\frac{\partial R}{\partial\beta} - \frac{1}{2(\alpha+\beta)}\frac{\partial R}{\partial\alpha} + \frac{2R}{(\alpha+\beta)^2} = 0 \quad (49.10)$$

of (49.08), therefore, taking $R(\alpha,\beta;\bar{\alpha},\bar{\beta})$ as the Riemannian function we have the following conditions on R.

$$R(\alpha,\bar{\beta};\bar{\alpha},\bar{\beta}) = \frac{(\alpha+\bar{\beta})^{3/2}[\alpha+N^2]^{1/2}}{(\bar{\alpha}+\bar{\beta})^{3/2}[\bar{\alpha}+N^2]^{1/2}}$$

$$\text{(49.11)}$$

on the characteristic $\beta = \bar{\beta}$

$$R(\bar{\alpha},\beta;\bar{\alpha},\bar{\beta}) = \frac{(\bar{\alpha}+\beta)^{1/2}}{(\bar{\alpha}+\bar{\beta})^{1/2}}$$

$$\text{(49.12)}$$

on the characteristic $\alpha = \bar{\alpha}$

and

$$R(\bar{\alpha},\bar{\beta};\bar{\alpha},\bar{\beta}) = 1 \quad (49.13)$$

Now conditions (49.11), (49.12) and (49.13) on $R(\alpha,\beta;\bar{\alpha},\bar{\beta})$ are satisfied if

$$R(\alpha,\beta;\bar{\alpha},\bar{\beta}) = \frac{(\alpha+\beta)^{1/2}(\alpha+\bar{\beta})(\alpha+N^2)^{1/2}}{(\bar{\alpha}+\bar{\beta})^{3/2}(\bar{\alpha}+N^2)^{1/2}} F[a,b,c;z] \quad (49.14)$$

wherein $z = -\frac{(\alpha-\bar{\alpha})(\beta-\bar{\beta})}{(\alpha+\beta)(\bar{\alpha}+\bar{\beta})}$ and a,b,c may be any constant numbers. Now we wish to fix these a,b and c so that equation (49.10) is satisfied.

Substituting (49.14) in equation (49.10), we get

$$\frac{\partial z}{\partial \alpha} \frac{\partial z}{\partial \beta} F''(z) + \left\{ \frac{\partial^2 z}{\partial \alpha \partial \beta} + \frac{1}{2(\alpha+\beta)} \frac{\partial z}{\partial \beta} + \frac{1}{\alpha+\bar{\beta}} \frac{\partial z}{\partial \beta} - \frac{3}{2(\alpha+\beta)} \frac{\partial z}{\partial \beta} \right\} F'(z)$$

$$+ \frac{3}{4(\alpha+\beta)^2} F(z) = 0 \qquad (49.15)$$

Since $z = -\frac{(\alpha-\bar{\alpha})(\beta-\bar{\beta})}{(\bar{\alpha}+\bar{\beta})(\alpha+\beta)}$, we get

$$\frac{\partial z}{\partial \alpha} \frac{\partial z}{\partial \beta} = \frac{z(z-1)}{(\alpha+\beta)^2} \qquad (49.16)$$

$$\frac{\partial^2 z}{\partial \alpha \partial \beta} = \frac{2z-1}{(\alpha+\beta)^2} \qquad (49.17)$$

and

$$\left(\frac{1}{\alpha+\bar{\beta}} - \frac{1}{\alpha+\beta} \right) \frac{\partial z}{\partial \beta} = \frac{z}{(\alpha+\beta)^2} \qquad (49.18)$$

Substituting (49.16) - (49.18) in equation (49.15), we get

$$z(1-z)F''(z) + (1-3z)F'(z) - \frac{3}{4} F(z) = 0 \qquad (49.19)$$

This equation (49.19) is the Gaussian differential equation

$$z(1-z)F''(z) + [c-(a+b+1)z] F'(z) - \alpha\beta F(z) = 0$$

with $a = 1/2$, $b = 3/2$ and $c = 1$.

Therefore, equation (49.19) possesses a unique solution which is regular at $z = 0$ and which does not vanish there. Assuming $F(0) = 1$, this solution is denoted by

$$F(z) = F \left[\frac{1}{2}, \frac{3}{2}, 1, \frac{(\alpha-\bar{\alpha})(\bar{\beta}-\beta)}{(\alpha+\beta)(\bar{\alpha}+\bar{\beta})} \right] \qquad (49.20)$$

Using (49.20) in (49.14), we get

$$R(\alpha,\beta;\bar{\alpha},\bar{\beta}) = \frac{(\alpha+\beta)^{1/2}(\alpha+\bar{\beta})(\alpha+N^2)^{1/2}}{(\bar{\alpha}+\beta)^{3/2}(\bar{\alpha}+N^2)^{1/2}} F\left[\frac{1}{2}, \frac{3}{2}, 1, \frac{(\alpha-\bar{\alpha})(\bar{\beta}-\beta)}{(\alpha+\beta)(\bar{\alpha}+\bar{\beta})}\right] \qquad (49.21)$$

Therefore, the solution of equations (49.08), (49.09) and, thereby, by the help of (49.06) of equation (49.05) subject to the boundary conditions

(49.07) is

$$p(M) = p(\bar{\alpha},\bar{\beta}) = p(T)R(\alpha_2,-\beta_2;\bar{\alpha},\bar{\beta})$$

$$+\int_T^P R(\gamma,-\beta_1;\bar{\alpha},\bar{\beta}) \left\{ a'(\gamma) + \frac{a(\gamma)}{2(\gamma+\beta_1)} \right\} d\gamma$$

$$+\int_T^Q R(\alpha_2,\lambda;\bar{\alpha},\bar{\beta}) \left\{ b'(\lambda) + \frac{3b(\lambda)}{2(\alpha_2+\lambda)} + \frac{b(\lambda)}{2(k_2-\psi^2)} \right\} d\lambda$$

Let

$$p = p(\xi,\eta,\psi) \tag{49.23}$$

be the solution (49.22).

Substituting (49.23) in (49.04), we get

$$\frac{\partial p}{\partial \psi} = \frac{(\xi^2-\eta^2)}{\eta} \frac{\psi}{\xi^2-\psi^2} \frac{\partial P}{\partial \eta}$$

or

$$p(\xi,\eta,\psi) = \frac{(\xi^2-\eta^2)}{\eta} \int \frac{\partial P}{\partial \eta} \frac{\psi}{(\xi^2-\psi^2)} d\psi + \phi(\eta,\xi) \tag{49.23'}$$

wherein $\phi(\eta,\xi)$ is an arbitrary function which we find by prescribing the pressure along a surface $\psi = $ constant i.e. $p(\xi,\eta,\psi_o) = f(\xi,\eta)$.

Thus we get

$$p = p(\xi,\eta,\psi) \tag{49.24}$$

Equation (49.24) gives us the solution for pressure function at any point insdie the range of influence of the characteristic hyper-surfaces $\xi^2 = \gamma^2$ and $\eta^2 = a^2$ when the pressure is also prescribed on a surface $\psi = \psi_o$ in the region.

We can also obtain some particular solution of pressure by first finding the general form of pressure satisfied by (49.04) and then finding the restriction on the form by (49.05).

Let

$$p = (\xi^2-\eta^2)^m(\xi^2-\psi^2)^n \qquad (49.25)$$

be the form of p.

Substituting (49.25) in equation (49.04), we get $m = n$ and, therefore,

$$p = (\xi^2-\eta^2)^m(\xi^2-\psi^2)^m \qquad \text{satisfies (49.04)}$$

Substituting this expression for p in (49.05), we get $m = -1$ if $2\xi^2-\eta^2-\psi^2 \neq 0$.

Therefore, we get an exact solution of (49.04) and (49.05) as given by

$$p = \frac{A}{(\xi^2-\eta^2)(\xi^2-\psi^2)} \qquad (49.26)$$

for the region where $2\xi^2-\eta^2-\psi^2 \neq 0$. Here A is an arbitrary constant.

Section 10. Theorem. The curves of intersection of the two families of surfaces, the hyperboloids of one sheet and the hyperboloids of two sheets, cannot be the stream lines of the three dimensional compressible fluid flow problem. The only fluid motion possible with the above curves as the stream lines is the incompressible and irrotational fluid motion.

Proof: We first consider the compressible fluid motion and choose the ellipsoidal coordinates for the natural coordinate system as done in section 9.

Equations of these families of surfaces, the squared element of arc length for the system and the metric coefficients of this system are given by the equations (49.01), (49.02) and (49.03) respectively. The compressible fluid motion is then given by the equations (41.04) to (41.09) and the pressure function of the flow is the solution of the equations (49.04) and (49.05).

Taking

$$\xi^2 = x,$$
$$\eta^2 = -y,$$
and $$\psi^2 = -z$$

(410.01)

as the transformation, equations (49.04), (49.05) become

$$(x + y) \frac{\partial p}{\partial y} = (x + z) \frac{\partial p}{\partial z}$$

(410.02)

and

$$2 \frac{\partial^2 p}{\partial x \partial y} + \left(\frac{3}{x+y} + \frac{1}{x+z}\right) \frac{\partial p}{\partial y} + \frac{1}{x+y} \frac{\partial p}{\partial x} = 0$$

(410.03)

From equation (410.02), we get

$$p = F[x, (x+y)(x+z)]$$

or

$$p = F[\alpha, \beta] \qquad (410.04)$$

wherein

$$\left. \begin{array}{l} \alpha = x \\[2mm] \beta = (x+y)(x+z) \end{array} \right\} \qquad (410.05)$$

Substituting for p from (410.04) in (410.03), we get

$$\beta^{3/2} \frac{\partial}{\partial \beta} \left[\beta^{1/2} \frac{\partial F}{\partial \alpha} \right] + (2x+y+z) \frac{\partial}{\partial \beta} \left[\beta^2 \frac{\partial F}{\partial \beta} \right] = 0 \qquad (410.06)$$

Letting $2x+y+z = \gamma$, we find that

$$\left| \frac{\partial(\alpha, \beta, \gamma)}{\partial(x, y, z)} \right| = z - y = \eta^2 - \psi^2 \neq 0$$

i.e. we can take α, β, γ as the three independent variables. Using the fact that $p = F(\alpha, \beta)$ only and α, β, γ are the independent variables in equation (410.06), we get

$$\frac{\partial}{\partial \beta} \left[\beta^{1/2} \frac{\partial F}{\partial \alpha} \right] = 0,$$

$$\frac{\partial}{\partial \beta} \left[\beta^2 \frac{\partial F}{\partial \beta} \right] = 0$$

or

$$\left. \begin{array}{l} \dfrac{\partial F}{\partial \alpha} = \dfrac{G(\alpha)}{\beta^{1/2}} , \\[4mm] \dfrac{\partial F}{\partial \beta} = \dfrac{H(\alpha)}{\beta^2} \end{array} \right\} \qquad (410.07)$$

wherein $G(\alpha)$, $H(\alpha)$ are the two unknown functions of α.

From equations (410.07), we get

$$G(\alpha) + \frac{2}{\beta^{1/2}} H'(\alpha) = 0$$

Since α, β are the independent variables,

$$G(\alpha) = 0,$$

$$H'(\alpha) = 0$$

or

$$\left. \begin{array}{l} G(\alpha) = 0, \\[2mm] H(\alpha) = H \end{array} \right\} \tag{410.08}$$

wherein H is some unknown constant.

Using equations (410.08) in (410.07), we get

$$\left. \begin{array}{l} \dfrac{\partial F}{\partial \alpha} = 0, \\[4mm] \dfrac{\partial F}{\partial \beta} = \dfrac{H}{\beta^2} \end{array} \right\}$$

or

$$p = F(\alpha,\beta) = \frac{-H}{\beta} + J \tag{410.09}$$

where J is the unknown constant also.

Using equation (49.03) in (41.07) and equation (41.09) in (41.05), we get

$$pu = \frac{k(\eta,\psi)}{(\xi^2-\eta^2)^{1/2}(\xi^2-\psi^2)^{1/2}} \tag{410.10}$$

and

$$pu^2 = \frac{2H}{(\xi^2-\eta^2)(\xi^2-\psi^2)} \tag{410.11}$$

From equations (410.10) and (410.11), we get

$$u = \frac{2H}{k(\eta,\psi)\,(\xi^2-\eta^2)^{1/2}(\xi^2-\psi^2)^{1/2}} \tag{410.12}$$

and

$$\rho = \frac{k^2(\eta,\psi)}{2H} \tag{410.13}$$

Substituting for p, u and ρ from equations (410.09), (410.12) and (410.13) in equation (41.04), we get

$$H = 0 \tag{410.14}$$

Putting (410.14) in (410.09) and (410.12), we get

$$p = J \text{ (const)}$$

and

$$u = 0$$

Therefore, we cannot have the family of curves considered in this theorem as the stream lines of a compressible fluid flow.

We now take the incompressible fluid flow with the chosen family of curves as the stream lines.

Let $\rho = \rho_0$ be the density of the fluid. From equations (41.04) and (41.07), we get

$$\frac{1}{2} \rho_0 u^2 + p = A(\eta, \psi) \tag{410.15}$$

and

$$u = \frac{B(\eta, \psi)}{\rho_0 (\xi^2 - \eta^2)^{1/2} (\xi^2 - \psi^2)^{1/2}} \tag{410.16}$$

wherein $A(\eta, \psi)$, $B(\eta, \psi)$ are the two unknown functions.

From (410.15) and (41.05), we get

$$\frac{\partial}{\partial \eta} \left[\frac{1}{2} g_1^2 \rho_0 u^2 \right] = g_1^2 \frac{\partial A}{\partial \eta} (\eta, \psi)$$

Substituting for u^2 from (410.16) and for g_1^2 from (49.03), we get

$$\frac{\partial}{\partial \eta} \left[\frac{B^2(\eta, \psi)}{2\rho_0} \right] = (\xi^2 - \eta^2)(\xi^2 - \psi^2) \frac{\partial A(\eta, \psi)}{\partial \eta}$$

Since the left hand side is a function of η, ψ and the right hand side is a function of ξ also, we get

$$\frac{\partial A}{\partial \eta} = 0 \ , \quad \frac{\partial B}{\partial \eta} = 0 \tag{410.17}$$

Likewise using (41.06) for (41.05), we get

$$\frac{\partial A}{\partial \psi} = 0, \qquad \frac{\partial B}{\partial \psi} = 0 \qquad\qquad (410.18)$$

From (410.17) and (410.18), we get

$A = $ constant,

$B = $ constant

Therefore, the flow variables are:

$$u = \frac{B}{\rho_o (\xi^2 - \eta^2)^{1/2}(\xi^2 - \psi^2)^{1/2}},$$

$$p = A - \frac{1}{2} \frac{B^2}{\rho_o (\xi^2 - \eta^2)(\xi^2 - \psi^2)}$$

$$\text{Finally, } \nabla \times q = \frac{1}{g_1 g_2 g_3} \cdot \begin{vmatrix} g_1 e_1 & g_2 e_2 & g_3 e_3 \\ \dfrac{\partial}{\partial \xi} & \dfrac{\partial}{\partial \eta} & \dfrac{\partial}{\partial \psi} \\ \dfrac{B}{\rho_o (\xi^2 - b^2)^{\frac{1}{2}}(\xi^2 - c^2)^{\frac{1}{2}}} & 0 & 0 \end{vmatrix} = 0$$

or

the incompressible flow is an irrotational flow.

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119

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