The behaviour and ultimate carrying capacity of beam-columns.

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UMI®
THE BEHAVIOUR AND ULTIMATE CARRYING
CAPACITY OF BEAM-COLUMNS

A Thesis
submitted to the Faculty of Graduate Studies through
The Department of Civil Engineering in Partial Fulfillment
of the Requirements for the Degree of
Doctor of Philosophy at the
University of Windsor

by

Ahmed Afifi Aglan

Windsor, Ontario
1972
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ABSTRACT

This study is divided into two parts: Part I deals with the determination of the lateral torsional buckling loads of beam-columns of wide flange cross-sections under a uniaxially eccentric thrust. The analysis takes into account the residual stresses, pre-buckling displacements and strain-hardening. The effects of these factors are discussed in detail.

The investigation is carried out in four steps to determine:

1. The moment-thrust-curvature relationship
2. The cross-sectional mechanical properties
3. The end-moment versus end-rotation relationship
4. The value of the lateral torsional buckling load

A good agreement was found between the obtained results and the previously available experimental data.

The problem of lateral buckling of beams, under equal end moments, is treated as a special case of the uniaxially loaded beam-columns with a zero axial load.
Part II deals with the determination of the ultimate carrying capacity of wide-flange beam-columns loaded biaxially. A modified numerical procedure to obtain the column deflection curves is presented. Some simplifying assumptions, which do not violate the boundary conditions of the problem, are made.

The investigation is performed in three steps:


2. Determination of the cross-sectional mechanical properties.

3. Determination of the end-moment versus end-rotation relationship.

In this analysis the strain-hardening is neglected, whereas the residual stresses are considered in the mathematical formulations and neglected in the numerical calculations.

The results obtained are compared with the previous experimental data and the CRC interaction equation.
ACKNOWLEDGEMENTS

The writer wishes to express his great appreciation to his supervisor, Dr. George Abdel-Sayed, for his guidance, helpful comments and suggestions during the project. The comments of the members of his thesis committee at the time of the comprehensive examination were also appreciated.

The writer is very grateful to the Department of Civil Engineering and the University of Windsor for the opportunity of carrying out this research.

The writer is indebted to Dr. W. W. McVinnie for providing him with the main program and its subroutine used to obtain the column deflection curves for the case of the biaxially loaded columns, when both the twist and residual stresses are neglected.

Thanks are also due to the Computer Centre staff, Dr. S. Tang and Mr. H. Toews for their advice and help.

Appreciation is expressed to the National Research Council of Canada for the financial support for this work.

Finally, the writer would like to thank his wife and family for their patience and moral support during the course of this work.
LIST OF NOTATIONS

\( A_c \) = area of cross section;
\( B_x, B_y \) = bending rigidity about the \( x \)- and \( y \)-axes respectively;
\( b \) = width of flange;
\( [c] \) = square matrix;
\( C_T \) = St. Venant's torsional rigidity;
\( C_W \) = Warping rigidity;
\( d \) = depth of wide-flange section;
\( D \) = half depth of the web;
\( D_T \) = lateral-torsional buckling coefficient;
\[ D_T = \frac{K_T \times 10^5}{A_c \, d^2} \]
\( E \) = modulus of elasticity;
\( E_{st} \) = strain-hardening modulus;
\( e_x \) = eccentricity of the load \( P \) in the \( x \)-direction;
\( e_y \) = eccentricity of the load \( P \) in the \( y \)-direction;
\( [I] \) = identity matrix;
\( I_x, I_y \) = moment of inertia about the \( x \)- and \( y \)-axes respectively;
\( I_p \) = polar moment of inertia;
\( I_{ft}, I_{fb} \) = the moment of inertia about the \( y \)-axis for the elastic parts of the top and bottom flanges respectively;
\( K_1, K_2, K_3 \) = factors defining the dimensions of the cross-section (Figure I.4);
\( K_T \) = St. Venant's torsional constant \( \frac{1}{3} \, G \left[ bt^3 + (d-2t)w^3 \right] \);
L, i/ = length of the beam-column;
L/rx = major axis slenderness ratio;
L/ry = minor axis slenderness ratio;
Mo = applied end moment about the major axis (uniaxially loaded beam-columns);
Mocr = critical end moment about the major axis (uniaxially loaded beam-columns);
Moi = calculated critical end moment about the major axis (uniaxially loaded beam-columns);
Moi' = assumed critical end-moment about the major axis (uniaxially loaded beam-columns);
Momax = maximum end-moment about the major axis due to excessive bending (uniaxially loaded beam-columns);
Mx,My,Mz = applied moments on a section about the x-, y- and z-axes respectively;
Mξ,Μζ,Μρ = applied moments on a section about the ξ-, ζ- and ρ-axes respectively;
Mx = applied moment on a section about the x-axis at which yielding first occurs in flexure (P=0);
MA, MY = applied end moments at A about the x- and y-axes respectively;
P = axial load;
PE = Euler load = \( \frac{\pi^2 EI_y}{L^2} \);
Py = axial load causing yielding over the entire cross-section;
rx = radius of gyration about the x-axis;
ry = radius of gyration about the y-axis;
r = distance of any point on the cross-section from shear-centre;
u, v = displacements in the x- and y-directions;
Vi = pre-buckling displacements in the plane of the applied moment;
W = web thickness;
x, y, z = system of axes used;
$|X| = \text{matrix containing eigenvectors; }$

$X_0, Y_0 = \text{distance between the shear centre and the centroid of the section in the } x \text{- and } y \text{- directions respectively; }$

$\beta = \text{rotation of the cross-section about the shear centre; }$

$\beta_c = \text{the angle of twist of the section at mid-span; }$

$\varepsilon = \text{normal strain; }$

$\varepsilon_0 = \text{strain corresponding to the axial force, } P;$

$\varepsilon_R = \text{residual strain; }$

$\varepsilon_{rc} = \text{maximum compressive residual strain; }$

$\varepsilon_{rt} = \text{maximum tensile residual strain; }$

$\varepsilon_{st} = \text{Strain at the onset of the strain-hardening }$

$\varepsilon_w = \text{warping strain; }$

$\varepsilon_y = \text{yield strain; }$

$\varepsilon = \text{normal strain divided by yield strain; }$

$\varepsilon = \text{bending strain divided by yield strain; }$

$\varepsilon^* = \text{bending and residual strains divided by yield strain; }$

$\varepsilon^* = \text{bending and warping strains divided by yield strain; }$

$\varepsilon^* = \text{bending, residual and warping strains divided by yield strain; }$

$\sigma = \text{stress; }$

$\sigma_R = \text{residual stress; }$

$\sigma_{rc} = \text{maximum compressive residual stress; }$

$\sigma_{rt} = \text{maximum tensile residual stress; }$

$\sigma_y = \text{yield stress; }$

$\alpha_1 = \text{the ratio of the length of yielded part in compression flange to half the flange width (Figure I.11); }$

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$\alpha_p$ = the ratio of the length of strain-hardened part in compression flange to half the flange width (Figure I.11);

$\varphi_1$ = the ratio of the length of yielded part in tension flange to half the flange width (Figure I.11);

$\varphi_2$ = the ratio of the length of strain-hardened part in tension flange to half the flange width (Figure I.11);

$\gamma_1$ = the ratio of the length of yielded part in the compression side of the web to the depth of the web (Figure I.11);

$\gamma_2$ = the ratio of the length of strain-hardened part in the compression side of the web to the depth of the web (Figure I.11);

$\gamma_3$ = the ratio of the length of yielded part in the tension side of the web to the depth of the web (Figure I.11);

$\gamma_4$ = the ratio of the length of strain-hardened part in the tension side of the web to the depth of the web (Figure I.11);

$\gamma$ = the ratio of the end-moment about the $x$-direction to the end-moment about the $y$-direction (biaxially loaded beam-column);

$\psi$ = curvature;

$\psi_x$ = curvature about the $x$-axis corresponding to initial outer fiber yielding ($P=0$, $MY=0$);

$\varepsilon, \zeta, \rho$ = system of axes used in the twisted position of the cross-section;

$[\lambda]$ = matrix containing eigenvalues;

$\theta$ = end rotation;

$\int \sigma r^2 dA_c$ = Wagner's effect.
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PART I

ELASTIC AND INELASTIC LATERAL TORSIONAL BUCKLING

OF BEAM-COLUMNS UNDER UNIAXIALLY ECCENTRIC THRUST
CHAPTER 1

INTRODUCTION

A perfectly straight wide flange beam-column subjected to an axial force and end moments about its major axis may fail because of:

1. Local buckling.
2. Excessive bending in the plane of applied moments.
3. Lateral torsional buckling.

The local buckling can be avoided by satisfying specified width to thickness ratios of the cross-sectional components (23, 61) which, fortunately, are usually met in most wide flange shapes. By eliminating the local buckling, the relationship between the applied end moment, $M_0$, and the resulting end rotation, $\theta_0$, of a wide flange member follows one of the curves shown in Figure I.1 in which the length as well as the axial force, $P$, are assumed to remain constant. The optimum performance of the beam-column is reached if failure is due to excessive bending in the plane of the applied moment (16, 30), i.e. point $C_1$ on curve (a) with maximum moment $M_{0\text{max}}$. 
If no adequate lateral bracing is provided, the beam-column deflects laterally out of the plane of bending with simultaneous rotation before acquiring its maximum bending capacity. This behaviour starts at a bifurcation point $C$, with a corresponding moment $M_{ocr}$, (Figure I.1) beyond which the $M_o-\theta_o$ curve slopes upwards until point $C_2$ (curve b) indicating the capability of the member to carry additional loading in the post-buckling range. However, this additional loading is generally small (14, 17) and the ultimate strength of the beam-column is considered to be reached at the bifurcation point.

The problem of lateral torsional buckling of beam-columns has been extensively studied in the elastic range (9, 49, 54). In the inelastic range, it was examined by Miranda (40, 41, 43), who took into account the pre-buckling displacements of the beam-column (i.e. the deflection in the plane of applied moment before the initiation of the lateral torsional buckling), but he neglected the residual stresses and the strain-hardening. Fukumoto (14, 15) also examined this problem considering the residual stresses but neglecting the pre-buckling displacements as well as the strain-hardening. While the present analysis was in progress, Lim (36, 37) was studying the same problem considering the pre-buckling displacements and the residual stresses but neglecting the strain-hardening. The effect of the strain-hardening on the lateral torsional buckling of beam-columns appears to have been always disregarded.

This study presents a solution for the problem of the elastic and inelastic lateral-torsional buckling of perfectly straight wide-flange beam-columns subjected to axial force and equal end moments about the
major axis. The residual stresses and pre-buckling displacements are considered together with the strain-hardening.
CHAPTER 2

FORMULATION OF THE GOVERNING DIFFERENTIAL EQUATIONS

1.2.1 Assumptions

The differential equations, governing the lateral torsional buckling of beam-columns under uniaxially eccentric thrust, have been treated extensively (9, 14, 41, 54) based on the following assumptions:

1. The beam-column is perfectly straight. The wide flange cross-section does not vary along the length of the member and retains its original shape without distortion prior to and during buckling.

2. The deflections and rotations are small compared to the dimensions of the beam-column.

3. The axial force and the end moments are applied at the ends. The moments are equal and act about the major axis of the cross-section. The axial force acts along the direction of the original centroidal...
axis of the cross-section and retains its direction until failure.

4. The material is homogeneous and isotropic in both elastic and inelastic stages.

5. Axial shortening of the beam-column is neglected.

6. Unloading is not allowed in the inelastic parts of the beam-column.

7. Yielding is governed by normal stresses only, i.e. the effect of shear stresses is neglected.

I.2.2 Material properties

The stress strain relation for specimens of structural materials, such as mild steel, is shown in Figure I.2. This is an idealized diagram which fairly represents the actual stress-strain curve. This diagram is valid for both compression and tension stresses which can be formulated as follows (39):

\[ \sigma = E \varepsilon - E \left[ \varepsilon + \frac{E_{st}}{E} \varepsilon_y \right] + \frac{E_{st}}{E} \varepsilon_y \]  

(I.2.1)

where:

\( \sigma \) = the stress;

E and \( E_{st} \) = the modulus of elasticity and the strain-hardening modulus respectively;
\( \varepsilon_y \) and \( \varepsilon_{st} \) = the absolute value of the yield strain and the strain at the onset of the strain-hardening respectively.

The two brackets have the special significance as follows:

1. when \(|\varepsilon| < \varepsilon_y\), both brackets disappear;
2. when \(\varepsilon_y < |\varepsilon| < \varepsilon_{st}\), only the second bracket disappears;
3. when \(|\varepsilon| < \varepsilon_{st}\), both brackets are taken into consideration.

The sign inside the brackets are positive if \(\varepsilon\) is negative and vice versa.

The proposed method of analysis can be applied to any beam-column whose actual stress-strain curve can be idealized to the curve given in Figure I.2. However, the computations are carried out in this study for the case of mild steel A36 which has the following properties:

\[
\begin{align*}
\sigma_y &= 36 \text{ Ksi} \\
E &= 30 \times 10^3 \text{ Ksi} \\
\frac{E_{st}}{E} &= 0.022 \\
G &= 11.5 \times 10^3 \text{ Ksi}
\end{align*}
\]

(Yield stress) (shear modulus)
1.2.3 Residual Stresses

Residual stresses are induced in structural members as a result of plastic deformations due to the following (4, 17, 27, 32):

1. During the process of cooling from rolling temperature to room temperature, plastic deformations occur, due to the fact that some parts of the member cool more rapidly than the others.

2. During cold bending of the specimen while being straightened or fabricated, plastic deformations may occur.

Residual stresses are also induced in structural members of built-up sections because of the differential heating and cooling of the metal during and after welding.

Tests on rolled wide flange sections indicate that the residual stress pattern can be assumed, with sufficient accuracy, as shown in Figure I.3. The value of the residual stress at any point on the section is given by the equation (4, 17, 27, 32):

\[
\sigma_R = \sigma_{rt} + \frac{\sigma_{rc} - \sigma_{rt}}{(b/2)} \left| x \right|
\]

(1.2.2)

where:

\(\sigma_R\) is the residual stress;

\(\sigma_{rc}\) and \(\sigma_{rt}\) = maximum compressive and tensile residual stresses respectively;
b = the width of the flange;

|x| = the absolute value of the co-ordinate x  (Figure I.3)

The assumed residual stresses in the section are in static equilibrium with respect to axial force and moments. However, they are not in equilibrium with respect to torsion (34); i.e.,

\[ \int \sigma_R r^2 \, dA_c \neq 0 \]  \hspace{1cm} (I.2.3)

in which:

\[ r^2 = x^2 + y^2 \]

\[ A_c = \text{area of the cross-section} \]

The value of the term \( \int \sigma_R r^2 \, dA_c \) depends on the dimensions of the cross-sectional components. An expression for this term will be given later in the determination of "the Wagner effect".

Neglecting the rounded corners, fillets and tapers in the cross-section of the beam-column, the conditions of statical equilibrium require that the ratio \( \sigma_{rt} \) to \( \sigma_{rc} \) be:

\[ \frac{\sigma_{rt}}{\sigma_{rc}} = \frac{bt}{bt + w(d - 2t)} \]  \hspace{1cm} (I.2.4)

where:

d = the depth of the wide-flange section;

t and w = the thickness of the flange and of the web respectively

In rolled wide-flange sections, the residual stresses are always less than the yield stresses, therefore, a corresponding residual strain can be introduced by dividing both sides of Equation I.2.2 by \( E \) leading to:
\[ \varepsilon_R = \varepsilon_{rt} + \frac{\varepsilon_{rc} - \varepsilon_{rt}}{(b/2)} \left| x \right| \quad (1.2.5) \]

in which:

\( \varepsilon_R \) = the residual strain;

\( \varepsilon_{rc} \) and \( \varepsilon_{rt} \) = maximum compressive and tensile residual strains respectively.

Using the non-dimensional factors \( K_1, K_2 \) and \( K_3 \) to describe the cross-section as shown in Figure 1.4, Equation 1.2.4 can be written as follows:

\[ \frac{\sigma_{rt}}{\sigma_{rc}} = \frac{\varepsilon_{rt}}{\varepsilon_{rc}} = K_0 = \frac{K_1 K_2}{K_1 K_2 + K_3} \quad (1.2.6) \]

In this analysis a value of \( \sigma_{rc} = 0.3 \sigma_y \) (27) was used in the numerical calculations.

1.2.4 The governing differential equations

The beam-column, prescribed by the previously mentioned assumptions, is shown in Figure 1.5. It is subjected to eccentric load \( P \) at its ends. The fixed axes \( x, y \) and \( z \) are orthogonal. The eccentricity of \( P \) at \( z = 0 \) and \( z = L \) is \( e_y \) which is taken positive in the positive direction of \( y \). Figure 1.6 shows a cross-section in the initial and buckled positions. The \( x- \) and \( y- \) axes are the principal axes of the cross-sections in the initial position and at the ends of the beam-column. In the buckled position, the principal axes of any cross-section are denoted by the \( \xi- \) and the \( \zeta- \) axes. The \( \rho- \) axis, which is orthogonal with the \( \xi \) and \( \zeta \) axes, is tangent to the deflected beam-column axis at any cross-section along its length.
In figure 1.6, the displacements are defined as follows:

\[ u, v = \text{the displacements of the shear centre} \]

in the x- and y- directions respectively;

\[ \bar{u}, \bar{v} = \text{the displacements of the centroid in the} \]

x- and y- directions;

\[ v_i = \text{pre-buckling displacement}; \]

\[ \beta = \text{a torsional rotation about the shear centre.} \]

The displacements \( u, v, \bar{u}, \bar{v} \) and \( v_i \) are positive in the positive direction of x- and y- axes, whereas \( \beta \) is taken positive by the right-hand screw rule.

A positive moment or torque will be represented by a vector pointing towards the positive direction of the appropriate co-ordinate axis using the right-hand rule convention on the front face and the left-hand rule on the back face (Figure 1.7).

Since the angle \( \beta \) is assumed to be small, the displacements \( \bar{v} \) and \( \bar{U} \) can be expressed as:

\[ \bar{v} = v \]

\[ \bar{u} = u + v_0 \beta \]

where \( v_0 = \text{distance from centroid, } C, \text{ to shear centre, } S. \)

The bending moments, at any point along the member, about the x- and y- axes are

\[ M^x = -P (e_y - v_i - v) \]

\[ M^y = -P (u + v_0 \beta) \]
and the twisting moment about the z-axis is

$$M^z = 0$$

(I.2.11)

The lateral displacements, resulting from the rotation of the cross-section, are assumed to be small compared to the lateral displacements caused by bending. Therefore,

$$\frac{d\bar{u}}{dz} = \frac{d\bar{u}}{dz}$$

(I.2.12)

The bending moments about the $\xi$, $\zeta$, and $\rho$ axes are:

$$M_{\xi} = M_x + M_y \beta$$

(I.2.13)

$$M_{\zeta} = M_y - M_x \beta$$

(I.2.14)

$$M_{\rho} = M_x \frac{du}{dz} + M_y \frac{dv}{dz} + M_{\rho 1} + M_{\rho 2}$$

(I.2.15)

The first torsional moment, $M_{\rho 1}$, is due to the torsional effect of the components of the axial force with respect to the shear centre. Since the axial thrust is constant and equal to $P$, its components in the $\xi$ and $\zeta$ directions are: (Figure I.8)

$$p_{\xi} = P \frac{du}{dz}$$

(I.2.16)

$$p_{\zeta} = P \frac{dv}{dz}$$

(I.2.17)

The components $p_{\xi}$ and $p_{\zeta}$ are acting through the centroid of the section and produce a torque about the shear centre, $\xi$, given by:
The second additional torsional moment, $M^{\theta 2}$, is due to the effect of the normal compressive stresses on the warped cross-section (57). This effect is usually referred to as "Wagner's effect." The torsional moment $M^{\theta 2}$ is equal to:

\[ M^{\theta 2} = \int \sigma r^2 \, dA_C \frac{d\theta}{dz} \]  \hspace{1cm} (I.2.19)

in which:

\[ r^2 = x^2 + (y - y_o)^2 \]  \hspace{1cm} (I.2.20)

Substituting $M^{\theta 1}$ of $M^{\theta 2}$ of Equations I.2.18 and I.2.19 into Equation I.2.15, the total torsional moment, $M^\theta$, is given by:

\[ M^\theta = (M^X + P y_o) \frac{du}{dz} + M^Y \frac{dv}{dz} + \left( \int \sigma r^2 \, dA_C \right) \frac{d\theta}{dz} \]  \hspace{1cm} (I.2.21)

Substituting the expressions for $M^X$ and $M^Y$ from Equations I.2.9 and I.2.10 into Equations I.2.13, I.2.14 and I.2.21, yields the following expressions for $M^\xi$, $M^\zeta$ and $M^\theta$, after neglecting small quantities of higher orders:

\[ M^\xi = M_o + P \left( v_i + v \right) \]  \hspace{1cm} (I.2.22)

\[ M^\zeta = -[M_o + P \left( v_i + y_0 \right)] \beta - Pu \]  \hspace{1cm} (I.2.23)

\[ M^\theta = [M_o + P \left( v_i + y_0 \right)] \frac{du}{dz} + \left( \int \sigma r^2 \, dA_C \right) \frac{d\theta}{dz} \]  \hspace{1cm} (I.2.24)

The bending moments, $M^\xi$ and $M^\zeta$, can be related to the corresponding curvatures of the member about the principal axes of the cross-section.
by (54):

\[ M_x = -B_x \frac{d^2 (v + v_z)}{dz^2} \quad (I.2.25) \]

and

\[ M_y = B_y \frac{d^2 u}{dz^2} \quad (I.2.26) \]

in which:

\[ B_x \text{ and } B_y = \text{The bending rigidities about the } x- \text{ and } y- \text{axes respectively.} \]

The relationship between the torsional moment, \( M_p \), and the angle of rotation, \( \beta \), has the form (9, 20, 54):

\[ M_p = C_T \frac{d\beta}{dz} - C_W \frac{d^3 \beta}{dz^3} \quad (I.2.27) \]

where:

\[ C_T = \text{The St. Venant torsional rigidity of the cross-section.} \]
\[ C_W = \text{The warping rigidity of the cross-section.} \]

By mathematical manipulation, Equations I.2.22 to I.2.27 yield the following differential equations:

\[ B_x \cdot (v_i'' + v'') + M_0 + P (v_i + v) = 0 \quad (I.2.28) \]
\[ B_y \cdot u'' + P u'' + [M_0 + P (v_i + y_o)] \beta = 0 \quad (I.2.29) \]
\[ C_W \beta'' = \left( C_T - \int_\sigma r^2 dA_C \right) \beta' + [M_0 + P (v_i + y_o)] u' = 0 \]

\[ ........ (I.2.30) \]

The ', " and "" indicate the first, second and third differentiations with respect to \( z \).
Equation I.2.28 governs the moment-thrust-curvature relationship \((M-P-\phi)\) occurring among the moment about the \(x\)-axis, \(M^x = M_0 + P (v_1 + v)\), the thrust, \(P\), and the curvature \(\phi^x = v_1'' + v''\). This equation is independent of the lateral displacement, \(u\), and the torsional displacement, \(\beta\). Therefore, it bears no effect on the lateral-torsional buckling of elastic beam-columns. However, for inelastic beam-columns, the \(M-P-\phi\) relationship does affect the lateral-torsional buckling as it governs the strain distribution in cross-sections and the corresponding rigidity coefficients. These coefficients appear in equations I.2.29 and I.2.30 which govern the lateral torsional buckling strength of beam-columns.

I.2.5 Boundary Conditions

The ends of the beam-column are free to rotate about either principal axis with no displacement in either direction. They are also free to warp but their rotation about the longitudinal axis (\(z\)-axis) is prevented.

These boundary conditions can be written as follows:

\[
\text{at } z = 0 \text{ and } z = L: \quad u = u'' = \beta = \beta'' = 0
\]  
(I.2.31)
Based on the previous definitions and boundary conditions, the investigation of the lateral torsional buckling of beam-columns, (i.e. the detection of the bifurcation point, \( c \), on the \( M_0 - \theta_0 \) curve, Figure I.1) requires the determination of the following four steps:

1. The moment-thrust-curvature relationship.
2. The cross-sectional mechanical properties.
3. The end moment versus end rotation relationship.
4. The lateral torsional buckling strength.

I.3.1 Determination of the moment-thrust-curvature relationship

This relationship, known as \( M-P-\phi \) curves, is established for a cross-section subjected to axial force, \( P \), and a moment acting about the \( x \)-axis, \( M^X \). Plane cross-sections are assumed to remain plane after
beam-column deformations; therefore, the strain at any point \((x, y)\) on the cross-section can be written as follows:

\[
\varepsilon = \varepsilon_0 + \phi^x y + \varepsilon_R
\]  

(1.3.1)

in which \(\varepsilon_0\) = the average normal strain, i.e.

\[
\varepsilon_0 = \frac{1}{A_c} \int \varepsilon \, dA_c
\]  

(1.3.2)

\(\phi^x\) = the curvature about the \(x\)-axis

\(\phi^x y\) = the flexural strain referred to as \(\varepsilon (\varepsilon = \phi^x y)\)

The thicknesses of the flanges and web are small compared to the dimensions of the cross-section. Hence, the strains are assumed to be constant over these thicknesses. For a uniaxially loaded beam-column, of wide-flange cross-section containing residual stresses, the strains across any section are symmetric about the \(y\)-axis and can take any one of the sixteen configurations shown in Figure I.9. In case the section is free from residual stresses or if the residual stresses are neglected, the strain configurations become six (Figure I.10).

The equilibrium between the external and internal forces requires:

\[
P = \int \sigma \, dA_c
\]  

(1.3.3)

and

\[
M^x = \int \sigma 
\cdot y \, dA_c
\]  

(1.3.4)
Substituting Equation 1.2.1 in Equation 1.3.3 and dividing by the axial thrust \( P_y \) which corresponds to the yield stress level, leads to:

\[
\frac{P}{P_y} = \frac{1}{A_c} \int \frac{e}{e_y} dA_c - \frac{1}{A_c} \int \left[ \frac{e}{e_y} \pm 1 \right] dA_c + \frac{1}{A_c} \frac{E_{st}}{E} \int \left[ \frac{e}{e_y} \pm \frac{e_{st}}{e_y} \right] dA_c
\]

......... (1.3.5)

where:

\[
P_y = E \frac{e_y}{A_c} ;
\]

\[
A_c = 2 (K_3 + 2K_1K_2) D^2
\]

The first integral of Equation 1.3.5 is equal to \( \frac{e_0}{e_y} \) (Equation 1.3.2 in the non-dimensional form), while the second and third integrals are the sectional volumes of \( \left[ \frac{e}{e_y} \pm 1 \right] \) and \( \left[ \frac{e}{e_y} \pm \frac{e_{st}}{e_y} \right] \) respectively and can be calculated numerically for any assumed strain configuration. Therefore, Equation 1.3.5 can be written in the form (38, 39)

\[
Q \left( \frac{e_0}{e_y} \right)^2 + R \left( \frac{e_0}{e_y} \right) + S = 0
\]

......... (1.3.7)

in which \( Q, R \) and \( S \) are coefficients depending on the thrust, \( P \), flexural strains, residual strains, geometry of the cross section and the assumed strain configuration.

The following equations illustrate the coefficients \( Q, R, \) and \( S \) for the strain configurations No. 2, 3, 8 in Figure 1.9 which will be shown later to be the most prevalent ones for the case when residual stresses are considered. In these equations, the coefficients \( Q, R \) and \( S \) are expressed in terms of the strains at the points 1 to 6 (Figure 1.4). The subscripted number in the strain notations refers to
the point at which the strain is considered.

For the strain configuration No. 2 (Figure I.9).

\[ Q = \frac{A_1}{\varepsilon_{rc} (1 + K_0)} \]  \hspace{1cm} (I.3.8.a)

\[ R = 2A_1\left[\frac{\bar{e}_3^* + 1}{\varepsilon_{rc} (1 + K_0)}\right] - 1 \]  \hspace{1cm} (I.3.8.b)

\[ S = \frac{A_1 (\bar{e}_3^* + 1)^2}{\varepsilon_{rc} (1 + K_0)} + \frac{p}{P_y} \]  \hspace{1cm} (I.3.8.c)

For the strain configuration No. 3 (Figure I.9):

\[ Q = \frac{B_1}{2\bar{e}_6} \]  \hspace{1cm} (I.3.9.a)

\[ R = 2A_1 + \frac{B_1 (\bar{e}_6 + 1)}{\bar{e}_6} - 1 \]  \hspace{1cm} (I.3.9.b)

\[ S = A_1 (\bar{e}_3^* + \bar{e}_6^* + 2) + \frac{B_1 (\bar{e}_6^* + 1)^2}{2\bar{e}_6} + \frac{p}{P_y} \]  \hspace{1cm} (I.3.9.c)

For the strain configuration No. 8 (Figure I.9)

\[ Q = \frac{B_1 - B_2}{2\bar{e}_6} \]  \hspace{1cm} (I.3.10.a)

\[ R = 2(A_1 - A_2) + \frac{B_1 (\bar{e}_6^* + 1) - B_2 (\bar{e}_6^* + \bar{e}_{st})}{\bar{e}_6} \]  \hspace{1cm} (I.3.10.b)

\[ S = (A_1 - A_2) (\bar{e}_3^* + \bar{e}_6^*) + 2(A_1 - A_2 \bar{e}_{st}) + \frac{B_1 (\bar{e}_6^* + 1)^2 - B_2 (\bar{e}_6^* + \bar{e}_{st})}{2\bar{e}_6} + \frac{p}{P_y} \]  \hspace{1cm} (I.3.10.c)

where:

\[ A_1 = \frac{K_1K_2}{2 (K_3 + 2K_1K_2)} \]  \hspace{1cm} (I.3.11.a)
\[ A_2 = \left( \frac{E_{st}}{E} \right) A_1 \]  
(I.3.11.b)

\[ B_1 = \frac{K_3}{2(K_3 + 2 K_1 K_2)} \]  
(I.3.11.c)

\[ B_2 = \left( \frac{E_{st}}{E} \right) B_1 \]  
(I.3.11.d)

In which:

\[ \bar{\varepsilon} = \text{bending strain divided by yield strain;} \]

\[ \bar{\varepsilon}^* = \text{bending and residual strain divided by yield strain;} \]

\[ \varepsilon_{rc} = \text{the maximum compressive residual strain, } \varepsilon_{rc}, \text{ divided by yield strain;} \]

\[ \varepsilon_{st} = \text{the absolute value of the strain at the onset of the strain-hardening divided by the yield strain;} \]

Expressions for Q, R and S were derived for each of the strain configurations shown in Figures 1.9 and 1.10 and are given in Tables 1 and 2.

The solution of Equation 1.3.7 is:

\[ \frac{\varepsilon_0}{\varepsilon_y} = \frac{-R \pm \sqrt{R^2 - 4QS}}{2Q} \]  
(I.3.12)

Examination of Equation I.3.12 shows that the negative sign is the one to be taken. This is because \( \varepsilon_0 \) must increase algebraically with \( P \). Therefore,
$$\frac{\varepsilon_0}{\varepsilon_y} = - \frac{R - \sqrt{R^2 - 4QS}}{2Q} \quad (1.3.13)$$

Substituting Equation 1.2.1 in Equation 1.3.4 and dividing by the moment $M_y$, at which yielding first occurs due to flexure, leads to:

$$\frac{M_x}{M_y} = \frac{D(1 + K_2)}{I_x} \int y \left( \frac{\varepsilon}{\varepsilon_y} \right) dA_c - \int y \left[ \frac{\varepsilon}{\varepsilon_y} \right] dA_c +$$

$$+ \left( \frac{E_{st}}{E} \right) \int y \left[ \frac{\varepsilon}{\varepsilon_y} \pm \frac{E_{st}}{\varepsilon_y} \right] dA_c \quad \ldots \ldots \ (1.3.14)$$

where:

$$M_y = \frac{E \varepsilon_y I_x}{D (1 + K_2)} \quad (1.3.15)$$

$I_x$ = moment of inertia of the section about the $x$-axis.

$D$ = half depth of the web

The first integral of Equation 1.3.14 represents the first moment of the volume of the $\varepsilon_{xy}$ distribution and is equal to:

$$\frac{D(1 + K_2)}{I_x} \int y \left( \frac{\varepsilon}{\varepsilon_y} \right) dA_c = \frac{\phi_x}{\phi_y} \quad (1.3.16)$$

in which $\phi_y$ is the curvature about the $x$-axis corresponding to the initial outer fiber yielding ($P = 0$). It is equal to:
The second and third integrals represent the first moment of the volume of the \( \left[\frac{e}{e_y} \pm 1\right] \) and \( \left[\frac{e}{e_y} \pm \frac{e_{st}}{e_y}\right] \) distributions and can be expressed in terms of the strains at the points 1 to 6 (Figure I.4).

For the most prevalent strain configurations (Figure I.9), Equation I.3.14 takes the following forms:

For the strain configuration No. 2 (Figure I.9)

\[
\frac{M^x}{M^y} = \frac{\phi^x}{\phi^y} + F_1 \gamma_1 (e_3 + 1) \quad (I.3.18.a)
\]

For the strain configuration No. 3 (Figure I.9)

\[
\frac{M^x}{M^y} = \frac{\phi^x}{\phi^y} + F_1 (e_3 + e_6 + 2) + F_2 \gamma_1 (3 - 2\gamma_1) (e_6 + 1)^{1/3} \quad (I.3.18.b)
\]

For the strain configuration No. 8 (Figure I.9)

\[
\frac{M^x}{M^y} = \frac{\phi^x}{\phi^y} + F_1 \left[1 - \left(\frac{E_{st}}{E}\right)\right] (e_3 + e_6) + 2 \left[1 - \left(\frac{E_{st}}{E}\right) e_{st}\right] + \\
+ F_2 \gamma_1 (3 - 2\gamma_1) (e_6 + 1). - \left(\frac{E_{st}}{E}\right) \gamma_3 (3 - 2\gamma_3) (e_6 + e_{st})^{1/3} \\
------------- (I.3.18.c)
\]

where:

\[
F_1 = \frac{K_1 K_2 (1 + K_2)}{[\frac{K_2}{3} K_3 + 4 K_1 K_2 (1 + K_2/2)^2]} \quad (I.3.19.a)
\]
\[ F_2 = \frac{K_3 (1 + K_2)}{\left[ \frac{2}{3} K_3 + 4 K_1 K_2 (1 + K_2/2)^2 \right]} \]  

(I.3.19.b)

\[ \alpha_1 = \text{the extent of yielding in each part of the compression flange as illustrated in Figure I.11,} \]

\[ \alpha_1 = \frac{\varepsilon_3 + 1}{\varepsilon_3 - \varepsilon_6} \]  

(I.3.19.c)

\[ \gamma_1 \text{ and } \gamma_3 = \text{the extent of yielding and strain-hardening in the compression part of the web (Figure I.11),} \]

\[ \gamma_1 = \frac{\varepsilon_6 + 1}{\varepsilon_6 - \varepsilon_5} \]  

(I.3.19.d)

\[ \gamma_3 = \frac{\varepsilon_6 + \varepsilon_{st}}{\varepsilon_6 - \varepsilon_5} \]  

(I.3.19.e)

\[ \varepsilon = \text{the total strains (bending strains, residual strains and longitudinal strains) divided by the yield strain.} \]

The forms of Equation I.3.14 derived for all the strain configurations in Figures I.9 and I.10 are shown in Tables 3 and 4.

For a specified thrust, \( P \), and residual stress distribution, a trial and error procedure is applied (39) to calculate the \( M-P-\phi \) curves utilising Equations I.3.13 and I.3.14 and the corresponding \( Q, R \) and \( S \) coefficients as follows:

1. A curvature \( \phi^x \) is assumed and the flexural strains, \( \varepsilon = \phi^x y \), are calculated at the points 1 to 6.
2. A strain configuration is assumed and with the flexural strains, $\bar{\varepsilon}$, calculated in step No. 1, the coefficients $Q$, $R$ and $S$ are calculated for the specified thrust $P$ and residual stresses as shown in the examples of Equations 1.3.8 to 1.3.10. Figures I.12, I.13 show the sequences for assuming the strain configuration for the two cases of considering residual stresses and neglecting them respectively.

3. $Q$, $R$ and $S$ are substituted in Equation 1.3.13 and $\left(\frac{\varepsilon_0}{\varepsilon_y}\right)$ is calculated.

4. The strains are calculated using Equation 1.3.1. The corresponding strain configuration is found and compared with the assumed one.

5. The steps 1 to 4 are repeated until the calculated configuration becomes the same as the assumed one. Then the moment $M^X$ is calculated using the appropriate form of Equation 1.3.14. Since this moment corresponds to the assumed curvature, $\phi^X$, and the thrust, $P$, it represents a point on the $M$-$P$-$\phi$ curve.

6. Steps 1 to 5 are repeated for different curvatures $\phi^X$ within a desirable range and the corresponding moments $M^X$ are found.
1.3.2 Determination of the cross-sectional mechanical properties

The coefficients $B_y$, $C_T$, $C_M$, $f_o$ and $\int \sigma r^2 \, dA_c$, which appear in the two governing differential equations (1.2.29 and 1.2.30), represent the cross-sectional mechanical properties. For an assumed curvature $\phi^X$, the corresponding moment and strain configuration are defined in step No. 1. Thereafter, the mechanical properties of the section can be calculated. These mechanical properties are constants in the elastic range and become variables in the inelastic range because of partial yielding or strain-hardening, and since yielding and strain-hardening are functions of the bending moment acting on the section, these properties will vary along the beam column.

In the elastic range, the coefficients $B_y$, $C_M$, $f_o$ and $\int \sigma r^2 \, dA_c$, are equal to:

$B_y$ (minor axis bending rigidity) = $EI_y$

$C_T$ (St. Venant torsional rigidity) = $GK_T$

$$= \frac{1}{3} G \left[ 2bt^3 + (d - 2t)w^3 \right];$$

$y_o$ (distance from centroid to shear centre) = 0;

$\int \sigma r^2 \, dA_c$ (the Wagner effect) = $\frac{P_{IP}}{A_c} + \int \sigma_R r^2 \, dA_c$

(I.3.20)

in which:

$I_y$ = moment of inertia of the section about the $y$-axis.
\[ I_p = \text{moment of inertia of the section about the polar axis;} \]

\[ K_T = \text{St. Venant torsional constant;} \]

\[ \int \sigma_R r^2 \, dA_c = \text{the contribution of the residual stresses which are not in equilibrium with respect to torsion.} \text{(as indicated in Section I.2.3). It's value is given by the following equation:} \]

\[ \int \sigma_R r^2 \, dA_c = \frac{2}{3} K_3 D^4 \sigma_{rt} + 2 K_1 K_3 D^2 \left( D + \frac{K_2}{2} \right)^2 \left( \sigma_{rc} + \sigma_{rt} \right) + \]

\[ + K_1^3 K_2 D^4 \left( \sigma_{rc} + \frac{\sigma_{rt}}{3} \right) \quad \ldots \ldots \text{(I.3.21)} \]

In the inelastic range, these mechanical properties vary with the different patterns of yielding and strain-hardening. The variations of these properties are discussed as follows:

**I.3.2.1 The minor axis bending rigidity, B_y**

This is defined as the resistance of the section to lateral bending. Up to the instant of lateral-torsional buckling, there is no moment acting about the y-axis (minor axis). At the initiation of buckling, the moment about the y-axis is indeterminate. The value \( B_y \) can be calculated by applying an infinitesimal moment, \( dM^y \), on the cross-section which has already strained due to a bending moment \( M^X \) and an axial thrust \( P \). The resistance to this infinitesimal moment determines the minor axis bending rigidity at the initiation of buckling. In other words, \( B_y \) can be defined as the initial slope of the moment-curvature...
relationship for the y-axis, when a virtual bending moment is applied about this axis (14, 17); that is:

\[ B_y = \lim_{\Delta M^y \to 0} \frac{\Delta M^y}{d} \left( \frac{d^2 u}{dz^2} \right) \]  

(1.3.22)

When the infinitesimal moment, \( dM^y \), is applied, unloading will take place on some parts of the section which will change the strain configuration by increasing or decreasing in the yielded zones and/or strain-hardened areas. By assuming no unloading of the already strained section, the infinitesimal moment, \( dM^y \), will only be resisted by the parts of the cross-section which are elastic or strain-hardened. This assumption applies the tangent modulus concept which gives a lower bound to the buckling load (14, 17, 36).

The reduced modulus, which is another approach allowing for unloading of the strained fibres that will contribute to the stiffening of the section, provides an upper bound to the buckling load (36). The actual carrying capacity of the member lies between the tangent modulus and the reduced modulus. However, the tangent modulus concept proved to be more realistic and on the safe side (14, 17, 36).

On this basis, the tangent modulus concept is applied in the present analysis and the bending rigidity, \( B_y \), is the elastic rigidities of the unyielded parts of the cross-section plus the rigidities of the strain-hardened areas. The equations for \( B_y \) for the most prevalent configurations in Figure 1.9 are:
1. For the strain configuration No. 2

\[ B_y = \frac{1}{2} EI_y [1 + (1 - \alpha_1)^3] \]  \hspace{1cm} (I.3.23.a)

2. For the strain configuration No. 3

\[ B_y = \frac{1}{2} EI_y \]  \hspace{1cm} (I.3.23.b)

3. For the strain configuration No. 8

\[ B_y = \frac{1}{2} EI_y [1 + \left( \frac{E_{st}}{E} \right)^2] \]  \hspace{1cm} (I.3.23.c)

Equations for \( B_y \), derived for each strain configuration for the case when considering residual stresses (Figure I.9), are illustrated in Table 5. Similar equations, have been derived for the case when neglecting residual stresses, are shown in Table 6.

I.3.2.2 The St. Venant torsional rigidity, \( C_T \)

The St. Venant torsional rigidity, \( C_T \), is defined as the initial slope of the relation between the torque and the first derivative of the unit angle of twist, \( \beta \), when an infinitesimal torque, \( dM^T \), is applied about the \( \rho \) axis, which has already strained due to \( M^X \) and \( P \) (14, 17); that is:

\[ C_T = \lim_{dM^T \to 0} \frac{dM^T}{d\left( \frac{d\beta}{dz} \right)} \]  \hspace{1cm} (I.3.24)

in which the infinitesimal torque, \( dM^T \), is to be resisted by St. Venant torsional rigidity only.
Neal (42) showed, theoretically and experimentally, that the torsional rigidity of the section, when part of it has yielded, is the same as the elastic torsional rigidity. He justified this on the basis that at the initiation of lateral torsional buckling, there are no shear stresses acting on the plastic zones of the cross-section, and hence there are no plastic components of shear strain. The relationship between shear stress and shear strain, in the whole cross-section, is identical with that for elastic material, provided the shear stresses remain small. Neal's theoretical and experimental works were confined to mild steel bars in the elastic and plastic ranges, but his argument applies equally well for the strain-hardening range (58). This concept has been accepted by many investigators (3, 5, 17, 25, 36, 40, 42, 51, 58, 59) in their studies of the inelastic lateral instability problems.

Therefore, the value of $C_T$ in this investigation is taken constant and equal to the full elastic torsional rigidity of the section, irrespective of the extent of the inelastic action of the cross-section; that is:

$$C_T = G K_T$$

$$= \frac{1}{3} G [2 bt^3 + (d - 2t)w^3]$$

(I.3.25)

in which:

$$G = \frac{E}{2(1 + \nu)}$$

(I.3.26)
The above argument holds only at the instant of lateral torsional buckling. By slight increase in the loading, the value of $G$ will deteriorate rapidly to its strain-hardening value, $G_{st}$, in the inelastic zones of the cross section (17).

1.3.2.3 The Warping torsional rigidity, $C_w$

The warping torsional rigidity, $C_w$, is the resistance of the section to a warping torque. In a similar procedure to that discussed in the determination of the minor axis bending rigidity, $B_y$, it can be shown that (14, 17):

$$C_w = \lim_{dM^W \to 0} \frac{dM^W}{d(z^3)}$$  \hspace{1cm} (I.3.27)

in which the warping torque, $M^W$, is the part of the twisting moment about the $p$-axis resisted by the warping of the section.

Adopting the tangent modulus concept, $C_w$ can be taken as the warping constant of the unyielded parts of the cross-section times the modulus of elasticity, $E$, plus the warping constant of the strain-hardened areas times the strain-hardening modulus, $E_{st}$.

The equations for $C_w$ for the most prevalent strain-configurations in Figure I.9 are:

1. For the strain configuration No. 2, $C_w$ is taken as the greater value of the following two equations:
\[ C_w = \frac{ED (2 + K_2)^2 I_y (1 - \alpha_1)^3}{2 \left[ 1 + (1 - \alpha_1)^3 \right]} ; \]

or

\[ C_w = \frac{1}{18} E K_1 \frac{3}{2} K_2^3 D^6 + \frac{2}{9} E K_3 \frac{3}{2} D^6 + \frac{1}{18} E K_1 \frac{3}{2} K_2^3 D^6 (1 - \alpha_1)^3 \]  

(I.3.28.a)

2. For the strain configuration No. 3.

\[ C_w = \frac{1}{18} E k_1^3 k_2^3 d^6 + \frac{2}{9} E k_3^3 d^6 (1 - \gamma_1)^3 \]  

(I.3.28.b)

3. For the strain configuration No. 8.

\[ C_w = \frac{E_{st} D^2 (2 + K_2)^2 I_y}{2 \left[ 1 + (\frac{E_{st}}{E}) \right]} \]  

(I.3.28.c)

Equations for \( C_w \), derived for each strain configuration in Figures I.9 and I.10 are shown in Tables 7 and 8.

I.3.2.4 The shear centre distance, \( y_0 \)

The shear centre distance, \( y_0 \), is defined as the distance from the original centroid, \( c \), to the shear centre of the unyielded areas, \( a \).

The equations for \( y_0 \) for the most prevalent strain configurations in Figure I.9 are:

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1. For the strain configuration No. 2

\[ \psi_0 = D (2 + K_2) \left( \frac{1}{2} - \frac{(1 - \alpha_1)^3}{[1 + (1 - \alpha_1)T_2]^3} \right) \]  

(I.3.29.a)

2. For the strain configuration No. 3

\[ \psi_0 = D (1 + K_2/2) \]  

(I.3.29.b)

3. For the strain configuration No. 8

\[ \psi_0 = D (2 + K_2) \left( \frac{1}{2} - \frac{E_{st}}{E} \right) \]  

(I.3.29.c)

Equations for \( \psi_0 \), for each strain configuration of Figure 1.9 and 1.10, were derived and are given in Tables 9 and 10.

I.3.2.5 The Wagner effect, \( \int \sigma r^2 \, dA_c \)

The Wagner effect, \( \int \sigma r^2 \, dA_c \), is attributed to the normal compressive stresses on the warped cross-section. The term

\[ 1 - \frac{\int \sigma r^2 \, dA_c}{C_T} \]

can be considered as a reduction factor to St. Venant torsional rigidity. In the elastic range, the coefficient \( \int \sigma r^2 \, dA_c \) is determined from Equation 1.3.20. The derivation of mathematical expressions for the Wagner effect becomes too laborious for inelastic sections. Therefore, it is calculated numerically by dividing each flange and web to twenty equal parts (Figure 1.14) as follows:
\[
\int \sigma r^2 \, dA_c = \sum_{n=1}^{n=20} \sigma \{x_n^2 + [D (1 + \frac{K_2}{2}) - y_n] \} \, dA_c
\]
(upper flange)
\[
+ \sum_{n=1}^{n=20} \sigma \{x_n^2 + [D (1 + \frac{K_2}{2}) + y_n] \} \, dA_c
\]
(lower flange)
\[
+ \sum_{n=1}^{n=20} \sigma \{y_n - y_0\} \, dA_c
\]
(web) \hspace{1cm} \text{(I.3.30)}

in which:

\(x_n, y_n\) = the x and y coordinates of the nth part respectively.

### I.3.3 Determination of the end-moment versus end-rotation relationship

For a beam-column with specified length, \(L\), thrust, \(P\), and end moment, \(M_0\), a deflection curve can be obtained using numerical integration and the moment-thrust-curvature relationship developed in step No. 1. The end moments are equal; therefore, the deflection curve is symmetrical with zero slope at the middle where the integration starts \((41, 43, 44)\). From the column deflection curve, the values of the deflections, moments and curvatures at evenly spaced discrete points along the length of the beam-column are known.

From a group of column deflection curves, the curves of end-moment versus end-rotation can be obtained \((41, 43)\).
I.3.4 Determination of the lateral torsional buckling strength

Each point on the $M_o \cdot \Theta_o$ curve, Figure I.1, represents a stable, uniaxially loaded beam-column deflected in the plane of applied end-moments. The deflection curve and bending moments diagram is defined in step No. 3. With specified thrust and moment, any cross-sectional strain configuration can be obtained from step No. 1 and the mechanical properties calculated in step No. 2.

The lateral torsional buckling (bifurcation point, C, Figure I.1) is examined using Equation I.2.29 and Equation I.2.30 where the coefficients (beam-column's sectional-properties) are variable with respect to $z$. Therefore, a direct solution is rather difficult and a finite difference approximation is applied. This leads to a set of simultaneous algebraic equations in the lateral displacement $u$ and the rotations $\beta$ (14) at a number of discrete points $n$ spaced at $h = \frac{L}{n}$, in which $n$ is an odd number to which the beam is divided (Figure I.15):

$$u_{i-1} - (2 - \frac{Ph^2}{By}) u_i + u_{i+1} + \left[ M_o + P(\psi_i + \psi_o) \right] \frac{h^2}{By} \beta_i = 0$$

.........(I.3.31)

$$\left[ \frac{M_o - P(\psi_i + \psi_o)}{C_T} \right] h^2 u_{i+1} - [0] u_i - \left[ \frac{M_o - P(\psi_i + \psi_o)}{C_T} \right] i-1$$

$$- \frac{C_w}{C_T} \beta_{i-2} + \left[ 2 \frac{C_w}{C_T} + \left( 1 - \frac{\int \alpha r^2 dA}{C_T} \right) h^2 \right] \beta_{i-1}$$

$$+ [0] \beta_i - \left[ 2 \frac{C_w}{C_T} + \left( 1 - \frac{\int \alpha r^2 dA}{C_T} \right) h^2 \right] \beta_{i+1} + \frac{C_w}{C_T} \beta_{i+2} = 0$$

.........(I.3.32)
In the present investigation, \( n \) is taken equal to twenty one. A discussion on the percentage of error resulting from choosing this value of \( n \) will be given later.

A boundary conditions of Equations 1.3.31 and 1.3.32, are applied as follows:

\[
\begin{align*}
\upsilon_0 &= \upsilon_n = \beta_0 = \beta_n = 0 \\
\beta_{-1} &= -\beta_1 \text{ and } \beta_{n+1} = -\beta_{n-1} \quad (I.3.33)
\end{align*}
\]

At the critical limit, i.e. at the lateral torsional buckling load, the determinant of the coefficients of the finite difference equations is equal to zero, and Equations 1.3.31 and 1.3.32 can be arranged in a matrix form as follows:

\[
(C - \lambda I) X = 0 \quad (I.3.34)
\]

where;

\[
C = \text{a (20 x 20) real non-symmetric matrix}
\]

\[
\lambda = \text{eigenvalues} = 1 - M_{01}, \text{ in which } M_{01} \text{ is the critical end moments}
\]

\[
I = \text{identity matrix}
\]

\[
X = \text{eigenvectors}
\]

The point of onset of lateral-torsional buckling and the rigidity coefficients \((B_y, C_T, C_W, Y_0 \text{ and } \int \sigma r^2 \, \text{d}A_c)\) are interdependent. Therefore, a trial and error approach is applied to solve the problem.
A critical end moment, \( M_{0i}^{l} \), is assumed using an approximate method for a beam-column with specified length and axial load, \( P \). The deflection and the moment at the evenly spaced discrete points along the entire length of the beam-column can be calculated using the column deflection curves (step No. 3). The corresponding mechanical properties are calculated from step No. 2 for the section at each of the discrete points. The matrix \( C \) can then be calculated and also the corresponding minimum critical end moment, \( M_{0i} \), from the eigenvalues of matrix \( C \). \( M_{0i} \) is then compared with the assumed \( M_{0i}^{l} \). If the ratio \( \frac{M_{0i}}{M_{0i}^{l}} \) is different from unity, the procedure is repeated using increased or decreased value of \( M_{0i} \) until \( M_{0i}^{l} \) assumed becomes close enough to the calculated \( M_{0i} \).

To obtain the eigenvalues of matrix \( C \) (non-symmetric) a method suggested by Wilkinson (60) is used: In this method, the matrix \( C \) is reduced into an equivalent matrix \( D \) of real almost upper triangular form. The eigenvalues of the latter matrix \( D \) can then be computed which are the same as those of matrix \( C \).

All the theoretical work of the four steps have been programmed in fortran IV on the 360/50 IBM computer at the University of Windsor. A flow chart for this program is given in Figure I.16.
CHAPTER 4

OBSERVATIONS

1.4.1 Summary and discussion of the results

An example of an 8 WF 31 is explored. This section is chosen because it has, also, been investigated in similar works (14, 36, 41) which will be used for comparison. The following results are obtained:

1. The moment-thrust curvature relationship executed by step No. 1 is shown in Figure 1.17 for \( \frac{P}{P_y} = 0.2 \) to 0.8. The domains of the different strain configurations are shown in this figure. They indicate that for \( \frac{P}{P_y} = 0.4 \) to 0.8, the prevalent configurations are numbers 1, 2 and 3 for the elastic and plastic ranges followed by configuration 8 for the strain-hardening range. With low ratio of \( \frac{P}{P_y} (\frac{P}{P_y} = 0.2) \) the configurations are numbers 1, 2, 3, 5 and 13. These results help in obtaining the M.P.\( \phi \) curves by assuming the most probable strain configuration.
2. The relations between the moment applied on the section and each of the cross-sectional mechanical properties, $B_y$, $C_w$, $x_0$ and $\int \sigma r^2 \, dA_c$ for $\frac{P}{P_y} = 0.2$ to 0.8 are given in Figures 1.18 to 1.21. These curves illustrate the variation of these mechanical properties in the inelastic range since they are constants in the elastic range.

3. The effect of residual stresses, pre-buckling displacements and strain-hardening can be seen in Figures I.22, I.23 and I.24, for $\frac{P}{P_y} = 0.2$, 0.4 and 0.6. The curves in each figure relate the $(M_{oc,r})$ to the slenderness ratio $(L_{x})$. Curve 1 is for failure due to excessive bending and considering residual stresses. Curves 2, 3 and 4 illustrate failure due to lateral-torsional buckling with curve 2 for neglecting residual stresses and considering pre-buckling displacements, curve 3 for considering residual stresses and neglecting pre-buckling displacements, and curve 4 for both considering residual stresses and pre-buckling displacements.

Each of these curves covers the elastic, plastic and strain-hardening ranges. The effect of the residual stresses, pre-buckling displacements, and strain-hardening can be discussed further as follows:

a - Effect of pre-buckling displacements

The pre-buckling displacements, $\nu_i$, appear in the governing differential equations I.2.29 and I.2.30 added to $x_0$ and multiplied by $P$, $P(\nu_i + x_0)$. This term is proportional to the loading; therefore, calculating the pre-buckling displacements has relatively small effect.
for small ratios of \( \frac{P}{P_y} \) (Figures I.22, I.23 and I.24). Its effect disappears completely when \( \frac{P}{P_y} = 0 \). For a constant ratio of \( \frac{P}{P_y} \), the displacements \( v_1 \) are much greater than \( y_o \) in the elastic range while \( v_1 \) are too small when compared to \( y_0 \) in the inelastic range. Therefore, the buckling load calculated with the pre-buckling displacements \( v_1 \) has maximum reduction in the elastic range. This can be illustrated by comparing curve 4 with curve 3 of Figures I.22, I.23 and I.24.

b - Effect of residual stresses

Figures I.22, I.23 and I.24 show that residual stresses have negligible effect on the lateral-torsional buckling in the elastic range. However, they reduce the maximum moment that can be reached before the start of the inelastic range (compare point \( a_2 \) with point \( a_3 \) in Figure I.23). Within the inelastic range, the residual stresses have considerable effect on the lateral torsional buckling because they extend the yielding zone in the cross-section leading to reduction in the bending and warping rigidities.

c - Effect of strain hardening

The present analysis is for mild steel A36, with \( \frac{\varepsilon_{st}}{\varepsilon_y} = 12 \) and \( \frac{E_{st}}{E} = 0.022 \), the maximum strain in the strain-hardening range is limited to twelve times the yield strain, \( \varepsilon_y \). This is taken arbitrarily but the computer program is developed as a general one which can consider any magnitude of strain in the section. It can be seen from Figures I.22, I.23 and I.24 that the strain-hardening has an effect only in short columns where \( \frac{L}{r_x} < 8 \). Such short columns are of little practical interest.
However, for high tensile steel or aluminum alloy with stress-strain relationship different from that of mild steel; i.e., $\frac{E_{st}}{E} < 12$ and/or $\frac{E_{st}}{E} > 0.022$, the effect of strain-hardening will be considerable and extends to slender columns.

To investigate the effect of the ratio $\frac{E_{st}}{E}$ on the lateral torsional buckling strength, an aluminum alloy section (Alcan 28021 (53)) is examined, considering $\frac{E_{st}}{E} = 0.1$ and $0.0$ with $\frac{E_{st}}{E_{y}}$ are taken from Figure I.25 after idealizing the stress-strain curve of the material used into two straight lines of different slopes. Normally this section, as most of the aluminum alloy sections, is free from residual stresses. The properties of the material and the dimensions of the cross-section are indicated in Figure I.25 and I.26 (53).

A comparison of the two curves in Figure I.26 shows that the ratio, $\frac{E_{st}}{E}$, has a relatively appreciable effect on the lateral torsional buckling strength in the strain-hardening range.

I.4.2 Comparison with previous theoretical and experimental work

A - The theoretically obtained results by Miranda (41), Fukumoto (17) and Lim (36) for an 8 WF 31 and $\frac{p}{p_{y}} = 0.4$ are in good agreement with the results of the present investigation when the same assumptions are taken into consideration (Figure I.27).

B - The experimentally obtained results of reference (17) for $\frac{p}{p_{y}} = 0.12$ and $\sigma_{y} = 40$ Ksi are adjusted for $\sigma_{y} = 36$ Ksi using the following formula (17).
\[
\left( \frac{L}{r_x} \right)_{36} = \left( \frac{L}{r_x} \right)_{\sigma_y} \sqrt{\frac{\sigma_y}{36}}
\]  

(I.4.1)

in which:

\[ \sigma_y \] = the yield stress of the tested specimen.

These results agree with the results of the present work based on \( \sigma_y = 36 \text{ Ksi} \), and \( \frac{E_{st}}{E} = 0.022 \) (Figure I.28)

C - Comparison between the present investigation and the test results reported in Reference (17) are shown in the following table:

<table>
<thead>
<tr>
<th>Specimen No.</th>
<th>Section Size</th>
<th>( \frac{P}{P_y} )</th>
<th>Yield stress of the Specimen KSi</th>
<th>( \frac{L}{r_x} ) (adjusted)</th>
<th>( \frac{M_{ocr}}{M_{x,y}} ) (observed)</th>
<th>( \frac{M_{ocr}}{M_{x,y}} ) (calculated)</th>
<th>% (diff.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4WF13</td>
<td>0.12</td>
<td>40</td>
<td>54</td>
<td>0.809</td>
<td>0.822</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>12WF27</td>
<td>0.218</td>
<td>38.5</td>
<td>26.5</td>
<td>0.76</td>
<td>0.76</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>12WF27</td>
<td>0.162</td>
<td>36.5</td>
<td>31.4</td>
<td>0.708</td>
<td>0.63</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>12WF27</td>
<td>0.066</td>
<td>37.9</td>
<td>52.5</td>
<td>0.455</td>
<td>0.455</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>12WF27</td>
<td>0.0275</td>
<td>38.5</td>
<td>80.0</td>
<td>0.288</td>
<td>0.273</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>18WF50</td>
<td>0.265</td>
<td>39.2</td>
<td>18.75</td>
<td>0.66</td>
<td>0.697</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>18WF50</td>
<td>0.255</td>
<td>39.2</td>
<td>23.3</td>
<td>0.795</td>
<td>0.72</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>18WF50</td>
<td>0.048</td>
<td>39.8</td>
<td>56.2</td>
<td>0.353</td>
<td>0.33</td>
<td>6</td>
</tr>
</tbody>
</table>

The deviation between the theoretical and experimental results may be attributed to the following factors:
1. The distribution and magnitude of residual stresses might have been subjected to a considerable variation from the assumed values (17).

2. The difficulty in testing a beam-column with the idealized boundary conditions assumed at the two ends.

3. The variation of the yielded stress between the flanges and the web.

4. The variation of the cross-sectional properties along the member and the initial imperfection of the beam columns as, practically, members are not perfectly straight.

Yet, it can be concluded from the above table and Figure I.28. that the theoretical results show good agreement with the experimental ones.

I.4.3 The number of discrete points to be considered in the finite difference approximations

The number of discrete points, n, to be considered in the finite difference approximations must be chosen such that the error in the calculated buckling load remains within a permissible limit. Furthermore, the execution time is a factor which must be taken into account. A small increase in n would substantially increase the computer time needed to solve a given problem. A study was made to determine the influence of n on the accuracy of the calculation of the lateral torsional
buckling load.

A beam-column, of an 8 WF 31 cross-section with \( \frac{P}{F_y} = 0.40 \), and \( \frac{L}{r_x} = 20, 40, 60 \) and 81, is investigated. The relationship between \( \frac{M_{ocr}}{M_y} \) and the number of discrete points, \( n \), is shown in Figures 1.29 and 1.30 for \( n = 7, 9, 21, 31 \). For \( \frac{L}{r_x} = 81 \), the beam-column is elastic, and the problem has an exact solution if the prebuckling displacements are neglected. This solution is also shown in Figure I.30.

The sources of error in applying the finite difference approximations are:

1. The truncation error, resulting from choosing the value of \( n \). The greater the value of \( n \) the smaller the truncation error.

2. The round off error, occurring from the fact that computers have no infinite arithmetic precision.

The effect of the last type of error on the lateral-torsional buckling loads, was checked by running the computer program with different values of \( \frac{L}{r_x} \), and \( n \) ranging from 7 to 31. Such runs were first made with single precision and secondly with double precision. The results obtained were the same up to three decimal points. This indicates that the effect of the round off error for values of \( n \) up to 31 is insignificant and can be ignored.

The effect of the truncation error is illustrated in Figures 1.29 and I.30. From these figures, it can be seen that the calculated critical end moments tend to increase very slowly when \( n = 21 \). Furthermore, this
value of \( n \) gives a calculated critical end moment close to the exact one (Figure 1.30). Hence, in the present analysis, \( n \) is taken equal to twenty one in all the calculations. This value may not be the optimum, yet it gives buckling loads within a tolerable error, and needs a relatively shorter computer time. The same value of \( n \) has also been used in a previous similar work (36).
CHAPTER 5

THE LATERAL BUCKLING OF BEAMS

1.5.1 Introduction and literature review

A perfectly straight wide-flange beam, simply supported at its ends and subjected to equal end moments about its major axis, is a special case of the uniaxially loaded beam-column when the axial load is equal to zero. Being simpler in nature, as the moments and the mechanical properties of the section are constant along the length of the beam, this case can be solved directly without resorting to finite difference techniques.

Thus, the treatment of this problem is prescribed separately herein. Under pure end moments, the member generally does not fail by bending in the plane of the applied moments if it is not adequately braced against lateral displacements. This occurs particularly if the beam is narrow, compared to its length. In this case the beam will deflect in the plane of the applied moments as long as these moments are below a certain critical value. However, when the critical moment is reached, bifurcation of the equilibrium takes place and the beam will deflect laterally out of the plane of bending with simultaneous rotation before the ultimate carrying capacity of the section is attained. This phenomenon is known as lateral buckling.
The problem of lateral buckling has been thoroughly studied in the elastic range by many investigators. Lee (35) summarized the work done on this problem until 1960.

In the inelastic range, Neal (42) examined this problem for rectangular, mild steel sections, while Horne (25) extended the same concept to include wide flange sections. In their analysis, they neglected residual stresses and strain-hardening.

Solutions for the lateral buckling of beams in the strain-hardening range were developed for rectangular and wide-flange cross-sections by Wittrick (58) and White (59) respectively. In both analyses, the beams were assumed to be made of a metal having a monotonically increasing stress-strain curve as in the case of aluminum alloys. Galambos (18) examined this problem for mild steel beams using the assumption suggested by White (59), i.e. he considered the material to be either elasto-plastic or strain-hardened and partially neglected the ductile range. He (18) also did not consider the reduction in the St. Venant torsional rigidity due to Wagner's effect (57).

The problem of lateral buckling of rolled wide flange beams, subjected to equal end moments, is re-examined here for the elastic and inelastic ranges. The present approach considers the effect of the strain-hardening, without neglecting any part of the ductile range, and takes into account the reduction in the St. Venant torsional rigidity due to Wagner's effect.
### 1.5.2 Governing differential equations and boundary conditions

The differential equations, governing the lateral buckling of beams, can be established by setting \( P = 0 \) in Equations 1.2.29 and 1.2.30. Thus,

\[
B_y u'' + M_o \beta = 0 \quad (1.5.1)
\]

\[
C_w \beta'' - (C_T - \int \sigma r^2 dA_c) \beta' + M_o u' = 0 \quad (1.5.2)
\]

The beam is simply supported with respect to the lateral buckling having the following boundary conditions (9, 54):

\[
u = u'' = \beta = \beta'' = 0 \quad \text{at } z = 0 \text{ and } z = L \quad (1.5.3)
\]

### 1.5.3 Method of solution

By satisfying the governing differential equations (Equations 1.5.1, 1.5.2) and the boundary conditions (Equation 1.5.3), a quartic equation is obtained (9,18,48). This equation governs the relation between the critical end moments at the onset of lateral buckling, \( M_{o_c r} \), and the dimensions and sectional properties of the beam. It can be written in a non-dimensional form as follows:

\[
\left( \frac{M_{o_c r}}{M_p^2} \right)^2 \left( \frac{L}{r_y} \right)^4 - \left[ \frac{2 B_y (C_T - \int \sigma r^2 dA_c)}{(M_p^2)^2 (r_y)^2} \right] \left( \frac{L}{r_y} \right)^2 - \frac{4 B_y C_w}{(r_y)^4 \left( \frac{M_p^2}{M_x} \right)^2} = 0 \quad (1.5.4)
\]
in which:

\[
\frac{L}{r_y} = \text{the slenderness ratio about the minor axis}
\]

\[M^X_p = \text{the plastic moment of resistance of the section.}\]

Solving equation I.6.4 leads to:

\[
\left( \frac{L}{r_y} \right)_{cr} = \sqrt{F + \frac{F^2 + 2HG}{H}} \tag{I.5.5}
\]

in which:

\[
F = \frac{\pi^2 B_Y (C_T - \int \sigma r^2 \, dA_C)}{(M^X_p)^2 (r_y)^2} ;
\]

\[
G = \frac{4 B_Y C_w}{(r_y)^4 (M^X_p)^2} ;
\]

\[
H = 2 \left( \frac{M_{ocr}}{M^X_p} \right)^2 \tag{I.5.6}
\]

Equation I.6.5 represents a direct relation between the slenderness ratio \(\frac{L}{r_y}\) and the moment \(M_{ocr}\) at the onset of lateral buckling. This relation is governed by the cross sectional mechanical properties and is calculated using the following three steps:
(1) Determination of the moment-curvature relationship (M-\phi curve); this is obtained by a trial and error procedure as in the case of uniaxially loaded beam-columns (Section I.3.1) in which the axial thrust \( P \) is equal to zero and by considering eight possible strain configurations. Figure 1.31 shows the sequence used for checking the strain-configurations.

(2) Determination of the mechanical properties of the section, \( B_y, C_T, C_w, Y_0 \) and \( \int \sigma r^2 \, dA_c \). \( B_y, C_w, Y_0 \) are obtained using the tables, developed earlier, with \( P = 0 \) whereas \( C_T \) and \( \int \sigma r^2 \, dA_c \) are calculated as before.

(3) Determination of the lateral buckling. For a specified value of \( M_{ocr} \), the sectional properties are calculated from step 2 and substituted in Equation I.6.5. By solving Equation I.6.5 the corresponding slenderness ratio \( \frac{L}{r_y} \) is then calculated.

A computer program covering all the theoretical aspects of the problem has been developed for the IBM 360/50 at the University of Windsor. A flow chart of this program is shown in Figure I.32.

I.5.4 Observations

Four beams of mild steel A36 of sizes 8WF 31,10 WF 89,14 WF 142, 14 WF 287 were examined. These beams are popular sizes and represent thin-walled, moderately thin-walled and thick-walled sections. The
following results are observed:

1. The relationships between \( \frac{M^X}{M^Y} \) and \( \frac{\sigma^x}{\sigma^y} \) calculated by step 1 are found to be almost identical for the four sections (Figure I.33). Figures I.34 to I.37 show the relation between the moment applied on the section about its major axis and each of the cross-sectional mechanical properties, \( B_y \), \( C_w \), \( Y_o \) and \( \int \sigma r^2 \, dA_c \), represented in a non-dimensional form. These curves show that these relationships are not direct functions of the section's size but depend primarily on the dimensions of its components.

2. The final results obtained here relating the ratio \( \frac{M_{ocr}}{M_p^X} \) to the slenderness ratio \( \frac{L}{r_y} \) (Figures I.38 to I.42) agree with those already published in the elastic and plastic ranges (18) when the same assumptions are considered, (i.e. \( \sigma_y = 33 \) KsI and neglecting Wagner's effect). However, the strain-hardening effect is found to be shifted to shorter beams with lengths ranging between \( 5r_y \) and \( 10r_y \) rather than about \( 20r_y \) as previously reported (18). This deviation is attributable to the consideration of the ductility of the material which leads to considerable reduction in the rigidities. Therefore, the beam buckles laterally before reaching strain-hardening, i.e. before regaining some of its rigidities.

When neglecting the ductile range, the cutoff point for the start of strain-hardening effect takes place with \( \frac{L}{r_y} \) ranging between
20 and 40 (Figures I.39 to I.42). The assumption that the section under study is partially elastic and partially strain-hardened rather than fully strain-hardened accounts for the differences between these results and those previously published (18). Other factors causing the divergence in the results are the consideration of Wagner's effect and the allowance for full constant torsional rigidity $C_T$.

3. Residual stresses have a negligible effect on the lateral buckling strength in the elastic range. However, they do reduce the maximum moment attainable before the start of the inelastic range (compare points $a_1$ and $a_2$ in figure I.40). This conclusion agrees with that previously reported (18).

4. Figure I.43 shows the relationship between $\frac{M_{ocr}}{M_p}$ and the parameter $D_T$ defined as follows (17, 18):

$$D_T = \frac{K_T \times 10^6}{A_c q^2}$$

It can be seen from this figure that sections with high $D_T$ value have more resistance to lateral buckling and the ratio $\frac{M_{ocr}}{M_p}$ is almost linearly proportional to $D_T$ when $\frac{M_{ocr}}{M_p} \leq 0.95$.

5. Some structural materials, such as aluminum, have no plastic flow range (ductile range), a high ratio of $\frac{\varepsilon_{st}}{\varepsilon_Y}$, and a low ratio of $\frac{\varepsilon_{st}}{\varepsilon_Y}$ ($\frac{\varepsilon_{st}}{\varepsilon_Y} = 0.1$ and $\frac{\varepsilon_{st}}{\varepsilon_Y} = 1.0$ compared to $\frac{\varepsilon_{st}}{\varepsilon_Y} = 0.022$ and $\varepsilon_Y = 12$ in steel A 36 (1)). To investigate the lateral buckling strength for beams made from such material, an aluminum alloy section (Alcan 28021),
which was analysed before for the case of the uniaxially loaded beam-columns is examined here also. The final results relating the critical end moment with the slenderness ratio, $\frac{L}{r_y}$, (Figure I.44), indicate that the strain-hardening is effective in slender beams with $\frac{L}{r_y} = 35$. 

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CHAPTER 6

CONCLUSIONS

The preceding chapters examine the problem of lateral-torsional buckling of uniaxially loaded beam-columns. The main points of conclusion are:

1. The lateral torsional buckling reduces the strength of beam-columns considerably in the inelastic range.

2. Residual stresses have negligible effect on the lateral torsional buckling load in the elastic range, but they reduce the buckling load considerably in the inelastic range.

3. The effect of pre-buckling displacements is not always insignificant and its effect is greater in the elastic range than in the inelastic range and being neglected may lead to considerable errors on the unsafe side.
4. The strain-hardening of mild steel has appreciable effect only for short columns (where \( \frac{L}{r_x} \leq 8 \)).

5. For the case of beams under pure end moments (\( p=0 \)), it can be seen that:

(a) The strain-hardening effect is operative in mild steel beams with \( \frac{L}{r_y} \) ranging between 5 and 10. This finding differs from the reported \( \frac{L}{r_y} = 20 \) in an earlier study (18).

(b) For constant value of \( \frac{L}{r_y} \) and with \( \frac{M_{cr}}{M_p} \) = 0.95, the ratio \( \frac{M_{cr}}{M_p} \) is almost linearly proportional to the non-dimensional parameter \( n_p \).

6. The present calculations yield a lower bound value of the critical end moments. These values are based on the assumption that no unloading of the already strained section occurs at the initiation of lateral torsional buckling; i.e. the tangent modulus concept is applied. Furthermore, the post-buckling effect is neglected, in order to obtain safer values for the ultimate moments.
PART II

INELASTIC BEAM-COLUMNS UNDER

BIAXIAL LOADING
CHAPTER 1

INTRODUCTION

The columns in a three-dimensional framework, are frequently subjected to bending moments acting about its two principle axes in addition to the axial load, i.e. biaxially loaded.

The elasto-plastic behaviour of a beam-column of wide flange cross-section loaded biaxially can be illustrated by the curve of end moment versus a displacement parameter (end rotation) shown in Figure II.1. If the stresses are within the elastic limit, the displacements of the column increase slowly with the loading. However, when yielding starts, the displacements increase at a faster rate compared to the load. The ultimate end moment is reached when stable equilibrium is no longer possible between internal and external forces. This corresponds to point C on the curve, beyond which, the load starts to decrease with further increase in the displacements.
The developments in the study of biaxially loaded beam-columns, were surveyed by Chen and Santathadaporn (10) and by Pillai (45). The following is a brief review of the previous research work on this problem.

A. Elastic beam-columns: Wagner (57) and Kappus (29) established the differential equations for the torsional buckling of thin-walled open sections. These equations were extended by Goodier (20, 21, 22.) to include biaxially loaded beam-columns with the assumptions that the displacements and twisting are small compared to the eccentricities of the loading. Approximate solutions of Goodier's equations were given by Thurlimann (55) and Prawel and Lee (46). In 1966, these equations were solved exactly by Culver (12).

B. Inelastic beam-columns: Birnstiel and Michalos (5) presented a general procedure for determining the ultimate carrying capacity of biaxially loaded beam-columns in the inelastic range. This procedure was further simplified and extended (6, 24) to cover the deformational response of the column from the zero loading up to the ultimate load and during unloading. They include the effect of residual stresses and strain reversal and confirmed their theory by experiments (7, 8). However, their procedure requires successive trials and corrections and needs considerable computational work. Sharma and Gaylord (40, 41) proposed an approximate solution for the problem but it does not satisfy all the boundary conditions. McVinnie (39) and his associates, Scott, Razzag and Masterson (38, 47, 50) presented an analytical solution for
beam-columns of tubular, rectangular and wide flange sections. Load-deflection curves were set up using a numerical integration procedure. In their analysis they neglected the effect of twist and residual stresses.

The present analysis is also based on a numerical integration procedure to find the column deflection curve in which the effect of twist is considered. The residual stresses are taken into account in the mathematical formulations. However, they are neglected in the numerical calculations as their inclusion would be time consuming.
CHAPTER 2

FORMULATION OF THE GOVERNING DIFFERENTIAL EQUATIONS

II.2.1 Assumptions

The differential equations governing the equilibrium of biaxially loaded beam-columns are treated (20, 21, 54, 56) on the basis of the following assumptions:

1. The beam-column is perfectly straight and of uniform wide-flange cross-section.

2. The deflections and rotations are too small compared to the dimensions of the beam-columns.

3. No external forces are applied on the beam-column between the supports.

4. The column is subjected to biaxially eccentric load placed symmetrically at both ends.
5. The stress-strain relationship of the material is ideally elastic-plastic (Figure II.2). It is the same for tension and compression. Strain-hardening is neglected.

6. The material is homogenous and isotropic in both the elastic and inelastic ranges.

7. Unloading is not allowed in the yielded parts of the beam-columns.

8. Yielding is governed by normal stresses only, i.e. the effect of shear stresses is neglected.

9. Axial shortening of the beam-column is neglected.

10. The thicknesses of the flanges and the web are small in comparison with the dimensions of the cross-section. Therefore, the strains are assumed to be uniform over these thicknesses and equal to the strains at their respective centre lines.

II.2.2 The governing differential equations

Figures II.3 and II.4 show a beam-column under biaxially eccentric thrust $P$. The fixed co-ordinate axes are $x, y$, and $z$; the eccentricities of $P$ are $e_x$ and $e_y$ at $z = 0$ and $z = L$. Figure II.5 shows a wide-flange cross-section in the initial and displaced positions.
The displacements, moments, torques and their sign conventions are defined in the same manner as in the case of the uniaxially loaded beam-column. Moments and curvatures are designated as $M_A^X$ and $\phi_A^X$ respectively, in which the superscripts indicate the axis about which they act; whereas, the subscripts show their position along the beam-column. If no subscripts are used, this indicates that the expressions of moments and/or curvatures are valid for any point along the length of the beam-column.

From Figure II.4, the displacements of the centroid in the x- and y- directions are given by:

$$\bar{u} = u + y_0 \beta$$  \hspace{1cm} (II.2.1)

$$\bar{v} = v - x_0 \beta$$  \hspace{1cm} (II.2.2)

in which:

$u, v$ = displacements of the shear centre in the x- and y- directions respectively;

$x_0, y_0$ = distance between the shear centre and the centroid of the section in the x- and y- directions respectively.

The bending moments at any sections along the beam-column about the x-, y- and z- axes are:
The relation between moments and curvatures in the initial and displaced position are (54):

\[ M^x = -P \left( e_y - v + x_0 \beta \right) \]  

(II.2.3)

\[ M^y = P \left( e_x - u - y_0 \beta \right) \]  

(II.2.4)

\[ M^z = 0 \]  

(II.2.5)

\[ M^x = M^x + M^y \beta \]  

(II.2.6)

\[ M^y = M^y - M^x \beta \]  

(II.2.7)

\[ M^r = M^x \frac{du}{dz} + M^y \frac{dv}{dz} + M^{r1} + M^{r2} \]  

(II.2.8)

and:

\[ \phi^r = \phi^x + \phi^y \beta \]  

(II.2.9)

\[ \phi^c = \phi^y - \phi^x \beta \]  

(II.2.10)

\[ M^{r1} \] is the torsional moment caused by the components of the axial force with respect to the shear centre. It is equal to:

\[ M^{r1} = P \left( y_0 \frac{du}{dz} - x_0 \frac{dv}{dz} \right) \]  

(II.2.11)

\[ M^{r2} \] is the torsional moment due to the effect of the normal compressive stresses on the warped cross-section (57),

\[ M^{r2} = (f_0 v^2 dA_c) \frac{d\beta}{dz} \]  

(II.2.12)

The relationships between the moments \( M^c \) and \( M^g \) and the curvatures about the \( x- \) and \( y- \) axes are:
\[ M^C = -B_x \frac{d^2v}{dz^2} \]  
(II.2.13)

\[ M^E = B_y \frac{d^2u}{dz^2} \]  
(II.2.14)

The torsional moment \( M^P \) and the angle \( \beta \) are related by the following equation (9, 20, 54):

\[ M^P = C_T \frac{d\beta}{dz} - C_w \frac{d^3\beta}{dz^3} \]  
(II.2.15)

The eccentricities of the load \( P \), at the ends of the beam-column are equal in each direction. Therefore, the end moments at \( A \) and \( B \) (Figure II.3) are:

\[ M^X_A = M^X_B = P e_y \]  
(II.2.16)

\[ M^Y_A = M^Y_B = P e_x \]  
(II.2.17)

By mathematical manipulation, equations II.2.3 to II.2.17 yield the following differential equations:

\[ B_x \frac{d^2v}{dz^2} + Pu + (M^Y_A - Px_O)\beta - M^X_A = 0 \]  
(II.2.18)

\[ B_y \frac{d^2u}{dz^2} + Pu - (M^X_A - Py_O)\beta - M^Y_A = 0 \]  
(II.2.19)

\[ C_w \frac{d^3\beta}{dz^3} - (C_T - \int_\sigma r^2 dA_c) \frac{d\beta}{dz} - (M^X_A - Py_O) \frac{dv}{dz} + \\
+ (M^Y_A - Px_O) \frac{dv}{dz} = 0 \]  
(II.2.20)
II.2.3 Boundary Conditions

The beam-column is assumed to be simply supported at its ends. Therefore, at \( z = 0 \) and \( z = L \) the boundary conditions are as follows (12):

\[
\begin{align*}
\mathbf{u} = \mathbf{v} = \beta &= 0 \text{ (lateral and twisting displacements are prevented)}; \\
\beta'' &= 0 \quad \text{(warping is permitted)}; \\
u'' &= \frac{M_A}{B_Y}; \\
v'' &= -\frac{M_A}{B_X}.
\end{align*}
\]
(II.2.21)
CHAPTER 3

METHOD OF SOLUTION

The differential equations II.2.18 to II.2.20 encounter three interdependent displacement components \( u, \tau \) and \( \beta \). The coefficients of these equations are constants along the length of the beam-column within the elastic range and become variables in the inelastic range. This makes the problem too difficult to have an exact solution. Hence, a solution based on a numerical integration procedure, to obtain the column deflections curves, is applied (39) and modified to take into account the effect of twist. To simplify the analysis; the following two assumptions are made:

1. The distance between the shear centre and the centroid of the cross-section is insignificant and can be neglected. This is because under biaxial loading, yielding usually occurs in both flanges on opposite sides of the web (Figure II.6). Thus,

\[
x_0 = y_0 = 0 \quad \text{(II.3.1)}
\]
2. The angle of rotation $\beta$ along the beam-column is assumed as follows:

$$\beta = \beta_c \sin \frac{\pi z}{L}$$  \hspace{1cm} (II.3.2)

in which:

$\beta_c$ = the angle of rotation at mid-span.

This assumption for $\beta$ is very reasonable since it satisfies the boundary conditions.

Differentiating Equation II.2.20 with respect to $z$, and using Equations II.2.6 to II.2.10 and Equations II.2.13 and II.2.14, and considering the equilibrium at mid-span of the beam-column, the following expression is obtained for the angle of rotation $\beta_c$:

$$\beta_c = \frac{M_A^x \phi_c^y + M_A^y \phi_c^x}{I_c \left( \frac{\pi}{L} \right)^4 + (C_T - \int \sigma r^2 \, dA_c) \left( \frac{\pi}{L} \right)^2 + M_A^x \phi_c^y - M_A^y \phi_c^x}$$  \hspace{1cm} .......(II.3.3)

where:

$\phi_c^x$ and $\phi_c^y$ = curvatures at mid span of the beam-column about the x- and y- axes respectively.

The values of $C_T$, $C_W$, and $\int \sigma r^2 \, dA_c$, in Equation II.3.3 are those of the cross-section at the middle of the beam-column.
The solution is performed in three steps:

1- Determination of the moment-thrust-twist-curvature relationship.

2- Determination of the mechanical properties of the cross-section.

3- Determination of the end-moment versus end-rotation relationship.

II.3.1 Determination of the moment-thrust-twist-curvature relationship

This is carried out in the same manner as in the case of the uniaxially loaded members. For a beam-column, under biaxially eccentric load $P$, each section along the length of the column, in the displaced position is acted upon by a thrust $P$, as well as a moment about each principal axis.

The normal strain at any point, $\varepsilon$, on the cross-section may be written as:

$$\varepsilon = \phi \varepsilon + \phi \varepsilon + \varepsilon_D + \varepsilon_W + \varepsilon_R \quad (II.3.4)$$

in which:

$\varepsilon_W = \text{warping strain}$

Since it is assumed that shifting of the shear centre can be neglected and the strains across the flanges and web thicknesses are uniform, it can be concluded that the web is free from warping strains (51, 54).
Thus, at any point on the flanges, the warping strains can be determined by the following equation:

\[ \varepsilon_w = W_s \frac{d^2 \rho}{dz^2} \]  

(II.3.5)

in which:

\( W_s \) represents the double sectorial area that a radius vector, joining the shear centre, \( S \), and a point of zero warping, sweeps up to that point on the flange (51, 54). The sectorial area is taken positive when the radius vector rotates counter-clockwise about the shear centre.

\[ \therefore W_s = \pm \xi D (1 + K_2/2) \]  

(II.3.6)

The distribution of the residual stresses is assumed as shown in Figure I.3. Therefore the residual strain, \( \varepsilon_R \), can be calculated using Equation I.5 in the form:

\[ \varepsilon_R = \varepsilon_{rt} + \frac{\varepsilon_{rc} - \varepsilon_{rt}}{(b/2)} |\xi| \]  

(II.3.7)

Hence, the normal strain, \( \varepsilon \), at any point \((\xi, \zeta)\) can be written as:

\[ \varepsilon = \phi \xi + \phi \zeta + \varepsilon_0 \pm \xi D (1 + K_2/2)\varepsilon^n + \]

\[ + \varepsilon_{rt} + \frac{\varepsilon_{rc} - \varepsilon_{rt}}{(b/2)} |\zeta| \]  

(II.3.8)

If the residual stresses are neglected, Equation II.3.8 will be reduced to:
\[ \epsilon = \phi \xi + \phi \zeta + \epsilon_0 \mp \xi D (1 + K_2/2) \beta'' \quad (II.3.9) \]

With reference to Figure II.2, the corresponding stress distribution may be written as:

\[ \sigma = E \varepsilon - E [\xi \pm \epsilon_y] \quad (II.3.10) \]

The brackets in Equation II.3.10 have the same special significance as indicated before (Section I.2.2).

The equilibrium between external and internal forces requires that:

\[ P = \int \sigma \, dA \quad (II.3.11) \]

\[ M^E = \int \sigma \zeta \, dA \quad (II.3.12) \]

\[ M^F = \int \sigma \xi \, dA \quad (II.3.13) \]

Substituting the value of \( \sigma \) from Equation II.3.10 and dividing by the appropriate factors the following non-dimensional equations are obtained (39):

\[ \frac{P}{P_y} = \frac{1}{A_c} \int \frac{\varepsilon}{\epsilon_y} \, dA_c - \frac{1}{A_c} \int \frac{[\varepsilon - \epsilon_y \pm 1]}{\epsilon_y} \, dA_c \quad (II.3.14) \]

\[ \frac{M^E}{M^I} = \frac{D(1 + K_2)}{I^I} \left\{ \int \zeta \left(\frac{\varepsilon}{\epsilon_y}\right) \, dA_c - \int \zeta \left[\frac{\epsilon - \epsilon_y \pm 1}{\epsilon_y}\right] \, dA_c \right\} \]

\[ \quad \cdots \cdots (II.3.15) \]

\[ \frac{M^E}{M^I} = \frac{D(1 + K_2)}{I^I} \left\{ \int \zeta \left(\frac{\varepsilon}{\epsilon_y}\right) \, dA_c - \int \zeta \left[\frac{\epsilon - \epsilon_y \pm 1}{\epsilon_y}\right] \, dA_c \right\} \]

\[ \quad \cdots \cdots (II.3.16) \]
The first integral in Equation II.3.14, II.3.15 and II.3.16 represents the volume of the distribution or its first moment about the $\xi$ or $\zeta$ axes (39) and is equal to:

$$\frac{1}{A_c} \int \frac{\varepsilon}{\varepsilon_y} dA_c = \frac{e_0}{\varepsilon_y}; \quad (\text{II.3.17})$$

$$\frac{D(1 + K_2)}{I_x} \int \zeta \left(\frac{\varepsilon}{\varepsilon_y}\right) dA_c = \frac{\phi_\xi}{\phi_y}; \quad (\text{II.3.18})$$

$$\frac{D(1 + K_2)}{I_x} \int \varepsilon \left(\frac{\varepsilon}{\varepsilon_y}\right) dA_c = \frac{\phi_\zeta}{\phi_y}; \quad (\text{II.3.19})$$

The second integral in Equations II.3.14, II.3.15 and II.3.16 is the volume or the first moment of the volume of the $[\frac{\varepsilon}{\varepsilon_y} \pm 1]$ distribution about the $\xi$ and $\zeta$ axes. It can be calculated for any assumed strain configuration in terms of the cross-sectional dimensions and the strains at the points 1 to 6 (Figure I.4)

For a wide flange section biaxially loaded in a beam-column, there are twenty-six possible strain-configurations for the case when residual stresses are neglected. These are shown in Figure II.7. Considering the residual stresses, the possible configurations will increase to forty-three as illustrated in Figure II.8.

In a similar way to that indicated before, for the case of uniaxially loaded beam-columns (Chapter 3, Part I), Equation II.3.14 can be reduced to the following form:

$$Q \left(\frac{e_0}{\varepsilon_y}\right)^2 + R \left(\frac{e_0}{\varepsilon_y}\right) + S = 0; \quad (\text{II.3.20})$$
which can be solved for \( \frac{\varepsilon_0}{\varepsilon_y} \):

\[
\frac{\varepsilon_0}{\varepsilon_y} = -R - \frac{\sqrt{R^2 - 4QS}}{2Q}
\]  

(II.3.21)

Expressions for \( Q \), \( R \) and \( S \) were derived for each strain configuration in Figures II.7 and II.8 and are shown in Tables 11 and 12.

Once the value of \( \frac{\varepsilon_0}{\varepsilon_y} \) has been known, the strain distribution across the section is determined. Thereafter, the moment volume calculations in Equations II.3.15 and II.3.16 can be performed. The final expressions for Equations II.3.15 and II.3.16, neglecting residual stresses are shown in Tables 15 and 16.

The moment-thrust-twist-curvature relationships (The curves of Figures II.10 and I.11) are obtained for specified values of \( P \) and \( \beta \) as follows:

1. \( \phi^e \) and \( \phi^\zeta \) are assigned certain values. A strain configuration is assumed and with these specified values of \( P \), \( \beta \), \( \phi^e \) and \( \phi^\zeta \), the coefficients \( Q \), \( R \) and \( S \) of Equation II.3.20 are calculated using the appropriate expressions in Tables 11 and 12.

2. \( Q \), \( R \) and \( S \) are substituted in Equation II.3.21 and \( \frac{\varepsilon_0}{\varepsilon_y} \) is calculated.

3. The strain is calculated using Equation II.3.14.

4. The corresponding strain configuration is found and compared with the assumed one.
5. The steps 1 to 4 are repeated until the calculated configuration becomes the same as the assumed one. The moments $M^\xi$ and $M^\zeta$ are calculated using the appropriate forms of Equations II.3.15 and II.3.16.

6. By varying $\phi^\xi$ and $\phi^\zeta$ over the desired range, the required curves are determined.

7. Similar curves are obtained by varying the angle of rotation $\phi$.

Using these curves the curvatures $\phi^\xi$ and $\phi^\zeta$ are obtained for any combination of $M^\xi$, $M^\zeta$ and $\phi$ as follows:

1. For a specified value of $M^\xi$ (for example, $M^{\xi_0}$) the curvatures, resulting in this moment, are found at the intersections of $M^{\xi_0}$ with the constant $\phi^\zeta$ and $\phi$ curves Figure II.9. A plot of these intersections, curve A, is shown in Figure II.12.

2. For any value of $M^\zeta$ (for example, $M^{\zeta_0}$) the curvatures resulting in this moment are found at the intersections of $M^{\zeta_0}$ and the constant $\phi^\xi$ and $\phi$ curves. Curve B of Figure II.12 is a plot of these intersections.

3. The intersection of curve A with Curve B determine the curvatures $\phi^{\xi_0}$ and $\phi^{\zeta_0}$.

4. Should the specified value of $\phi$ be not the same as the one assumed in Figures II.9 and II.10, steps 1 to 3 must be carried out for $\phi$ smaller and greater than the specified one and the curvatures
\( \phi_0 \) and \( \phi^0 \) are to be determined by interpolation.

In order to find the intersection of curve A and curve B, second-degree polynomials derived from three data points, bounding the value of the independent variable, are used for all interpolations (50).

**II.3.2 Determination of the mechanical properties of the cross-section**

These are the coefficients \( C_T, C_N \), and \( \int \sigma r^2 \, dA_c \) which appear in equation II.3.3. For specified values of curvatures \( \phi^c \) and \( \phi^c \) and a specified angle of rotation, \( \beta \), the corresponding moments and strain configuration are defined in step No. 1. Thereafter, the mechanical properties are calculated. The St. Venant torsional rigidity, \( C_T \), is assumed to be equal to its full elastic torsional rigidity regardless of the extent of yielding. A detailed discussion on this assumption is given in chapter 3, part I.

The warping rigidity of the cross-section, \( C_W \), for the different strain configurations, is calculated using the tangent modulus concept; i.e. it is equal to the warping rigidity of the elastic parts of the cross-section. Expressions for \( C_W \), for the most prevalent cases, in which the bottom flange or the top and bottom flanges are partially yielded, are calculated approximately by utilizing the assumption of neglecting the shifting of the shear centre as follows (51, 52):

\[
C_W = ED^2 (1 + K_2/2)^2 (I_{tf} + I_{bf}) \quad \text{(II.3.22)}
\]
in which:

\[ I_{tf} \text{ and } I_{bf} = \text{ the moments of inertia about the minor axis of} \]

the section (y-axis) for the elastic parts of the top and bottom flanges respectively.

In this equation the contribution of the web to the warping rigidity of the section is neglected because it is very small compared to the rigidity of the flanges.

Formulae were derived for each of the strain configurations of Figures II.7 and II.8 in terms of the strains at the points 1 to 6 (Figure I.4). These are given in Tables 17 and 18 for the cases when the residual stresses are neglected and taken into account respectively.

Wagner's effect, \( \int \sigma r^2 \, dA_e \), is calculated numerically by dividing the flanges and the web into twenty equal parts and following the same procedure outlined in chapter 3, part I.

II.3.3 Determination of the End-Moment Versus End-Rotation Relationship

The relationship between the end moments and end rotations (Figure II.1) can be obtained from a set of column deflection curves. These curves are constructed utilizing the same numerical technique used for the uniaxially loaded beam-columns after being modified for the biaxially loaded members (39). In the present analysis the effect of twist is also taken into account.

Figure II.13 shows a typical column deflection curve for a beam-column under biaxially eccentric thrust with equal eccentricity at both ends in each of the principal directions. The ratio of the end-
moment about the x-direction to the end moment about the y-direction is designated as $\gamma$, hence:

$$\gamma = \frac{M_Y}{M_X} = \frac{M_Y}{M_X} = \frac{\dot{u}_A}{\dot{v}_A} = \frac{\dot{u}_B}{\dot{v}_B}$$  \hspace{1cm} (II.3.23)

The beam-column is divided into a number of small segments of equal length, $a$. The projections of a segment onto the y-z plane and x-z plane are shown in Figure II.14 and II.15. The x and y displacements at $i$ are denoted by $u_i$ and $v_i$, respectively, and those at $i+1$ by $u_{i+1}$ and $v_{i+1}$. The curvature and rotation at $i$ are represented by $\phi_i$ and $\beta_i$. Assuming that the projection of the element onto each of the y-z and x-z planes is a flat circular arc (Figures II.14 and II.15) and considering the geometry of the curves, the $u$ and $v$ displacements and the slope $\theta$ at $i+1$ can be written in terms of $u, v$ and $\theta$ at $i$.

$$\frac{v_{i+1}}{D} = \frac{v_i}{D} + \frac{a}{D} \frac{\pi}{3} \sqrt{\frac{e}{1 + K_2}} r_x \frac{\phi_{i+1}^x}{\phi_y}$$  \hspace{1cm} (II.3.24)

$$\frac{u_{i+1}}{D} = \frac{u_i}{D} + \frac{a}{D} \frac{\pi}{3} \sqrt{\frac{e}{1 + K_2}} r_x \frac{\phi_{i+1}^y}{\phi_y}$$  \hspace{1cm} (II.3.25)
\[
\frac{\phi_i^x + 1}{\phi_y^x} = \left( \frac{\phi_i^x}{\phi_y^y} \right) - \frac{3}{\pi} \left( \frac{a}{r_x} \right) \sqrt{\gamma} \left( \frac{\phi_i^x}{\phi_y^y} \right) \tag{II.3.26}
\]

\[
\frac{\phi_i^y + 1}{\phi_y^y} = \left( \frac{\phi_i^y}{\phi_y^y} \right) - \frac{3}{\pi} \left( \frac{a}{r_x} \right) \sqrt{\gamma} \left( \frac{\phi_i^y}{\phi_y^y} \right) \tag{II.3.27}
\]

where:

\( \phi_y^x \) is the rotation caused by a moment, \( M_y^x \), acting at one end of a simply supported beam of length \( \pi r_x / \sqrt{\gamma} \). The \( \phi_y^x \) is chosen in such a way that the values of \( \phi_i^x / \phi_y^x \) and \( \phi_i^y / \phi_y^y \) are numbers varying from zero to unity or little greater (39). The length \( \pi r_x / \sqrt{\gamma} \) is determined by equating the Euler Load, \( P_E \), to the yield load, \( P_y \), in which \( P_E \) is equal to:

\[
P_E = \frac{\pi^2 EI_y}{L^2} \tag{II.3.28}
\]

At any point \( i \) on the column deflection curve, the moments about the \( x \)- and \( y \)-axes can be calculated using the following equations:

\[
\frac{M_i^x}{M_y^x} = \frac{p}{P_y} \left( 1 + K_2 \right) \left( \frac{v_i}{r_x} \right) \left( \frac{P}{r_x} \right) \tag{II.3.29}
\]

\[
\frac{M_i^y}{M_y^x} = \frac{p}{P_y} \left( 1 + K_2 \right) \left( \frac{u_i}{r_x} \right) \left( \frac{D}{r_x} \right) \tag{II.3.30}
\]

The moments about the \( \xi \) and \( \zeta \) axes are found using Equations II.2.6 and II.2.7 in their non-dimensional forms. Hence,
\[ \frac{M_i^C}{M_i^X} = \left( \frac{M_i^X}{M_i^Y} \right) + \left( \frac{M_i^Y}{M_i^Y} \right) \beta_i \]  

**(II.3.31)**

\[ \frac{M_i^C}{M_i^Y} = \left( \frac{M_i^Y}{M_i^Y} \right) - \left( \frac{M_i^X}{M_i^Y} \right) \beta_i \]  

**(II.3.32)**

The curvatures \( \phi_i^X \) and \( \phi_i^C \) are determined from the moments \( M_i^C \) and \( M_i^Y \) by the procedure outlined in the first step (Section II.3.1), from which the curvatures about the x- and y- axes are calculated using the following equations:

\[ \frac{\phi_i^X}{\phi_i^Y} = \frac{1}{(1 + \beta_i)^2} \left\{ \frac{\phi_i^C}{\phi_i^X} \beta_i - \frac{\phi_i^C}{\phi_i^Y} \beta_i \right\} \]  

**(II.3.33)**

\[ \frac{\phi_i^Y}{\phi_i^X} = \frac{1}{(1 + \beta_i)^2} \left\{ \frac{\phi_i^C}{\phi_i^X} \beta_i + \frac{\phi_i^C}{\phi_i^Y} \right\} \]  

**(II.3.34)**

To construct the end-moment versus end-rotation curve for a specified value \( \gamma \), the mid-span point of the beam column, \( C \), has been selected to start the numerical integration process as \( \phi_C^X \) and \( \phi_C^Y \) are both equal to zero (Figure II.13). Taking advantage of the symmetry condition, only half of the column length is considered. The procedure is outlined in the following steps:
1. The displacements $v_C$ and $u_C$ are specified and the angle of twist, $\beta_C$, is assumed. At the beginning of the calculations, it is convenient to assume $\beta_C$ to be equal to zero. Later on, this value of $\beta_C$ will be corrected by trial and error.

2. The moments and curvatures are then calculated using Equations II.3.29 to II.3.34.

3. Deflections and rotations at the next panel point are found using Equations II.3.24 to II.3.27, where the subscript "i" refers to point C. The procedure continues to find the displacements at the next panel taking $i + 1$ to be $i$. Then the process is repeated to cover the full length of the column.

4. The ratio $\frac{M^y_A}{M^x_A}$ is calculated and compared to the specified value $\gamma$.

5. If the calculated ratio is equal to $\gamma$ plus or minus, a permissible error, the procedure is followed immediately by step No. 6. If not, $u_C$ is corrected, first by the method outlined in references (38, 47, 50).

6. The angle of twist $\beta_C$ is then calculated using Equation II.3.3.

7. The calculated angle of twist is used as the assumed one and the procedure is repeated from step No. 2 until the difference between the calculated angle of twist and the assumed one comes within a tolerable error. This would give a point on the end-moment versus end-rotation curve.
The value of $U_c$, $V_c$, and $P_c$ are incremented by trial and error until sufficient points on the end-moment end-rotation curve are obtained.

In solving the problem, the distribution of the residual stresses changes the mathematical expressions in steps one and two whereas those in step three remain unaltered. The mathematical expressions were derived for both the cases with and without residual stresses. However, only the case without residual stresses is solved in detail here while the expressions for the other case are presented for future reference.

All the numerical techniques of this work have been programmed in fortran IV, on the 360/50 IBM Computer at the University of Windsor. A flow chart for the first two steps is given in Figure II.16. The flow chart for the third step is indicated in Figure II.17.
II.4.1 Summary and discussion of the results.

Several factors affect the behaviour of biaxially loaded beam-columns. Among these are the cross-sectional dimensions, the slenderness ratio, \( \frac{L}{r_y} \), the ratio of the moments at each end, \( \gamma \), and the axial load \( P \). The consideration of all these factors would require the solution of an extremely large number of problems. Since the prime purpose of this analysis is to develop a method of solution, rather than presenting a numerous sets of interaction curves, only an 8 WF 31 beam-column is examined. The slenderness ratio, \( \frac{L}{r_y} \), is considered in the range from 20 to 140 and \( \frac{P}{P_y} \) from 0.2 to 0.8 with the values of \( \gamma \) 0.2, 0.4 and 0.6.

In the development of the column deflection curves, the panel length, \( a \), is taken with a maximum value of two and half times the radius
of gyration about the minor axis. The ratio \( \frac{\phi_x}{\phi_y} \) is taken in the range from 0 to 10.

The interaction curves, relating the axial force and the ultimate end moment, constructed in a non-dimensional form, for different values of \( \frac{L}{r_y} \) are shown in Figures II.18 to II.20. On the same curves, the results, obtained from the following Column Research Council (CRC) Equation (19), are plotted:

\[
\frac{P}{P_0} + \frac{P_{ex}}{S_x \sigma_y (1 - P/P_{ex})} + \frac{P_{ey}}{S_y \sigma_y (1 - P/P_{ey})} = 1.0
\]

\[\text{......... (II.4.1)}\]

where:

\[
\frac{P_0}{P_y} = 1.0 - \frac{\sigma_y}{4\pi^2E} \left( \frac{L}{r_y} \right)^2 ;
\]

\[
\frac{P_{ex}}{P_y} = \frac{\pi^2E}{\sigma_y} \left( \frac{1}{L/r_x} \right)^2 ;
\]

\[
\frac{P_{ey}}{P_y} = \frac{\pi^2E}{\sigma_y} \left( \frac{1}{L/r_y} \right)^2 ;
\]

Equation II.4.1 can be written as (19):

\[
\frac{P}{P_y} \left( \frac{1}{(P_0/P_y)} + \frac{m_x}{a_x} + \frac{m_y}{a_y} \right) = 1.0
\]

\[\text{......... (II.4.3)}\]

where the eccentricity ratios \( m_x \) and \( m_y \) are:

\[
m_x = \frac{e_y \cdot d}{2 \cdot r_x^2} ;
\]

\[
m_y = \frac{e_x \cdot b}{2 \cdot r_y^2} \quad \text{(II.4.4)}
\]
and the amplification factors $a_x$ and $a_y$ are:

$$a_x = 1 - \frac{P}{P_{ex}}$$

$$a_y = 1 - \frac{P}{P_{ey}}$$

(Galambos (19) proposed the interaction equation:

$$\frac{P}{P_y} = \frac{1.0}{(P_o/P_y)} + m_x + m_y$$

This equation contains the following unconservative assumptions:

1. The amplification factors $a_x$ and $a_y$ are assumed equal 1.0.

2. No reduction for lateral buckling is allowed for the component of the equation dealing with bending about the strong axis.

The interaction curves calculated by Equation II.4.6 are also shown in Figures II.18 to II.20. From these figures the following results are observed.

1a - For a constant value of the axial load $P$ and the slenderness ratio $\frac{L}{r_y}$, the ultimate end moment about the x-axis decreases, appreciably, with the increase in the ratio $\gamma$.

1b - For a constant value of $\gamma$ and $L/r_y$, the ultimate end moments decrease, considerably, with the increase in the value of $P$. 

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For a constant value of $\gamma$ and $P$, there is a significant decrease in the values of the ultimate end moments as the slenderness ratio, $L/r_y$, increases.

2. The comparison between the interaction curves, developed in this analysis, with the interaction curves of the CRC equation show that the latter equation is too conservative especially for short columns. This deviation is attributed to the fact that the CRC equation is based on elastic analysis whereas, for short columns a large part of the section yields before the ultimate carrying capacity of the column is reached.

3. The comparison between the interaction equation II.4.6 with the present analysis shows a better agreement. Despite the uncon­servative assumptions, Equation II.4.6 gives conservative values for the ultimate load.

II.4.2 Comparison with previous theoretical and experimental work

The following table shows the comparison between the ultimate strength predicted by the present analysis with those of the experimental works conducted in Germany and Russia (11, 33) and reported in reference (19).
<table>
<thead>
<tr>
<th>Specimen No.</th>
<th>Section size</th>
<th>( \frac{P}{P_y} )</th>
<th>Yield Stress Ksi</th>
<th>( \frac{L}{r_y} )</th>
<th>( \gamma )</th>
<th>Ultimate ( \frac{M_A}{M_y} ) Previous work</th>
<th>Ultimate ( \frac{M_A}{M_y} ) present work</th>
<th>% diff</th>
<th>Type of previous work</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>b = 16 cm, d = 16 cm</td>
<td>0.549</td>
<td>38.2</td>
<td>57</td>
<td>0.365</td>
<td>0.340</td>
<td>0.366</td>
<td>7.7</td>
<td>Experimental</td>
</tr>
<tr>
<td>2</td>
<td>t = 1.5 cm, w = 0.8 cm</td>
<td>0.390</td>
<td>37.2</td>
<td>57</td>
<td>0.365</td>
<td>0.480</td>
<td>0.474</td>
<td>1.33</td>
<td>(Ref. (33))</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0.214</td>
<td>37.4</td>
<td>114</td>
<td>0.365</td>
<td>0.338</td>
<td>0.350</td>
<td>3.5</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>b = 9.4 cm, d = 18 cm</td>
<td>0.521</td>
<td>35.6</td>
<td>50</td>
<td>0.285</td>
<td>0.260</td>
<td>0.280</td>
<td>7.8</td>
<td>Experimental</td>
</tr>
<tr>
<td>5</td>
<td>t = 0.6 cm, ( A_c = 30.6 \text{ cm}^2 )</td>
<td>0.343</td>
<td>35.4</td>
<td>100</td>
<td>0.285</td>
<td>0.170</td>
<td>0.180</td>
<td>5.9</td>
<td>(Ref. (11))</td>
</tr>
<tr>
<td>6</td>
<td>5WF 18:5</td>
<td>0.280</td>
<td>35.7</td>
<td>75</td>
<td>0.50</td>
<td>0.495</td>
<td>0.560</td>
<td>13.2</td>
<td>Theoretical</td>
</tr>
<tr>
<td>7</td>
<td>5WF 18.5</td>
<td>0.326</td>
<td>35.7</td>
<td>94</td>
<td>0.305</td>
<td>0.482</td>
<td>0.525</td>
<td>9.0</td>
<td>(Ref. (5))</td>
</tr>
</tbody>
</table>

This table also shows a comparison between the present solution with the theoretical results of reference (6). In view of the shortage of computer time, a limited number of comparisons are made.

The comparison exhibit a reasonable agreement between these results. A better agreement is observed when the slenderness ratio \( \frac{L}{r_y} \) increases, since the effect of the residual stresses, which are neglected in this analysis, is predominant for columns with smaller \( \frac{L}{r_y} \) ratios.

The discrepancy between the results of the present solution with those of the experimental work may be attributed to the same factors that have been mentioned before for the case of uniaxially loaded beam-columns (Section I.4.2).
CHAPTER 5

CONCLUSIONS

From this study, the following conclusions can be drawn:

1. The present procedure for obtaining the ultimate carrying capacity of biaxially loaded beam-columns, developed primarily for wide flange sections, provides a simplified and relatively fast method for predicting the ultimate strength of these members. It can be extended to cover other cross-sections such as angles, channels, T-sections, etc. In general, the method is applicable to any member whose stress-strain curve can be idealized into a number of straight lines of different slopes.

This analysis, which is restricted to prismatic columns with identical loading conditions at the ends, could, with some modifications, be applied to different end loading conditions.
2. The present analysis, in which the residual stresses are neglected, tends to overestimate the ultimate load. This can be shown from the comparison between the predicted strengths with the test results. By introducing the residual stresses, the disagreement between the theoretical and experimental values is expected to be very small and the method can then be considered very reliable for practical purposes.

3. It is confirmed in this analysis that:
   a) The increase in the axial load $P$ and/or the slenderness ratio $\frac{L}{r_\gamma}$ will appreciably reduce the ultimate end moments.
   b) An increase in the ratio $\gamma$ will considerably decrease the value of the end-moment about the major axis at collapse.

4. The CRC interaction equation provides too conservative values for the ultimate load and this becomes more evident with the decrease in the slenderness ratio $\frac{L}{r_\gamma}$.

5. A set of interaction curves could be developed for most of the broad flange sections as well as for other sections that can be used for design purposes.

6. The development of a new interaction equation, that can encounter the plasticity of the section as well as the residual stresses, is recommended.
REFERENCES


41. Miranda, Constancio, "The Occurrence of Lateral-Torsional Buckling in Beam-Columns", thesis presented to the Ohio State University, at Columbus, Ohio in 1964, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.


TABLES
Table 1: The Coefficients $Q$, $R$ and $S$ of the quadratic equation 1.3.7 for the different strain configurations (Uniaxial loading considering residual stresses)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$Q$</th>
<th>$R$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.6$</td>
<td>$-1$</td>
<td>$P^{**}$</td>
</tr>
<tr>
<td>2</td>
<td>$A/E_e(1+\kappa)$</td>
<td>$2A \left[ (\varepsilon_e^*+1)/E_e(1+\kappa) \right] - 1$</td>
<td>$A \left[ (\varepsilon_e^*+1)^2/E_e(1+\kappa) \right] + P$</td>
</tr>
<tr>
<td>3</td>
<td>$B/2\varepsilon_e$</td>
<td>$2A + B \left( \varepsilon_e^* + 1 \right) /\varepsilon_e - 1$</td>
<td>$A \left( \varepsilon_e^* + \varepsilon_e^* + 2 \right) + B \left( \varepsilon_e^* + 1 \right)^2 /2\varepsilon_e + P$</td>
</tr>
<tr>
<td>4</td>
<td>$B/2\varepsilon_s$</td>
<td>$2A \left[ 2(\varepsilon_s^*+1)/E_s(1+\kappa) \right]$</td>
<td>$A \left[ \left{ (\varepsilon_s^<em>+1)^2 - \varepsilon_s^</em> - 1 \right} /E_s(1+\kappa) \right] +$ $+ B \left( \varepsilon_s - 1 \right)^2 /2\varepsilon_s + P$</td>
</tr>
</tbody>
</table>
Table 1 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>(-A/\varepsilon_c(1+\kappa_0))</td>
<td>2A - 2B ((\tilde{\varepsilon}_s-1)/\varepsilon_c (1+\kappa_0) + 2B ((\tilde{\varepsilon}_s-1)/\tilde{\varepsilon}_s-1)</td>
<td>A ((\tilde{\varepsilon}_3 + \tilde{\varepsilon}_c + 2) - A ((\tilde{\varepsilon}_s-1)^3) /\varepsilon_c (1+\kappa_0) + B[(\tilde{\varepsilon}_s-1) - ((\tilde{\varepsilon}_c+1))^2] /2\tilde{\varepsilon}_s + P</td>
</tr>
<tr>
<td>6</td>
<td>0.0</td>
<td>4A + 2B ((\tilde{\varepsilon}_s-1)/\tilde{\varepsilon}_s-1)</td>
<td>2A [\varepsilon_c (1-\kappa_0)] + B [(\tilde{\varepsilon}_3 - 1) - ((\tilde{\varepsilon}_c+1))^2] /2\tilde{\varepsilon}_s + P</td>
</tr>
<tr>
<td>7</td>
<td>8/2\varepsilon_c - 1/A/\varepsilon_c(1+\kappa_0)</td>
<td>2A + B ((\tilde{\varepsilon}_c+1)) /\varepsilon_c - 2A_1 ((\tilde{\varepsilon}<em>c + \varepsilon</em>{st}))/\varepsilon_c (1+\kappa_0)</td>
<td>A ((\tilde{\varepsilon}_3 + \tilde{\varepsilon}_c + 2) + B ((\tilde{\varepsilon}_c+1))^2) /2\varepsilon_c - A_1 [((\tilde{\varepsilon}<em>3 + \varepsilon</em>{st}))^2 + B_2 ((\tilde{\varepsilon}<em>3 + \varepsilon</em>{st}))/\varepsilon_c (1+\kappa_0)] + P</td>
</tr>
<tr>
<td>8</td>
<td>(8 - B) /2\varepsilon_c</td>
<td>2(A - A_1) + B ((\tilde{\varepsilon}_c+1)) - B_2 ((\tilde{\varepsilon}<em>c + \varepsilon</em>{st}))/\varepsilon_c - 1</td>
<td>(A - A_1) ((\tilde{\varepsilon}_3 + \tilde{\varepsilon}<em>c)) + 2(A - A_1 \varepsilon</em>{st}) + B ((\tilde{\varepsilon}_c+1))^2 - B_2 ((\tilde{\varepsilon}<em>c + \varepsilon</em>{st})^2) /2\varepsilon_c + P</td>
</tr>
<tr>
<td>Case</td>
<td>Q</td>
<td>R</td>
<td>S</td>
</tr>
<tr>
<td>------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>9</td>
<td>(1 + A) / (1 + mn)</td>
<td>(cE^* + cE_0) / (cE^* - 1)</td>
<td>2A - [2A (cE^* - 1) + 2A_c] / (cE^* + cE_0) + 2A_c (cE_0) + [cE^* - 1]</td>
</tr>
<tr>
<td>10</td>
<td>- A / cE_0 (1 + mn)</td>
<td>- 2A / (cE^* - 1) / E_0</td>
<td>2A + 2B (cE^* - 1) / E_0</td>
</tr>
<tr>
<td>11</td>
<td>- A / cE_0 (1 + mn)</td>
<td>- 2A / (cE^* - 1) / E_0</td>
<td>2A + 2B (cE^* - 1) / E_0</td>
</tr>
</tbody>
</table>

Table 1 (Cont'd.)
<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>(- \frac{B}{2\bar{E}_5})</td>
<td>(4A + 2B \left( \bar{E}_5 - 1 \right) / \bar{E}_5 - \frac{2A}{1 + \eta_0} \left[ (\bar{E}<em>3 + \bar{E}</em>{st}) / \bar{E}_c (1 + \eta_0) \right] - 2A \left[ (\bar{E}<em>3 + \bar{E}</em>{st}) / \bar{E}_c (1 + \eta_0) + 1 \right] - 2B, (\bar{E}<em>3 - \bar{E}</em>{st}) / \bar{E}_5 - 1 - 2A \left[ (\bar{E}<em>3 + \bar{E}</em>{st}) / \bar{E}_c (1 + \eta_0) - B, (\bar{E}<em>3 - \bar{E}</em>{st}) / \bar{E}_5 + \bar{P} \right] )</td>
<td>(2A \left[ \bar{E}_c (1 - \eta_0) \right] + B \left[ (\bar{E}_3 - 1)^2 - (\bar{E}_3 + 1)^2 \right] / 2 \bar{E}_5 - A \left[ (\bar{E}<em>3 + \bar{E}</em>{st})^2 - (\bar{E}<em>3 - \bar{E}</em>{st})^2 / (\bar{E}<em>5 - \bar{E}</em>{st}) \right] / 2 \bar{E}_5 + \bar{P} )</td>
</tr>
<tr>
<td>13</td>
<td>(- \frac{B}{2\bar{E}_6})</td>
<td>(2(2A - A_c) + 2B \left( \bar{E}_5 - 1 \right) / \bar{E}_5 - B, (\bar{E}<em>6 + \bar{E}</em>{st}) / \bar{E}_6 - 1 )</td>
<td>(2A \left[ \bar{E}_c (1 - \eta_0) \right] - A \left[ (\bar{E}_3 + \bar{E}_6) + (\bar{E}<em>3 + \bar{E}</em>{st}) \right] + 2) + B \left[ (\bar{E}_5 - 1)^2 - (\bar{E}_5 + 1)^2 \right] / 2 \bar{E}_5 - B, (\bar{E}<em>5 + \bar{E}</em>{st}) / 2 \bar{E}_6 + \bar{P} )</td>
</tr>
<tr>
<td>14</td>
<td>(A / \bar{E}_c (1 + \eta_0))</td>
<td>(2(2A - A_c) + 2B \left( \bar{E}_5 - 1 \right) / \bar{E}_5 - 2B, (\bar{E}<em>5 - \bar{E}</em>{st}) / \bar{E}_5 + 2A \left[ (\bar{E}<em>3 - \bar{E}</em>{st}) / \bar{E}_c (1 + \eta_0) - \bar{E}_5 + 2A \left[ (\bar{E}<em>3 - \bar{E}</em>{st}) / \bar{E}_c (1 + \eta_0) - 1 \right] \right] / 2 \bar{E}_5 - B, (\bar{E}<em>5 - \bar{E}</em>{st})^2 / (\bar{E}<em>5 + \bar{E}</em>{st})^3 / 2 \bar{E}_5 + \bar{P} )</td>
<td>(2A \left[ \bar{E}_c (1 - \eta_0) \right] - A \left[ (\bar{E}_3 + \bar{E}_6) + (\bar{E}<em>3 + \bar{E}</em>{st}) \right] + \bar{P} - A \left[ (\bar{E}<em>3 - \bar{E}</em>{st}) / \bar{E}_c (1 + \eta_0) + B \left[ (\bar{E}_5 - 1)^2 - (\bar{E}_5 + 1)^2 \right] / 2 \bar{E}_5 - B, (\bar{E}<em>5 - \bar{E}</em>{st})^2 - (\bar{E}<em>5 + \bar{E}</em>{st})^3 / 2 \bar{E}_5 + \bar{P} )</td>
</tr>
</tbody>
</table>
Table 1 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
</table>
| 15    | 0.0                | $4 \left( A - A_1 \right) + 2B \left( \bar{E}_5 - 1 \right)$ | $2A \left[ E_{rc} \left( 1 - \kappa_0 \right) \right] - A_1 \left( \bar{E}_3^{+} + \bar{E}_6^{+} + 2 \right)$  
  $- A_1 \left( \bar{E}_5^* - \varepsilon_{st} \right)^2 / E_{rc} (1 + \kappa_0) + B \left[ \left( \bar{E}_5 - 1 \right) - (\bar{E}_6^* + 1)^2 \right] / 2 \bar{E}_5 - B_1 \left[ \left( \bar{E}_5 - \varepsilon_{st} \right) - \left( \bar{E}_6^* + \varepsilon_{st} \right) \right] / 2 \bar{E}_5$ |
| 16    | $(A - A_1)/E_{rc} (1 + \kappa_0)$ | $2 \left[ A \left( \bar{E}_3^{+} + 1 \right) - A_1 \left( \bar{E}_3^{+} + \varepsilon_{st} \right) \right] / E_{rc} (1 + \kappa_0) - 1$ | $\left[ A \left( \bar{E}_3^{+} + 1 \right) - A_1 \left( \bar{E}_3^{+} + \varepsilon_{st} \right) \right] / E_{rc} (1 + \kappa_0) + P$ |

Where:

$A = K_1 K_2 / 2 \left( K_3 + 2 K_1 K_2 \right)$, $A_1 = \left( \varepsilon_{st} / E \right) A$, $K_0 = K_1 K_2 / (K_1 K_2 + K_3)$

$B = K_3 / 2 \left( K_3 + 2 K_1 K_2 \right)$, $B_1 = \left( \varepsilon_{st} / E \right) B$

* Numbers refer to Figure I.9.

** P is in terms of $P_y$. \n
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Table 2: The Coefficients Q, R and S of the quadratic equation I.3.7 for the different strain configurations (Uniaxial loading neglecting residual stresses)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>-1</td>
<td>$P^{**}$</td>
</tr>
<tr>
<td>2</td>
<td>$8/2\tilde{e}_3$</td>
<td>$2A + B(\tilde{e}_3+1)/\tilde{e}_3 - 1$</td>
<td>$2A(\tilde{e}_3+1) + B(\tilde{e}_3+1)^{3/2}\tilde{e}_3 + P$</td>
</tr>
<tr>
<td>3</td>
<td>0.0</td>
<td>$4A + 2B(\tilde{e}_3+1)/\tilde{e}_3 - 1$</td>
<td>$P$</td>
</tr>
<tr>
<td>4</td>
<td>$(B-B_1)/2\tilde{e}_3$</td>
<td>$2(A-A_1) + (B-B_1)\times$ $\tilde{e}_3 + 1$</td>
<td>$2(A-A_1)(\tilde{e}_3+1) + (B-B_1)(\tilde{e}_3+1)^{3/2}\tilde{e}_3 + P$</td>
</tr>
<tr>
<td>5</td>
<td>$-B_1/2\tilde{e}_3$</td>
<td>$2(2A-A_1) + (2B-B_1)\times$ $\tilde{e}_3 + 1$</td>
<td>$-2A_1(\tilde{e}_3+1) - B_1(\tilde{e}_3+1)^{3/2}\tilde{e}_3 + P$</td>
</tr>
</tbody>
</table>
Table 2 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td></td>
<td>$4(A-A_1) + z(B-B_1) + \left(\bar{E}_0 + 1\right) / \bar{E}_0 - 1$</td>
<td>$P$</td>
</tr>
</tbody>
</table>

* Numbers refer to Figure I.10.
** $P$ is in terms of $P_Y$

The expressions for $A, B, A_1, B_1$, are shown in Table 1.
Table 3: Equations for $M^X/M^Y$ for the different strain configurations
(Uniaxial loading considering residual stresses)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$M^X/M^Y**$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\phi^x$</td>
</tr>
<tr>
<td>2</td>
<td>$\phi^x + F_1 \alpha_1 (\varepsilon_3 + 1)$</td>
</tr>
<tr>
<td>3</td>
<td>$\phi^x + F_1 (\varepsilon_3 + \varepsilon + 2) + F_2 \varepsilon_1 (3 - 2 \varepsilon_1)(\varepsilon + 1)/3$</td>
</tr>
<tr>
<td>4</td>
<td>$\phi^x + F_1 \alpha_1 (\varepsilon_3 + 1) - F_1 \psi_1 (\varepsilon_5 - 1) - F_2 \varepsilon_2 (3 - 2 \varepsilon_2)(\varepsilon_5 - 1)/3$</td>
</tr>
<tr>
<td>5</td>
<td>$\phi^x + F_1 [\varepsilon_3 + \varepsilon + 2 - \psi_1 (\varepsilon_5 - 1)] + F_2 \varepsilon_1 (3 - 2 \varepsilon_1) \times (\varepsilon_3 + 1) - \varepsilon_2 (3 - 2 \varepsilon_2)(\varepsilon_5 - 1)]/3$</td>
</tr>
</tbody>
</table>
Table 3 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$M_x/M_y^{**}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$\phi^x + f_1 \left( \varepsilon_3 + \varepsilon_4 - \varepsilon_1 - \varepsilon_5 + 4 \right) + f_2 \left[ \varepsilon_1 (3 - 2 \delta_1)(\varepsilon_3 + 1) - \varepsilon_2 (3 - 2 \delta_2)(\varepsilon_5 - 1) \right]/3$</td>
</tr>
<tr>
<td>7</td>
<td>$\phi^x + f_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 - \left( \frac{E_{st}}{E} \right) \delta_2 (\varepsilon_3 + \varepsilon_{st}) \right] + f_2 \delta_1 (3 - 2 \delta_1)x (\varepsilon_5 + 1)/3$</td>
</tr>
<tr>
<td>8</td>
<td>$\phi^x + f_1 \left[ \left( 1 - \frac{E_{st}}{E} \right) (\varepsilon_3 + \varepsilon_4) + 2 \left( 1 - \left( \frac{E_{st}}{E} \varepsilon_{st} \right) \right) \right] + f_2 \left[ \delta_1 x (3 - 2 \delta_1)(\varepsilon_5 + 1) - \left( \frac{E_{st}}{E} \right) \delta_3 (3 - 2 \delta_3)(\varepsilon_5 + \varepsilon_{st}) \right]/3$</td>
</tr>
<tr>
<td>9</td>
<td>$\phi^x + f_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 - \delta_1 (\varepsilon_5 - 1) - \delta_2 \left( \varepsilon_3 + \varepsilon_{st} \right) \right] + f_2 \left[ \delta_1 (3 - 2 \delta_1)(\varepsilon_5 + 1) - \delta_2 (3 - 2 \delta_2)(\varepsilon_5 - 1) \right]/3$</td>
</tr>
</tbody>
</table>
Table 3 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$M^X/M^Y$ **</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$\phi^X + F_1 \left[ \left( 1 - \frac{E_{st}}{E} \right) (\varepsilon_3 + \varepsilon_4) + 2 \left{ 1 - \left( \frac{E_{st}}{E} \right) \varepsilon_{st} \right} - \Psi_y (\varepsilon_5 - 1) \right] +$</td>
</tr>
<tr>
<td></td>
<td>$+ F_2 \left[ \delta_1 (3 - 2 \delta_1) (\varepsilon_3 + 1) - \delta_2 (3 - 2 \delta_2) (\varepsilon_5 - 1) - \left( \frac{E_{st}}{E} \right) \delta_3 \right.$</td>
</tr>
<tr>
<td></td>
<td>$\left( \varepsilon_4 + \varepsilon_{st} \right) / 3$</td>
</tr>
<tr>
<td>11</td>
<td>$\phi^X + F_1 \left[ \varepsilon_3 + \varepsilon_4 - \varepsilon_1 - \varepsilon_5 + 4 - \left( \frac{E_{st}}{E} \right) \alpha_2 \left( \varepsilon_3 + \varepsilon_{st} \right) +$</td>
</tr>
<tr>
<td></td>
<td>$+ F_2 \left[ \delta_1 (3 - 2 \delta_1) (\varepsilon_3 + 1) - \delta_2 (3 - 2 \delta_2) (\varepsilon_5 - 1) \right] / 3$</td>
</tr>
<tr>
<td>12</td>
<td>$\phi^X + F_1 \left[ \varepsilon_3 + \varepsilon_4 - \varepsilon_1 - \varepsilon_5 + 4 - \left( \frac{E_{st}}{E} \right) \left{ \alpha_2 \left( \varepsilon_3 + \varepsilon_{st} \right) - \Psi_y (\varepsilon_5 - \varepsilon_{st} \right} \right] + F_2 \left[ \delta_1 (3 - 2 \delta_1) (\varepsilon_3 + 1) - \delta_2 (3 - 2 \delta_2) (\varepsilon_5 - 1) + \left( \frac{E_{st}}{E} \right) \delta_4 \right.$</td>
</tr>
<tr>
<td></td>
<td>$\left( 3 - 2 \delta_4 \right) (\varepsilon_5 - \varepsilon_{st}) \right] / 3$</td>
</tr>
<tr>
<td>Case</td>
<td>Expression 1</td>
</tr>
<tr>
<td>------</td>
<td>--------------</td>
</tr>
<tr>
<td>13</td>
<td>$\phi^* + F_1 \left[ \epsilon_3 + \epsilon_2 - \epsilon_1 - \epsilon_5 + \frac{4}{E} \left( \epsilon_3 + \epsilon_2 + 2\epsilon_5 \right) \right] + \frac{1}{E} \left[ \frac{1}{E} \left( \epsilon_3 + \epsilon_2 + 2\epsilon_5 \right) \right]^3 \left( 3 - 2\gamma_2 \right) \left( \gamma_3 + \gamma_2 \right) + \frac{1}{E} \left[ \frac{1}{E} \left( \epsilon_3 + \epsilon_2 + 2\epsilon_5 \right) \right]^3 \left( 3 - 2\gamma_2 \right) \left( \gamma_3 + \gamma_2 \right) $</td>
</tr>
</tbody>
</table>

Table 3 (cont'd.)

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Table 3 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$M^x/M^y$ **</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>$\phi^x + F_1 \alpha_1 (\varepsilon_3 + 1) - \left( \frac{\varepsilon_{st}}{2} \right) F_2 \alpha_2 (\varepsilon_3 + \varepsilon_{st})$</td>
</tr>
</tbody>
</table>

where:

$F_1 = \left( 1 + K_2 \right) \left( 1 + K_2 / 2 \right) K_1 K_2 / \left[ \left( 2/3 \right) K_3 + 4 K_1 K_2 \left( 1 + K_2 / 2 \right)^2 \right]$

$F_2 = \left( 1 + K_2 \right) K_3 / \left[ \left( 2/3 \right) K_3 + 4 K_1 K_2 \left( 1 + K_2 / 2 \right)^2 \right]$

$\alpha_1 = (\varepsilon_3 + 1) / (\varepsilon_3 - \varepsilon_0)$

$\alpha_2 = (\varepsilon_3 + \varepsilon_{st}) / (\varepsilon_3 - \varepsilon_0)$

$\gamma_1 = (\varepsilon_3 + 1) / (\varepsilon_3 - \varepsilon_0)$

$\gamma_2 = (\varepsilon_3 + \varepsilon_{st}) / (\varepsilon_3 - \varepsilon_0)$

$\phi^x = (\varepsilon_3 - \varepsilon_1) / (\varepsilon_3 - \varepsilon_0)$

$\phi_y = (\varepsilon_3 - \varepsilon_1) / (\varepsilon_3 - \varepsilon_0)$

* Numbers refer to Figure I.9.

** All curvatures are in terms of $\phi^x y$.

*** Parameters refer to Figure I.11.
Table 4: Equations for \( \frac{M^x}{M^x_{y}} \) for the different strain configurations
(Uniaxial loading neglecting residual stresses)

<table>
<thead>
<tr>
<th>Case*</th>
<th>( \frac{M^x}{M^x_{y}} )**</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \phi^x )</td>
</tr>
<tr>
<td>2</td>
<td>( \phi^x + 2 \frac{f_1}{2} (\varepsilon_3 + 1) + \frac{f_2}{2} \left{ \delta_1 (3 - 2 \delta_1) (\varepsilon_3 + 1) - \delta_2 (3 - 2 \delta_2) (\varepsilon_1 - 1) \right} / 3 )</td>
</tr>
<tr>
<td>3</td>
<td>( \phi^x + 4 \frac{f_1}{2} (\varepsilon_3 + 1) + \frac{f_2}{2} \left{ \delta_1 (3 - 2 \delta_1) (\varepsilon_3 + 1) - \delta_2 (3 - 2 \delta_2) (\varepsilon_1 - 1) \right} / 3 )</td>
</tr>
<tr>
<td>4</td>
<td>( \phi^x + 2 \frac{f_1}{2} \left[ \varepsilon_3 + 1 - \left( \frac{E_{st}}{E} \right) (\varepsilon_3 + \varepsilon_{st}) \right] + \frac{f_2}{2} \left{ \delta_1 (3 - 2 \delta_1) (\varepsilon_3 + 1) - \left( \frac{E_{st}}{E} \right) \delta_3 (3 - 2 \delta_3) (\varepsilon_3 + \varepsilon_{st}) \right} / 3 )</td>
</tr>
<tr>
<td>5</td>
<td>( \phi^x + 2 \frac{f_1}{2} \left[ 2 \varepsilon_3 + 2 - \left( \frac{E_{st}}{E} \right) (\varepsilon_3 + \varepsilon_{st}) \right] + \frac{f_2}{2} \left{ \delta_1 (3 - 2 \delta_1) (\varepsilon_3 + 1) - \delta_2 (3 - 2 \delta_2) (\varepsilon_1 - 1) - \left( \frac{E_{st}}{E} \right) \delta_3 (3 - 2 \delta_3) (\varepsilon_3 + \varepsilon_{st}) \right} / 3 )</td>
</tr>
</tbody>
</table>
Table 4 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$M_x^y / M_x^y$ **</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$\phi^x + 4 \psi \left[ E_8 + 1 - \frac{E_5 + 1}{E} \right] + F_2 \left[ \psi (3 - 2 \psi) (E_5 + 1) - \psi (3 - 2 \psi) \right] \times (\psi - E) \left[ \psi_3 (3 - 2 \psi_3) (E_3 + E_5) - \psi_4 (3 - 2 \psi_5) (E_1 - E_5) \right] / 3$</td>
</tr>
</tbody>
</table>

* Numbers refer to Figure I.10.

** All Curvatures are in terms of $\phi^x$. 

*** Parameters refer to Figure I.11.
<table>
<thead>
<tr>
<th>Case*</th>
<th>By</th>
<th>By</th>
<th>By</th>
<th>By</th>
<th>By</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$EJ_{Y}$</td>
<td>$EJ_{Y} [1 + (1 - \alpha Z_{c})^{-3}] / 2$</td>
<td>$EJ_{Y} [1 + (1 - \alpha Z_{c})^{-3}] / 2$</td>
<td>$EJ_{Y} [1 + (1 - \alpha Z_{c})^{-3}] / 2$</td>
<td>$EJ_{Y} [1 + (1 - \alpha Z_{c})^{-3}] / 2$</td>
</tr>
<tr>
<td>2</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
</tr>
<tr>
<td>3</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
</tr>
<tr>
<td>4</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
</tr>
<tr>
<td>5</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
</tr>
<tr>
<td>6</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
<td>$EJ_{Y} / 2$</td>
</tr>
</tbody>
</table>
Table 5 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>By</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>$EI_y [1 + \left( \frac{E_{st}}{E} \right) {1 - (1 - \alpha_z)^3}]$</td>
</tr>
<tr>
<td>8</td>
<td>$EI_y [1 + \left( \frac{E_{st}}{E} \right)]/2$</td>
</tr>
<tr>
<td>9</td>
<td>$EI_y [1 - \psi_1^3 + \left( \frac{E_{st}}{E} \right) {1 - (1 - \alpha_z)^3}]/2 + E \kappa_3^3 D^4 \left[1 - (\xi_1 + \xi_2)\right]/6$</td>
</tr>
<tr>
<td>10</td>
<td>$EI_y [1 + \left( \frac{E_{st}}{E} \right) - \psi_1^3] / 2 + E \kappa_3^3 D^4 \left[1 - (\xi_1 + \xi_2)\right]/6 + E_{st} \kappa_3^3 D^4 \xi_3/6$</td>
</tr>
<tr>
<td>11</td>
<td>$E_{st} I_y [1 - (1 - \alpha_z)^3] / 2 + E \kappa_3^3 D^4 \left[1 - (\xi_1 + \xi_2)\right]/6$</td>
</tr>
<tr>
<td>12</td>
<td>$E_{st} I_y [1 + \psi_1^3 - (1 - \alpha_z)^3] / 2 + E \kappa_3^3 D^4 \left[1 - (\xi_1 + \xi_2)\right]/6 + E_{st} \kappa_3^3 D^4 \xi_4/6$</td>
</tr>
</tbody>
</table>
Table 5 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>By</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>$E_{st} I_y/2 + E K_3^D 3 D^4 [1 - (X_1 + X_2)]/6 + E_{st} X_3^4 D^4 \delta_3/6$</td>
</tr>
<tr>
<td>14</td>
<td>$E_{st} I_y (1 - Y_b^3)/2 + E K_3^D 3 D^4 [1 - (y_1 + y_2)]/6 + E_{st} K_3^D 4 (\delta_3 + \delta_4)/6$</td>
</tr>
<tr>
<td>15</td>
<td>$E_{st} I_y + E K_3^D 4 D^4 [1 - (y_1 + y_2)]/6 + E_{st} K_3^D 4 (\delta_3 + \delta_4)/6$</td>
</tr>
<tr>
<td>16</td>
<td>$E I_y [1 + (1 - \alpha_1)^3 + (E_{st}/E) {1 - (1 - \alpha_2)^3}]$</td>
</tr>
</tbody>
</table>

* Numbers refer to Figure I.9.

** Parameters refer to Figure I.11 and their expressions are shown in Table 3.
Table 6: Equations for $B_y$ for the different strain configurations
(Uniaxial loading neglecting residual stresses)

<table>
<thead>
<tr>
<th>Case *</th>
<th>$B_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$EI_y$</td>
</tr>
<tr>
<td>2</td>
<td>$EI_y/z$</td>
</tr>
<tr>
<td>3</td>
<td>$E K_3^3 D^4 / 2 \bar{e}_1$</td>
</tr>
<tr>
<td>4</td>
<td>$EI_y \left[ 1 + \left( \frac{E_{st}}{E} \right) \right] / 2$</td>
</tr>
<tr>
<td>5</td>
<td>$E_{st} I_y / 2 + E K_3^3 D^4 \left[ 1 - \left( \bar{e}_1 + \bar{e}<em>2 \right) \right] / 16 + E</em>{st} K_3^3 D^4 \bar{e}_3 / 16$</td>
</tr>
</tbody>
</table>
Table 6 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>By</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$E_s I_y + EM_3^3 D^4 \left[1 - (\delta_1 + \delta_2)\right]/6 + E_s M_3 D^4 (\delta_3 + \delta_4)/6$</td>
</tr>
</tbody>
</table>

* Numbers refer to Figure I.10.

** Parameters refer to Figure I.11 and their expressions are shown in Table 3.
Table 7: Equations for \( C_W \) for the different strain configurations
(Uniaxial loading considering residual stresses)

<table>
<thead>
<tr>
<th>Case*</th>
<th>( C_W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( EI_y D^2 \frac{(2 + \kappa_z)^3}{4} )</td>
</tr>
</tbody>
</table>
| 2     | The larger value of (a) and (b) \[
\frac{ED(2 + \kappa_z)^3 I_y}{2} \left(1 - \alpha_1^3\right)^3 \right] \left[1 + \left(1 - \alpha_1^3\right)^3\right] \] (a) \[
ER^3 \kappa_2^3 D^6/18 + 2 ER^3 D^5/9 + ER^3 \kappa_2^3 D^6 (1 - \alpha_1^3) / 18 \] (b) |
| 3     | \[
ER^3 \kappa_2^3 D^6/18 + 2 ER^3 D^5 (1 - \delta_z^3) / 9 \] |
| 4     | The larger value of (a) and (b) \[
\frac{ED^2 (2 + \kappa_z)^3 I_y}{2} \left(1 - \psi_1^3\right) (1 - \alpha_1^3)^3 \right] \left[1 + \left(1 - \alpha_1^3\right) - \psi_1^3\right] \] (a) \[
ER^3 \kappa_2^3 D^6 (1 - \psi_1^3) / 18 + 2 ER^3 D^5 (1 - \delta_z^3) / 9 + ER^3 \kappa_2^3 D^6 (1 - \alpha_1^3) / 18 \] (b) |
Table 7 (Cont'd.)

<table>
<thead>
<tr>
<th>Case *</th>
<th>( C_W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( E K_1^3 K_z D^6 (1 - \eta_1^3) /18 + 2 E K_3^3 D^6 \left[ 1 - (\eta_1 + \eta_2) \right]^3 / q )</td>
</tr>
<tr>
<td>6</td>
<td>( 2 E K_3^3 D^6 \left[ 1 - (\eta_1 + \eta_2) \right]^3 / q )</td>
</tr>
<tr>
<td>7</td>
<td>The larger value of (a) and (b) ( E_{st} D^2 \left( 2 + K_z^2 \right) I_y \left[ 1 - (1 - \alpha_2)^3 \right] / 2 \left{ 1 + \left( \frac{E_{st}}{E} \right) \left[ 1 - (1 - \alpha_2)^3 \right] \right} ) (a) or ( E K_1^3 K_z^3 D^6 / 18 + 2 E K_3^3 D^6 (1 - \eta_1)^3 / q + E_{st} K_1^3 K_z^3 D^6 \left[ 1 - (1 - \alpha_2)^3 \right] / 18 ) (b)</td>
</tr>
<tr>
<td>8</td>
<td>( E_{st} D^2 \left( 2 + K_z^2 \right) I_y / 2 \left[ 1 + \left( \frac{E_{st}}{E} \right) \right] )</td>
</tr>
<tr>
<td>Case*</td>
<td>( C_W )</td>
</tr>
<tr>
<td>-------</td>
<td>------------------</td>
</tr>
</tbody>
</table>
| 9     | The larger value of (a) and (b)  
\[
E_{st} D^2 (2 + \alpha_e) \gamma \left[ 1 - (1 - \alpha_e)^3 \right] \left( 1 - \psi_1^3 \right)/2 \left\{ 1 - \psi_1^3 + \left( \frac{E_t}{E} \right) \left[ 1 - (1 - \alpha_e)^3 \right] \right\} 
\]  
\[
E K_i^3 K_e^3 D^6 \left[ 1 - (\psi_1 + \delta_e) \right]^3/9 + E_{st} K_i^3 K_e^3 D^6 \left[ 1 - (1 - \alpha_e)^3 \right]/18 
\] |
| 10    | The larger value of (a) and (b)  
\[
E_{st} D^2 (2 + \alpha_e) \gamma \left[ 1 - (1 - \alpha_e)^3 \right] \left( 1 - \psi_1^3 \right)/2 \left\{ 1 - \psi_1^3 + \left( \frac{E_t}{E} \right) \right\} 
\]  
\[
E K_i^3 K_e^3 D^6 \left[ 1 - (\psi_1 + \delta_e) \right]^3/9 + E_{st} K_i^3 K_e^3 D^6 \left[ 1 - (1 - \alpha_e)^3 \right]/18 
\] |
|       | (neglecting the strain hardening of the web) |
| 11    | \[
E_{st} K_i^3 K_e^3 D^6 \left[ 1 - (1 - \alpha_e)^3 \right]/18 + 2 E K_i^3 D^6 \left[ 1 - (\psi_1 + \delta_e) \right]^3/9 
\] |
<table>
<thead>
<tr>
<th>Case</th>
<th>( C_W )</th>
</tr>
</thead>
</table>
| 12   | The larger value of (a) and (b)  
\[
\frac{E_{st} D^2 (2+K_s)^2 I_y \left[ 1 - (1 - \alpha_x)^3 \right] \psi_x^3 / 2 \left[ 1 + \psi_x^3 - (1 - \alpha_x)^3 \right]}{2 + 2 E K_s^3 D^6 \left[ 1 - (\gamma_1 + \gamma_2) \right]^3 / 9}
\]  
(neglecting the strain hardening of the web) |
| 13   |  
\[
\frac{E_{st} K_1^3 K_2^3 D^6 / 18 + 2 E K_3^3 D^6 \left[ 1 - (\gamma_1 + \gamma_2) \right]^3 / 9}{2 + 2 E K_s^3 D^6 \left[ 1 - (\gamma_1 + \gamma_2) \right]^3 / 9}
\]  
(neglecting the strain hardening of the web) |
| 14   | The larger value of (a) and (b)  
\[
\frac{E_{st} D^2 (2+K_s)^2 I_y \psi_x^3 / 2 \left[ 1 + \psi_x^3 \right]}{2 + 2 E K_s^3 D^6 \left[ 1 - (\gamma_1 + \gamma_2) \right]^3 / 9}
\]  
(neglecting the strain hardening of the web) |
Table 7 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$C_W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>$E_{st} D^2 (2 + n^2)^2 I_x / 4$</td>
</tr>
<tr>
<td>16</td>
<td>$E D^2 (2 + n^2)^2 I_x [ (1 - \alpha_1)^3 + (E_{st}) [1 - (1 - \alpha_2) \frac{3}{2}] / 2 [1 + (1 - \alpha_1)^3 + (E_{st}) [1 - (1 - \alpha_2) \frac{3}{2}]]$</td>
</tr>
</tbody>
</table>

* Numbers refer to Figure I.9.

** Parameters refer to Figure I.11 and their expressions are shown in Table 3.
Table 8: Equations for $C_W$ for the different strain configurations
(Uniaxial loading neglecting residual stresses)

<table>
<thead>
<tr>
<th>Case *</th>
<th>$C_W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$EI_y D^2 (2 + \kappa_z)^2 / 4$</td>
</tr>
<tr>
<td>2</td>
<td>$E K_1 K_2 D^6 / 18 + E K_3 D^6 \left[ (\varepsilon_1 + \Delta) / \varepsilon_1 + \kappa_z / 2 \right]^3$</td>
</tr>
<tr>
<td>3</td>
<td>$2 E K_3 D^6 / 9 \varepsilon_1^3$</td>
</tr>
<tr>
<td>4</td>
<td>$E_{st} D^2 (2 + \kappa_z)^2 I_y / 2 \left[ 1 + \left( \frac{E_{st}}{E} \right) \right]$</td>
</tr>
</tbody>
</table>
| 5      | $E_{st} K_3 D^6 / 18 + 2 E K_3 D^6 \left[ 1 - (\varepsilon_1 + \varepsilon_2) \right]^3 / 9$

(neglecting the strain hardening of the web)
Table 8 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$c_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$E_{st} D^2 (2 + \kappa_z)^2 I y / 4$</td>
</tr>
</tbody>
</table>

* Numbers refer to Figure I.10.

** Parameters refer to Figure I.11 and their expressions are shown in Table 3.
Table 9: Equations for $\gamma_o$ for the different strain configurations
(Uniaxial loading considering residual stresses)

<table>
<thead>
<tr>
<th>Case *</th>
<th>$\gamma_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>$D (2 + \kappa_2) \left[ \frac{1}{2} - \frac{\left(1 - \alpha_1^3\right)}{\left(1 + \alpha_1^3\right)} \right]$</td>
</tr>
<tr>
<td>3</td>
<td>$D \left(1 + \kappa_2/2\right)$</td>
</tr>
<tr>
<td>4</td>
<td>$D (2 + \kappa_2) \left[ \frac{1}{2} - \frac{\left(1 - \alpha_1^3\right)}{\left(1 + \alpha_1^3 - \psi_1^3\right)} \right]$</td>
</tr>
<tr>
<td>5</td>
<td>$D \left(1 + \kappa_2/2\right)$</td>
</tr>
<tr>
<td>6</td>
<td>$D \left(\gamma_1 - \gamma_2\right)$</td>
</tr>
<tr>
<td>Case</td>
<td>$v_0$</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td></td>
<td>$D(2+\alpha_2) [\sqrt[2]{2} - (\xi_{0z} + \frac{1}{2}) [1 - (\alpha_2)^2] / {1 + (\xi_{0z} + \frac{1}{2}) } }]$</td>
</tr>
</tbody>
</table>
Table 9 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Yo</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>$-D (z + k_z)/2$</td>
</tr>
<tr>
<td>14</td>
<td>$D (z + k_z) \left[ \eta/2 - \eta/(1 + \eta_z^2) \right]$</td>
</tr>
<tr>
<td>15</td>
<td>$D (\eta - \eta_z)$</td>
</tr>
<tr>
<td>16</td>
<td>$D (z + k_z) \left[ \eta/2 - \left{ (1 - \alpha_z^3 + \frac{\xi_{zt}}{\xi_{zt}})[1 - (1 - \alpha_z^3)] \right} / \left{ 1 + (1 - \alpha_z^3 + \frac{\xi_{zt}}{\xi_{zt}})[1 - (1 - \alpha_z^3)] \right}$</td>
</tr>
</tbody>
</table>

* Numbers refer to Figure I.9

** Parameters refer to Figure I.11 and their expressions are shown in Table 3.
Table 10: Equations for $Y_0$ for the different strain configurations

<table>
<thead>
<tr>
<th>Case*</th>
<th>$Y_0$</th>
<th>$D_{1} (1 + \frac{E_2}{E_1})$</th>
<th>$D_{2} \left( \frac{E_1 + E_2}{E_1} \right) / 2 \bar{\varepsilon}_3$</th>
<th>$D_{3} \left(\frac{E_2}{E_1}\right) \left[ V_2 - \left( \frac{E_2}{E_1} \right) \left( \frac{\bar{\varepsilon}_3}{\bar{\varepsilon}_1} \right) \right]$</th>
<th>$-D_{4} (2 + \frac{E_2}{E_1}) / 2$</th>
<th>$D_{5} \left( \frac{E_1}{E_2} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>$D_{1}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$D_{2} (1 + \frac{E_2}{E_1})$</td>
<td>$D_{3} \left( \frac{E_1 + E_2}{E_1} \right) / 2 \bar{\varepsilon}_3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$D_{4} (2 + \frac{E_2}{E_1}) / 2$</td>
<td></td>
<td></td>
<td>$D_{5} \left( \frac{E_1}{E_2} \right)$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$D_{6}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>$D_{7}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>$D_{8}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Numbers refer to Figure 1.10.

** Parameters refer to Figure 1.11 and their expressions are shown in Table 3.
Table 11: The Coefficients $Q$, $R$ and $S$ of the quadratic equation II.3.20

for different strain patterns (Biaxial loading, neglecting residual stresses)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>-1</td>
<td>$P^{**}$</td>
</tr>
<tr>
<td>2</td>
<td>$A/(\varepsilon_3 - \varepsilon_4)$</td>
<td>$2A(\varepsilon_3 + 1)/(\varepsilon_3 - \varepsilon_4) - 1$</td>
<td>$A(\varepsilon_3 + 1)^2/(\varepsilon_3 - \varepsilon_4) + P$</td>
</tr>
<tr>
<td>3</td>
<td>$A/(\varepsilon_2 - \varepsilon_1) + A/(\varepsilon_3 - \varepsilon_4)$</td>
<td>$2A(\varepsilon_2 + 1)/(\varepsilon_2 - \varepsilon_1) + 2A(\varepsilon_3 + 1)/(\varepsilon_3 - \varepsilon_4) - 1$</td>
<td>$A[(\varepsilon_2 - 1)^2/(\varepsilon_2 - \varepsilon_1) + (\varepsilon_3 + 1)^2/(\varepsilon_3 - \varepsilon_4)] + P$</td>
</tr>
</tbody>
</table>
Table 11 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$A/(\bar{e}_1 - \bar{e}_4)$ + $A/(\bar{e}_3 - \bar{e}_5)$</td>
<td>$2A \left( (\bar{e}_1 + i)/(\bar{e}_1 - \bar{e}_2) + (\bar{e}_3 + i)/(\bar{e}_3 - \bar{e}_4) \right)$</td>
<td>$A \left[ (\bar{e}_1 + i)^2/(\bar{e}_1 - \bar{e}_2) + (\bar{e}_3 + i)^2/(\bar{e}_3 - \bar{e}_4) \right]$ + $\bar{P}$</td>
</tr>
<tr>
<td>5</td>
<td>$A/(\bar{e}_3 - \bar{e}_4)$</td>
<td>$2A \left( (\bar{e}_1 + i)/(\bar{e}_1 - \bar{e}_2) + (\bar{e}_3 - i)/(\bar{e}_3 - \bar{e}_5) \right)$</td>
<td>$A \left[ (\bar{e}_1 + i)^2/(\bar{e}_1 - \bar{e}_2) + (\bar{e}_3 - i)^2/(\bar{e}_3 - \bar{e}_5) \right]$ + $\bar{P}$</td>
</tr>
<tr>
<td>6</td>
<td>$A/(\bar{e}_3 - \bar{e}_4)$ + $B/(\bar{e}_3 - \bar{e}_5)$</td>
<td>$2A \left( (\bar{e}_3 + i)/(\bar{e}_3 - \bar{e}_4) + (\bar{e}_3 + i)/(\bar{e}_3 - \bar{e}_5) \right)$</td>
<td>$A \left( \bar{e}_3 + i \right)^2/(\bar{e}_3 - \bar{e}_4) + B \left( \bar{e}_3 + i \right)^2/(\bar{e}_3 - \bar{e}_5) + \bar{P}$</td>
</tr>
</tbody>
</table>
Table 11 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>$A/(E_3 - E_4) + A/(E_2 - E_1)$ + $B/(E_6 - E_5)$</td>
<td>$2A(E_3) , (E_2 + 1)/(E_3 - E_4) + 2A(E_2 - 1)/(E_2 - E_1)$ + $2B(E_6 + 1)/(E_6 - E_5) - 1$</td>
<td>$A[(E_2 - U)^2/(E_2 - E_1) + (E_3 + 1)^2/(E_3 - E_4)] + B(E_6 + 1)^2/(E_6 - E_5) + P$</td>
</tr>
<tr>
<td>8</td>
<td>$A/(E_3 - E_4) + A/(E_2 - E_1)$ + $B/(E_6 - E_5)$</td>
<td>$2A[(E_4 + 1)/(E_3 - E_2) + (E_3 + 1)/(E_3 - E_4)] + 2B(E_6 + 1)/(E_6 - E_5) - 1$</td>
<td>$A[(E_1 + U)^2/(E_1 - E_2) + (E_3 + 1)^2/(E_3 - E_4)] + B[(E_6 + 1)^2/(E_6 - E_5)] + P$</td>
</tr>
<tr>
<td>9</td>
<td>$A/(E_3 - E_4) + B/(E_6 - E_5)$</td>
<td>$2A[(E_4 + 1)/(E_3 - E_2) + (E_2 - 1)/(E_2 - E_1) + E_3 + 1)/(E_3 - E_4)] + (E_2 - E_1) + (E_3 + 1)^2/(E_3 - E_4)] + 2B(E_6 + 1)/(E_6 - E_5) - 1$</td>
<td>$A[(E_1 + U)^2/(E_1 - E_2) + (E_2 - 1)^2/(E_2 - E_1) + (E_3 + 1)^2/(E_3 - E_4)] + B(E_6 + 1)^2/(E_6 - E_5) + P$</td>
</tr>
</tbody>
</table>
Table 11 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$B/(\tilde{E}_6 - \tilde{E}_5)$</td>
<td>$2A + 2B(\tilde{E}_6 + 1)/(\tilde{E}_6 - \tilde{E}_5) - 1$</td>
<td>$A(\tilde{E}_3 + \tilde{E}_4 + 2) + B(\tilde{E}_6 + 1)/\tilde{E}_5 + P$</td>
</tr>
<tr>
<td>11</td>
<td>$A/(\tilde{E}_2 - \tilde{E}_1) + B/(\tilde{E}_6 - \tilde{E}_5)$</td>
<td>$2A + 2A(\tilde{E}_2 - 1)/(\tilde{E}_2 - \tilde{E}_1) + 2B(\tilde{E}_6 + 1)/(\tilde{E}_6 - \tilde{E}_5) - 1$</td>
<td>$A(\tilde{E}_2 - 1)^2/(\tilde{E}_2 - \tilde{E}_1) + A(\tilde{E}_3 + \tilde{E}_4 + 2) + B(\tilde{E}_6 + 1)/\tilde{E}_5 + P$</td>
</tr>
<tr>
<td>12</td>
<td>0.0</td>
<td>$2A[(\tilde{E}_1 + 1)/(\tilde{E}_1 - \tilde{E}_2) + (\tilde{E}_2 - 1)/(\tilde{E}_2 - \tilde{E}_1) + (\tilde{E}_3 + 1)/(\tilde{E}_3 - \tilde{E}_4) + (\tilde{E}_4 - 1)/(\tilde{E}_4 - \tilde{E}_3)] - 1$</td>
<td>$A[(\tilde{E}_1 + 1)^2/(\tilde{E}_1 - \tilde{E}_2) + (\tilde{E}_2 - 1)^2/(\tilde{E}_2 - \tilde{E}_1) + (\tilde{E}_3 + 1)^2/(\tilde{E}_3 - \tilde{E}_4) + (\tilde{E}_4 - 1)^2/(\tilde{E}_4 - \tilde{E}_3)] + P$</td>
</tr>
</tbody>
</table>
Table 11 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>$\frac{A}{(\bar{E}_2 - \bar{E}_4)} + $</td>
<td>$2A + 2A(\bar{E}_1 + 1)/(\bar{E}_1 - \bar{E}_2) + $</td>
<td>$A(\bar{E}_1 + 1)^2/(\bar{E}_1 - \bar{E}_2) + A(\bar{E}_3 + \bar{E}_4 + $</td>
</tr>
<tr>
<td></td>
<td>$+ B/(\bar{E}_6 - \bar{E}_5)$</td>
<td>$+ 2B(\bar{E}_6 + 1)/(\bar{E}_6 - \bar{E}_5) - 1$</td>
<td>$+ 2) + B(\bar{E}_6 + 1)^2/(\bar{E}_6 - \bar{E}_5) + P$</td>
</tr>
<tr>
<td>14</td>
<td>$B/(\bar{E}_6 - \bar{E}_5)$</td>
<td>$2A + 2A[(\bar{E}_1 + 1)/(\bar{E}_1 - $</td>
<td>$A(\bar{E}_3 + \bar{E}_4 + 2) + A[(\bar{E}_1 + 1)^2/(\bar{E}_1 - $</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$- \bar{E}_2) + (\bar{E}_2 - i)/(\bar{E}_2 - \bar{E}_1)] + $</td>
<td>$- \bar{E}_2) + (\bar{E}_2 - i)^2/(\bar{E}_2 - \bar{E}_1)] + B(\bar{E}_6 + 1)^2 $</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ 2B(\bar{E}_6 + 1)/(\bar{E}_6 - \bar{E}_5) - 1$</td>
<td>$/(\bar{E}_6 - \bar{E}_5) + P$</td>
</tr>
<tr>
<td>15</td>
<td>$A/(\bar{E}_2 - \bar{E}_1) +$</td>
<td>$2A[ (\bar{E}_2 - i)/(\bar{E}_2 - \bar{E}_1) + (\bar{E}_3 + 1)/(\bar{E}_3 - \bar{E}_4)] + 2B[(\bar{E}_5 - 1) $</td>
<td>$A[ (\bar{E}_2 - i)^2/(\bar{E}_2 - \bar{E}_1) + (\bar{E}_3 + 1)^2/(\bar{E}_3 - $</td>
</tr>
<tr>
<td></td>
<td>$+ A/(\bar{E}_3 - \bar{E}_4) +$</td>
<td>$/(\bar{E}_5 - \bar{E}_6) + (\bar{E}_5 + 1)/(\bar{E}_5 - \bar{E}_5)$]</td>
<td>$\bar{E}_4)] + B[(\bar{E}_5 - 1)^2/(\bar{E}_5 - \bar{E}_6) + (\bar{E}_6 + 1)^2 $</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$/(\bar{E}_6 - \bar{E}_5)] + P$</td>
</tr>
<tr>
<td>S</td>
<td>R</td>
<td>Q</td>
<td>Case*</td>
</tr>
<tr>
<td>-----------------------</td>
<td>----------------------------------------</td>
<td>---</td>
<td>-------</td>
</tr>
<tr>
<td>( (\varepsilon_3 - \varepsilon_2) + A (\varepsilon_4 + \varepsilon_5 + \varepsilon_6) + \varepsilon_7 )</td>
<td>( 2A + 2B + 2 \sqrt{A + B} = 0 )</td>
<td>0.0</td>
<td>16</td>
</tr>
<tr>
<td>( (\varepsilon_3 - \varepsilon_2) + \varepsilon_7 )</td>
<td>( 4A + 2B = 0 )</td>
<td>0.0</td>
<td>17</td>
</tr>
<tr>
<td>( \varepsilon_6 - \varepsilon_5 )</td>
<td>( A (\varepsilon_7 + \varepsilon_4 + \varepsilon_6) + \varepsilon_3 )</td>
<td>0.0</td>
<td>18</td>
</tr>
</tbody>
</table>

Table 11 (Cont'd.)
Table 11 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>( A/(\bar{e}_2 - \bar{e}_1) )</td>
<td>[2A \left[ 1 + \frac{(\bar{e}_2 - 1)/(\bar{e}_2 - \bar{e}_1)}{(\bar{e}_2 + 1)/(\bar{e}_2 - \bar{e}_1)} \right] + 2B \left[ (\bar{e}_5 - 1)/(\bar{e}_5 - \bar{e}_6) + \frac{(\bar{e}_6 + 1)/(\bar{e}_6 - \bar{e}_5)}{1 + (\bar{e}_6 + 1)/(\bar{e}_6 - \bar{e}_5)} \right] - 1]</td>
<td>[A \left( \bar{e}_2 - 1 \right)^2/(\bar{e}_2 - \bar{e}_1) + \bar{e}_3 + \bar{e}_4 + 2 + B \left[ (\bar{e}_5 - 1)^2/(\bar{e}_5 - \bar{e}_6) + (\bar{e}_6 + 1)^2/(\bar{e}_6 + 1)/(\bar{e}_6 - \bar{e}_5) \right] + P ]</td>
</tr>
<tr>
<td>20</td>
<td>( A/(\bar{e}_3 - \bar{e}_4) )</td>
<td>[2A \left[ (\bar{e}_3 + 1)/(\bar{e}_3 - \bar{e}_4) + (\bar{e}_2 - 1)/(\bar{e}_2 - \bar{e}_1) \right] + 2B \left[ (\bar{e}_5 - 1)/(\bar{e}_5 - \bar{e}_6) + (\bar{e}_6 + 1)/(\bar{e}_6 - \bar{e}_5) \right] - 1]</td>
<td>[A \left( \bar{e}_1 + 1 \right)^2/(\bar{e}_1 - \bar{e}_2) + (\bar{e}_2 - 1)^2/(\bar{e}_2 - \bar{e}_1) + (\bar{e}_3 + 1)^2/(\bar{e}_3 - \bar{e}_4) + B \left[ (\bar{e}_5 - 1)^2/(\bar{e}_5 - \bar{e}_6) + \bar{e}_3 + \bar{e}_4 + 2 \right] + P ]</td>
</tr>
<tr>
<td>21</td>
<td>( A/(\bar{e}_1 - \bar{e}_2) )</td>
<td>(2A \left[ 1 + \frac{(\bar{e}_1 + 1)/(\bar{e}_1 - \bar{e}_2)}{(\bar{e}_1 - 1)/(\bar{e}_1 - \bar{e}_2)} \right] - 1]</td>
<td>[A \left( \bar{e}_1 + 1 \right)^2/(\bar{e}_1 - \bar{e}_2) + \bar{e}_3 + \bar{e}_4 + 2 + B(\bar{e}_5 + \bar{e}_6 + 2) + P ]</td>
</tr>
<tr>
<td>Case</td>
<td>Q</td>
<td>R</td>
<td>S</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>22</td>
<td>(\frac{A}{(\bar{e}_1 - \bar{e}_2)} + \frac{A}{(\bar{e}_3 - \bar{e}_4)})</td>
<td>(2A\left[\frac{1}{(\bar{e}_1 - \bar{e}_2)} + \frac{1}{(\bar{e}_3 - \bar{e}_4)}\right] + 2B - 1)</td>
<td>(A\left[\frac{1}{(\bar{e}_1 + 1)} + \frac{1}{(\bar{e}_3 - \bar{e}_4)}\right] + B(\bar{e}_5 + \bar{e}_6 + 2) + P)</td>
</tr>
<tr>
<td>23</td>
<td>(\frac{A}{(\bar{e}_3 - \bar{e}_4)})</td>
<td>(2A\left[\frac{1}{(\bar{e}_1 - \bar{e}_2)} + \frac{1}{(\bar{e}_3 - \bar{e}_4)}\right] + 2B - 1)</td>
<td>(A\left[\frac{1}{(\bar{e}_1 + 1)} + \frac{1}{(\bar{e}_3 - \bar{e}_4)}\right] + B(\bar{e}_5 + \bar{e}_6 + 2) + P)</td>
</tr>
<tr>
<td>24</td>
<td>...</td>
<td>(2A\left[\frac{1}{(\bar{e}_1 - \bar{e}_2)} + \frac{1}{(\bar{e}_3 - \bar{e}_4)}\right] + 2B - 1)</td>
<td>(A\left[\frac{1}{(\bar{e}_1 + 1)} + \frac{1}{(\bar{e}_3 - \bar{e}_4)}\right] + B(\bar{e}_5 + \bar{e}_6 + 2) + P)</td>
</tr>
</tbody>
</table>
Table 11 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>0.0</td>
<td>$2A\left[ (\bar{\varepsilon}_1 + 1)/(\bar{\varepsilon}_1 - \bar{\varepsilon}_2) + (\bar{\varepsilon}_2 - 1)/ (\bar{\varepsilon}_2 - \bar{\varepsilon}_3) + (\bar{\varepsilon}_3 + 1)/(\bar{\varepsilon}_3 - \bar{\varepsilon}_4) + (\bar{\varepsilon}_4 - 1)/(\bar{\varepsilon}_4 - \bar{\varepsilon}_5) \right] + 2B \left[ (\bar{\varepsilon}_5 - 1)/ (\bar{\varepsilon}_5 - \bar{\varepsilon}_6) + (\bar{\varepsilon}_6 + 1)/(\bar{\varepsilon}_6 - \bar{\varepsilon}_5) \right] - 1$</td>
<td>$A\left[ (\bar{\varepsilon}_1 + 1)^2/(\bar{\varepsilon}_1 - \bar{\varepsilon}_2) + (\bar{\varepsilon}_2 - 1)^2/ (\bar{\varepsilon}_2 - \bar{\varepsilon}_3) + (\bar{\varepsilon}_3 + 1)^2/(\bar{\varepsilon}_3 - \bar{\varepsilon}_4) + (\bar{\varepsilon}_4 - 1)^2/(\bar{\varepsilon}_4 - \bar{\varepsilon}_5) \right] + B \left[ (\bar{\varepsilon}_5 - 1)^2/(\bar{\varepsilon}_5 - \bar{\varepsilon}_6) + (\bar{\varepsilon}_6 + 1)^2/(\bar{\varepsilon}_6 - \bar{\varepsilon}_5) \right] + P$</td>
</tr>
<tr>
<td>26</td>
<td>$B/(\bar{\varepsilon}_6 - \bar{\varepsilon}_5)$</td>
<td>$2A\left[ (\bar{\varepsilon}_1 + 1)/(\bar{\varepsilon}_1 - \bar{\varepsilon}_2) + (\bar{\varepsilon}_2 - 1)/(\bar{\varepsilon}_2 - \bar{\varepsilon}_3) + (\bar{\varepsilon}_3 + 1)/(\bar{\varepsilon}_3 - \bar{\varepsilon}_4) + (\bar{\varepsilon}_4 - 1)/(\bar{\varepsilon}_4 - \bar{\varepsilon}_5) \right] + 2B \left[ (\bar{\varepsilon}_5 - 1)/(\bar{\varepsilon}_5 - \bar{\varepsilon}_6) + (\bar{\varepsilon}_6 + 1)/(\bar{\varepsilon}_6 - \bar{\varepsilon}_5) \right] - 1$</td>
<td>$A\left[ (\bar{\varepsilon}_1 + 1)^3/(\bar{\varepsilon}_1 - \bar{\varepsilon}_2) + (\bar{\varepsilon}_2 - 1)^3/ (\bar{\varepsilon}_2 - \bar{\varepsilon}_3) + (\bar{\varepsilon}_3 + 1)^3/(\bar{\varepsilon}_3 - \bar{\varepsilon}_4) + (\bar{\varepsilon}_4 - 1)^3/(\bar{\varepsilon}_4 - \bar{\varepsilon}_5) \right] + B \left[ (\bar{\varepsilon}_5 - 1)^3/(\bar{\varepsilon}_5 - \bar{\varepsilon}_6) + (\bar{\varepsilon}_6 + 1)^3/(\bar{\varepsilon}_6 - \bar{\varepsilon}_5) \right] + P$</td>
</tr>
</tbody>
</table>

where: $A = K_1 K_2/2 \left( K_3 + 2K_1 K_2 \right)$, $B = K_3/2 \left( K_3 + 2 K_1 K_2 \right)$

* Numbers refer to Figure II.7

** P is in terms of $P_y$

$\bar{\varepsilon}_1$ to $\bar{\varepsilon}_6$ denote bending and warping strains divided by yield strain at points 1 to 6 (Figure I.4)
Table 12: The Coefficients Q, R, and S of the quadratic equation II.3.20 for the different strain configurations (Biaxial loading considering residual stresses)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.0$</td>
<td>$-1$</td>
<td>$P^{**}$</td>
</tr>
<tr>
<td>2</td>
<td>$A/2(\tilde{\varepsilon}_3^{<em>} - \tilde{\varepsilon}_6^{</em>})$</td>
<td>$A(\tilde{\varepsilon}_3^{<em>} - l)/(\tilde{\varepsilon}_3^{</em>} - \tilde{\varepsilon}_6^{*}) - 1$</td>
<td>$A(\tilde{\varepsilon}_3^{<em>} + l)/2(\tilde{\varepsilon}_3^{</em>} - \tilde{\varepsilon}_6^{*}) + P$</td>
</tr>
<tr>
<td>3</td>
<td>$A/2(\tilde{\varepsilon}_2^{<em>} - \tilde{\varepsilon}_5^{</em>}) + A/2(\tilde{\varepsilon}_3^{<em>} - \tilde{\varepsilon}_6^{</em>})$</td>
<td>$A(\tilde{\varepsilon}_2^{<em>} - l)/(\tilde{\varepsilon}_2^{</em>} - \tilde{\varepsilon}_5^{<em>}) + A(\tilde{\varepsilon}_3^{</em>} + l)/(\tilde{\varepsilon}_3^{<em>} - \tilde{\varepsilon}_6^{</em>}) - 1$</td>
<td>$A(\tilde{\varepsilon}_2^{<em>} - l)/2(\tilde{\varepsilon}_2^{</em>} - \tilde{\varepsilon}_5^{<em>}) + A(\tilde{\varepsilon}_3^{</em>} + l)/2(\tilde{\varepsilon}_3^{<em>} - \tilde{\varepsilon}_6^{</em>}) + P$</td>
</tr>
</tbody>
</table>
Table 12 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>( \frac{A}{2}(\bar{\varepsilon}_1 - \bar{\varepsilon}_5) ) + ( \frac{A}{2}(\bar{\varepsilon}_3 - \bar{\varepsilon}_2) ) + ( A(\bar{\varepsilon}_1 + 1)/(\bar{\varepsilon}_1 - \bar{\varepsilon}_5) ) + ( A(\bar{\varepsilon}_3 + 1)/(\bar{\varepsilon}_3 - \bar{\varepsilon}_6) ) - 1</td>
<td>( A(\bar{\varepsilon}_1 + 1)/2(\bar{\varepsilon}_1 - \bar{\varepsilon}_5) + A(\bar{\varepsilon}_3 + 1)/2(\bar{\varepsilon}_3 - \bar{\varepsilon}_6) + P )</td>
<td>( \frac{A}{2}(\bar{\varepsilon}_3 - \bar{\varepsilon}_2) ) + P</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{A}{2}(\bar{\varepsilon}_3 - \bar{\varepsilon}_5) ) + ( \frac{A}{2}(\bar{\varepsilon}_2 - \bar{\varepsilon}_3) ) + ( (\bar{\varepsilon}_2 + 1)/(\bar{\varepsilon}_2 - \bar{\varepsilon}_3) ) + ( \bar{\varepsilon}_3 + 1)/(\bar{\varepsilon}_3 - \bar{\varepsilon}_6) ) ] - 1</td>
<td>( A[(\bar{\varepsilon}_3 + 1)/2(\bar{\varepsilon}_1 - \bar{\varepsilon}_5) + (\bar{\varepsilon}_3 - \bar{\varepsilon}_5)] + P )</td>
<td>( \frac{1}{2}(\bar{\varepsilon}_3 - \bar{\varepsilon}_5) + (\bar{\varepsilon}_3 + 1)/2(\bar{\varepsilon}_3 - \bar{\varepsilon}_6) ) + P</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{A}{2}(\bar{\varepsilon}_3 - \bar{\varepsilon}_6) ) + ( \frac{A}{2}(\bar{\varepsilon}_4 - \bar{\varepsilon}_5) ) + ( (\bar{\varepsilon}_3 + 1)/(\bar{\varepsilon}_3 - \bar{\varepsilon}_6) ) + ( (\bar{\varepsilon}_4 + 1)/(\bar{\varepsilon}_4 - \bar{\varepsilon}_6) ) ] - 1</td>
<td>( A[(\bar{\varepsilon}_3 + 1)/2(\bar{\varepsilon}_3 - \bar{\varepsilon}_6) + (\bar{\varepsilon}_4 + 1)/2(\bar{\varepsilon}_4 - \bar{\varepsilon}_6)] + P )</td>
<td>( \frac{1}{2}(\bar{\varepsilon}_4 - \bar{\varepsilon}_6) ) + P</td>
</tr>
</tbody>
</table>
Table 12 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>$A/2(\varepsilon_1 - \varepsilon_5)$</td>
<td>$A[(\varepsilon_1^* + 1)/(\varepsilon_1^* - \varepsilon_5^* + (\varepsilon_3^* + 1)^2)/$</td>
<td>$A[(\varepsilon_1^* + 1)/(\varepsilon_1^* - \varepsilon_5^* + (\varepsilon_3^* + 1)^2)/$</td>
</tr>
<tr>
<td></td>
<td>$+ A/2(\varepsilon_3 - \varepsilon_2^*)$</td>
<td>$+(\varepsilon_3^* + 1)/(\varepsilon_3^* - \varepsilon_2^*) +$</td>
<td>$+(\varepsilon_3^* + 1)/(\varepsilon_3^* - \varepsilon_2^*) +$</td>
</tr>
<tr>
<td></td>
<td>$+ A/2(\varepsilon_4^* - \varepsilon_6^*)$</td>
<td>$+(\varepsilon_4^* + 1)/(\varepsilon_4^* - \varepsilon_6^*)] - 1$</td>
<td>$+(\varepsilon_4^* + 1)/(\varepsilon_4^* - \varepsilon_6^*)] - 1$</td>
</tr>
<tr>
<td></td>
<td>$+ A/2(\bar{\varepsilon}_3 - \bar{\varepsilon}_2)$</td>
<td></td>
<td>$/2 + P$</td>
</tr>
<tr>
<td>8</td>
<td>$A/2(\bar{\varepsilon}_3^* - \bar{\varepsilon}_6^*)$</td>
<td>$A\left[1 + (\varepsilon_6^* + 1)/(\varepsilon_6^* - \varepsilon_4)\right] + 2B(\varepsilon_6^* + 1)/$</td>
<td>$A\left[(\bar{\varepsilon}_3^* + \bar{\varepsilon}_6^* + 2 + (\bar{\varepsilon}_6^* + 1)^2)/$</td>
</tr>
<tr>
<td></td>
<td>$+ B/(\varepsilon_6^* - \varepsilon_5)$</td>
<td>$(\bar{\varepsilon}_6^* - \varepsilon_5^*)] - 1$</td>
<td>$+(\bar{\varepsilon}_6^* - \varepsilon_5^*)] - 1$</td>
</tr>
<tr>
<td>9</td>
<td>$A/2(\bar{\varepsilon}_2^* - \bar{\varepsilon}_5^*)$</td>
<td>$A\left[(\bar{\varepsilon}_2^* - 1)/(\varepsilon_2^* - \varepsilon_5^*) +$</td>
<td>$A\left[(\bar{\varepsilon}_3^* + \bar{\varepsilon}_6^* + 2 + (\bar{\varepsilon}_6^* - 1)^2)/(\varepsilon_2^* - \varepsilon_5^*) +$</td>
</tr>
<tr>
<td></td>
<td>$+ A/2(\bar{\varepsilon}_6^* - \bar{\varepsilon}_4^*)$</td>
<td>$+(\bar{\varepsilon}_6^* + 1)/(\varepsilon_6^* - \varepsilon_5^*) +$</td>
<td>$+(\bar{\varepsilon}_6^* + 1)/(\varepsilon_6^* - \varepsilon_4^<em>)\right] + B(\varepsilon_6^</em> + 1)/$</td>
</tr>
<tr>
<td></td>
<td>$+ B/(\varepsilon_6^* - \varepsilon_5)$</td>
<td>$+ 2B(\varepsilon_6^* + 1)/(\varepsilon_6^* - \varepsilon_5^*)] - 1$</td>
<td>$(\varepsilon_6^* - \varepsilon_5^*)] - 1$</td>
</tr>
</tbody>
</table>

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Table 12 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
<th>S</th>
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</thead>
<tbody>
<tr>
<td>10</td>
<td>( A/2(\bar{E}_5^<em>-\bar{E}_3^</em>) + A/2(\bar{E}_5^-\bar{E}_2^-) + A/2(\bar{E}_3^-\bar{E}_4^-) + B/2(\bar{E}_5^-\bar{E}_4^-) )</td>
<td>( A\left[ (\bar{E}_5^*-1)/(\bar{E}_3^-) + (\bar{E}_5^-1) \right] )</td>
<td>( A\left[ (\bar{E}_5^*-1)/(\bar{E}_3^-) + (\bar{E}_5^-1) \right] )</td>
</tr>
<tr>
<td>11</td>
<td>( A/2(\bar{E}_2^-\bar{E}_5^-) + A/2(\bar{E}_3^-\bar{E}_4^-) + A/2(\bar{E}_4^-\bar{E}_5^-) )</td>
<td>( A\left[ (\bar{E}_2^-1)/(\bar{E}_2^-) + (\bar{E}_3^-1)/(\bar{E}_4^-) + (\bar{E}_4^-) \right] + 1 )</td>
<td>( A\left[ (\bar{E}_2^-1)/(\bar{E}_2^-) + (\bar{E}_3^-1)/(\bar{E}_4^-) + (\bar{E}_4^-) \right] )</td>
</tr>
<tr>
<td>12</td>
<td>( A/2(\bar{E}_3^-\bar{E}_5^-) + A/2(\bar{E}_5^-\bar{E}_3^-) + A/2(\bar{E}_3^-\bar{E}_4^-) + A/2(\bar{E}_4^-\bar{E}_5^-) )</td>
<td>( A\left[ (\bar{E}_3^- + 1)/(\bar{E}_3^-) + (\bar{E}_3^-) \right] )</td>
<td>( A\left[ (\bar{E}_3^- + 1)/(\bar{E}_3^-) + (\bar{E}_3^-) \right] )</td>
</tr>
<tr>
<td>Case*</td>
<td>Q</td>
<td>R</td>
<td>S</td>
</tr>
<tr>
<td>-------</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>13</td>
<td>$A/2(\tilde{E}^*_5 - \tilde{E}_5)$</td>
<td>$A[ (\tilde{E}_5^* - 1)/(\tilde{E}^<em>_5 - \tilde{E}_5^</em>) + (\tilde{E}_5^* - 1)/(\tilde{E}^<em>_5 - \tilde{E}_5^</em>)]$</td>
<td>$A[ (\tilde{E}_5^* - 1)/(\tilde{E}^<em>_5 - \tilde{E}_5^</em>) + (\tilde{E}_5^* - 1)/(\tilde{E}^<em>_5 - \tilde{E}_5^</em>)]$</td>
</tr>
<tr>
<td></td>
<td>$+ A/2(\tilde{E}^<em>_5 - \tilde{E}_2^</em>)$</td>
<td>$+ (\tilde{E}_5^* - 1)/(\tilde{E}^<em>_5 - \tilde{E}_2^</em>)$</td>
<td>$+ B(\tilde{E}_5^* - 1)/(\tilde{E}^<em>_5 - \tilde{E}_5^</em>) - 1$</td>
</tr>
<tr>
<td></td>
<td>$+ A/2(\tilde{E}^<em>_3 - \tilde{E}_6^</em>)$</td>
<td>$+ (\tilde{E}_3^* + 1)/(\tilde{E}^<em>_3 - \tilde{E}_6^</em>)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ B/(\tilde{E}^<em>_5 - \tilde{E}_6^</em>)$</td>
<td>$2B(\tilde{E}_5^* - 1)/(\tilde{E}^<em>_5 - \tilde{E}_6^</em>) - 1$</td>
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<tr>
<td>14</td>
<td>$A/2(\tilde{E}^*_1 - \tilde{E}_5)$</td>
<td>$A[ 1 + (\tilde{E}^<em>_1 - 1)/(\tilde{E}^</em>_1 - \tilde{E}_5^*)$</td>
<td>$A[ 1 + (\tilde{E}^<em>_1 - 1)/(\tilde{E}^</em>_1 - \tilde{E}_5^*)$</td>
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</tbody>
</table>
|       | $+ A/2(\tilde{E}^*_6 - \tilde{E}_4^*)$ | $+ (\tilde{E}_6^* + 1)/(\tilde{E}^*_6 - \tilde{E}_4^*)$ | $+ B(\tilde{E}^*_1 - \tilde{E}_5^*) + (\tilde{E}_6^* + 1)/(\tilde{E}^*_6 - \tilde{E}_4^*)$]
|       | $+ B/(\tilde{E}^*_6 - \tilde{E}_5^*)$ | $+ 2B(\tilde{E}_6^* + 1)/(\tilde{E}^*_6 - \tilde{E}_5^*)$ | $+ 2B(\tilde{E}_6^* + 1)/(\tilde{E}^*_6 - \tilde{E}_5^*)$ |

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Table 12 (Cont'd.)

<table>
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<tr>
<th>Case*</th>
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<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>$A/2(\varepsilon_{1}^* - \varepsilon_{5}^*)$</td>
<td>$A[(\varepsilon_{1}^* + 1)/(\varepsilon_{1}^* - \varepsilon_{5}^<em>) + (\varepsilon_{2}^</em> - 1)/ (\varepsilon_{2}^* - \varepsilon_{5}^<em>) + (\varepsilon_{6}^</em> + 1)/ (\varepsilon_{6}^* - \varepsilon_{5}^<em>) + 2B(\varepsilon_{6}^</em> + 1)]$</td>
<td>$A[(\varepsilon_{1}^* + 1)/(\varepsilon_{1}^* - \varepsilon_{5}^<em>) + (\varepsilon_{2}^</em> - 1)/ (\varepsilon_{2}^* - \varepsilon_{5}^<em>) + (\varepsilon_{6}^</em> + 1)/ (\varepsilon_{6}^* - \varepsilon_{5}^<em>) + 2B(\varepsilon_{6}^</em> + 1)]$</td>
</tr>
<tr>
<td>16</td>
<td>$B/(\varepsilon_{6}^* - \varepsilon_{5}^*)$</td>
<td>$2A + 2B(\varepsilon_{6}^* + 1)/(\varepsilon_{6}^* - \varepsilon_{5}^*) - 1$</td>
<td>$A(\varepsilon_{3}^* + \varepsilon_{4}^* + 2\varepsilon_{6}^* + 4)/2 + B(\varepsilon_{6}^* + 1)^2/(\varepsilon_{6}^* - \varepsilon_{5}^*) + P$</td>
</tr>
<tr>
<td>17</td>
<td>$A/2(\varepsilon_{5}^* - \varepsilon_{2}^*)$</td>
<td>$A[(\varepsilon_{1}^* + 1)/(\varepsilon_{1}^* - \varepsilon_{5}^<em>) + (\varepsilon_{2}^</em> - 1)/ (\varepsilon_{2}^* - \varepsilon_{5}^<em>) + (\varepsilon_{3}^</em> + 1)/ (\varepsilon_{3}^* - \varepsilon_{5}^<em>) + (\varepsilon_{4}^</em> + 1)/ (\varepsilon_{4}^* - \varepsilon_{5}^<em>) + 2B(\varepsilon_{5}^</em> - 1)/(\varepsilon_{5}^* - \varepsilon_{6}^*) - 1]$</td>
<td>$A[(\varepsilon_{1}^* + 1)/(\varepsilon_{1}^* - \varepsilon_{5}^<em>) + (\varepsilon_{2}^</em> - 1)/ (\varepsilon_{2}^* - \varepsilon_{5}^<em>) + (\varepsilon_{6}^</em> + 1)/ (\varepsilon_{6}^* - \varepsilon_{5}^*)]$</td>
</tr>
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</table>
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<th>R</th>
<th>S</th>
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</thead>
<tbody>
<tr>
<td>18</td>
<td>( \frac{A}{2} (E_5^* - E_1^*) )</td>
<td>( \frac{A}{2} \left[ 1 + \frac{(E_5^* - 1)}{(E_5^* - E_1^<em>)} + \frac{(E_5^</em> + 1)}{(E_3^* - E_6^*)} \right] )</td>
<td>( \frac{A}{2} \left[ E_2 + E_5 - 2 + \frac{(E_5^* - 1)^2}{(E_5^* - E_1^<em>)} + \frac{(E_5^</em> + 1)}{(E_3^* - E_6^<em>)} \right] / 2 + B(E_5^</em> - 1)/2 + P )</td>
</tr>
<tr>
<td>19</td>
<td>( \frac{A}{2} (E_5^* - E_1^*) )</td>
<td>( A \left[ 2 + \frac{(E_5^* - 1)}{(E_5^* - E_1^<em>)} \right] + 2B \left( \frac{(E_5^</em> + 1)}{(E_5^* - E_6^*)} - 1 \right) )</td>
<td>( \frac{A}{2} \left[ E_3 + E_4 + 2E_6 + 4 + \frac{(E_2^* - 1)^2}{(E_2^* - E_5^<em>)} \right] / 2 + B(E_6^</em> + 1)/2 + P )</td>
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<td>20</td>
<td>( \frac{A}{2} (E_3^* - E_6^*) )</td>
<td>( \frac{A}{2} \left[ 1 + \frac{(E_1^* + 1)}{(E_1^* - E_5^<em>)} + \frac{(E_3^</em> - 1)}{(E_5^* - E_1^<em>)} + \frac{(E_3^</em> + 1)}{(E_3^* - E_6^<em>)} \right] + 2B(E_5^</em> - 1)/2 + P )</td>
<td>( \frac{A}{2} \left[ E_2 + E_5 - 2 + \frac{(E_1^* + 1)}{(E_1^* - E_5^<em>)} + \frac{(E_5^</em> - 1)}{(E_5^* - E_6^<em>)} + \frac{(E_3^</em> + 1)}{(E_3^* - E_6^<em>)} \right] / 2 + B(E_5^</em> - 1)/2 + P )</td>
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Table 12 (Cont'd.)

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<thead>
<tr>
<th>Case*</th>
<th>Q</th>
<th>R</th>
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<tbody>
<tr>
<td>21</td>
<td>$\frac{A}{2}(\bar{E}_5 - \bar{E}_1)$ + $\frac{A}{2}(\bar{E}_3 - \bar{E}_2)$ + $B/(\bar{E}_5 - \bar{E}_6)$</td>
<td>$A\left[ \frac{(\bar{E}_1 + 1)/(\bar{E}_1 - \bar{E}_5)}{(\bar{E}_5 - 1)/(\bar{E}_5 - \bar{E}_1) + (\bar{E}_5 - 1)/(\bar{E}_5 - \bar{E}_1)} \right] + B(\bar{E}_5 - \bar{E}_6)/(\bar{E}_5 - \bar{E}_1)$</td>
<td>$A\left[ \frac{(\bar{E}_1 + 1)/(\bar{E}_1 - \bar{E}_5) + (\bar{E}_5 - 1)/(\bar{E}_5 - \bar{E}_1)}{(\bar{E}_5 - \bar{E}_1) + (\bar{E}_5 - 1)/(\bar{E}_5 - \bar{E}_1)} \right] / 2 + B(\bar{E}_5 - \bar{E}_1) + P$</td>
</tr>
<tr>
<td>22</td>
<td>$\frac{A}{2}(\bar{E}_5 - \bar{E}_1)$ + $\frac{A}{2}(\bar{E}_3 - \bar{E}_2)$ + $\frac{A}{2}(\bar{E}_3 - \bar{E}_6)$ + $\frac{A}{2}(\bar{E}_4 - \bar{E}_6)$</td>
<td>$A\left[ \frac{(\bar{E}_1 + 1)/(\bar{E}_1 - \bar{E}_5) + (\bar{E}_3 - 1)/(\bar{E}_3 - \bar{E}_5) + (\bar{E}_3 - 1)/(\bar{E}_3 - \bar{E}_5)}{(\bar{E}_3 - \bar{E}_5) + (\bar{E}_3 - 1)/(\bar{E}_3 - \bar{E}_5) + (\bar{E}_4 - 1)} \right] / (\bar{E}_4 - \bar{E}_6)$</td>
<td>$A\left[ \frac{(\bar{E}_1 + 1)/(\bar{E}_1 - \bar{E}_5) + (\bar{E}_3 - 1)/(\bar{E}_3 - \bar{E}_5) + (\bar{E}_3 - 1)/(\bar{E}_3 - \bar{E}_5)}{(\bar{E}_3 - \bar{E}_5) + (\bar{E}_3 - 1)/(\bar{E}_3 - \bar{E}_5) + (\bar{E}_4 - 1)} \right] / 2 + P$</td>
</tr>
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</table>
| 23    | $A/2(\bar{\epsilon}_3 - \bar{\epsilon}_4)$  
       + $A/2(\epsilon_4 - \bar{\epsilon}_5)$  
       + $B/(\epsilon_5 - \bar{\epsilon}_6)$ | $A\left[ 1 + \left( \bar{\epsilon}_1 + 1 \right)/(\epsilon_1 - \bar{\epsilon}_1) + (\bar{\epsilon}_3 - 1)/(\epsilon_3 - \bar{\epsilon}_3) + (\epsilon_4 - 1)/(\epsilon_4 - \bar{\epsilon}_4) \right]$  
       + $A\left[ \bar{\epsilon}_2 + \epsilon_5 + 2 + (\bar{\epsilon}_5 - 1)^2/(\epsilon_5 - \bar{\epsilon}_5) + (\bar{\epsilon}_3 - 1)^2/(\epsilon_3 - \bar{\epsilon}_3) + (\epsilon_4 + 1)^2/(\epsilon_4 - \bar{\epsilon}_4) \right]/2 + B(\epsilon_5 - \bar{\epsilon}_6)^2/(\epsilon_5 - \bar{\epsilon}_6) + \bar{P}$ | $A\left[ \bar{\epsilon}_2 + \epsilon_5 + 2 + (\bar{\epsilon}_5 - 1)^2/(\epsilon_5 - \bar{\epsilon}_5) + (\bar{\epsilon}_3 - 1)^2/(\epsilon_3 - \bar{\epsilon}_3) + (\epsilon_4 + 1)^2/(\epsilon_4 - \bar{\epsilon}_4) \right]/2 + B(\epsilon_5 - \bar{\epsilon}_6)^2/(\epsilon_5 - \bar{\epsilon}_6) + \bar{P}$ |
| 24    | $A/2(\bar{\epsilon}_5 - \bar{\epsilon}_4)$  
       + $A/2(\epsilon_5 - \bar{\epsilon}_2)$  
       + $A/2(\epsilon_6 - \bar{\epsilon}_4)$ | $A\left[ 1 + (\bar{\epsilon}_5 - 1)/(\epsilon_5 - \bar{\epsilon}_5) + (\bar{\epsilon}_3 - 1)/(\epsilon_3 - \bar{\epsilon}_3) + (\epsilon_4 + 1)/(\epsilon_4 - \bar{\epsilon}_4) \right] + 2B[\epsilon_5 - \bar{\epsilon}_6] + (\bar{\epsilon}_6 + 1)/(\epsilon_5 - \bar{\epsilon}_5) - 1$ | $A\left[ 1 + (\bar{\epsilon}_5 - 1)/(\epsilon_5 - \bar{\epsilon}_5) + (\bar{\epsilon}_3 - 1)/(\epsilon_3 - \bar{\epsilon}_3) + (\epsilon_4 + 1)/(\epsilon_4 - \bar{\epsilon}_4) \right] + 2B[\epsilon_5 - \bar{\epsilon}_6] + (\bar{\epsilon}_6 + 1)/(\epsilon_5 - \bar{\epsilon}_5) - 1$ |
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<td>25</td>
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<td>$A \left[ 1 + \frac{(\bar{E}_5 - 1)(E_5 - E_1)}{E_5} + \frac{(\bar{E}_5 - 1)(E_5 - E_2)(E_5 - E_3)}{E_5} + \frac{(\bar{E}_5 - 1)(E_5 - E_3)(E_5 - E_4)}{E_5} \right] + 2B \left[ \frac{E_5 - E_3}{E_5 - E_4} \right]$</td>
<td>$A \left[ \frac{\bar{E}_3 + \bar{E}_5}{E_3} + \frac{2 + \frac{(E_5 - 1)^2}{E_5}}{1} \right] + P$</td>
</tr>
<tr>
<td>26</td>
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<td>$A \left[ 1 + \frac{(\bar{E}_5 - 1)(E_5 - E_1)}{E_5} + \frac{(\bar{E}_5 - 1)(E_5 - E_2)(E_5 - E_3)}{E_5} + \frac{(\bar{E}_5 - 1)(E_5 - E_3)(E_5 - E_4)}{E_5} \right] + 2B \left[ \frac{E_5 - E_3}{E_5 - E_4} \right]$</td>
<td>$A \left[ \frac{\bar{E}_3 + \bar{E}_5}{E_3} + \frac{2 + \frac{(E_5 - 1)^2}{E_5}}{1} \right] + P$</td>
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<td>( R )</td>
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<tr>
<td>27</td>
<td>( A/2(\bar{e}_s - \bar{e}_d) + )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>( A/2(\bar{e}_s - \bar{e}_d) + )</td>
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<tbody>
<tr>
<td>29</td>
<td>$A/2(\varepsilon_6^<em>-\varepsilon_2^</em>)$</td>
<td>$A\left[2 + (\varepsilon_1^<em>+1)/(\varepsilon_1^</em> - \varepsilon_5^<em>) + (\varepsilon_5^</em>-1)/(\varepsilon_5^* - \varepsilon_1^<em>) + (\varepsilon_6^</em>-1)/(\varepsilon_6^* - \varepsilon_2^<em>)\right] + 2B\left[(\varepsilon_5^</em>+1)/(\varepsilon_5^<em>-\varepsilon_6^</em>) + (\varepsilon_6^<em>+1)/(\varepsilon_6^</em>-\varepsilon_5^*)\right] - 1$</td>
<td>$A\left[\varepsilon_3^* + \varepsilon_4^* + 2\varepsilon_6^* + 4 + (\varepsilon_1^<em>+1)^2/(\varepsilon_1^</em> - \varepsilon_5^<em>) + (\varepsilon_5^</em>-1)^2/(\varepsilon_5^* - \varepsilon_1^<em>) + (\varepsilon_6^</em>-1)^2/(\varepsilon_6^* - \varepsilon_2^<em>)\right] / 2 + B\left[(\varepsilon_5^</em>+1)^2/(\varepsilon_5^<em>-\varepsilon_6^</em>) + (\varepsilon_6^<em>+1)^2/(\varepsilon_6^</em>-\varepsilon_5^*)\right] + P$</td>
</tr>
<tr>
<td>30</td>
<td>$A/2(\varepsilon_1^<em>-\varepsilon_5^</em>) + A/2(\varepsilon_6^<em>-\varepsilon_4^</em>) + B/(\varepsilon_6^* - \varepsilon_5^*)$</td>
<td>$A\left[1 + (\varepsilon_1^<em>+1)/(\varepsilon_1^</em> - \varepsilon_5^<em>) + (\varepsilon_5^</em>-1)/(\varepsilon_5^* - \varepsilon_1^<em>)\right] + 2B\left[(\varepsilon_6^</em>+1)/(\varepsilon_6^<em>-\varepsilon_4^</em>)\right] - 1$</td>
<td>$A\left[\varepsilon_3^* + \varepsilon_4^* + 2 + (\varepsilon_1^<em>+1)^2/(\varepsilon_1^</em> - \varepsilon_5^<em>) + (\varepsilon_5^</em>-1)^2/(\varepsilon_5^* - \varepsilon_1^<em>) + (\varepsilon_6^</em>-1)^2/(\varepsilon_6^* - \varepsilon_2^<em>)\right] / 2 + B\left[(\varepsilon_6^</em>+1)^2/(\varepsilon_6^<em>-\varepsilon_4^</em>)\right] + P$</td>
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<th>S</th>
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<tr>
<td>31</td>
<td>( A/2(\bar{\varepsilon}_3 - \bar{\varepsilon}_5) )  \ + ( A/2(\bar{\varepsilon}_2 - \bar{\varepsilon}_5) )  \ + ( B/(\bar{\varepsilon}_6 - \bar{\varepsilon}_5) )</td>
<td>[ A \left( 2 + (\bar{\varepsilon}_1^* - 1) / (\bar{\varepsilon}_5^* - \bar{\varepsilon}_6^<em>) \right) + 2B(\bar{\varepsilon}_6^</em> + 1) / (\bar{\varepsilon}_6^* - \bar{\varepsilon}_5^*) - 1 ]</td>
<td>[ A \left[ \bar{\varepsilon}_3^* + \bar{\varepsilon}_4^* + 2(\bar{\varepsilon}_6^* + 1) + (\bar{\varepsilon}_1^* + 1)^2 \right] / (\bar{\varepsilon}_1^* - \bar{\varepsilon}_5^<em>) + (\bar{\varepsilon}_2^</em> - 1)^2 / (\bar{\varepsilon}_2^* - \bar{\varepsilon}_5^*) ]</td>
</tr>
<tr>
<td>32</td>
<td>( A/2(\bar{\varepsilon}_5^* - \bar{\varepsilon}_4^<em>) )  \ + ( A/2(\bar{\varepsilon}_6^</em> - \bar{\varepsilon}_4^*) )</td>
<td>[ A \left( 2 + (\bar{\varepsilon}_5^* - 1) / (\bar{\varepsilon}_5^* - \bar{\varepsilon}_6^<em>) \right) + (\bar{\varepsilon}_6^</em> + 1) / (\bar{\varepsilon}_6^* - \bar{\varepsilon}_4^*) ]</td>
<td>[ A \left[ \bar{\varepsilon}_3^* + \bar{\varepsilon}_4^* + 2(\bar{\varepsilon}_6^* + 1) + (\bar{\varepsilon}_5^* - 1)^2 / (\bar{\varepsilon}_5^* - \bar{\varepsilon}_4^<em>) \right] + (\bar{\varepsilon}_6^</em> + 1)^2 / (\bar{\varepsilon}_6^* - \bar{\varepsilon}_5^*) ]</td>
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<td>R</td>
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<td>---</td>
</tr>
<tr>
<td>33</td>
<td>0.0</td>
<td>( A \left[ 3 + (\bar{E}_1 + 1)/(\bar{E}_1 - \bar{E}_5) + (\bar{E}_5 - 1)/(\bar{E}_5 - \bar{E}_1^<em>) \right] + 2 B \left[ (\bar{E}_5 - 1)/(\bar{E}_5 - \bar{E}_1^</em>) \right] + (\bar{E}_6 + 1)/(\bar{E}_6 - \bar{E}_5) ) - 1</td>
<td>( A \left[ \bar{E}_2^* + \bar{E}_3^* + \bar{E}_4^* + \bar{E}_5^* + 2 \bar{E}_6^* + 2 + (\bar{E}_1^* + 1)/(\bar{E}_1^* - \bar{E}_5^<em>) + (\bar{E}_5^</em> - 1)/(\bar{E}_5^* - \bar{E}_1^*) \right] + P )</td>
</tr>
<tr>
<td>34</td>
<td>0.0</td>
<td>( A \left[ 3 + (\bar{E}_2^* - 1)/(\bar{E}_2^* - \bar{E}_5^<em>) + (\bar{E}_5^</em> - 1)/(\bar{E}_5^* - \bar{E}_2^*) \right] + 2 B ) - 1</td>
<td>( A \left[ \bar{E}_1^* + \bar{E}_3^* + \bar{E}_4^* + \bar{E}_5^* + 2 \bar{E}_6^* + 6 + (\bar{E}_2^* - 1)/(\bar{E}_2^* - \bar{E}_5^<em>) + (\bar{E}_5^</em> - 1)/(\bar{E}_5^* - \bar{E}_2^<em>) \right] + B (\bar{E}_5^</em> + \bar{E}_6^* + 2) + P )</td>
</tr>
<tr>
<td>Case*</td>
<td>Q</td>
<td>R</td>
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</tr>
<tr>
<td>35</td>
<td>0.0</td>
<td>$4A + 2B \left[ (\varepsilon_5^* - 1)/(\varepsilon_5^* - \varepsilon_6^<em>) + (\varepsilon_6^</em> + 1)/(\varepsilon_6^* - \varepsilon_5^*) \right] - 1$</td>
<td>$A \left[ \varepsilon_1^* + \varepsilon_2^* + \varepsilon_3^* + \varepsilon_4^* + 2 \varepsilon_5^* + 2 \varepsilon_6^* \right] / 2 + B \left[ \varepsilon_5^* - \varepsilon_6^* \right] / 2 + \varepsilon_6^* \left[ (\varepsilon_5^* + 1)/(\varepsilon_6^* - \varepsilon_5^*) \right] + P$</td>
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<td>36</td>
<td>$A/2(\varepsilon_5^* - \varepsilon_1^*)$</td>
<td>$A \left[ 3 + (\varepsilon_5^* - 1)/(\varepsilon_5^* - \varepsilon_6^<em>) \right] + 2B \left[ (\varepsilon_5^</em> - 1)/(\varepsilon_5^* - \varepsilon_6^*) \right] - 1$</td>
<td>$A \left[ \varepsilon_2^* + \varepsilon_3^* + \varepsilon_4^* + \varepsilon_5^* + 2 \varepsilon_6^* \right] + \varepsilon_6^* \left[ (\varepsilon_5^* + 1)/(\varepsilon_6^* - \varepsilon_5^*) \right] + P$</td>
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<tr>
<td>37</td>
<td>$A/2 (\tilde{E}_6 - \tilde{E}_4)$</td>
<td>$A \left[ 2 + (\tilde{E}_1 + 1)/(\tilde{E}_3 - \tilde{E}_5) + (\tilde{E}_4 - 1)/(\tilde{E}_5 - \tilde{E}_1) + \tilde{E}_3 + \tilde{E}_4 + \tilde{E}_5 + \tilde{E}_6 + 4 + (\tilde{E}_1 + 1)^2/(\tilde{E}_1 - \tilde{E}_5) + (\tilde{E}_5 - \tilde{E}_4)(\tilde{E}_4 - \tilde{E}_3)\right] + 2B$</td>
<td>$A \left[ \tilde{E}_1 + \tilde{E}_3 + \tilde{E}_5 + \tilde{E}_6 + 4 + (\tilde{E}_1 + 1)^2/(\tilde{E}_1 - \tilde{E}_5) + (\tilde{E}_5 - \tilde{E}_4)(\tilde{E}_4 - \tilde{E}_3)\right] + 2 + B \left[ (\tilde{E}_5 - 1)/(\tilde{E}_5 - \tilde{E}_4)(\tilde{E}_5 - \tilde{E}_3)\right] + P$</td>
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<td>38</td>
<td>$A/2 (\tilde{E}_5 - \tilde{E}_2)$</td>
<td>$A \left[ 3 + (\tilde{E}_5 + 1)/(\tilde{E}_5 - \tilde{E}_2)\right] + 2B - 1$</td>
<td>$A \left[ \tilde{E}_1 + \tilde{E}_3 + \tilde{E}_4 + \tilde{E}_5 + \tilde{E}_6 + 6 + (\tilde{E}_5 + 1)^2/(\tilde{E}_5 - \tilde{E}_2)\right] /2 + B \left( \tilde{E}_5 + \tilde{E}_6 + 2\right) + P$</td>
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<td>39</td>
<td>(\frac{A}{2}(\bar{E}_5^* - \bar{E}_2^<em>)) + (\frac{A}{2}(\bar{E}_6^</em> - \bar{E}_4^*))</td>
<td>(A\left[2 + (\bar{E}_5^* + 1)/(\bar{E}_5^* - \bar{E}_2^<em>) + (\bar{E}_6^</em> + 1)/(\bar{E}_6^* - \bar{E}_4^*)\right] + 2B - 1)</td>
<td>(A\left[\bar{E}_1^* + \bar{E}_3^* + \bar{E}_5^* + \bar{E}_6^* + 4 + (\bar{E}_5^* - \bar{E}_2^<em>)^2 / (\bar{E}_5^</em> - \bar{E}_2^<em>) + (\bar{E}_6^</em> + 1)^2 / (\bar{E}_6^* - \bar{E}_4^<em>)\right] / 2 + B(\bar{E}_5^</em> + \bar{E}_6^* + 2) + P)</td>
</tr>
<tr>
<td>4Q</td>
<td>(\frac{A}{2}(\bar{E}_6^* - \bar{E}_4^*))</td>
<td>(A\left[2 + (\bar{E}_2^* - 1)/(\bar{E}_2^* - \bar{E}_5^<em>) + (\bar{E}_5^</em> + 1)/(\bar{E}_5^* - \bar{E}_2^<em>) + (\bar{E}_6^</em> + 1)/(\bar{E}_6^* - \bar{E}_4^*)\right] + 2B - 1)</td>
<td>(A\left[\bar{E}_1^* + \bar{E}_3^* + \bar{E}_5^* + \bar{E}_6^* + 4 + (\bar{E}_5^* - \bar{E}_2^<em>)^2 / (\bar{E}_5^</em> - \bar{E}_2^<em>) + (\bar{E}_6^</em> + 1)^2 / (\bar{E}_6^* - \bar{E}_4^<em>)\right] / 2 + B(\bar{E}_5^</em> + \bar{E}_6^* + 2) + P)</td>
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</tr>
<tr>
<td>41</td>
<td>0.0</td>
<td>$A\left[ 2 + (E_1^* - 1)/(E_2^* - E_5^<em>) + (E_5^</em> + 1)/(E_5^* - E_2^<em>) + \frac{(E_4^</em> - 1)}{(E_4^* - E_5^<em>)} + \frac{(E_6^</em> + 1)}{(E_6^* - E_4^*)} \right] + 2B - 1$</td>
<td>$A\left[ E_1^* + E_3^* + E_5^* + E_6^* + 4 + \frac{(E_2^* - 1)^2}{(E_2^* - E_5^<em>)} + \frac{(E_5^</em> + 1)^2}{(E_5^* - E_2^<em>)} + \frac{(E_4^</em> - E_5^<em>)^2}{(E_4^</em> - E_5^<em>)} + \frac{(E_6^</em> + 1)^2}{(E_6^* - E_4^<em>)} \right]/2 + B\left[ E_5^</em> + E_6^* + 2 \right] + P$</td>
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<td>42</td>
<td>0.0</td>
<td>$A\left[ 2 + (E_1^* + 1)/(E_2^* - E_5^<em>) + (E_5^</em> - 1)/(E_5^* - E_2^<em>) + \frac{(E_4^</em> - 1)}{(E_4^* - E_6^<em>)} + \frac{(E_6^</em> + 1)}{(E_6^* - E_4^<em>)} \right] + 2B\left[ \frac{(E_5^</em> - 1)}{(E_5^* - E_2^<em>)} + \frac{(E_6^</em> + 1)}{(E_6^* - E_4^*)} - 1 \right]$</td>
<td>$A\left[ E_2^* + E_3^* + E_5^* + E_6^* + \frac{(E_1^* + 1)^2}{(E_1^* - E_5^<em>)} + \frac{(E_5^</em> + 1)^2}{(E_5^* - E_1^<em>)} + \frac{(E_4^</em> + 1)^2}{(E_4^* - E_6^<em>)} + \frac{(E_6^</em> - E_5^<em>)^2}{(E_6^</em> - E_5^<em>)} \right]/2 + B\left[ \frac{(E_5^</em> - 1)^2}{(E_5^* - E_2^<em>)} + \frac{(E_6^</em> + 1)^2}{(E_6^* - E_5^*)} \right] + P$</td>
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<td>$A/2(\bar{\varepsilon}_1 - \bar{\varepsilon}_5)$</td>
<td>$A\left[ 1 + (\bar{\varepsilon}_1 + 1)/(\bar{\varepsilon}_1 - \bar{\varepsilon}_5) \right]$</td>
<td>$A\left[ \bar{\varepsilon}_3 + \bar{\varepsilon}_2 + (\bar{\varepsilon}_1 + 1)^2/(\bar{\varepsilon}_1 - \bar{\varepsilon}_5) \right]$</td>
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<td>$+ A/2(\bar{\varepsilon}_6 - \bar{\varepsilon}_5)$</td>
<td>$(\bar{\varepsilon}_1 + 1)/(\bar{\varepsilon}_1 - \bar{\varepsilon}_5) + (\bar{\varepsilon}_4 - 1)/(\bar{\varepsilon}_4 - \bar{\varepsilon}_6) + (\bar{\varepsilon}^<em>_6 + 1)/(\bar{\varepsilon}^</em>_6 - \bar{\varepsilon}_5)$</td>
<td>$+ (\bar{\varepsilon}^<em>_1 - 1)^2/(\bar{\varepsilon}^</em>_2 - \bar{\varepsilon}^*_5) + (\bar{\varepsilon}_4 - 1)^2/(\bar{\varepsilon}_4 - \bar{\varepsilon}_5)$</td>
</tr>
<tr>
<td></td>
<td>$+ B/(\bar{\varepsilon}_6 - \bar{\varepsilon}_5)$</td>
<td>$(\bar{\varepsilon}_6 - \bar{\varepsilon}_5) + B(\bar{\varepsilon}_1 + 1)/(\bar{\varepsilon}^*_6 - \bar{\varepsilon}_5) - 1$</td>
<td>$(\bar{\varepsilon}^<em>_6 - \bar{\varepsilon}_5) + B(\bar{\varepsilon}^</em>_6 + 1)/(\bar{\varepsilon}^*_6 - \bar{\varepsilon}_5) + P$</td>
</tr>
</tbody>
</table>

where $A = K_3^1 K_2^1 (K_3 + 2 K_1 K_2)$, $B = K_3^2 / (K_3 + 2 K_1 K_2)$

* Numbers refer to Figure II.8.

**P is in terms of $P_y$

$\bar{\varepsilon}^*_1$ to $\bar{\varepsilon}^*_6$ denote bending, residual and warping strains divided by yield strain at points 1 to 6 (Figure I.4)
Table 13: Equations for $M^X/M^X_y$ for the different strain configurations  
(Biaxial loading neglecting residual stresses)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$M^X/M^X_y$ **</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\phi^X$</td>
</tr>
<tr>
<td>2</td>
<td>$\phi^X + f_i \left( \frac{(\varepsilon_3 + 1)^2}{(\varepsilon_3 - \varepsilon_4)} \right)$</td>
</tr>
<tr>
<td>3</td>
<td>$\phi^X + f_i \left[ \frac{(\varepsilon_3 + 1)^2}{(\varepsilon_3 - \varepsilon_4)} + \frac{(\varepsilon_2 - 1)^2}{(\varepsilon_2 - \varepsilon_1)} \right]$</td>
</tr>
<tr>
<td>4</td>
<td>$\phi^X + f_i \left[ \frac{(\varepsilon_3 + 1)^2}{(\varepsilon_3 - \varepsilon_4)} - \frac{(\varepsilon_i + 1)^2}{(\varepsilon_i - \varepsilon_2)} \right]$</td>
</tr>
<tr>
<td>5</td>
<td>$\phi^X + f_i \left[ \frac{(\varepsilon_3 + 1)^2}{(\varepsilon_3 - \varepsilon_4)} - \frac{(\varepsilon_i + 1)^2}{(\varepsilon_i - \varepsilon_2)} - \frac{(\varepsilon_2 - 1)^2}{(\varepsilon_2 - \varepsilon_1)} \right]$</td>
</tr>
</tbody>
</table>
Table 13 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$M^X/M_y^{**}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$\phi^x + F_1 (\varepsilon_3 + 1)/(\varepsilon_3 - \varepsilon_4) + F_2 \left( (\varepsilon_3 + 1)^2 (\varepsilon_6 - 3\varepsilon_5 - 2) / 3 (\varepsilon_6 - \varepsilon_5)^2 \right)$</td>
</tr>
<tr>
<td>7</td>
<td>$\phi^x + F_1 \left[ (\varepsilon_3 + 1)^2 / (\varepsilon_3 - \varepsilon_4) - (\varepsilon_3 - \varepsilon_4) / (\varepsilon_2 - \varepsilon_1) \right] + F_2 \left[ (\varepsilon_6 + 1)^2 (\varepsilon_6 - 3\varepsilon_5 - 2) / 3 (\varepsilon_6 - \varepsilon_5)^2 \right]$</td>
</tr>
<tr>
<td>8</td>
<td>$\phi^x + F_1 \left[ (\varepsilon_3 + 1)^2 / (\varepsilon_3 - \varepsilon_4) - (\varepsilon_3 + 1)^2 / (\varepsilon_3 - \varepsilon_1) \right] + F_2 \left[ (\varepsilon_6 + 1)^2 (\varepsilon_6 - 3\varepsilon_5 - 2) / 3 (\varepsilon_6 - \varepsilon_5)^2 \right]$</td>
</tr>
<tr>
<td>9</td>
<td>$\phi^x + F_1 \left[ (\varepsilon_3 + 1)^2 / (\varepsilon_3 - \varepsilon_4) - (\varepsilon_3 - \varepsilon_4) / (\varepsilon_2 - \varepsilon_1) - (\varepsilon_3 - \varepsilon_1) / (\varepsilon_3 - \varepsilon_1) \right] + F_2 \left[ (\varepsilon_6 + 1)^2 (\varepsilon_6 - 3\varepsilon_5 - 2) / 3 (\varepsilon_6 - \varepsilon_5)^2 \right]$</td>
</tr>
<tr>
<td>10</td>
<td>$\phi^x + F_1 (\varepsilon_3 + \varepsilon_4 + 2) + F_2 \left[ (\varepsilon_6 + 1)^2 (\varepsilon_6 - 3\varepsilon_5 - 2) / 3 (\varepsilon_6 - \varepsilon_5)^2 \right]$</td>
</tr>
<tr>
<td>11</td>
<td>$\phi^x + F_1 \left[ (\varepsilon_3 + \varepsilon_4 + 2 - (\varepsilon_2 - 1)^2 / (\varepsilon_2 - \varepsilon_1) \right] + F_2 \left[ (\varepsilon_6 + 1)^2 (\varepsilon_6 - 3\varepsilon_5 - 2) / 3 (\varepsilon_6 - \varepsilon_5)^2 \right]$</td>
</tr>
<tr>
<td>Case*</td>
<td>( \frac{M^X}{M^Y} ) **</td>
</tr>
<tr>
<td>-------</td>
<td>-----------------</td>
</tr>
<tr>
<td>12</td>
<td>( \phi^X + F_i \left[ \frac{(\varepsilon_3 + 1)^2}{\varepsilon_3 - \varepsilon_4} + \frac{(\varepsilon_4 - 1)^2}{\varepsilon_4 - \varepsilon_3} - \frac{(\varepsilon_1 + 1)^2}{\varepsilon_1 - \varepsilon_2} - \frac{(\varepsilon_2 - 1)^2}{\varepsilon_2 - \varepsilon_1} \right] )</td>
</tr>
<tr>
<td>13</td>
<td>( \phi^X + F_i \left[ \frac{\varepsilon_3 + \varepsilon_4 + 2 - (\varepsilon_1 + 1)^2}{\varepsilon_1 - \varepsilon_2} \right] + F_2 \left[ \frac{(\varepsilon_6 + 1)^2}{3(\varepsilon_6 - 3\varepsilon_5 - 2)} \right] )</td>
</tr>
<tr>
<td>14</td>
<td>( \phi^X + F_i \left[ \frac{\varepsilon_3 + \varepsilon_4 + 2 - (\varepsilon_1 + 1)^2}{\varepsilon_1 - \varepsilon_2} - \frac{(\varepsilon_2 - 1)^2}{\varepsilon_2 - \varepsilon_1} \right] + F_2 \left[ \frac{(\varepsilon_6 + 1)^2}{3(\varepsilon_6 - 3\varepsilon_5 - 2)} \right] )</td>
</tr>
<tr>
<td>15</td>
<td>( \phi^X + F_i \left[ \frac{(\varepsilon_3 + 1)^2}{\varepsilon_3 - \varepsilon_4} - \frac{(\varepsilon_2 - 1)^2}{\varepsilon_2 - \varepsilon_1} \right] + F_2 \left[ \frac{(\varepsilon_6 + 1)^2}{3(\varepsilon_6 - 3\varepsilon_5 - 2)} \right] )</td>
</tr>
<tr>
<td>16</td>
<td>( \phi^X + F_i \left[ \frac{\varepsilon_3 + \varepsilon_4 + 2 - (\varepsilon_1 + 1)^2}{\varepsilon_1 - \varepsilon_2} - \frac{(\varepsilon_2 - 1)^2}{\varepsilon_2 - \varepsilon_1} \right] + F_2 \left[ \frac{(\varepsilon_6 + 1)^2}{3(\varepsilon_6 - 3\varepsilon_5 - 2)} \right] )</td>
</tr>
<tr>
<td>17</td>
<td>( \phi^X + F_i \left[ \frac{\varepsilon_3 + \varepsilon_4 + 2 - (\varepsilon_1 + 1)^2}{\varepsilon_1 - \varepsilon_2} - \frac{(\varepsilon_2 - 1)^2}{\varepsilon_2 - \varepsilon_1} \right] + F_2 \left( \varepsilon_6 - \varepsilon_5 \right) / 3 )</td>
</tr>
<tr>
<td>Case*</td>
<td>$\frac{M^x}{M^y}$ **</td>
</tr>
<tr>
<td>-------</td>
<td>-------------------</td>
</tr>
<tr>
<td>18</td>
<td>$\phi^x + F_1 \left( \varepsilon_3 + \varepsilon_4 - \varepsilon_1 - \varepsilon_2 + 4 \right) + F_2 \left[ \left( \varepsilon_6 + 1 \right)^2 \left( \varepsilon_6 - 3 \varepsilon_5 - 2 \right) / 3 \left( \varepsilon_6 - \varepsilon_5 \right)^2 - \left( \varepsilon_5 - 1 \right)^2 \left( \varepsilon_5 - 3 \varepsilon_6 + 2 \right) / 3 \left( \varepsilon_5 - \varepsilon_6 \right)^2 \right]$</td>
</tr>
<tr>
<td>19</td>
<td>$\phi^x + F_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 - \left( \varepsilon_2 - 1 \right)^2 / \left( \varepsilon_2 - \varepsilon_1 \right) \right] + F_2 \left[ \left( \varepsilon_6 + 1 \right)^2 \left( \varepsilon_6 - 3 \varepsilon_5 - 2 \right) / 3 \left( \varepsilon_6 - \varepsilon_5 \right)^2 - \left( \varepsilon_5 - 1 \right)^2 \left( \varepsilon_5 - 3 \varepsilon_6 + 2 \right) / 3 \left( \varepsilon_5 - \varepsilon_6 \right)^2 \right]$</td>
</tr>
<tr>
<td>20</td>
<td>$\phi^x + F_1 \left[ \left( \varepsilon_3 + 1 \right)^2 / \left( \varepsilon_3 - \varepsilon_4 \right) - \left( \varepsilon_1 + 1 \right)^2 / \left( \varepsilon_1 - \varepsilon_2 \right) - \left( \varepsilon_2 - 1 \right)^2 / \left( \varepsilon_2 - \varepsilon_1 \right) \right] + F_2 \left[ \left( \varepsilon_6 + 1 \right)^2 \left( \varepsilon_6 - 3 \varepsilon_5 - 2 \right) / 3 \left( \varepsilon_6 - \varepsilon_5 \right)^2 - \left( \varepsilon_5 - 1 \right)^2 \left( \varepsilon_5 - 3 \varepsilon_6 + 2 \right) / 3 \left( \varepsilon_5 - \varepsilon_6 \right)^2 \right]$</td>
</tr>
<tr>
<td>21</td>
<td>$\phi^x + F_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 - \left( \varepsilon_1 + 1 \right)^2 / \left( \varepsilon_1 - \varepsilon_2 \right) \right] + F_2 \left( \varepsilon_6 - \varepsilon_5 \right) / 3$</td>
</tr>
<tr>
<td>22</td>
<td>$\phi^x + F_1 \left[ \left( \varepsilon_3 + 1 \right)^2 / \left( \varepsilon_3 - \varepsilon_4 \right) - \left( \varepsilon_1 + 1 \right)^2 / \left( \varepsilon_1 - \varepsilon_2 \right) \right] + F_2 \left( \varepsilon_6 - \varepsilon_5 \right) / 3$</td>
</tr>
<tr>
<td>23</td>
<td>$\phi^x + F_1 \left[ \left( \varepsilon_3 + 1 \right)^2 / \left( \varepsilon_3 - \varepsilon_4 \right) - \left( \varepsilon_1 + 1 \right)^2 / \left( \varepsilon_1 - \varepsilon_2 \right) - \left( \varepsilon_2 - 1 \right)^2 / \left( \varepsilon_2 - \varepsilon_1 \right) \right] + F_2 \left( \varepsilon_6 - \varepsilon_5 \right) / 3$</td>
</tr>
</tbody>
</table>
### Table 13 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$\frac{M^X}{M^X_{y}}$ **</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>$\phi^X + F_1 \left[ \frac{(E_3 + 1)^2}{E_3 - E_4} + \frac{(E_4 - 1)^2}{E_4 - E_3} - \frac{(E_1 + 1)^2}{E_1 - E_2} - \frac{(E_2 - 1)^2}{E_2 - E_1} \right] + F_2 \frac{(E_6 - E_5)}{3}$</td>
</tr>
<tr>
<td>25</td>
<td>$\phi^X + F_1 \left[ \frac{(E_3 + 1)^2}{E_3 - E_4} + \frac{(E_4 - 1)^2}{E_4 - E_3} - \frac{(E_1 + 1)^2}{E_1 - E_2} - \frac{(E_2 - 1)^2}{E_2 - E_1} \right] + F_2 \left[ \frac{(E_6 + 1)^2}{E_6 - 3E_5 - 2} \right] / 3 \left( E_5 - E_6 \right)^2$</td>
</tr>
<tr>
<td>26</td>
<td>$\phi^X + F_1 \left[ \frac{(E_3 + 1)^2}{E_3 - E_4} + \frac{(E_4 - 1)^2}{E_4 - E_3} - \frac{(E_1 + 1)^2}{E_1 - E_2} - \frac{(E_2 - 1)^2}{E_2 - E_1} \right] + F_2 \left[ \frac{(E_6 + 1)^2}{E_6 - 3E_5 - 2} \right] / 3 \left( E_5 - E_6 \right)^2$</td>
</tr>
</tbody>
</table>

where:  
- $F_1 = (1 + K_2) \left[ (1 + K_2/2) K_1 K_2 / \left[ 2/3 K_3 + 4 K_1 K_2 \left( 1 + K_2/2 \right)^2 \right] \right]$  
- $F_2 = (1 + K_2) \left[ (1 + K_2/2) K_3 / \left[ 2/3 K_3 + 4 K_1 K_2 \left( 1 + K_2/2 \right)^2 \right] \right]$  

* Numbers refer to Figure II.7.  
** All curvatures are in terms of $\phi^X_y$.  

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Table 14: Equations for \( \frac{M^Y}{M^X_y} \) for the different strain configurations
(Biaxial loading neglecting residual stresses)

<table>
<thead>
<tr>
<th>Case*</th>
<th>( \frac{M^Y}{M^X_y} )**</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( F_3 \phi^y )</td>
</tr>
<tr>
<td>2</td>
<td>( F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3+1)^2(\varepsilon_3 - 3 \varepsilon_4 - 2)}{3(\varepsilon_3 - \varepsilon_4)^2} \right] )</td>
</tr>
<tr>
<td>3</td>
<td>( F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3+1)^2(\varepsilon_3 - 3 \varepsilon_4 - 2)}{3(\varepsilon_3 - \varepsilon_4)^2} - (\varepsilon_2-1)^2(\varepsilon_2 - 3 \varepsilon_1 - 2) / 3(\varepsilon_2 - \varepsilon_1)^2 \right] )</td>
</tr>
<tr>
<td>4</td>
<td>( F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3+1)^2(\varepsilon_3 - 3 \varepsilon_4 - 2)}{3(\varepsilon_3 - \varepsilon_4)^2} + (\varepsilon_3+1)^2(\varepsilon_3 - 3 \varepsilon_4 - 2) / 3(\varepsilon_3 - \varepsilon_4)^2 \right] )</td>
</tr>
<tr>
<td>5</td>
<td>( F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3+1)^2(\varepsilon_3 - 3 \varepsilon_4 - 2)}{3(\varepsilon_3 - \varepsilon_4)^2} - (\varepsilon_2-1)^2(\varepsilon_2 - 3 \varepsilon_1 + 2) / 3(\varepsilon_2 - \varepsilon_1)^2 \right] )</td>
</tr>
<tr>
<td>Case*</td>
<td>$\frac{W_y}{M_y}**$</td>
</tr>
<tr>
<td>-------</td>
<td>------------------</td>
</tr>
<tr>
<td>6</td>
<td>$F_3 \phi^y + F_4 \left[ (\varepsilon_3 + 1)^2 (\varepsilon_3 - 3 \varepsilon_4 - 2) / 3 (\varepsilon_3 - \varepsilon_4)^2 \right]$</td>
</tr>
<tr>
<td>7</td>
<td>$F_3 \phi^y + F_4 \left[ (\varepsilon_3 + 1)^2 (\varepsilon_3 - 3 \varepsilon_4 - 2) / 3 (\varepsilon_3 - \varepsilon_4)^2 - (\varepsilon_2 - 1)^2 (\varepsilon_2 - 3 \varepsilon_1 + 2) / 3 (\varepsilon_2 - \varepsilon_1)^2 \right]$</td>
</tr>
<tr>
<td>8</td>
<td>$F_3 \phi^y + F_4 \left[ (\varepsilon_1 + 1)^2 (\varepsilon_1 - 3 \varepsilon_2 - 2) / 3 (\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_3 + 1)^2 (\varepsilon_3 - 3 \varepsilon_4 - 2) / 3 (\varepsilon_3 - \varepsilon_4)^2 \right]$</td>
</tr>
<tr>
<td>9</td>
<td>$F_3 \phi^y + F_4 \left[ (\varepsilon_1 + 1)^2 (\varepsilon_1 - 3 \varepsilon_2 - 2) / 3 (\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_3 + 1)^2 (\varepsilon_3 - 3 \varepsilon_4 - 2) / 3 (\varepsilon_3 - \varepsilon_4)^2 - (\varepsilon_2 - 1)^2 (\varepsilon_2 - 3 \varepsilon_1 + 2) / 3 (\varepsilon_2 - \varepsilon_1)^2 \right]$</td>
</tr>
<tr>
<td>10</td>
<td>$F_3 \phi^y + F_4 \left( \varepsilon_3 - \varepsilon_4 \right) / 3$</td>
</tr>
<tr>
<td>11</td>
<td>$F_3 \phi^y + F_4 \left[ (\varepsilon_3 - \varepsilon_4) / 3 - (\varepsilon_2 - 1)^2 (\varepsilon_2 - 3 \varepsilon_1 + 2) / 3 (\varepsilon_2 - \varepsilon_1)^2 \right]$</td>
</tr>
<tr>
<td>Case*</td>
<td>$\frac{M_y}{M_{x,y}}$ **</td>
</tr>
<tr>
<td>-------</td>
<td>-------------------</td>
</tr>
<tr>
<td>12</td>
<td>$F_3 \phi_1^y + F_4 \left[ \frac{(\varepsilon_1 + 1)^2 (\varepsilon_1 - 3 \varepsilon_2 - 2)}{3 (\varepsilon_1 - \varepsilon_2)^2} - \frac{(\varepsilon_2 - 3 \varepsilon_1 + 2)}{3 (\varepsilon_1 - \varepsilon_2)^2} (\varepsilon_2 - \varepsilon_1)^2 + \frac{(\varepsilon_3 + 1)^2 (\varepsilon_3 - 3 \varepsilon_4 - 2)}{3 (\varepsilon_3 - \varepsilon_4)^2} - \frac{(\varepsilon_4 - 1)^2 (\varepsilon_4 - 3 \varepsilon_3 + 2)}{3 (\varepsilon_4 - \varepsilon_3)^2} \right]$</td>
</tr>
<tr>
<td>13</td>
<td>$F_3 \phi_1^y + F_4 \left[ \frac{(\varepsilon_3 - \varepsilon_4)/3 + (\varepsilon_1 + 1)^2 (\varepsilon_1 - 3 \varepsilon_2 - 2)/3 (\varepsilon_1 - \varepsilon_2)^2} \right]$</td>
</tr>
<tr>
<td>14</td>
<td>$F_3 \phi_1^y + F_4 \left[ \frac{(\varepsilon_3 - \varepsilon_4)/3 + (\varepsilon_3 + 1)^2 (\varepsilon_3 - 3 \varepsilon_4 - 2)/3 (\varepsilon_3 - \varepsilon_4)^2} - (\varepsilon_4 - 1)^2 (\varepsilon_4 - 3 \varepsilon_3 + 2)/3 (\varepsilon_4 - \varepsilon_3)^2 \right]$</td>
</tr>
<tr>
<td>15</td>
<td>$F_3 \phi_1^y + F_4 \left[ \frac{(\varepsilon_3 + 1)^2 (\varepsilon_3 - 3 \varepsilon_4 - 2)/3 (\varepsilon_3 - \varepsilon_4)^2} - (\varepsilon_2 - 1)^2 (\varepsilon_2 - 3 \varepsilon_1 + 2)/3 (\varepsilon_2 - \varepsilon_1)^2 \right]$</td>
</tr>
<tr>
<td>16</td>
<td>$F_3 \phi_1^y + F_4 \left[ \frac{(\varepsilon_3 - \varepsilon_4)/3 + (\varepsilon_3 + 1)^2 (\varepsilon_3 - 3 \varepsilon_4 - 2)/3 (\varepsilon_3 - \varepsilon_4)^2} - (\varepsilon_4 - 1)^2 (\varepsilon_4 - 3 \varepsilon_3 + 2)/3 (\varepsilon_4 - \varepsilon_3)^2 \right]$</td>
</tr>
<tr>
<td>17</td>
<td>$F_3 \phi_1^y + F_4 \left[ \frac{(\varepsilon_3 - \varepsilon_4)/3 + (\varepsilon_3 + 1)^2 (\varepsilon_3 - 3 \varepsilon_4 - 2)/3 (\varepsilon_3 - \varepsilon_4)^2} - (\varepsilon_4 - 1)^2 (\varepsilon_4 - 3 \varepsilon_3 + 2)/3 (\varepsilon_4 - \varepsilon_3)^2 \right]$</td>
</tr>
<tr>
<td>Case*</td>
<td>$\frac{M^y}{M^x}$ **</td>
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<tr>
<td>-------</td>
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</tr>
<tr>
<td>18</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3 - \varepsilon_4)}{3} - \frac{(\varepsilon_2 - \varepsilon_1)}{3} \right]$</td>
</tr>
<tr>
<td>19</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3 - \varepsilon_4)}{3} - \frac{(\varepsilon_2 - 1)^2}{3} \left( \frac{\varepsilon_2 - 3 \varepsilon_1 + 2}{3} \right)^2 \right]$</td>
</tr>
<tr>
<td>20</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_1 + 1)^2}{3} \left( \frac{\varepsilon_1 - 3 \varepsilon_2 - 2}{3} - \frac{(\varepsilon_3 - \varepsilon_2)^2}{3} - \frac{(\varepsilon_3 + 1)^2}{3} \left( \frac{\varepsilon_3 - 3 \varepsilon_4 - 2}{3} \right)^2 \right) \right]$</td>
</tr>
<tr>
<td>21</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_1 + 1)^2}{3} \left( \frac{\varepsilon_1 - 3 \varepsilon_2 - 2}{3} - \frac{(\varepsilon_3 - \varepsilon_2)^2}{3} - \frac{(\varepsilon_3 + 1)^2}{3} \left( \frac{\varepsilon_3 - 3 \varepsilon_4 - 2}{3} \right)^2 \right) \right]$</td>
</tr>
<tr>
<td>22</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_1 + 1)^2}{3} \left( \frac{\varepsilon_1 - 3 \varepsilon_2 - 2}{3} - \frac{(\varepsilon_3 - \varepsilon_2)^2}{3} - \frac{(\varepsilon_3 + 1)^2}{3} \left( \frac{\varepsilon_3 - 3 \varepsilon_4 - 2}{3} \right)^2 \right) \right]$</td>
</tr>
<tr>
<td>23</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_1 + 1)^2}{3} \left( \frac{\varepsilon_1 - 3 \varepsilon_2 - 2}{3} - \frac{(\varepsilon_3 - \varepsilon_2)^2}{3} - \frac{(\varepsilon_3 + 1)^2}{3} \left( \frac{\varepsilon_3 - 3 \varepsilon_4 - 2}{3} \right)^2 \right) \right]$</td>
</tr>
</tbody>
</table>
Table 14 (Cont'd.)

<table>
<thead>
<tr>
<th>Case</th>
<th>$\frac{M^y}{M^x}$ **</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_1 + 1)^2(\varepsilon_1 - 3 \varepsilon_2 - 2)}{3(\varepsilon_1 - \varepsilon_2)^2} - \frac{(\varepsilon_2 - 1)^2(\varepsilon_2 - 3 \varepsilon_1 + 2)}{3(\varepsilon_2 - \varepsilon_1)^2} + \frac{(\varepsilon_3 + 1)^2(\varepsilon_3 - 3 \varepsilon_4 - 2)}{2(\varepsilon_3 - \varepsilon_4)^2} - \frac{(\varepsilon_4 - 1)^2(\varepsilon_4 - 3 \varepsilon_3 + 2)}{3(\varepsilon_4 - \varepsilon_3)^2} \right]$</td>
</tr>
<tr>
<td>25</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_1 + 1)^2(\varepsilon_1 - 3 \varepsilon_2 - 2)}{3(\varepsilon_1 - \varepsilon_2)^2} - \frac{(\varepsilon_2 - 1)^2(\varepsilon_2 - 3 \varepsilon_1 + 2)}{3(\varepsilon_2 - \varepsilon_1)^2} + \frac{(\varepsilon_3 + 1)^2(\varepsilon_3 - 3 \varepsilon_4 - 2)}{3(\varepsilon_3 - \varepsilon_4)^2} - \frac{(\varepsilon_4 - 1)^2(\varepsilon_4 - 3 \varepsilon_3 + 2)}{3(\varepsilon_4 - \varepsilon_3)^2} \right]$</td>
</tr>
<tr>
<td>26</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_1 + 1)^2(\varepsilon_1 - 3 \varepsilon_2 - 2)}{3(\varepsilon_1 - \varepsilon_2)^2} - \frac{(\varepsilon_2 - 1)^2(\varepsilon_2 - 3 \varepsilon_1 + 2)}{3(\varepsilon_2 - \varepsilon_1)^2} + \frac{(\varepsilon_3 + 1)^2(\varepsilon_3 - 3 \varepsilon_4 - 2)}{3(\varepsilon_3 - \varepsilon_4)^2} - \frac{(\varepsilon_4 - 1)^2(\varepsilon_4 - 3 \varepsilon_3 + 2)}{3(\varepsilon_4 - \varepsilon_3)^2} \right]$</td>
</tr>
</tbody>
</table>

where:

$F_3 = \frac{(8 K_1^3 K_2 + K_3^3)}{6} \left[ \frac{2}{3} K_3 + 4K_1 K_2 (1 + K_2/2)^2 \right]$ 

$F_4 = \frac{(1 + K_2) K_1^2 K_2}{\left[ \frac{2}{3} K_3 + 4K_1 K_2 (1 + K_2/2)^2 \right]}$

* Numbers refer to Figure II.7.
* All curvatures are in terms of $\phi^y$. 

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Table 15: Equations for $M^{x}/M^{y}$ for the different strain configurations
(Biaxial loading considering residual stresses)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$M^{x}/M^{y}$**</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\phi^{x}$</td>
</tr>
<tr>
<td>2</td>
<td>$\phi^{x} + F_{1} \frac{(\varepsilon_{3} + 1)^{2}}{2} (\varepsilon_{3} - \varepsilon_{c})$</td>
</tr>
<tr>
<td>3</td>
<td>$\phi^{x} + F_{1} \left[ \frac{(\varepsilon_{3} + 1)^{2}}{(\varepsilon_{3} - \varepsilon_{c})} - \frac{(\varepsilon_{2} + 1)^{2}}{(\varepsilon_{2} - \varepsilon_{s})} \right] / 2$</td>
</tr>
<tr>
<td>4</td>
<td>$\phi^{x} + F_{1} \left[ \frac{(\varepsilon_{3} + 1)^{2}}{(\varepsilon_{3} - \varepsilon_{c})} - \frac{(\varepsilon_{1} + 1)^{2}}{(\varepsilon_{1} - \varepsilon_{s})} \right] / 2$</td>
</tr>
<tr>
<td>5</td>
<td>$\phi^{x} + F_{1} \left[ \frac{(\varepsilon_{3} + 1)^{2}}{(\varepsilon_{3} - \varepsilon_{c})} - \frac{(\varepsilon_{1} + 1)^{2}}{(\varepsilon_{1} - \varepsilon_{s})} - \frac{(\varepsilon_{2} + 1)^{2}}{(\varepsilon_{2} - \varepsilon_{s})} \right] / 2$</td>
</tr>
<tr>
<td>Case*</td>
<td>( \phi^x + F_{11} \left[ (e_3 + e_3)^2 / (e_3 - e_3) + (e_4 + e_4)^2 / (e_4 - e_4) \right] / 2 )</td>
</tr>
<tr>
<td>-------</td>
<td>--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>6</td>
<td>( \phi^x + F_{11} \left[ (e_3 + e_3 + 2 + (e_3 + e_3)^2 / (e_3 - e_3) + (e_4 + e_4)^2 / (e_4 - e_4) \right] / 2 )</td>
</tr>
<tr>
<td>7</td>
<td>( \phi^x + F_{11} \left[ (e_3 + e_3 + 2 + (e_3 + e_3)^2 / (e_3 - e_3) + (e_4 + e_4)^2 / (e_4 - e_4) \right] / 2 )</td>
</tr>
<tr>
<td>8</td>
<td>( \phi^x + F_{11} \left[ (e_3 + e_3 + 2 + (e_3 + e_3)^2 / (e_3 - e_3) + (e_4 + e_4)^2 / (e_4 - e_4) \right] / 2 )</td>
</tr>
<tr>
<td>9</td>
<td>( \phi^x + F_{11} \left[ (e_3 + e_3 + 2 + (e_3 + e_3)^2 / (e_3 - e_3) + (e_4 + e_4)^2 / (e_4 - e_4) \right] / 2 )</td>
</tr>
<tr>
<td>10</td>
<td>( \phi^x + F_{11} \left[ (e_3 + e_3 + 2 + (e_3 + e_3)^2 / (e_3 - e_3) + (e_4 + e_4)^2 / (e_4 - e_4) \right] / 2 )</td>
</tr>
<tr>
<td>11</td>
<td>( \phi^x + F_{11} \left[ (e_3 + e_3 + 2 + (e_3 + e_3)^2 / (e_3 - e_3) + (e_4 + e_4)^2 / (e_4 - e_4) \right] / 2 )</td>
</tr>
</tbody>
</table>

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
<table>
<thead>
<tr>
<th>Case*</th>
<th>M^X/M^Y **</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>$\phi^x + F_1^x \left[ (e_3 + e_5) + (e_4 + e_2) + (e_5 - e_5) \right] / 2$</td>
</tr>
<tr>
<td>13</td>
<td>$\phi^x + F_2^x \left[ (e_3 + e_5) + (e_4 + e_2) + (e_5 - e_5) \right] / 2$</td>
</tr>
<tr>
<td>14</td>
<td>$\phi^x + F_1^x \left[ (e_3 + e_5) + (e_4 + e_2) + (e_5 - e_5) \right] / 2 + F_2^x \left[ (e_3 + e_5) + (e_4 + e_2) + (e_5 - e_5) \right] / 3$</td>
</tr>
<tr>
<td>15</td>
<td>$\phi^x + F_1^x \left[ (e_3 + e_5) + (e_4 + e_2) + (e_5 - e_5) \right] / 2 + F_2^x \left[ (e_3 + e_5) + (e_4 + e_2) + (e_5 - e_5) \right] / 3$</td>
</tr>
<tr>
<td>16</td>
<td>$\phi^x + F_1^x \left[ (e_3 + e_5) + (e_4 + e_2) + (e_5 - e_5) \right] / 2 + F_2^x \left[ (e_3 + e_5) + (e_4 + e_2) + (e_5 - e_5) \right] / 3$</td>
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</table>

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Table 15 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>( M_X^*/M_Y^{**} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>( \phi_X + F_{1}[(\varepsilon_3 + 1)^2/(\varepsilon_3 - \varepsilon_6) - \varepsilon_2 - \varepsilon_5 + 2 - (\varepsilon_5 - 1)^2/(\varepsilon_5 - \varepsilon_1)]/2 - F_2[(\varepsilon_5 - 1)^2/(\varepsilon_5 - 3 \varepsilon_6 + 2) / 3 (\varepsilon_5 - \varepsilon_6)^2] )</td>
</tr>
<tr>
<td>19</td>
<td>( \phi_X + F_{1}[(\varepsilon_3 + \varepsilon_4 + 2 \varepsilon_6 + 4) - (\varepsilon_2 - 1)^2/(\varepsilon_2 - \varepsilon_5)]/2 + F_{2}[(\varepsilon_6 + 1)^2/(\varepsilon_6 - 3\varepsilon_5 - 2) / 3 (\varepsilon_6 - \varepsilon_5)^2] )</td>
</tr>
<tr>
<td>20</td>
<td>( \phi_X + F_{1}[(\varepsilon_3 + 1)^2/(\varepsilon_3 - \varepsilon_6) - \varepsilon_2 - \varepsilon_5 + 2 - (\varepsilon_1 + 1)^2/(\varepsilon_1 - \varepsilon_5) - (\varepsilon_5 - 1)^2/(\varepsilon_5 - \varepsilon_1)]/2 - F_{2}[(\varepsilon_5 - 1)^2/(\varepsilon_5 - 3 \varepsilon_6 + 2) / 3 (\varepsilon_5 - \varepsilon_6)^2] )</td>
</tr>
<tr>
<td>21</td>
<td>( \phi_X + F_{1}[(\varepsilon_3 + 1)^2/(\varepsilon_3 - \varepsilon_6) - (\varepsilon_1 + 1)^2/(\varepsilon_1 - \varepsilon_5) - (\varepsilon_5 - 1)^2/(\varepsilon_5 - \varepsilon_2)]/2 - F_{2}[(\varepsilon_5 - 1)^2/(\varepsilon_5 - 3 \varepsilon_6 + 2) / 3 (\varepsilon_5 - \varepsilon_6)^2] )</td>
</tr>
<tr>
<td>22</td>
<td>( \phi_X + F_{1}[(\varepsilon_3 + 1)^2/(\varepsilon_3 - \varepsilon_6) + (\varepsilon_4 - 1)^2/(\varepsilon_4 - \varepsilon_6) - (\varepsilon_1 + 1)^2/(\varepsilon_1 - \varepsilon_5) - (\varepsilon_2 - 1)^2/(\varepsilon_2 - \varepsilon_5)]/2 )</td>
</tr>
<tr>
<td>23</td>
<td>( \phi_X + F_{1}[(\varepsilon_3 + 1)^2/(\varepsilon_3 - \varepsilon_6) + (\varepsilon_4 - 1)^2/(\varepsilon_4 - \varepsilon_6) - (\varepsilon_1 + 1)^2/(\varepsilon_1 - \varepsilon_5) - (\varepsilon_5 - 1)^2/(\varepsilon_5 - \varepsilon_1) - \varepsilon_2 - \varepsilon_5 + 2]/2 - F_{2}[(\varepsilon_5 - 1)^2/(\varepsilon_5 - 3 \varepsilon_6 + 2) / 3 (\varepsilon_5 - \varepsilon_6)^2] )</td>
</tr>
</tbody>
</table>
### Table 15 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$M^x/M^x_y$ **</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>$\phi^x + F_1 \left[ (\epsilon_3 + \epsilon_5 + 2 \epsilon_6 + \epsilon_4 + (\epsilon_5 - 1)^2) / (\epsilon_5 - \epsilon_3) - (\epsilon_5 - 1) / (\epsilon_5 - \epsilon_1) - (\epsilon_5 - 1) / (\epsilon_5 - \epsilon_2) \right] / 2 - F_2 \left[ (\epsilon_5 - 1)^2 (\epsilon_5 - 3 \epsilon_6 + 2) / 3 (\epsilon_5 - \epsilon_2) \right]^2</td>
</tr>
<tr>
<td>Case*</td>
<td>( \frac{M^X}{M^X} ) **</td>
</tr>
<tr>
<td>-------</td>
<td>-----------------</td>
</tr>
<tr>
<td>29</td>
<td>( \phi^X + F_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 \varepsilon_6 + 4 - (\varepsilon_1 + 1)^2 / (\varepsilon_1 - \varepsilon_5) - (\varepsilon_5 - 1)^2 / (\varepsilon_5 - \varepsilon_1) - (\varepsilon_5 - 1)^2 / (\varepsilon_5 - \varepsilon_2) \right] / 2 - F_2 \left[ (\varepsilon_5 - 1)^2 (\varepsilon_5 - 3 \varepsilon_6 + 2) / 3 (\varepsilon_5 - \varepsilon_6)^2 \right] )</td>
</tr>
<tr>
<td>30</td>
<td>( \phi^X + F_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 + (\varepsilon_6 + 1)^2 / (\varepsilon_6 - \varepsilon_4) - (\varepsilon_1 + 1)^2 / (\varepsilon_1 - \varepsilon_5) \right] / 2 + F_2 \left[ (\varepsilon_6 + 1)^2 (\varepsilon_6 - 3 \varepsilon_5 - 2) / 3 (\varepsilon_6 - \varepsilon_5)^2 \right] )</td>
</tr>
<tr>
<td>31</td>
<td>( \phi^X + F_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 \varepsilon_6 + 4 - (\varepsilon_1 + 1)^2 / (\varepsilon_1 - \varepsilon_5) - (\varepsilon_2 - 1)^2 / (\varepsilon_2 - \varepsilon_5) \right] / 2 + F_2 \left[ (\varepsilon_6 + 1)^2 (\varepsilon_6 - 3 \varepsilon_5 - 2) / 3 (\varepsilon_6 - \varepsilon_5)^2 \right] )</td>
</tr>
<tr>
<td>32</td>
<td>( \phi^X + F_1 \left[ \varepsilon_3 + \varepsilon_4 - \varepsilon_2 - \varepsilon_5 + 4 + (\varepsilon_6 + 1)^2 / (\varepsilon_6 - \varepsilon_4) - (\varepsilon_5 - 1)^2 / (\varepsilon_5 - \varepsilon_1) \right] / 2 + F_2 \left[ (\varepsilon_6 + 1)^2 (\varepsilon_6 - 3 \varepsilon_5 - 2) / 3 (\varepsilon_6 - \varepsilon_5)^2 - (\varepsilon_5 - 1)^2 (\varepsilon_5 - 3 \varepsilon_6 + 2) / 3 (\varepsilon_5 - \varepsilon_6)^2 \right] )</td>
</tr>
<tr>
<td>33</td>
<td>( \phi^X + F_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 \varepsilon_6 - \varepsilon_2 - \varepsilon_5 + 6 - (\varepsilon_1 + 1)^2 / (\varepsilon_1 - \varepsilon_5) - (\varepsilon_5 + 1)^2 / (\varepsilon_5 - \varepsilon_1) \right] / 2 + F_2 \left[ (\varepsilon_6 + 1)^2 (\varepsilon_6 - 3 \varepsilon_5 - 2) / 3 (\varepsilon_6 - \varepsilon_5)^2 - (\varepsilon_5 - 1)^2 (\varepsilon_5 - 3 \varepsilon_6 + 2) / 3 (\varepsilon_5 - \varepsilon_6)^2 \right] )</td>
</tr>
<tr>
<td>Case*</td>
<td>$M^x/M^y_{y}$</td>
</tr>
<tr>
<td>-------</td>
<td>----------------</td>
</tr>
<tr>
<td>34</td>
<td>$\phi^x + f_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 \varepsilon_6 - \varepsilon_1 - \varepsilon_5 + 2 - (\varepsilon_2 - 1)^2 / (\varepsilon_2 - \varepsilon_5) - (\varepsilon_5 + 1)^2 / (\varepsilon_5 - \varepsilon_2) \right] / 2 + \frac{f_2(\varepsilon_6 - \varepsilon_5)}{3}$</td>
</tr>
<tr>
<td>35</td>
<td>$\phi^x + f_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 \varepsilon_6 - \varepsilon_1 - \varepsilon_2 - 2 \varepsilon_5 + 4 \right] / 2 + \frac{f_2(\varepsilon_6 + 1)^2}{(\varepsilon_6 - 3 \varepsilon_5 - 2) / 3 (\varepsilon_6 - 3 \varepsilon_5)^2 - (\varepsilon_5 - 1)^2 (\varepsilon_5 - 3 \varepsilon_6 + 2) / 3 (\varepsilon_5 - \varepsilon_6)^2}$</td>
</tr>
<tr>
<td>36</td>
<td>$\phi^x + f_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 \varepsilon_6 - \varepsilon_1 - \varepsilon_2 - 2 \varepsilon_6 + 6 - (\varepsilon_5 - 1)^2 / (\varepsilon_5 - \varepsilon_1) \right] / 2 + \frac{f_2(\varepsilon_6 + 1)^2}{(\varepsilon_6 - 3 \varepsilon_5 - 2) / 3 (\varepsilon_6 - \varepsilon_5)^2 - (\varepsilon_5 - 1)^2 (\varepsilon_5 - 3 \varepsilon_6 + 2) / 3 (\varepsilon_5 - \varepsilon_6)^2}$</td>
</tr>
<tr>
<td>37</td>
<td>$\phi^x + f_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 \varepsilon_6 - \varepsilon_1 - \varepsilon_2 - 2 - (\varepsilon_5 + 1)^2 / (\varepsilon_5 - \varepsilon_2) \right] / 2 + \frac{f_2(\varepsilon_6 - \varepsilon_5)}{3}$</td>
</tr>
<tr>
<td>38</td>
<td>$\phi^x + f_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 \varepsilon_6 - \varepsilon_1 - \varepsilon_5 + 2 - (\varepsilon_5 + 1)^2 / (\varepsilon_5 - \varepsilon_2) \right] / 2 + \frac{f_2(\varepsilon_6 - \varepsilon_5)}{3}$</td>
</tr>
<tr>
<td>39</td>
<td>$\phi^x + f_1 \left[ \varepsilon_3 + \varepsilon_4 + 2 \varepsilon_6 - \varepsilon_1 - \varepsilon_5 + (\varepsilon_6 + 1)^2 / (\varepsilon_6 - \varepsilon_4) - (\varepsilon_5 + 1)^2 / (\varepsilon_5 - \varepsilon_4) \right] / 2 + \frac{f_2(\varepsilon_6 - \varepsilon_5)}{3}$</td>
</tr>
</tbody>
</table>
Table 15 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>[ M^x/M^y_y ]**</th>
</tr>
</thead>
</table>
| 40    | \[
\phi^x + F_1 \left[ \varepsilon_3 + \varepsilon_6 - \varepsilon_1 - \varepsilon_5 + (\varepsilon_6 + 1)^2/(\varepsilon_6 - \varepsilon_4) - (\varepsilon_5+1)^2/(\varepsilon_5 - \varepsilon_2) - (\varepsilon_2 - 1)^2/(\varepsilon_2 - \varepsilon_5) \right] / 2 + F_2 (\varepsilon_6 - \varepsilon_5)/3
\] |
| 41    | \[
\phi^x + F_1 \left[ \varepsilon_3 + \varepsilon_6 - \varepsilon_1 - \varepsilon_5 + (\varepsilon_6 + 1)^2/(\varepsilon_6 - \varepsilon_4) + (\varepsilon_4 - 1)^2/(\varepsilon_4 - \varepsilon_6) - (\varepsilon_5+1)^2/(\varepsilon_5 - \varepsilon_2) - (\varepsilon_2 - 1)^2/(\varepsilon_2 - \varepsilon_5) \right] / 2 + F_2 (\varepsilon_6 - \varepsilon_5)/3
\] |
| 42    | \[
\phi^x + F_1 \left[ \varepsilon_3 + \varepsilon_6 - \varepsilon_2 - \varepsilon_5 + 4 + (\varepsilon_6 + 1)^2/(\varepsilon_6 - \varepsilon_4) + (\varepsilon_4 - 1)^2/(\varepsilon_4 - \varepsilon_6) - (\varepsilon_1+1)^2/(\varepsilon_1 - \varepsilon_5) - (\varepsilon_5 - 1)^2/(\varepsilon_5 - \varepsilon_1) \right] / 2 + F_2 (\varepsilon_6 + 1)^2 (\varepsilon_6 - 3 \varepsilon_5 - 2)/3 (\varepsilon_6 - \varepsilon_5)^2 - (\varepsilon_5 - 1)^2 (\varepsilon_5 - 3 \varepsilon_6 + 2)/3 (\varepsilon_5 - \varepsilon_6)^2
\] |
| 43    | \[
\phi^x + F_1 \left[ \varepsilon_3 + \varepsilon_6 + 2 + (\varepsilon_6 + 1)^2/(\varepsilon_6 - \varepsilon_4) + (\varepsilon_4 - 1)^2/(\varepsilon_4 - \varepsilon_6) - (\varepsilon_1+1)^2/(\varepsilon_1 - \varepsilon_5) - (\varepsilon_2 - 1)^2/(\varepsilon_2 - \varepsilon_5) + F_2 \left[ (\varepsilon_6 + 1)^2 (\varepsilon_6 - 3 \varepsilon_5 - 2)/3 (\varepsilon_6 - \varepsilon_5)^2 \right] \right]
\] |

where: \[ F_1 = (1 + K_2) (1 + K_2/2) K_1 K_2 / [2/3 K_3 + 4 K_1 K_2 (1 + K_2/2)^2] \]
\[ F_2 = (1 + K_2) (1 + K_2/2) K_3 / [2/3 K_3 + 4 K_1 K_2 (1 + K_2/2)^2] \]

* Numbers refer to Figure II.8.

** All curvatures are in terms of \( \phi^x_y \).
Table 16: Equations for $M^y/M^x_y$ for the different strain configurations
(Biaxial loading considering residual stresses)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$M^y/M^x_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$F_3 \phi^y$</td>
</tr>
<tr>
<td>2</td>
<td>$F_3 \phi^y + F_4 \left[ (\varepsilon_3 + 1)^2 (2 \varepsilon_3 - 3 \varepsilon_6 - 1) / 6 (\varepsilon_3 - \varepsilon_6)^2 \right]$</td>
</tr>
<tr>
<td>3</td>
<td>$F_3 \phi^y + F_4 \left[ (\varepsilon_3 + 1)^2 (2 \varepsilon_3 - 3 \varepsilon_6 - 1) / 6 (\varepsilon_3 - \varepsilon_6)^2 - (\varepsilon_2 - 1)^2 (2 \varepsilon_2 - 3 \varepsilon_5 + 1) / 6 (\varepsilon_2 - \varepsilon_5)^2 \right]$</td>
</tr>
<tr>
<td>4</td>
<td>$F_3 \phi^y + F_4 \left[ (\varepsilon_3 + 1)^2 (2 \varepsilon_3 - 3 \varepsilon_6 - 1) / 6 (\varepsilon_3 - \varepsilon_6)^2 + (\varepsilon_1 + 1)^2 (2 \varepsilon_1 - 3 \varepsilon_5 - 1) / 6 (\varepsilon_1 - \varepsilon_5)^2 \right]$</td>
</tr>
<tr>
<td>5</td>
<td>$F_3 \phi^y + F_4 \left[ (\varepsilon_3 + 1)^2 (2 \varepsilon_3 - 3 \varepsilon_6 - 1) / 6 (\varepsilon_3 - \varepsilon_6)^2 - (\varepsilon_2 - 1)^2 (2 \varepsilon_2 - 3 \varepsilon_5 + 1) / 6 (\varepsilon_2 - \varepsilon_5)^2 \right]$</td>
</tr>
</tbody>
</table>
Table 16 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>( \frac{M^y/M_x}{**} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>( F_\phi^\gamma + F_4 \left[ \frac{(\epsilon_3+1)^2(2\epsilon_3-3\epsilon_4-1)}{6(\epsilon_3-\epsilon_4)^2} + \frac{(\epsilon_4+1)^2(2\epsilon_4-3\epsilon_4-1)}{6(\epsilon_4-\epsilon_4)^2} \right] )</td>
</tr>
<tr>
<td>7</td>
<td>( F_\phi^\gamma + F_4 \left[ \frac{(\epsilon_3+1)^2(2\epsilon_3-3\epsilon_4-1)}{6(\epsilon_3-\epsilon_4)^2} + \frac{(\epsilon_4+1)^2(2\epsilon_4-3\epsilon_4-1)}{6(\epsilon_4-\epsilon_4)^2} \right] )</td>
</tr>
<tr>
<td>8</td>
<td>( F_3 \phi^\gamma + F_4 \left[ \frac{(2\epsilon_3+\epsilon_4+3)}{6} - \frac{(\epsilon_4+1)^3}{6(\epsilon_4-\epsilon_4)^2} \right] )</td>
</tr>
<tr>
<td>9</td>
<td>( F_3 \phi^\gamma + F_4 \left[ \frac{(2\epsilon_3+\epsilon_4+3)}{6} - \frac{(\epsilon_2-1)^3}{6(\epsilon_2-\epsilon_2)^2} \right] )</td>
</tr>
<tr>
<td>10</td>
<td>( F_\phi^\gamma + F_4 \left[ \frac{(\epsilon_3+1)^2(2\epsilon_3-3\epsilon_4-1)}{6(\epsilon_3-\epsilon_4)^2} + \frac{(\epsilon_4+1)^2(2\epsilon_4-3\epsilon_4-1)}{6(\epsilon_4-\epsilon_4)^2} \right] )</td>
</tr>
<tr>
<td>11</td>
<td>( F_3 \phi^\gamma + F_4 \left[ \frac{(\epsilon_3+1)^2(2\epsilon_3-3\epsilon_4-1)}{6(\epsilon_3-\epsilon_4)^2} + \frac{(\epsilon_4+1)^3(2\epsilon_4-3\epsilon_4-1)}{6(\epsilon_4-\epsilon_4)^2} \right] )</td>
</tr>
<tr>
<td>Case</td>
<td>( M^y / M^x )</td>
</tr>
<tr>
<td>------</td>
<td>----------------</td>
</tr>
<tr>
<td>12</td>
<td>[ F_3 \phi^y + F_4 \left[ \frac{(E_1 + 1)^2 (2E_1 - 3E_5 - 1)}{6(E_1 - E_5)^2} + \frac{(E_3 + 1)^2 (2E_3 - 3E_6 - 1)}{6(E_3 - E_6)^2} - \frac{(E_2 - 1)^2 (2E_2 - 3E_5 + 1)}{6(E_2 - E_5)^2} - \frac{(E_4 + 1)^2 (2E_4 - 3E_6 - 1)}{6(E_4 - E_6)^2} \right] ]</td>
</tr>
<tr>
<td>13</td>
<td>[ F_3 \phi^y + F_4 \left[ \frac{(E_3 + 1)^2 (2E_3 - 3E_6 - 1)}{6(E_3 - E_6)^2} - \frac{(E_5 - 1)^2}{6(E_5 - E_2)^2} \right] ]</td>
</tr>
<tr>
<td>14</td>
<td>[ F_3 \phi^y + F_4 \left[ \frac{(E_1 + 1)^2 (2E_1 - 3E_5 - 1)}{6(E_1 - E_5)^2} + \frac{(2E_3 + E_6 + 3)}{6} - \frac{(E_2 - 1)^2 (2E_2 - 3E_5 + 1)}{6(E_2 - E_5)^2} - \frac{(E_4 + 1)^2 (2E_4 - 3E_6 - 1)}{6(E_4 - E_6)^2} \right] ]</td>
</tr>
<tr>
<td>15</td>
<td>[ F_3 \phi^y + F_4 \left[ \frac{(E_1 + 1)^2 (2E_1 - 3E_5 - 1)}{6(E_1 - E_5)^2} + \frac{(2E_3 + E_6 + 3)}{6} - \frac{(E_2 - 1)^2 (2E_2 - 3E_5 + 1)}{6(E_2 - E_5)^2} - \frac{(E_4 + 1)^2 (2E_4 - 3E_6 - 1)}{6(E_4 - E_6)^2} \right] ]</td>
</tr>
<tr>
<td>16</td>
<td>[ F_3 \phi^y + F_4 \left( \frac{E_3 - E_4}{3} \right) ]</td>
</tr>
<tr>
<td>17</td>
<td>[ F_3 \phi^y + F_4 \left[ \frac{(E_1 + 1)^2 (2E_1 - 3E_5 - 1)}{6(E_1 - E_5)^2} + \frac{(E_3 + 1)^2 (2E_3 - 3E_6 - 1)}{6(E_3 - E_6)^2} - \frac{(E_4 + 1)^2 (2E_4 - 3E_6 - 1)}{6(E_4 - E_6)^2} + \frac{(E_5 - 1)^3}{6(E_5 - E_2)^2} \right] ]</td>
</tr>
<tr>
<td>Case*</td>
<td>$\frac{\gamma^y}{M^x_y}$</td>
</tr>
<tr>
<td>-------</td>
<td>-------------------</td>
</tr>
<tr>
<td>18</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3 + 1)^2 (2 \varepsilon_3 - 3 \varepsilon_6 - 1)}{6 (\varepsilon_3 - \varepsilon_5)^2 + (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_7)^2} - (2 \varepsilon_2 + \varepsilon_5 - 3) / 6 \right]$</td>
</tr>
<tr>
<td>19</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3 - \varepsilon_4) / 3 - (\varepsilon_2 - 1)^2 (2 \varepsilon_2 - 3 \varepsilon_5 - 1) / 6 (\varepsilon_2 - \varepsilon_5)^2}{3} \right]$</td>
</tr>
<tr>
<td>20</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{\varepsilon_3 + 1)^2 (2 \varepsilon_3 - 3 \varepsilon_6 + 1) / 6 (\varepsilon_3 - \varepsilon_5)^2 + (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_7)^2} - (2 \varepsilon_2 + \varepsilon_5 - 3) / 6 \right]$</td>
</tr>
<tr>
<td>21</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3 + 1)^2 (2 \varepsilon_3 - 3 \varepsilon_6 - 1) / 6 (\varepsilon_3 - \varepsilon_5)^2 + (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_7)^2} - (2 \varepsilon_2 + \varepsilon_5 - 3) / 6 \right]$</td>
</tr>
<tr>
<td>22</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3 + 1)^2 (2 \varepsilon_3 - 3 \varepsilon_6 - 1) / 6 (\varepsilon_3 - \varepsilon_5)^2 + (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_7)^2} - (2 \varepsilon_2 + \varepsilon_5 - 3) / 6 \right]$</td>
</tr>
<tr>
<td>23</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3 + 1)^2 (2 \varepsilon_3 - 3 \varepsilon_6 - 1) / 6 (\varepsilon_3 - \varepsilon_5)^2 + (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_7)^2} - (2 \varepsilon_2 + \varepsilon_5 - 3) / 6 \right]$</td>
</tr>
</tbody>
</table>
Table 16 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$W^y/M^x_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(2 \varepsilon_3 + \varepsilon_6 + 3)/6 + (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_1)^2 - (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_2)^2}{(\varepsilon_6 + 1)^3/6 (\varepsilon_6 - \varepsilon_4)^2} \right]$</td>
</tr>
<tr>
<td>25</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(2 \varepsilon_3 + \varepsilon_6 + 3)/6 - (\varepsilon_6 + 1)^3/6 (\varepsilon_6 - \varepsilon_4)^2 + (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_1)^2 - (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_2)^2}{(\varepsilon_6 - \varepsilon_5 - 1)^3/6 (\varepsilon_6 - \varepsilon_5 - 1)^2 + (\varepsilon_5 - \varepsilon_1)^2} \right]$</td>
</tr>
<tr>
<td>26</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3 + 1)^2(2 \varepsilon_3 - 3 \varepsilon_6 - 1)/6 (\varepsilon_3 - \varepsilon_6)^2 - (\varepsilon_4 + 1)^2(2 \varepsilon_4 - 3 \varepsilon_6 - 1)/6 (\varepsilon_4 - \varepsilon_6)^2}{(\varepsilon_5 - \varepsilon_1)^2} \right]$</td>
</tr>
<tr>
<td>27</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3 + 1)^2(2 \varepsilon_3 - 3 \varepsilon_6 - 1)/6 (\varepsilon_3 - \varepsilon_6)^2 + (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_1)^2}{(\varepsilon_6 - \varepsilon_5 - 1)^3/6 (\varepsilon_6 - \varepsilon_5 - 1)^2 + (\varepsilon_5 - \varepsilon_1)^2} \right]$</td>
</tr>
<tr>
<td>28</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_3 - \varepsilon_4)/3 + (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_1)^2 - (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_2)^2}{(\varepsilon_6 - \varepsilon_5 - 1)^3/6 (\varepsilon_6 - \varepsilon_5 - 1)^2 + (\varepsilon_5 - \varepsilon_1)^2} \right]$</td>
</tr>
<tr>
<td>29</td>
<td>$F_3 \phi^y + F_4 \left[ \frac{(\varepsilon_1 + 1)^2(2 \varepsilon_1 - 3 \varepsilon_5 - 1)/6 (\varepsilon_1 - \varepsilon_5)^2 + (\varepsilon_3 - \varepsilon_6)/3 + (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_1)^2 - (\varepsilon_5 - 1)^3/6 (\varepsilon_5 - \varepsilon_2)^2}{(\varepsilon_6 - \varepsilon_5)^2} \right]$</td>
</tr>
<tr>
<td>Case*</td>
<td>( M^Y / M_X )</td>
</tr>
<tr>
<td>-------</td>
<td>-----------------</td>
</tr>
<tr>
<td>30</td>
<td>[ F_3 \phi^Y + F_4 [ (\varepsilon_1 + 1)^2 (2\varepsilon - 3\varepsilon_5 - 1) / 6 (\varepsilon_1 - \varepsilon_5)^2 + (2\varepsilon_3 + \varepsilon_6 + 3) / 6 - (\varepsilon_6 + 1)^2 / 6 (\varepsilon_6 - \varepsilon_4)^2 ] ]</td>
</tr>
<tr>
<td>31</td>
<td>[ F_3 \phi^Y + F_4 [ (\varepsilon_3 - \varepsilon_4) / 4 + (\varepsilon_1 + 1)^2 (2\varepsilon - 3\varepsilon_5 - 1) / 6 (\varepsilon_1 - \varepsilon_5)^2 -(\varepsilon_6 - 1)^2 ] [ (2\varepsilon_2 - 3\varepsilon_5 + 1) / 6 (\varepsilon_2 - \varepsilon_5)^2 ]</td>
</tr>
<tr>
<td>32</td>
<td>[ F_3 \phi^Y + F_4 [ (\varepsilon_3 + \varepsilon_6 + 3) / 6 + (\varepsilon_5 - 1)^3 / 6 (\varepsilon_5 - \varepsilon_1)^2 - (\varepsilon_6 + 1)^2 / 6 (\varepsilon_6 - \varepsilon_4)^2 -(2\varepsilon_2 + \varepsilon_5 - 3) / 6 ] ]</td>
</tr>
<tr>
<td>33</td>
<td>[ F_3 \phi^Y + F_4 [ (\varepsilon_3 - \varepsilon_4) / 3 + (\varepsilon_1 + 1)^2 (2\varepsilon - 3\varepsilon_5 - 1) / 6 (\varepsilon_1 - \varepsilon_5)^2 + (\varepsilon_5 - 1)^3 ] [ / 6 (\varepsilon_5 - \varepsilon_1)^2 -(2\varepsilon_2 + \varepsilon_5 - 3) / 6 ] ]</td>
</tr>
<tr>
<td>34</td>
<td>[ F_3 \phi^Y + F_4 [ (\varepsilon_3 + \varepsilon_4) / 3 + (2\varepsilon_1 + \varepsilon_5 + 3) / 6 - (\varepsilon_2 - 1)^2 (2\varepsilon_2 - 3\varepsilon_5 + 1) + (\varepsilon_5 + 1)^3 ] [ / 6 (\varepsilon_5 - \varepsilon_1)^2 ] ]</td>
</tr>
<tr>
<td>35</td>
<td>[ F_3 \phi^Y + F_4 (\varepsilon_1 + \varepsilon_3 - \varepsilon_2 - \varepsilon_4) / 3 ]</td>
</tr>
<tr>
<td>Case*</td>
<td>( M^Y_{\lambda}/M^X_{\lambda} )</td>
</tr>
<tr>
<td>-------</td>
<td>----------------------------------</td>
</tr>
<tr>
<td>36</td>
<td>( F_3 \phi^Y + F_4 \left[ \frac{(\varepsilon_3 - \varepsilon_4) + (\varepsilon_5 - 1)^3/6(\varepsilon_5 - \varepsilon_4)^2 - (2\varepsilon_2 + \varepsilon_5 - 3)/6}{(\varepsilon_5 - \varepsilon_4)^2} \right] )</td>
</tr>
<tr>
<td>37</td>
<td>( F_3 \phi^Y + F_4 \left[ \frac{(\varepsilon_1 + 1)^2(2\varepsilon_1 - 3\varepsilon_5 - 1)/6(\varepsilon_1 - \varepsilon_5)^2 + (\varepsilon_5 - 1)^3/6(\varepsilon_5 - \varepsilon_1)^2 + (2\varepsilon_3 + \varepsilon_6 + 3)/6 - (2\varepsilon_2 + \varepsilon_5 - 3)/6 - (\varepsilon_6 + 1)^3/6(\varepsilon_6 - \varepsilon_4)^2}{(\varepsilon_5 - \varepsilon_4)^2} \right] )</td>
</tr>
<tr>
<td>38</td>
<td>( F_3 \phi^Y + F_4 \left[ \frac{(\varepsilon_3 - \varepsilon_4)/3 + (2\varepsilon_1 + \varepsilon_5 + 3)/6 - (\varepsilon_5 + 1)^3/6(\varepsilon_5 - \varepsilon_2)^2}{(\varepsilon_5 - \varepsilon_4)^2} \right] )</td>
</tr>
<tr>
<td>39</td>
<td>( F_3 \phi^Y + F_4 \left[ \frac{(2\varepsilon_1 + \varepsilon_5 + 3)/6 + (2\varepsilon_3 + \varepsilon_6 + 3)/6 - (\varepsilon_5 + 1)^3/6(\varepsilon_5 - \varepsilon_2)^2}{(\varepsilon_6 + 1)^3/6(\varepsilon_6 - \varepsilon_4)^2} \right] )</td>
</tr>
<tr>
<td>40</td>
<td>( F_3 \phi^Y + F_4 \left[ \frac{(2\varepsilon_1 + \varepsilon_5 + 3)/6 + (2\varepsilon_3 + \varepsilon_6 + 3)/6 - (\varepsilon_5 + 1)^3/6(\varepsilon_5 - \varepsilon_2)^2 - (\varepsilon_6 + 1)^3/6(\varepsilon_6 - \varepsilon_4)^2 - (\varepsilon_2 - 1)^2 (2\varepsilon_2 - 3\varepsilon_5 - 1)/6 (\varepsilon_2 - \varepsilon_5)^2}{(\varepsilon_5 - \varepsilon_4)^2} \right] )</td>
</tr>
<tr>
<td>41</td>
<td>( F_3 \phi^Y + F_4 \left[ \frac{(2\varepsilon_1 + 2\varepsilon_3 + \varepsilon_5 + \varepsilon_6 + 6)/6 - \left{ \frac{(\varepsilon_2 - 1)^2 (2\varepsilon_2 - 3\varepsilon_5 + 1) + (\varepsilon_5 + 1)^3}{6(\varepsilon_5 - \varepsilon_2)^2} - \left{ (\varepsilon_4 - 1)^2 (2\varepsilon_4 - 3\varepsilon_6 + 1) + (\varepsilon_6 + 1)^3}{6(\varepsilon_6 - \varepsilon_4)^2} \right} \right} (\varepsilon_5 - \varepsilon_4)^2}{(\varepsilon_5 - \varepsilon_5)^2} )</td>
</tr>
</tbody>
</table>
Table 16 (Cont'd)

<table>
<thead>
<tr>
<th>Case*</th>
<th>( \frac{M_Y}{N_X} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>42</td>
<td>( F_3 \phi^Y + F_4 \left[ \frac{(2 \varepsilon_3 + \varepsilon_6 - 2 \varepsilon_2 - \varepsilon_5)}{6} + \left( \varepsilon_1 + 1 \right)^2 \left( \frac{2 \varepsilon_1 - 3 \varepsilon_5 - 1}{6} (\varepsilon_1 - \varepsilon_5) + (\varepsilon_5 - 1)^3 + \varepsilon_1 - \varepsilon_5 \right) \right] )</td>
</tr>
<tr>
<td>43</td>
<td>( F_3 \phi^Y + F_4 \left[ \frac{(2 \varepsilon_3 + \varepsilon_6 + 3)}{6} + \left( \varepsilon_1 + 1 \right)^2 \left( \frac{2 \varepsilon_1 - 3 \varepsilon_5 - 1}{6} (\varepsilon_1 - \varepsilon_5)^3 + \varepsilon_2 - \varepsilon_5 \right) \right] )</td>
</tr>
</tbody>
</table>

where \( F_3 = \frac{(8K_1^3 K_2 + K_3^3)}{6} \left[ \frac{2/3 K_3 + 4K_1 K_2 (1 + K_2/2)^2}{2/3 K_3 + 4K_1 K_2 (1 + K_2/2)^2} \right] \)

\( F_4 = \frac{(1 + K_2) K_1^2 K_2}{2/3 K_3 + 4K_1 K_2 (1 + K_2/2)^2} \)

* Numbers refer to Figure II.8.

** All curvatures are in terms of \( \phi^Y \).
Table 17: Equations for $C_w$ for the different strain configurations

(Biaxial loading neglecting residual stresses)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$C_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$4EK_1^3 K_2 D_6^2 (1 + K_2/2)^2 / 3$</td>
</tr>
<tr>
<td>2</td>
<td>$2EK_1^3 K_2 D_6^2 (1 + K_2/2)^2 / 3 + 2EK_1^3 K_2 D_6^2 (1 + K_2/2)^2 (E_4 + 1)[(E_4 + 1)^2 + 3(E_3 + 1)^2] / 3(E_4 - E_3)^3$</td>
</tr>
<tr>
<td>3</td>
<td>$2EK_1^3 K_2 D_6^2 (1 + K_2/2)^2 \left{ (E_4 + 1)[(E_4 + 1)^2 + 3(E_3 + 1)^2] / 3(E_4 - E_3)^3 + (E_1 - 1)[(E_1 - 1)^2 + 3(E_2 - 1)^2] / 3(E_2 - E_1)^3 \right}$</td>
</tr>
<tr>
<td>4</td>
<td>$2EK_1^3 K_2 D_6^2 (1 + K_2/2)^2 \left{ (E_4 + 1)[(E_4 + 1)^2 + 3(E_3 + 1)^2] / 3(E_4 - E_3)^3 + (E_2 + 1)[(E_2 - 1)^2 + 3(E_2 - 1)^2] / 3(E_2 - E_1)^3 \right}$</td>
</tr>
<tr>
<td>5</td>
<td>$2EK_1^3 K_2 D_6^2 (1 + K_2/2)^2 \left{ (E_1 - 1)[(E_1 - 1)^2 + 3(E_1 - 1)^2] / 3(E_1 - E_2)^3 + (E_2 + 1)[(E_2 + 1)^2 + 3(E_1 + 1)^2] / 3(E_2 - E_1) + (E_4 + 1)[(E_4 + 1)^2 + 3(E_3 + 1)^2] / 3(E_4 - E_3)^3 - 1/3 \right}$</td>
</tr>
<tr>
<td>Case*</td>
<td>( C_w )</td>
</tr>
<tr>
<td>-------</td>
<td>------------------</td>
</tr>
<tr>
<td>6</td>
<td>[ 2E_k^3 \kappa_2 D^6 (1 + \kappa_z / 2)^2 / 3 + 2E_k^3 \kappa_2 D^6 (1 + \kappa_z / 2)^2 (\epsilon_4 + 1) \left[ (\epsilon_4 + 1)^3 + 3(\epsilon_4 + 1) \right] / 3 (\epsilon_4 - \epsilon_3)^3 ]</td>
</tr>
<tr>
<td>7</td>
<td>[ 2E_k^3 \kappa_2 D^6 (1 + \kappa_z / 2)^2 \left[ (\epsilon_4 + 1)^2 + 3(\epsilon_3 + 1)^2 \right] / 3 (\epsilon_4 - \epsilon_3)^3 ]</td>
</tr>
<tr>
<td>8</td>
<td>[ 2E_k^3 \kappa_2 D^6 (1 + \kappa_z / 2)^2 \left[ (\epsilon_4 + 1)^2 + 3(\epsilon_3 + 1)^2 \right] / 3 (\epsilon_4 - \epsilon_3)^3 ]</td>
</tr>
<tr>
<td>9</td>
<td>[ 2E_k^3 \kappa_2 D^6 (1 + \kappa_z / 2)^2 \left[ (\epsilon_4 + 1)^2 + 3(\epsilon_3 + 1)^2 \right] / 3 (\epsilon_4 - \epsilon_3)^3 ]</td>
</tr>
<tr>
<td>10</td>
<td>[ E_k^3 \kappa_2^3 D^6 / 18 + 2E_k^3 \kappa_3 D^6 \left[ 1 - \left( (\epsilon_6 + 1) / (\epsilon_6 - \epsilon_5) \right) \right]^3 / 9 ]</td>
</tr>
<tr>
<td>11</td>
<td>[ E_k^3 \kappa_2^3 D^6 \left[ 1 - (\epsilon_2 - \epsilon_1) / (\epsilon_2 - \epsilon_3) \right]^3 / 18 + 2E_k^3 \kappa_3 D^6 \left[ 1 - (\epsilon_6 + 1) / (\epsilon_6 - \epsilon_5) \right]^3 / 9 ]</td>
</tr>
</tbody>
</table>
Table 17 (Cont'd.)

<table>
<thead>
<tr>
<th>Case</th>
<th>$C_W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>$2EK_x^3K_zD^6(1+K_z/2)^2\left{(\epsilon_1-1)/(\epsilon_1-\epsilon_2)\right}^{3/2}/3 \left(\epsilon_1-\epsilon_2\right)^3+\left(\epsilon_2+1\right)[\left(\epsilon_2+1\right)^2+3\left(\epsilon_2+1\right)^2]/3 \left(\epsilon_2-\epsilon_3\right)^3+\left(\epsilon_4+1\right)[\left(\epsilon_4+1\right)^2+3\left(\epsilon_4+1\right)^2]/3 \left(\epsilon_3-\epsilon_4\right)^3-2/3\right}$</td>
</tr>
<tr>
<td>13</td>
<td>$EK_x^3K_zD^6\left{1-(\epsilon_1+1)/(\epsilon_1-\epsilon_2)\right}^{3/2}/3 + 2EK_x^3D^6\left{1-(\epsilon_6+1)/(\epsilon_6-\epsilon_5)\right}^{3/2}/9$</td>
</tr>
<tr>
<td>14</td>
<td>$EK_x^3K_zD^6\left{1-(\epsilon_1+1)/(\epsilon_1-\epsilon_2)+\epsilon_1/(\epsilon_1-\epsilon_2)\right}^{3/2}/3 + 2EK_x^3D^6\left{1-(\epsilon_6+1)/(\epsilon_6-\epsilon_5)\right}^{3/2}/9$</td>
</tr>
<tr>
<td>15</td>
<td>$2EK_x^3K_zD^6(1+K_z/2)^2\left{(\epsilon_4+1)[\left(\epsilon_4+1\right)^2+3\left(\epsilon_3+1\right)^2]/3 \left(\epsilon_1-\epsilon_2\right)^3+\left(\epsilon_1-1\right)[\left(\epsilon_1-1\right)^2+3\left(\epsilon_2-1\right)^2]/3 \left(\epsilon_1-\epsilon_2\right)^3\right}$</td>
</tr>
<tr>
<td>16</td>
<td>$EK_x^3K_zD^6\left{1-(\epsilon_1+1)/(\epsilon_1-\epsilon_2)+\epsilon_1/(\epsilon_1-\epsilon_2)\right}^{3/2}/3 + 2EK_x^3D^6\left{1-(\epsilon_6+1)/(\epsilon_6-\epsilon_5)-\epsilon_1/(\epsilon_1-\epsilon_2)\right}^{3/2}/9$</td>
</tr>
<tr>
<td>17</td>
<td>$EK_x^3K_zD^6\left{1-(\epsilon_1+1)/(\epsilon_1-\epsilon_2)-\epsilon_1/(\epsilon_1-\epsilon_2)\right}^{3/2}/3 + 2EK_x^3D^6\left{1-(\epsilon_6+1)/(\epsilon_6-\epsilon_5)-\epsilon_1/(\epsilon_1-\epsilon_2)\right}^{3/2}/9$</td>
</tr>
</tbody>
</table>
Table 17 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{n} )</td>
<td>( 2E_{1}^{3}D_{n}^{3} {1 - \left( \varepsilon_{n} + \frac{1}{2} \right) / (\varepsilon_{5} - \varepsilon_{2}) } / (\varepsilon_{5} - \varepsilon_{3}) } / 9 )</td>
<td>( E_{1}^{3}D_{n}^{3} {1 - \left( \varepsilon_{n} + \frac{1}{2} \right) / (\varepsilon_{5} - \varepsilon_{2}) } / 9 )</td>
<td>( 2E_{1}^{3}D_{n}^{3} {1 - \left( \varepsilon_{n} + \frac{1}{2} \right) / (\varepsilon_{5} - \varepsilon_{2}) } / 9 )</td>
<td>( E_{1}^{3}D_{n}^{3} {1 - \left( \varepsilon_{n} + \frac{1}{2} \right) / (\varepsilon_{5} - \varepsilon_{2}) } / 9 )</td>
<td>( 2E_{1}^{3}D_{n}^{3} {1 - \left( \varepsilon_{n} + \frac{1}{2} \right) / (\varepsilon_{5} - \varepsilon_{2}) } / 9 )</td>
<td>( E_{1}^{3}D_{n}^{3} {1 - \left( \varepsilon_{n} + \frac{1}{2} \right) / (\varepsilon_{5} - \varepsilon_{2}) } / 9 )</td>
</tr>
<tr>
<td>Case*</td>
<td>$C_w$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>$2 E K_1^3 K_2 D^6 (1 + K_2 / 2)^2 \left{ (\varepsilon_1 - 1) \left[ (\varepsilon_1 - 1)^2 + 3 (\varepsilon_2 - 1)^3 \right] / 3 (\varepsilon_1 - \varepsilon_2)^3 + (\varepsilon_2 + 1) \times \left[ (\varepsilon_2 + 1)^2 + 3 (\varepsilon_1 + 1)^2 \right] / 3 (\varepsilon_2 - \varepsilon_1)^3 + (\varepsilon_4 + 1) \left[ (\varepsilon_4 + 1)^2 + 3 (\varepsilon_3 + 1)^2 \right] / 3 (\varepsilon_4 - \varepsilon_3)^3 + (\varepsilon_3 - 1) \left[ (\varepsilon_3 - 1)^2 + 3 (\varepsilon_4 - 1)^2 \right] / 3 (\varepsilon_3 - \varepsilon_4)^3 - 2 / 3 \right}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>$2 E K_1^3 K_2 D^6 (1 + K_2 / 2)^3 \left{ (\varepsilon_1 - 1) \left[ (\varepsilon_1 - 1)^2 + 3 (\varepsilon_2 - 1)^3 \right] / 3 (\varepsilon_1 - \varepsilon_2)^3 + (\varepsilon_2 + 1) \times \left[ (\varepsilon_2 + 1)^2 + 3 (\varepsilon_1 + 1)^2 \right] / 3 (\varepsilon_2 - \varepsilon_1)^3 + (\varepsilon_4 + 1) \left[ (\varepsilon_4 + 1)^2 + 3 (\varepsilon_3 + 1)^2 \right] / 3 (\varepsilon_4 - \varepsilon_3)^3 + (\varepsilon_3 - 1) \left[ (\varepsilon_3 - 1)^2 + 3 (\varepsilon_4 - 1)^2 \right] / 3 (\varepsilon_3 - \varepsilon_4)^3 - 2 / 3 \right}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>$2 E K_1^3 K_2 D^6 (1 + K_2 / 2)^2 \left{ (\varepsilon_1 - 1) \left[ (\varepsilon_1 - 1)^2 + 3 (\varepsilon_2 - 1)^3 \right] / 3 (\varepsilon_1 - \varepsilon_2)^3 + (\varepsilon_2 + 1) \times \left[ (\varepsilon_2 + 1)^2 + 3 (\varepsilon_1 + 1)^2 \right] / 3 (\varepsilon_2 - \varepsilon_1)^3 + (\varepsilon_4 + 1) \left[ (\varepsilon_4 + 1)^2 + 3 (\varepsilon_3 + 1)^2 \right] / 3 (\varepsilon_4 - \varepsilon_3)^3 + (\varepsilon_3 - 1) \left[ (\varepsilon_3 - 1)^2 + 3 (\varepsilon_4 - 1)^2 \right] / 3 (\varepsilon_3 - \varepsilon_4)^3 - 2 / 3 \right}$</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

*Numbers refer to Figure II.7.
Table 18: Equations for $C_W$ for the different strain configurations
(Biaxial loading considering residual stresses)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$C_W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$4E\kappa_1^3 \kappa_2 D^6 (1+\kappa_2/2)^2/3$</td>
</tr>
<tr>
<td>2</td>
<td>$E\kappa_1^3 \kappa_2 D^6 (1+\kappa_2/2)^2 + E\kappa_1^3 \kappa_2 D^6 (1+\kappa_2/2)^3 (\varepsilon_6+1)^3/3 (\varepsilon_6 - \varepsilon_3)^3$</td>
</tr>
<tr>
<td>3</td>
<td>$2E\kappa_1^3 \kappa_2 D^6 (1+\kappa_2/2)^2/3 + E\kappa_1^3 \kappa_2 D^6 (1+\kappa_2/2)^2 [(\varepsilon_5-1)^3/3 (\varepsilon_5 - \varepsilon_2) + (\varepsilon_6+1)^3/3 (\varepsilon_6 - \varepsilon_3)^3]$</td>
</tr>
<tr>
<td>4</td>
<td>$2E\kappa_1^3 \kappa_2 D^6 (1+\kappa_2/2)^2/3 + E\kappa_1^3 \kappa_2 D^6 (1+\kappa_2/2)^2 [(\varepsilon_5+1)^3/3 (\varepsilon_5 - \varepsilon_2)^3 + (\varepsilon_6+1)^3/3 (\varepsilon_6 - \varepsilon_3)^3]$</td>
</tr>
<tr>
<td>5</td>
<td>$E\kappa_1^3 \kappa_2 D^6 (1+\kappa_2/2)^2/3 + E\kappa_1^3 \kappa_2 D^6 (1+\kappa_2/2)^2 [(\varepsilon_5+1)^3/3 (\varepsilon_5 - \varepsilon_2)^3 + (\varepsilon_6+1)^3/3 (\varepsilon_6 - \varepsilon_3)^3]$</td>
</tr>
<tr>
<td>Case</td>
<td>( C_W )</td>
</tr>
<tr>
<td>-------</td>
<td>------------</td>
</tr>
<tr>
<td>6</td>
<td>( 2 E \kappa_1^3 \kappa_2^3 D^6 (1 + \kappa_1/2)^2/3 + E \kappa_1^3 \kappa_2^3 D^6 (1 + \kappa_2/2)^2 \left[ (E_6 + 1)^3/3 (E_6 - E_3)^3 + (E_6 + 1)^3/3 (E_6 - E_4)^3 \right] )</td>
</tr>
<tr>
<td>7</td>
<td>( E \kappa_1^3 \kappa_2^3 D^6 (1 + \kappa_1/2)^2/3 + E \kappa_1^3 \kappa_2^3 D^6 (1 + \kappa_2/2)^2 \left[ (E_5 + 1)^3/3 (E_5 - E_3)^3 + (E_6 + 1)^3/3 (E_6 - E_4)^3 \right] )</td>
</tr>
<tr>
<td>8</td>
<td>( 2 E \kappa_1^3 \kappa_2^3 D^6 (1 + \kappa_2/2)^2/3 + E \kappa_1^3 \kappa_2^3 D^6 (1 + \kappa_2/2)^2 (E_4 + 1) \left[ (E_4 + 1)^2 + 3 (E_4 - 2 E_6 - 1)^2 \right] / 12 (E_4 - E_6)^3 )</td>
</tr>
<tr>
<td>9</td>
<td>( E \kappa_1^3 \kappa_2^3 D^6 (1 + \kappa_1/2)^2/3 + E \kappa_1^3 \kappa_2^3 D^6 (1 + \kappa_2/2)^2 \left[ (E_5 - 1)^3/3 (E_5 - E_3)^3 + (E_4 + 1) \left[ (E_4 + 1)^2 + 3 (E_4 - 2 E_6 - 1)^2 \right] / 12 (E_4 - E_6)^3 \right] )</td>
</tr>
<tr>
<td>10</td>
<td>( E \kappa_1^3 \kappa_2^3 D^6 (1 + \kappa_2/2)^2 \left[ (E_1 - 1) \left[ (E_1 - 1)^2 + 3 (E_1 - 2 E_5 + 1)^2 \right] / 12 (E_1 - E_5)^3 + (E_2 - 1) \left[ (E_2 - 1)^2 + 3 (E_2 - 2 E_5 + 1)^2 \right] / 12 (E_2 - E_5)^3 \right] + (E_6 + 1)^3/3 (E_6 - E_3)^3 + (E_6 + 1)^3/3 (E_6 - E_4)^3 )</td>
</tr>
<tr>
<td>11</td>
<td>( E \kappa_1^3 \kappa_2^3 D^6 (1 + \kappa_2/2)^2/3 + E \kappa_1^3 \kappa_2^3 D^6 (1 + \kappa_2/2)^2 \left[ (E_5 - 1)^3/3 (E_5 - E_3)^3 + (E_6 + 1)^3/3 (E_6 - E_3)^3 + (E_6 + 1)^3/3 (E_6 - E_4)^3 \right] )</td>
</tr>
<tr>
<td>Case*</td>
<td>$C_W$</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>12</td>
<td>$E K_1^3 K_2 D^6 (1 + \frac{N_2}{2})^2 \left[ \frac{(E_5 - 1)}{3} + \frac{(E_5 - E_1)}{3} + \frac{(E_5 - E_2)}{3} \right] + \frac{(E_6 + 1)}{3} \left[ \frac{(E_6 - E_1)}{3} + \frac{(E_6 - E_2)}{3} + \frac{(E_6 + 1)}{3} \right]$</td>
</tr>
<tr>
<td>13</td>
<td>$E K_1^3 K_2 D^6 (1 + \frac{N_2}{2})^2/3 + E K_1^3 K_2 D^6 (1 + \frac{N_2}{2})^2 \left[ \frac{(E_5 - 1)}{3} + \frac{3(E_5 - E_2)}{3} \right] / 12 \left[ \frac{(E_4 - E_5)}{3} + \frac{(E_4 + 1)}{3} \right]$</td>
</tr>
<tr>
<td>14</td>
<td>$E K_1^3 K_2 D^6 (1 + \frac{N_2}{2})^2 \left[ \frac{(E_5 - 1)}{3} + \frac{(E_5 - E_1)}{3} + \frac{(E_5 - E_2)}{3} \right] / 12 \left[ \frac{(E_4 - E_5)}{3} + \frac{(E_4 + 1)}{3} \right]$</td>
</tr>
<tr>
<td>15</td>
<td>$E K_1^3 K_2 D^6 (1 + \frac{N_2}{2})^2 \left[ \frac{(E_5 - 1)}{3} + \frac{(E_5 - E_1)}{3} + \frac{(E_5 - E_2)}{3} \right] / 12 \left[ \frac{(E_4 - E_5)}{3} + \frac{(E_4 + 1)}{3} \right]$</td>
</tr>
<tr>
<td>16</td>
<td>$E K_1^3 K_2 D^6/18 + 2E K_3^3 D^6 \left[ 1 - \frac{(E_6 + 1)}{3} \right] / \left[ \frac{(E_6 - E_5)}{9} \right]$</td>
</tr>
<tr>
<td>17</td>
<td>$E K_1^3 K_2 D^6 (1 + \frac{N_2}{2})^2 \left[ \frac{(E_5 - 1)}{3} + \frac{3(E_5 - E_1)}{3} \right] / 12 \left[ \frac{(E_2 - E_5)}{3} + \frac{(E_2 - 1)}{3} \right]$</td>
</tr>
</tbody>
</table>

Table 18 (Cont'd.)
Table 18 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$C_W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>$E K_1^3 K_2 D^6 (1 + K_2/2)^2/3 + E K_1^3 K_2 D^6 (1 + K_2/2)^2 { (E_1 - 1) [ (E_1 - 1)^2 + 3(E_1 - 2E_5 + 1)^2] / 12 (E_1 - E_5)^3 + (E_6 + 1)^3 / 3 (E_6 - E_3)^3 }$</td>
</tr>
<tr>
<td>19</td>
<td>$E K_1^3 K_2 D^6 { 1 - (E_2 - 1) / 2 (E_2 - E_5) }^3 / 18 + 2 E K_1^3 D^6 { 1 - [ (E_6 + 1) / (E_6 - E_5) ] }^3 / 9$</td>
</tr>
<tr>
<td>20</td>
<td>$E K_1^3 K_2 D^6 (1 + K_2/2)^2 { (E_1 - 1) [ (E_1 - 1)^2 + 3 (E_1 - 2E_5 + 1)^2 ] / 12 (E_1 - E_5)^3 + (E_1 + 1)^3 / 3 (E_1 - E_5)^3 + (E_6 + 1)^3 / (E_6 - E_3)^3 }$</td>
</tr>
<tr>
<td>21</td>
<td>$E K_1^3 K_2 D^6 (1 + K_2/2)^2 { (E_3 - 1) [ (E_3 - 1)^2 + 3 (E_3 - 2E_5 + 1)^2 ] / 12 (E_3 - E_5)^3 + (E_3 - 2E_5 + 1)^2 / 12 (E_3 - E_5) + (E_6 + 1)^3 / (E_6 - E_3)^3 - (E_5 + 1)^3 / (E_5 - E_3)^3 }$</td>
</tr>
<tr>
<td>22</td>
<td>$E K_1^3 K_2 D^6 (1 + K_2/2)^2 { (E_5 + 1)^3 / 3 (E_5 - E_3)^3 + (E_5 - 1)^3 / 3 (E_5 - E_3)^3 + (E_6 + 1)^3 / 3 (E_6 - E_3)^3 + (E_6 - 1)^3 / 3 (E_6 - E_4)^3 }$</td>
</tr>
<tr>
<td>23</td>
<td>$E K_1^3 K_2 D^6 (1 + K_2/2)^2 { (E_1 - 1) [ (E_1 - 1)^2 + 3 (E_1 - 2E_5 + 1)^2 ] / 12 (E_1 - E_5)^3 - (E_5 + 1)^3 / 3 (E_5 - E_1)^3 + (E_6 + 1)^3 / 3 (E_6 - E_3)^3 + (E_6 - 1)^3 / 3 (E_6 - E_4)^3 - 1/3 }$</td>
</tr>
<tr>
<td>Case*</td>
<td>$C_w$</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>24</td>
<td>$E K_1^3 K_2 D^6 (1+K_2/2)^2 \left{ (E_1-1)\left[ (E_1-1)^2 + 3(E_1-2E_5+1)^2 \right] / 12 \left( E_1 - E_5 \right)^3 + (E_2-1)\left[ (E_2-1)^2 + 3(E_2-2E_5+1)^2 \right] / 12 \left( E_2 - E_5 \right)^3 + (E_4+1)\left[ (E_4+1)^2 + 3(E_4-2E_6-1)^2 \right] / 12 \left( E_4 - E_6 \right)^3 \right}$</td>
</tr>
<tr>
<td>25</td>
<td>$E K_1^3 K_2 D^6 (1+K_2/2)^2 \left{ (E_1-1)\left[ (E_1-1)^2 + 3(E_1-2E_5+1)^2 \right] / 12 \left( E_1 - E_5 \right)^3 + (E_2-1)\left[ (E_2-1)^2 + 3(E_2-2E_5+1)^2 \right] / 12 \left( E_2 - E_5 \right)^3 + (E_4+1)\left[ (E_4+1)^2 + 3(E_4-2E_6-1)^2 \right] / 12 \left( E_4 - E_6 \right)^3 \right}$</td>
</tr>
<tr>
<td>26</td>
<td>$E K_1^3 K_2 D^6 (1+K_2/2)^2 \left{ (E_1-1)\left[ (E_1-1)^2 + 3(E_1-2E_5+1)^2 \right] / 12 \left( E_1 - E_5 \right)^3 \right}$</td>
</tr>
<tr>
<td>27</td>
<td>$E K_1^3 K_2 D^6 (1+K_2/2)^2 \left{ (E_1-1)\left[ (E_1-1)^2 + 3(E_1-2E_5+1)^2 \right] / 12 \left( E_1 - E_5 \right)^3 \right}$</td>
</tr>
<tr>
<td>28</td>
<td>$E K_1^3 K_2 D^6 \left{ 1 - (E_5 - 1)^3 / 8 \left( E_5 - E_1 \right)^3 \right} / 18 + 2 E K_3^3 D^6 \left{ 1 - (E_6 + 1) / (E_6 - E_5) \right} / 9$</td>
</tr>
<tr>
<td>29</td>
<td>$E K_1^3 K_2 D^6 \left{ 1 - (E_1 + 1) / 2(E_1 - E_2) \right} - (E_5 - 1)^3 / 8 \left( E_5 - E_1 \right)^3 - (E_5 - 1)^3 / 8 \left( E_5 - E_2 \right)^3 \right} / 18 + 2 E K_3^3 D^6 \left{ 1 - (E_5 - 1) / (E_5 - E_6) - (E_6 + 1) / (E_6 - E_5) \right} / 9$</td>
</tr>
<tr>
<td>Case*</td>
<td>( C_W )</td>
</tr>
<tr>
<td>-------</td>
<td>------------------</td>
</tr>
<tr>
<td>30</td>
<td>( E k_1^3 k_2 D^6 (1 + \eta_2/ \pi)^2 \left{ \frac{(\xi_1+i) - (\xi_1-\xi) - 3(\xi_1-\xi) + (\xi_1+i) - 3(\xi_1-\xi)}{2} \right} \frac{(\xi_1+i) - 3(\xi_1-\xi) + (\xi_1+i) - 3(\xi_1-\xi)}{2} \frac{2 \xi_1}{2} \frac{3 \xi_1}{2} \frac{3 \xi_1}{2} )</td>
</tr>
<tr>
<td>31</td>
<td>( E k_1^3 k_3 D^6 \left{ \frac{1 - (\xi_1+i) - (\xi_1-\xi) - (\xi_1-\xi) - (\xi_1-\xi)}{2} \right} \frac{1 - (\xi_1+i) - (\xi_1-\xi) - (\xi_1-\xi) - (\xi_1-\xi)}{2} \frac{1 - (\xi_1+i) - (\xi_1-\xi) - (\xi_1-\xi) - (\xi_1-\xi)}{2} )</td>
</tr>
<tr>
<td>32</td>
<td>( E k_1^3 k_3 D^6 (1 + \eta_2/ \pi)^2 \left{ \frac{(\xi_1+i) - (\xi_1-\xi) - (\xi_1-\xi) + (\xi_1+i) - (\xi_1-\xi) + (\xi_1+i) - (\xi_1-\xi) + (\xi_1+i)}{2} \right} \frac{(\xi_1+i) - (\xi_1-\xi) + (\xi_1+i) - (\xi_1-\xi) + (\xi_1+i)}{2} \frac{2 \xi_1}{2} \frac{3 \xi_1}{2} )</td>
</tr>
<tr>
<td>33</td>
<td>( E k_1^3 k_3 D^6 \left{ \frac{1 - (\xi_1+i) - (\xi_1-\xi) - (\xi_1-\xi) - (\xi_1-\xi)}{2} \right} \frac{1 - (\xi_1+i) - (\xi_1-\xi) - (\xi_1-\xi) - (\xi_1-\xi)}{2} \frac{1 - (\xi_1+i) - (\xi_1-\xi) - (\xi_1-\xi) - (\xi_1-\xi)}{2} )</td>
</tr>
<tr>
<td>34</td>
<td>( E k_1^3 k_3 D^6 \left{ \frac{1 - (\xi_1+i) - (\xi_1-\xi) - (\xi_1-\xi) - (\xi_1-\xi)}{2} \right} \frac{1 - (\xi_1+i) - (\xi_1-\xi) - (\xi_1-\xi) - (\xi_1-\xi)}{2} \frac{1 - (\xi_1+i) - (\xi_1-\xi) - (\xi_1-\xi) - (\xi_1-\xi)}{2} )</td>
</tr>
<tr>
<td>35</td>
<td>( 2E k_3^3 D^6 \left{ \frac{1 - (\xi_1+i) - (\xi_1-\xi) - (\xi_1-\xi)}{2} \right} \frac{1 - (\xi_1+i) - (\xi_1-\xi) - (\xi_1-\xi) - (\xi_1-\xi)}{2} \frac{1 - (\xi_1+i) - (\xi_1-\xi) - (\xi_1-\xi) - (\xi_1-\xi)}{2} )</td>
</tr>
</tbody>
</table>
Table 18 (Cont'd.)

<table>
<thead>
<tr>
<th>Case*</th>
<th>$C_W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>$E_k^3 K_2 D^6 \left(1 - (E_5 - 1)/(E_5 - E_7)\right)^2/144 + 2 E_k^3 D^6 \left(1 - (E_5 - 1)/(E_5 - E_6) - (E_6 + 1)/(E_6 - E_5)\right)^2/9$</td>
</tr>
<tr>
<td>37</td>
<td>$E_k^3 K_2 D^6 \left(1 + K_2/2\right)^2 \left{ (E_1 - 1) [ (E_1 - E_1)^2 + 3 (E_2 - 2 E_5 - 1)^2 ] /12 (E_1 - E_5)^3 + (E_4 + 1) [ (E_4 + 1)^2 + 3 (E_4 - 2 E_6 - 1)^2 ] /12 (E_4 - E_6)^3 + (E_5 + 1)^2 / 3 (E_5 - E_5)^3 / 3 \right}$</td>
</tr>
<tr>
<td>38</td>
<td>$E_k^3 K_2 D^6 \left(1 - (E_5 + 1)/(E_5 - E_6)\right)^3 / 144$</td>
</tr>
<tr>
<td>39</td>
<td>$E_k^3 K_2 D^6 \left(1 + K_2/2\right)^2 \left{ (E_2 + 1) [ (E_2 + 1)^2 + 3 (E_2 - 2 E_5 - 1)^2 ] /12 (E_2 - E_5)^3 + (E_4 + 1) [ (E_4 + 1)^2 + 3 (E_4 - 2 E_6 - 1)^2 ] /12 (E_4 - E_6)^3 \right}$</td>
</tr>
<tr>
<td>40</td>
<td>$E_k^3 K_2 D^6 \left(1 + K_2/2\right)^2 \left{ (E_2 + 1) [ (E_2 + 1)^2 + 3 (E_2 - 2 E_5 - 1)^2 ] /12 (E_2 - E_5)^3 + (E_4 + 1) [ (E_4 + 1)^2 + 3 (E_4 - 2 E_6 - 1)^2 ] /12 (E_4 - E_6)^3 + (E_5 + 1)^2 / 3 (E_5 - E_5)^3 / 3 \right}$</td>
</tr>
<tr>
<td>41</td>
<td>$E_k^3 K_2 D^6 \left(1 + K_2/2\right)^2 \left{ (E_2 + 1) [ (E_2 + 1)^2 + 3 (E_2 - 2 E_5 - 1)^2 ] /12 (E_2 - E_5)^3 + (E_4 + 1) [ (E_4 + 1)^2 + 3 (E_4 - 2 E_6 - 1)^2 ] /12 (E_4 - E_6)^3 + (E_5 + 1)^2 / 3 (E_5 - E_5)^3 / 3 \right}$</td>
</tr>
<tr>
<td>Case*</td>
<td>( C_W )</td>
</tr>
<tr>
<td>-------</td>
<td>------------</td>
</tr>
<tr>
<td>42</td>
<td>( E K_1^3 K_2 D^6 (1+K_2/2)^2 \left{ \left( E_1 - 1 \right) \left[ \left( E_1 - 1 \right)^2 + 3 \left( E_1 - 2 E_5 + 1 \right)^2 \right] / 12 \left( E_1 - E_5 \right)^3 \left( E_4 + 1 \right) \left[ (E_4+1)^2 + 3 (E_4 - 2 E_6 - 1)^2 \right] / 12 (E_4 - E_6)^3 + (E_5 + 1)^3 / 3 (E_5 - E_1)^3 + (E_6 - 1)^3 / 3 (E_6 - E_4)^3 - 2/3 \right} )</td>
</tr>
<tr>
<td>43</td>
<td>( E K_1^3 K_2 D^6 (1+K_2/2)^2 \left{ (E_4 + 1) \left[ (E_4+1)^2 + 3 (E_4 - 2 E_6 - 1)^2 \right] / 12 (E_4 - E_6)^3 + (E_5 + 1)^3 / 3 (E_5 - E_1)^3 + (E_5 - 1)^3 / 3 (E_5 - E_2)^3 + (E_6 + 1)^3 / 3 (E_6 - E_4)^3 - 1/3 \right} )</td>
</tr>
</tbody>
</table>

* Numbers refer to Figure II.8.
Figure 1.1: Typical $M_0$-$\theta_0$ curves for a beam column under uniaxially eccentric load.
Figure I.3: Assumed residual stress pattern
Figure I.4: Parameters defining the beam column cross section
Figure I.5: A beam column under uniaxially eccentric load
A - Prior to loading
B - After loading in unbuckled position
C - After lateral-torsional buckling

c = Centroid of the section
s = Shear centre

Figure 1.6: The displacements of the cross section of the beam column
Figure 1.7: Sign convention

Figure 1.8: Components of $P$ in the $\xi$ and $\zeta$ directions

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Figure 1.9: Possible strain-configurations for beam-column cross section under uniaxial loading (residual stresses are considered)
Figure 1.10: Possible strain configurations for beam-column cross section under uniaxial loading (residual stresses are neglected)
Figure I.11: Properties of strained cross section
Figure 1.12: Sequence for checking the assumed strain configurations (Uniaxial loading considering residual stresses)
Figure I.13: Sequence for checking the assumed strain configurations

(Uniaxial loading neglecting residual stress)
Figure I.14: Derivation of an expression for $\int_{A} \sigma r^2 \, dA$
Figure 1.15
Figure 16. A Flow chart for a computer programme used to determine the lateral-torsional buckling loads for beam-columns loaded uniaxially.

Start

Read $\frac{q^*}{f_y}$

Read $b, t, d, v, c_y, E, G, (E_E/E), (K_E/E_y)$

compute sectional properties like $K_1, K_2, K_3, K_0, D, h, s, z, F_y, H_y, M_y, \phi_y, e_y, I_x, I_y, I_x, I_y, X, X, Y, C_p$

Call subroutine astrn
compute $N^x/N_y^x, \psi^x/\psi^x_y, F_y, C_k, Y_0$

3
Figure 1.16 (Cont'd.)

Specify a value for $(M_{L})$

Determine an upper bound for the critical end moment $M_{E}^{\max}$ and a lower bound using approx. solution.

Specify a value $(R_i)$ to determine the increment by which the end moment will be increased

$I = 1$
Figure I.16 (Cont'd.)

Assumed critical end moment $N_{0i}(I) = \frac{M_0}{M_x} \min + \frac{\Delta M_{00}}{M_y} I$

1. Call subroutine curve
   develop the column deflection curves and construct the end-moment end-rotation relationship

2. Call subroutine curve
   Construct the matrix $[C]$ (20,20) using the data obtained from the column deflection curves and the sectional properties of the beam-column at $n$ discrete points

3. Call HSEGC
   transfer the matrix $[C]$ into an almost upper triangular matrix $[D]$
Figure I.16 (Cont'd.)

Call ACHFG
Calculate the eigenvalues \( \lambda \) of
\[(D - \lambda I)X = 0\]

The calculated critical end moments \( M_{oi}^{1} \)
\[M_{oi}^{1} = 1 - \lambda\]

Call AINDI
Determine a lowest value for
\[M_{oi}^{1}\]

Compute the ratio \[\frac{M_{oi}}{M_{oi}^{1}}\]

\[AL = \frac{M_{oi}}{M_{oi}^{1}} (1) - \frac{M_{oi}}{M_{oi}^{1}} (I - 1)\]
Figure I.17: Moment-curvature-thrust relationship for 8WF31
($\alpha_y = 36, \frac{E}{E \cdot E} = 0.022$ & $\frac{E \cdot E}{E \cdot y} = 12$)
$B_{Yo} = \text{bending rigidity about the weak axis for the full section}$

Figure 118: $M^X/M^X_y$ Versus $B_y/B_{Yo}$ Curves for 8 WF 31
Figure 11.9: $M^x/M_y^x$ Versus $C_W/C_T$ Curves for $P/P_y=0.2$, 0.4, 0.6, 0.8.

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Figure 120 $\frac{M_x^y}{M_y^x}$ Versus $\frac{y_0}{d}$ Curves for 8 WF 31
Figure 2.21: $M_X^*/M_Y^*$ Versus $(1 - \frac{\int r^2 dA}{C_T})$ Curves for 8 WF 31
Figure 1.22: Lateral torsional buckling strength curves for an 8WF31 \((\gamma = 36, \varepsilon_{st}/\varepsilon = 0.022 \& \varepsilon_{st}/\varepsilon = 12)\)

1. Failure by excessive bending considering residual stresses
2. Failure by lateral torsional buckling
   - 2. Neglecting residual stresses
   - 3. Neglecting pre-buckling displacements
   - 4. Considering residual stresses and pre-buckling displacements
Figure I.23: Lateral torsional buckling strength curves for an 8WF31 (\(\sigma_y = 36, \frac{E_{st}}{E} = 0.022 \) & \(\frac{E_{st}}{\varepsilon_y} = 12\))

1. Failure by excessive bending considering residual stresses
2. Failure by lateral torsional buckling neglecting residual stresses
3. Failure by lateral torsional buckling neglecting pre-buckling displacements
4. Failure by lateral torsional buckling considering residual stresses and pre-buckling displacements
Maximum strain at the mid-span section = 25 yield strain

strain hardening

\[ \frac{P}{P_y} = 0.6 \]

Figure I-24: Lateral torsional buckling strength curves for an 8WF31 (\(\alpha_y = 36, \frac{E}{E_y} = 0.022 \) & \(\frac{\varepsilon_y}{E_y} = 12\))

1. Failure by excessive bending considering residual stresses
2. Failure by lateral torsional buckling
3. Failure by lateral torsional buckling neglecting pre-buckling displacements
4. Failure by lateral torsional buckling considering residual stresses and pre-buckling displacements

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Figure I.25: Typical stress-strain curves for Aluminum alloys

(Alcan Section - D545 - H11A (53))
Figure 1.26 Lateral torsional buckling strength curves for an aluminum alloy section Alcan No. 28021. Comparing the calculations with $E_{st}/E = 0.00$.

(Considering pre-buckling displacement and assuming no residual stresses, $\sigma_y = 27$ ksi, $E = 10,000$ ksi, $\varepsilon_{st}/\varepsilon_y = 1.0$)
Figure 127: Comparison with previous theoretical results for a 8WF31 (\(\alpha_y=33, \frac{\varepsilon_{st}E}{\varepsilon_y} = 0.022 \) & \(\frac{\varepsilon_{st}E}{\varepsilon_y} = 12\))

Results neglecting residual stresses (\(E=29,000\) ksi)
(1a) Ref. (41), (1b) present investigation

Results neglecting pre-buckling displacement (\(E=30,000\) ksi)
(2a) Ref. (14), (2b) present investigation
Figure 128: Comparison of theory with Lehigh tests for 8WF31

\( \alpha_y = 36, \frac{E_{st}}{E} = 0.022 \quad \text{and} \quad \frac{E_{st}}{\varepsilon_y} = 12 \)
FIGURE I.30 The relationship between $M_{ocr}/M_p^x$ and the number of discrete points $n$ for 8 WF 31

- $P/P_y = 0.4$
- Prebuckling displacements are neglected

$\ell/r_x = 81$
Figure 13: Sequence for checking the strain patterns.
Figure I.32 - A flow chart for a computer program used to determine the lateral buckling loads for beams under equal end moments.

Start

Read $\frac{\phi_x}{\phi_y}(i), i = 1, 50$

Read $b, t, d, w, c_y, E, G, (E_{st}/E_0), (E_{st}/E_p)$

Compute Sectional properties like $K_x, K_y, K_z, K_0, D_x, A, S, Z, P_y, M_y, P_{sy}, \phi_y, \gamma_y, I_x, I_y, I_z, I_x, I_y, C_T$

Call subroutine STRN
Compute $c_0, c_1, c_2, \ldots, c_6$

Call subroutine CHECK
check the values $M_x/M_y$
and compute $\int \sigma x^2 dx_c$

Call subroutine ASTRN
Compute $M_x/K_y, \phi_x/\phi_y, P_y, C_w, Y_o$

1
Figure 1.32 (cont'd.)

\[
\begin{align*}
(1) &= \frac{E_{d}^{2}}{E_{d}^{2} + 2 CE_{d} G_{d}} \\
\text{compute } &G_{d}(1), G_{d}(1), H_{d}(1) \text{ corresponding to } \frac{1}{X_{d}} \\
I &= I + 1 \\
\text{if } &I > 50 \text{ stop}
\end{align*}
\]
Figure I-33: Moment-Curvature relationship for section under pure moment ($\sigma_y = 36.0$ ksi, $E_{st}/E = 0.022$, $\varepsilon_{st}/\varepsilon_y = 12$)
Figure 134: \( \frac{M_x}{M_y} \) Versus \( \frac{B_y}{B_{y0}} \) Curves

(\( P/P_y = 0.0 \))

\[ B_{y0} = \text{bending rigidity about the weak axis for the full section.} \]
Figure I36: $\frac{M_x}{M_y}$ Versus $y_o/d$ Curves

($P/P_y = 0.0$)

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Figure I.37: $M_x^X/M_y^X$ Versus $(1 - \frac{\int r^2 dA}{C_T})$ Curves 
($P/P_y = 0.0$)

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Figure I.38: Lateral buckling strength curves

\( \alpha_y = 36 \text{ ksi, } E_{st}/E = 0.022, \ \varepsilon_{st}/\varepsilon_y = 12 \)
Figure 3.9: Lateral buckling strength curves for 8WF31
\( \sigma_y = 350 \text{ksi} \), \( E_s / E = 0.022 \)
Figure I.41: Lateral buckling strength curves for 14 WF 142
($\sigma_y = 36\cdot0$ ksi, $E_{st}/E = 0\cdot022$)
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Figure I-44: Lateral buckling strength curves for beams made of aluminum (Alcan shape no. 28021, $\sigma_y = 27.0$ ksi, $E = 10,000$ ksi)
Figure II.1: Typical end moment versus end rotation curve for a biaxially loaded column
Figure II.2: Idealized stress-strain diagram.
Figure II.3: Beam-column under biaxially eccentric load (equal eccentricities)
Figure II.4: Beam-column under biaxially eccentric load (equal eccentricities)
\[ \bar{u} = u + y_0 \beta \]
\[ \bar{v} = v - x_0 \beta \]

Figure II.5: The displacements of a cross-section of a beam column under biaxially eccentric load.
Figure II.6: Usual pattern of yielding in a cross section of a beam-column loaded biaxially.
Figure II.7: Possible strain-configurations for a beam-column's cross-section under biaxial loading (residual stresses are neglected)

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Figure II.8: Possible strain-configurations for a beam-column's cross-section under biaxial loading (residual stresses are considered.)
Figure II.9: Diagram showing the sequence for checking the assumed strain configurations. (Biaxial loading neglecting residual stresses)

Compute bonding strains plus wrapping strains plus \( P/P_y \) at points 1 to 6 (\( \varepsilon_{10} \) to \( \varepsilon_{60} \))

IF \( \varepsilon_{10} > 1, \varepsilon_{20} > 1 \)  
IF \( \varepsilon_{20} > 1, \varepsilon_{50} > 1 \)  
IF \( \varepsilon_{20} > 1, \varepsilon_{40} > 1 \)  
IF \( \varepsilon_{20} > 1 \)  

Case A  
Case B  
Case C  
Case D  
Case E

where \( \varepsilon_{i0} = \varepsilon_i + \varepsilon_w + P/P_y \)

\( \varepsilon_{60} = \varepsilon_6 + \varepsilon_w + P/P_y \)
Figure II. 9: (Cont'd.)

- Yielded compression zone
- Yielded tension zone
- x: Number between parentheses refer to Figure II. 7

Case A

- If $\varepsilon_i > 1$ or $\varepsilon_n > 0$
  - $A_1(10)$

- If $\varepsilon_i < 1$
  - $A_2(16)$
    - If $\varepsilon_n < 1$
      - $A_3(14)$
    - If $\varepsilon_n > 1$
      - $A_4(18)$

Go To $A_9$

If $\varepsilon_i < 1$ or $\varepsilon_n < 0$

- $A_n(14)$
  - If $\varepsilon_n > 1$
    - $A_5(17)$
      - If $\varepsilon_n < 1$
        - $A_6(21)$
      - If $\varepsilon_n > 1$
        - $A_7(21)$
  - If $\varepsilon_n < 1$
    - $A_8(13)$

- $A_9(18)$

Go To $A_9$
Figure II.9: (Contd.)

Yielded compression zone

Yielded tension zone

Refer to Figure II.7

Case B

If $\epsilon_3 < \epsilon_1$

If $\epsilon_3 > \epsilon_1$

If $\epsilon_4 < \epsilon_1$

If $\epsilon_4 > \epsilon_1$

If $\epsilon_5 < \epsilon_1$

If $\epsilon_5 > \epsilon_1$

If $\epsilon_6 < \epsilon_1$

If $\epsilon_6 > \epsilon_1$

If $\epsilon_7 < \epsilon_1$

If $\epsilon_7 > \epsilon_1$

If $\epsilon_8 < \epsilon_1$

If $\epsilon_8 > \epsilon_1$

If $\epsilon_9 < \epsilon_1$

If $\epsilon_9 > \epsilon_1$

If $\epsilon_{10} < \epsilon_1$

If $\epsilon_{10} > \epsilon_1$

If $\epsilon_{11} < \epsilon_1$

If $\epsilon_{11} > \epsilon_1$

If $\epsilon_{12} < \epsilon_1$

If $\epsilon_{12} > \epsilon_1$

If $\epsilon_{13} < \epsilon_1$

If $\epsilon_{13} > \epsilon_1$

If $\epsilon_{14} < \epsilon_1$

If $\epsilon_{14} > \epsilon_1$

If $\epsilon_{15} < \epsilon_1$

If $\epsilon_{15} > \epsilon_1$

If $\epsilon_{16} < \epsilon_1$

If $\epsilon_{16} > \epsilon_1$

If $\epsilon_{17} < \epsilon_1$

If $\epsilon_{17} > \epsilon_1$

If $\epsilon_{18} < \epsilon_1$

If $\epsilon_{18} > \epsilon_1$

If $\epsilon_{19} < \epsilon_1$

If $\epsilon_{19} > \epsilon_1$

If $\epsilon_{20} < \epsilon_1$

If $\epsilon_{20} > \epsilon_1$

If $\epsilon_{21} < \epsilon_1$

If $\epsilon_{21} > \epsilon_1$

If $\epsilon_{22} < \epsilon_1$

If $\epsilon_{22} > \epsilon_1$

If $\epsilon_{23} < \epsilon_1$

If $\epsilon_{23} > \epsilon_1$

If $\epsilon_{24} < \epsilon_1$

If $\epsilon_{24} > \epsilon_1$

If $\epsilon_{25} < \epsilon_1$

If $\epsilon_{25} > \epsilon_1$

Go To C_1

Go To C_1

Go To D_5

Go To D_5

Go To D_4

Go To D_4

Go To D_3

Go To D_3

Go To C_1

Go To C_1

Go To B_3 (25)

Go To B_3 (25)

Go To B_2 (20)

Go To B_2 (20)

Go To B_3 (16)

Go To B_3 (16)

Go To B_2 (15)

Go To B_2 (15)

Go To C_1

Go To C_1

Go To C_1

Go To C_1

Go To C_1

Go To C_1

Go To C_1

Go To C_1

Go To C_1

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Go To C_1

Go To C_1

Go To C_1
Figure II.9: (Cont'd.)

Yielded compression zone

Yielded tension zone

x Number between parenthesis
refer to Figure II.7

If $\epsilon_4 > 1$

If $\epsilon_4 \leq 1$

Go To $D_1$

Go To $D_1$
Figure II. 9: (Contd.)

Yielded compression zone
Yielded tension zone

X Numbers between parenthesis refer to Figure II. 7

If $\varepsilon_2 > 1$

If $\varepsilon_1 < \varepsilon_6$

If $\varepsilon_2 < 1$

Go To $D_4$

Case D

$D_1(3)^X$

If $\varepsilon_2 \leq 1$

Go To $E$

If $\varepsilon_1 = \varepsilon_6$

Go To $D_6$

If $\varepsilon_6 < \varepsilon_1$

$D_2(5)$

If $\varepsilon_2 > 1$

If $\varepsilon_2 < 1$

Go To $D_4$

Go To $E_1$

$D_3(7)$

If $\varepsilon_2 > 1$

If $\varepsilon_2 < 1$

If $\varepsilon_2 > 1$

Go To $D_3$

If $\varepsilon_1 = \varepsilon_4$

Go To $D_6$

If $\varepsilon_1 < \varepsilon_4$

Go To $D_4$

If $\varepsilon_6 < \varepsilon_1$

Go To $D_6$

If $\varepsilon_6 < \varepsilon_1$

Go To $D_4$

If $\varepsilon_2 < -1$

If $\varepsilon_2 > -1$

Go To $E_1$
Figure 2.9: (Contd.)

Yielded tension zero

$D_E(9.5) = x$

$D_E(17.5)$

$D_E(25)$

If $C_2 < C_5$, Go To $D_6$

If $C_2 > 1$, Go To $E_6$

If $C_1 < C_6$, Go To $E_5$

If $C_1 > 1$, Go To $E_4$

If $C_2 > 1$, Go To $E_7$

X Number between parenthesis

Refer to Figure 2.7

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Figure II.9: (Cont'd)

Yielded compression zone
Yielded tension zone
× Number between parenthesis
refer to Figure II.7

Case D (Cont'd)

If $\varepsilon_2 \leq 1$
Go To $E_2$

If $\varepsilon_2 > 1$

If $\varepsilon_4 < \varepsilon_1$
Go To $E_4$

If $\varepsilon_4 = \varepsilon_1$
Go To $E_6$

If $\varepsilon_4 > \varepsilon_1$
Go To $E_5$

If $\varepsilon_1 < \varepsilon_4$
Go To $E_6$
The load $P$ and the rotation $\beta$ are constants for all curves.

Figure II.10: Moment-Curvature curves about $\tau$-axis

The load $P$ and the rotation $\beta$ are constants for all curves.

Figure II.11: Moment-Curvature curves about $\zeta$-axis
The load $P$ and the angle of rotation $\beta$ are constant.

Figure II.12: Curvature relationships for constant moment about each axis and constant rotation about the shear center.
Figure II.13: Column deflection curve

$U_A = U_B$

$V_A = V_B$
Figure II.14

\[ \delta_v = \frac{\alpha}{2} \tan \psi^x = \frac{\alpha^2}{2} \phi^x \]

Figure II.15

\[ \delta_u = \frac{\alpha}{2} \tan \psi^y = \frac{\alpha^2}{2} \phi^y \]
Figure II.16. A flow chart for a computer program to determine the N-P relationships and the mechanical properties of the section.
Figure II:17: A Flow chart for a computer program to determine the column deflection curve

START

Read $b$, $t$, $d$, $w$, $\frac{P}{F_Y}$, $\sigma_y$,
$E$, $G$, $E_T$, IMAX, JMAX, KMAX.

Read $x$
$\phi_x^{(I)}$, $I = 1$, IMAX
$\phi_y^{(J)}$, $J = 1$, JMAX
$\phi_y^{(K)}$, $K = 1$, KMAX

Read the moment data from the first programme
i.e. $\frac{M_x}{M_y}$, $\frac{M_x}{M_y}$, $C_{WD}$, $\int \sigma r^2 \, dA$
$\phi_x$, $\phi_y$, $\phi'_x$, $\phi'_y$
correspond to
$\phi_x^{(I)}$, $\phi_y^{(J)}$, $\phi_{y'}^{(K)}$

Calculate the mechanical properties of the section like
$K_1$, $K_2$, $K_3$, $A$, $Z$, $S$, $P_Y$, $M_y$, $C_T$, $\phi'^X$

Calculate the constants $C_1$ to $C_6$,
$\frac{P}{E I y}$, $\frac{P}{E I y}$

Read $\delta$, $A_A$, $B_B$, $C_C$

Read $\gamma$

Read $\frac{L}{F_Y}$, $a$

Calculate Npts

264
Figure II.17 (Cont'd.)

8

Initialize boundary conditions
UAS = 0.0
VAS = 0.0
Bo = 0.0

V(1) = delta

1 U(1) = V(1)/C_3
   \( \gamma = 0.0 \quad U_0 = 0.0 \)

4 \( \sigma^x(1) = 0.0 \)
   \( \gamma(1) = 0.0 \)

I = 1

Calculate \( M_C^x(I), M_C^y(I), M_C^w(I) \), \( M_C^z(I) \), \( M_C^\gamma(I) \)

Call subroutine \( G_{cuv} \)
Calculate \( \phi^x(I), \phi^y(I) \)

Call subroutine \( H_{cuv} \)
Calculate \( C_{wc}, \int \delta \rho \chi^2 d\alpha_c \)

Call Subroutine Curve
Find the interaction of the two curves
From Fig. II.12 for given values of \( \rho_c, \rho_c^\gamma, \beta^\gamma \)
Figure 11.17 (cont'd.)

1. Calculate \( x(t), \) \( y(t) \)
   \[ \phi_2 = \frac{\dot{x}(t)}{y(t)} \]
   \[ \phi_3 = \frac{\dot{y}(t)}{x(t)} \]

2. \( \phi \leq \phi_{max} \)
   \[ \delta = \frac{\phi_2}{2} \]
   \[ v(t) = v(t) - \delta \]

3. \( \theta Y_{max} \)
   \( \text{No} \)

4. \( \text{Yes} \)
   \( \text{No} \)

5. \( \text{Yes} \)
   \( \text{No} \)

6. \( \text{Yes} \)
   \( \text{No} \)

7. \( \text{Yes} \)
   \( \text{No} \)

8. \( I = I + 1 \)
   \( V(t) + 0.0030 \)

9. \( \text{Yes} \)
   \( \text{No} \)

10. \( \text{No} \)

Call subroutine \( \phi_2(t) \)

Call subroutine \( M_1(t), M_2(t), B(t) \)

Call subroutine curve of two curves for given values of \( M, M_1, B, \text{etc.} \)

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Figure II.17 (Cont'd.)

Calculate $U_A, V_A, \phi_A^x, \phi_A^y$

$\gamma_{10} = |\gamma_1 - \hat{\gamma}| - BB \cdot \gamma$

$\gamma_{10} \leq 0$

$\gamma_{10} > 0$

Calculate $M_A^x, M_A^y$

Calculate $B_\phi$

from equation (II.3.3)

$B_{10} = |B_\phi - B_0| - CC \cdot B_0$

$B_{10} > 0$

$B_{10} \leq 0$

$S < 0$

$S > 0$

Calculate $\gamma_0, S$, $U_0, U_1$

$\phi_0 = B_\phi$

$4$

$2$

$3$

$12$
Figure II.17 (Cont'd.)

Write $V_A, V (\ell), U_A, U (\ell), \frac{X_1}{X}, X_2, X, X_3, R, L_1, L_2, L_3, L_4$.

$V_A = V_A - V_{AS}$

$U_A = U_A - U_{AS}$

$V_A > 0$

$V_A < 0$

$U_A > 0$

$U_A < 0$

$AL = \frac{L}{h} - Euler$ buckling length

$AY = Y - max Y$

$AL > 0$

$AL < 0$

$AY > 0$

$AY < 0$

$\delta (1) = V (\ell) - (1)$

$\delta = (1) + \text{delta}$

Stop

End
Figure II.18 (a): Interaction curve for 8WF3I relating $P/P_y$, $M_A^x/M_y^x$ and $l/r_y$ for $\gamma = 0.20$
Figure II.18(b): Interaction curves for $8W_3F_3$ relating $P/P_y$, $M_x^f/M_y^f$ and $v/v_y$ for $y = 0.20$

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Figure II. 19(a): Interaction curves for 8WF31 relating $P/P_y$, $M_A^x/M_y^x$ and $\nu/\nu_y$ for $\gamma = 0.40$.
Figure II. 19 (b): Interaction curves for 8WF3I relating $P/P_y$, $M_A^x/M_y^x$, and $l/r_y$ for $\gamma = 0.40$
Figure II. 20 (a): Interaction curves for 8WF3I relating $P/P_y$, $M_A^X/M_y^X$ and $l/r_y$ for $\gamma = 0.60$. 

Present analysis 

CRC Equation 

$l/r_y = 20$ 

$l/r_y = 100$ 

$l/r_y = 60$ 

$l/r_y = 140$
Figure II.20 (b): Interaction curves for 8WF3l relating $P/P_y$, $M_A^x/M_y^x$ and $l/r_y$ for $\gamma = 0.60$
1939: Born on the 21st of March in Cairo, Egypt.

1960: Graduated with a B.Sc. in Civil Engineering from Ain Shams University, Cairo, Egypt.

1960: Appointed as a structural engineer at the Ministry of Public Works, Egypt.

1964: Completed a Post-Graduate Diploma Course in Advanced Studies in Structural Engineering from the Faculty of Science at the University of Manchester, England.

1965: Obtained a M.Sc. degree in Civil Engineering (Structures) from the Faculty of Science at the University of Manchester, England.

1966: Appointed as a Research and Structural Engineer at the Ministry of Irrigation, Egypt.

1968: Enrolled in a Ph.D. program in Civil Engineering in September at the University of Windsor.