1-1-1969

A generalization of pi-structures.

K. L. Duggal

University of Windsor

A GENERALIZATION OF π-STRUCTURES

by

K.L. Duggal

A Thesis
Submitted to the Faculty of Graduate Studies through the Department
of Mathematics in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy
at the University of Windsor

Windsor, Ontario

1969

UMI Number: DC52633

INFORMATION TO USERS

Approved;

ABSTRACT

The aim of the present work is to generalize the π-structures of G. Legrand (Thése; Rendiconti del circolo Matematico di Palermo; Serie 2, t. vii, 1958, pp. 323-354; t. viii, 1959, pp. 5-48) by considering a linear operator J acting on the complexified tangent space T_x^c of a differentiable manifold V_n satisfying a relation of the form $J^{r+1}=\lambda^{r+1}$ (identity), where $r \geq 1$ is an integer and λ a nonzero complex constant. Such structures will be called Almost r-Product Structures, briefly a.r.p.s.

We introduce the subject by giving the necessary historical background as well as some comments on important results.

We define an a.r.p.s. on a differentiable manifold V_n (of class C^∞) and introduce bases adapted to this structure. This helps us to obtain a characterization of the infinitesimal connections (defined on the set of adapted bases which has a natural structure of principal fibre bundle) in terms of J. Further we generalize the concepts of curvature tensor and the holonomy group of these connections.

Next we consider a complex symmetric tensor G on V_n equipped with a.r.p.s. Introducing the compatability condition $JG = \lambda G$, we obtain a singular Riemannian structure subordinate to the a.r.p.s. By defining special adapted bases and special connections, we are able to get a characterization of these connections by conditions on J and G. We also obtain a characterization of these singular Riemannian structures in

terms of the holonomy groups of these connections.

In order to investigate the conditions for complete integrability of a.r.p.s., we give a short introduction on completely integrable systems and construct a tensor determined on this structure which we call the torsion tensor.

The operators C and M of Lichnerowicz are generalized by defining the operators $\overset{s}{C}$ and $\overset{s}{M}$ as follows:

$$\overset{s}{C}\phi(v_1,\ldots,v_t) = \phi(\overset{s}{J}v_1,\ldots,\overset{s}{J}v_t)$$

$$\overset{s}{M}\phi(v_1,\ldots,v_t) = \sum_{k=1}^{t}\phi(v_1,\ldots,\overset{s}{J}v_k,\ldots,v_t)$$

where $v_1,\ldots,v_t \in T_x^c$, ϕ is a t-form and $1 \leq s \leq r+1$.

The following are the main results on the study of these operators.

(a) Let r be an odd integer, $s = \frac{r+1}{2}$, ϕ a linear form, and T the torsion form of an a.r.p.s. Then,

$$\lambda^{r+1}d\phi + \overset{s}{C}d\phi - \overset{s}{M}d\overset{s}{C}\phi = 4\lambda^{r+1}\phi . T$$

Consideration of these operators $\overset{s}{M}, \overset{s}{C}$ also gives a local result for the torsion form T:

(b) $T(u,v) = -\frac{1}{4\lambda^{r+1}} \cdot \underset{s}{N}(u,v)$

where $T(u,v) = t^i_{jk}\, u^j v^k$; t^i_{jk} are the components of the torsion tensor and $\underset{s}{N}(u,v)$ is a generalization of the Nijenhuis tensor.

We again consider the complex symmetric tensor G and say that G is hermitian with respect to J if

$$JG + {}^{t}(JG) = 0.$$

where ${}^{t}(JG)$ is the transpose of JG.

The resulting structure is called an almost r-product hermitian structure subordinate to the a.r.p.s., briefly H-structure. Such structures may exist on a differentiable manifold of a dimension which has to be a multiple of $(r+1)$ (where $r \geq 1$ is an integer). The manifold is not necessarily of even dimension as stated in the study of π-structures. Most of the other properties of the almost hermitian structures in the broad sense generalize in a natural way to the H-structures.

Finally we examine some details which appear in the study of H-structures by generalizing the concepts of hermitian and pseudohermitian structures, almost kählerian structures, kählerian and pseudokählerian structures.

ACKNOWLEDGMENTS

The author wishes to express his deep appreciation and sincere thanks to Professor Hermes A. Eliopoulos for suggesting the topic and for his able guidance, help and encouragement during the course of the present investigation.

Acknowledgments are made to Rev. D.T. Faught, Chairman of the department of Mathematics, to the Government of Ontario and to the National Research Council of Canada for providing sufficient financial support.

TABLE OF CONTENTS

CHAPTER I

General Introduction

(a) A new approach in classical geometry was initiated by Eli Cartan[13], S.S. Chern[14] and A. Weil[9]. From their work has been developed a new technique of creating structures over the object of study as the foundation, and the derivation of the needed properties from these structures.

These structural ideas have been responsible for the growth of many new concepts, such as vectors and tensor fields, algebras of various sorts, fibre spaces and fibre bundles. The background of this development is the history of Differential Geometry. Differential Geometry in its general sense is a study of relations between global and local properties of a differential geometric object. The spaces under consideration are not only topological spaces but are also considered to be differentiable manifolds, so that methods of differential calculus may be applied. Thus, one studies the existence of different structures on a differentiable manifold. Such an existence brings in analysis which linearizes a problem by replacing the study of an object by the study of its infinitesimal (or linear) parts. For example, differentiable manifolds are replaced by tangent spaces (differentiable), differential maps by Jacobians, and Lie groups by Lie algebras. For the global study of the problem, all these linear parts are pieced together over each point of the object under study, and end up as what is known as a fibre space. This fibre space is equipped with more structure to obtain the notion of a fibre bundle. In this way by building more and more structures over the object of investigation, more information about the object can be obtained.

(b) A. Weil[9] pointed out in 1947 that there exists in a complex space a tensor field F of type (1,1) whose square is minus unity. C. Ehresmann[10,11] defined in 1947 an almost complex space as an even dimensional manifold which carries a tensor field F whose square is minus unity. The present work is based on special types of structures called G-structures of the first kind[12], which are defined by linear operators satisfying some algebraic relations. Such structures have been extensively studied by S.S. Chern[14], C. Ehresmann[10,11], A. Frolicher[15], A. Lichnerowicz[2], G. Legrand[1], D. Bernard[12], H.A. Eliopoulos[3-6], D.C. Spencer[20], K. Yano[19] and many others. We are particularly interested in the work of A. Lichnerowicz and G. Legrand. G. Legrand[1] studied a generalization of the almost complex structures by considering a linear operator J acting on the complexified tangent space T_x^c at any point $x \in V_n$ satisfying a relation of the form $J^2 = \lambda^2$ (identity) where λ is a nonzero complex constant. Such structures were called π-structures. For the remaining case, $\lambda = 0$, H.A. Eliopoulos introduced almost tangent structures[3]. The object of the present work is to generalize π-structures by considering a linear operator J acting on V_n satisfying a relation of the form $J^{r+1} = \lambda^{r+1}$ (identity) where $r \geq 1$ is an integer and λ is a nonzero complex constant. We call such structures almost r-product structures, briefly <u>a.r.p.s</u>. An attempt on similar lines was made in

1960 by C.J. Hsu[7] but very few of the properties were discussed. For the remaining case, $\lambda = 0$, H.A. Eliopoulos[4] generalized almost tangent structures by considering nilpotent operators of degree $r \geq 2$. It is also worth mentioning that the study of affine connections on a differentiable manifold with a system of r distributions ($r \geq 2$) has been extensively made by several authors:[8;16;17;18].

(c) In this work we assume that the differentiable manifold V_n as well as the subspaces $T_0, \ldots T_r$ of T_x^c are of class C^∞ unless we state it to the contrary. It is also assumed that the manifolds introduced are of dimension at least equal to 2, arc-wise connected and the second countability axiom is satisfied.

Most of the properties concerning π-structures generalize in a natural way to <u>a.r.p.s</u>. However, while generalizing the notion of the almost hermitian structures in the broad sense (Chapter 6), we observe that such structures are able to exist on differentiable manifolds of a dimension which has to be a multiple of $(r+1)$, where r is any integer ≥ 1.

For the remaining case, $\lambda = 0$, H.A. Eliopoulos discussed 'Euclidean structures compatible with almost tangent structures'[5] and generalized this conception to r-tangent structures[6]. A similar attempt has been made to study 'Singular Riemannian structures compatible with π-structures'[25], and furthermore, we have generalized this conception to <u>a.r.p.s.</u> (Chapter 3). This topic was not discussed by G. Legrand[1]. In a way, it can be said that Chapter 3 constitutes one of the additional contributions to the usual generalization of π-structures.

The introduction of the operators $\overset{s}{C}$ and $\overset{s}{M}$ [7] on the <u>a.r.p.s.</u> was a great success in the sense of natural generalization of the operators C and M considered by Lichnerowicz[2] and Legrand[1], except for the restriction that r is odd and $s = \frac{r+1}{2}$.

It has been considered advisable to give a short account of necessary basic concepts at the beginning of some of the chapters.

CHAPTER 2

Connections and the Holonomy Group of Almost r-Product Structures

2.0. Basic Concepts

(A) Infinitesmial Connections. Let E be a principal fibre bundle, differentiable of class C^{∞}, of which the base is a differentiable manifold V_n of dimension n and the structure group is a Lie group G, operating on itself by the left translation. We denote by p the canonical mapping $E \to V_n$. Let θ_z be the tangent vector space to E at a point z. A vector of θ_z will be called vertical if it belongs to the subspace V_z of θ_z tangent to the fibre.

For each $z \in E$ let lh_z be a subspace of θ_z with the following properties:

a) lh_z depends differentiably on z.

b) lh_z is supplementary to V_z. Any vector α of θ_z is then the sum of a vertical vector $V\alpha$ and of a vector $lh\alpha \in lh_z$. $V\alpha$ (respectively $lh\alpha$) is the vertical part (respectively horizontal) of α. If $V\alpha = 0$, α is called horizontal.

c) lh_z is invariant under operation by G on E, i.e. $lh_{zg} = D_g lh_z$ where D_g denotes the operation of right translations by the elements g of G.

If for each θ_z such a space lh_z exists we say that an infinitesimal connection is defined on E.

To an infinitesimal connection is canonically associated a 1-form w with values in the Lie algebra L of G, that is to say, for any

$z \in E$, a linear mapping of θ_z into L: for $\alpha \in \theta_z$, $w(\alpha)$ is the element of L generated by $V\alpha$. The 1-form w possesses the following properties:

(a') w depends differentiably on z.

(b') if α is vertical, $w(\alpha)$ is the element of L generated by α.

(c') $w(D_g \alpha) = (\mathrm{adj}g^{-1})\, w(\alpha)$ where $(\mathrm{adj}g^{-1})$ denotes the image of the element g^{-1} under the adjoint representation.

Conversely, let w be a 1-form on E with values in L having the three preceding properties. Let us denote by lh_z the subspace of θ_z consisting of the vectors α such that $w(\alpha) = 0$. The field lh_z defines an infinitesimal connection and w is the associated 1-form.

(B) <u>Complex Linear Connections</u>. We consider a differentiable manifold V_n. Let T_x^c be the complexified vector space of the tangent vector space T_x at the point $x \in V_n$. Let us say that a base of the vector space T_x^c is a complex base relative to x. Let $E_c(V_n)$ be the set of complex bases relative to the different points of V_n and p the mapping $E_c(V_n) \to V_n$ such that a complex base relative to x is made to correspond to the point x itself. The set $E_c(V_n)$ admits a natural structure of a principal fibre bundle with base V_n and structure group GL(n,c). We will call a complex linear connection any infinitesimal connection on $E_c(V_n)$. One is able to determine such a connection by a 1-form w on $E_c(V_n)$ with values in the Lie algebra of GL(n,c). The 1-form w may be represented by an nxn matrix of which the elements w_j^i are pfaffian forms on $E_c(V_n)$ with complex values.

(C) Curvature Form of the Complex Linear Connection. The curvature form of a complex linear connection (w^i_j) is the tensor 2-form on $E_c(V_n)$ with values in the Lie algebra of GL(n,c) defined by

$$\Omega^i_j = d(w^i_j) + w^i_h \wedge w^h_j \qquad (2.0.1)$$

For any vector σ tangent to $E_c(V_n)$ at the point (e_i), let us put $\theta^{oi}(\sigma) = \theta^i(p\sigma)$; the θ^{oi} are pfaffian forms on $E_c(V_n)$. One is then able to write

$$\Omega^i_j = 1/2\, R^i_{j,k\ell}\, \theta^{0k} \wedge \theta^{0\ell} \qquad (R^i_{j,k\ell} = - R^i_{j,\ell k})$$

and the $R^i_{j,k\ell}$ defines a tensor on V_n. It is called the curvature tensor of the complex linear connection.

(D) Holonomy Groups of the Complex Linear Connection. The paths which we will examine in the present work will be supposed differentiable (of class C^∞) piece-wise, that is to say formed by the product of a finite number of differentiable paths.

A path z(t) of the principal fibre bundle E equipped with an infinitesimal connection is called 'horizontal' if all its tangents are horizontal.

The holonomy group at z is the set Ψ_z of the elements $g \in G$ such that z and zg^{-1} may be connected by a horizontal path.

It can be shown that Ψ_z is a subgroup of G and that the holonomy groups Ψ_z, $\Psi_{z'}$ at two points z,z' are two conjugate subgroups.

We call the restricted holonomy group at z the set σ_z of elements $g \in G$ such that z and zg^{-1} may be connected by a horizontal path of which the projection onto the base V_n will be a loop homotopic to 0.

One can show that σ_z is an invariant subgroup of ψ_z. It is also easy to prove that σ_z is the connected component of the identity of ψ_z.

Suppose that we have associated to each point x of V_n a neighbourhood u(x) of x. A loop L_x at x will be called small if it is contained in u(x). Let $L_{(x,y)}$ be a path joining x to a point y of V_n and L_y a small loop at y. The loop at x $\hat{L}_x = L(x,y)^{-1}.L_y.L(x,y)$ will be called a 'Lasso' with origin x. The factorization lemma of Lichnerowicz allows us to replace any loop at x homotopic to 0 by a loop formed with a finite product of lassos with origin x, of which the development (the solution of the differential equation $g^{-1}dg = w(dz)$ such that $g(0) = g_0$ is called the development of the path z(t) on G beginning with g_0) leads to the same element of the holonomy group σ_z at an arbitrary point z above x [2].

If V_n is equipped with a complex linear connection, the holonomy group of this connection turns out to be a group of linear transformations of T_x^c. It is this group which is usually called the homogeneous holonomy group of the complex linear connection at x. Similarly one can find the restricted homogeneous holonomy group of the connection at x.

For further details one is referred to [1] and [2].

2.1. Almost r-Product Structures (a.r.p.s.)

(A) Let V_n be a differentiable manifold of class C^∞. We will denote by T_x^c the complexified space of the tangent space T_x at

$x \in V_n$. An almost r-product structure on V_n is defined by the knowledge of (r+1) proper subspaces $T_o, \ldots, T_r$ of T_x^c such that

$T_x^c = T_0 \oplus \ldots \oplus T_r$ and $\dim(T_k) = n_k \neq 0$; $\sum_o^r n_k = n$.

Any vector v of T_x^c is the sum of vectors $P_k v \in T_k$. If λ is a nonzero complex constant and $r \geq 1$ is a positive integer, let us set

$$Jv = \lambda(P_0 v + wP_1 v + \ldots + w^r P_r v) \tag{2.1.1}$$

where $1, w, w^2, \ldots, w^r$ are (r+1) roots of unity.

We thus define on T_x^c a linear operator J such that

$$J^{r+1} = \lambda^{r+1} \text{ (identity)} \tag{2.1.2}$$

To this operator J, there corresponds a complex tensor defined by

$$(Jv)^i = F_j^i v^j \qquad v \in T_x^c$$

From the relation (2.1.2), we obtain

$$F_j^{k_1} F_{k_1}^{k_2} \ldots F_{k_r}^i = \lambda^{r+1} \delta_j^i \tag{2.1.3}$$

where δ_j^i is kronecker delta.

Conversely, let us suppose given on a differentiable manifold V_n, a field of tensors (F_j^i) of class C^∞, satisfying (2.1.3) at each point of V_n. We disregard the case where F_j^i is proportional to the kronecker tensor δ_j^i. At a point $x \in V_n$ the linear operator on T_x^c defined by the tensor (F_j^i) has eigenvalues $\lambda, \lambda w^1, \ldots, \lambda w^r$. Let T_k be the eigenspace of T_x^c generated by the eigenvectors corresponding to λw^k $(k = 0, \ldots, r)$; v being any vector whatsoever of T_x^c,

$$v_0 = v + \frac{1}{\lambda} Jv + \frac{1}{\lambda^2} J^2 v + \ldots + \frac{1}{\lambda^r} J^r v$$

is a vector of T_0. Indeed,

$$v_0 = (r+1)\ P_0 v + P_1 v \sum_0^r w^k + \ldots + P_r v \sum_0^r w^{rk}$$

$$= (r+1)\ P_0 v + 0 + \ldots + 0$$

Similarly, in general, one can say that

$$v_\ell = v + w^{\ell r} . Jv/\lambda + \ldots + w^{\ell r-(\ell-1)\ell}\ J^\ell v/\lambda^\ell + \ldots\ldots$$

$$+ w^\ell J^r v/\lambda^r$$

$$= (r+1) P_\ell v + 0 + \ldots + 0 + \ldots + 0$$

is a vector of T_ℓ, $0 \leq \ell \leq r$

Moreover, $v_0 + \ldots + v_r = (r+1)\ (P_0 v + \ldots + P_r v)$

$$= (r+1) v, \text{ i.e.,}$$

$$v = \frac{1}{r+1} (v_0 + \ldots + v_r)$$

Hence $T_x^c = T_0 \oplus \ldots \oplus T_r$

V_n is thus equipped with <u>a.r.p.s.</u>

(B) <u>Adapted Bases for an</u> <u>a.r.p.s.</u> Given V_n equipped with <u>a.r.p.s.</u>, let us consider a basis (e_{α_k}) of T_k such that $Je_{\alpha_k} = \lambda w^k e_{\alpha_k}$, $0 \leq k \leq r$ and $n_{k-1} + 1 \leq \alpha_k \leq n_k$, $n_{-1} = 0$. As T_x^c is a direct sum of $T_0, \ldots, T_r$, one can deduce that there exists a basis $(e_i) = (e_{\alpha_0}; \ldots; e_{\alpha_r})$ of T_x^c such that (e_{α_k}) is the basis of T_k and $Je_{\alpha_k} = \lambda w^k e_{\alpha_k}$. Such a basis of

T^c_x is called a basis adapted to <u>a.r.p.s.</u>

In the sequel, we assume the following notations.

We set $P_{-1} = 0$ and $P_k = n_0 + \dots + n_k$. Then $P_{k-1} < \alpha_k, \beta_k, \dots \leq P_k$. We further denote by N_k, the set of indices $(\alpha_k, \beta_k, \dots)$.

(C) <u>Matrix Representation of F^i_j.</u> Let us assume that F^i_j is referred to an adapted base. We know that if $v \in T^c_x$, then $(Jv)^i = F^i_j v^j$ where v^j are the components of v. Let us set $v = e_i$; we have $(Je_{\alpha_m})^i = F^i_j v^j$, where v^j are the components of e_{α_m}. $0 \leq m \leq r$

Also $(Je_{\alpha_m}) = \lambda w^m e_{\alpha_m}$.

Therefore

$$(Je_{\alpha_m})^i = (\lambda w^m e_{\alpha_m})^i = \lambda w^m (e_{\alpha_m})^i = F^i_{\alpha_0} v^{\alpha_0} + \dots + F^i_{\alpha_m} v^{\alpha_m} + \dots + F^i_{\alpha_r} v^{\alpha_r}.$$

As $v^{\alpha_m} = 1$ and $v^{\alpha_s} = 0$ for $s \neq m$ so we have

$$\lambda w^m (e_{\alpha_m})^i = F^i_{\alpha_m} \tag{2.1.5}$$

Now $(e_{\alpha_m})^{\beta_m} = \delta^{\beta_m}_{\alpha_m}$ etc and $(e_{\alpha_m})^{\beta_s} = 0$ for $(s = 0, \dots, \widehat{m}, \dots, r)$, where $\widehat{\ }$ denotes the missing integer.

Hence $F^{\beta_m}_{\alpha_m} = \lambda\, w^m \delta^{\beta_m}_{\alpha_m}$ etc and $F^{\beta_s}_{\alpha_m} = 0$ for $(s = 0, \dots, \widehat{m}, \dots, r)$.

We conclude that F^i_j is represented by a matrix of the form

$$F^i_j = \begin{vmatrix} \lambda w^0 I_{00} & 0_{01} & \cdots & 0_{0r} \\ 0_{10} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0_{r0} & \cdots & \cdots & \lambda w^r I_{rr} \end{vmatrix} \tag{2.1.6}$$

where (I_{mm}) is the $n_m \times n_m$ unit matrix and (0_{sm}) is the $n_s \times n_m$ zero matrix $(s \neq m;\ s,m = 0,\ldots,r)$.

(D) Let (e_i) and $(e_{j'})$ be adapted basis at $x \in V_n$. Then

$$e_{j'} = A^{i}_{j'} e_i \tag{2.1.7}$$

Since $T^c_x = T_0 \oplus \ldots \oplus T_r$ and each T_k is invariant under J, we have:

$$e_{\beta'_k} = A^{\alpha_k}_{\beta'_k} e_{\alpha_k}, \quad k = 0,\ldots,r \tag{2.1.7a}$$

and setting $(A^{\alpha_k}_{\beta'_k}) = A_{kk} \in GL(n_k,c)$ we have that the matrix $A = (A^i_{j'})$ is of the form

$$A = \begin{vmatrix} A_{00} & 0_{01} & \cdots & 0_{0r} \\ 0_{10} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0_{r0} & \cdots & \cdots & A_{rr} \end{vmatrix} \tag{2.1.8}$$

We shall denote the set of all matrices of the form A by $G(n_r)$

<u>LEMMA 1:</u> $G(n_r)$ is a Lie subgroup of $GL(n,c)$

PROOF: We must prove that (a) $G(n_r)$ is a multiplicative subgroup (abstract) and (b) $G(n_r)$ is an analytic subgroup of $GL(n,c)$.

(a) Let $A, A' \in G(n_r)$. Using multiplication by blocks we have

$$A.A' = \begin{vmatrix} A_{00} & & 0 \\ & \ddots & \\ 0 & & A_{rr} \end{vmatrix} \cdot \begin{vmatrix} A'_{00} & & 0 \\ & \ddots & \\ 0 & & A'_{rr} \end{vmatrix} = \begin{vmatrix} A_{00}A'_{00} & & 0 \\ & \ddots & \\ 0 & & A_{rr}A'_{rr} \end{vmatrix} \in G(n_r)$$

$$\text{and } A^{-1} = \begin{vmatrix} A^{-1}_{00} & & 0 \\ & \ddots & \\ 0 & & A^{-1}_{rr} \end{vmatrix} \in G(n_r).$$

Hence $G(n_r)$ is a multiplicative subgroup (abstract) of $GL(n,c)$.

(b) $G(n_r)$ is closed because the equations (2.1.7a) are satisfied. Also any closed subgroup of a Lie group G is an analytic subgroup of G.[22] Hence $G(n_r)$ is an analytic subgroup of $GL(n,c)$.

It is also very easy to prove the following lemma.

<u>LEMMA 2:</u> $G(n_r)$ is composed of all the elements of $GL(n,c)$ which commute with the matrix F^i_j.

2.2 G_p-Connections

(A) Let $E_p(V_n)$ be the set of all the bases adapted to <u>a.r.p.s.</u> relative to the different points of V_n, and p be the canonical mapping

$$p : E_p(V_n) \rightarrow V_n$$

such that an adapted basis at x is made to correspond to the point x itself. $E_p(V_n)$ has, with respect to p, a natural structure of a principal fibre bundle of base space V_n whose structure group is the subgroup $G(n_r)$ of $GL(n,c)$... (For more details we refer to the Appendix I).

Any infinitesimal connection defined on $E_p(V_n)$ will be called almost r-product connection, briefly G_p-connection.

Given a covering of V_n by neighbourhoods endowed with local cross sections of $E_p(V_n)$, a G_p-connection may be defined in each neighbourhood u by a local form with values in the Lie algebra of $G(n_r)$; such a form may be represented at $x \in V_n$ by means of n x n matrices whose elements are local pfaffian forms (with complex values) denoted by

$$\pi_u = (\pi^i_j) \qquad (2.2.1)$$

Hence a G_p-connection is represented by the matrix

$$\begin{vmatrix} \pi_{00} & 0_{01} & \cdot & \cdot & \cdot & \cdot & 0_{0r} \\ 0_{10} & \cdot & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & & \cdot \\ 0_{r0} & \cdot & \cdot & \cdot & \cdot & \cdot & \pi_{rr} \end{vmatrix} \qquad (2.2.2)$$

where π_{kk} is the matrix of the same kind as A_{kk} in (2.1.8) but without the restriction of the non-singularity of A_{kk}'s.

<u>THEOREM 1</u>: With respect to G_p-connection the absolute differential of (F^i_j) is equal to zero i.e. $\nabla(F^i_j) = 0$.

PROOF: We refer the tensor (F^i_j) to bases adapted to the G_p-structure. The absolute differential of (F^i_j) is given by

$$\nabla(F^i_j) = d(F^i_j) + \pi^i_s F^s_j - \pi^s_j F^i_s$$

Since (F^i_j) is given by (2.1.6), $d(F^i_j) = 0$. Also, taking into consideration the form of the matrix (2.1.6) and the matrix (2.2.2), we have

$$\nabla F^{\beta_o}_{\alpha_o} = \pi^{\beta_o}_a F^a_{\alpha_o} - \pi^a_{\alpha_o} F^{\beta_o}_a = \pi^{\beta_o}_{\alpha_o} F^{\alpha_o}_{\alpha_o} - \pi^{\alpha_o}_{\alpha_o} F^{\beta_o}_{\alpha_o} = \pi^{\beta_o}_{\alpha_o} \lambda \delta^{\alpha_o}_{\alpha_o} - \pi^{\alpha_o}_{\alpha_o} \lambda \delta^{\beta_o}_{\alpha_o} = \lambda \pi^{\beta_o}_{\alpha_o} - \lambda \pi^{\beta_o}_{\alpha_o} = 0.$$

$\vdots$

$$\nabla F^{\beta_r}_{\alpha_r} = \pi^{\beta_r}_a F^a_{\alpha_r} - \pi^a_{\alpha_r} F^{\beta_r}_a = \pi^{\beta_r}_{\alpha_r} F^{\alpha_r}_{\alpha_r} - \pi^{\alpha_r}_{\alpha_r} F^{\beta_r}_{\alpha_r} = \pi^{\beta_r}_{\alpha_r} \lambda w^r \delta^{\alpha_r}_{\alpha_r} - \pi^{\alpha_r}_{\alpha_r} \lambda w^r \delta^{\beta_r}_{\alpha_r}$$

$$= \lambda w^r \left(\pi^{\beta_r}_{\alpha_r} - \pi^{\beta_r}_{\alpha_r}\right) = 0.$$

$$\nabla F^{\beta_a}_{\alpha_l} = \pi^{\beta_a}_c F^c_{\alpha_l} - \pi^c_{\alpha_l} F^{\beta_a}_c = \pi^{\beta_a}_{\alpha_l} F^{\alpha_l}_{\alpha_l} - \pi^{\alpha_a}_{\alpha_l} F^{\beta_a}_{\alpha_a} = \pi^{\beta_a}_{\alpha_l} \lambda w^l \delta^{\alpha_l}_{\alpha_l} - \pi^{\alpha_a}_{\alpha_l} \lambda w^a \delta^{\beta_a}_{\alpha_a}$$

$$= \pi^{\beta_a}_{\alpha_l} \lambda w^l - \pi^{\beta_a}_{\alpha_l} \lambda w^a$$

$$= \lambda \pi^{\beta_a}_{\alpha_l} (w^l - w^a) \quad \text{for} \quad 0 \leq a \neq l \leq r$$

$$= 0 \quad \text{as} \quad \pi^{\beta_a}_{\alpha_l} = 0, \quad \lambda \neq 0 \quad \text{and} \quad w^l \neq w^a.$$

Hence $\nabla(F^i_j) = 0$ (2.2.3)

Conversely, let us consider a complex linear connection and a covering of V_n by neighbourhoods equipped with local cross-sections of $E_p(V_n)$. This connection may be defined on each neighbourhood by a local form, with values in the Lie algebra of GL(n,c) represented by a matrix (w^i_j) whose elements are complex-valued local pfaffian forms. We will say that (w^i_j) defines the connection relative to the adapted bases of the considered local section. In order that the given connection can be identified with a G_p-connection, it is necessary and sufficient that (w^i_j) belongs to the Lie algebra of the structure group $G(n_r)$ of $E_p(V_n)$ i.e. to be given by the matrix of the form (2.2.2). Comparing with the relations obtained in theorem 1, we obtain the following theorem:

<u>THEOREM 2</u>: In order that a complex linear connection may be identified with a G_p-connection, it is necessary and sufficient that the tensor (F^i_j) have a zero absolute differential with respect to this connection.

2.3. Curvature Tensor of a G_p-Connection

Suppose that a G_p-connection is given on V_n equipped with <u>a.r.p.s</u>. The curvature of this connection is defined by

$$\Omega^i_j = d\pi^i_j + \pi^i_\ell \wedge \pi^\ell_j \qquad (2.3.1)$$

where the tensor 2-form (2.3.1.) is the form of the connection. From (2.3.1) we get, by using the matrix (2.2.2)

$$\Omega^{\beta_0}_{\alpha_0} = d\pi^{\beta_0}_{\alpha_0} + \pi^{\beta_0}_{\lambda_0} \wedge \pi^{\lambda_0}_{\alpha_0} + \cdots + \pi^{\beta_0}_{\lambda_r} \wedge \pi^{\lambda_r}_{\alpha_0} = d\pi^{\beta_0}_{\alpha_0} + \pi^{\beta_0}_{\lambda_0} \wedge \pi^{\lambda_0}_{\alpha_0}$$

$$\vdots$$

$$\Omega^{\beta_r}_{\alpha_r} = d\pi^{\beta_r}_{\alpha_r} + \pi^{\beta_r}_{\lambda_0} \wedge \pi^{\lambda_0}_{\alpha_r} + \cdots + \pi^{\beta_r}_{\lambda_r} \wedge \pi^{\lambda_r}_{\alpha_r} = d\pi^{\beta_r}_{\alpha_r} + \pi^{\beta_r}_{\lambda_r} \wedge \pi^{\lambda_r}_{\alpha_r}$$

$$\Omega^{\beta_b}_{\alpha_a} = d\pi^{\beta_b}_{\alpha_a} + \pi^{\beta_b}_{\lambda_0} \wedge \pi^{\lambda_0}_{\alpha_a} + \cdots + \pi^{\beta_b}_{\lambda_r} \wedge \pi^{\lambda_r}_{\alpha_a}$$

$$= 0 + 0 + \cdots + 0.$$

$$= 0 \qquad \text{for} \quad 0 \leq a \neq b \leq r$$

Hence we have the following matrix representation of Ω^i_j:

$$(\Omega^i_j) = \begin{vmatrix} \Omega_{00} & 0_{01} & \cdots & 0_{0r} \\ 0_{10} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0_{r0} & \cdots & \cdots & \Omega_{rr} \end{vmatrix} \qquad (2.3.2)$$

where $(\Omega^{\beta_k}_{\alpha_k}) = \Omega_{kk}$.

By contraction on the upper and lower indices one obtains

$$\Omega^{\beta_0}_{\beta_0} = d\pi^{\beta_0}_{\beta_0};\ldots\ldots;\Omega^{\beta_r}_{\beta_r} = d\pi^{\beta_r}_{\beta_r} .$$

Let us put $\Psi_m = \lambda w^m \Omega^{\beta_m}_{\beta_m}$ for each $(m = 0,\ldots,r)$.

This defines Ψ_0, Ψ_1, ...,Ψ_r 2-forms with scalar values (complex). We will say that Ψ_m is the m-th characteristic form of the G_p-connection. One has $\Psi_m = \lambda w^m \, d\pi^{\beta_m}_{\beta_m}$ for every m. It is easy to see from these results that Ψ_m's are closed forms.

Let us assume that there is given on V_n a linear connection (real or complex), say (w^i_j). Let $(\overset{0}{\Omega}{}^i_j)$ be its curvature form. The scalar 2-form $(\overset{0}{\Omega}{}^i_j) = d(w^i_j)$ is closed and homologous to 0.

Let us set $X = \pi^i_j - d(w^i_j)$. X defines a scalar 1-form. We have

$$
\begin{aligned}
(\Psi_0 + \frac{\Psi_1}{w^1} + \cdots + \frac{\Psi_r}{w^r}) - \lambda \overset{0}{\Omega}{}^i_i &= \lambda \Omega^{\beta_0}_{\beta_0} + \cdots + \lambda \Omega^{\beta_r}_{\beta_r} - \lambda \overset{0}{\Omega}{}^i_i \\
&= \lambda (\Omega^i_i - \overset{0}{\Omega}{}^i_i) \\
&= \lambda (d\pi^i_i - d\omega^i_i) \\
&= \lambda d(\pi^i_i - \omega^i_i) \qquad (2.3.3) \\
&= \lambda d(X) \\
&= d(\lambda X).
\end{aligned}
$$

Hence $(\Psi_0 + \frac{\Psi_1}{w^1} + \ldots + \frac{\Psi_r}{w^r}) - \lambda(\overset{0}{\Omega}{}^i_j)$ is homologous to 0.

Finally if one takes a Riemannian connection, $\overset{0}{\Omega}{}^i_j = 0$, which means that $(\Psi_0 + \frac{\Psi_1}{w^1} + \ldots + \frac{\Psi_r}{w^r})$ is homologous to 0.

One can further prove that the homology class in V_n of the forms Ψ_m does not depend on the considered G_p-connection. Indeed, let us suppose given another G_p-connection defined relative to adapted bases by $(\hat{\pi}^{\gamma_0}_{\beta_0}, \ldots \hat{\pi}^{\gamma_r}_{\beta_r})$. Let $\hat{\Psi}_m$ be the m-th characteristic form of this connection.

Let us define $\phi = \pi^{\gamma_m}_{\gamma_m} - \hat{\pi}^{\gamma_m}_{\gamma_m}$, a scalar 1-form. We have

$$
\begin{aligned}
\Psi_m - \hat{\Psi}_m &= \lambda \overset{m}{\omega} \Omega^{\alpha_m}_{\alpha_m} - \lambda \overset{m}{\omega} \hat{\Omega}^{\alpha_m}_{\alpha_m} \\
&= \lambda \overset{m}{\omega} (d\pi^{\alpha_m}_{\alpha_m} - d\hat{\pi}^{\alpha_m}_{\alpha_m}) \qquad (2.3.4) \\
&= \lambda \overset{m}{\omega} d(\pi^{\alpha_m}_{\alpha_m} - \hat{\pi}^{\alpha_m}_{\alpha_m}) \\
&= \lambda \overset{m}{\omega} d\phi = d(\lambda \overset{m}{\omega} \phi).
\end{aligned}
$$

This means that the cohomology class of $\hat{\Psi}_m$ is the same as that of Ψ_m. Obviously, this result is true for $(m = 0,\ldots,r)$. Hence the statement is justified. One calls such a class the characteristic cohomology class of the a.r.p.s. determined by the operator J. This leads to the following theorem:

THEOREM 3: The characteristic 2-forms of all the G_p-connections belong to the same cohomology class of degree 2.

2.4 The Holonomy Group of a G_p-Connection

(A) We shall prove the following theorem:

THEOREM 4: A necessary and sufficient condition in order that a complex linear connection in a manifold V_n be a G_p-connection of an a.r.p.s. is that the holonomy group of the connection be a subgroup of $G(n_r)$.

PROOF: If V_n is endowed with a G_p-connection, any horizontal path constructed on $E_c(V_n)$ relative to the complex linear connection identifies with the G_p-connection, and, starting at an adapted base s, ends at an adapted base. One deduces from this that the holonomy group at s of this connection is a subgroup[2] of the structure group $G(n_r)$ of the fibre bundle $E_p(V_n)$.

Conversely, let V_n be a differentiable manifold endowed with a complex linear connection. Let us consider the point $x \in V_n$, and assume that there exists at x a complex basis s such that the holonomy group Ψ_s of the connection at s is a subgroup of $G(n_r)$; the elements of Ψ_s are matrices of the form (2.1.8). Let λ be a nonzero complex constant, and let us consider at the point x the tensor whose components with respect to the base s are

$$F_{\gamma_m}^{\beta_m} = \lambda w\ \delta_{\gamma_m}^{\beta_m} \text{ etc}$$

and $F_{\gamma_m}^{\beta_\ell} = 0$ for $(\ell \neq m;\ \ell, m = 0,\ldots,r)$.

This is the tensor represented by the matrix (2.1.6). It will be invariant under the transformations by the elements of ψ_s because $\alpha J = J\alpha$ is trivially true. On the other hand if one computes the powers $J^2,\ldots,J^r$, J^{r+1}, one obtains

$$J^2 = \begin{vmatrix} \lambda^2 w^0 I_{00} & 0_{01} & \cdots & 0_{0r} \\ 0_{10} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0_{r0} & \cdots & \cdots & \lambda^2 w^{2r} I_{rr} \end{vmatrix}, \ldots\ J^r = \begin{vmatrix} \lambda^r w^0 I_{00} & 0_{01} & \cdots & 0_{0r} \\ 0_{10} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0_{r0} & \cdots & \cdots & \lambda^r w^{rr} I_{rr} \end{vmatrix}$$

$$J^{r+1} = \begin{vmatrix} \lambda^{r+1} w^0 I_{00} & 0_{01} & \cdots & 0_{0r} \\ 0_{10} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0_{r0} & \cdots & \cdots & \lambda^{r+1} w^{r(r+1)} I_{rr} \end{vmatrix} = \lambda^{r+1} \text{ . Identity} \qquad (2.3.5)$$

The latter of the results (2.3.5) provides

$$F_j^{k_1} F_{k_1}^{k_2} \ldots F_{k_r}^{i} = \lambda^{r+1} . \delta_j^i \qquad (2.3.6)$$

From the tensor F^i_j we deduce by parallel transport in V_n a tensor F^i_j defined over the whole manifold V_n with absolute differential equal to zero, [2]. Moreover, the relations (2.3.5) and (2.3.6) remain true at every point of V_n. Since $\nabla(F^i_j) = 0$, then by the theorem 2, the given connection may be identified with a G_p-connection.

(B) The restricted holonomy group

Let $\tilde{V}_n$[1)] be the universal cover[2] of the differentiable manifold V_n equipped with a.r.p.s., and q the cannonical mapping $q:\tilde{V}_n \to V_n$. Each point of $\tilde{V}_n$ admits an open neighbourhood V such that q is a homeomorphism of V onto q(V). One can thus define on $\tilde{V}_n$ an a.r.p.s. by the inverse image under q of the a.r.p.s. given on V_n. In a similar way, one can define a G_p-connection on $\tilde{V}_n$. Its homogeneous holonomy group[2] at the point $\tilde{x} \in \tilde{V}_n$ may be identified with the restricted homogeneous holonomy group of the given connection at the point $x = q\tilde{x}$.

LEMMA 3: Let $S_kG(n_r)$ denote the set of matrices of $G(n_r)$ for which $A_{kk} = I_{kk}$ for each $k = 0,1,..,r$. Then each $S_kG(n_r)$ is an invariant subgroup of $G(n_r)$.

PROOF: If B and B' belong to $S_kG(n_r)$, then by definition $\det|A_{kk}| = \det|A'_{kk}| = 1$. We must prove that $(B'^{-1})B$ and $(B'^{-1}).B.B'$ also belong to $S_kG(n_r)$. We shall prove for a fixed k only since the other cases can be proved analogously.

---

1) In the sequel we shall replace V_n by its universal cover $\tilde{V}_n$ without changing the notations.

$$(B'^{-1})\,B = \begin{vmatrix} A'^{-1}_{00} \cdot A_{00} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & A'^{-1}_{rr} \cdot A_{rr} \end{vmatrix}$$

where $\det\left| A'^{-1}_{kk} \cdot A_{kk} \right| = \det\left| A'^{-1}_{kk} \right| \cdot \det\left| A_{kk} \right| = (\det\left| A'_{kk} \right|)^{-1} \cdot \det\left| A_{kk} \right| = 1$

which implies that $(B'^{-1})B$ belongs to $S_kG(n_r)$.

Also

$$(B'^{-1})B.B' = \begin{vmatrix} A'^{-1}_{00} \cdot A_{00} \cdot A'_{00} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & A'^{-1}_{rr} \cdot A_{rr} \cdot A'_{rr} \end{vmatrix}$$

where $\det\left| A'^{-1}_{kk} \cdot A_{kk} \cdot A'_{kk} \right| = \det\left| A'^{-1}_{kk} \right| \cdot \det\left| A_{kk} \right| \cdot \det\left| A'_{kk} \right|$

$$= (\det\left| A_{kk} \right|)^{-1} \cdot \det\left| A_{kk} \right| \cdot \det\left| A'_{kk} \right|$$

$$= 1 \quad . \quad 1 \quad . \quad 1 = 1$$

Hence $(B'^{-1})B.B'$ belongs to $S_kG(n_r)$.

THEOREM 5: In order that the restricted homogeneous holonomy group of a G_p-connection be a subgroup of each $S_kG(n_r)$, it is necessary and sufficient that the characteristic forms Ψ_k,s of the connection be zero at any point.

PROOF: Let s be an adapted basis at the point $x_0 \in V_n$. Let us assume that the restricted holonomy group σ_s is a subgroup of $S_kG(n_r)$ for a fixed k. This assumption will be true at every point of $E_p(V_n)$. We introduce at the point x_0 the covariant tensor t_0 of order n_k, whose components with respect to the basis s are

$$t_{i_1,\ldots,i_{n_k}} = \varepsilon_{i_1,\ldots,i_{n_k}}^{n_{k-1}+1,\ldots,n_{k-1}+p,\ldots,n_k} \qquad (2.3.7)$$

It can be shown that the tensor t_0 is invariant under σ_s. Indeed, $t_{i_1,\ldots,i_{n_k}}$ are different from zero only when $i_1,\ldots,i_{n_k}$ is a permutation of $n_{k-1}+1,\ldots,n_k$. On the other hand

$$t_{j_1',\ldots,j_{n_k}'} = A_{j_1'}^{i_1}\cdots A_{j_{n_k}'}^{i_{n_k}}\varepsilon_{i_1,\ldots,i_{n_k}}^{n_{k-1}+1,\ldots,n_k},$$

where $t_{\alpha'_{(k-1)_1},\ldots,\alpha'_{(k-1)_{n_k}}} = A_{\alpha'_{(k-1)_1}}^{\beta_{(k-1)_1}}\cdots A_{\alpha'_{(k-1)_{n_k}}}^{\beta_{(k-1)_{n_k}}}\cdot\varepsilon_{\beta_{(k-1)_1},\ldots,\beta_{(k-1)_{n_k}}}^{n_{k-1}+1,\ldots,n_k}$

and $t_{\alpha'_{(k-1)_1},\ldots,\alpha'_{m_{(p)}},\ldots,\alpha'_{(k-1)_{n_k}}} = 0$

for $(m = 0,\ldots,\widehat{(k-1)},\ldots,r)$ and $1 \leq p \leq n_k$.

Now $t_{\alpha'_{(k-1)_1},\ldots,\alpha'_{(k-1)_{n_k}}} = A^{\beta_{(k-1)_1}}_{\alpha'_{(k-1)_1}} \cdots A^{\beta_{(k-1)_{n_k}}}_{\alpha'_{(k-1)_{n_k}}} \cdot \mathcal{E}^{n_{k-1}+1,\ldots,n_k}_{\beta_{(k-1)_1},\ldots,\beta_{(k-1)_{n_k}}}$

$$= \det \begin{vmatrix} A^{n_{k-1}+1}_{\alpha'_{(k-1)_1}} & \cdots & A^{n_k}_{\alpha'_{(k-1)_1}} \\ \vdots & & \vdots \\ A^{n_{k-1}+1}_{\alpha'_{(k-1)_{n_k}}} & \cdots & A^{n_k}_{\alpha'_{(k-1)_{n_k}}} \end{vmatrix}$$

$$= \mathcal{E}^{n_{k-1}+1,\ldots,n_k}_{\alpha'_{(k-1)_1},\ldots,\alpha'_{(k-1)_{n_k}}},$$

Hence $t_{j'_1,\ldots,j'_{n_k}} = \mathcal{E}^{n_{k-1}+1,\ldots\ldots,n_k}_{j'_1,\ldots,j'_{n_k}}$.

This justifies the statement.

By parallel transport, t_0 generates a tensor defined on the whole V_n, which we denote by t. We have $\nabla t = 0$. If U is an open neighbourhood of V_n endowed with a local cross-section of $E_p(V_n)$, there exists a differentiable function e^f with complex values $\neq 0$ defined on U such that we have with respect to U,

$$t_{i_1,\ldots,i_{n_k}} = \mathcal{E}^{n_{k-1}+1,\ldots,n_k}_{i_1,\ldots,i_{n_k}} \cdot e^f$$

$$\nabla t_{i_1,\ldots,i_{n_k}} = (d e^f) \cdot \mathcal{E}^{n_{k-1}+1,\ldots,n_k}_{i_1,\ldots,i_{n_k}} + e^f \nabla \mathcal{E}^{n_{k-1}+1,\ldots,n_k}_{i_1,\ldots,i_{n_k}} .$$

On the other hand,

$$\nabla \mathcal{E}_{i_1,\ldots,i_{n_k}}^{n_{k-1}+1,\ldots,n_k} = -\pi_{i_1}^{a}\,\mathcal{E}_{a,i_2,\ldots,i_{n_k}}^{n_{k-1}+1,\ldots,n_k} - \cdots - \pi_{i_{n_k}}^{a}\,\mathcal{E}_{i_1,\ldots,a}^{n_{k-1}+1,\ldots,n_k}$$

or

$$\nabla \mathcal{E}_{\alpha_{(k-1)_1},\ldots,\alpha_{(k-1)_{n_k}}}^{n_{k-1}+1,\ldots,n_k} = -\pi_{\alpha_{(k-1)_1}}^{a} \cdot \mathcal{E}_{a,\alpha_{(k-1)_2},\ldots,\alpha_{(k-1)_{n_k}}}^{n_{k-1}+1,\ldots,n_k} - \cdots$$

$$- \pi_{\alpha_{(k-1)_{n_k}}}^{a} \cdot \mathcal{E}_{\alpha_{(k-1)_1},\ldots,a}^{n_{k-1}+1,\ldots,n_k} .$$

$$= -\left(\pi_{\alpha_{(k-1)_1}}^{\alpha_{(k-1)_1}} + \cdots + \pi_{\alpha_{(k-1)_{n_k}}}^{\alpha_{(k-1)_{n_k}}}\right) \cdot \mathcal{E}_{\alpha_{(k-1)_1},\ldots,\alpha_{(k-1)_{n_k}}}^{n_{k-1}+1,\ldots,n_k}$$

$$= -\left(\pi_{\alpha_k}^{\alpha_k}\right) \mathcal{E}_{\alpha_{(k-1)_1},\ldots,\alpha_{(k-1)_{n_k}}}^{n_{k-1}+1,\ldots,n_k} .$$

Thus we have

$$\nabla t_{i_1,\ldots,i_{n_k}} = e^{f}\left(df - \pi_{\alpha_k}^{\alpha_k}\right) \cdot \mathcal{E}_{i_1,\ldots,i_{n_k}}^{n_{k-1}+1,\ldots,n_k} .$$

Since $\nabla t_{i_1,\ldots,i_{n_k}} = 0$, we have $df = \pi_{\alpha_k}^{\alpha_k}$. Also

$$\Psi_k = \lambda w^k \Omega_{\alpha_k}^{\alpha_k} = \lambda w^k d\pi_{\alpha_k}^{\alpha_k} = \lambda w^k d^2 f = 0 \text{ as } \lambda w^k \neq 0.$$

Hence the characteristic form ψ_k is everywhere zero.

We would have been able to make an analogous argument by varying k and $S_k G(n_r)$ such that $\det|A_{kk}| = 1$ for $0 \leq k \leq r$. Finally we say that all ψ_k's are everywhere zero.

Conversely, let us consider a differentiable manifold V_n, simply connected, equipped with a G_p-connection, and let us assume that ψ_k is zero at any point of V_n. Relative to each local section of $E_p(V_n)$ one has $d\pi_{\alpha_k}^{\alpha_k} = 0$. Let x be a point of V_n. Then one is able to find an open neighbourhood U of x equipped with a local section of $E_p(V_n)$ and a function f with complex values $\neq 0$ defined on U such that with respect to the cross-section, $\pi_{\alpha_k}^{\alpha_k} = df$.

We consider the covariant tensor t of the order n_k, defined on U, whose components relative to the local section are

$$t_{i_1,\ldots,i_{n_k}} = \varepsilon_{i_1,\ldots,i_{n_k}}^{n_{k-1}+1,\ldots,n_k} \cdot e^{f} .$$

Its absolute differential is determined by

$$\nabla t_{i_1,\ldots,i_{n_k}} = \varepsilon_{i_1,\ldots,i_{n_k}}^{n_{k-1}+1,\ldots,n_k} \cdot e^{f}\left(df - \pi_{\alpha_k}^{\alpha_k}\right) = 0 .$$

Given an adapted base s at the point x, the holonomy group σ_s of the connection at s is, as we have seen, a subgroup of $G(n_r)$. Since t is zero on U, the elements of σ_s obtained by development of the loops at x situated in U leave t invariant. One deduces from this that they belong to an invariant subgroup $S_kG(n_r)$. Since we may associate with every point x such a neighbourhood U, it follows from the Lichnerowicz factorization lemma[2], that for every $s \in E_p(V_n)$, σ_s is a subgroup of $S_kG(n_r)$. Proof is the same, by varying k from 0 to r.

CHAPTER 3

Singular Riemannian Structures Compatible with the a.r.p.s.

3.0 Introduction

Differential geometry is concerned with the study of geometric objects defined on differentiable manifolds. One of the simplest geometric objects is a field of non-singular, symmetric, second order covariant tensors, and the branch of differential geometry which studies the structures associated with this object is called Riemannian geometry.

A differentiable manifold V_n of class C^b is said to admit a structure of a Riemannian manifold of class $C^a (a < b-1)$ if there exists on V_n a symmetric tensor G of class C^a such that, if g_{ij} are the components of this tensor for the arbitrary frames then the associated quadratic form is $ds^2 = g_{ij}\ \theta^i . \theta^j$.

We shall assume that the quadratic form is positive definite (which is of greater interest from the geometric point of view) which implies that $\det|G| > 0$. The more general case of indefinite G, with $\det |G| \neq 0$ is important for the theory of relativity. Alternatively, G is given by associating with the tangent space T_x^c at $x \in V_n$ a scalar product:

$$(u,v) = g_{ij} u^i v^j, \quad \text{where } u,v \in T_x^c$$

The well-known theorem of Whitney[2] states that a differentiable manifold of class C^b(b is positive or $b = \infty$) always admits a structure of a Riemannian manifold of class C^{b-1}. On the other hand, such a result is not true in general for the real analytic manifolds. We can introduce only one Riemannian metric of class C^{∞}. This classical result is accepted.

The object of this chapter is to investigate some properties of G defined on a differentiable manifold V_n, equipped with a.r.p.s., by constructing over it a singular Riemannian structure.

3.1 R_p-Structures

Let us suppose that one has defined on V_n, equipped with a.r.p.s., a complex metric of class C^{∞}, that is, a symmetric tensor $G = (g_{ij})$ for which the components, in a system of local co-ordinates (x^i), are complex functions of the (x^i) of class C^{∞}, with the condition that the rank of $G = (g_{ij})$ is n_0. We will say that the metric G is compatible with a.r.p.s. if the scalar product of two arbitrary vectors of T_x^c is proportional to the scalar product of one of the vectors with the transform of the other by J. This means that for any pair of vectors $u, v \in T_x^c$, one has

$$(u,Jv) = \lambda(u,v) \tag{3.1.1}$$

where (u,v) denotes the scalar product $g_{ij}u^i v^j$

The condition (3.1.1) can be expressed as

$$g_{ij}u^i F^j_k v^k = \lambda g_{ik} u^i v^k$$

or

$$g_{ij}F^j_k = \lambda g_{ik}$$

or

$$JG = \lambda G. \tag{3.1.2}$$

We will say, in the above case, that V_n is endowed with a singular Riemannian structure subordinate to the a.r.p.s.; we call such a structure an R_p-structure.

With respect to a basis adapted to a.r.p.s., (3.1.2) can be written as

$$\begin{vmatrix} \lambda w^0 I_{00} & & 0 \\ & \ddots & \\ 0 & & \lambda w^r I_{rr} \end{vmatrix} \cdot \begin{vmatrix} G_{00} & \cdots & G_{0r} \\ \vdots & \ddots & \vdots \\ G_{r0} & \cdots & G_{rr} \end{vmatrix} = \lambda \begin{vmatrix} G_{00} & \cdots & G_{0r} \\ \vdots & \ddots & \vdots \\ G_{r0} & \cdots & G_{rr} \end{vmatrix}$$

It is easy to see from above that G has the form:

$$G = \begin{vmatrix} G_{00} & 0_{01} & \cdots & 0_{0r} \\ 0_{10} & 0_{11} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0_{r0} & \cdots & \cdots & 0_{rr} \end{vmatrix}$$

where $G_{00} = (G_{\alpha_0\beta_0})$ is an $n_0 x n_0$ matrix of rank n_0 (3.1.3)

THEOREM 6: Given an arbitrary quadratic form on V_n defined by a tensor $M = (m_{ij})$ of rank n and a linear operator J on T^c_x such that $J^{r+1} = \lambda^{r+1}$ (identity), one can always obtain from M an R_p-structure

PROOF: Let us set $G = J^r.M + J^{r-1}\lambda M + \ldots + \lambda^r M$ (3.1.4)

We shall prove that one can take for G the matrix defined by (3.1.4). Operating by J on both sides of (3.1.4) we have

$$JG = J^{r+1}.M + J^r\lambda M + J^{r-1}\lambda^2 M + \ldots + \lambda^r JM$$

$$= \lambda^{r+1}.M + \lambda J^r M + \lambda^2 J^{r-1} M + \ldots + \lambda^r JM$$

$$= \lambda(\lambda^r M + J^r M + \lambda J^{r-1} M + \ldots + \lambda^{r-1} JM)$$

$$= \lambda G.$$

Hence we see that (3.1.2) is satisfied.

Moreover, from (3.1.4) with respect to a basis adapted to a.r.p.s., we have

$$G = \begin{vmatrix} (r+1)\lambda^r M_{00} & 0_{01} \cdots & 0_{0r} \\ 0_{10} & 0_{11} & \vdots \\ \vdots & \ddots & \vdots \\ 0_{r0} & \cdots & 0_{rr} \end{vmatrix}$$

where $M_{00} = (m_{\alpha_0\beta_0})$ is an $n_0 \times n_0$ matrix. (3.1.5)

This means that $g_{\alpha_0\beta_0} = (r+1)\lambda^r m_{\alpha_0\beta_0}$. Since M is of rank n, we have det $|g_{\alpha_0\beta_0}| \neq 0$. Moreover, we note that under a change of basis

$m_{j'k'} = A^h_{j'} A^i_{k'} m_{hi}$, in particular we have

$$m_{\alpha'_0\beta'_0} = A^h_{\alpha'_0} A^i_{\beta'_0} m_{hi} = A^{\lambda_0}_{\alpha'_0} A^{\mu_0}_{\beta'_0} m_{\lambda_0\mu_0}$$

so that det $|M'_{00}| = (\det|A_{00}|)^2.(\det|M_{00}|) \neq 0$

Hence $G = (g_{ij})$ is of rank n_0.

3.2 R_p-Adapted Bases

We consider at a point x of V_n a basis (e_i) adapted to an a.r.p.s. and the corresponding dual basis (θ^i). We have

$$ds^2 = g_{ij}\theta^i . \theta^j = g_{\alpha_0\beta_0}\theta^{\alpha_0} . \theta^{\beta_0}$$

Since the quadratic form is of rank n_0, one can always find an orthonormal base $(e_{\alpha'_0})$ for T_0 by taking suitable linear combinations of (e_{α_0}). By doing so ds^2 can be written as

$$ds^2 = \sum_{1}^{n_0} (\theta^{\alpha_0})^2.$$

One can also find families of vectors $(e_{\alpha'_a})$, $1 \leq a \leq r$ by taking suitable linear combinations of (e_{α_a}) respectively, such that $Je_{\alpha'_a} = \lambda w^a e_{\alpha'_a}$. It is quite clear that the new vectors $(e_{i'}) = (e_{\alpha'_0}; e_{\alpha'_1}; \ldots ; e_{\alpha'_r})$ form an adapted basis for which $(e_{\alpha'_0})$ are orthonormal. In this case we will say that such a basis is adapted to the subordinate R_p-structure. Such a basis will be called R_p-adapted basis.

Suppose now that (e_i) and $(e_{j'})$ are two R_p-adapted bases. Then we have:

$$g_{k'l'} = A^i_{k'} A^j_{l'} g_{ij} \qquad (3.2.1)$$

where $(A^i_{k'}) = A = \begin{vmatrix} A_{00} & & 0 \\ & \ddots & \\ 0 & & A_{rr} \end{vmatrix}$ and $(g_{k'l'}) = G = \begin{vmatrix} I_{n_0} & 0 \cdots 0 \\ 0 & 0 \\ \vdots & \ddots \\ 0 \cdots & 0 \end{vmatrix}$

In the sequel we shall use A_r instead of A_{rr}. We may write (3.2.1) in the form

$$G = A^{t}(AG) \tag{3.2.2}$$

where ${}^{t}(AG)$ stands for the transpose of (AG),

or

$$\begin{vmatrix} I_{n_0} & 0 \cdots 0 \\ 0 & 0 \ \ddots \\ \vdots & \ddots \\ 0 & \cdots 0 \end{vmatrix} = \begin{vmatrix} A_0 & & 0 \\ & \ddots & \\ 0 & & A_r \end{vmatrix} \cdot \begin{vmatrix} {}^{t}(A_0) & 0 \cdots 0 \\ 0 & 0 \ \ddots \\ \vdots & \ddots \\ 0 & \cdots 0 \end{vmatrix} = \begin{vmatrix} A_0{}^{t}(A_0) & 0 \cdots 0 \\ 0 & 0 \ \ddots \\ \vdots & \ddots \\ 0 & \cdots 0 \end{vmatrix}$$

or $A_0^{t}(A_0) = I_{n_0}$ which implies that A_0 is orthonormal. We thus see that a transformation matrix between any two R_p-adapted bases is of the form

$$R = \begin{vmatrix} A_0 & & 0 \\ & \ddots & \\ 0 & & A_r \end{vmatrix} \qquad \text{where } A_0 \text{ is orthonormal}$$

Let $O(n_r)$ be the set of matrices of the form R. This set is a subset of $G(n_r)$ such that its elements satisfy the relation $R^{t}(RG) = G$

<u>THEOREM 7</u>: $O(n_r)$ is a Lie subgroup of $G(n_r)$

PROOF: Let R and $R_1 \in O(n_r)$. Then we have

$$(RR_1)^{t}(RR_1G) = (RR_1)^{t}(R_1G)^{t}(R) = (R)\left\{(R_1)^{t}(R_1G)\right\}^{t}(R)$$
$$= RG^{t}(R) = R^{t}(G)^{t}(R) = R^{t}(RG) = G$$

and

$$\begin{aligned}(R^{-1})^t(R^{-1}G) &= (R^{-1})^t(G)^t(R^{-1}) = R^{-1}(G)^t(R^{-1}) \\ &= (R^{-1})(R)^t(RG)^t(R^{-1}) \\ &= {}^t(RG)^t(R^{-1}) = G^t(R)^t(R^{-1}) \\ &= G.\end{aligned}$$

Hence RR_1 and R^{-1} belong to $O(n_r)$. This means that $O(n_r)$ is an abstract subgroup of $G(n_r)$. Using the condition $A_0^t(A_0) = I_{n_0}$, one can say that $O(n_r)$ is a closed subgroup of the Lie group $G(n_r)$. Hence we deduce that $O(n_r)$ is a Lie subgroup[22] of $G(n_r)$.

Let $E_R(V_n)$ be the set of all the R_p-adapted bases at different points of V_n and let $p:E_R(V_n)\rightarrow V_n$ be the mapping, such that each R_p-adapted basis at a point $x \in V_n$, is made to correspond to the point x itself. Then $E_R(V_n)$ has, with respect to p, a natural structure of a principal fibre bundle with base space V_n and structure group $O(n_r)$, (for more detail we refer to Appendix I).

3.3 R_p-Connections

Any infinitesimal connection defined on $E_R(V_n)$ will be called an R_p-connection. Given a covering of V_n by neighbourhoods endowed with the local cross-sections of $E_R(V_n)$, an R_p-connection may be defined in each neighbourhood u by a form W_u with values in the Lie algebra $LO(n_r)$ of $O(n_r)$. Such a form may be represented at $x \in V_n$ by means of a matrix of order n whose elements are complex valued linear forms at x. The form W_u will be locally denoted by $W_u = (W^i_j)$, where $W^i_j = LO(n_r)$.

To determine the form of the elements of $LO(n_r)$, we recall that $O(n_r)$ consists of matrices R of GL(n,c) which commute with J and are such that $R^t(RG) = G$. The Lie algebra of $O(n_r)$ consists of the set of all the infinitesimal right translations of $O(n_r)$ defined by a tangent vector at the identity element of $O(n_r)$. Thus one can show that $LO(n_r)$ consists of matrices

$$R = \begin{vmatrix} A_0 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & A_r \end{vmatrix} \quad \text{such that } \overline{RG} + {}^t(RG) = 0. \qquad (3.3.1)$$

For more details we refer to Appendix II.

With respect to R_p-adapted basis the condition (3.3.1) means that

$$\begin{vmatrix} \bar{A}_0 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & \cdots & \cdots & 0 \end{vmatrix} + \begin{vmatrix} {}^t(A_0) & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{vmatrix} = \begin{vmatrix} 0 \\ \\ \\ 0 \end{vmatrix}$$

or

$$\bar{A}_0 + {}^t(A_0) = 0. \qquad (3.3.2)$$

Clearly, $E_R(V_n)$ may be considered as a sub-bundle of the fibre bundle $E_c(V_n)$ of all bases. Thus any R_p-connection defines canonically a linear connection with which it can be identified.

Conversely, let us consider a complex linear connection and a covering of V_n by open sets, each equipped with a local section of $E_R(V_n)$. This connection may be defined on each neighbourhood by a local form, with values in the Lie algebra of GL(n,c), represented by a matrix (w^i_j) whose elements are complex-valued local Pfaffian forms. In order that the given connection be identified with an R_p-connection, it is necessary and sufficient that (w^i_j) belong to the Lie algebra of the structure group $O(n_r)$ of $E_R(V_n)$, that is to say that the following conditions be satisfied:

$$w^{\alpha_a}_{\beta_b} = w^{\beta_b}_{\alpha_a} = 0 \qquad 0 \leq a \neq b \leq r \tag{3.3.3}$$

$$w^{\alpha_0}_{\beta_0} + w^{\beta_0}_{\alpha_0} = 0 \tag{3.3.4}$$

The condition (3.3.3) expresses that the absolute differential of the tensor (F^i_j) is zero (a necessary and sufficient condition that one has a G_p-connection). The condition (3.3.4) means that the sub-matrix $(w^{\alpha_0}_{\beta_0})$ belongs to the Lie algebra of the orthogonal group $O(n_0,c)$. In order to interpret (3.3.4), we introduce the absolute differential of the metric tensor in the given connection as follows:

$$\nabla g_{ij} = dg_{ij} - g_{kj}\, w^k_i - g_{ik} w^k_j.$$

We recall that with respect to an R_p-adapted basis

$$(g_{ij}) = \begin{vmatrix} g_{\alpha_0\beta_0} & 0 & \cdot & \cdot & \cdot & 0 \\ & & & & & \cdot \\ 0 & 0 & & & & \cdot \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \\ 0 & \cdot & \cdot & & \cdot & 0 \end{vmatrix} \quad \text{and } g_{\alpha_0\beta_0} = \delta_{\alpha\beta} .$$

Hence we have

$$\nabla g_{\alpha_a\beta_b} = dg_{\alpha_a\beta_b} - g_{k\beta_b}\omega^k_{\alpha_a} - g_{\alpha_a k}\omega^k_{\beta_b} \qquad 0 \leq a \neq b \leq r .$$

$$= 0 - (g_{\lambda_0\beta_b}\omega^{\lambda_0}_{\alpha_a} + \cdots + g_{\lambda_r\beta_b}\omega^{\lambda_r}_{\alpha_a}) - (g_{\alpha_a\lambda_0}\omega^{\lambda_0}_{\beta_b} + \cdots + g_{\alpha_a\lambda_r}\omega^{\lambda_r}_{\beta_b})$$

$$= 0 - (0) - (0) = 0 .$$

$$\nabla g_{\alpha_0\beta_0} = 0 - (g_{\lambda_0\beta_0}\omega^{\lambda_0}_{\alpha_0} + \cdots + g_{\lambda_r\beta_0}\omega^{\lambda_r}_{\alpha_0}) - (g_{\alpha_0\lambda_0}\omega^{\lambda_0}_{\beta_0} + \cdots + g_{\alpha_0\lambda_r}\omega^{\lambda_r}_{\beta_0})$$

$$= 0 - (\omega^{\beta_0}_{\alpha_0} + 0 + \cdots + 0) - (\omega^{\alpha_0}_{\beta_0} + 0 + \cdots + 0)$$

$$= -(\omega^{\beta_0}_{\alpha_0} + \omega^{\alpha_0}_{\beta_0}) = 0 .$$

$$\nabla g_{\alpha_m\beta_m} = -(g_{\lambda_0\beta_m}\omega^{\lambda_0}_{\alpha_m} + \cdots + g_{\lambda_r\beta_m}\omega^{\lambda_r}_{\alpha_m}) - (g_{\alpha_m\lambda_0}\omega^{\lambda_0}_{\beta_m} + \cdots + g_{\alpha_m\lambda_r}\omega^{\lambda_r}_{\beta_m})$$

$$= -(0) - (0) = 0 \qquad 1 \leq m \leq r$$

This leads to the following theorem:

THEOREM 8: The absolute differential of the metric tensor in an R_p-connection is zero.

Combining this result with $\nabla(F^i_j) = 0$, we have

THEOREM 9: In order that a complex linear connection be able to be identified with an R_p-connection, it is necessary and sufficient that the tensors (F^i_j) and (g_{ij}) have zero absolute differential.

3.4 The Holonomy Group of an R_p-Connection

Let us consider an R_p-connection. Any horizontal path constructed on $E_c(V_n)$ relative to the complex linear connection coincides with the R_p-connection, and, beginning at an R_p-adapted basis b ends at an R_p-adapted basis. One concludes from this that the holonomy group at b of the the complex linear connection is a subgroup of $O(n_r)$.

Conversely, let V_n be a differentiable manifold endowed with a complex linear connection, and let us suppose that at the point x of V_n there exists a complex basis b such that the holonomy group Ψ_b of the connection at b is a subgroup of $O(n_r)$. Let us consider at the point x the tensors (g_{ij}) and (F^i_j) for which the components relative to b are given by

$$g_{\alpha_\ell \beta_m} = 0 \text{ for } 1 \leqslant \ell, m \leqslant r; \; g_{\alpha_0 \beta_0} = \delta_{\alpha\beta}$$

and

$$F^{\alpha_m}_{\beta_m} = \lambda w^m \delta^{\alpha_m}_{\beta_m}; \; F^{\alpha_m}_{\beta_s} = 0; \; 0 \leqslant m \neq s \leqslant r.$$

These tensors are invariant under Ψ_b. By parallel transport on V_n one obtains the tensors (g_{ij}) and (F^i_j) defined on the whole manifold. Now, at the point x, we have:

$$F^{k_1}_j F^{k_2}_{k_1} \dots F^i_{k_r} = \lambda^{r+1}.\delta^i_j, \text{ which implies that } J^{r+1} = \lambda^{r+1} I.$$

Also

$$F^j_k g_{ji} - \lambda g_{ik} = F^{\alpha_0}_{\beta_0} g_{\alpha_0 \gamma_0} - \lambda g_{\gamma_0 \beta_0}$$

$$= \lambda \delta^{\alpha_0}_{\beta_0} \delta_{\alpha_0 \gamma_0} - \lambda \delta_{\gamma_0 \beta_0}$$

$$= \lambda \delta_{\beta_0 \gamma_0} - \lambda \delta_{\gamma_0 \beta_0}$$

$$= 0$$

Hence $F^j_k g_{ji} = \lambda g_{ik}$ or $JG = \lambda G$.

These relations remain true at any point of V_n. Thus V_n may be endowed with an R_p-structure. Since the tensors (g_{ij}) and (F^i_j) are invariant under Ψ_b, they therefore have zero absolute differential[2]. Thus the given connection may be identified with an R_p-connection. This leads to the following theorem:

<u>THEOREM 10</u>: In order that a differentiable manifold has an R_p-structure, it is necessary and sufficient that there exists a complex linear connection whose holonomy group is a subgroup of $O(n_r)$.

3.5 A Note on Characteristic Forms

An R_p-connection determines canonically a G_p-connection. We can thus associate with it characteristic forms defined by

$$\Psi_k = \lambda w^k \Omega_{\alpha_k}^{\alpha_k} \qquad 0 \leq k \leq r$$

where $\Omega_j^i = d\pi_j^i + \pi_h^i \wedge \pi_j^h$ is a tensor 2-form.

If the connection is defined with respect to the G_p-adapted basis by (π_j^i), we have

$$\Psi_k = \lambda w^k \, d\pi_{\alpha_k}^{\alpha_k}$$

Since the given connection is an R_p-connection, we have:

$$\pi_{\alpha_0}^{\beta_0} + \pi_{\beta_0}^{\alpha_0} = 0 \quad \text{or} \quad \pi_{\alpha_0}^{\alpha_0} + \pi_{\alpha_0}^{\alpha_0} = 0 \quad \text{or} \quad \pi_{\alpha_0}^{\alpha_0} = 0.$$

Hence $\Psi_0 = \lambda w^0 d\pi_{\alpha_0}^{\alpha_0} = 0$.

We thus state the following theorem:

<u>THEOREM 11</u>: The first characteristic form Ψ_0 is zero for any R_p-connection.

<u>REMARK 1</u>: One can easily verify that the rest of the theory on this topic is the same as given in Chapter 2.

CHAPTER 4

Integrability of an a.r.p.s.

4.0 Completely Integrable Systems

We consider the space R^m, and a differentiable system of linear equations in R^m of the form

$$\theta^\alpha \equiv A_i^\alpha dx^i = 0 \tag{4.0.1}$$

where $(\alpha = 1,\ldots,a)$ and $a \leq m$,
assuming that θ^α are linearly independent, where the variables (x^i) in the coefficients A_i^α take generic values.

NOTE: A point (x^i) is generic for the system (4.0.1) when the matrix (A_i^α) is of rank a (as θ^α are linearly independent so (A_i^α) are of rank a).

DEFINITION: A manifold defined by the equations $f_i(x^1,\ldots,x^m)=0$; $(i = 1,\ldots,m)$ is called an <u>integral manifold</u> of the system (4.0.1) if the θ^α's are zero on the manifold when the equations $\frac{\partial f_j}{x^i} dx^i = 0$; $(j = 1,\ldots,m)$ hold.

We are interested, in particular, in integral manifolds of the dimension (m-a). Let us examine whether there exist manifolds passing through a generic point M_0 with co-ordinates (x_0^i). Let us put $\beta = (A_i^\alpha)$. Let us consider the sub-matrix $\beta_{(1,2\ldots,a,m-a+1,\ldots,m)}$ which is a matrix of the order a x a being different than zero at M_0, as the rank of $\beta = (A_i^\alpha)$ is a. This implies that $\det\beta_{(1,2\ldots,a,m-a+1,\ldots m)} \neq 0$

at M_0 which means that there exists a neighbourhood where it is $\neq 0$. Then in this neighbourhood of M_0 we can solve the equation (4.0.1) for the a-differentials in terms of the other m-a differentials. Calling these differentials $dz^1, dz^2, \ldots, dz^a$, we can write

$$dz^1 = \sum_{k=m-a+1}^{m} B_k^1 \, dz^k = \sum_{k=1}^{m-a} B_k^1 \, dx^k$$

$$dz^a = \sum_{k=m-a+1}^{m} B_k^a \, dx^k = \sum_{k=1}^{m-a} B_k^a \, dx^k.$$

We thus see that the integral manifold, if it exists, can be defined by expressing $z^1, z^2, \ldots, z^a$ as functions of $x^1, \ldots, x^{m-a}$ in a suitable way.

<u>THEOREM 12</u>: If there exists an integral manifold passing through a generic point, we may obtain it by integrating a system of ordinary differential equations and this integral is unique.

PROOF: Let $M_0(x^1_0, \ldots, x_0^{m-a}, x_0^{m-a+1}, \ldots, x_0^m)$ be a generic point. Let us set $m-a=h$. Let us consider the point $(x^1, \ldots, x^h)$ as a point of R^h. Let $0 = (x_0^1, \ldots, x_0^h)$. We take the sphere S of R^m defined by $\sum_{A=1}^{h} (x^A - x_0^A)^2 \leq R^2$. Any radius of this sphere is given by the h parameters $(c^1, \ldots, c^h)$ such that $(c^1)^2 + \ldots + (c^h)^2 = 1$. Then any point in the sphere is given by

$$x^1 = c^1 t; \ldots ; x^h = c^h t; \qquad 0 \leq t \leq R$$

For every integral manifold which we try to find, the $z^\alpha (\alpha = 1, \ldots, a)$ will be the functions of the coordinates of the point inside S and when we move along a radius of the sphere the unknown functions z^{α} satisfy the equation (4.0.2). By substituting, we find

$$dz^{i} = \sum_{k=1}^{h} B_{k}^{i}(c^{1},\ldots,c^{h},t)c^{k}dt$$

or

$$dz^{i} = \phi^{i}(c^{1},\ldots,c^{h},t) \text{ with the initial conditions}$$

$Z^{i} = x_{0}^{h+i}$ for $x^{A} = x_{0}^{A}$ $(A=1,\ldots,h)$

For each h-tuple $c^{1},\ldots,c^{h}$ there exists a unique solution in some interval $(0,t)$, where t_{0} depends continuously on $(c^{1},\ldots,c^{h})$, so that t_{0} attains its minimum. Let R be this minimum. We thus see that if there is an integral manifold of dimension h passing through M_{0} it is given in the interior of S by integration of a system of ordinary differential equations and furthermore it is unique.

DEFINITION: The system (4.0.1) is called completely integrable if it passes by each generic point and is an integral manifold of dimension (m-a) in a neighbourhood of that point.

Let us investigate the necessary and sufficient condition for the complete integrability of the given system. We can first make the following remark:

If a form θ vanishes for a manifold, then its exterior differential also vanishes for the manifold. Then for every integral manifold, $d\theta^{\alpha}$,s vanish. We have

$$d\theta^{\alpha} = \frac{1}{2} b_{ij}^{\alpha} dx^{i} \wedge dx^{j} \tag{4.0.3}$$

$dx^{1},\ldots,dx^{h},dz^{1},\ldots,dz^{a}$ forms a base for θ^{α}, where $h = m-a$. One can easily show that the sequence $dx^1,\ldots,dx^h,\theta^1,\ldots,\theta^a$ is linearly independent. By taking this sequence as the base of the linear differential forms, we have

$$d\theta^\alpha = \frac{1}{2}C^\alpha_{ij}\,dx^i\wedge dx^j + D^\alpha_{i\lambda}\,dx^i\wedge dx^\lambda + \frac{1}{2}\,E^\alpha_{\lambda\mu}\,\theta^\lambda\wedge\theta^\mu. \tag{4.0.4}$$

For any integral manifold passing through M_0, the L.H.S. of (4.0.4) vahishes and the second and third term of its R.H.S. vanish. Hence

$$C^\alpha_{ij}\,dx^i\wedge dx^j = 0 \Rightarrow C^\alpha_{ij} = 0 \quad \text{as } dx^i\wedge dx^j$$

is a part of the base. Hence for every point of an integral manifold in the vicinity of M_0, the coefficients C^α_{ij} vanish. This leads to the following statement:

In order that the system (4.0.1) be completely integrable, it is necessary and sufficient that for every generic point the coefficients C^α_{ij} vanish. Thus one must have

$$d\theta^\alpha = \theta_1\wedge w^1 + \ldots + \theta_a\wedge w^a.$$

(For more details we refer to E. Cartan,[13]).

This introduction will help us to investigate the condition for complete integrability of an a.r.p.s. on V_n which is the aim of this chapter.

4.1 Torsion Tensor of an a.r.p.s.

Let us consider a covering of the differentiable manifold V_n, of class C^∞, endowed with an a.r.p.s., by open sets each having a local section of $E_p(V_n)$. The local section above some open set U associates to each point $x \in U$ an adapted base (e_i) for which we denote the dual cobase by (θ^i). Let us put

$$d\theta^i = \frac{1}{2} c^i_{jk}\theta^j \wedge \theta^k, \text{ where } C^i_{jk} + C^i_{kj} = 0 \tag{4.1.1}$$

Let U' be another open set of the cover; $(\theta^{i'})$ and $C^{i'}_{j'k'}$ are defined in an analogous manner for the local section above U'. At each point $x \in U \cap U'$, there exist some matrices $(A^{\alpha_s}_{\beta'_s}) \in GL(n_s, c)$ for $(0 \leq s \leq r)$ such that

$$\overset{\alpha_0}{\theta} = A^{\alpha_0}_{\beta'_0}\overset{\beta'_0}{\theta} ; \quad \ldots \quad ; \overset{\alpha_r}{\theta} = A^{\alpha_r}_{\beta'_r}\overset{\beta'_r}{\theta} \tag{4.1.2}$$

We will write by putting $A^{\alpha_s}_{\beta'_t} = 0$ for $s \neq t$

$$\theta^i = A^i_{j'}\theta^{j'} \tag{4.1.3}$$

from which $d\theta^i = dA^i_{j'} \wedge \theta^{j'} + A^i_{j'}\, d\theta^{j'}$

or $\frac{1}{2} C^i_{jk}\theta^j \wedge \theta^k = d A^i_{j'} \wedge \theta^{j'} + \frac{1}{2} A^i_{j'} C^{j'}_{k'\ell'}\theta^{k'} \wedge \theta^{\ell'}$

or $\frac{1}{2} C^i_{jk} A^j_{k'} A^k_{\ell'}\theta^{k'} \wedge \theta^{\ell'} = \partial_{s'}(A^i_{j'})\theta^{s'} \wedge \theta^{j'} + \frac{1}{2} A^i_{j'} C^{j'}_{k'\ell'}\theta^{k'} \wedge \theta^{\ell'}$

or $\partial_{s'}(A^i_{j'})\,\theta^{s'} \wedge \theta^{j'} + \frac{1}{2}\left(A^i_{j'} C^{j'}_{k'\ell'} - C^i_{jk} A^j_{k'} A^k_{\ell'}\right)\theta^{k'} \wedge \theta^{\ell'} = 0 .$

As $\theta^{s'} \wedge \theta^{j'}$ and $\theta^{k'} \wedge \theta^{l'}$ are linearly independent, so one deduces in particular

$$C^i_{jk} A^j_{k'} A^k_{l'} = A^i_{j'} C^{j'}_{k'l'} . \tag{4.1.4}$$

We will use the following notation in the sequel:

If $\alpha_s \in N_s$ then $\bar{\alpha}_s \in \bar{N}_s$; $\bar{N}_s = N_0 + \ldots + N_{s-1} + N_{s+1} + \ldots N_r$.

Hence we have

$$C^{\alpha_s}_{\bar{\beta}_s \bar{\gamma}_s} A^{\bar{\beta}_s}_{\bar{\nu}'_s} A^{\bar{\gamma}_s}_{\bar{\mu}'_s} = A^{\alpha_s}_{\beta'_s} C^{\beta'_s}_{\bar{\nu}'_s \bar{\mu}'_s} \tag{4.1.5}$$

This is true for every $(0 \leq s \leq r)$.

Let $(A^{\beta'_s}_{\alpha_s})$ be the inverse matrices of $(A^{\alpha_s}_{\beta'_s})$ respectively. Then the equations (4.1.5) are equivalent to

$$C^{\beta'_s}_{\bar{\nu}'_s \bar{\mu}'_s} = A^{\beta'_s}_{\alpha_s} A^{\bar{\beta}_s}_{\bar{\nu}'_s} A^{\bar{\gamma}_s}_{\bar{\mu}'_s} .$$

Using the notation (2.1.4), we put

$$t^{\alpha_k}_{ij} = 0 \quad \text{for} \quad P_{k-1} < i, j \leq P_k \quad \text{and}$$

$$t^{\alpha_s}_{\bar{\beta}_s \bar{\gamma}_s} = C^{\alpha_s}_{\bar{\beta}_s \bar{\gamma}_s} \qquad \text{for every } (0 \leq s \leq r).$$

It is seen that (t^i_{jk}) are the components of a tensor of type $\binom{1}{2}$. We call it the torsion tensor of the <u>a.r.p.s</u>. The associated vector 2-form, defined by $T^i = t^i_{jk}\, \theta^j \wedge \theta^k$ is called the torsion form.

REMARK 2: Let us suppose that the torsion form is zero on V_n equipped with an <u>a.r.p.s</u>. The formula (4.1.1) becomes

$$d\theta^{\alpha_s} = \frac{1}{2} C^{\alpha_s}_{\beta_s \gamma_s} \theta^{\beta_s} \wedge \theta^{\gamma_s} + \sum_{C^2_{r+1}} C^{\alpha_s}_{\beta_\ell \gamma_m} \theta^{\beta_\ell} \wedge \theta^{\gamma_m}, \qquad \ell \neq m .$$

One is then able to say, according to the terminology used by E. Cartan, that $d\theta^{\alpha_0}, \ldots, d\theta^{\alpha_r}$ belong to the ring determined by the forms $(\theta^{\alpha_0}), \ldots, (\theta^{\alpha_r})$ respectively.

Given a linear connection (complex or real) without torsion, for example a Riemannian connection, defined relative to some adapted basis by (w^i_j). Let (θ^i) be the dual cobase of the considered adapted base (e_i). Let us put $w^i_j = \gamma^i_{jk}.\theta^k$.

The assumption of the vanishing of the torsion form of the connection leads to

$$\begin{aligned} d\theta^{\alpha_s} &= \theta^{\beta_0} \wedge w^{\alpha_s}_{\beta_0} + \cdots + \theta^{\beta_s} \wedge w^{\alpha_s}_{\beta_s} + \cdots + \theta^{\beta_r} \wedge w^{\alpha_s}_{\beta_r} \\ &= \theta^{\beta_s} \wedge w^{\alpha_s}_{\beta_s} + \sum_{\ell} \gamma^{\alpha_s}_{\beta_\ell \beta_s} \theta^{\beta_\ell} \wedge \theta^{\beta_s} + \sum_{\ell, m} \gamma^{\alpha_s}_{\beta_\ell \beta_m} \theta^{\beta_\ell} \wedge \theta^{\beta_m} \\ &\quad + \frac{1}{2} \left(\gamma^{\alpha_s}_{\bar{\beta}_s \bar{\mu}_s} - \gamma^{\alpha_s}_{\bar{\mu}_s \bar{\beta}_s} \right) \theta^{\bar{\beta}_s} \wedge \theta^{\bar{\mu}_s} \quad \text{where } (\ell \neq m \neq s) \\ &= \theta^{\beta_s} \wedge \left(w^{\alpha_s}_{\beta_s} - \sum_{\ell} \gamma^{\alpha_s}_{\beta_\ell \beta_s} \theta^{\beta_\ell} \right) + \sum_{\ell, m} \gamma^{\alpha_s}_{\beta_\ell \beta_m} \theta^{\beta_\ell} \wedge \theta^{\beta_m} \\ &\quad + \frac{1}{2} \left(\gamma^{\alpha_s}_{\bar{\beta}_s \bar{\mu}_s} - \gamma^{\alpha_s}_{\bar{\mu}_s \bar{\beta}_s} \right) \theta^{\bar{\beta}_s} \wedge \theta^{\bar{\mu}_s}. \end{aligned} \tag{4.1.6}$$

Let (T^i) be the torsion form of the <u>a.r.p.s.</u> We have

$$T^{\alpha_s} = \frac{1}{2} \left(\gamma^{\alpha_s}_{\bar{\beta}_s \bar{\mu}_s} - \gamma^{\alpha_s}_{\bar{\mu}_s \bar{\beta}_s} \right) \theta^{\bar{\beta}_s} \wedge \theta^{\bar{\mu}_s}.$$

Also let us set

$$\hat{\pi}^{\alpha_s}_{\beta_s} = \omega^{\alpha_s}_{\beta_s} - \sum_{\ell} \gamma^{\alpha_s}_{\beta_\ell \beta_s} \theta^{\beta_\ell} . \tag{4.1.7}$$

One is then able to define G_p-connection by (4.1.7). By substituting the values of $\hat{\pi}^{\alpha_s}_{\beta_s}$ and T^{α_s} in (4.1.6), we get

$$d\theta^{\alpha_s} = \theta^{\beta_s} \wedge \hat{\pi}^{\alpha_s}_{\beta_s} + \sum_{\ell,m} \gamma^{\alpha_s}_{\beta_\ell \beta_m} \theta^{\beta_\ell} \wedge \theta^{\beta_m} + T^{\alpha_s} . \tag{4.1.8}$$

By comparing (4.1.8) with the result of remark 2, one sees that the torsion form of this G_p-connection coincides with (T^i). This leads to the following theorem:

<u>THEOREM 13</u>: In order that an <u>a.r.p.s.</u> be without torsion, it is necessary and sufficient that there exists a G_p-connection without torsion.

4.2 Integrability Conditions

We will say that an <u>a.r.p.s.</u> on V_n is completely integrable if the fields of subspaces $T_0, T_1, \ldots, T_r$ are completely integrable in the neighbourhood of each point of V_n. We shall first of all study the integrability conditions of a fixed subspace T_s of T^c_x. The subspace T_s may be defined by the system of equations $dz^{\alpha_k} = 0$ where $0 \leq k \neq s \leq r$ and (z^{α_k}) are complex-valued functions of class C^∞. If T_s is integrable, then one can choose $\theta^{\alpha_k} \equiv dz^{\alpha_k}$ in an adapted basis of the cotangent space $(T^c_x)^*$. It then follows that $d\theta^{\alpha_k} \equiv d(dz^{\alpha_k}) = 0$ and so that components $C^{\alpha_k}_{ij}$ are zero. Thus the torsion tensor T^{α_k} will be zero. This is the necessary condition for T_s to be integrable.

Conversely, suppose that an a.r.p.s. is given on V_n for which the components $C^{\alpha_k}_{ij}$ are zero. Let us further assume that the structure is of class C^w. Consequently it is the same on the field of subspace T_s defined by

$$\left(F^i_j - \lambda \overset{s}{\omega} \delta^i_j\right) dx^j = 0 . \tag{4.2.1}$$

Indeed, first of all, these equations can be written as the following (r+1) equations:

$$\left(F^i_{\alpha_\ell} - \lambda \overset{s}{\omega} \delta^i_{\alpha_\ell}\right) dx^{\alpha_\ell} = 0 \qquad 0 \leq \ell \leq r$$

For $i = \beta_s$ and $\ell = s$, we have

$$\left(F^{\beta_s}_{\alpha_s} - \lambda \overset{s}{\omega} \delta^{\beta_s}_{\alpha_s}\right) dx^{\alpha_s} = 0 \Rightarrow \left(\lambda \overset{s}{\omega} - \lambda \overset{s}{\omega}\right) dx^{\alpha_s} = 0 \Rightarrow 0\left(dx^{\alpha_s}\right) = 0;$$

for $i = \beta_k$; $\ell = k$ where $0 \leq k \neq s \leq r$; we have

$$\left(F^{\beta_k}_{\alpha_k} - \lambda \overset{s}{\omega} \delta^{\beta_k}_{\alpha_k}\right) dx^{\alpha_k} = 0 \Rightarrow \left(\lambda \overset{k}{\omega} - \lambda \overset{s}{\omega}\right) dx^{\alpha_k} = 0 \Rightarrow dx^{\alpha_k} = 0 \text{ as } \overset{k}{\omega} \neq \overset{s}{\omega}.$$

Hence $dx^{\alpha_k} = 0$.

This system must be of rank $(n-n_k)$ and is equivalent to a system of $(n-n_s)$ equations:

$$\theta^{\alpha_k} \equiv B^{\alpha_k}_j \; dx^j = 0 \tag{4.2.2}$$

where $B_j^{\alpha_k}$ are real valued functions of class C^w.

We thus see that the system (4.2.1) satisfies the conditions of Frobenius for the complete integrability of T_s. Indeed, for $0 \leq k \neq s \leq r$, we have

$$d\theta^{\alpha_k} \wedge \left\{ (\wedge\theta^{\alpha_0})(\wedge\theta^{\alpha_1}) \cdots (\wedge\theta^{\alpha_{s-1}})(\wedge\theta^{\alpha_{s+1}}) \cdots (\wedge\theta^{\alpha_r}) \right\}$$

$$= \left\{ \frac{1}{2} C^{\alpha_s}_{\beta_k \gamma_k} \theta^{\beta_k} \wedge \theta^{\gamma_k} + \sum C^{\alpha_k}_{\beta_\ell \gamma_m} \theta^{\beta_\ell} \wedge \theta^{\gamma_m} \right\} \cdot \left\{ (\wedge\theta^{\alpha_0}) \cdots (\wedge\theta^{\alpha_{s-1}})(\wedge\theta^{\alpha_{s+1}}) \cdots (\wedge\theta^{\alpha_r}) \right\} = 0,$$

since terms of the first bracket contain θ^{α_k} for every $0 \leq k \neq s \leq r$, where $(\wedge\theta^{\alpha_k}) = \theta^{\alpha_k(1)} \wedge \ldots\ldots \wedge \theta^{\alpha_k(n_k)}$.

It follows (since the Pfaffian system (4.2.2) is completely integrable) that there exist locally, functions (z^{α_k}) of class C^w such that

$$\theta^{\alpha_k} = B^{\alpha_k}_{\beta_k} dz^{\beta_k} = 0 \text{ where } (B^{\alpha_k}_{\beta_k}) \text{ is regular.}$$

Thus the system (4.2.2) is equivalent to the system $dz^{\alpha_k} = 0$. Hence T_s is integrable. This leads to the following theorem:

<u>THEOREM 14</u>: Let V_n admit an <u>a.r.p.s</u>. In order that a subspace T_s of T_x^c is integrable, it is necessary that the components $C^{\alpha_k}_{ij} = 0$. $(0 \leq k \neq s \leq r)$. This condition is sufficient only if the manifold and its structure both are real analytic.

COR: Under the same assumption of the above theorem, all the subspaces T_k $(0 \leq k \leq r)$ are integrable if the torsion tensor is zero.

Finally, combining the above two results we have the following theorem:

THEOREM 15: In order that an a.r.p.s. on V_n is completely integrable, it is necessary that the torsion tensor is zero. In case the manifold and its structure are real analytic, then this condition is also sufficient.

CHAPTER 5

The Operators $\overset{s}{C}$ and $\overset{s}{M}$ on the a.r.p.s.

5.0 The Operators $\overset{s}{C}$ and $\overset{s}{M}$

Assuming that the given manifold V_n is equipped with a.r.p.s., we generalize the operators C and M considered by Lichnerowicz[2] and Legrand[1] as follows:

Let us denote by Λ_t^c, the vector space of exterior t-forms with complex values defined on V_n. If $v_1, \ldots, v_t$ are any t vectors of T_x^c and ϕ is a t-form, we define

$$\overset{s}{C}\phi(v_1,\ldots,v_t) = \phi(J^s v_1,\ldots,J^s v_t)$$

$$\overset{s}{M}\phi(v_1,\ldots,v_t) = \sum_{k=1}^{t}\phi(v_1,\ldots,J^s v_k,\ldots,v_t) \qquad (5.0.1)$$

where $(1 \leq s \leq r+1)$.

In the sequel, the following notation will be used for the sake of convenience:

$$F_i^{h_1} F_{h_1}^{h_2} \ldots F_{h_{s-1}}^j \equiv \overset{s}{F}{}_i^j \; ; \; F_i^j \equiv \overset{1}{F}{}_i^j .$$

Using these notations, we have

$$F_i^{h_1} \ldots F_{h_r}^j \equiv \overset{r+1}{F}{}_i^j = \lambda^{r+1}\,\delta_i^j .$$

If $i_1, \ldots, i_t$ are the components of ϕ with respect to a fixed base at a point x, then the components of $\overset{s}{C}\phi$ and $\overset{s}{M}\phi$ are respectively given by

$$(\overset{s}{C}\phi)_{i_1,\dots,i_t} = \overset{s}{F}{}^{j_1}_{i_1} \cdots \overset{s}{F}{}^{j_t}_{i_t} \phi_{j_1,\dots,j_t}.$$

$$(\overset{s}{M}\phi)_{i_1,\dots,i_t} = \sum_{k=1}^{t} \overset{s}{F}{}^{h}_{i_k} \phi_{i_1,\dots,i_{k-1},h,i_{k+1},\dots,i_t}.$$

Proposition 1. For any t-form ϕ, we have

(a) $\overset{r+1}{C} = \lambda^{(r+1)t}$ (b) $\overset{r+1}{M} = t\lambda^{r+1}$.

(a) $(\overset{s}{C}\phi)_{i_1,\dots,i_t} = \overset{s}{F}{}^{j_1}_{i_1} \dots \overset{s}{F}{}^{j_t}_{i_t} \phi_{j_1,\dots,j_t}$;

setting s = r+1, we have

$$(\overset{r+1}{C}\phi)_{i_1,\dots,i_t} = (\overset{r+1}{F}{}^{j_1}_{i_1})(\overset{r+1}{F}{}^{j_2}_{i_2}) \cdots (\overset{r+1}{F}{}^{j_t}_{i_t})\phi_{j_1,\dots,j_t}$$

$$= (\lambda^{r+1}\delta^{j_1}_{i_1})(\lambda^{r+1}\delta^{j_2}_{i_2}) \cdots (\lambda^{r+1}\delta^{j_t}_{i_t})\phi_{j_1,\dots,j_t}$$

$$= \lambda^{(r+1)t} \cdot \phi_{i_1,\dots,i_t}.$$

Hence $\overset{r+1}{C} = \lambda^{(r+1)t}$.

(b) $(\overset{s}{M}\phi)_{i_1,\dots,i_t} = \sum_{k=1}^{t} \overset{s}{F}{}^{h}_{i_k} \phi_{i_1,\dots,i_{k-1},h,i_{k+1},\dots,i_t}$;

setting s = r+1, we get

$$(\overset{r+1}{M}\phi)_{i_1,\dots,i_t} = \sum_{k=1}^{t} \overset{r+1}{F}{}^h_{i_k}\phi_{i_1,\dots,i_{k-1},h,i_{k+1},\dots,i_t}.$$

$$= \sum_{k=1}^{t} \overset{r+1}{\lambda}\,\delta^h_{i_k}\phi_{i_1,\dots,i_{k-1},h,i_{k+1},\dots,i_t}.$$

$$= t\overset{r+1}{\lambda}.$$

It is easy to deduce that if ϕ is any 1-form then C = M and $C^{r+1} = M^{r+1} = \lambda^{r+1}$.

DEFINITION: For fixed k, the following t-form

$$\phi(a,b) = \phi_{\alpha_{k(1)},\dots,\alpha_{k(a)},\bar{\alpha}_{k(1)},\dots,\bar{\alpha}_{k(b)}}\,\theta^{\alpha_{k(1)}}\wedge\dots\wedge\theta^{\alpha_{k(a)}}\wedge\theta^{\bar{\alpha}_{k(1)}}\wedge\dots\wedge\theta^{\bar{\alpha}_{k(b)}}$$

will be called a t-form of type (a,b) where a+b=t.[8] This definition has intrinsic meaning. Indeed, if $(\theta^{j'})$ is the dual cobasis of another adapted basis $(e_{j'})$ then

$\theta^i = A^i_{j'}.\theta^{j'}$; we see that

$$\theta^{\alpha_k} = A^{\alpha_k}_{j'}\theta^{j'} = A^{\alpha_k}_{\beta'_k}\theta^{\beta'_k} \quad \text{and}$$

$$\theta^{\bar{\alpha}_k} = A^{\bar{\alpha}_k}_{j'}\theta^{j'} = A^{\bar{\alpha}_k}_{\bar{\beta}_k}\theta^{\bar{\beta}_k}.$$

Hence the statement is justified.

THEOREM 16: C^s and M^s transform a t-form of type (a,b) into a t-form of the same type.

PROOF: $(\overset{s}{C}\phi)_{i_1,\dots,i_t} = \overset{s}{F}{}_{i_1}^{j_1}\,\overset{s}{F}{}_{i_2}^{j_2}\cdots \overset{s}{F}{}_{i_t}^{j_t}\,\phi_{j_1,\dots,j_t}$;

for a t-form of type (a,b) we have

$$(\overset{s}{C}\phi)_{\alpha_{k(1)},\dots,\alpha_{k(a)},\bar\alpha_{k(1)},\dots,\bar\alpha_{k(b)}}$$

$$= \overset{s}{F}{}_{\alpha_{k(1)}}^{\beta_{k(1)}}\cdots \overset{s}{F}{}_{\alpha_{k(a)}}^{\beta_{k(a)}}\cdot \overset{s}{F}{}_{\bar\alpha_{k(1)}}^{\bar\beta_{k(1)}}\cdots \overset{s}{F}{}_{\bar\alpha_{k(b)}}^{\bar\beta_{k(b)}}\,\phi_{\beta_{k(1)},\dots,\beta_{k(a)},\bar\beta_{k(1)},\dots,\bar\beta_{k(b)}}.$$

Now
$$\overset{s}{F}{}_{\alpha_{k(1)}}^{\beta_{k(1)}} \text{ etc} = \overset{s}{\lambda}\,\overset{sk}{\omega}\,\delta_{\alpha_{k(1)}}^{\beta_{k(1)}} \text{ etc}$$

and
$$\overset{s}{F}{}_{\bar\alpha_{k(1)}}^{\bar\beta_{k(1)}} \text{ etc} = \sum_m \overset{s}{\lambda}\,\overset{sm}{\omega}\,\delta_{\bar\alpha_{k(1)}}^{\bar\beta_{k(1)}} \text{ etc}, \quad (m=0,\dots,\hat{k},\dots,r).$$

Moreover, $w^s + \dots + w^{sr} = 0$ implies that $\sum_m w^{sm} = -w^{sk}$.

Hence we have
$$\overset{s}{F}{}_{\bar\alpha_{k(1)}}^{\bar\beta_{k(1)}} = -\overset{s}{\lambda}\,\overset{sk}{\omega}\,\delta_{\bar\alpha_{k(1)}}^{\bar\beta_{k(1)}} = -\overset{s}{\lambda}\,\overset{sk}{\omega}\,\delta_{\alpha_{k(1)}}^{\beta_{k(1)}}$$

Using these results and simplifying, we get
$$(\overset{s}{C}\phi)_{(a,b)} = \overset{as}{\lambda}\,\overset{ask}{\omega}\,(-1)^{b}\,\overset{bs}{\lambda}\,\overset{bsk}{\omega}\,\phi_{(a,b)}$$

Therefore
$$(\overset{s}{C}\phi)_{(a,b)} = (-1)^{b}\,\overset{ts}{\lambda}\,\overset{tsk}{\omega}\,\phi(a,b),$$
where a+b=t.

(b) $$(\overset{s}{M}\phi)_{i_1,\dots,i_t} = \sum_{\ell=1}^{t} F_{i_\ell}^{h} \phi_{i_1,\dots,i_{\ell-1},h,i_{\ell+1},\dots,i_t}.$$

For a t-form of type (a,b), we have

$$(\overset{s}{M}\phi)_{\alpha_{k(1)},\dots,\alpha_{k(a)},\bar{\alpha}_{k(1)},\dots,\bar{\alpha}_{k(b)}}$$

$$= \sum_{p=1}^{a} \overset{s}{F}{}_{\alpha_{k(p)}}^{\beta_{k(p)}} \phi_{\alpha_{k(1)},\dots,\alpha_{k(p-1)},\beta_{k(p)},\dots,\alpha_{k(a)},\bar{\alpha}_{k(1)},\dots,\bar{\alpha}_{k(b)}}$$

$$+ \sum_{p=1}^{b} \overset{s}{F}{}_{\bar{\alpha}_{k(p)}}^{\bar{\beta}_{k(p)}} \phi_{\alpha_{k(1)},\dots,\alpha_{k(a)},\bar{\alpha}_{k(1)},\dots,\bar{\alpha}_{k(p-1)},\bar{\beta}_{k(p)},\dots,\bar{\alpha}_{k(b)}}.$$

As shown previously in (a), we have

$$F_{\alpha_{k(p)}}^{\beta_{k(p)}} = \overset{s}{\lambda} \omega^{sk} \delta_{\alpha_{k(p)}}^{\beta_{k(p)}} \quad \text{and} \quad F_{\bar{\alpha}_{k(p)}}^{\bar{\beta}_{k(p)}} = -\overset{s}{\lambda} \omega^{sk} \delta_{\alpha_{k(p)}}^{\beta_{k(p)}}$$

for every $1 \leq p \leq a$ and every $1 \leq p \leq b$, respectively. Substituting these values and simplifying, we get

$$(\overset{s}{M}\phi)_{(a,b)} = \sum_{p=1}^{a} \overset{s}{\lambda} \omega^{sk} \delta_{\alpha_{k(p)}}^{\beta_{k(p)}} \phi_{\alpha_{k(1)},\dots,\alpha_{k(p-1)},\beta_{k(p)},\dots,\alpha_{k(a)},\bar{\alpha}_{k(1)},\dots,\bar{\alpha}_{k(b)}}$$

$$- \sum_{p=1}^{b} \overset{s}{\lambda} \omega^{sk} \delta_{\bar{\alpha}_{k(p)}}^{\bar{\beta}_{k(p)}} \phi_{\alpha_{k(1)},\dots,\alpha_{k(a)},\bar{\alpha}_{k(1)},\dots,\bar{\alpha}_{k(p-1)},\bar{\beta}_{k(p)},\dots,\bar{\alpha}_{k(b)}}.$$

Hence $$(\overset{s}{M}\phi)_{(a,b)} = \overset{s}{\lambda} \omega^{sk} (a-b)\, \phi_{(a,b)}$$

The theorem is therefore proved.

5.1 Torsion Tensor in Local Coordinates

Let us consider a form of type (1,0), $\phi = \phi_{\alpha_k} \theta^{\alpha_k}$ (k fixed); we put

$$d\phi = \phi_{(2,0)} + \phi_{(1,1)} + \phi_{(0,2)}; \tag{5.1.1}$$

we assume that r is odd and furthermore we set $s = \frac{r+1}{2}$. With these conditions, we have

$$\overset{s}{C} d\phi = \overset{2s}{\lambda}\, \omega^{2sk} \{ \phi_{(2,0)} - \phi_{(1,1)} + \phi_{(0,2)} \}$$

$$= \overset{r+1}{\lambda} \{ \phi_{(2,0)} - \phi_{(1,1)} + \phi_{(0,2)} \} \tag{5.1.2}$$

$$\overset{s}{C} \phi = \overset{s}{\lambda}\, \omega^{sk} \phi .$$

$$d \overset{s}{C} \phi = \overset{s}{\lambda}\, \omega^{sk} \{ \phi_{(2,0)} + \phi_{(1,1)} + \phi_{(0,2)} \}$$

Finally,

$$\overset{s}{M} d \overset{s}{C} \phi = \overset{s}{\lambda}\, \omega^{sk} [\overset{s}{\lambda}\, \omega^{sk} (2\phi_{(2,0)} - 2\phi_{(0,2)})]$$

$$= 2 \overset{r+1}{\lambda} (\phi_{(2,0)} - \phi_{(0,2)}) \tag{5.1.3}$$

Hence we have the following result:

$$\overset{r+1}{\lambda} d\phi + \overset{s}{C} d\phi - \overset{s}{M} d \overset{s}{C} \phi = 4 \overset{r+1}{\lambda} \phi_{(0,2)} . \tag{5.1.4}$$

On the other hand

$$d\phi = \phi_{\alpha_k} d\theta^{\alpha_k} + d\phi_{\alpha_k} \wedge \theta^{\alpha_k}$$

$$= \frac{1}{2} \phi_{\alpha_k} C^{\alpha_k}_{ij}\, \theta^i \wedge \theta^j + d\phi_{\alpha_k} \wedge \theta^{\alpha_k} .$$

Now we know from chapter 4 that $C^{\alpha_k}_{\bar{\beta}_k \bar{\gamma}_k} = t^{\alpha_k}_{\bar{\beta}_k \bar{\gamma}_k}$ and all others are zero, where $t^{\alpha_k}_{\bar{\beta}_k\bar{\gamma}_k}$ are the components of the torsion form T^{α_k} (k:fixed). Hence we have

$$d\phi = \frac{1}{2}\phi_{\alpha_k} t^{\alpha_k}_{\bar{\beta}_k\bar{\gamma}_k}\theta^{\bar{\beta}_k}\wedge\theta^{\bar{\gamma}_k} + d\phi_{\alpha_k}\wedge\theta^{\alpha_k};$$

Using previous results it is easy to say that

$$\phi_{(0,2)} = \frac{1}{2}\phi_{\alpha_k} t^{\alpha_k}_{\bar{\beta}_k\bar{\gamma}_k}\theta^{\bar{\beta}_k}\wedge\theta^{\bar{\gamma}_k}.$$

Substituting this value of $\phi_{(0,2)}$ in (5.1.4) we get

$$\overset{r+1}{\lambda} d\phi + \overset{s}{C} d\phi - \overset{s}{M} d\overset{s}{C}\phi = 2\overset{r+1}{\lambda}\phi_{\alpha_k} t^{\alpha_k}_{\bar{\beta}_k\bar{\gamma}_k}\theta^{\bar{\beta}_k}\wedge\theta^{\bar{\gamma}_k}. \tag{5.1.5}$$

In case we consider a form of type (0,1) $\phi = \phi_{\bar{\alpha}_k}\theta^{\bar{\alpha}_k}$ and proceed exactly as above with r odd and $s = \frac{r+1}{2}$, we get

$$\overset{r+1}{\lambda} d\phi + \overset{s}{C} d\phi - \overset{s}{M} d\overset{s}{C}\phi = 2\overset{r+1}{\lambda}\phi_{\bar{\alpha}_k} t^{\bar{\alpha}_k}_{\beta_k\gamma_k}\theta^{\beta_k}\wedge\theta^{\gamma_k}.$$

Combining these two results, we say that for any linear form $\phi = \phi_i\theta^i$ and r odd, we have

$$\overset{r+1}{\lambda} d\phi + \overset{s}{C} d\phi - \overset{s}{M} d\overset{s}{C}\phi = 2\overset{r+1}{\lambda}\left(\phi_{\alpha_k} t^{\alpha_k}_{\bar{\beta}_k\bar{\gamma}_k}\theta^{\bar{\beta}_k}\wedge\theta^{\bar{\gamma}_k} + \phi_{\bar{\alpha}_k} t^{\bar{\alpha}_k}_{\beta_k\gamma_k}\theta^{\beta_k}\wedge\theta^{\gamma_k}\right)$$
$$= 2\overset{r+1}{\lambda}\phi\cdot T^{\alpha_k} \tag{5.1.6}$$

where T^{α_k} is the torsion form (k:fixed) of the <u>a.r.p.s</u>. Proceeding in a similar way by varying k from 0 to r, one can easily deduce the following theorem:

<u>THEOREM 17</u>: Let r be an odd integer, $s = \frac{r+1}{2}$, a linear form, and T the torsion form of the <u>a.r.p.s</u>. Then,

$$\overset{r+1}{\lambda} d\phi + \overset{s}{C} d\phi - \overset{s}{M} d\overset{s}{C}\phi = 4\overset{r+1}{\lambda}\phi\cdot T \tag{5.1.7}$$

We may use this result to obtain an expression for the torsion tensor in local co-ordinates in terms of the tensor $\overset{s}{F}{}^i_j$. Let us take a system of local co-ordinates (x^i). The linear form ϕ is defined by $\phi = \phi_i dx^i$.

We shall represent by the symbol ∂_j, the partial derivative operator $\frac{\partial}{\partial x^i}$. One has

$$(d\phi)_{ij} = \partial_i \phi_j - \partial_j \phi_i$$

$$(\overset{s}{C} d\phi)_{ij} = \overset{s}{F}{}_i^{\ell} \overset{s}{F}{}_j^{m} (\partial_\ell \phi_m - \partial_m \phi_\ell)$$

$$(d\overset{s}{C}\phi)_{ij} = \partial_i (\overset{s}{F}{}_j^{m} \phi_m) - \partial_j (\overset{s}{F}{}_i^{m} \phi_m)$$

$$(\overset{s}{M} d \overset{s}{C} \phi)_{ij} = \overset{s}{F}{}_j^{\ell} \left[\partial_\ell (\overset{s}{F}{}_i^{m} \phi_m) - \partial_i (\overset{s}{F}{}_\ell^{m} \phi_m) \right]$$

By taking into account the relation

$\overset{s}{F}{}_i^{\ell} \overset{s}{F}{}_\ell^{m} = \overset{s+1}{F}{}_i^{m} = \overset{s+1}{\lambda} \delta_i^m$, one is able to write

$$\begin{aligned}
(\overset{s}{M} d \overset{s}{C} \phi)_{ij} = {} & \overset{s}{F}{}_i^{\ell} \overset{s}{F}{}_j^{m} (\partial_\ell \phi_m) + \overset{s}{F}{}_i^{\ell} (\partial_\ell \overset{s}{F}{}_j^{m}) \phi_m \\
& - \overset{s}{F}{}_i^{\ell} \overset{s}{F}{}_\ell^{m} \partial_j \phi_m - \overset{s}{F}{}_i^{\ell} (\partial_j \overset{s}{F}{}_\ell^{m}) \phi_m \\
& - \overset{s}{F}{}_j^{\ell} \overset{s}{F}{}_i^{m} (\partial_\ell \phi_m) - \overset{s}{F}{}_j^{\ell} (\partial_\ell \overset{s}{F}{}_i^{m}) \phi_m \\
= {} & \overset{s}{F}{}_i^{\ell} \overset{s}{F}{}_j^{m} \partial_\ell \phi_m - \overset{s}{F}{}_j^{\ell} \overset{s}{F}{}_i^{m} (\partial_\ell \phi_m) - \overset{s+1}{\lambda} (\partial_j \phi_i - \partial_i \phi_j) \\
& + \left\{ \overset{s}{F}{}_i^{\ell} (\partial_\ell \overset{s}{F}{}_j^{m} - \partial_j \overset{s}{F}{}_\ell^{m}) - \overset{s}{F}{}_j^{\ell} (\partial_\ell \overset{s}{F}{}_i^{m} - \partial_i \overset{s}{F}{}_\ell^{m}) \right\} \phi_m ,
\end{aligned}$$

from which we have

$$\left(\overset{r+1}{\lambda} d\phi + \overset{s}{C} d\phi - \overset{s}{M} d \overset{s}{C} \phi\right)_{ij} = \left[\overset{s}{F}{}_i^{\ell} \left(\partial_\ell \overset{s}{F}{}_j^{m} - \partial_j \overset{s}{F}{}_\ell^{m}\right) - \overset{s}{F}{}_j^{\ell} \left(\partial_\ell \overset{s}{F}{}_i^{m} - \partial_i \overset{s}{F}{}_\ell^{m}\right)\right] \phi_m$$

$$= 4 \overset{r+1}{\lambda} (\phi \cdot T)_{ij} = 4 \overset{r+1}{\lambda} t_{ij}^{m} \phi_m ; \quad (5.1.8)$$

$$t_{ij}^{m*} = \frac{1}{4 \lambda^{r+1}} \left[\overset{s}{F}{}_i^{\ell} \left(\partial_\ell \overset{s}{F}{}_j^{m} - \partial_j \overset{s}{F}{}_\ell^{m}\right) - \overset{s}{F}{}_j^{\ell} \left(\partial_\ell \overset{s}{F}{}_i^{m} - \partial_i \overset{s}{F}{}_\ell^{m}\right)\right]$$

5.2 Relation between the Torsion and the Brackets of certain Vector Fields

Let the infinitesimal transformation defined by the vector field x be denoted by xf where f is a function. Let u,v be any two vector fields, ϕ be any 1-form, then it is well-known that

$$d\phi(u,v) = u\phi(v) - v\phi(u) - \phi([u,v]),$$

where $[u,v]$ is the Poisson's bracket of two vector fields u,v. Making use of this formula, we have

$$\overset{s}{C} d\phi(u,v) = d\phi(\overset{s}{J}u, \overset{s}{J}v)$$

$$= \overset{s}{J}u\, \phi(\overset{s}{J}v) - \overset{s}{J}v\, \phi(\overset{s}{J}u) - \phi([\overset{s}{J}u, \overset{s}{J}v]).$$

$$d\overset{s}{C}\phi(u,v) = u\, \phi(\overset{s}{J}v) - v\, \phi(\overset{s}{J}u) - \phi(\overset{s}{J}[u,v]).$$

---

*By setting r=s=1, one gets the definition of torsion tensor given by Legrand[1].

$$\overset{s}{M} d \overset{s}{C}\phi(u,v) = d\overset{s}{C}\phi(\overset{s}{J}u,v) + d\overset{s}{C}\phi(u,\overset{s}{J}v)$$

$$= \overset{s}{J}u\,\phi(\overset{s}{J}v) - v\,\lambda^{r+1}\phi(u) - \phi(\overset{s}{J}[\overset{s}{J}u,v])$$

$$+ \lambda^{r+1} u\,\phi(v) - \overset{s}{J}v\,\phi(\overset{s}{J}u) - \phi(\overset{s}{J}[u,\overset{s}{J}v]).$$

Also $\phi.T = \phi.(T(u,v)$, where $T(u,v)$ is the vector with components $(T(u,v))^i = t^i_{jk}u^j v^k$. Substituting all these values in (5.1.7), we get

$$\lambda^{r+1}[u,v] + [\overset{s}{J}u,\overset{s}{J}v] - \overset{s}{J}[\overset{s}{J}u,v] - \overset{s}{J}[u,\overset{s}{J}v] = -4\lambda^{r+1}.T(u,v).$$

Let us set

$$\underset{s}{N}(u,v) = [\overset{s}{J}u,\overset{s}{J}v] + \lambda^{r+1}[u,v] - \overset{s}{J}[\overset{s}{J}u,v] - \overset{s}{J}[u,\overset{s}{J}v]$$

where we shall call $\underset{s}{N}(u,v)$ the generalized Nijenhuis tensor, as it is obviously true that by setting r=s=1 we get the definition of Nijenhuis tensor[15]:

$$N(u,v) = [Ju,Jv] + J^2[u,v] - J[Ju,v] - J[u,Jv].$$

Using this, we get

$$T(u,v) = \frac{-1}{4\lambda^{r+1}} \cdot \underset{s}{N}(u,v) \tag{5.2.1}$$

Hence we state the following theorem:

THEOREM 18: In order that the a.r.p.s. be without Torsion, it is necessary and sufficient that the generalized Nijenhuis tensor $\underset{S}{N}(u,v) = 0$.

Connecting the result of the above theorem with the integrability conditions (Chapter 4), we state the following theorem:

THEOREM 19: In order that an a.r.p.s. be completely integrable, it is necessary and sufficient that the generalized Nijenhuis tensor be equal to zero.

CHAPTER 6

Hermitian Structures Subordinate to a.r.p.s.

6.0 Almost r-Product Hermitian Structures, briefly H-Structures

Let V_n be a differentiable manifold endowed with a.r.p.s. Let us assume that we have defined in V_n a complex symmetric tensor $G = (g_{ij})$, whose components in a system of local co-ordinates (x^i) are complex functions of the (x^i), of class C^∞, with determinant everywhere different from zero. We will say that G is hermitian with respect to J if one has at each point x and for any pair of vectors v,w of T_x^c

$$(\overset{r}{J}v, \overset{r}{J}w) = -\lambda^{r+1}(v,w) \tag{6.0.1}$$

where (v,w) denotes the scalar product $g_{ij}v^iw^j$.

We will say that V_n is endowed with an almost r-product hermitian structure subordinate to the a.r.p.s. The equation (6.0.1) can be written as

$$F_i^k \overset{r}{F}{}_j^\ell g_{k\ell} = -\lambda^{r+1} g_{ij} \tag{6.0.2}$$

Multiplying both sides by F_h^j, we get

$$F_i^k \overset{r}{F}{}_j^\ell F_h^j g_{k\ell} = -\lambda^{r+1} F_h^j g_{ij}$$

or

$$F_i^k \overset{r+1}{F}{}_h^\ell g_{k\ell} = -\lambda^{r+1} F_h^j g_{ij}$$

or

$$F_i^k \lambda^{r+1} \delta_h^\ell g_{k\ell} = -\overset{r+1}{\lambda} F_h^j g_{ij}$$

or

$$F_i^k \; g_{kh} = -F_h^j \; g_{ij}.$$

By making suitable changes of indices we have

$$F_i^k \, g_{kh} = -F_h^k \, g_{ki}$$

which means that

$$JG + {}^t(JG) = 0 \qquad (6.0.3)$$

where ${}^t(JG)$ is the transpose of (JG).

Let us set $F_{ij} = F_i^k \, g_{kj}$. This means that

$$F_{ij} = F_i^k g_{kj} = -F_j^k \, g_{ki} = -F_{ji};$$

therefore $F_{ij} + F_{ji} = 0$ and (F_{ij}) are the components of an exterior 2-form F which we will call the fundamental form of H-structure. Let the matrix (g^{ij}) be the inverse of (g_{ij}). Then from (6.0.2), we get

$$F_{ia} g^{ka} \overset{r}{F}_{jb} \overset{r}{g}{}^{lb} g_{kl} = -\overset{r+1}{\lambda} g_{ij}$$

where $$\overset{r}{g}{}^{lb} = g^{lb_1} g^{j_1 b_2} \cdots g^{j_{r-1} b_r}$$

or $$F_{ia} \overset{r}{F}_{jb} \overset{r}{g}{}^{lb} g^{ka} g_{kl} = -\overset{r+1}{\lambda} g_{ij}$$

or $$F_{ia} \overset{r}{F}_{jb} \overset{r}{g}{}^{lb} \delta_l^a = -\overset{r+1}{\lambda} g_{ij}$$

or $$F_{ia} \overset{r}{F}_{jb} \overset{r}{g}{}^{ab} = -\overset{r+1}{\lambda} g_{ij} \qquad (6.0.4)$$

Let us define a linear operator J on T_x^c by the tensor

$$F^i_j = F_{jk} g^{ki}; \tag{6.0.5}$$

we have $F^i_a \overset{r}{F}{}^a_j = F_{ak} g^{ik} \overset{r}{F}_{jl} g^{al} = F_{ak} \overset{r}{F}_{jl} g^{al} g^{ik} = \overset{r+1}{\lambda} g_{kj} g^{ik}$

or

$$\overset{r+1}{F}{}^i_j = \overset{r+1}{\lambda} \delta^i_j \tag{6.0.6}$$

Moreover, we have

$$F^k_i \overset{r}{F}{}^\ell_j g_{k\ell} = F_{ik} \overset{r}{F}_{j\ell} \overset{r}{g}{}^{k\ell} = -\overset{r+1}{\lambda} g_{ij}. \tag{6.0.7}$$

F^i_j is certainly not proportional to the kronecker tensor δ^i_j, (as one would then have from (6.0.6) and (6.0.7): $g_{ij} = 0$). The relation (6.0.6) shows that J determines an a.r.p.s. and it follows from (6.0.7) that the metric (g_{ij}) is hermitian with respect to J. Hence we state the following theorem:

THEOREM 20: Given an exterior 2-form (F_{ij}) of rank n and a Riemannian metric (g_{ij}) both of class C^∞ and defined on V_n such that (6.0.4) is satisfied, one can always define an a.r.p.s. and a subordinate H-structure.

THEOREM 21: G is Hermitian with respect to J iff for any

$u,v \in T^c_x$ $(u,Jv) + (Ju,v) = 0$. (6.0.8)

where (,) is the inner product defined by G.

PROOF: Let us suppose that G is hermitian with respect to J. Let $u, v \in T_x^c$, then

$$(u, Jv) + (Ju, v) = g_{ik} u^i (Jv)^k + g_{kh} (Ju)^k v^h$$

$$= g_{ik} u^i F_h^k v^h + g_{kh} F_i^h u^i v^h$$

$$= (g_{ik} F_h^k + g_{kh} F_i^k) u^i v^h$$

or $(u, Jv) + (Ju, v) = 0$ because of (6.0.3)

Conversely, let us suppose that $(u, Jv) + (Ju, v) = 0$ for every $u, v \in T_x^c$. Let $v, w \in T_x^c$, then we must prove that

$$(Jv, J^r w) + \lambda^{r+1} (v, w) = 0$$

One can always find a vector $u \in T_x^c$ for every vector $w \in T_x^c$ such that $w = Ju$. Assuming this we have

$$(Jv, J^r w) + \lambda^{r+1} (v, w) = (Jv, J^{r+1} u) + \lambda^{r+1} (v, Ju)$$

$$= \lambda^{r+1} \left[(Jv, u) + (v, Ju) \right]$$

$$= 0.$$

The theorem is, therefore, proved.

If we set $v = u$ in (6.0.8), then we get $(Ju, u) + (u, Ju) = 0$ or $(Ju, u) = 0$ for every $u \in T_x^c$.

We may thus state the following theorem:

THEOREM 22: If G is hermitian with respect to J, then any vector of T_x^c is orthogonal to its transform by J.

In order to obtain an expression for G relative to a basis adapted to a.r.p.s., we have from (6.0.1)

$$g_{\alpha_m\beta_m} = (e_{\alpha_m}, e_{\beta_m}) = -\frac{1}{\lambda^{s+1}}\left(Je_{\alpha_m}, \overset{s}{J}e_{\beta_m}\right)$$

$$= -\frac{1}{\lambda^{s+1}}\left(\lambda \omega^{m} e_{\alpha_m}, \lambda^{s} \omega^{sm} e_{\beta_m}\right)$$

$$= -\left(\omega^{m} e_{\alpha_m}, \omega^{sm} e_{\beta_m}\right) = -\omega^{m(s+1)} g_{\alpha_m\beta_m} = -g_{\alpha_m\beta_m}.$$

Therefore $g_{\alpha_m\beta_m} = 0$ for every $0 \le m \le r$.

This condition is equivalent to the condition that

$$G = \begin{vmatrix} 0 & G_{01} & \cdots & G_{0r} \\ G_{10} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ G_{r0} & \cdots & \cdots & 0 \end{vmatrix} \qquad (6.0.9)$$

Now if $n_0, n_1, \ldots, n_r$ are all different from each other, then (6.0.9) is incompatible with the assumption that det $|G| \neq 0$. In order that an a.r.p.s. on V_n admits an H-structure, it is necessary that $n_0 = n_1 = \ldots = n_r = m$ (say). This means that $n = (r+1)m$. In the sequel we will assume this condition.

The general transformation equations of the tensor field g_{ij}, in an adapted basis, are given by

$$g_{\alpha'_s \beta'_s} = A^{\lambda_s}_{\alpha'_s} A^{\mu_s}_{\beta'_s} g_{\lambda_s \mu_s} = 0 \qquad \text{for} \quad 0 \leq s \leq r$$

and

$$g_{\alpha'_s \beta'_\ell} = A^{\lambda_s}_{\alpha'_s} A^{\mu_\ell}_{\beta'_\ell} g_{\lambda_s \mu_\ell} \qquad \text{for} \quad 0 \leq s \neq \ell \leq r.$$

One can also deduce that

$$\begin{aligned} F_{\alpha_s \beta_s} = F^{k}_{\alpha_s} g_{k \beta_s} &= F^{\gamma_0}_{\alpha_s} g_{\gamma_0 \beta_s} + \cdots + F^{\gamma_r}_{\alpha_s} g_{\gamma_r \beta_s} \\ &= 0 + \cdots + 0 \\ &= 0 \end{aligned} \qquad (6.0.10)$$

and

$$\begin{aligned} F_{\alpha_s \beta_\ell} = F^{k}_{\alpha_s} g_{k \beta_\ell} &= F^{\gamma_0}_{\alpha_s} g_{\gamma_0 \beta_\ell} + \cdots + F^{\gamma_s}_{\alpha_s} g_{\gamma_s \beta_\ell} + \cdots + F^{\gamma_r}_{\alpha_s} g_{\gamma_r \beta_\ell} \\ &= 0 + \cdots + \lambda \overset{s}{\omega} g_{\alpha_s \beta_\ell} + \cdots + 0 \\ &= \lambda \overset{s}{\omega} g_{\alpha_s \beta_\ell} \end{aligned} \qquad (6.0.10')$$

Now one is able to say that the fundamental form F and the quadratic form ds^2 can be written as

$$F = \lambda \sum_{C^2_{r+1}} \omega^s g_{\alpha_s \beta_\ell} \theta^{\alpha_s} \wedge \theta^{\beta_\ell}$$

for $0 \leq s \neq \ell \leq r$

$$ds^2 = 2 \sum_{C^2_{r+1}} g_{\alpha_s \beta_\ell} \theta^{\alpha_s} \cdot \theta^{\beta_\ell} \qquad (6.0.11)$$

6.1 G_H-Adapted Bases

THEOREM 23: If G is a Riemannian metric hermitian with respect to J, there always exists a basis adapted to an a.r.p.s., such that G will have the form

$$G = \begin{vmatrix} O_m & I_m & \cdots & I_m \\ I_m & O_m & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ I_m & \cdot & \cdot & O_m \end{vmatrix} \qquad (6.1.1)$$

where I_m is an m x m unit matrix, O_m is an m x m zero matrix and G is an n x n matrix; also n = (r+1)m

PROOF: Let V_n be a differentiable manifold equipped with an H-structure. Let us choose at the point x an adapted base (e_i) and consider two subspaces T_s and T_m where T_m is fixed and $(s=0,..,\hat{m},...,r)$ is arbitrary. There exists a vector $v_{\beta_{m(1)}} = A^{\beta_m} e_{\beta_m}$ of T_m such that

$$(e_{\alpha_{s(1)}}, v_{\beta_{m(1)}}) = 1 \text{ and } (e_{\alpha_{s(\gamma)}}, v_{\beta_{m(1)}}) = 0 \text{ for } \gamma \neq 1.$$

This condition can be written as

$$g_{\alpha_{s(\gamma)}\beta_m} A^{\beta_m} = \delta_{\gamma 1} \qquad \text{(Kronecker symbol)}$$

where one can determine A^{β_m} in a unique manner by a Cramer system of linear equations. In a general way, for a given α_s, there exists a vector $\boldsymbol{v}_{\beta_m}$ such that

$$(e_{\alpha_{s(a)}}, v_{\beta_{m(b)}}) = \delta_{ab} \qquad 1 \leqslant a,b \leqslant n.$$

The n vectors v_{β_m} are linearly independent; indeed, let us consider a linear relation

$$A^b v_{\beta_{m(b)}} = 0 \ .$$

Multiplying the two parts by $e_{\alpha_{s(a)}}$ we have,

$$A^b v_{\beta_{m(b)}} . e_{\alpha_{s(a)}} = 0, \text{ which implies that}$$

$$A^b(e_{\alpha_{s(a)}}, v_{\beta_{m(b)}}) = 0 \quad \text{or } A^b\delta_{ab} = 0 \text{ or } A^a = 0.$$

Hence v_{β_m} form a bases of T_m.

Let us set $e_{\alpha'_s} = e_{\alpha_s}$ for every $(s = 0,\ldots,\hat{m},\ldots,r)$ and $e_{\alpha'_m} = v_{\beta_m}$; then it is easy to see that the set of vectors $(e_{i'}) = (e_{\alpha'_0},\ldots,e_{\alpha'_r})$ constitute an adapted bases in the sense of Chapter 2. It is also quite obvious that under these conditions G will have the following form (6.1.1).

One can now deduce that (6.0.11) can be written as:

$$ds^2 = 2 \sum_{C^2_{r+1}} \theta^{\alpha_{s(a)}} \cdot \theta^{\beta_{\ell(a)}} \tag{6.1.2}$$

and $$F = \lambda \cdot \sum_{C^2_{r+1}} \theta^{\alpha_{s(a)}} \wedge \theta^{\beta_{\ell(a)}} \qquad 0 \leq a \leq m$$

Any base, with respect to which one has the relation (6.1.2), will be called adapted to H-structure, briefly G_H-adapted bases.

Suppose now that (e_i) and $(e_{j'})$ are two G_H-adapted bases, then

$$g_{k'\ell'} = A^i_{k'} A^j_{\ell'} g_{ij} \tag{6.1.3}$$

where

$$(A^i_{k'}) = \begin{vmatrix} A_{00} & & & O \\ & \cdot & & \\ & & \cdot & \\ O & & & A_{rr} \end{vmatrix} \quad ; \qquad A_{kk} = (A^{\alpha_k}_{\beta'_k}) \in GL(n_k, c)$$

and $(g_{k'\ell'}) = G$ is as given by (6.1.1).

In the sequel we will use A_k instead of A_{kk} and A for $(A^i_{k'})$. We may thus write (6.1.3) in the following form:

$$G = A^t(AG) \tag{6.1.4}$$

where ${}^t(AG)$ is the transpose of (AG),

or

$$\begin{vmatrix} 0 & I_m & \cdots & I_m \\ I_m & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ I_m & \cdots & & 0 \end{vmatrix} = \begin{vmatrix} A_0 & & 0 \\ & \ddots & \\ 0 & & A_r \end{vmatrix} \cdot \begin{vmatrix} 0 & {}^t(A_1) & \cdots & {}^t(A_r) \\ {}^t(A_0) & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ {}^t(A_0) & {}^t(A_1) & \cdots & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0, & (A_0).{}^t(A_1), & \cdots & ,(A_0)^t.(A_r) \\ (A_1)^t.(A_0), & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ (A_r).{}^t(A_0), & \cdots & \cdots & 0 \end{vmatrix}$$

Hence we have the following $\frac{r(r+1)}{2}$ relations:

$$\begin{aligned} (A_0)^t(A_s) &= I_m \qquad 1 \leq s \leq r \\ (A_1)^t(A_s) &= I_m \qquad 2 \leq s \leq r \\ &\vdots \\ (A_{r-1})^t(A_r) &= I_m \end{aligned} \tag{6.1.5}$$

These results can be combined into the following single result:

$(A_k)^t(A_s) = I_m$ for $0 \leq k \leq r-1$ and $k+1 \leq s \leq r$, which implies that ${}^t(A_s) = (A_k^{-1})$ or $(A_s) = {}^t(A_k^{-1})$. (6.1.6)

We thus see that a transformation matrix between two G_H-adapted bases is of the form

$$H = \begin{vmatrix} A_0 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & A_r \end{vmatrix}$$ where each A_k satisfies the condition (6.1.6).

Let u(m) be the set of matrices of the form H. This set is the subset of G(m) such that its elements satisfy the relation $(H)^t(HG) = G$, where

$$G(n_r) = G(m) \quad \text{when } n_0 = n_1 = \dots = n_r = m$$

THEOREM 24: u(m) is a Lie subgroup of G(m).

PROOF: Same as given in Theorem 7 (page 33).

Let $E_H(V_n)$ be the set of all the G_H-adapted bases at the different points of V_n and let $p: E_H(V_n) \longrightarrow V_n$ be the mapping such that a G_H-adapted basis at a point $x \in V_n$ is made to correspond to x itself. $E_H(V_n)$ has then with respect to p, a natural structure of a principal fibre bundle with base space V_n and the structure group u(m). (for more details we refer to Appendix I).

6.2 G_H-Connections

Any infinitesimal connection [2] defined on the fibre bundle $E_H(V_n)$ will be called a G_H-connection. Given a covering of V_n by neighbourhoods endowed with the local sections of $E_H(V_n)$, a G_H-connection may be defined in each neighbourhood v by a form w_v with values in the Lie algebra Lu(m) of the group u(m). The form w_v may be represented at $x \in V_n$ by means of a matrix of the order n, whose elements are complex-valued linear forms at x; it will be denoted locally by $w_v = (w^i_j)$, where $(w^i_j) \in Lu(m)$.

To determine the form of the elements of Lu(m), we recall that u(m) consists of matrices H of GL(n,c) which commute with J and are such that $(H)^t(HG) = G$. The Lie algebra of u(m) consists of the set of all the infinitesimal right translations of u(m) defined by a tangent vector at the identity element of u(m). Thus, one can show that Lu(m) consists of n x n matrices which commute with J and are skew-hermitian with respect to G [22]. Explicitly, it means that Lu(m) consists of matrices of the form H such that $\overline{HG} + {}^t(HG) = 0$, where $\overline{HG}$ is the conjugate of HG. (For more details we refer to Appendix II). (6.2.1)

With respect to a G_H-adapted basis the condition (6.2.1) can be written as

$$\begin{vmatrix} 0 & {}^t(A_1) & \cdots & {}^t(A_r) \\ {}^t(A_0) & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ {}^t(A_0) & \cdots & & 0 \end{vmatrix} + \begin{vmatrix} 0 & \bar{A}_0 & \cdots & \bar{A}_0 \\ \bar{A}_1 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \bar{A}_r & \cdots & & 0 \end{vmatrix} = \begin{vmatrix} 0 \\ \\ \\ 0 \end{vmatrix}$$

or

$$\begin{vmatrix} 0, & {}^t(A_1) + A_0, \ldots\ldots, & {}^t(A_r) + A_0 \\ {}^t(A_0)+A_1, & 0 & \cdot \\ \vdots & \ddots & \vdots \\ {}^t(A_0) + A_r, & \cdots & 0 \end{vmatrix} = \begin{vmatrix} 0 \\ \\ \\ 0 \end{vmatrix}$$

Hence we have the following relations:

$$\bar{A}_0 + {}^t(A_s) = 0 \quad \text{for } 1 \leq s \leq r$$

$$\bar{A}_1 + {}^t(A_s) = 0 \quad \text{for } 2 \leq s \leq r$$

$$\vdots \qquad\qquad \vdots$$

$$A_{r-1} + {}^t(A_r) = 0$$

which mean that

$$\bar{A}_k + {}^t(A_s) = 0 \tag{6.2.2}$$

$0 \leq k \leq r-1;\ k+1 \leq s \leq r$.

Clearly, $E_H(V_n)$ may be considered as a sub-bundle of the fibre bundle $E_c(V_n)$ of all bases. Thus any G_H-connection defines canonically a linear connection with which it may be identified.

Conversely, given a complex linear connection and a covering of V_n by open sets, each equipped with a local-section of $E_H(V_n)$. This connection may be defined on each neighbourhood by a local form, with values in the Lie algebra of GL(n,c), represented by a matrix (w^i_j) whose elements are complex-valued local Pfaffian forms. In order that the given connection may be identified with a G_H-connection, it is necessary and sufficient that (w^i_j) belong to Lu(m); that is,

$$\omega^{\beta_\ell}_{\alpha_m} = \omega^{\alpha_m}_{\beta_\ell} = 0 \qquad \text{for } (0 \leq m \neq \ell \leq r) \qquad (6.2.2)$$

$$\omega^{\beta_k}_{\alpha_k} + \omega^{\alpha_s}_{\beta_s} = 0 \qquad \text{for } (0 \leq k \leq r-1;\ k+1 \leq s \leq r) \quad (6.2.3)$$

The condition (6.2.2) expresses that $\nabla(F^i_j) = 0$ (necessary and sufficient condition that one has a G_p-connection). In order to interpret the conditions (6.2.3), we introduce the absolute differential of (g_{ij}), assuming that (6.2.2) is true. We have

$$\nabla g_{ij} = -g_{kj}\, w^k_i - g_{ik}\, w^k_j.$$

With respect to a G_H-adapted basis, we have

$$\nabla g_{\alpha_m \beta_m} = -g_{k\beta_m}\omega^k_{\alpha_m} - g_{\alpha_m k}\,\omega^k_{\beta_m}$$

$$= -\left(g_{\lambda_0 \beta_m}\omega^{\lambda_0}_{\alpha_m} + \cdots + g_{\lambda_m \beta_m}\omega^{\lambda_m}_{\alpha_m} + \cdots + g_{\lambda_r \beta_m}\omega^{\lambda_r}_{\alpha_m}\right)$$

$$- \left(g_{\alpha_m \lambda_0}\omega^{\lambda_0}_{\beta_m} + \cdots + g_{\alpha_m \lambda_m}\omega^{\lambda_m}_{\beta_m} + \cdots + g_{\alpha_m \lambda_r}\omega^{\lambda_r}_{\beta_m}\right)$$

$$= -\left(\omega^{\beta_0}_{\alpha_m} + \cdots\cdots + 0 + \cdots + \omega^{\beta_r}_{\alpha_r}\right)$$

$$- \left(\omega^{\alpha_0}_{\beta_m} + \cdots + 0 + \cdots + \omega^{\alpha_r}_{\beta_r}\right)$$

$$\text{or } \nabla g_{\alpha_m \beta_m} = -\left(w^{\beta_\ell}_{\alpha_m} + w^{\alpha_m}_{\beta_\ell} \right) = 0 \quad \text{as} \quad 0 \leq m \neq \ell \leq r$$

$$\nabla g_{\alpha_k \beta_s} = - g_{m\beta_s} w^{m}_{\alpha_k} - g_{\alpha_k m} w^{m}_{\beta_s} \quad \text{for } (0 \leq k \leq r-1);\ k+1 \leq s \leq r$$

$$= -\left(g_{\lambda_0 \beta_s} w^{\lambda_0}_{\alpha_k} + \cdots + g_{\lambda_s \beta_s} w^{\lambda_s}_{\alpha_k} + \cdots + g_{\lambda_r \beta_s} w^{\lambda_r}_{\alpha_k} \right)$$

$$-\left(g_{\alpha_k \lambda_0} w^{\lambda_0}_{\beta_s} + \cdots + g_{\alpha_k \lambda_k} w^{\lambda_k}_{\beta_s} + \cdots + g_{\alpha_k \lambda_r} w^{\lambda_r}_{\beta_s} \right)$$

$$= -\left(0 + \cdots + w^{\beta_k}_{\alpha_k} + \cdots + 0 \right)$$

$$-\left(0 + \cdots + w^{\alpha_s}_{\beta_s} + \cdots + 0 \right)$$

$$= -\left(w^{\beta_k}_{\alpha_k} + w^{\alpha_k}_{\beta_k} \right) = 0 .$$

If the condition (6.2.2) is already satisfied, then (6.2.3) is equivalent to $\nabla(g_{ij}) = 0$. This leads to the following theorem:

THEOREM 25: In order that a complex linear connection may be identified with a G_H-connection, it is necessary and sufficient that the tensors (F^i_j) and (g_{ij}) have zero absolute differential.

We will say that a complex linear connection defined on a differentiable manifold equipped with a complex matric (g_{ij}) is Euclidean if $\nabla(g_{ij}) = 0$. The preceding theorem expresses that one is able to identify the G_H-connection with the Euclidean G_p-connection, (for the metric defining H-structure).

Let us consider on V_n a linear connection with respect to the G_p-adapted basis by the matrix (w^i_j). The forms (w^i_j) defined

$$\hat{w}^{\beta_\ell}_{\alpha_s} = w^{\alpha_s}_{\beta_\ell} = 0 \qquad \text{for } 0 \leq s \neq \ell \leq r \tag{6.2.4}$$

$$\text{and} \quad \hat{w}^{\beta_k}_{\alpha_k} = w^{\beta_k}_{\alpha_k} \qquad 0 \leq k \leq r$$

define a linear connection which we can identify with a G_p-connection. We will say that it is the G_p-connection induced by the given connection.

THEOREM 26: The G_p-connection induced by a Euclidean connection is again Euclidean.

PROOF: Let us define a Euclidean connection on V_n relative to the G_H-adapted basis by (w^i_j); then we know that

$$\nabla g_{ij} = -\omega^k_i g_{kj} - \omega^k_j g_{ik} = 0 \text{ , that is}$$

$$\nabla g_{\alpha_m \beta_m} = -\left(\omega^{\lambda_1}_{\alpha_m} g_{\lambda_1 \beta_m} + \cdots + \omega^{\lambda_{m-1}}_{\alpha_m} g_{\lambda_{m-1}\beta_m} + 0 + \cdots + \omega^{\lambda_r}_{\alpha_m} g_{\lambda_r \beta_m}\right)$$

$$-\left(\omega^{\lambda_1}_{\beta_m} g_{\alpha_m \lambda_1} + \cdots + \omega^{\lambda_{m-1}}_{\beta_m} g_{\alpha_m \lambda_{m-1}} + 0 + \cdots + \omega^{\lambda_r}_{\beta_m} g_{\alpha_m \lambda_r}\right)$$

$$= 0 \qquad \text{for} \quad (0 \leq m \leq r)$$

$$\nabla g_{\alpha_m \beta_s} = -\left(\omega^{\lambda_1}_{\alpha_m} g_{\lambda_1 \beta_s} + \cdots + \omega^{\lambda_{s-1}}_{\alpha_m} g_{\lambda_{s-1}\beta_s} + 0 + \cdots + \omega^{\lambda_r}_{\alpha_m} g_{\lambda_r \beta_s}\right)$$

$$-\left(\omega^{\lambda_1}_{\beta_s} g_{\alpha_m \lambda_1} + \cdots + \omega^{\lambda_{m-1}}_{\beta_s} g_{\alpha_m \lambda_{m-1}} + 0 + \cdots + \omega^{\lambda_r}_{\beta_s} g_{\alpha_m \lambda_r}\right)$$

$$= 0 \qquad \text{for} \quad (0 \leq m \neq s \leq r)$$

Let the induced G_p-connection be represented by the matrix (w^i_j), then in this connection

$$\nabla g_{ij} = -\hat{w}^k_i g_{kj} - \hat{w}^k_j g_{ik}$$

and we have

$$\begin{aligned}\nabla g_{\alpha_m\beta_m} &= -\left(\hat{\omega}^{\lambda_1}_{\alpha_m} g_{\lambda_1\beta_m} + \cdots + \hat{\omega}^{\lambda_r}_{\alpha_m} g_{\lambda_r\beta_m}\right)\\ &\quad -\left(\hat{\omega}^{\lambda_1}_{\beta_m} g_{\alpha_m\lambda_1} + \cdots + \hat{\omega}^{\lambda_r}_{\beta_m} g_{\alpha_m\lambda_r}\right)\\ &= 0 \qquad\qquad \text{for} \quad 0 \leq m \leq r\end{aligned}$$

Also

$$\begin{aligned}\nabla g_{\alpha_m\beta_s} &= -\left(\hat{\omega}^{\lambda_1}_{\alpha_m} g_{\lambda_1\beta_s} + \cdots + \hat{\omega}^{\lambda_r}_{\alpha_m} g_{\lambda_r\beta_s}\right)\\ &\quad -\left(\hat{\omega}^{\lambda_1}_{\beta_s} g_{\alpha_m\lambda_1} + \cdots + \hat{\omega}^{\lambda_r}_{\beta_s} g_{\alpha_m\lambda_r}\right)\\ &= -\left(\hat{\omega}^{\lambda_m}_{\alpha_m} g_{\lambda_m\beta_s} + \hat{\omega}^{\lambda_s}_{\beta_s} g_{\alpha_m\lambda_s}\right)\\ &= -\left(\omega^{\lambda_m}_{\alpha_m} g_{\lambda_m\beta_s} + \omega^{\lambda_s}_{\beta_s} g_{\alpha_m\lambda_s}\right)\\ &= 0 .\end{aligned}$$

Hence the G_p-connection induced by (w^i_j) is again Euclidean. We may thus state the following theorem:

THEOREM 27: Any given Euclidean connection can be identified with a Euclidean G_p-connection and hence with a G_H-connection.

V_n can be endowed with the structure of Riemannian manifold (from the theorem of Whitney) and then from the fundamental theorem of Riemannian geometry, we may assert the existence of a unique Euclidean connection of the manifold V_n. The induced connection is then a G_H-connection which we call the first canonical connection[2].

(a) Let (w^i_j), respectively $(\hat{w}^i_j)$, be the forms defining the Riemannian connection, respectively the first canonical connection, relative to some local section of $E_c(V_n)$. Let us put

$$w^i_j = \gamma^i_{jk}\,\theta^k; \quad \hat{w}^i_j = \hat{\gamma}^i_{jk}\,\theta^k$$

where (θ^i) is, at each point, the dual basis of the basis defined by the local section. We will say that the (γ^i_{jk}), respectively $(\hat{\gamma}^i_{jk})$, are the components of the Riemannian connection, respectively first canonical connection, relative to the local section considered. Let $a^i_{jk} = \hat{\gamma}^i_{jk} - \gamma^i_{jk}$, then the a^i_{jk} are the components of a tensor[12]. With respect to the G_p-adapted basis we have

$$a^{\alpha_s}_{\beta_s k} = \hat{\gamma}^{\alpha_s}_{\beta_s k} - \gamma^{\alpha_s}_{\beta_s k} = \gamma^{\alpha_s}_{\beta_s k} - \gamma^{\alpha_s}_{\beta_s k} = 0$$

$$a^{\alpha_s}_{\beta_s k} = \hat{\gamma}^{\alpha_s}_{\bar{\beta}_s k} - \gamma^{\alpha_s}_{\bar{\beta}_s k} = -\gamma^{\alpha_s}_{\bar{\beta}_s k} \qquad 0 \leqslant s \leqslant r$$

Let $\nabla(F^i_j)$ be the absolute differential of the tensor (F^i_j) in the Riemannian connection. Let us introduce the covariant derivative $\nabla_k(F^i_j)$ defined by

$$\nabla(F^i_j) = \nabla_k(F^i_j).\theta^k$$

with respect to the G_p-adapted basis, we have

$$\nabla F^i_j = dF^i_j + \omega^i_k F^k_j - \omega^k_j F^i_k$$

or $\nabla_k F^i_j = \partial_k F^i_j + \gamma^i_{hk} F^h_j - \gamma^h_{jk} F^i_h$, then

$$\nabla_k F^{\alpha_s}_{\beta_s} = \gamma^{\alpha_s}_{hk} F^h_{\beta_s} - \gamma^h_{\beta_s k} F^{\alpha_s}_h = \gamma^{\alpha_s}_{\mu_s k} F^{\mu_s}_{\beta_s} - \gamma^{\mu_s}_{\beta_s k} F^{\alpha_s}_{\mu_s}$$

$$= \gamma^{\alpha_s}_{\mu_s k} \lambda \overset{s}{\omega} \delta^{\mu_s}_{\beta_s} - \gamma^{\mu_s}_{\beta_s k} \lambda \overset{s}{\omega} \delta^{\alpha_s}_{\mu_s} = 0$$

$$\nabla_k F^{\alpha_s}_{\bar{\beta}_s} = \gamma^{\alpha_s}_{hk} F^h_{\bar{\beta}_s} - \gamma^h_{\bar{\beta}_s k} F^{\alpha_s}_h$$

$$= \gamma^{\alpha_s}_{\bar{\mu}_s k} F^{\bar{\mu}_s}_{\bar{\beta}_s} - \gamma^{\mu_s}_{\bar{\beta}_s k} F^{\alpha_s}_{\mu_s}$$

$$= \gamma^{\alpha_s}_{\bar{\mu}_s k} \Big\{ \lambda \Big(\overset{0}{\omega} \delta^{\mu_0}_{\beta_0} + \dots + \overset{s-1}{\omega} \delta^{\mu_{s-1}}_{\beta_{s-1}} + \overset{s+1}{\omega} \delta^{\mu_{s+1}}_{\beta_{s+1}} + \dots$$

$$\dots + \overset{s}{\omega} \delta^{\mu_r}_{\beta_r} \Big) \Big\} - \gamma^{\mu_s}_{\bar{\beta}_s k} F^{\alpha_s}_{\mu_s}$$

or

$$\nabla_k F^{\alpha_s}_{\bar{\beta}_s} = \gamma^{\alpha_s}_{\bar{\mu}_s k} \lambda \left(- \overset{s}{\omega} \delta^{\bar{\mu}_s}_{\bar{\beta}_s}\right) - \gamma^{\mu_s}_{\bar{\beta}_s k} \lambda \overset{s}{\omega} \delta^{\alpha_s}_{\mu_s}$$

$$= -\lambda \overset{s}{\omega} \gamma^{\alpha_s}_{\beta_s k} - \lambda \overset{s}{\omega} \gamma^{\mu_s}_{\bar{\beta}_s k}$$

$$= -2\lambda \overset{s}{\omega} \gamma^{\alpha_s}_{\beta_s k} . \tag{6.2.5}$$

Similarly $$\nabla_k F^{\bar{\alpha}_s}_{\beta_s} = 2\lambda \overset{s}{\omega} \gamma^{\bar{\alpha}_s}_{\beta_s k} . \tag{6.2.6}$$

One can deduce from this that

$$a^i_{jk} = \frac{1}{2\lambda^2} F^i_j \nabla_k (F^h_j) \tag{6.2.7}$$

and this relation remains valid with respect to any base.

Finally

$$\hat{\gamma}^i_{jk} = \gamma^i_{jk} + \frac{1}{2\lambda^2} F^i_h \nabla_k (F^h_j) \tag{6.2.8}$$

Let us note that since the Riemannian connection has zero torsion we have

$$d\theta^i = \theta^j \wedge w^i_j = \gamma^i_{jk}\, \theta^j \wedge \theta^k$$

and so the components of the torsion tensor for the a.r.p.s. are given by

$$t^{\alpha_s}_{\bar{\beta}_s \bar{\gamma}_s} = \gamma^{\alpha_s}_{\bar{\beta}_s \bar{\gamma}_s} - \gamma^{\alpha_s}_{\bar{\gamma}_s \bar{\beta}_s} = -\frac{1}{2\lambda \omega^s} \left(\nabla_{\bar{\gamma}_s} F^{\alpha_s}_{\bar{\beta}_s} - \nabla_{\bar{\beta}_s} F^{\alpha_s}_{\bar{\gamma}_s}\right)$$

and

$$t^{\bar{\alpha}_s}_{\beta_s \gamma_s} = \frac{1}{2\lambda \omega^s} \left(\nabla_{\gamma_s} F^{\bar{\alpha}_s}_{\beta_s} - \nabla_{\beta_s} F^{\bar{\alpha}_s}_{\gamma_s}\right). \tag{6.2.9}$$

The torsion form of the first canonical connection is defined by

$$
\begin{aligned}
\hat{\Sigma}^i &= d\theta^i + \hat{w}^k_j \wedge \theta^j \\
&= -w^i_j \wedge \theta^j + \hat{w}^i_j \wedge \theta^j \\
&= (\hat{w}^i_j - w^i_j) \wedge \theta^j \qquad (6.2.10) \\
&= a^i_{jk}\, \theta^k \wedge \theta^j \\
&= \frac{1}{4\lambda^2} F^i_h (\nabla_j F^h_k - \nabla_k F^h_j)\, \theta^j \wedge \theta^k .
\end{aligned}
$$

Relative to the G_p-adapted bases, we have

$$
\begin{aligned}
\hat{\Sigma}^{\alpha_s} &= a^{\alpha_s}_{\bar{\beta}_s k}\, \theta^k \wedge \theta^{\bar{\beta}_s} \qquad \text{for } 0 \leq s \leq r \\
&= -\gamma^{\alpha_s}_{\bar{\beta}_s k}\, \theta^k \wedge \theta^{\bar{\beta}_s} \\
&= -\gamma^{\alpha_s}_{\beta_s \gamma_s}\, \theta^{\gamma_s} \wedge \theta^{\bar{\beta}_s} - \gamma^{\alpha_s}_{\bar{\beta}_s \bar{\gamma}_s}\, \theta^{\bar{\gamma}_s} \wedge \theta^{\bar{\beta}_s} \qquad (6.2.11) \\
&= -\frac{1}{2\lambda\omega^s} \nabla_{\gamma_s} F^{\alpha_s}_{\bar{\beta}_s}\, \theta^{\alpha_s} \wedge \theta^{\bar{\beta}_s} - \frac{1}{4\lambda\omega^s} (\nabla_{\bar{\beta}_s} F^{\alpha_s}_{\bar{\gamma}_s} - \nabla_{\bar{\gamma}_s} F^{\alpha_s}_{\bar{\beta}_s})\, \theta^{\bar{\beta}_s} \wedge \theta^{\bar{\gamma}_s}
\end{aligned}
$$

The second term of this result represents the torsion form of the a.r.p.s.

We thus note that the first canonical connection is a G_H-connection for which the vanishing of the torsion form implies the vanishing of the torsion form of the a.r.p.s. This leads to the following theorem:

THEOREM 28: On a manifold V_n equipped with an a.r.p.s., there always exists a G_H-connection whose torsion form vanishes only if the torsion form of the a.r.p.s. vanishes.

The curvature form of the first canonical connection is defined by

$$\hat{\Omega}^i_j = d\,\hat{w}^i_j + \hat{w}^i_h \wedge \hat{w}^h_j \,.$$

Let (Ω^i_j) be the curvature form of the Riemannian connection

$$\Omega^i_j = dw^i_j + w^i_h \wedge w^h_j;$$

according to (6.2.8),

$$\hat{w}^i_j = w^i_j + \frac{1}{2\lambda^2} F^i_j \nabla(F^h_j);$$

one deduces from this

$$\hat{\Omega}^i_j = \Omega^i_j + \frac{1}{4\lambda^2}\nabla(F^i_h \wedge \nabla(F^h_j) + \frac{1}{2\lambda^2} F^i_h \nabla\nabla(F^h_j) \qquad (6.2.12)$$

where $\nabla\nabla(F^h_j)$, the absolute differential of the tensor 1-form (∇F^h_j) in the Riemannian connection, is defined by $\nabla\nabla(F^h_j) = d(\nabla F^h_j) + w^h_\ell \nabla F^\ell_j - w^\ell_j \wedge \nabla(F^h)$.

(b) We will characterize the second canonical connection or the connection of Chern-Libermann by the following properties:

(i) It is a G_H-connection.

(ii) Relative to the G_H-adapted basis, its torsion form is expressed by

$$\Sigma^{\alpha_s} = \frac{1}{2} b^{\alpha_s}_{\beta_s \gamma_s} \theta^{\beta_s} \wedge \theta^{\gamma_s} + T^{\alpha_s}, \qquad 0 \leq s \leq r,$$

where $(T^{\alpha_0}, \ldots, T^{\alpha_r})$ is the torsion form of the a.r.p.s. and $(b^{\alpha_0}_{\beta_0\gamma_0}, \ldots \ldots, b^{\alpha_r}_{\beta_r\gamma_r})$ is an antisymmetric tensor with respect to the lower indices.

Let us show that there exists one, and only one, connection satisfying these conditions. Let (C^i_{jk}) be the components of the desired connection, (γ^i_{jk}), those of the Riemannian connection. Let us set

$$\sigma^i_{jk} = C^i_{jK} - \gamma^i_{jk} \qquad (6.2.13)$$

The (σ^i_{jk}) are the components of a tensor. The tensor form of the connection is defined by

$$\Sigma^i = \sigma^i_{jk}\, \theta^k \wedge \theta^j$$

$$= 1/2\ (\sigma^i_{kj} - \sigma^i_{jk})\ \theta^j \wedge \theta^k.$$

Relative to the G_H-adapted bases, the conditions (i) and (ii) become

$$C^{\alpha_s}_{\bar{\beta}_s k} = C^{\alpha_s}_{\beta_s k} = 0\,;$$

$$C^{\alpha_s}_{\beta_s k} + C^{\beta_m}_{\alpha_m k} = 0 \qquad (0 \leq s \leq r-1);(s+1 \leq m \leq r) \tag{6.2.14}$$

$$\sigma^{i}_{\alpha_s \bar{\beta}_s} - \sigma^{i}_{\bar{\beta}_s \alpha_s} = 0 \tag{6.2.15}$$

We can write (6.2.14) as

$$\sigma^{\alpha_s}_{\bar{\beta}_s k} = -\gamma^{\alpha_s}_{\bar{\beta}_s k}\,; \quad \sigma^{\bar{\alpha}_s}_{\beta_s k} = -\gamma^{\bar{\alpha}_s}_{\beta_s k} \tag{6.2.16}$$

$$\sigma^{\alpha_s}_{\beta_s k} + \sigma^{\beta_m}_{\alpha_m k} = -\left(\gamma^{\alpha_s}_{\beta_s k} + \gamma^{\beta_m}_{\alpha_m k}\right) \tag{6.2.17}$$

Since the Riemannian connection is Euclidean, the second part of (6.2.17) is zero. Using (6.2.15) one then obtains

(i) $$\sigma^{\alpha_s}_{\beta_s \bar{\gamma}_s} = \sigma^{\alpha_s}_{\bar{\gamma}_s \beta_s} = C^{\alpha_s}_{\bar{\gamma}_s \beta_s} - \gamma^{\alpha_s}_{\bar{\gamma}_s \beta_s} = 0 - \gamma^{\alpha_s}_{\bar{\gamma}_s \beta_s} = -\gamma^{\alpha_s}_{\bar{\gamma}_s \beta_s}$$

(ii) $$\sigma^{\bar{\alpha}_s}_{\beta_s \gamma_s} = \sigma^{\bar{\alpha}_s}_{\gamma_s \bar{\beta}_s} = C^{\bar{\alpha}_s}_{\gamma_s \bar{\beta}_s} - \gamma^{\bar{\alpha}_s}_{\gamma_s \bar{\beta}_s} = 0 - \gamma^{\bar{\alpha}_s}_{\gamma_s \bar{\beta}_s} = -\gamma^{\bar{\alpha}_s}_{\gamma_s \bar{\beta}_s}$$

(iii) $$\sigma^{\alpha_s}_{\beta_s \gamma_s} = -\sigma^{\beta_m}_{\alpha_m \gamma_s} = -\sigma^{\beta_m}_{\gamma_s \alpha_m} = -C^{\beta_m}_{\gamma_s \alpha_m} + \gamma^{\beta_m}_{\gamma_s \alpha_m} = \gamma^{\beta_m}_{\gamma_s \alpha_m}$$

If one introduces the covariant derivative $\nabla_k(F^i_j)$ of the tensor (F^i_j) in the Riemannian connection and if one puts

$$\nabla^k(F^i_j) = g^{kh}\nabla_h(F^i_j),$$

one is able to express the preceding results in the forms

$$
\begin{aligned}
&\text{(i)} && \sigma^{\alpha_s}_{\beta_s\gamma_s} = -\frac{1}{2\lambda\omega^s}\nabla^{\alpha_s}F_{\beta_s\gamma_s}\\
&\text{(ii)} && \sigma^{\alpha_s}_{\beta_s\bar\gamma_s} = \frac{1}{2\lambda\omega^s}\nabla_{\beta_s}F^{\alpha_s}_{\bar\gamma_s}\\
&\text{(iii)} && \sigma^{\alpha_s}_{\bar\beta_s\gamma_s} = \frac{1}{2\lambda\omega^s}\nabla_{\gamma_s}F^{\alpha_s}_{\bar\beta_s}\\
&\text{(iv)} && \sigma^{\alpha_s}_{\bar\beta_s\bar\gamma_s} = \frac{1}{2\lambda\omega^s}\nabla_{\bar\gamma_s}F^{\alpha_s}_{\bar\beta_s}\\
&\text{(v)} && \sigma^{\bar\alpha_s}_{\beta_s\gamma_s} = -\frac{1}{2\lambda\omega^s}\nabla_{\gamma_s}F^{\bar\alpha_s}_{\beta_s}\\
&\text{(vi)} && \sigma^{\bar\alpha_s}_{\beta_s\bar\gamma_s} = -\frac{1}{2\lambda\omega^s}\nabla_{\bar\gamma_s}F^{\bar\alpha_s}_{\beta_s}\\
&\text{(vii)} && \sigma^{\bar\alpha_s}_{\bar\beta_s\gamma_s} = -\frac{1}{2\lambda\omega^s}\nabla_{\bar\beta_s}F^{\bar\alpha_s}_{\gamma_s}\\
&\text{(viii)} && \sigma^{\bar\alpha_s}_{\bar\beta_s\bar\gamma_s} = \frac{1}{2\lambda\omega^s}\nabla^{\bar\alpha_s}F_{\bar\beta_s\bar\gamma_s}
\end{aligned}
\qquad (6.2.18)
$$

Conversely, these formulas define a tensor on V_n and the connection which is associated with it by (6.2.13) satisfies the condition set down. Let us note that formulas (6.2.18) remain valid with respect to any G_p-adapted base.

The torsion form of the connection may be expressed, relative to the G_p-adapted bases, by

$$\Sigma^{\alpha_s} = \frac{1}{2\lambda\omega^s}\nabla^{\alpha_s}F_{\beta_s\gamma_s}\theta^{\beta_s}\wedge\theta^{\gamma_s} + \frac{1}{4\lambda\omega^s}\left(\nabla_{\bar\beta_s}F^{\alpha_s}_{\bar\gamma_s} - \nabla_{\bar\gamma_s}F^{\alpha_s}_{\bar\beta_s}\right)\theta^{\bar\beta_s}\wedge\theta^{\bar\gamma_s} \tag{6.2.19}$$

from which one deduces

$$b^{\alpha_s}_{\beta_s\gamma_s} = \frac{1}{\lambda\omega^s}\nabla^{\alpha_s}F_{\beta_s\gamma_s}, \text{ for } 0 \leq s \leq r;$$

with respect to any base whatsoever, one is able to write

$$\sigma^i_{jk} = \frac{1}{2\lambda^2}F^i_h\nabla_h F^h_j + \frac{1}{4\lambda^2}\left(F^i_h\nabla_j F^h_k + F^h_j\nabla_h F^i_k - F^i_h\nabla^h F_{jk} - F^h_k\nabla^i F_{jh}\right) \tag{6.2.20}$$

Indeed, this formula actually defines a tensor, for which the components with respect to a G_p-adapted basis coincide with those which are defined by (6.2.17). One deduces from this that, relative to any base whatever, that is to say an arbitrary local section of $E_c(V_n)$, the components of the connection of Chern-Libermann are

$$C^i_{jk} = \gamma^i_{jk} + \frac{1}{2\lambda^2}F^i_h\nabla_k F^h_j \tag{6.2.21}$$

$$+ \frac{1}{4\lambda^2}\left(F^i_h\nabla_j F^h_k + F^h_j\nabla_h F^i_k - F^i_h\nabla^h F_{jk} - F^h_k\nabla^i F_{jh}\right).$$

6.3 The Holonomy Group of the G_H-Connections

Let us consider a G_H-connection; any horizontal path constructed on $E_c(V_n)$ relative to the linear connection identified with the given G_H-connection and beginning at a G_H-adapted basis z, ends at a G_H-adapted basis z'. One deduces from this that the holonomy group[2] at z of this connection is a subgroup of u(m).

Conversely, let V_n be a differentiable manifold equipped with a linear connection and let us suppose that at a point x of V_n there is a basis z such that the holonomy group ψ_z of the connection at z is a subgroup of u(m). Let us consider, at the point x, the tensors (g_{ij}) and (F^i_j) for which the components with respect to the basis z are defined by

$$(g_{ij}) = \begin{vmatrix} 0_m & I_m & \cdots & I_m \\ I_m & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ I_m & \cdots & \cdots & 0_m \end{vmatrix} \quad \text{and} \quad (F^i_j) = \begin{vmatrix} \lambda w^0 I_{00} & & & 0 \\ & \lambda w^1 I_{11} & & \\ & & \ddots & \\ 0 & & & \lambda w^r I_{rr} \end{vmatrix}$$

These tensors are invariant under ψ_z (a subgroup of u(m)). By parallel transport on V_n, we obtain the tensors (g_{ij}) and (F^i_j) defined on the whole manifold. Now at the point x we have

$$F^{k_1}_j F^{k_2}_{k_1} \cdots F^i_{k_s} = \lambda^{s+1} \delta^i_j \tag{6.3.1}$$

Also $$F^k_i g_{kh} + F^k_h g_{ki} = F^{\alpha_\ell}_i g_{\alpha_\ell \beta_m} + F^{\alpha_\ell}_{\beta_m} g_{\alpha_\ell i}$$

$$\text{or} \quad F_i^k g_{kh} + F_h^k g_{ki} = F_{\gamma_s}^{\alpha_\ell} \delta_{\alpha\beta} + F_{\beta_m}^{\alpha_\ell} \delta_{\alpha\gamma}$$
$$= F_{\gamma_s}^{\beta_\ell} + F_{\beta_m}^{\gamma_\ell}$$
$$= 0 + 0 = 0 \quad \text{where } 0 \leq \ell \neq m \neq s \leq r$$

Hence

$$F_j^k g_{kh} = -F_h^k g_{ki}, \tag{6.3.2}$$

and these two relations remain true at any point of V_n. Thus V_n may be endowed with a H-structure subordinate to an a.r.p.s. Since the tensors (g_{ij}) and (F_j^i) are invariant under ψ_z, they have zero absolute differential,[2]; thus the given connection may be identified with a G_H-connection. Hence the following theorem:

THEOREM 29: A necessary and sufficient condition that a linear connection on V_n be a G_H-connection of a H-structure subordinate to a.r.p.s., is that the holonomy group of the connection is a subgroup of u(m).

Suppose now that V_n is a differential manifold equipped with a metric (g_{ij}). We will say that a basis z at the point x of V_n is adapted to the metric if the components of the metric tensor with respect to z are

$$(g_{ij}) \doteq \begin{vmatrix} O_m & I_m & \cdots & I_m \\ I_m & \cdot & & \cdot \\ \vdots & & \cdot & \vdots \\ I_m & \cdots & & O_m \end{vmatrix};$$

Further suppose that $n = (r+1)m$. Given on V_n a Euclidean connection and let us suppose that there exists at the point x of V_n a basis z, adapted to the metric, such that the holonomy group Ψ_z is a subgroup of G(m). By assumption $\nabla g_{ij} = 0$, the metric tensor is thus invariant under Ψ_z. It follows that Ψ_z is a subgroup of U(m). Then as above, one may equip V_n with a H-structure subordinate to a.r.p.s. for which the metric coincides with the initial metric. The given connection can then be identified with a G_H-connection. We have thus proved:

THEOREM 30: A necessary and sufficient condition that a Euclidean connection in V_n be a G_H-connection of a H-structure subordinate to the a.r.p.s. is that the holonomy group of the connection be a subgroup of G(m).

6.4 The Characteristic Forms of a G_H-Connection

Let V_n be a differentiable manifold endowed with a H-structure. Any G_H-connection determines canonically a G_p-connection. We can thus associate with it characteristic forms as defined in Chapter 2. If the connection is defined relative to the G_p-adapted basis by $(\pi^{\alpha_0}_{\beta_0}, \ldots\ldots, \pi^{\alpha_r}_{\beta_r})$, one has $\Psi_k = \lambda\, \omega^k\, d(\pi^{\alpha_k}_{\alpha_k})$, $0 \leq k \leq r$.

Since the given connection is a G_H-connection, one is able to write, relative to the G_H-adapted basis,

$$\pi^{\beta_k}_{\alpha_k} + \pi^{\alpha_s}_{\beta_s} = 0 \qquad (0 \leq k \leq r-1,\ k+1 \leq s \leq r).$$

One deduces from this that

$$w^{s}\psi_{k} + w^{k}\psi_{s} = 0 \qquad (6.4.1)$$

Except for the relation (6.4.1) between characteristic forms, there will be no basic change in the results which we developed in Chapter 2.

NOTE: If we put $r = 1$, then $k = 0$, $s = 1$, $w = -1$;

$$-\psi_0 + \psi_1 = 0 \qquad \psi_0 = \psi_1 = \psi^{1} .$$

CHAPTER 7

Particular Cases of H-Structures

7.0 Hermitian and Pseudo-Hermitian Structures

Given a H-structure on V_n, we will say that it is hermitian if the underlying a.r.p.s. is integrable, that is, for each $x_0 \in V_n$ there exist n complex-valued functions (z^i), defined in an open neighbourhood v of x_0, such that at each point x of v, the subspaces $T_s (0 \leq s \leq r)$ determining the a.r.p.s. may be defined by $dz^{\alpha_k} = 0$ where $(0 \leq k \neq s \leq r)$. The basis of T^c_x dual of (dz^i) is adapted to the a.r.p.s. With respect to this basis one thus has

$$ds^2 = 2 \sum_{C^2_{r+1}} g_{\alpha_{s(a)} \beta_{\ell(a)}} \theta^{\alpha_{s(a)}} \cdot \theta^{\beta_{\ell(a)}} \qquad \text{where } (1 \leq a \leq m)$$

and

$$F = \lambda \sum_{C^2_{r+1}} \overset{s}{\omega} g_{\alpha_{s(a)} \beta_{\ell(a)}} \theta^{\alpha_{s(a)}} \wedge \theta^{\beta_{\ell(a)}}, \qquad (1 \leq s \neq \ell \leq r).$$

The a.r.p.s. underlying the hermitian structure is necessarily without torsion.

Conversely, suppose we have a H-structure for which the underlying a.r.p.s. is without torsion. Such a structure will be called a Pseudo-Hermitian structure, briefly P-H-structure. In an analytic case, that is to say when V_n and the a.r.p.s. are of class C^w, a P-H-structure is hermitian. We recall that in the Riemannian connection we have, relative to the G_p-adapted basis,

$$\nabla_k F^{\alpha_s}_{\beta_s} = 0$$

$$\nabla_k F^{\alpha_s}_{\bar{\beta}_s} = -2\lambda\omega^s \gamma^{\alpha_s}_{\bar{\beta}_s k}$$

$$\nabla_k F^{\bar{\alpha}_s}_{\beta_s} = 2\lambda\omega^s \gamma^{\bar{\alpha}_s}_{\beta_s k}$$

and the components of the torsion tensor of the a.r.p.s. are given by

$$t^{\alpha_s}_{\bar{\beta}_s\bar{\gamma}_s} = \gamma^{\alpha_s}_{\bar{\beta}_s\bar{\gamma}_s} - \gamma^{\alpha_s}_{\bar{\gamma}_s\bar{\beta}_s} = -\frac{1}{2\lambda\omega^s}\left(\nabla_{\bar{\gamma}_s} F^{\alpha_s}_{\bar{\beta}_s} - \nabla_{\bar{\beta}_s} F^{\alpha_s}_{\bar{\gamma}_s}\right).$$

Since $F_{ij} = F^k_i\, g_{kj}$, we obtain

$$\nabla_h(F_{ij}) = \nabla_h(F^k_i\, g_{kj}) = g_{kj}\,\nabla_h(F^k_j) + F^k_i\nabla_h(g_{kj})$$

$$= g_{kj}\nabla_h(F^k_i).$$

Let us set $r = 1$, that is we consider a π-structure[1].

We have

$$\nabla_h F_{\alpha_0\beta_\ell} = g_{k\beta_\ell}\nabla_h(F^k_{\alpha_0})$$

$$= g_{\gamma_0\beta_1}\nabla_h(F^{\gamma_0}_{\alpha_0}) + g_{\gamma_1\beta_1}\nabla_h(F^{\gamma_1}_{\alpha_0})$$

$$= \quad 0 \qquad + \qquad 0$$

$$= 0$$

$$\nabla_h F_{\alpha_1\beta_0} = g_{k\beta_0}\nabla_h(F^k_{\alpha_0}) = g_{\gamma_0\beta_0}\nabla_h(F^{\gamma_0}_{\alpha_0}) + g_{\gamma_1\beta_0}\nabla_h(F^{\gamma_1}_{\alpha_1})$$

$$= 0$$

Hence we state the following theorem:

THEOREM 31: For an almost hermitian structure in the broad sense [1] we have:

$$\nabla_h F_{\alpha_0 \beta_1} = \nabla_h F_{\alpha_1 \beta_0} = 0$$

with respect to a π-adapted basis, where ∇_h denotes the covariant derivative in the Riemannian connection and (F_{ij}) is the fundamental form of the almost hermitian structure.

NOTE: One is no longer able to generalize this result in the case of a.r.p.s.

Let us now consider a G_H-adapted basis; with respect to this basis, we have

$$\begin{aligned}
\gamma^{\alpha_s}_{\bar{\beta}_s \bar{\mu}_s} &= -\frac{1}{2\lambda\omega^s} \nabla_{\bar{\mu}_s} F_{\bar{\beta}_s}^{\alpha_s} \qquad \text{for } 0 \leq s \leq r \\
&= -\frac{1}{2\lambda\omega^s} \nabla_{\bar{\mu}_s} \left(F_{\bar{\beta}_s k}\, g^{k\alpha_s} \right) \\
&= -\frac{1}{2\lambda\omega^s} \left(\nabla_{\bar{\mu}_s} F_{\bar{\beta}_s \gamma_s}\, g^{\gamma_s \alpha_s} + \nabla_{\bar{\mu}_s} F_{\bar{\beta}_s \bar{\gamma}_s}\, g^{\bar{\gamma}_s \alpha_s} \right) \\
&= -\frac{1}{2\lambda\omega^s} \left(0 + \nabla_{\bar{\mu}_s} F_{\bar{\beta}_s \bar{\gamma}_s} \delta_{\gamma\alpha} \right) \\
&= -\frac{1}{2\lambda\omega^s} \nabla_{\bar{\mu}_s} F_{\bar{\beta}_s \bar{\gamma}_s} \delta_{\gamma\alpha} \\
&= \frac{1}{2\lambda\omega^s} \nabla_{\bar{\mu}_s} F_{\bar{\alpha}_s \bar{\beta}_s} .
\end{aligned} \tag{7.0.1}$$

Similarly
$$\gamma^{\bar{\alpha}_s}_{\beta_s \mu_s} = -\frac{1}{2\lambda\omega^s} \nabla_{\mu_s} F_{\alpha_s \beta_s} \qquad (7.0.2)$$

Then we get
$$t^{\alpha_s}_{\bar{\beta}_s \bar{\mu}_s} = \left(\gamma^{\alpha_s}_{\bar{\beta}_s \bar{\mu}_s} - \gamma^{\alpha_s}_{\bar{\mu}_s \bar{\beta}_s}\right) = \frac{1}{2\lambda\omega^s}\left(\nabla_{\bar{\mu}_s} F_{\bar{\alpha}_s \bar{\beta}_s} - \nabla_{\bar{\beta}_s} F_{\bar{\alpha}_s \bar{\mu}_s}\right) \qquad (7.0.3)$$

and
$$t^{\bar{\alpha}_s}_{\beta_s \mu_s} = \left(\gamma^{\bar{\alpha}_s}_{\beta_s \mu_s} - \gamma^{\bar{\alpha}_s}_{\mu_s \beta_s}\right) = -\frac{1}{2\lambda\omega^s}\left(\nabla_{\mu_s} F_{\alpha_s \beta_s} - \nabla_{\beta_s} F_{\alpha_s \mu_s}\right) \qquad (7.0.4)$$

We see from the above results that the a.r.p.s. underlying H-structure is without torsion if and only if $\nabla_{\bar{\mu}_s} F_{\bar{\alpha}_s \bar{\beta}_s}$ (respectively $\nabla_{\mu_s} F_{\alpha_s \beta_s}$) is symmetric with respect to $\bar{u}_s$ and $\bar{\beta}_s$ (respectively u_s and β_s). But since $\nabla_{\bar{u}_s} F_{\bar{\alpha}_s \bar{\beta}_s}$ (respectively $\nabla_{u_s} F_{\alpha_s \beta_s}$) is antisymmetric with respect to $\bar{\alpha}_s$ and $\bar{\beta}_s$ (respectively α_s and β_s) such a symmetry condition implies that $\nabla_{\bar{u}_s} F_{\bar{\alpha}_s \bar{\beta}_s}$ (respectively $\nabla_{u_s} F_{\alpha_s \beta_s}$) = 0.

Indeed,
$$\nabla_{\mu_s} F_{\alpha_s \beta_s} = \nabla_{\beta_s} F_{\alpha_s \mu_s} = -\nabla_{\beta_s} F_{\mu_s \alpha_s} = -\nabla_{\alpha_s} F_{\mu_s \beta_s},$$

also
$$\nabla_{\mu_s} F_{\alpha_s \beta_s} = -\nabla_{\mu_s} F_{\beta_s \alpha_s} = -\nabla_{\alpha_s} F_{\beta_s \mu_s} = \nabla_{\alpha_s} F_{\mu_s \beta_s},$$

therefore
$$2\nabla_{\mu_s} F_{\alpha_s \beta_s} = 0$$

or
$$\nabla_{\mu_s} F_{\alpha_s \beta_s} = 0.$$

Similarly
$$\nabla_{\bar{\mu}_s} F_{\bar{\alpha}_s \bar{\beta}_s} = 0.$$

Hence we state the following theorem:

THEOREM 32: In order that a H-structure by Pseudohermitian, it is necessary and sufficient that, relative to the $G_{\bar{H}}$-adapted basis, one has at any point

$$\nabla_{\mu_s} F_{\alpha_s \beta_s} = 0 \; ; \; \nabla_{\bar{\mu}_s} F_{\bar{\alpha}_s \bar{\beta}_s} = 0 \tag{7.0.5}$$

or equivalently

$$\gamma^{\bar{\alpha}_s}_{\beta_s \mu_s} = 0 \quad ; \quad \gamma^{\alpha_s}_{\bar{\beta}_s \bar{\mu}_s} = 0 \tag{7.0.6}$$

The condition (7.0.6) remains valid with respect to any G_H-adapted basis. With respect to any base whatsoever, the P-H-structures may be characterized by

$$F^h_k \nabla_h F_{ij} + F^h_j \nabla_k F_{ih} = 0 \tag{7.0.7}$$

Indeed, the first part of (7.0.7) defines a tensor (P_{ijk}), such that one has with respect to a G_p-adapted basis

$$\begin{aligned}
P_{\alpha_s \beta_s \gamma_s} &= F^h_{\gamma_s} \nabla_h F_{\alpha_s \beta_s} + F^h_{\beta_s} \nabla_{\gamma_s} F_{\alpha_s h} \quad \text{for} \quad (0 \le s \le r) \\
&= F^{\mu_s}_{\gamma_s} \nabla_{\mu_s} F_{\alpha_s \beta_s} + F^{\mu_s}_{\beta_s} \nabla_{\gamma_s} F_{\alpha_s \mu_s} \\
&= \lambda \overset{s}{\omega} \delta^{\mu_s}_{\gamma_s} \nabla_{\mu_s} F_{\alpha_s \beta_s} + \lambda \overset{s}{\omega} \delta^{\mu_s}_{\beta_s} \nabla_{\gamma_s} F_{\alpha_s \mu_s} \\
&= \lambda \overset{s}{\omega} \left(\nabla_{\gamma_s} F_{\alpha_s \beta_s} + \nabla_{\gamma_s} F_{\alpha_s \beta_s} \right) \\
&= 2 \lambda \overset{s}{\omega} \nabla_{\gamma_s} F_{\alpha_s \beta_s} ,
\end{aligned} \tag{7.0.8}$$

the other components being zero.

Given a P-H-structure on a differentiable manifold V_n, the torsion form of the Chern-Libermann connection is determined, relative to the G_p-adapted basis, by the formulas

$$\Sigma^{\alpha_s} = \frac{1}{2\lambda w^s} \nabla^{\alpha_s} F_{\beta_s \gamma_s} \theta^{\beta_s} \wedge \theta^{\gamma_s} \tag{7.0.9}$$

$$\Sigma^{\bar{\alpha}_s} = -\frac{1}{2\lambda w^s} \nabla^{\bar{\alpha}_s} F_{\bar{\beta}_s \bar{\gamma}_s} \theta^{\bar{\beta}_s} \wedge \theta^{\bar{\gamma}_s} \tag{7.0.10}$$

Since the <u>a.r.p.s</u>. is without torsion, one obtains by exterior differentiation of the two parts of (7.0.9) and (7.0.10), expressions of $d\Sigma^{\alpha_s}$ and $d\Sigma^{\bar{\alpha}_s}$ not containing any term of type (1,2). By introducing the curvature form (Ω^i_j) of the connection, let us write Bianchi's identity

$$d\Sigma^i = \Omega^i_j \wedge \theta^j - C^i_{jk}\, \theta^k \wedge \Sigma^j \tag{7.0.11}$$

where (C^i_{jk}) are the components of the connection.

Relative to the G_p-adapted basis one has

$$\begin{aligned} C^{\alpha_s}_{\bar{\beta}_s k} &= C^{\bar{\alpha}_s}_{\beta_s k} = 0 \\ \Omega^{\alpha_s}_{\bar{\beta}_s} &= \Omega^{\bar{\alpha}_s}_{\beta_s} = 0 . \end{aligned} \tag{7.0.12}$$

It follows then from (7.0.11) that $\Omega^{\alpha_s}_{\beta_s} \wedge \theta^{\beta_s}$ for $(0 \leq s \leq r)$ does not contain any term of type (1,2). Let $(R^i_{j,k\ell})$ be the curvature tensor of the connection defined by

$$\Omega^i_j = \frac{1}{2} R^i_{j,k\ell}\, \theta^k \wedge \theta^\ell, \quad (R^i_{j,k\ell} = -R^i_{j,\ell k})$$

one thus has

$$R^{\alpha_s}_{\beta_s,\bar{\gamma}_s\bar{\delta}_s} = 0 \qquad (7.0.13)$$

and

$$R^{\bar{\alpha}_s}_{\bar{\beta}_s,\gamma_s\delta_s} = 0 \qquad (7.0.14)$$

Let us set

$$R_{ij,k\ell} = g_{ih} R^h_{j,k\ell}. \qquad (7.0.15)$$

$R_{ij,k\ell}$ is antisymmetric not only with respect to the indices k and ℓ, but also with respect to the indices i and j. Let us agree to say that two indices are of the same type if they are both between (km) +1 and (km) + m for $0 \leq k \leq r$. Then one sees that, with respect to any G_p-adapted base, $R_{ij,k\ell}$ is zero as soon as the indices i and j, or the indices k and ℓ, are of the same type.

7.1 Almost r-Product Kählerian Structures

An almost r-product hermitian structure on a differentiable manifold V_n $(n=(r+1)m)$ will be called almost r-product Kählerian, briefly rk-structure, if the fundamental form F is closed. With respect to any base whatsoever, this condition is written as

$$\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0 \qquad (7.1.1)$$

and this relation must be satisfied at any point of V_n, whatever i,j,k, may be.

If one takes the G_p-adapted basis, one is able to decompose the relation (7.1.1) in the following way:

$$\nabla_{\gamma_s} F_{\bar{\alpha}_s \bar{\beta}_s} = 0 \quad , \quad \nabla_{\bar{\gamma}_s} F_{\alpha_s \beta_s} = 0 \tag{7.1.2}$$

$$\nabla_{\alpha_s} F_{\beta_s \gamma_s} + \nabla_{\beta_s} F_{\gamma_s \alpha_s} + \nabla_{\gamma_s} F_{\alpha_s \beta_s} = 0 . \tag{7.1.3}$$

Relative to the G_H-adapted bases, these conditions may be written

$$\gamma^{\alpha_s}_{\bar{\beta}_s \gamma_s} = 0 \quad ; \quad \gamma^{\bar{\alpha}_s}_{\beta_s \bar{\gamma}_s} = 0 \tag{7.1.4}$$

$$\gamma^{\alpha_s}_{\bar{\beta}_s \bar{\gamma}_s} + \gamma^{\beta_s}_{\bar{\gamma}_s \bar{\alpha}_s} + \gamma^{\gamma_s}_{\bar{\alpha}_s \bar{\beta}_s} = 0 \tag{7.1.5}$$

One can regard the condition (7.1.2) as expressing the nullity of the tensor (r_{ijk}) defined by

$$r_{\bar{\alpha}_s \bar{\beta}_s \gamma_s} = \nabla_{\gamma_s} F_{\bar{\alpha}_s \bar{\beta}_s} \quad ; \quad r_{\alpha_s \beta_s \bar{\gamma}_s} = -\nabla_{\bar{\gamma}_s} F_{\alpha_s \beta_s} \tag{7.1.6}$$

the other components being zero. With respect to any base whatsoever,

$$r_{ijk} = \frac{1}{2\lambda} (F^h_k \nabla_h F_{ij} - F^h_j \nabla_k F_{ih}) . \tag{7.1.7}$$

Similarly the conditions (7.1.3) express the nullity of the tensor (S_{ijk}) defined by

$$S_{\alpha_s \beta_s \gamma_s} = \nabla_{\alpha_s} F_{\beta_s \gamma_s} + \nabla_{\beta_s} F_{\gamma_s \alpha_s} + \nabla_{\gamma_s} F_{\alpha_s \beta_s} , \tag{7.1.8}$$

the other components being zero. With respect to any base whatsoever one has

$$S_{ijk} = \frac{1}{2\lambda} (P_{ijk} + P_{jki} + P_{kij}) \tag{7.1.9}$$

where we set

$$P_{ijk} = F^h_k \nabla_h (F_{ij}) + F^h_j \nabla_k (F_{ih}).$$

7.2 A Note on Kähleri an and Pseudokähleri an Structures

A H-structure on V_n will be called Kähleri an if the underlying <u>a.r.p.s.</u> is integrable and the fundamental form F is closed. It will be called Pseudokähleri an if the underlying <u>a.r.p.s.</u> is without torsion and F is closed. In other words, a H-structure is Kähleri an (respectively Pseudokähleri an) if it is at the same time hermitian (respectively Pseudohermitian) and almost r-product Kähleri an.

In order that a H-structure be Pseudokähleri an it is necessary and sufficient that, at any point of V_n, one has with respect to the G_p-adapted bases

$$\nabla_{\gamma_s} F_{\alpha_s \beta_s} = 0 ; \nabla_{\gamma_s} F_{\bar{\alpha}_s \bar{\beta}_s} = 0 ; \nabla_{\bar{\gamma}_s} F_{\alpha_s \beta_s} = 0 \tag{7.2.1}$$

This is to say that the covariant derivative of (F_{ij}) in the Riemanni an connection is zero. The following theorem may be deduced from this.

<u>THEOREM 33</u>: In order that a H-structure be Pseudokäheri an it is necessary and sufficient that the Riemanni an connection be a G_p-connection, that is to say it coincides with the connection of Lichnerowicz.

APPENDIX I

We show that $E_p(V_n)$ is a differentiable principal subfibre bundle of $E_c(V_n)$ with base V_n and structure group $G(n_r)$.

Let us consider $E_c(V_n)$ as the set of all complex bases at different points of V_n. Let $p':E_c(V_n)\longrightarrow V_n$ be the canonical mapping such that a base relative to $x \in V_n$ is made to correspond to the point x itself. It is well-known [2] that under this mapping $E_c(V_n)$ has, with respect to p', a natural structure of a principal fibre bundle with base V_n and the structure group GL(n,c). Furthermore, let us consider a canonical mapping p: $E_p(V_n)\longrightarrow V_n$ such that a base adapted to a.r.p.s. relative to $x \in V_n$ is made to correspond to the point x itself. We also assume that the mapping p is the restriction of the mapping p'. Previously we have proved that $G(n_r)$ is a Lie subgroup of GL(n,c). Hence the right translation by $g \in G(n_r)$ is the restriction to $E_p(V_n)$ of the right translation operated on $E_c(V_n)$. From this it is obviously true that for every $x \in V_n$, there exists a neighbourhood v of x and a differentiable section of $E_c(V_n)$ with values in $E_p(V_n)$. Hence one can deduce from the proposition 1,5,2 of D. Bernard[12] that $E_p(V_n)$ is a differentiable principal subfibre bundle of $E_c(V_n)$ with base V_n and structure group $G(n_r)$.

NOTE: A similar proof follows for references given under Appendix I, Chapter 3 and Chapter 6.

APPENDIX II

We show that the set $LO(n_r)$ of matrices R satisfying the identity $\overline{RG} + {}^t(RG) = 0$ is a Lie algebra of $O(n_r)$.

Let us assume that $\overline{RG} + {}^t(RG) = 0$ and $\overline{R_1G} + {}^t(R_1G) = 0$. For simplicity, we set $RG = X$ and $R_1G = Y$. Also set

$$Z = [X,Y] = XY - YX.$$

$$\begin{aligned} {}^t(Z) &= {}^t(XY) - {}^t(YX) \\ &= {}^t(Y)\,{}^t(X) - {}^t(X)\,{}^t(Y) \\ &= (-\overline{Y})(-\overline{X}) - (-\overline{X})(-\overline{Y}) \\ &= \overline{Y}\,\overline{X} - \overline{X}\,\overline{Y} \\ &= -\overline{Z}. \end{aligned}$$

Hence ${}^t(Z) + \overline{Z} = 0$, which implies that $[X,Y] \in L\,O\,(n_r)$.

NOTE: A similar proof follows for the reference under Appendix II in Chapter 6

REFERENCES

1 G. Legrand, Etude d'une généralization des structures presque complexes sur les variétés différentiables; Thèse; Rendiconti del circolo Matematico di Palermo; Serie 2, t. vii, 1958, pp 323-354; t. viii, 1959, pp 5-48.

2 A. Lichnerowicz, 'Théorie Globale des connexions et des groupes d'holonomie', Edizioni Cremonese, Rome; 1962.

3 H.A. Eliopoulos, On the general theory of differentiable manifolds with almost tangent structures, Canadian Math. Bull. Vol. 8, No. 6, 1965, pp. 721-748.

4 H.A. Eliopoulos, Some properties of differentiable structures defined by nilpotent operators of degree r, Bull. Soc. Math de Greece.

5 H.A. Eliopoulos, Euclidean structures compatible with almost tangent structures, Bull. de l'Acad. Royale de Belgique, 5[e] Serie, t.L; 1964-10, pp 1174-1182.

6 H.A. Eliopoulos, Structures riemanniennes dégénérées sur les variétés r-tangentes, Bull. de l'Acad. Royale de Belgique 5[e] Serie-Tome L III pp 794-88, 1967-68.

7 C.J. Hsu, On some properties of π-structures on Differentiable Manifolds, Tohoku Maths. J, 12(1960), pp 429-454.

8 C.S. Houch, Affine connections in an Almost Product Manifold, Tensor New Series, Vol. 12, No. 2, 1962, pp 167-184.

9 A. Weil, Sur la théorie des formes différentielles attachées a une variété analytique complexe(Comm. Math, Helv 20, 1947, pp 110-16).

10 C. Ehresmann, Sur la théorie des espaces fibres (Coll. Int. C.N.R.S. Topologie algebraic, Paris, 1947, pp 3-35).

11 C. Ehresmann, Sur las variétés presque complexes, Proc. Int. Congr. Math, 1950, t, 11, pp 412-419.

12 D. Bernard, Sur la géométric différentielle des G-structures; Ann. Inst. Fourier, Grenoble, 10; 1960, pp 153-273.

13 E. Cartan, Les systems differentielles exterieures et leurs applications géométriques (Hermann et C[ie]., France, 1945).

14 S.S. Chern, Differentiable Manifolds (mimeographed notes) Instituto de Física e Matemática, Universidade do Recife, Recife, Brazil, 1959.

15 A. Frolicher and A. Nijenhuis, Theory of vector valued differential forms, Part I, K. Ned. Akad Wet. Proc. Math. Sc. v. 59, pp 338-359, (1956).

16 T. Fukami, Affine connections in almost product manifolds with some structures, Tohoku Maths J., ii, 1959, 430-446.

17 A.G. Walkar, Connection for parallel distributions in the large, (1) Quart. J. Maths Oxford (2). 6 (1955) 301-308; (2) Quart. J. Maths Oxford (2) 9 (1958) 221-231.

18 K. Yano, Affine connections in an almost product space, Kodai Math Sem. Rep., 11 (1959), 1-24.

19 K. Yano, Differential Geometry on Complex and Almost Complex Spaces, The Macmillan Company, New York 1965.

20 D.C. Spencer, Differentiable manifolds, Notes, Princeton University (1954).

21 N. Steenrod, The Topology of fibre bundles, Princeton University Press, 1951.

22 C. Chevalley, Theory of Lie Groups, Princeton University Press, Princeton, 1946.

23 H. Schwerdtfeger, 'Introduction to Linear Algebra and the Theory of Matrices', 2nd edition, Noordhoff, N.V. 1961, Groningen.

24 S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. 1, Interscience Publishers, 1963.

25 K.L. Duggal, Singular Riemannian structures compatible with π-structures, to be published in Can. Math. Bull.

26 K.L. Duggal, Singular Riemannian structures subordinate to $r\pi$-structures. Presented at the Seventy-fifth annual meeting of A.M.S. held at New Orleans, U.S.A. on January 23, 1969.

VITA AUCTORIS

The author was born in the Punjab (India), on December 21, 1929. He received his education in that area, up to the B.A. level.

His childhood was spent in a happy family atmosphere, which was conducive to study and encouraged an interest in the academic and cultural aspects of life. He obtained his M.A. in Mathematics in 1954 from Agra University (India) coming third in the University's list of M.A. graduates.

He was married in 1956, and has three children. He has taught at various levels, up to M.A. seminars, in three colleges affiliated to the Punjab University, leaving India in September, 1966 to pursue higher studies in Mathematics at the University of Windsor, Ontario, Canada, where he completed the requirements for his M.Sc. in Mathematics in May 1967.

Impressed and encouraged by the ability for research demonstrated by Professor H.A. Eliopoulos, the author began his doctoral studies under his able guidance in June 1967, receiving financial assistance from Rev. D.T. Faught, Chairman of the department of Mathematics, the Government of Ontario, and the National Research Council. Among those to whom the author is indebted for aid and support, his wife must be mentioned, for having created and maintained an atmosphere of optimistic encouragement which was of great help to him in completing his research.

The author is, at present, employed by the University of Windsor as a part-time Lecturer, and has been promoted to Assistant Professor beginning from September 1, 1969.