Radiation conditions and uniqueness theorems for \(n\)-dimensional wave equation in an infinite domain.

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RADIATION CONDITIONS AND UNIQUENESS THEOREMS
FOR n-DIMENSIONAL WAVE EQUATION
IN AN INFINITE DOMAIN

by

SADANAND SRIVASTAVA

A Dissertation
Submitted to the Faculty of Graduate Studies through the Department
of Mathematics in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy at the
University of Windsor.

Windsor, Ontario
1968
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ABSTRACT

The present work is intended to be a comprehensive treatment of the Radiation Conditions, a particular case being Sommerfeld's Radiation conditions, which guarantee the unique solution of the exterior boundary value problems for the second order linear elliptic differential equations (Chapter 4)

\[ L(u) = \sum_{i,j=1}^{n} a_{ij}(X) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(X) \frac{\partial u}{\partial x_i} + c(X)u = 0 \]

in n dimensional space \( E^n \).

First of all, Sobolev's Integral formula has been generalized. In order to do so, the concept of retarded argument and auxiliary functions \( \sigma_s \) and \( \tau \) are introduced. These auxiliary functions have been discussed in the Appendix B.

Secondly, using this generalized integral formula, the radiation conditions are found which are generalization of the classical Sommerfeld's Conditions.

Then maximum principle for the solution in an unbounded domain is established, which finally leads to uniqueness theorem for the exterior boundary value problems.

In Chapter 2, the wave equation with constant coefficients is discussed, which reduces to Helmholtz equation in \( E^n \), and uniqueness of the solution is established without making use of the maximum principle (though it is possible otherwise also).
Next in Chapter 3, the reduced wave equation with variable refractive index has been considered. The radiation conditions which lead to unique solution, have been obtained. Again the maximum principle, though applicable, has not been used to prove the uniqueness of the problem. Furthermore Huyghens' Principle has been discussed in Appendix A.
ACKNOWLEDGEMENTS

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<table>
<thead>
<tr>
<th>TABLE OF CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>v</td>
</tr>
<tr>
<td>CHAPTER 1 - INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>(a) Related Literature</td>
<td>1</td>
</tr>
<tr>
<td>(b) Scope of the Present Work</td>
<td>6</td>
</tr>
<tr>
<td>CHAPTER 2 - WAVE EQUATION WITH CONSTANT COEFFICIENTS</td>
<td>8</td>
</tr>
<tr>
<td>Section 1. A Few Definitions and Statements</td>
<td>8</td>
</tr>
<tr>
<td>Section 2. Generalized Kirchhoff's Formula</td>
<td>10</td>
</tr>
<tr>
<td>Section 3. Radiation Condition</td>
<td>15</td>
</tr>
<tr>
<td>Section 4. The Uniqueness Theorem</td>
<td>18</td>
</tr>
<tr>
<td>CHAPTER 3 - WAVE EQUATION WITH VARIABLE REFRACTIVE INDEX</td>
<td>21</td>
</tr>
<tr>
<td>Section 1. Generalized Sobolev's formula</td>
<td>21</td>
</tr>
<tr>
<td>Section 2. Radiation Condition</td>
<td>25</td>
</tr>
<tr>
<td>Section 3. The Uniqueness Theorem</td>
<td>28</td>
</tr>
<tr>
<td>CHAPTER 4 - GENERAL WAVE EQUATION WITH VARIABLE COEFFICIENTS</td>
<td>33</td>
</tr>
<tr>
<td>Section 1. Estimates and Existence</td>
<td>33</td>
</tr>
<tr>
<td>Section 2. Generalization of Sobolev's Formula</td>
<td>44</td>
</tr>
<tr>
<td>Section 3. Radiation Condition</td>
<td>49</td>
</tr>
<tr>
<td>Section 4. The Maximum Principle</td>
<td>52</td>
</tr>
<tr>
<td>Section 5. The Uniqueness Theorems</td>
<td>57</td>
</tr>
</tbody>
</table>
(a) A Discussion of Related Literature

The correct formulation of boundary value problems for elliptic equations in a bounded domain has been very well investigated[1]. This is not so for an infinite domain. In this case, apart from conditions on the boundary of the domain, some conditions, at infinity have to be imposed.

The original uniqueness theorem for the reduced wave-equation:

$$\Delta u + k^2 u = 0$$  \hspace{1cm} (1.1)

where $\Delta \equiv \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ and $k$ real; in an infinite domain was stated by Sommerfeld [24,25], and along with the radiation condition, an additional 'finiteness-condition' was assumed, uniformly with respect to direction.

$$\lim_{r \to \infty} r \cdot (\frac{\partial u}{\partial r} - iku) = 0$$  \hspace{1cm} (1.2)

$$\lim_{r \to \infty} r \cdot |u| = \text{const}$$  \hspace{1cm} (1.3)

The theorem was first proved in this form by Magnus [14].

The elimination of the extraneous condition (1.3) was accomplished shortly afterwards by Rellich [21] resulting in the theorem;

'Let $G$ be the exterior of a finite surface $B$. There exists at most one function $u = u(x_1,x_2,...,x_n)$ defined in $\overline{G} = G \cup \partial B$, such that;

(a) $u \in C^2(G)$

(b) $\Delta u + k^2 u = 0$, $k$ real
(c) $u$ assumes given values on $B$

(d) \[ \lim_{r \to \infty} r^{n-1} \left( \frac{\partial u}{\partial r} - iku \right) = 0, \] uniformly along all rays from the origin and where $r^2 = x_1^2 + x_2^2 + \cdots + x_n^2$.

(e) $B$ and $u$ satisfy such regularity conditions as to ensure the validity of the following case of Green's formula:

\[
\iiint_{G_{\rho}} \left\{ \nabla u \cdot u \nabla u - u \nabla u \cdot \nabla u \right\} dV = \iint_{\partial(G_{\rho}, \chi)} \left( \frac{\partial u}{\partial \nu} - u \frac{\partial u}{\partial \nu} \right) \, ds
\]

$G_{\rho}$ is the domain exterior to $B$ and interior to the hypersphere with centre at the origin and radius $\rho$ is so large that the hypersphere contains $B$ in its interior.

Later [21], he replaced the radiation condition (d) by a modified radiation condition in integral form.

\[
\lim_{r=\rho \to \infty} \iint_{\Sigma(\rho, \chi)} \left| \frac{\partial u}{\partial r} - iku \right|^2 \, ds = 0
\]

(1.4)

where $\Sigma$ is the hyper-surface of a sphere of radius $\rho$ with some fixed point in space as centre and $ds$ is a surface element of the class $C^2$.

He obtained his proof by showing that

\[
\lim_{r=\rho \to \infty} \iint_{\Sigma(\rho, \chi)} |u|^2 \, ds \neq 0 \text{ a.e. unless } u \equiv 0
\]

(1.5)

This result is referred to below as the 'Rellich Growth Estimate'. His result seems to have settled the uniqueness problem for (1.1) in the case of finite boundary. We might remark here that Magnus dealt with the exterior Dirichlet problem for smooth closed boundaries, and assumed the existence and continuity of $\frac{\partial u}{\partial \nu}$ on the boundary. Rellich also restricted himself to the Dirichlet problem, but did not give explicit conditions for the boundary surface, or for the behaviour of the
wave-function at the boundary. Instead, he gave the implicit condition (e). Both authors assumed \( k \) to be real. This restriction was removed by several authors of whom Atkinson [2] appears to have been the first.

Magnus [15], demonstrated that the radiation condition implied the 'finite-ness' condition. Thus implicitly contained in his paper is a proof of the representation theorem, using only the radiation condition.

Miranker [18] presented a new proof of the 'Rellich Growth Estimate' and showed how the uniqueness theorem followed from it. His proof makes no use of expansion in spherical harmonics and seems to be simpler than Rellich's original proof. The method of proof is similar to one which is used in deriving the mean value theorem for solutions of (1.1). He obtained the representation of \( u \) as a sum of single and double layers by means of the same method. Miranker's proof of this theorem has the advantage that apart from the radiation condition it makes no use of asymptotic properties of \( u \).

When \( k \) is no longer a constant, but a function of space variables, Eq. (1.1) changes to

\[
\Delta u + k^2(x)u = 0
\]

(1.6)

where \( X = (x_1, x_2, x_3, \ldots, x_n) \)

Miranker [19] considered the equation (1.6) in three-dimensional space, in which \( k^2(x) \) was locally square integrable and was such that:

\[
|k^2(x) - h^2| = O(|x|^\mu)
\]

(1.7)

where \( h^2 > 0 \) and \( \mu > 3 \) are constants.
He proved that if, \( u \) was defined and bounded in whole-space and if

\[
\frac{1}{4\pi} \iiint_{G} \frac{|k^2(x) - h^2|}{|x|} \, dV < 1
\]

(1.8)

then

\[
\lim_{|x| \to \infty} \iint_{S} |u|^2 \, ds \neq 0 \quad \text{unless } u \equiv 0.
\]

(1.9)

Equation (1.9) is an equivalent of the 'Rellich's Growth Estimate', [21]. Miranker concluded that under (1.8) no non-trivial bounded solution of (1.6) could exist which was quadratically integrable over the whole space. Furthermore he asserted that no non-trivial bounded solution of (1.6), could exist satisfying the radiation-conditions.

\[
\lim_{|x| \to \infty} \iint_{S} \left| \frac{\partial u}{\partial |x|} - ik(x)u \right|^2 \, ds = 0
\]

(1.10)

and

\[
\lim_{|x| \to \infty} \iint_{S} \left| \frac{\partial u}{\partial |x|} - hu \right|^2 \, ds = 0
\]

(1.11)

He also considered the case in which \( u \) was defined in the complement of \( G \) and demonstrated the existence of a solution of \( \Delta v + h^2 v = 0 \), such that \( v \to u \) as \( |x| \to \infty \). The rate of this approach depends on the rate with which \( k^2(x) \to h^2 \) at infinity. This 'Osculation Theorem' leads to validity of the 'Rellich's Growth Estimate' and a consequence is the uniqueness theorem for the exterior Dirichlet and Neumann problem for \( u \).
Consider now, the general elliptic equation of second order which we write in the form.

\[ L(u) = \sum_{i,j=1}^{n} a_{ij}(\chi) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\chi) \frac{\partial u}{\partial x_i} + c(\chi)u = 0 \quad (1.12) \]

By the exterior boundary problem we mean the following:

Let \( G \) be a given open region in the \( n \)-dimensional space \( \mathbb{E}^n \), containing a neighbourhood of infinity and having a smooth internal boundary \( \partial G \). A continuous function \( \phi(\chi) \) is prescribed on \( \partial G \). We wish then to find a function \( u = u(\chi) \) which is of class \( C^2 \) and satisfies.

\[ L(u) = 0 \text{ in } G \quad (1.13) \]
\[ u = \phi \text{ on } \partial G. \quad (1.14) \]

The problem is further specified by certain conditions on the behaviour of \( u \) at \( \infty \), such as

(i) The function \( u \) shall tend to an assigned limit 1 as \( x \to \infty \).

A problem is said to be well-posed for a given equation, if for any assigned data, there exists exactly one corresponding solution.

It is simple to observe that problem (i) is well-set for the Laplace equation in \( n(n\geq 3) \) variables while this does not hold for the general linear elliptic equation

\[ L(u) = 0 \]

nor even for this special form

\[ \sum_{i,j=1}^{n} a_{ij}(\chi) \frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \quad (1.15) \]

It is necessary to impose some restrictions on the behaviour of the solutions at infinity (Meyers and Serrin [16], Krzyanski [11]).
It is apparent from the survey of the literature that not much attention has so far been directed to exterior boundary value problems. Furthermore, the usual method of solution, particularly for the reduced wave equations \( \Delta u + k^2 u = 0 \) and \( \Delta u + k^2(\chi)u = 0 \) has the drawback that it assumes radiation conditions a priori.

(b) Scope of Present Work

The present work is intended to be a systematic treatment of a generalization of Sommerfeld's Radiation Conditions* which guarantee the unique solution of the problem.

In Section 1 of Chapter 2, we recall a few definitions which are often useful in this connection.

In Section 2, we generalize the Kirchhoff's formula [23] for the wave equation in n-dimensional euclidean space; and Section 3 deals with the formulation of Sommerfeld's Radiation Condition.

In Section 4, we state and prove the uniqueness theorem in an unbounded region, for the wave equation in n-dimensional space.

Chapter 3 deals with the wave equations with variable refractive indices, in n-dimensional space.

Section 1, is concerned with the generalization of Sobolev's formula [22].

The next two sections deal with the deduction of Sommerfeld's Radiation Condition and the uniqueness theorem in an unbounded region.

* Here Sommerfeld's condition becomes a particular case.
Finally Chapter 4, deals with the general elliptic equation (reduced wave equation) in n-dimensional space.

In Section 1, we establish some estimates and give a complete statement of the uniqueness theorem.

Section 2 deals with the formula which is similar to generalized Sobolev's formula of Chapter 3.

Section 3 deals with the deduction of Radiation conditions.

Sections 4 and 5 deal with the Maximum Principle in an unbounded region and with uniqueness theorems for the exterior boundary value problems.

Furthermore, we have added two appendices. The first appendix deals with Huyghen's Principle and the second one with the auxiliary functions $\sigma$ and $\tau$; which are of significance in obtaining the generalized Sobolev's formula in Chapters 3 and 4.
CHAPTER 2

This chapter deals with the n-dimensional wave equation in an infinite domain $G$ with a finite boundary $\partial G$.

$$\Delta \phi = \frac{1}{c^2} \phi_{tt}$$

where

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2} .$$

and $C$ is an arbitrary constant.

1. A Few Definitions and Statements

Definition 1: Let $R$ be a region of euclidean n-dimensional space $\mathbb{E}_n$. A complex-valued function $u$ on $R$ is in $C^{(m)}(R)$ for integral $m \geq 0$, if

(a) for $0 \leq k \leq m$, the $k^{th}$ order partial derivatives $D^k u$, exist and are continuous in $R$, $\partial u = u$, being continuous in $R$

(b) for $x \in R^c$, $\lim_{x \to x'} D^k u$ exists where $R^c$ denotes the interior of $R$.

Definition 2: Let $A$ be a subset of $\mathbb{E}_n$. A complex-valued function $u$ on $A$ is said to satisfy a Hölder condition with exponent $\lambda$, $0 < \lambda < 1$, if

$$\sup_{x \neq y \in A} \frac{|u(x) - u(y)|}{|x - y|^\lambda} < \infty$$

This least upper bound is called Hölder Constant; $u$ is said to be 'Hölder Continuous' with exponent $\lambda$ in a region $R$, if it satisfies a 'Hölder Condition' in every compact subset of $R$. For integral $m \geq 0$ and $0 < \lambda < 1$, $C^{(m+\lambda)}(R)$ will denote the sub-class of functions in $C^{(m)}(R)$ whose derivatives of order $m$ when continuously extended to $R$, are 'Hölder continuous' with exponent $\lambda$ in $R$.

Definition 3: A $C^{(m+\lambda)}$ hyper-surface element is a set of points in $\mathbb{E}_n$. 

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which for some system of cartesian co-ordinates \((x_1, x_2, \ldots, x_n)\) admits a representation; 
\[ x_n = F(x_1, x_2, \ldots, x_{n-1}) \]
where \(x_1, x_2, \ldots, x_{n-1} \in \mathbb{G}\) and \(F \in C^{m+\lambda}(G)\). Here \(G\) is a domain in \(\mathbb{E}_{n-1}\) such that boundary \(\partial G\) is a rectifiable Jordan curve. In addition we require that it be possible to extend \(F\) to a function \(\widetilde{F}\) defined in a domain \(\widetilde{G}\) containing \(G\) where 
\[ \widetilde{F} \in C^{m+\lambda}(\widetilde{G}) \].

**Definition 4:** A subset \(B\) of \(\mathbb{E}_n\) is a closed surface if it is a connected; compact \((n-1)\)-manifold i.e., every point of \(B\) has a neighbourhood whose intersection with \(B\) is a \((n-1)\)-cell.

A closed surface \(B\) in \(\mathbb{E}_n\) separates the set \((\mathbb{E}_n - B)\) into two components, one of which is un-bounded and called the exterior of \(B\). The bounded component is the interior of \(B\).

**Definition 5:** For functions defined in some set \(A\) we introduce the norms:
\[
\|u\|_0^A = \sup_{A} |u(\chi)| \\
\|u\|_p^A = \sum_{j \leq p} |D^j u|_0^A \\
\|u\|_\lambda^A = |u|_0^A + [u]_\lambda^A \\
\|u\|_{p+\lambda}^A = \sum_{j \leq p} |D^j u|_\lambda^A
\]

where
\[
[u]_\lambda^A = \sup_{x \neq y \in A} \frac{|u(x) - u(y)|}{|x-y|^\lambda}.
\]
and \(D^j u\) denotes any of the \(j^{th}\) partial derivatives of \(u\). \([u]_\lambda^A\) is often referred to as the Hölder coefficients of \(u\) in \(A\). We also define the norm \(\|\phi\|_q^A, q \leq p\), to be the supremum of the \(q\)-norms of \(u\) on each portion of \(A\) admitting the local representation mentioned in Definition 3.
Definition 6: The boundary \( \partial G \) of the domain \( G \) will be said to enjoy the property \((P)\) if there exist positive constants \( a \) and \( \theta \in (0,1) \) such that the inequality

\[
\text{mes}(K(r) \cap G) \leq (1-\theta)\text{mes}K(r).
\]

holds for any sphere \( K(r) \) with centre on \( \partial G \) and radius \( r < a \), where \( \text{mes} \) \( A \) denotes the Lebesgue measure of the set \( A \) in \( E^n \).

In addition the following notation will be adopted unless otherwise mentioned.

The open sphere of radius \( \rho \) and centre \( X \) will be denoted by \( S(\rho,X) \) its boundary by \( \partial(\rho,X) \). \( \overline{A} \) and \( \overset{0}{A} \) denote the closure and interior of \( A \) respectively.

2. Generalized Kirchhoff's Formula

This section deals with an integral formula* which is very useful in our future discussion.

We consider the equation:

\[
\Delta \phi = \frac{1}{c^2} \phi_{tt}
\]

(2.1)

where \( C \) is a constant in an infinite domain with finite boundary. Let \( P \) denote the point in \( n \)-dimensional euclidean space \( E^n \), with co-ordinates \((x_1, x_2, \ldots, x_n)\). We shall consider in addition, the space \( E^{n+1} \) with co-ordinates \((x_1, x_2, \ldots, x_n, t)\) or \((p, t)\). The characteristic cone, for (2.1) with vertex \((p, t_0)\) has the equation

\[
t_0 = t - r/c,
\]

(2.2)

where \( r^2 = (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + \ldots + (x_n - x_n^0)^2 \).

* This formula is valid for all \( n \geq 3 \), though dimensionality of the space plays a significant role, particularly with respect to Huyghens' Principle (See Appendix A).

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We denote by [ \_ ], retarded value of the function, e.g., the retarded value of \( \varphi \) is denoted by

\[
[\varphi(P,t)] = \varphi(P,t - \frac{r}{c})
\]  

(2.3)

Certain relationships between the function satisfying (2.1) and its derivatives hold on its characteristics-surface. We establish these relations for the derivatives of \( \hat{\varphi} \) with respect to \( t \) defined by

\[
\hat{\varphi} = \left[ \frac{\partial \varphi}{\partial t} \right] \quad \text{and} \quad \hat{\varphi}_0 = [\hat{\varphi}]
\]  

(2.4)

where \( \hat{\varphi} \) will be regarded as functions in \( E_n \).

If we construct a central field of variation with centre \( p_0 \), we have the fundamental equation

\[
(\text{grad} \frac{\varphi}{c})^2 = \frac{1}{c^2} \frac{\Delta \varphi}{c^2}
\]  

(2.5)

On carrying out the differentiation of \( \hat{\varphi} \) with respect to co-ordinates both directly and via the argument \( (t-r/c) \), we can use the fairly obvious expression:

\[
\text{grad} \hat{\varphi}_{p+1} \cdot \text{grad} \frac{\varphi}{c} = [\text{grad} \frac{\partial \varphi^{p+1}}{\partial \text{tp}^{p+1}}] \cdot \text{grad} \frac{\varphi}{c} - [\frac{\partial \varphi^{p+2}}{\partial \text{tp}^{p+2}}] \cdot (\text{grad} \frac{\varphi}{c})^2
\]  

(2.6)

where the dot (\( \cdot \)) denotes the scalar product in \( n \)-dimensional space \( E_n \).

Then we obtain the equation:

\[
\Delta \hat{\varphi}_p = -2 \text{grad} \hat{\varphi}_{p+1} \cdot \text{grad} \frac{\varphi}{c} - \hat{\varphi}_{p+1} \Delta \frac{\varphi}{c}
\]  

(2.7)

We introduce an operator \( L \), defined by:

\[
L(v) = -2 \text{grad} v \cdot \text{grad} \frac{\varphi}{c} - v \Delta \frac{\varphi}{c}
\]  

(2.8)

Applying (2.8), Eq. (2.7) can be re-written as

\[
\Delta \hat{\varphi}_p = L(\hat{\varphi}_{p+1})
\]  

(2.9)
These are in fact the relationships which are satisfied on the cone (2.2) (Smirnov [22]). The operator \( L \) satisfies
\[
vL(\omega) + \omega L(v) = -2 \text{div}(\omega \text{grad} \frac{L}{c})
\] (2.10)

We have for powers of \( r \):
\[
\Delta(r^p) = (n+p-2)p r^{p-2}
\] (2.11)
\[
L(r^p) = \frac{1}{c} (n+2p-1)r^{p-1}
\] (2.12)

Auxiliary functions \( \sigma_m \) are being defined as:
\[
\sigma_m = r^{-n+2}
\]
which implies
\[
\Delta(\sigma_m) = 0
eq m = n
\] (2.13a)
\[
\sigma_m = \frac{(n-3)(n-5)\cdots(2m-4)(2m-2) r^{n-1}}{\Gamma(-\frac{n-1}{2})(n-3)(n-4)\cdots(n-1+3m-3)}
\] (2.13b)
which implies
\[
L(\sigma_3) = 0
\] (2.14a)
and
\[
L(\sigma_{m+1}) = \Delta(\sigma_m) \quad \text{for} \quad 3 < m < n
\] (2.14b)

Let \( G \) be a domain in \( \mathbb{E}^n \), bounded by \( S \). We form an integral of multiplicity \( n, (n \geq 3) \)
\[
\iint_G \mathbf{P}^{1-1} \left\{ \Phi_{p-1} \Delta \sigma_{n-p+1} - \sigma_{n-p+1} \Delta \Phi_{p-1} \right\} + \left( \frac{\Phi_{p-1} L(\sigma_{n-p+1})}{p} + \sigma_{n-p+1} L(\Phi_{p-1}) \right) \, dV
\]
where
\[
dV = dx_1 \, dx_2 \cdots dx_n
\]
Applying divergence theorem, we obtain
\[
\iiint_G \sum_{p=1}^{n-2} (-1)^{p-1} \left\{ (\partial_{n-p+1} \sigma_{n-p+1} - \sigma_{n-p+1} \partial_{n-p+1}) \right\} d\mathbf{v} \\
+ \left\{ \partial_p \sigma_{n-p+1} \right\} d\mathbf{v} \\
= \oint_{\partial G} \sum_{p=1}^{n-2} (-1)^{p-1} \left\{ (\partial_{n-p+1} \sigma_{n-p+1} - \sigma_{n-p+1} \partial_{n-p+1}) \right\} d\mathbf{s}
\]
(2.15)

where \( \frac{\partial}{\partial \mathbf{v}} \) denotes the out-going normal derivative at a boundary point in which \( \phi \) is defined and
\[
ds^2 = dx_1^2 + dx_2^2 + \ldots + dx_n^2.
\]

It is easy to show that
\[
\frac{\partial \phi}{\partial \mathbf{v}} = \left[ \frac{\partial \phi}{\partial \mathbf{v}} \right] + \left[ \frac{\partial \phi}{\partial t} \right] \frac{\partial \mathbf{r}}{\partial \mathbf{v}}.
\]
(2.16)

and since left-hand side of (2.15) vanishes because of (2.13) and (2.14) we obtain
\[
\int_{\partial G} \sum_{p=1}^{n-2} (-1)^{p-1} \left\{ (\partial_{n-p+1} \sigma_{n-p+1} - \sigma_{n-p+1} \partial_{n-p+1}) \right\} d\mathbf{s} = 0
\]
(2.17)

In particular, when \( n=3, p=1 \)
\[
\int_{\partial G} \left\{ \frac{\partial \phi}{\partial \mathbf{v}} - \frac{1}{r} \left[ \frac{\partial \phi}{\partial \mathbf{v}} \right] - \frac{1}{r} \left[ \frac{\partial \phi}{\partial t} \right] \frac{\partial r}{\partial \mathbf{v}} \right\} d\mathbf{s} = 0
\]

Let us consider the interior of \( G \) i.e., in \( \mathbf{G} \). We draw a sphere with radius \( \epsilon \), arbitrarily small. Then applying formula (2.17)
over the deleted surface \((B-\Sigma_{\epsilon})\), we obtain
\[
\int \int_{B-\Sigma_{\epsilon}} (-1)^{p-1} \left\{ \frac{\partial \gamma_{n-p+1}}{\partial \nu} \phi_{n-p+1} - \frac{\partial \phi_{n-p+1}}{\partial \nu} \right\} ds = 0 \quad \text{(2.18)}
\]

In order to evaluate the first integral of (2.18), we set

\[
\begin{align*}
\mathbf{x}_1 - \mathbf{x}_1^0 &= r \cos \theta_1 \\
\mathbf{x}_2 - \mathbf{x}_2^0 &= r \sin \theta_1 \cos \theta_2 \\
\mathbf{x}_3 - \mathbf{x}_3^0 &= r \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
&\vdots \\
\mathbf{x}_{n-1} - \mathbf{x}_{n-1}^0 &= r \sin \theta_1 \sin \theta_2 \sin \theta_3 \\
\mathbf{x}_n - \mathbf{x}_n^0 &= r \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_{n-1} \sin \phi
\end{align*}
\]

where

\[
0 \leq \theta_n \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad n = 1, 2, \ldots, (n-1).
\]

which implies

\[
ds = r^{n-1} \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \sin^{n-2} \theta_3 \ldots \sin \theta_{n-2} \sin \theta_{n-1} \sin \phi \nu_1 \nu_2 \ldots \nu_{n-1}
\]

Then we have

\[
\lim_{\epsilon \to 0} \int_{\Sigma_{\epsilon}} (-1)^{p-1} \left\{ \frac{\partial \gamma_{n-p+1}}{\partial \nu} \phi_{n-p+1} - \frac{\partial \phi_{n-p+1}}{\partial \nu} \right\} ds = 2(n-2)(\pi)^{n/2} \frac{\nu(x_1^0, x_2^0, \ldots, x_n^0, t_0)}{\Gamma(n/2)}
\]

Finally from (2.18) we obtain
\[ \tilde{\psi}(p_0, t_0) = \frac{\Gamma(B)}{2(n-2) \pi^{n/2}} \int \int_{B} \sum_{p=1}^{n-2} (-1)^{p-1} \left\{ \tilde{\psi}_{p-1} \frac{\partial \sigma_{n-p+1}}{\partial v} \right\} ds \]

- \sigma_{n-p+1} \left[ \frac{\partial}{\partial v} \tilde{\psi}_{p-1} \right] - \sigma_{n-p+1} \frac{\partial \tilde{\psi}}{\partial v} \left[ \frac{\partial}{\partial v} \right] ds \quad (2.19)\]

3. Sommerfeld's Radiation Condition (Generalized)

Let us suppose we have a steady-state outside some bounding surface \( \partial G = B \). We draw a sphere \( S(\rho, X) \) with centre \( P \), lying outside \( B \), radius \( \rho \) so large that the hyper-sphere contains \( B \) in its interior.

Now we apply formula (2.19) over the surface between \( B \) and interior of \( S(\rho, X) \), and we obtain

\[ \tilde{\psi}(p_0, t_0) = \frac{\Gamma(B)}{2(n-2) \pi^{n/2}} \int \int_{B+p = \rho} \sum_{p=1}^{n-2} (-1)^{p-1} \left\{ \tilde{\psi}_{p-1} \right\} \frac{\partial \sigma_{n-p+1}}{\partial v} \]

- \sigma_{n-p+1} \left[ \frac{\partial \tilde{\psi}_{p-1}}{\partial v} \right] - \sigma_{n-p+1} \frac{\partial \tilde{\psi}}{\partial v} \left[ \frac{\partial}{\partial v} \right] ds = 0 \]

If we put

\[ \tilde{\psi} = e^{-i\omega t} u(x) \]

then equation (2.1) changes to equation (2.20) i.e.

\[ \Delta u + k^2 u = 0 \quad (2.20) \]

where

\[ k^2 = \frac{\omega^2}{c^2}. \]

Using the concept of the retarded argument, we set:

* The constant term before the integral has some mis-print in the original paper of Sobolev [23].
\[
[\tilde{\phi}] = e^{-i\omega(t-x)}u
\] (2.21a)

\[
[\frac{\partial^P}{\partial t^P}] = (-i\omega)^P e^{-i\omega(t-r/c)}u
\] (2.21b)

\[
[\frac{\partial \tilde{\phi}}{\partial \nu}] = e^{-i\omega(t-r/c)} \frac{\partial u}{\partial \nu}
\] (2.21c)

Making use of relations (2.19) and (2.21), we obtain

\[
 u(P_0) = \frac{\Gamma(\frac{3}{2})}{2(n-2)n^2} \left( \int \int_{B} \sum_{p=1}^{n-2} (i\omega)^{p-1} e^{ikr} \left\{ \frac{\partial \sigma_{n-p+1}}{\partial \nu} u + ik \sigma_{n-p+1} \frac{\partial r}{\partial \nu} u \right\} ds \right)
\]

\[
\left\{ \sigma_{n-p+1} \frac{\partial u}{\partial \nu} + ik \sigma_{n-p+1} \frac{\partial r}{\partial \nu} u \right\} ds
\]

(2.22)

It is quite natural to require that the integral over \(\Sigma(\rho,\chi)\) tends to zero as \(r = \rho\) tends to infinity. This is fulfilled if we subject \(u\) to the following condition:

\[
\lim_{r=\rho \to \infty} \sum_{p=1}^{n-2} (i\omega)^{p-1} e^{ikr} \left\{ \frac{\partial \sigma_{n-p+1}}{\partial \nu} u + ik \sigma_{n-p+1} \frac{\partial r}{\partial \nu} u \right\} + ik \sigma_{n-p+1} u = 0
\] (2.23)

which is essentially Sommerfeld's Radiation Condition* that (in its particular form) was intuitively imposed in order to get a unique solution for (2.20).

Now equation (2.23) can be written as:

---

* This condition suffices to prove the uniqueness theorem, but further simplifications help deduce the classical Sommerfeld's Radiation Condition.
If
\[ \lim_{\rho \to \infty} \left\{ e^{ikr\rho^{-1}} \left( u \frac{\partial \sigma_n}{\partial v} - \sigma_n \frac{\partial u}{\partial v} + ik \sigma_n \frac{\partial r}{\partial v} + (i\omega) e^{ikr} \right) \right\} = 0 \]

the whole expression (2.23) also tends to zero. Therefore it suffices to consider (2.24) as radiation condition.

If \( u \) is a solution of
\[ \Delta u + k^2 u = 0. \]
then \( \overline{u} \) conjugate of \( u \) is also a solution of
\[ \Delta \overline{u} + k^2 \overline{u} = 0 \]
and if \( u \) satisfies the radiation condition
\[ \lim_{\rho \to \infty} e^{ikr\rho^{-1}} \left\{ r^{\frac{n-1}{2}} \left( u \frac{\partial \sigma_n}{\partial v} - \sigma_n \frac{\partial u}{\partial v} + ik \sigma_n \frac{\partial r}{\partial v} \right) \right\} = 0 \]
so does \( \overline{u} \) satisfy the corresponding radiation condition
\[ \lim_{\rho \to \infty} e^{-ikr\rho^{-1}} \left\{ r^{\frac{n-1}{2}} \left( \overline{u} \frac{\partial \sigma_n}{\partial v} - \sigma_n \frac{\partial \overline{u}}{\partial v} - ik \sigma_n \frac{\partial r}{\partial v} \right) \right\} = 0 \]

Multiplying (2.24) and (2.25) we have
\[ \lim_{r \to \infty} r^{\frac{n-1}{2}} \left\{ r^{\frac{n-1}{2}} \left( u \frac{\partial \sigma_n}{\partial v} - \sigma_n \frac{\partial u}{\partial v} + ik \sigma_n \frac{\partial r}{\partial v} \right) \right\}^2 = 0 \]

And we have the modified radiation condition in the integral form:
\[ \lim_{r \to \infty} \int_{\Sigma \rho} r^{\frac{n-1}{2}} \left( u \frac{\partial \sigma_n}{\partial v} - \sigma_n \frac{\partial u}{\partial v} + ik \sigma_n \frac{\partial r}{\partial v} \right)^2 ds = 0 \]

(2.26)
It may be remarked that the radiation condition (2.26) is weaker condition i.e., equation (2.24) implies equation (2.26) but converse is not true in general. Moreover, equation (2.26) may be further simplified to

$$\lim_{r \to \infty} \int_{\Sigma_p} \left\{ \frac{1}{2} \left( n-1 \right) \frac{1}{r} u - \frac{\partial u}{\partial v} + i ku \right\}^2 ds = 0$$

or following Atkinson [2].

$$\lim_{r \to \infty} \int_{\Sigma_p} \left\{ \frac{\partial u}{\partial r} - i ku \right\}^2 ds = 0$$

(2.27)

4. The Uniqueness Theorem

Theorem 2.1 Let \( G \) be the exterior of a regular bounded surface \( B \). There exists at most one function \( u(x) = u(x_1, x_2, \ldots, x_n) \), defined in \( \bar{G} = G \cup B \) such that

(a) \( u \in C^2(G) \)

(b) any one of the following assumes a prescribed value on \( B \):

(i) \( u \)

(ii) \( \frac{\partial u}{\partial v} + \beta u \)

(c) \( \Delta u + k^2 u = 0, \ k \neq 0 \text{ and } \text{Re } k \geq 0; \text{ Im } k \geq 0 \)

(d) \( \lim_{r = \rho \to \infty} \left\{ \frac{n-1}{r^2} \left( \frac{\partial u}{\partial v} - i ku \right) \right\} = 0 \)

(d') \( \lim_{r = \rho \to \infty} \int_{\Sigma_p} \left\{ \frac{\partial u}{\partial r} - i ku \right\}^2 ds = 0 \)

Proof of Theorem 2.1 The radiation condition (2.27) may be written as

$$\lim_{r = \rho \to \infty} \int_{\Sigma_p} \left\{ \frac{\partial u}{\partial r}^2 + |k|^2 |u|^2 + iku \frac{\partial u}{\partial v} - iku \frac{\partial u}{\partial r} \right\} ds = 0$$

(2.28)
We can easily show that
\[
\iint_{S} \iint \bar{u} \bar{v} dV + \iint_{S} \iint \bar{u} \bar{v} dV = \iint_{S} \iint \frac{\partial u}{\partial v} ds
\]
then
\[
\iint_{S} \iint \bar{u} (-k^2 u) dV + \iint_{S} \iint \bar{v} \bar{v} dV = \iint_{S} \iint \frac{\partial u}{\partial v} ds
\]
or
\[
\iint_{S} \iint \bar{u} \frac{\partial u}{\partial v} ds = -\iint_{S} \iint \left( |k|^2 \iint_{S} \iint (|u|^2 - |v|^2) dV \right)
\]
(2.29)
The corresponding conjugate equation for (2.29) is (2.30)
\[
-\iint_{S} \iint u \frac{\partial u}{\partial v} ds = i\kappa |k|^2 \iint_{S} \iint |u|^2 dV - i\kappa \iint_{S} \iint |v|^2 dV (2.30)
\]
Adding (2.29) and (2.30) we obtain:
\[
\iint_{S} \iint \left( (\iint u \frac{\partial u}{\partial v} - i\kappa \frac{\partial u}{\partial v}) ds = -i \kappa |k|^2 \iint_{S} \iint (k-k) \iint_{S} \iint |u|^2 dV - i\kappa \iint_{S} \iint |v|^2 dV \right)
\]
\[
= 2 i\kappa |k| \left\{ |k|^2 \iint_{S} \iint |u|^2 dV + \iint_{S} \iint |v|^2 dV \right\}
\]
(2.31)
Upon substituting (2.31) into (2.28) we get as \( r = \rho \) tends to \( \infty \)
\[
\iint_{S} \iint \left\{ \frac{\partial u}{\partial r} |^2 ds + \kappa |^2 \iint_{S} \iint |u|^2 dV + 2 i\kappa |k| \iint_{S} \iint |u|^2 dV \right\}
\]
\[
+ \iint_{S} \iint |v|^2 dV = 0
\]
(2.32)
Since all the terms of (2.32) are non-negative, it follows...
that if, \( \text{Im} \ (k) > 0 \), then
\[
\iint_{\Omega} |u|^2 \, dV = 0
\]
whence
\[
u \equiv 0
\]
If \( \text{Im} \ (k) = 0 \), then we have
\[
\lim_{|r| \to \infty} \int_{\Gamma} |u|^2 \, ds = 0 \tag{2.33}
\]
But by Rellich's well-known Growth Estimate [21] (2.33) implies
\[
u \equiv 0, \text{ in } \overline{\Omega},
\]
which proves the theorem 2.1.
CHAPTER 3

The third chapter is concerned with the n-dimensional wave equation* with variable refractive index

$$\Delta \Phi = \frac{1}{C(X)} \Phi_{tt}$$

Here $C(X)$ is a function of space variables, and which plays an important role. We find a representation formula in terms of auxiliary functions $\sigma_m$ and $\tau$. The auxiliary functions are discussed in Appendix B.

1. A Generalized Sobolev's Formula

We consider the equation,

$$\Delta \Phi = \frac{1}{C^2(X)} \Phi_{tt} \tag{3.1}$$

where $C(X)$ is a function of space variables only, in an infinite domain with finite boundary. Let $P$ denote the point in n-dimensional euclidean space $E_n$, with co-ordinates $(x_1, x_2, \ldots, x_n)$. We shall consider in addition, the space $E_{n+1}$ with co-ordinates $(x_1, x_2, \ldots, x_n, t)$ or $(P, t)$.

We can construct a field for the variational problem. Let $\tau(P, P_0)$ be the basic function of the central field with centre $P_0$. Then

$$\tau(P, P_0) = \text{constant} \tag{3.2}$$

* Discussion of Appendix A is applicable to this chapter also with a slight transformation [6] but we are no longer so sure about the clean-cut character of the wave propagation.
gives quasi-spheres with centre \( P_0 \), and the metric defined by

\[
J = \int_{P_0} \frac{ds}{C(x)} = \int_{P_0} \sqrt{\sum_{i=1}^{n} (dx_i^2)} \cdot \frac{1}{C(x)}
\]  

(3.3)

We have auxiliary functions \( \tau \) and \( \sigma_m \) defined by the equations:

\[
\text{grad}^2 \tau(P, P_0) = \frac{1}{C^2(P)} \quad \tag{3.4}
\]

\[
\frac{n}{2} \sum_{i=1}^{n} \frac{\partial \sigma_0}{\partial x_i} \frac{\partial \tau}{\partial x_i} + \sigma_0 \sum_{i=1}^{n} \frac{\partial^2 \tau}{\partial x_i^2} = 0
\]

(3.5)

and

\[
\frac{n}{2} \sum_{i=1}^{n} \frac{\partial \sigma_m}{\partial x_i} \frac{\partial \tau}{\partial x_i} + \sigma_m \sum_{i=1}^{n} \frac{\partial^2 \tau}{\partial x_i^2} + \frac{n}{2} \sum_{i=1}^{n} \frac{\partial^2 \sigma_{m-1}}{\partial x_i^2} = 0
\]

(3.6)

where \( \sigma_m = 0 \) for \( m = -1, -2, -3, \ldots \).

Let \( \tilde{\psi}(P, t) \) be a solution of (3.1), then using the concept of retarded argument we have

\[
[\tilde{\psi}_{tt}] = C^2(P) [\Delta \tilde{\psi}] \quad \tag{3.7}
\]

We choose \( \sigma_m(P) \) so that

\[
\sigma_m(P) \Delta \tilde{\psi} = \text{div}(-2 \frac{\partial \tilde{\psi}}{\partial t} \sigma_m \text{grad} \tau)
\]

(3.8)

Let \( G \) be a domain in \( E_n \), bounded by \( B \). We form an integral of multiplicity \( n, (n \geq 3) \).

\[
\iint_{G} \sum_{p=1}^{n-2} (-1)^{p-1} \left\{ \tilde{\psi}_{p-1} \Delta \sigma_{n-p+1} - \sigma_{n-p+1} \Delta \tilde{\psi}_{p-1} \right\} d\nu
\]

where \( d\nu = dx_1 \cdot dx_2 \cdot dx_3 \ldots \ldots dx_n \).

Applying divergence theorem to above integral, we obtain

\[
* \quad \text{When } C(X) \rightarrow C \text{ of chapter 2, Eqs. (3.5) and (3.6) correspond to Eqs. (2.13) and (2.14), where } 3 < m < n.
\]
\[ \iint_0^{n-2} \sum_{p=1}^{(-1)^{p-1}} \left\{ \frac{\partial \phi}{\partial v} n-p+1 - \sigma_{n-p+1} \right\} \, dv \]

\[ \quad = \int \int_0^{n-2} \sum_{p=1}^{(-1)^{p-1}} \left\{ \frac{\partial \sigma_{n-p+1}}{\partial v} - \sigma_{n-p+1} \right\} \, ds \]

where \( \frac{\partial \phi}{\partial v} \) denotes the out-going normal derivative at a boundary point of a region in which \( \sigma \) is defined and

\[ ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + \ldots + dx_n^2. \]

Then with the help of relation (3.8), (3.9) yields

\[ \iint_0^{n-2} \sum_{p=1}^{(-1)^{p-1}} \left( \frac{\partial \sigma_{n-p+1}}{\partial v} - \sigma_{n-p+1} \right) \, dv = \int \int_0^{n-2} \sum_{p=1}^{(-1)^{p-1}} \left\{ 2 \frac{\text{div} \frac{\partial \phi}{\partial v} n-p+1 \text{grad} \tau} {\partial v} \right\} \, dv \]

which implies

\[ \int \int_0^{n-2} \sum_{p=1}^{(-1)^{p-1}} \left\{ \frac{\partial \sigma_{n-p+1}}{\partial v} - \sigma_{n-p+1} \right\} \, dv = \int \int_0^{n-2} \sum_{p=1}^{(-1)^{p-1}} \left( \frac{\partial \phi}{\partial v} n-p+1 \right) \, dv = 0 \]

(3.10)

It is easy to show that

\[ \frac{\partial \phi}{\partial v} = [\frac{\partial \phi}{\partial v} - \frac{\partial \phi}{\partial \tau}] \frac{\partial \tau}{\partial v} \]

(3.11)

Consequently, we have from (3.10)
All our discussions relate only to a neighbourhood of \( P \), in which the extremals of integral do not cut themselves and form a field satisfying the following conditions:

\[
\sigma_m(P, P_0) = \tau(P, P_0) c^2, \quad \text{such that} \quad \lim_{P \to P_0} \sigma_m(P, P_0) \tau(P, P_0) = \frac{1}{c^2(P_0)}
\]

\[
|\Delta \sigma_{n-p+1}(P, P_0)| \leq \frac{N}{\tau(P, P_0)}, \quad N \text{ is an arb. constant}
\]

\[
\lim_{P \to P_0} \int_{\partial B} \frac{\partial}{\partial n} \sigma_n \ ds = -\frac{2(n-2)\pi^{n/2}}{\Gamma(n/2)}
\]

Let \( \hat{\Phi}(P, t) \) be a solution of the equation (3.1) in \( \Omega \), bounded by \( B \). Let \( P_0 \) be an interior point of \( \Omega \). Suppose that a central field exists, containing the domain \( G \) and that we have functions \( \sigma_m \) satisfying (3.13). We exclude from \( \Omega \) a small sphere \( S_\varepsilon \) with centre \( P_0 \) and radius \( \varepsilon \). We now apply formula (3.12) over the surface \( (B - \Sigma P_0) \) to obtain

\[
\int_{\Sigma} (-1)^{p-1} \left\{ \sigma_{n-p+1} \left[ \frac{\partial \hat{f}_{p-1}}{\partial n} \right] - \frac{\partial \sigma_{n-p+1}}{\partial n} \right\} ds
\]

\[
+ \int_{\Sigma} (-1)^{p-1} \left\{ \sigma_{n-p+1} \left[ \frac{\partial \hat{f}_{p-1}}{\partial n} \right] - \frac{\partial \sigma_{n-p+1}}{\partial n} \right\} ds
\]

\[
+ \int_{D^*=(\Omega - S_\varepsilon)} (-1)^{p-1} \left[ \frac{\partial \sigma_{n-p+1}}{\partial n} \right] dv = 0
\]
Since \( [\phi], [\phi] \frac{\partial t}{\partial v} \) are bounded and
\[
\tau(p, p_0) = O(\varepsilon) \text{ on } \Sigma_e
\]
\[
\sigma_m(p, p_0) = O(1/\varepsilon) \text{ on } \Sigma_e
\]

We finally obtain [22], from Eq. (3.14) the generalized Sobolev's formula for equation (3.1).

\[
\frac{[\phi]}{[\phi]_\Sigma} \Omega = \frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}} \int_\Sigma \int_{p=1}^{n-2} (\Sigma (-1)^{p-1} \left\{ \frac{\partial \sigma_n-p+1}{\partial v} - \sigma_n-p+1 \right\} \frac{\partial t}{\partial v} \left[ [\phi] \frac{\partial \phi}{\partial v} \right] \right) \right) \right) ds + \frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}} \int_\Sigma \int_{p=1}^{n-2} (\Sigma (-1)^{p-1} \left\{ \frac{\partial \sigma_n-p+1}{\partial v} \right\} \right) \right) \right) ds
\]

\[
\left( [\phi]_p \right)_\Sigma \sigma_n-p+1 \right) \right) \right) dv \quad (3.15)
\]

2. Generalized Sommerfeld's Radiation Condition

Let us suppose we have a steady-state outside some surface \( B \). We draw a sphere \( S(p, X) \) with centre \( P \), lying outside \( B \) and radius \( p \) so large that the hypersurface contains \( B \) in its interior.

Now applying formula (3.15) over the surface between \( B \) and interior of \( S(p, X) \), we obtain

\[
\frac{[\phi]}{[\phi]_\Sigma} \Omega = \frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}} \int_\Sigma \int_{p=1}^{n-2} (\Sigma (-1)^{p-1} \left\{ \frac{\partial \sigma_n-p+1}{\partial v} - \sigma_n-p+1 \right\} \right) \right) \right) ds + \frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}} \int_\Sigma \int_{p=1}^{n-2} (\Sigma (-1)^{p-1} \left\{ \frac{\partial \sigma_n-p+1}{\partial v} \right\} \right) \right) \right) ds
\]

If we put
\[
\phi = e^{-i\omega(t)}u(x) \quad (3.16)
\]

into equation (3.1), we obtain

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\[ \Delta u + k^2(X)u = 0 \quad (3.17) \]

where \( k^2 = \left(\frac{\omega}{c}\right)^2 \).

Using the concept of the retarded argument we define following

\[ [\phi] = e^{-i\omega(t-\tau)}u \quad (3.18a) \]

\[ \left[ \frac{\partial \phi}{\partial t} \right] = (i\omega)P_e^{-i\omega(t-\tau)}u \quad (3.18b) \]

\[ \left[ \frac{\partial \phi}{\partial v} \right] = e^{-i\omega(t-\tau)}\frac{\partial u}{\partial v} \quad (3.18c) \]

which yields

\[ u(P_0) = \frac{\Gamma(n/2)}{2\pi^{n/2}} \int \int \int \frac{\Sigma(i\omega)^{p-1} e^{i\omega t}}{B + \Sigma(\rho,X)} \left\{ \frac{\partial \sigma_{n-p+1}}{\partial v} - \sigma_{n-p+1} \frac{\partial u}{\partial v} + i\omega \frac{\partial \sigma_{n-p+1}}{\partial v} u \right\} ds \]

\[ + \int \int \int \frac{\Sigma(i\omega)^{p-1} \left\{ u \Delta \sigma_{n-p+1} \right\}}{B + \Sigma(\rho,X)} dv \quad (3.19) \]

It is quite natural to require that the integral over \( \Sigma(\rho,X) \) tends to zero as \( r = \rho \) tends to infinity. Also that integral over \( S(\rho,X) \) is bounded as \( r = \rho \) tends to infinity.

This is fulfilled, if we subject \( u \) to following condition:

\[ \lim_{r = \rho \to \infty} \int \int \int \frac{\Sigma(i\omega)^{p-1} e^{i\omega t \rho^{-1}}}{B + \Sigma(\rho,X)} \left\{ \frac{\partial \sigma_{n-p+1}}{\partial v} - \sigma_{n-p+1} \frac{\partial u}{\partial v} + i\omega \frac{\partial \sigma_{n-p+1}}{\partial v} u \right\} ds = 0 \quad (3.20) \]

the general form of Sommerfeld's Radiation Condition, which was in its particular form, intuitively imposed by many authors to get a unique
solution for (3.17). Now in Eq. (3.20), if
\[
\lim_{r \to \infty} e^{i \omega r} n^{-1} \left\{ \frac{\partial \sigma}{\partial n} n - \frac{\sigma}{n} \frac{\partial u}{\partial n} + i \omega \frac{\sigma}{n} \frac{\partial}{\partial n} \right\} = 0
\]
then the whole expression (3.20) tends to zero and therefore it suffices to consider
\[
\lim_{r \to \infty} e^{i \omega r} r^{n-1} \left\{ \frac{n-1}{r^2} \left( u \frac{\partial \sigma}{\partial n} n - \frac{\sigma}{n} \frac{\partial u}{\partial n} + i \omega \frac{\sigma}{n} \frac{\partial}{\partial n} \right) \right\} = 0 \ (3.21)
\]
as the necessary radiation condition, instead of the whole expression (3.20).

If \( u \) is a solution of
\[
\Delta u + k^2(x)u = 0
\]
the conjugate of \( u, \overline{u} \) is also a solution of
\[
\Delta \overline{u} + k^2(x)\overline{u} = 0
\]
and it satisfies the corresponding radiation condition
\[
\lim_{r \to \infty} e^{-i \omega r} r^{n-1} \left\{ \frac{n-1}{r^2} \left( \overline{u} \frac{\partial \sigma}{\partial n} n - \frac{\sigma}{n} \frac{\partial \overline{u}}{\partial n} - i \omega \frac{\sigma}{n} \frac{\partial}{\partial n} \right) \right\} = 0 \ (3.22)
\]
Multiplying (3.21) and (3.22) we have:
\[
\lim_{r \to \infty} \left\{ \frac{n-1}{r^2} \left( u \frac{\partial \sigma}{\partial n} n - \frac{\sigma}{n} \frac{\partial u}{\partial n} + i \omega \frac{\sigma}{n} \frac{\partial}{\partial n} \right) \right\} = 0
\]
which implies
\[
\lim_{r \to \infty} \int_{r^2}^{n-1} \left| \frac{n-1}{r^2} \left( u \frac{\partial \sigma}{\partial n} n - \frac{\sigma}{n} \frac{\partial u}{\partial n} + i \omega \frac{\sigma}{n} \frac{\partial}{\partial n} \right) \right| ds = 0 \ (3.23)
\]
Now (3.23) is known as modified radiation condition, in the integral form. It may be noted that the radiation condition (3.23) is
weaker condition i.e., we can deduce equation (3.23) from Eq. (3.21) but converse is not true in general. Moreover, Eq. (3.23) may be further simplified to

\[ \lim_{r \to p^{-\infty}} \int_{\Sigma_p} \left| \frac{n-1}{2} \left( \frac{\partial u}{\partial v} - i\omega \frac{\partial}{\partial v} \sigma_n \right) \sigma_n \right|^2 ds = 0 \quad (3.24) \]

following Atkinson [2].

3. The Uniqueness Theorem

**Theorem 3.1** Let \( G \) be the exterior of a regular bounded surface \( B \). There exists at most one function \( u(x) = u(x_1, x_2, \ldots, x_n) \), defined in \( \overline{G} = G \cup B \) such that

(a) \( u \in C^2(G) \) \quad \( B = \partial G \)

(b) any one of the following assumes a prescribed value on \( B \\

(i) \( u \); or

(ii) \( \frac{\partial u}{\partial v} + \beta u \)

(c) \( \Delta u + k^2(x)u = 0, k \neq 0 \) and \( \text{Re } k > 0; \text{ Im } k > 0 \)

(d) \( \lim_{r \to p^{-\infty}} \left\{ \frac{e^{i\omega r}}{r^{n-1}} \left( u \frac{\partial \sigma_n}{\partial v} - \sigma_n \frac{\partial u}{\partial v} + i\omega \frac{\partial}{\partial v} \sigma_n \right) \right\} = 0 \)

or

\( \lim_{r \to p^{-\infty}} \int_{\Sigma_p} \left| \frac{n-1}{2} \left( \frac{\partial u}{\partial v} - i\omega \frac{\partial}{\partial v} \sigma_n \right) \sigma_n \right|^2 ds = 0 \)

We shall state and prove two lemmas which lead to the final proof of the theorem 3.1.

**Lemma 1.** Let \( u \) be a bounded solution of (3.17) and satisfy the radiation condition.
\[
\lim_{r=\rho \to \infty} \int_{\Sigma} \frac{n-1}{r^2} (u \frac{\partial u}{\partial n} - \frac{\partial u}{\partial n} + i\omega \frac{u}{\partial n} \frac{\partial u}{\partial n})^2 \, ds = 0
\]

then

\[
\lim_{r=\rho \to \infty} \int_{\Sigma} |u|^2 \, ds \neq 0 \text{ unless } u \equiv 0 \quad (3.25)
\]

**Proof:** We have to impose further restrictions [19] by assuming that

\[
k^2(x) = h^2 + \rho(x) \quad (3.26)
\]

and

\[
\lim_{|x| \to \infty} \rho(x) = 0
\]

if

\[
|k^2(x) - h^2| = 0 \quad (|x|^{-\mu}) \quad \mu > 3 \text{ and } \mu \text{ is function of } \nu.
\]

Then there exists a solution \( v \) of \( \Delta v + h^2 v = 0 \) such that

\[
u - v = 0(|x|^{-\mu + 2 + \varepsilon}) \quad v \text{ satisfying radiation condition;}
\]

\( \varepsilon \) is an arbitrary small positive number. Furthermore \( v \neq 0 \), if \( u \neq 0 \).

From these observations, easily we define

\[
u = v + v_0 \quad (3.27)
\]

where

\[
\Delta v + h^2 v = 0 \quad (3.28)
\]

and

\[
|v_0| = 0(|x|^{-\mu + 2 + \varepsilon})
\]

Let us suppose \( u \neq 0 \). If \( \overline{v} \) and \( \overline{v}_0 \) are complex conjugates of \( v \) and \( v_0 \) respectively, we have

\[
\int_{\Sigma} \int_{|r|=|x|} |u|^2 \, ds = \int_{\Sigma} \int_{|r|=|x|} |v + v_0|^2 \, ds = \int_{\Sigma} \int_{|r|=|x|} (v + v_0)(\overline{v} + \overline{v}_0) \, ds
\]

\[
= \int_{\Sigma} \int_{|r|=|x|} |v|^2 \, ds + \int_{\Sigma} \int_{|r|=|x|} (v \overline{v} + v_0 \overline{v}_0) \, ds + \int_{\Sigma} \int_{|r|=|x|} |v_0|^2 \, ds
\]

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Since \( u^0, v^0 \); therefore the 'Rellich's Growth-Estimate for solutions of the reduced wave equation

\[
\Delta v + \frac{h^2}{4} v = 0
\]

asserts [21] that

\[
\lim_{r \to \infty} \int \int_{|x| = r} |v|^2 \, ds = 0
\]

Moreover, since \( \mu > 3 \) and \( \epsilon \) is an arbitrary small positive number, there exists a \( \delta > 0 \) such that

\[
|v_0| = 0 \left( |x|^{-1-\delta} \right)
\]

Thus as \( p \) tends to \( \infty \), the magnitude of the last two integrals, (using Schwarz inequality we can easily show it), in the right hand side of Eq. (3.29) is of lower order than

\[
\int \int_{\Sigma|\omega|} |v|^2 \, ds.
\]

Hence Eq. (3.29) implies that

\[
\lim_{r \to \infty} \int \int_{\Sigma|\omega|} |u|^2 \, ds \neq 0
\]

**Lemma 2.** Let \( u \) be a bounded solution of (3.17) and satisfy the radiation condition

\[
\lim_{r \to \infty} \int \int_{\Sigma|\omega|} \left\{ \frac{n-1}{2r^2} \sigma_n \frac{\partial u}{\partial \omega} - i\omega \frac{\partial u}{\partial \omega} |u|^2 \right\} \, ds = 0
\]

Assume that either \( u \) or its normal derivative vanish on \( B \). Then

\[
\lim_{r \to \infty} \int \int_{\Sigma|\omega|} \left\{ \frac{n-1}{2r^2} \sigma_n \left( \left| \frac{\partial u}{\partial \omega} \right|^2 + \omega^2 |u|^2 \right)^2 \right\} \, ds = 0 \quad (3.30)
\]
Proof: Let $\bar{u}$ denote the complex conjugate of $u$, then by Green's Theorem

$$
\iint_{\Sigma|x|} \{ \bar{u}\Delta u - u\Delta \bar{u} \} \, dv = \iint_{\Sigma|x|} \left( \frac{\partial u}{\partial v} - u \frac{\partial \bar{u}}{\partial v} \right) ds + \iint_{\Sigma|x|} \left( \frac{\partial \bar{u}}{\partial \rho} - u \frac{\partial u}{\partial \rho} \right) ds
$$

In $s(\rho, X)$, $u = \bar{u} = 0$; on $B$ and we note

$$
\frac{\partial u}{\partial v} = \frac{\partial u}{\partial \rho}
$$
or

$$
\frac{\partial u}{\partial v} = \frac{\partial u}{\partial \rho} = 0, \text{ on } B.
$$

Therefore:

$$
\iint_{\Sigma|x|} \left( \bar{u} \frac{\partial u}{\partial \rho} - u \frac{\partial \bar{u}}{\partial \rho} \right) ds = 0 \tag{3.31}
$$

Expanding the radiation condition, we obtain,

$$
\lim_{\rho \to \infty} \int_{\Sigma|x|} \left( \frac{n-1}{r^2} \frac{1}{\sigma_0} \right)^2 \left\{ \left| \frac{\partial u}{\partial \rho} \right|^2 + \omega^2 \left( \frac{\partial \bar{u}}{\partial \rho} \right)^2 |u| \right. \left. ^2 + i\omega \frac{\partial \bar{u}}{\partial \rho} \cdot \right\} ds = 0
$$
or using equation (3.31) we have

$$
\lim_{r \to \rho \to \infty} \int_{\Sigma|x|} \left( \frac{n-1}{r^2} \frac{1}{\sigma_0} \right)^2 \left\{ \left| \frac{\partial u}{\partial \rho} \right|^2 + \omega^2 \left( \frac{\partial \bar{u}}{\partial \rho} \right)^2 |u| \right. \left. ^2 \right\} ds = 0
$$

Proof of Theorem 3.1

The proof of theorem 3.1 is an immediate consequence of Lemma 2. In fact, it follows from Schwarz's inequality that the difference of two solutions of (3.17), which also satisfy the radiation condition, itself satisfies the radiation condition, i.e., if we have $u_1, u_2$ as two solutions, satisfying
and
\[
\lim_{\rho \to \infty} \int \int_{x} \left\{ \frac{n-1}{r^2} \sigma_n \left( \frac{\partial u}{\partial \rho} - i\omega \frac{\partial \tau}{\partial \rho} \right) u_1 \right\} ds = 0
\]

If \( w = u_1 - u_2 \) then it is easy to show that:
\[
\lim_{\rho \to \infty} \int \int_{x} \left\{ \frac{n-1}{r^2} \sigma_n \left( \frac{\partial u_2}{\partial \rho} - i\omega \frac{\partial \tau}{\partial \rho} \right) u_2 \right\} ds = 0
\]

Thus if \( u \) is a solution of (3.17) which satisfies the radiation condition and which itself or whose normal derivative is zero along \( B \), Lemma 2 implies that
\[
\lim_{\rho \to \infty} \int_\Sigma |x| \left\{ \frac{n-1}{r^2} \sigma_n \left( \frac{\partial \tau}{\partial \rho} \right)^2 \left| u \right|^2 \right\} ds = 0
\]

Since
\[
\left( \frac{n-1}{r^2} \sigma_n \right)^2 = O(N) \quad N \text{ an arb. const.}
\]

and
\[
\omega^2 \left( \frac{\partial \tau}{\partial \rho} \right)^2 = O(k^2(x)) \quad \sigma_m, \tau \text{ real}
\]

then under the assumption (3.26) we conclude that
\[
\lim_{|x| = \rho \to \infty} \int |u|^2 ds = 0
\]

which implies that
\( u = 0 \)

and hence the theorem 3.1.
CHAPTER 4

The main purpose of this chapter is to generalize the results of Chapters 2 and 3 and to deduce the radiation condition for the second order linear elliptic (reduced wave) equation in n-dimensional euclidean space and to show that the exterior boundary value problem is well posed.

1. Estimates and Existence*

(a) Estimates

The Schauder Estimates: We state here the Schauder-type estimates pertaining to the solutions of linear elliptic equations with Hölder continuous coefficients which are needed in the existence proofs developed below. However, Schauder’s interior estimates and also the boundary data (Dirichlet) are so familiar to us that we omit here the description of these estimates, referring to the articles of Douglis and Nirenberg[4] and Graves[9] for the details. We shall describe therefore, only the Schauder type estimates near the boundary for solutions carrying non-Dirichlet boundary data, for the proof of this estimate we refer to Agmon, Douglis and Nirenberg[1].

Consider the equation

\[ L(u) = \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = 0 \]  

(4.1)

* Author does not claim any originality of this section. All these results have been introduced here mainly because of their newness and usefulness in our future discussion.
in the domain $G$, together with the boundary condition on $\partial G$.
\begin{equation}
M(u) = \frac{\partial u}{\partial N} + h(X)u = \varphi(X)
\end{equation}
(4.2)
and let the following conditions be satisfied:

(i) At all $X \in G$, and for all real $n$-tuples $\eta_i$
\begin{equation}
\sum_{i=1}^{n} a_{ij}(X)\eta_i \eta_j \geq \lambda \sum_{i=1}^{n} \eta_i^2; \lambda > 0
\end{equation}

(ii) The coefficients $a_{ij}, b_i, c(X)$ belong to $C^{1+\lambda}(\bar{G})$
and in addition $a_{ij}$ is in $C^{2+\lambda}(\bar{G})$.

(iii) The functions $h(X)$ and $\varphi(X)$ belong to $C^{1+\lambda}(\partial G)$.

(iv) The boundary $\partial G$ of $G$ belongs to $C^{2+\lambda}$.

Then we state our estimate as a lemma.

**Lemma 1.** Let $D$ be a sub-region of $\bar{G}$ such that $\partial D \cap \partial G$ is not void. If $u(X)$
is a bounded solution in $C^{2+\lambda}(\bar{G})$ of the equation (4.1) satisfying the
boundary condition (4.2) on $\partial G$ then we have the estimates:
\begin{equation}
|u|^D_{2+\lambda} \leq C'(|\varphi|^\partial G + |u|^G_{1+\lambda})
\end{equation}
where $C'$ is a constant depending on the geometry of $\partial G$ and $D$ and also
on the following quantities (assumed to be finite)
\begin{align*}
|a_{ij}|^G_{1+\lambda}, |b_i|^G_{1+\lambda}, |a_{ij}|^\partial G_{1+\lambda}, |h|^\partial G_{1+\lambda}
\end{align*}

**Ladyzhenskaya and Uraltseva's Estimates:** Recently
Ladyzhenskaya and Uraltseva [12,13] have established very powerful
theorems securing the assessment of the Hölder norms of solutions of a

*Proofs of lemmas and theorems of this section, are either omitted
or just sketched, for complete proof refer to [1], [4], [12] and [13].*
wide class of linear and non-linear elliptic equations.

Here we shall state some of them (Lemmas 2, 3 and 4) with the help of which we shall estimate the Hölder continuity of solutions of linear equations. These lemmas are called for because of their strength in controlling the behaviour of solutions both in the interior and near the boundary of the region.

We shall say that a function \( u(X) \) belongs to \( H^2(G,M,v,\delta) \) if the following conditions are fulfilled:

(i) \( u(X) \in W^1_2 \), locally in \( G \).

(ii) \( \max |u(X)|^2 \leq M \)

(iii) for any concentric spheres \( k(r) \subseteq k(R) \subseteq G \) with radii \( r \) and \( R (0 < r < R) \), \( u(X) \) satisfies the inequalities

\[
\int_{A_K,r} |\nabla u|^2 \, dX \leq v \, \text{mes}(A_K,R) \left\{ \max_{A_K,R} \left( -K u(X)^2 \right) \right\} \cdot (R-r)^{-2} + 1
\]

whenever

\[
K \geq \max_{k(R)} u(X) - \delta
\]

\[
\int_{B_K,r} |\nabla u|^2 \, dX \leq v \, \text{mes}(B_K,R) \left\{ (R-r)^{-2} \max_{B_K,R} \left( -u + K \right)^2 + 1 \right\}
\]

whenever

\[
K \leq \min_{P} u(X) + \delta
\]

Here \( M,v,\delta \) are fixed positive numbers independent of any particular choice of \( u(X) \) and \( A_K,\rho \) and \( B_K,\rho \) denote the sets \( \{ x \in K(\rho) \} \) and \( \{ x \in k(\rho) \ ; u(X) < K \} \) respectively.

**Lemma 2.** Let \( x^0 \) be an arbitrary interior point of \( G \), \( r_0 = \text{distance} (x^0, \partial G) \) and \( u(X) \) be any function of class \( H^2(G,M,v,\delta) \).
Then the following estimate

\[ \text{Osc} \{ u(x) ; k(r) \} \leq C \left( \frac{1 + r_0}{r_0} \right)^\lambda \cdot r^\lambda \]  

(4.6)

holds for any sphere \( k(r) \) with centre at \( x \) and radius \( r \leq r_0 \) where the constants \( C \) and \( \lambda \in (0,1) \) depend on the parameters of the class \( H_2 \) only. By \( \text{Osc} \{ u(x) ; k(r) \} \) we denote the oscillation of a function \( u(x) \) on a set \( k(r) \). This lemma determines a uniform interior Hölder continuity of all the functions belonging to class \( H_2(G,M,v,\delta) \), whereas a Hölder continuity near the boundary for such a class will be established in the following lemmas:

**Lemma 3.** Suppose that the boundary \( \partial G \) of \( G \) enjoys the property \( (P) \). Suppose a function \( u(x) \) in \( H_2(G,M,v,\delta) \) satisfies a Hölder condition with exponent \( \beta \) on \( \partial G \). Suppose further that for any concentric sphere \( k(r) \subset k(R), r \leq R \leq a \), centered on \( \partial G \), \( u(x) \) satisfies the inequality (4.4) for all \( k \) such that

\[ k \geq \max_{k(R)} u(x) - \delta \quad \text{and} \quad k \geq \max_{k(R) \cap \partial G} u(x) \]

and also the inequality (4.5) for all \( k \) such that

\[ k \leq \min_{k(R) \cap \partial G} u(x) + \delta \]

and

\[ k \leq \min_{k(R) \cap \partial G} u(x) \]

then we have

\[ \text{Osc} \{ u(x) ; k(r) \} \leq C r^\lambda \]

(4.7)

for every sphere \( k(r) \) with centre on \( \partial G \) of \( G \) and radius \( r < a \) where \( C \) and \( \lambda \in (0,1) \) depend only on the parameters \( M,v,\delta,\beta,a,\theta \).
**Lemma 4.** Let the boundary \( \partial G \) of \( G \) belong to \( C^1 \) and assume that a function \( u(x) \) in \( H_2(G;\Omega,\nu,\delta) \) possesses the following properties:

For any concentric spheres \( K(r) \subseteq K(R) \) with centre on \( \partial G \) and radii less than a fixed number \( a_0 \),

\[
\int_{G \cap A_{k,r}^G} |\nabla u|^2 \, dx \leq \nu \, \text{mes} \left( G \cap A_{k,r}^G \right) \left\{ (R-r)^{-2} \max_{G \cap A_{k,r}^G} (u(X)-k)^2 + 1 \right\}
\]

(4.8)

for all \( k \) such that

\[
k \geq \max_{G \cap K(R)} u(X) - \delta
\]

and

\[
\int_{G \cap B_{k,r}^G} |\nabla u|^2 \, dx \leq \nu \, \text{mes} \left( G \cap B_{k,r}^G \right) \left\{ (R-r)^{-2} \max_{G \cap B_{k,r}^G} (u(X)-k)^2 + 1 \right\}
\]

(4.9)

for all \( k \) such that

\[
k \leq \min_{G \cap K(R)} u(X) + \delta
\]

In this case \( u(X) \) satisfies the inequality (4.7) where \( C \) and \( \lambda \in (0,1) \) depend only on \( M, \nu, \delta, a_0 \) and the geometry of \( \partial G \).

It is of interest to note that in Lemma 4, no information is needed concerning the regularity behaviour of \( u(X) \) on the boundary \( \partial G \) of \( G \). Respecting the very difficult and sophisticated details of these estimates, we refer to the papers by Ladyzhenskaia and Ural'tseva [12,13].

**Hölder Estimates:** Consider a linear equation

\[
\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (A_{ij}(X) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^{n} B_i(X) \frac{\partial u}{\partial x_i} + C(X) u = F
\]

in an exterior region \( G \) and the boundary condition

\[
\frac{\partial u}{\partial \nu} + \lambda u = q
\]

(4.10)

* (4.10) transforms to

\[
\sum_{i,j=1}^{n} a_{ij}(X) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(X) \frac{\partial u}{\partial x_j} + C(X) u = q
\]

provided \( A_{ij} = a_{ij} \) and \( \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_{ij} + B_j(X) = b_j(X) \) etc.
on the boundary $S$ of $G$.

**Lemma 5.** Consider the first boundary value problem (4.10) and (4.11) for which the following hypotheses are assumed.

(i) There is a constant $A > 0$ such that

\[
\sum_{i,j=1}^{n} A_{ij} \eta_i \eta_j \geq A \sum_{i=1}^{n} \eta_i^2
\]

for all $x \in G$ and all n-tuples $\eta$.

(ii) $A_{ij}, B_i, C$ and $F$ are bounded and measurable in $G$.

(iii) The boundary function $\phi \in C^\beta(S)$.

(iv) The boundary $S$ satisfies property (P).

Under these assumptions a bounded solution $u(x)$ of the problem (4.10) - (4.11) is uniformly Hölder continuous in any interior sub-regions as well as in a boundary strip of $G$. The interior Hölder estimates of $u(x)$ depend only on the bounds on $A_{ij}, B_i, C, F$, the ellipticity $\lambda$, the dimension $n$ and $\max |u(x)|$, while the boundary estimates depend still on the geometry of $S, |\phi|^\beta$ and $\beta$.

The proof of this theorem will be omitted, for following closely Ladyzhenskaia and Uraltseva, we are immediately led to a situation where Lemmas 2 and 3 are applicable.

**Lemma 6.** Consider the linear equation (4.10) in an exterior region and the boundary condition

\[
M(u) = \frac{\partial u}{\partial N} + h(x) u = \psi(x)
\]

on the boundary $S$ of $G$, for which the following hypotheses are assumed:
(i) There is a constant \( \lambda > 0 \), such that
\[
\sum_{i,j=1}^{n} A_{ij}(x) \eta_i \eta_j \geq \lambda \sum_{i=1}^{n} \eta_i^2
\]
for all \( x \in G \) and all \( n \)-tuples \( \eta \).

(ii) \( A_{ij}(x) \) is bounded and belongs to \( C^2(\overline{G}) \). The functions \( B_i(x) \), \( C(x) \) and \( F(x) \) are bounded and measurable in \( \overline{G} \).

(iii) The boundary \( S \) belongs to \( C^3 \).

Then a bounded solution \( u(x) \) of the problem (4.10) - (4.12) satisfies a uniform Hölder condition not only in any interior sub-regions but in a neighborhood of the boundary \( S \). The Hölder continuity in the interior depends only on the same quantities as in Lemma 5, while the Hölder continuity near the boundary depends still on the geometry of \( S \) and on \( A_{ij}^G, |H|^G, |\Psi|^G \); which we assumed to be finite, where \( h(x) \) and \( \Psi(x) \) are the traces on \( S \) of the functions \( H(x) \) and \( \Psi(x) \).

Proof:

We have only to establish the estimates near the boundary, because the interior estimates can be obtained regardless of the particular boundary condition and are essentially included in Lemma 5. To do this, we proceed as follows:

Let \( S' \) be any portion of \( S \) having the parametric representation as below:
\[
x_i = g_i(t_1, t_2, \ldots, t_{n-1}), g_i \in C^3
\]

(4.13)

We straighten out the portion \( S' \) by the smooth one-to-one transformation \( x \leftrightarrow t \), defined by
\[
x_i = g_i(t_1, t_2, \ldots, t_{n-1}) + t h_i(t_1, t_2, \ldots, t_{n-1}), i = 1, 2, \ldots, n
\]

where
\[ h_i = \frac{1}{A(X)} \sum_{ij} A_{ij} \cos(v_j x_i) \bigg|_{x_i=g_i} \]

and

\[ A(X) = \sqrt{\sum_{ij} A_{ij} \cos(v_j x_i)^2} \]

This transformation takes a neighbourhood of \( S' \) into a neighbourhood \( T \) of a plane element \( \mu \) on \( t_n = 0 \) (of course \( \mu \) corresponds to \( S' \)). We observe that the direction of the conormal for the transformed equation agrees with that of the normal to \( \mu \). Under such change of variables the equation (4.10) goes into an equation of the form

\[ \sum_{p,q=1}^{n} \frac{\partial}{\partial t_p} (A_{pq} \frac{\partial u}{\partial t_q}) + \sum_{p=1}^{n} \frac{\partial u}{\partial t_p} + \bar{u}(t) u + \bar{H}(t) = 0 \quad (4.14) \]

and the boundary condition on \( S' \) transforms into

\[ \frac{\partial u}{\partial t_n} = \bar{H}(t) u + \bar{\Psi}(t), \quad t \in \mu. \quad (4.15) \]

where

\[ \bar{H}(t) = H(t)/A(t) \quad \text{and} \quad \bar{\Psi}(t) = \Psi(t)/A(t) \]

We now multiply both sides of (4.14) by a function \( \eta(t) \in W^1_2(T) \) vanishing on the boundary \( \partial T \) of \( (T-\mu) \), and integrate the resulting identity over \( T \). Then, by Stokes' Theorem

\[ -\int_\mu \frac{\partial u}{\partial t_n} \eta \, d \Sigma - \int_T A_{pq} \frac{\partial u}{\partial t_q} \, dt + \int_T (\bar{H} \frac{\partial u}{\partial t_p} + \bar{u} + \bar{T}) \eta \, dt = 0 \]

where \( d \Sigma \) denotes the area element on the hyperplane, \( t_n = 0 \) and the summation convention is assumed. The surface integral

\[ -\int_\mu \frac{\partial u}{\partial t_n} \eta \, d \Sigma = -\int_\mu (\bar{H} u + \bar{T}) \eta \, d \Sigma \]
in view of (4.15), which in turn equals to
\[ \int_T \frac{\partial^n}{\partial t^n} (\nabla u + \nabla \psi) \eta dt \]

Thus we obtain
\[ \int_T \left\{ - \frac{\partial^u}{\partial t^n} \frac{\partial u}{\partial t} + (\nabla u + \nabla \psi) \eta + \frac{\partial^2}{\partial t^n} (\nabla u + \nabla \psi) \eta \right\} dt = 0 \]  

(4.16)

Substituting in place of \( \eta(t) \) a function which equals \( (u(t) - k)p^2(t) \) when \( u(t) \geq k \) and vanishes when \( u(t) < k \) where \( p(t) \) is a non-negative function vanishing outside a sphere \( K(R) \) with centre on \( \mu \) and radius sufficiently small, we obtain the inequality:
\[ \int_{T \cap K(R)} |\nabla u|^2 \rho^2(t) dt \leq \nu \int_{T \cap K(R)} (u(t) - k)^2 |\nabla p|^2 + \rho^2(t) \int_{T \cap K(R)} dt \]

for all \( k \) such that
\[ k \geq \max_{T \cap K(R)} u(t) - \delta; \]
\( \delta \) being an appropriate constant. It is now easy to derive the inequality (4.8) from the above (4.17). The inequality (4.9) may be verified quite analogously. Hence we have the desired assertion of the Lemma 6 by virtue of Lemma 4.

(b) Existence Theorems:

(i) Existence theorem for the third boundary value problem:

Theorem 1. There exists a bounded solution \( \tilde{u}(X) \in C^{2+\lambda} (\Omega) \) of the exterior third boundary value problem for linear equations of the form
\[ L(u) = \sum_{i,j=1}^{n} a_{ij}(X) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(X) \frac{\partial u}{\partial x_i} + C(X)u = f(X) \]

and the boundary (non-Dirichlet) condition

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\[ M(u) = \frac{\partial u}{\partial n} + h(X) u = \psi(X), \quad x \in S, \]

with the following requirements:

(a) There is a positive constant \( \lambda \) such that
\[
\sum_{i,j} a_{ij}(X) \eta_i \eta_j \geq \lambda \sum_{i=1}^{n} \eta_i^2
\]
for every \( x \in \overline{G} \) and every \( n \)-tuple \( \eta \).

(b) The coefficients of \( L \) and \( f(X) \) are bounded and uniformly Hölder continuous with exponent \( \alpha \) in \( G \).

(c) \( c(X) \leq -m < 0 \) in \( G \) and \( h(X) < 0 \) on \( S \), \( m \) being a constant.

(d) The functions \( h(X), \psi(X) \) and \( a_{ij}(X) \) belong to \( C^{1+\alpha}(S) \).

(e) The boundary \( S \) belongs to \( C^{2+\alpha} \).

Proof: Let \( \{ K_n \} \) be a sequence of concentric spheres centered at the origin and with radii growing monotonically towards infinity as \( n \to \infty \) and with \( S \subseteq K_1 \). By \( S_n \) we mean the boundary of \( K_n \). Consider the sequence of functions \( \{ u_n(X) \} \) defined by the relations
\[
L(u_n) = f(X) \text{ in } G \setminus K_n
\]
\[ M(u_n) = \psi(X) \text{ on } S \]
\[ u_n(X) = 0 \quad \text{on } S_n \quad n = 1, 2, \ldots \]

The required assumptions guarantee the existence of such a sequence \( \{ u_n(X) \} \) which is uniformly bounded, because it follows from the ordinary maximum principle that for \( x \in G \setminus K_n \)
\[
|u_n(X)| \leq \max \left\{ \frac{\sup f}{\inf c}, \frac{\sup \psi}{\inf h} \right\}^* \]

Applying Schauder's interior estimates to the individual solutions \( u_n(X) \) we conclude that there exists a sequence of \( \{ u_n(X) \} \)

* Whenever \( |h| = 0 \), this becomes strict inequality.
that converges uniformly on any compact subregions of $G$ to a solution $u(X)$ of the equation (4.10). That this solution really satisfies the boundary condition (4.12) is implied by the Schauder type boundary estimates (Lemma 1), since the second derivatives of $u_n(X)$ are equi-continuous in $\overline{G}\cap K_1$ by Lemma 1. This completes the proof.

(ii) **Existence theorem for the first boundary value problem:**

**Theorem 2:** Under the assumptions stated below, the first boundary value problem for the linear equation

$$L(u) = \sum_{i,j=1}^{n} a_{ij}(X) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(X) \frac{\partial u}{\partial x_i} + C(X) u = f(X)$$

with the boundary condition

$$u(X) = \phi(X) \text{, on } S,$$

and the requirements:

(a) There exists a constant $\lambda > 0$, such that

$$\sum_{i,j=1}^{n} a_{ij}(X) \eta_i \eta_j \geq \lambda \sum_{i=1}^{n} \eta_i^2$$

for all $X \in \overline{G}$ and all real $n$-tuples $\eta$.

(b) The functions $a_{ij}(X)$, $b_i(X)$, $C(X)$ and $f(X)$ are bounded and uniformly Hölder continuous with exponent $\alpha$ in $\overline{G}$.

(c) $C(X) \leq -m^2 < 0$ in $\overline{G}$; $m$ being a constant.

(d) The function $\phi(X) \in C^{2+\alpha}(S)$.

(e) The boundary $S \in C^{2+\alpha}$.

possesses a bounded solution of class $C^{2+\alpha}(\overline{G})$.

**Proof:** Let $\{K_n\}$ be a sequence of concentric spheres (used in Theorem 1). We determine the sequence of functions $\{u_n(X)\}$ by solving the Dirichlet problem.
\[ L(u) = f(X) \text{ in } \Omega \]
\[ u_n(X) = \xi(X) \text{ on } S = \partial \Omega \]
\[ u_n(X) = 0 \text{ on } S_n \quad n = 1, 2, \ldots \]

Observing that the sequence thus obtained is uniformly bounded, the assertion of the theorem will directly follow from the Schauder type interior and boundary estimates.

2. Generalization of Sobolev's Formula

Let \( S \) be a geometrically closed surface with bounded curvature in the Euclidean \( n \)-dimensional space \( \mathbb{E}_n \) of real variables \( X = (x_1, x_2, x_3, \ldots, x_n) \) and \( G \) the exterior region lying outside \( S \) and containing a neighbourhood of infinity. Let \( \overline{G} = G \cup S \). For simplicity we suppose that the origin of the co-ordinates lies inside the bounded domain \( D = \mathbb{B} \) enclosed by \( S \).

We shall consider a linear second order equation of the form

\[ \sum_{i,j=1}^{n} a_{ij}(X) \phi_{x_i x_j} + \sum_{i=1}^{n} b_i(X) \phi_{x_i} + h(X) \phi - \frac{1}{c^2(X)} \phi_{tt} = 0 \quad (4.18) \]

or

\[ L'(\phi) + h \phi - \frac{1}{c^2(X)} \phi_{tt} = 0 \]

with the following conditions:

(a) All the coefficients are defined and bounded in \( \overline{G} \).

(b) \( a_{ij}(X) \) are continuously differentiable.

* Though the formula is true for any \( n \geq 3 \), Huyghens' Principle [Appendix A] fails for even \( n \geq 2 \), of course with a few exceptions [7].
(c) The symmetric matrix $\| a_{ij} \|$, $(a_{ij} = a_{ji})$ is positive definite for any $x$ in $G$, i.e. for all $x \in G$ and for all real $n$-tuples $\eta \neq 0$,

$$\sum_{i,j=1}^{n} a_{ij}(x) \eta_i \eta_j \geq \lambda \sum_{i=1}^{n} \eta_i^2 \quad ; \quad \lambda > 0$$

Let $\xi = \xi(x) = (\xi_1(x), \ldots, \xi_n(x))$ be a direction field assigned on $S$ which is inwardly directed in the sense $\xi \cdot \nu > 0$, $\nu$ being the interior normal to $S$ at $x$ (with respect to $G$). Given such a direction field $\xi$, we define the directional derivative along $\xi$ of a continuously differentiable function $\phi(x)$ as

$$\frac{\partial u}{\partial \xi} = \xi \cdot \nabla \phi$$

where

$$\nabla \equiv \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right)$$

The directional derivative of $\phi(x)$ along the particular direction $\xi$ given by

$$\xi_i = \sum_{j=1}^{n} a_{ij}(x) \cos(v, x_j) \quad i = 1, 2, 3, \ldots, n$$

is known to be the co-normal derivative of $\phi(x)$ for the operator $L$ that we denote by

$$\frac{\partial u}{\partial N} = \sum_{i,j=1}^{n} a_{ij}(x) \cos(v, x_j) \frac{\partial \phi}{\partial x_i} \quad \text{(4.19)}$$

Next we introduce auxiliary functions $\sigma_m$ and $\tau$ which simplify our work. The basic function $\tau(P, P_0)$ of the central field satisfies the equation

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial \tau}{\partial x_i} \cdot \frac{\partial \tau}{\partial x_j} = \frac{1}{C^2(x)} \quad \text{(4.20)}$$
We choose functions $\sigma_m$ such that

$$\sigma_m \cdot L'(\dot{\psi}_p) = \text{div} \left( -2 \sigma_m \frac{\partial}{\partial t} \cdot \frac{\partial \psi_p}{\partial t} \right)$$

satisfying the following equations

$$\sum_{i,j=1}^{n} a_{ij}(x) \left\{ \frac{\partial \sigma_m}{\partial x_i} \cdot \frac{\partial \tau}{\partial x_j} + \frac{\partial \sigma_m}{\partial x_i} \frac{\partial \tau}{\partial x_j} + \sigma_m \frac{\partial^2 \tau}{\partial x_i \partial x_j} \right\} + \sum_{i=1}^{n} b_i(x) \sigma_m \frac{\partial \tau}{\partial x_i} = 0 \quad (4.21)$$

and

$$\sum_{i,j=1}^{n} a_{ij}(x) \left\{ \frac{\partial \sigma_m}{\partial x_i} \cdot \frac{\partial \tau}{\partial x_j} + \frac{\partial \sigma_m}{\partial x_i} \frac{\partial \tau}{\partial x_j} + \sigma_m \frac{\partial^2 \tau}{\partial x_i \partial x_j} \right\} + \sum_{i=1}^{n} b_i(x) \sigma_m \frac{\partial \tau}{\partial x_i} + k \sigma_m = 0 \quad (4.22)$$

Let us denote by $L'^{*}$, the adjoint of $L'$ and $D$ be a bounded domain enclosed by $S = \partial G$.

We consider an expression of the form

$$\int \int \int_{\mathbb{C}} \sum_{p=1}^{n-2} \left\{ \phi_{p-1} \cdot \sum_{p=1}^{n-1} L'(\phi_{p-1}) - \phi_{p-1} \cdot L'^{*}(\sigma_{n-1}) \right\} \, dv$$

where

$$\phi_{p-1} = \frac{\partial^{p-1}}{\partial \tau^{p-1}} \psi_p,$$

and then we apply Green's formula to obtain

$$\sigma_m \approx O(r^{-\frac{n-1}{2}}) \quad \text{and} \quad \lim_{P \to P_0} \tau(P, P_0) \sigma_m(P, P_0) = \frac{1}{c^2(P_0)} \quad (F.46)$$

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\[ \int \int \int_{D} \sum_{p=1}^{n-2} (-1)^{p-1} \left\{ a_{n-p+1} L'(\dot{\phi}_{p-1}) - \dot{\phi}_{p-1} L'(\sigma_{n-p+1}) \right\} \, dv \]

\[ = \int \int \int_{D} \sum_{p=1}^{n-2} (-1)^{p-1} \left\{ a_{n-p+1} F(\dot{\phi}_{p-1}) - \dot{\phi}_{p-1} F(\sigma_{n-p+1}) \right\} \, dv 
+ \sigma_{n-p+1} \dot{\phi}_{p-1} \, Q \} \, ds \]  \hspace{1cm} (4.23)

where

\[ P(\dot{\phi}) = \sum_{i, j=1}^{n} \left\{ a_{i, j}(x) \cos (v, x_j) \frac{\partial \dot{\phi}}{\partial x_i} \right\} \]
\[ = \frac{\partial \dot{\phi}}{\partial N} \]

and

\[ Q = \sum_{i} \left\{ b_i - \sum_{j=1}^{n} \frac{\partial a_{i, j}}{\partial x_j} \cos (v, x_j) \right\} \]

With the help of (4.24) we can re-write (4.23) as

\[ \int \int \int_{D} \sum_{p=1}^{n-2} (-1)^{p-1} \left\{ a_{n-p+1} L'(\dot{\phi}_{p-1}) - \dot{\phi}_{p-1} L'(\sigma_{n-p+1}) \right\} \, dv \]

\[ = \int \int \int_{D} \sum_{p=1}^{n-2} (-1)^{p-1} \left\{ a_{n-p+1} \frac{\partial \dot{\phi}_{p-1}}{\partial N} - \dot{\phi}_{p-1} \frac{\partial \sigma_{n-p+1}}{\partial N} \right\} \, dv 
+ \sigma_{n-p+1} \dot{\phi}_{p-1} \, Q \} \, ds \]  \hspace{1cm} (4.25)

We denote hence onward by [ ], retarded value of the function.

Also it is easy to note that

\[ \frac{\partial \dot{\phi}_{p-1}}{\partial N} = \left[ \frac{\partial \dot{\phi}_{p-1}}{\partial N} \right] - \frac{\partial t}{\partial N} \left[ \frac{\partial \dot{\phi}_{p-1}}{\partial t} \right]. \]  \hspace{1cm} (4.26)

Using the [ ] notation and relations (4.22), (4.26), Eq. (4.25) becomes
\begin{align*}
\int \int_{S} \Sigma_{p=1}^{n-2} (-1)^{p-1} \left\{ \sigma_{n-p+1} \frac{\partial \phi_{p-1}}{\partial N} - \frac{\partial \sigma_{n-p+1}}{\partial N} \left[ \phi_{p-1} \right] \right\} ds \\
+ \sigma_{n-p+1} \left[ \phi_{p-1} \right] \frac{\partial T}{\partial N} + \left[ \phi_{p-1} \right] \sigma_{n-p+1} Q \right\} ds \\
+ \int \int_{G} \Sigma_{p=1}^{n-2} (-1)^{p-1} \left\{ \left[ \phi_{p-1} \right] L^{1,2}(\sigma_{n-p+1}) \right\} dv = 0 \quad (4.27)
\end{align*}

Let \( \phi(P,t) \) be a solution of the equation (4.18) in \( G = D \), bounded by \( S \). Let \( P_0 \) be an interior point of \( D \). Suppose that a central field exists with centre \( P_0 \) containing the domain \( D \) and that we have functions \( \sigma_m \) satisfying certain conditions (see Appendix B). We exclude from \( D \) a small sphere \( D_\varepsilon \) with centre \( P_0 \) and radius \( \varepsilon \). We can apply (4.27) to \( D' = D - D_\varepsilon \), to obtain

\begin{align*}
\int \int_{S-\varepsilon} \Sigma_{p=1}^{n-2} (-1)^{p-1} \left\{ \sigma_{n-p+1} \frac{\partial \phi_{p-1}}{\partial N} - \frac{\partial \sigma_{n-p+1}}{\partial N} \left[ \phi_{p-1} \right] \right\} ds \\
+ \sigma_{n-p+1} \left[ \phi_{p-1} \right] \frac{\partial T}{\partial N} + \left[ \phi_{p-1} \right] \sigma_{n-p+1} Q \right\} ds \\
+ \int \int_{G=\varepsilon} \Sigma_{p=1}^{n-2} (-1)^{p-1} \left\{ \left[ \phi_{p-1} \right] L^{1,2}(\sigma_{n-p+1}) \right\} dv = 0 \quad (4.28)
\end{align*}

Since \( \left[ \phi_{p} \right] \); \( \left[ \phi_{p} \right] \frac{\partial T}{\partial N} \) are bounded and as \( P \to P_0 \)

\begin{align*}
\tau(P,P_0) &= O(\varepsilon) \quad \text{on } D_\varepsilon \\
\sigma_{n}(P,P_0) &= O(1/\varepsilon) \quad \text{on } D_\varepsilon \quad (4.29)
\end{align*}

Then we obtain an integral representation for \( \phi(X,t) \), as given by
\[ \phi(P_0, t_0) = \frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}} \left\{ \int_S \sum_{p=1}^{n-2} \frac{(-1)^{p-1}}{\Sigma p^{n+1}} \frac{\partial \phi_{p-1}}{\partial N} \right\} \]

which is the generalized form of Sobolev's formula.

3. Sommerfeld's Radiation Conditions (Generalized)

Let us find the conditions that must be satisfied by the solutions of the generalized wave equation at infinity. Suppose we have a steady state outside some surface \( S \). We draw a sphere \( D \) with centre at a point \( P \), lying outside \( S \) with very large radius \( \rho \), so that \( S \) lies inside \( D \). Applying (4.30) we obtain

\[ \phi(P, t) = \frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}} \left\{ \int_S \sum_{p=1}^{n-2} \frac{(-1)^{p-1}}{\Sigma p^{n+1}} \frac{\partial \phi_{p-1}}{\partial N} \right\} \]

Let us define

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\[ \psi = e^{-\omega t} u(x) \text{ such that} \]

\[
\begin{align*}
\begin{bmatrix}
\partial_{\gamma} \\
\partial_{\tau} \\
\partial_{\Delta N}
\end{bmatrix}
= & \begin{bmatrix}
e^{-i\omega (t-\gamma)} u \\
(i\omega)(p - e^{-i\omega (t-\gamma)} u \\
e^{-i\omega (t-\gamma)} u
\end{bmatrix},
\end{align*}
\]  

(4.32)

Then integrating over \( D \), we obtain

\[
\frac{r(n/2)}{2(n-2)!^{n/2}} \int_S \sum_{p=1}^{n-2} \frac{(i\omega)^{p-1}}{p!} e^{i\omega \tau} \left\{ \sigma_{n-p+1} \frac{\partial u}{\partial N} - \frac{\partial \sigma_{n-p+1}}{\partial N} u \right\} ds
\]

In order that there is no contribution from the surface integral taking the normal along the radius \( r = \rho \), it is required that we subject \( u(x) \) to the following restrictions:

\[
\lim_{r=\rho \to \infty} \sum_{p=1}^{n-2} \frac{(i\omega)^{p-1}}{p!} e^{i\omega \tau} \left\{ \sigma_{n-p+1} \frac{\partial u}{\partial N} - u \frac{\partial \sigma_{n-p+1}}{\partial N} \right\} = 0
\]

(4.33)

the general form of Sommerfeld's radiation-condition, which was, in its particular form, intuitively imposed by many authors to get a unique solution of (4.1).

Now if

\[
\lim_{r=\rho \to \infty} e^{i\omega \tau} \left\{ \sigma_n \frac{\partial u}{\partial N} - u \frac{\partial \sigma_n}{\partial N} - i\omega u \sigma_n \frac{\partial \tau}{\partial N} + u \sigma_n Q \right\} = 0
\]

then obviously because of (F.46) other terms of (4.33) also tend to zero and, therefore, it suffices to consider

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as the radiation condition, instead of the whole expression (4.33).

If \( u(x) \) is a solution of

\[
L'(u) + (h + \frac{\omega^2}{c^2(x)}) u = 0,
\]

with \( k(x) = h + \frac{\omega^2}{c^2(x)} \),

then conjugate of \( u, \bar{u} \) is also a solution of

\[
L'(\bar{u}) + (h + \frac{\omega^2}{c^2(x)}) \bar{u} = 0
\]

and it satisfies the corresponding radiation condition

\[
\lim_{r \to \infty} e^{i \omega r} r^{n-1} \left\{ \sigma_n \frac{\partial u}{\partial N} - i \omega \sigma_n u \frac{\partial r}{\partial N} + u \sigma_n Q \right\} = 0
\]

as \( r = \rho \to \infty \)

(4.34)

Multiplying (4.34) and (4.35), we have

\[
\lim_{\rho \to \infty} \left\{ r^{n-1} \left| r^2 \left( \sigma_n \frac{\partial u}{\partial N} - u \frac{\partial \sigma_n}{\partial N} - i \omega \sigma_n \frac{\partial r}{\partial N} + u \sigma_n Q \right) \right|^2 \right\} = 0
\]

which implies

\[
\lim_{\rho \to \infty} \int \int \Sigma_{(\rho,x)} \left\{ \left| r^2 \left( \sigma_n \frac{\partial u}{\partial N} - u \frac{\partial \sigma_n}{\partial N} - i \omega \sigma_n \frac{\partial r}{\partial N} + u \sigma_n Q \right) \right|^2 \right\} ds = 0
\]

(4.36)

Now (4.36) is called a modified radiation condition, in the integral form. It may be noted that the radiation condition (4.36) is a weaker condition i.e., we can deduce equation (4.36) but converse is not in general true. However, equation (4.36) may be simplified further to

\[
\lim_{\rho \to \infty} \int \int \Sigma_{(\rho,x)} \left\{ \left| r^2 \left( \sigma_n \frac{\partial u}{\partial N} - i \omega u \frac{\partial r}{\partial N} + u \sigma_n Q \right) \right|^2 \right\} ds = 0
\]
4. The Maximum Principle

The uniqueness theorem which one can prove by means of the maximum principle applies to bounded domains. We shall establish a class of maximum principles on unbounded domains by imposing some restrictions on the growth of the function at infinity. In our case, this restriction is radiation condition. As a matter of fact, these would be slight variations of the classical Phragmen-Lindelöf theorem [20].

Given the domain $G$ and the finite boundary $\partial G$, we suppose that an increasing sequence of bounded regions $G_1 \subset G_2 \subset \ldots \subset G_k$ can be found with the properties

(i) Each $G_k$ is contained in $G$; for each point $x \in G_N$ and hence $x \in G_k$, for all $k \geq N$.

Theorem 4.1

Let $G$ be a domain, and let $u$ satisfy

$$\begin{align*}
(L' + k) u &\geq 0 \text{ in } G, \\
0 &\leq 0 \text{ on } S = \partial G.
\end{align*}$$

(4.37)

Suppose that there is an increasing sequence of domains $\{G_k\}$ with the property above and that there exists a radiation function $R(u)$ which satisfies

$$\begin{align*}
(L' + k) (R) &\leq 0 \text{ in } G, \\
\lim_{|x| \to \infty} R(u) &= 0
\end{align*}$$

(4.38)

Then $u \leq 0$ in $\overline{G}$. 

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Remark

To show that

$$|R(u)| = \left| e^{i\omega r} r^{-n-1} \left\{ \frac{\partial u}{\partial N} - u \frac{\partial \sigma}{\partial N} - i\omega \sigma \frac{\partial r}{\partial N} u + u \frac{\partial Q}{\partial N} \right\} \right|$$

satisfies (4.38)

(i) It is easy to show that

$$|R| \simeq O(r^{2-n}) \quad (1)$$

and

$$L' + k = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} \sum_{i=1}^{d} \frac{\partial u}{\partial x_i} + k$$

$$= \Delta + b \cdot \nabla + K$$

[Duff, 6]

Changing this to polar coordinates and operating on (1), we can demonstrate that

$$(L' + k) |R| \leq 0 \text{ in } G.$$  

(ii) It is true by definition (4.34)

Proof:

The theorem is proved by a simple reductio ad absurdum. To do this, let

$$(L' + k)(u) \geq 0 \text{ in } G$$

$$u(X) \leq 0 \text{ on } \partial G$$

and assume there is a point $x^0 \in G$ such that

$$u(x^0) > 0$$

Set $V(X) = \bar{u}(X) - \varepsilon |R(u)|$

where $\varepsilon > 0$, is a constant for which

$$V(x^0) > 0.$$
Since \( u(x) \) is bounded from above which tends to zero as 
\[ |x| \to \infty \] (this is proved in the following sequel) and \( R(u) \to 0 \) as 
\[ |x| \to \infty \] we have \( v(x) < 0 \) in some neighbourhood of \( \infty \).
Also \( V(x) < 0 \) on \( S \) since 
\[ u(x) \leq 0 \] on \( S \)
and 
\[ |R(u)| \geq 0 \text{ in } \overline{G} \]
Hence the function \( V(x) \) must achieve its maximum at some finite point \( x^* \in G \).
It follows from Hopf's strong maximum principle that
\[ V(x) = V(x^*) = \text{constant}, \]
because 
\[ L(V(x)) = L(u) - \varepsilon L(R) \geq 0 \]
This is apparently absurd and our assertion is proved.

Remark
This theorem becomes just another variation of the Corollary, [stated on page 99 Protter and Weinberger], provided we can decompose 
\[ R(u) = \frac{U(X)}{W(X)} \]
where \( W(X) \) has the following properties
\[ W > 0 \text{ in } \overline{G} \]
\[ (L^1 + K)(W) \leq 0 \text{ in } G \]
\[ \lim_{|x| \to \infty} W(X) = \infty \text{ in } G \]
Theorem 4.2

Let \( G \) be a domain and let \( u \) satisfy
\[
\begin{align*}
(L' + k) (u) &> 0 \quad \text{in} \ G \\
M(u) &\geq 0 \quad \text{on} \ \partial G = S
\end{align*}
\]  

(4.39)

Suppose that there is an increasing sequence of domains \( \{C_k\} \) with the property above and that there exists a radiation function \( R(u) \) which satisfies
\[
(L' + K) (R) \leq 0 \quad \text{in} \ G
\]
\[
\lim_{|x| \to \infty} R(u) = 0
\]

then we have
\[
u(x) \leq 0 \quad \text{in} \ G.
\]

Proof: Let \( L(u) \geq 0 \) in \( G \) and \( M(u) \geq 0 \) on \( \partial G \). Assume for contradiction that the conclusion of the theorem 4.2 is false.

Set \( \mu = \max_{\partial G} u(x) \)

Theorem 4.1 allows us to conclude that \( \mu > 0 \) for otherwise \( u(x) \) would be non-positive in \( \bar{G} \), contrary to what was just assumed. Furthermore, we define a function
\[
v(x) = u(x) - \mu
\]
which yields
\[
L(v) = L(u) - L(\mu)
\]
and therefore
\[
L(v) = L(u) - K(X)\mu \quad \text{where} \ K(X) \leq 0 \quad \geq 0 \quad \text{in} \ G
\]
and
\[
v(x) \leq 0 \quad \text{on} \ \partial G.
\]
Then from theorem 4.1 it follows that
\[ v(X) \leq 0 \text{ or } u(X) \leq \mu \text{ in } \overline{G} \]

Hence we must have

\[ \mu = \sup_{G} u(X) \]

i.e. \( u(X) \) must attain its absolute maximum at some boundary point \( X^* \in S \).

Noting that \( u(X) \) cannot be constant and applying the boundary point maximum principle of Giraud and Hopf (see Miranda [18]) we may assert that \( \frac{\partial u}{\partial \nu} < 0 \) at the point \( x^* \). Accordingly we obtain \( M(u) < 0 \) at \( x^* \), which contradicts the given boundary condition thereby completing the proof.

**Remark**

It is essential to relax the condition \( k(x) \leq 0 \), in order to deduce the results of previous chapters. Without any restriction on \( k(x) \), we shall point out the modification in our proof of theorem (4.2).

We set

\[ v(X) = u(X) - |R(u)| \]

to obtain

\[ L(v) \geq 0 \text{ in } G \]

and

\[ v(X) \leq 0 \text{ on } \partial G = S \]

\[ \implies v(X) \leq 0 \text{ in } \overline{G} \text{ (Theorem 4.1)} \]

\[ \implies u(X) \leq |R(u)| \text{ in } \overline{G} \]

Hence if we set

\[ \mu = \sup_{G} R(u) \]

\[ u(X) \leq \mu \text{ in } \overline{G} \]

Hence etc....
5. The Uniqueness Theorems

As an immediate consequence of the maximum principle there holds the following uniqueness theorem.

**Theorem 4.3**

Let $G$ be the exterior of a finite closed surface $S$. There exists at most one function $u$, defined in $\overline{G} = G \cup \partial G$ such that

(a) $u \in C^2(G)$

(b) $L(u) = 0$

(c) $u$ assumes given value on $S$, i.e.
$$u = \Phi$$

(d) There exists a radiation function $R(u)$, which satisfies (4.38)?

In order to prove this theorem, we need a Lemma and a definition.

**Definition** The function sequence $\{u_n\}$ will be called a $\Phi$-sequence for the equation $L(u) = 0$, if each function $u_n$ satisfies $L(u) = 0$, in the set $S \cap S_n$ and continuously takes on the values $\Phi(x)$ on $S$. Where $\{S_n\}$ is a sequence of open balls, centered at the origin, with radii $r_n$, $n = 1, 2, \ldots$ tending monotonically to infinity and with $\partial S_n$.

**Lemma:**

Every locally bounded $\Phi$-sequence has a sub-sequence which converges to a solution in $G$ taking on the boundary values $\Phi(x)$.

* In other words, if $u$ satisfies the radiation condition (4.38).
Proof:

Hypotheses imposed on the coefficients of the operator $L$ allow the application of Schauder's interior estimates to the individual function $u_n$. It follows that the sequence $\{u_n\}$ is locally equi-continuous as well as locally bounded, whence one can extract a sub-sequence which converges uniformly on compact sub-sets of $G$ to a solution $u$. That this solution takes on the boundary values $\Phi(x)$ may be proved with the aid of Schauder's boundary estimates.

Proof(Theorem 4.3):

Suppose $R(u)$ is a radiation function. We first extend $R(u)$ so that it is defined over all of $G$. To this end, let $\rho$ be a number so large that $R(u)$ is defined for $r \geq \rho$. Let $R_0 = \min R$; and define

$$R^* = \begin{cases} R_0 & \text{for } r < \rho, \\ \min(R_0, R) & \text{for } r \geq \rho. \end{cases}$$

It can easily be shown that $R^*$ has the property defined by (4.38).

Now we are in position to construct a solution $u$ such that

$$u(x) = \Phi(x) \text{ on } S,$$

and

$$\lim_{r \to 0} u(x) = 0.$$

First we define

$$\overline{R} = C R^* \text{ and } \underline{R} = -C R^*,$$

where $\overline{R}$ does not denote conjugate of $R$. 

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where $C$ is a constant chosen so large that
\[ \overline{R} \geq \Phi \text{ on } S \]
and
\[ \underline{R} \leq \Phi \text{ on } S. \]

Let the set \( \{u_n\} \) be a \( \Phi \)-sequence with \( u_n = 0 \), on \( r = \rho_n \).

Because \( R^* \) satisfies (4.38), it follows that
\[ \overline{R} > u_n > \underline{R}, \quad (4.40) \]

By Lemma, the sequence \( \{u_n\} \) determines a solution \( u \) in \( G \) with boundary values \( \Phi \). Finally, since (4.40) holds for each function \( u_n \), it must also hold for the limit function \( u \), which implies
\[ \lim_{r \to \infty} u = 0. \]

There cannot be more than one solution of the problem. This is an immediate consequence of the maximum principle. If the problem is not well-set, then let \( u, v \) be two distinct solutions of the problem, for some set of boundary value.

\[ u = \Phi \text{ on } S \]
\[ \lim_{r \to \infty} u = 0 \]

and

\[ v = \Phi \text{ on } S \]
\[ \lim_{r \to \infty} v = 0 \]

Since the difference of two solutions would have zero data on \( \partial G \) and would tend to zero as \( r \to \infty \). Therefore
\[ u \equiv v \]
the existence of which implies a unique solution and that completes the proof.

Theorem (4.4) (Non-Dirichlet Problem):

Let $G$ be the exterior of a finite closed surface $S$. There exists at most one function $u$, defined in $G \cup \partial G$ such that

(a) $u \in C^2(G)$,
(b) $L(u) = 0$,
(c) $u$ assumes given value on $S$, i.e.

$$M(u) = \frac{\partial u}{\partial n} + hu = \Phi$$

(d) There exists a radiation function $R(u)$, which satisfies (4.38).

As in theorem (4.3) we shall state a proof of the Lemma which satisfies the non-Dirichlet boundary condition (c) of theorem (4.4).

Proof (Lemma)

The fact that a sub-sequence $\{u_n\}$ exists which converges uniformly on compact subsets of $G$ to a solution $u$ follows exactly as in Lemma (Theorem 4.3). The only difficulty lies in showing that $u$ satisfies (Theorem 4.4-c), and this necessarily requires some degree of smoothness of $S$, as well as of the direction field $\xi$, the function $h(X)$ and the data $\Phi$. If the local representations of $\partial G$ are of class $C^{2+\mu}$, and $\xi, h$ and $\Phi$ are in $C^{1+\mu}$ on $\partial G = S$, then standard Schauder type boundary estimates imply that the second derivatives of $u$ are equicontinuous in $\overline{G\setminus S}$. From this it is immediate that $u$ satisfies (Theorem 4.4-c).

Suppose that a radiation function exists. Then by theorem 4.3 the exterior Dirichlet problem is solvable. Let $v_1$ be the solution.
such that
\[ L(v_1) = 0, \text{ in } G \]
\[ v_1 = 0 \text{ on } S = \partial G \]
and let \( v_2 \) be the solution defined by
\[ L(v_2) = 0 \text{ in } G \]
\[ v_2 = 1 \text{ on } S = \partial G \]
Evidently \( 0 < v_2 < 1 \) in \( G \). Moreover \( M(v_2) < 0 \), so that there exists a positive constant \( C \) with the property
\[ C \cdot M(v_2) \leq -|\phi - M(v_1)| \]
Now let \( \{u_n\} \) be a \( \Phi \)-sequence for the equation \( L(u) = 0 \), with \( u_n = v_1 \) on \( r = r_n \). Setting \( v = C \cdot v_2 - (u_n - v_1) \) we obtain easily
\[ L(v) = 0 \text{ in } G \]
\[ M(v) < 0 \text{ on } \partial G, \ v \geq 0 \text{ on } \partial G_n. \ \partial G = S \]
It follows then from the boundary point lemma that \( v \geq 0 \); thus we have proved
\[ |u_n| \leq C|v_2| + |v_1|, \quad v_2 \in (0,1). \quad (4.41) \]
This shows that the sequence \( \{u_n\} \) is uniformly bounded. Therefore by Lemma, there exists a solution \( u \) of \( L(u) = 0 \) which satisfies \( M(u) = \phi \text{ on } \partial G \). It remains to show that \( u \) tends to zero as \( r \) tends to infinity. This, however, follows directly from (4.41). The uniqueness of the solution is an immediate consequence of this maximum principle.
Huyghens' Principle states a curious property of the wave equation

$$\ddot{\phi} = \Delta \phi$$

for odd values of $n \geq 3$. The gist of Huyghens' principle is that the domain of dependence, for the initial value problem, is only the boundary of the region. There may be a hole in the putative domain of dependence. Such a hole is technically known as a 'lacuna'. It is well-known that in general there will be no lacuna in the domain of dependence for a boundary problem or initial value problem for the wave equation, if $n$ is even and $\geq 2$, for then Huyghens' principle does not apply.

We shall be much concerned with characteristic cones in what follows: so it is convenient to define some relevant notations at the outset.

If $P$ is a point in the $(n+1)$-dimensional space, we denote the co-ordinates of $P$ by $(t; x_1, x_2, \ldots, x_n)$ or $(t; X)$. By $C^P$ we shall mean the characteristic cone with vertex $P$, the $n$-dimensional manifold given by

$$t - t^P = \sqrt{\sum_{i=1}^{n} (x_i - x_i^P)^2}.$$  

Let $T$ be a domain on $C^P$, bounded in $C^P$ by a piecewise smooth $(n-1)$ dimensional manifold $S$. Let the vertex $P$ of $C^P$ be an interior point of $T$. Let $PR$ be a ray of $C^P$, (where $R$ is on $S$) which cuts $S$ only at $R$.

Further, let $\phi(t; X)$ be a function defined in some neighborhood of $T$, such that:

(i) $\phi$ and its first derivatives are continuous in $T$ and
S, and its second derivatives are piecewise continuous in T.

(ii) In T, \( \phi \) satisfies the wave equation

\[
\mathcal{L}(\phi) = \frac{\partial^2 \phi}{\partial t^2} - \Delta \phi = 0
\]

Then the following is true

**THEOREM 1:**

The \( \frac{1}{2}(n-3) \) fold iterated integral of \( \phi \) from R to P along the ray \( RP \) is uniquely determined if the values of \( \phi \) and its first derivatives are given at \( S \).

This iterated integral is to be understood in the sense of Riemann and Liouville [6]. That is, if \( r_1 \) is the distance of \( P \) from \( R \), and if \( \bar{\phi}(r) \) means the value of \( \phi \) at the point on the segment \( RP \) at distance \( r \) from \( R \), the \( \frac{1}{2}(n-3) \)-fold iterated integral is

\[
\frac{1}{\Gamma\left(\frac{n-3}{2}\right)} (RP, \phi) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \frac{\partial}{\partial r_1} \int_0^{r_1} (r_1 - r)^{\frac{1}{2}(n-3)} \bar{\phi}(r) dr
\]

\[
= \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} R_1^{\frac{1}{2}(n-3)} \phi(0) + \frac{1}{\Gamma\left(\frac{n-3}{2}\right)} \int_0^{r_1} (r_1 - r)^{\frac{1}{2}(n-3)} \frac{\partial}{\partial r} \bar{\phi}(r) dr
\]

(A.1)

provided that \( n \geq 2 \), and

\[
\frac{1}{\Gamma\left(\frac{n-3}{2}\right)} (RP, \phi) = \Gamma^{-1}(RP, \phi) = \frac{\partial}{\partial r_1} \phi(r_1)
\]

for \( n = 1 \).

The proof consists of an explicit computation of the iterated integral. The method is to integrate by Gauss's theorem a certain divergence in T obtained from the wave equation by writing it in terms of some
special new independent variables $a, u, \eta_i$ ($i = 1, 2, \ldots, n-1$). The variable $a$ is constant on each of a family of characteristic cones in which $C^P$ is embedded, and the variables $u, \eta_i$ are considered parameters on these cones. This device is, except for the particular choice of new independent variables, essentially the device used by Douglin[5].

We may assume that $P$ is the origin of the co-ordinates, and that the ray $RP$ is given by

$$t = x_1 \leq 0$$
$$x_{i+1} = 0 \quad i = 1, 2, \ldots, n-1.$$ 

Then if $x_1 - t > 0$, the new variables may be obtained by

$$\alpha = \frac{1}{x_1 - t} \left\{ t^{2} - x_1^{2} - \sum_{i=1}^{n-1} x_{i+1}^{2} \right\}$$
$$u = x_1 - t$$
$$\eta_i = x_{i+1} \quad i = 1, 2, \ldots, n-1$$

(A.2)

The co-ordinate surface $\alpha = 0$ is the characteristic cone

$$(t + \frac{\alpha_0}{2})^2 - (x_1 + \frac{\alpha_0}{2})^2 - \sum_{i=1}^{n-1} x_{i+1}^{2} = 0$$

with vertex $Q$ on the ray $RP$, where the co-ordinates of $Q$ are

$$t^Q = x_1^Q = -\frac{\alpha_0}{\alpha}$$
$$x_{i+1}^Q = 0 \quad i = 1, 2, \ldots, n-1$$

In particular, the surface $\alpha = 0$ is the cone $C^P$.

The co-ordinate surface $u = u_0$ is the characteristic plane $x_1 - t = u_0$ parallel to $RP$. Note that on $C^P$ we have $u > 0$ except at $RP$, where $u = 0$. 

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By a straightforward calculation it may be verified that at any point where $\phi$ has continuous second derivatives we have immediately,

$$
\frac{1}{4} L(\phi) = \frac{1}{4} \left\{ \phi_{tt} - \sum_{i=1}^{n-1} x_i x_i \phi_{i+1} \phi_{i+1} \right\}
$$

$$
= \phi_{\alpha u} + \frac{1}{u} \sum_{i=1}^{n-1} \eta_i \phi_{\alpha \eta_i} + \frac{1}{2u} \phi_{\alpha} - \frac{1}{4} \sum_{i=1}^{n-1} \eta_i \eta_i
$$

since it was assumed that $\phi$ satisfies the wave equation at $T$, it follows that at $T$, or, which is the same thing, for $\alpha = 0$ we have

$$
\frac{1}{4} u^{-1/2} (n-1) L(\phi)
$$

$$
= \frac{\partial}{\partial u} \left\{ u^{-1/2} (n-1) \phi_{\alpha} \right\} + \sum_{i=1}^{n-1} \frac{\partial}{\partial \eta_i} \left\{ u^{-1/2} (n-1) \frac{\phi_{\alpha}}{u} - \frac{1}{4} \phi \right\}
$$

$$
= 0 \tag{A.3}
$$

We now integrate this equation with respect to $u, \eta_i$ over the domain $T_\varepsilon$ of these variables, corresponding to that part of $T$ in which $u \geq \varepsilon > 0$. The expression on the right is the divergence of a vector; hence, by Gauss's theorem its integral over $T_\varepsilon$ is equal to the integral of the normal component of that vector over the boundary of $T_\varepsilon$. The boundary of $T_\varepsilon$ consists of the two parts $\gamma_\varepsilon$, where $u = \varepsilon$; $S_\varepsilon$ that part of $S$ in which $u \geq \varepsilon$. We may write the result of this integration symbolically as follows:

$$
I(\gamma_\varepsilon) + I(S_\varepsilon) = 0
$$

The integral over $\gamma_\varepsilon$, where $u = \varepsilon$, is

$$
I(\gamma_\varepsilon) = -\int_{\gamma_\varepsilon} u^{-1/2} (n-1) \phi_{\alpha} \eta_1 \cdots \eta_{n-1}
$$

$$
= -\varepsilon^{-1/2} (n-1) \int_{S_\varepsilon} \phi_{\alpha} \eta_1 \eta_2 \cdots \eta_{n-1} \tag{A.4}
$$
For at the boundary \( v \), the unit normal pointing out of \( T \) (as a domain in the \( u, \eta \) space) has the \( u \)-component \(-1\) and all \( \eta \)-components zero.

Our result will follow on allowing to tend to zero in the equation

\[
I(v) = -\varepsilon^{-1/2} (n-1) \int \hat{v}_\alpha \, d\eta_1 d\eta_2 \ldots d\eta_{n-1} = -I(s)
\]

We shall see that \( I(v) \) tends to a limit as \( \varepsilon \) tends to zero. To prove this, we transform the integral by introducing new variables of integration, as follows:

\[
\eta_i = \varepsilon^{1/2} v^{1/2} \omega; \quad \sum_{i=1}^{n-1} \omega_i^2 = 1
\]

Then

\[
d\eta_1 d\eta_2 \ldots d\eta_{n-1} = (\varepsilon^{1/2} v^{1/2})^{n-2} d(\varepsilon^{1/2} v^{1/2}) d\Omega
\]

\[
= 1/2 \varepsilon^{1/2} (n-1) v^{1/2} (n-2) d\nu d\Omega
\]

where \( d\Omega \) is the element of \((n-2)\)-dimensional measure on the surface of the unit sphere which is the domain of the quantities \( \omega_i \). (Of course, in introducing the above formalism, we tacitly assume that \( n \geq 3 \). We will not consider the cases, \( n = 1, 2 \).) On \( v \), we have

\[
t^2 - x_1^2 - \sum_{i=1}^{n-1} \eta_i^2 = t^2 - x_1^2 - \varepsilon v = 0
\]

\[
\varepsilon = u = x_1 - t
\]

Thus the arguments of \( \hat{v}_\alpha \) in (A.4) are

\[
t = -1/2 (v+\varepsilon), \quad x_1 = -1/2 (v-\varepsilon), \quad x_{i+1} = \eta_i \varepsilon^{1/2} v^{1/2} \omega_i
\]

Since, as may be computed from (A.2),
Thus
\[
I(v_\varepsilon) = 1/4 \int_{v_\varepsilon} (\hat{\Phi}_t + \hat{\Phi}_{x_1}) v^{1/2} (n-3) dv \, d\Omega
\]

As \( \varepsilon \) tends to zero, \( v_\varepsilon \) tends to the ray \( PR \). The arguments of \( \hat{\Phi}_t, \hat{\Phi}_{x_1} \) tends to
\[
t = -1/2 v, \quad x_1 = -1/2 v, \quad x_{i+1} = 0 \quad i = 1, 2, \ldots, n-1
\]
Since \( v = -1/2(t+x_1) \), it follows that the domain of \( v \) in \( \varepsilon \) tends to \( 0 \leq V \leq r_1/2 \), where \( r_1 \) is the distance of \( R \) from \( P \). The domain of the \( \omega_i \) tends to the entire surface of the unit sphere in \((n-1)\) dimensions. (Here is where we make use of our assumption that \( R \) is the only common point of \( S \) and the ray \( PR \)). Hence, because \( \hat{\Phi}_t \) and \( \hat{\Phi}_{x_1} \), are continuous functions in \( T \), it follows that
\[
\lim_{\varepsilon \to 0} I(v_\varepsilon) = 1/4 K(n-1) \int_0^{r_1/2} v^{n-3} \left( \hat{\Phi}_t + \hat{\Phi}_{x_1} \right) dv
\]

where
\[
K(n-1) = \frac{2\pi^{n/2}}{\Gamma(n/2)}
\]
is the \((n-2)\) dimensional measure of the unit sphere in \((n-1)\) dimensions.

Because \( I(v_\varepsilon) \) tends to a limit, so must \( -I(S_\varepsilon) \) tend to the same limit, which therefore is expressible as an improper integral over \( S \). The above expression for \( \lim(I(v_\varepsilon)) \) is evidently proportional to
\[
\frac{1}{\Gamma(n/2)} \int_0^{r_1} (r_1-r)^{1/2} (n-3) \frac{d}{dr} \hat{\Phi}(r) dr
\]
\[
= \frac{-r_1^{1/2} (n-3)}{\Gamma(n/2)} \hat{\Phi}(0) + \frac{1}{2} (n-3) (\Phi(R, \hat{\Phi})
\]
Hence we have

\[ I^{1/2} (n-3)(\text{RP}, \hat{\psi}) = h(S) \]

where \( h \) is a quantity determined by the values of \( \hat{\psi} \) and its derivatives at \( S \). Our theorem is thereby proved, for \( n \geq 3 \).

**Huyghens' Principle**

The algorithm of the preceding section may be applied to solve the initial-value problem for the wave equation. We may define the initial value problem as follows:

Let \( V \) be a bounded \((n+1)\)-dimensional region bounded by a \( n \)-dimensional region \( T \) on a characteristic cone \( C^P \) with vertex \( P \), where \( P \) is an interior point of \( T \); a region \( B^P \) on another \( n \)-dimensional manifold \( B \). \((B^P \) means that part of \( B \) which is inside or on \( C^P \)).

Let \( \hat{\psi}(t; x_1, x_2, \ldots, x_n) \) be continuous with continuous first derivatives in the closure of \( V \), and let the second derivatives of \( \hat{\psi} \) be piecewise continuous in the closure of \( V \). Let \( \hat{\psi} \) satisfy the wave equation in \( V \).

The problem we shall consider is to determine the value of \( \hat{\psi} \) at \( P \), given the values of \( \hat{\psi} \) and its first derivatives at \( B^P \). Let us denote the intersection of \( B \) and a characteristic cone with vertex \( Q \) by \( S^Q \).

Let \( R \) be a point of \( S^P \) and the only point of \( S^P \) on the ray \( PR \). Let \( P^1 \) be a point on the segment \( PR \) at a distance \( r^1 \) from \( R \). Then by Theorem 1, we have

\[ I^{1/2} (n-3)(\text{RP}^1, \hat{\psi}) = h(r^1) \]

where the value of \( h \) is determined by the values of \( \hat{\psi} \) and its first.
derivatives at $S^P$. This is an integral equation for $\xi(\mathbf{r}^1)$, and may be solved by differentiating both sides $1/2(n-3)$ times (in the sense of Riemann and Liouville). Thus we obtain $\xi(P)$ explicitly in terms of the data at $B^P$.

If $n$ is odd and greater than unity, then $1/2(n-3)$ is an integer, so that $\xi$ is determined by differentiating $h$ in the ordinary sense an integral number of times. Thus $\xi(P)$ is determined by the value of $h(\mathbf{r})$ in any neighbourhood of $\mathbf{r}^1$, or by the values of the data in any neighbourhood of $S^P$. Thus any closed subset of $B^P$ completely inside $C^P$ does not belong to the domain of dependence for the initial value problem; such a set is a 'lacuna'. This is Huyghens' Principle.

If $n$ is even, the $1/2(n-3)$ fold derivatives is no longer a local property; the above evaluation of $\xi(P)$ would depend on the values of the data at all points of $B^P$. This bears out the well-known fact that Huygen's Principle fails for even $n$. Nevertheless, it is shown by Gardner[7] that lacuna exists for certain special surfaces $S^P$. 

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APPENDIX B

BASIC FUNCTIONS $\tau$ and $\sigma_s$

Let us consider a real or complex valued function $\hat{\phi}(t,\chi)$ which satisfies

$$L(\hat{\phi}) = \frac{1}{c^2(\chi)} \hat{\phi}_{tt} \quad (B.1)$$

where

$$L = \sum_{i,j=1}^{n} a_{ij}(\chi) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\chi) \frac{\partial}{\partial x_i} + h(\chi)$$

We shall look for a product solution of (B.1) of the form

$$\hat{\phi}(t,\chi) = g(t) \ u(\chi) \quad (B.2)$$

If we insert this form into (B.1) and separate variables, we obtain

$$c^2(\chi) \frac{L(u)}{u} = \frac{g''(t)}{g(t)} = -\omega^2 \text{(say)} \quad (B.3)$$

which implies

$$L(u) + \frac{\omega^2}{c^2(\chi)} u = 0 \quad (B.4)$$

Equation (B.4) for $u$ is called the reduced wave equation.

Introducing

$$m(\chi) = \frac{1}{c(\chi)} \text{ and } k = \omega$$

we obtain from this equation (B.4)

$$L(u) + k^2 m^2(\chi) u = 0 \quad (B.5)$$

the constant $\omega$ is analogous to the angular frequency; because two linearly independent solutions of

$$\frac{g''(t)}{g(t)} = -\omega^2$$

are the periodic functions

$$g(t) = e^{\pm i\omega t}$$
It follows that every product solution of \( (B.1) \) is of the form
\[
\psi(t, \chi) = e^{\pm i\omega t} u(\chi)
\] (B.6)

It suffices to study with negative time factor of the form (B.6).

We shall now consider the solution of
\[
\Delta u + k^2 m^2(\chi)u = 0
\] (B.7)

for large values of \( k \). We begin with the observation that when \( m(\chi) \) is constant, (B.7) admits the plane wave equation
\[
u(\chi, k) = Z(k) e^{imK \cdot \chi}
\] (B.8)

here the propagation vector \( K \) is a real or complex vector of length \( |K| = k \) and amplitude \( Z(k) \) is a real or complex constant. The exponential \( e^{imK \cdot \chi} \) is called the phase factor of the solution. By analogy with (B.8) we shall seek solutions of (B.5) of the form
\[
u(\chi) = Z(\chi, k)e^{ik\pi(\chi)}
\] (B.9)

Upon inserting (B.9) into (B.5) and cancelling the phase factor \( e^{ik\pi(\chi)} \) we obtain,
\[
\sum_{i,j} a_{ij}(\chi) \left\{ \frac{\partial^2 u}{\partial x_i \partial x_j} + i k \left( \frac{\partial^2 u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{\partial^2 u}{\partial x_j} \frac{\partial u}{\partial x_i} \right) + ikZ \frac{\partial^2 u}{\partial x_i \partial x_j} \right\} + \frac{1}{\Omega} \sum_{j} b_{ij}(\chi) \left( \frac{\partial^2 u}{\partial x_i} + ikZ \frac{\partial u}{\partial x_i} \right) + h(\chi)Zk^2m^2(\chi)Z = 0
\] (B.10)

To solve (B.10) for large values of \( k \), we assume that \( Z(\chi, K) \) can be expanded in inverse powers of \( k^{10} \). It is convenient to write the expansion in terms of \( ik \) in the form
\[
Z(\chi, K) \approx \sum_{s=0}^{\infty} c_s(\chi)(ik)^{-s}
\] (B.11)
with $a_s(X) = 0$ for $s = -1, -2, \ldots$.

We have used the sign of asymptotic equality in (B.11) to indicate that the series must be an asymptotic expansion of $Z$ as $k \to \infty$.

This means that for each $N \geq 0$

$$Z(X, k) = \sum_{s=0}^{N} a_s(X)(ik)^{-s} + o(k^{-N}) \quad (B.12)$$

By definition the order symbol denotes a term for which

$$\lim_{k \to \infty} k^N \cdot [O(k^{-N})] = 0$$

We will assume the termwise differentiation of (B.12). Upon inserting (B.12) into (B.10) we obtain

$$\left\{ \sum_{i,j} a_{ij}(X) \frac{\partial \sigma_{s+1}}{\partial x_i} \frac{\partial \tau}{\partial x_j} + \sum_{i} b_i(X) \frac{\partial \sigma_{s+1}}{\partial x_i} \right\} + \sum_{i,j} a_{ij}(X) \frac{\partial \sigma_{s+1}}{\partial x_i} \frac{\partial \tau}{\partial x_j} + \sum_{i} b_i(X) \frac{\partial \sigma_{s+1}}{\partial x_i} + h \sigma_s = 0$$

(B.13)

From (B.13) it follows that the coefficients of each power of $(ik)$ must be zero. (We shall ignore the contribution from the term $O(k^{-N})$ i.e. $p^* \to 0$).

For $s = -2$, we obtain

$$\left\{ \sum_{i,j} a_{ij}(X) \frac{\partial \tau}{\partial x_i} \frac{\partial \tau}{\partial x_j} - m^2(X) \right\} \sigma_0 = 0 \quad (B.14)$$

Since $\sigma_0 = 0$ for $s = -1, -2, \ldots$ and if we assume $\sigma_0 \neq 0$,

Eq. (B.14) leads to an equation of the type of eiconal equation for $\tau$

$$\sum_{i,j} a_{ij}(X) \frac{\partial \tau}{\partial x_i} \frac{\partial \tau}{\partial x_j} = m^2(X) = \frac{1}{c^2(X)} \quad (B.15)$$
which is a first order non-linear partial differential equation for \( \tau(\chi) \).

We could obtain solutions of (B.15) by applying the general theory of first order partial differential equation [3]. For \( s = -1 \), we obtain

\[
\sum_{i,j} a_{ij}(\chi) \left( \frac{\partial \sigma_0}{\partial t} + \frac{\partial \sigma_0}{\partial x_i} + \frac{\partial^2 \tau}{\partial x_i \partial x_j} + \frac{\partial \sigma_0}{\partial x_1} \frac{\partial \tau}{\partial x_1} \right) + \sum_i b_i(\chi) \sigma_0 \frac{\partial \tau}{\partial x_1} = 0
\]

For \( s = 0,1,2, \ldots \) the vanishing of the coefficients implies

\[
\sum_{i,j} a_{ij}(\chi) \left( \frac{\partial \sigma_s}{\partial t} + \frac{\partial \sigma_s}{\partial x_i} + \frac{\partial \sigma_s}{\partial x_j} + \frac{\partial^2 \tau}{\partial x_1 \partial x_i} \right) + \sum_i b_i(\chi) \sigma_s \frac{\partial \tau}{\partial x_1} + n \sum_i a_{ij}(\chi) \frac{\partial \sigma_{s-1}}{\partial x_i} + \sum_i b_i(\chi) \frac{\partial \sigma_{s-1}}{\partial x_i} + h(\chi) = 0
\]

Equations (B.16) and (B.17) are parallel to the so-called transport equations. Some additional conditions are imposed on \( \sigma_s \) and \( \tau \), which are mentioned in Chapter 3 and 4.

**REMARK**

The auxiliary equations, used in Chapter 3, can be immediately obtained from the equations (B.15), (B.16) and (B.17).

Equation

\[
L(u) = \sum_{i,j} a_{ij}(\chi) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(\chi) \frac{\partial u}{\partial x_i} + h(\chi)u
\]

has to take the form

\[
\Delta u = \sum_i \frac{\partial^2 u}{\partial x_i^2}
\]

It is obvious then that

\[
a_{ij}(\chi) = \delta_{ij}, \quad b_i(\chi) = 0
\]

where \( \delta_{ij} \) is Kronecker delta.
Under these restrictions equations (B.15), (B.16) and (B.17) change to

\[ \sum_{i}^{n} \left( \frac{\partial \tau}{\partial x_i} \right)^2 = \mu^2(x) = \frac{1}{c^2(x)} \]  \hspace{1cm} (B.18)

\[ 2 \sum_{i}^{n} \frac{\partial \sigma_0}{\partial x_i} \frac{\partial \tau}{\partial x_i} + \sigma_0 \sum_{i}^{n} \frac{\partial^2 \tau}{\partial x_i^2} = 0 \]  \hspace{1cm} (B.19)

and

\[ 2 \sum_{i=1}^{n} \frac{\partial \sigma_s}{\partial x_i} \frac{\partial \tau}{\partial x_i} + \sigma_s \sum_{i=1}^{n} \frac{\partial^2 \tau}{\partial x_i^2} + \sum_{i=1}^{n} \frac{\partial^2 \sigma_{s-1}}{\partial x_i^2} = 0 \]  \hspace{1cm} (B.20)
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75


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