Adaptive boundary element method.

Weiwei Sun
University of Windsor
NOTE TO USERS

This reproduction is the best copy available.

UMI
ADAPTIVE BOUNDARY ELEMENT METHOD

BY

WEIWEI SUN

A Dissertation
submitted to the
Faculty of Graduate Studies and Research
through the Department of
Mathematics and Statistics in Partial Fulfillment
of the requirements for the Degree
of Doctor of Philosophy at
the University of Windsor

Windsor, Ontario, Canada
1991
Abstract

This thesis is devoted to the study of the adaptive boundary element method, a subject which has been explored intensively during the last few years. In this thesis, a systematic simulation for the adaptive r-method is presented for the Laplace’s equation with constant, linear and quadratic elements and the equations of linear elasticity with linear and quadratic elements. The stability, singularity, advantages, disadvantages and limitations of the algorithm are explored. Based on the present discussions, a new adaptive h-r algorithm is developed. In this combined algorithm, the residual error used as an error indicator is minimized such that the mesh achieves asymptotic optimality. Simultaneously, the number of grid points is adaptively increased to meet a specified user tolerance. The algorithm is successfully used in solving Laplace’s equation and Navier’s equation of linear elasticity.
Dedicated to my parents, wife and daughter
ACKNOWLEDGEMENT

I would like to express my gratitude to my supervisor, Professor N. G. Zamani for suggesting the topic of this dissertation and for his generous help throughout my studies at University of Windsor. His encouragement and kindness have been a great support which I will remember forever.
TABLE OF CONTENTS

Abstract........................................................................ iv
Dedication.......................................................................... v
Acknowledgement............................................................ vi

Chapter One Introduction
   1.1 Background................................................................. 1
      1.1.1 Adaptive methods
      1.1.2 Optimal mesh
      1.1.3 Grading function
   1.2 Carey and Dinh's algorithm............................................. 7
   1.3 Outline of the Thesis.................................................... 12

Chapter Two The Optimal Mesh Algorithm (r-method)
   2.1 Introduction............................................................... 15
   2.2 Error Indicator........................................................... 18
   2.3 Adaptive Optimal Mesh Algorithm................................. 20
   2.4 Potential................................................................. 25
   2.5 Linear Elasticity.......................................................... 43
   2.6 Conclusion............................................................... 88

Chapter Three An Adaptive h-r Boundary Element Algorithm
   For the Laplace Equation
   3.1 Introduction............................................................. 69
   3.2 Truncation Error Indicator.......................................... 70
   3.3 The Adaptive h-r algorithm......................................... 79
### Chapter Four: An Adaptive h-r Algorithm For the Equations of Linear Elasticity

- **4.1 Introduction** ............................................. 91
- **4.2 Truncation Error** ........................................... 91
- **4.3 The Adaptive Algorithm** .................................. 99
- **4.4 Numerical Results** ....................................... 104
- **4.5 Conclusion** .............................................. 112

### Chapter Five: Stability and Singularity

- **5.1 Introduction** .............................................. 113
- **5.2 Continuous Distribution Functions** .................... 117
- **5.3 Singular Distribution Functions** ......................... 120

### Chapter Six: Conclusion

- .......................................................... 132

### Chapter Seven: Appendix I

- **7.1 The Proof of Theorem 2.1** ............................... 135
- **7.2 The Extension of Mean Value Theorem** .................. 141
- **7.3 The Basic Formulas in Boundary Element Methods** .......................... 142
- **7.4 An Interpolation Error Analysis of the Cauchy Mean Value Integral** .......... 149

### Chapter Eight: Appendix II

- **8.1 Fundamental Equations in Linear Elasticity**
Chapter One

Introduction

1.1 Background

1.1.1 Adaptive Methods

Adaptive methods have become important tools in the numerical solution of ordinary and partial differential equations mainly due to the need for extensive analysis of computer output particularly when an initial solution is not satisfactory. It is very desirable to incorporate automatic (adaptive) decision making algorithms into software such as finite element, finite difference and boundary element programs. Adaptive techniques have been used for a long time in the stepsize control of ordinary differential equations. The recent advances in Expert Systems and Artificial Intelligence have triggered considerable research in the area of adaptive algorithms [4, 42, 49]. Four recent conferences on this topic reflect the major activities undertaken by mathematicians and engineers in this area [6, 7, 20, 52].

In the context of the numerical solution of partial differential equations, an adaptive method is an automatic adjustment of certain parameters based on the approximate solution obtained. In general, this adjustment takes place only after an initial solution has been obtained in parts of
the domain where the discretization is poor as identified by an \textit{a posteriori} error indicator and therefore, such methods are iterative in nature. Most of the work in this area is based on local methods. For this reason, our discussion is implicitly directed to such techniques. The term adjustment, which is fundamental in an adaptive method, has been applied in three different contexts as follows:

1. h-method (multigrid, local refinement and h-version);
2. p-method;
3. r-method (optimal mesh and mesh redistribution).

These concepts have been used in some commercial codes \cite{2,3,8,28,34,35,66}. In the h-method, the number of nodal points in the discretization is increased in the whole domain or only in the areas of large solution gradients. This procedure is perhaps the oldest technique for improving the quality of the numerical solution. In the p-method, the number of nodal points is kept fixed but the degree of the approximating polynomial is increased. The concept of p-method is still under intensive investigation. Finally, the r-method is the least explored subject in the numerical solution of partial differential equations among the three algorithms. In the r-method, the number of nodal points and the degree of the approximating polynomial are kept fixed but the positions of the nodes are altered. Naturally, there should be a large concentration of nodal points in regions of steep gradients.

The key concept in an adaptive strategy is the error indicator. The adjustment in the adaptive process is highly
dependent upon the error indicator. In most investigations, the numerical discretization error in solving differential equations is used as an error indicator. Because of the rigorous theoretical foundations, most adaptive techniques have been successfully and primarily applied to finite element methods. Unfortunately, the boundary element methods (BEMD) do not benefit from such rigour and, therefore, there is more heuristics involved. This is perhaps because of a fundamental difference between differential and integral equations. It is well known that elliptic differential operators are local in nature whereas the integral operators arising in regular elliptic boundary value problems do not possess this property. Recently, some theoretical analysis on the local character of the boundary integral equations was presented by Wendland and Yu [64], where the boundary integral equations were considered as a pseudo-differential operator acting on the boundary. The existence of residual estimates of the global error has also been demonstrated by Rank [40] for the Galerkin boundary element method. Thus, some of the results obtained for the adaptive finite element method have been transferred to the adaptive boundary element method. Until recently most of the research in adaptive boundary elements has focused on the h-method, such as the results presented by Rencis and Mullen [43], Rencis [44], Wendland and Yu [64], Rank [41], Guiggiani [23] and Gomez-Lera, Cerrolaza and Alarcon [21]. Very few articles have dealt with the adaptive p-method [1,14,38,61] and r-method [13,29,57]. In general, the latter is based on a

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
stricter error expression. The error analysis known in the boundary element method is still not precise enough in spite of the theoretical results and error estimates for boundary integral equations presented by Wendland [62,63], Hisao [25] and Costabel and Stephen [16]. There are numerous other references but due to lack of space only some are listed here [21,27,31,32,33,45,46,47,48,51,53,55,60].

Recently, some combined adaptive algorithms have been explored. Theoretically, any combination of the h, p and r methods can produce an adaptive algorithm. The only combined algorithms known in the literature are the adaptive h-p method used in finite elements by Gui and Babuska [22] and boundary elements by Guo, von Petersdorff and Stephen [24] and our adaptive h-r method to be presented in chapters 3 and 4.

1.1.2 Optimal Mesh

At the early stages of the development of digital computers, most algorithms for numerical solution of differential equations employed uniform meshes. As one of the early pioneers, Brown [9] used a non-uniform mesh in solving a nonlinear two-point boundary value problem where the truncation error on the non-uniform mesh was of higher order.

The concept of an optimal mesh was first used in the context of optimal approximation [10,18,19]. In general, the determination of optimal meshes can be formulated as a minimization of an objective function subject to certain constraints. For one dimensional problems, this can be
expressed as

\[
\text{min } R(u,x_i) \tag{1.1.1}
\]

subject to: \( x_0 < x_1 < \ldots < x_N, \ u \in H_N \)

where \( R(u,x_i) \) is an objective function denoting an error or a physical indicator. \( H_N \) is an \( N \)-dimensional space of the solutions of the problem and \( u \) is a solution with respect to the mesh \( \{x_i\} \). When the necessary conditions for minimization are used, the optimal mesh is characterized by the system of nonlinear equations,

\[
\frac{\partial R}{\partial u_i} = 0 \quad i = 1,2,\ldots,M, \tag{1.1.2}
\]

\[
\frac{\partial R}{\partial x_j} = 0 \quad j = 1,2,\ldots,N. \tag{1.1.3}
\]

where \( M \) is the number of nodal unknowns and \( N \) is the number of grid points. In general, (1.1.1) is a nonlinear programming problem with constraints which cannot be solved analytically. Although the standard constrained nonlinear programming algorithms were proposed to solve the optimal mesh problem [11,52,65], numerical computation can be very expensive. In these references, \( R(u,x) \) is defined as the energy functional allowing both the nodal values and coordinates to enter as unknowns. A practical method is to try to uncouple equations (1.1.2) and (1.1.3) and then solve them iteratively.

1.1.3 Grading Function

The concept of the grading function was originally
introduced by Babuska and Rheinboldt [5] and was defined as a transformation from the domain coordinate, \( x \), to a new coordinate where the mesh is uniform, i.e., \( \zeta \).

Let \( \{ x_i \} \) be a partition of \([a, b]\). The grading function \( \zeta(x) \) defined on \([a, b]\) is characterized by

\[
\zeta_i \equiv \zeta(x_i) = \frac{i}{N} \quad i=0,1,2,\ldots,N. \tag{1.1.4}
\]

The function \( \zeta(x) \) is schematically shown in figure 1.1. Note that by a careful choice of \( \zeta(x) \) it is possible to have a high concentration of nodal points in the \( x \)-space of selected regions.

![Fig. 1.1 Mapping from \( x \) to \( \zeta \) through the grading function](image)

In most theoretical studies, the objective function \( R(u, x_i) \) in section 1.1.2 is assumed to be an error indicator in the numerical solution of the differential equation. \( R \) can therefore be expressed as a function of the solution \( u(x) \) and \( h_i \), where \( h_i \) is the element length, i.e., \( h_i = x_{i+1} - x_i \). The optimization problem (1.1.1) can therefore be
expressed in terms of the grading function $\xi(x)$ as follows,

$$\min_{\xi} R(u, \xi) \quad (1.1.5)$$

The above problem is one of calculus of variations and, in some cases, closed form solutions can be obtained. The algorithms derived in this thesis will follow such a concept.

1.2 Carey and Dinh's Method

In 1985, Carey and Dinh [12] presented an analysis of optimal mesh and optimal grading function in various norms and seminorms for a piecewise polynomial of interpolation problem. This analysis was also applied to solve two-point boundary value problems by the finite difference and finite element methods. The key points of their work are presented in this section.

Although one is mainly interested in optimal mesh distribution schemes for the approximate solution of differential equations, Carey and Dinh began by constructing an optimal mesh for an interpolation problem as described below.

Let $W(x)$ be a function defined on $[a, b]$. For the given grid $\{x_i\}$ on $[a, b]$, let $P(x)$ be a piecewise polynomial of degree $k$ which interpolates $W(x)$ at the grid points. The global error in the $H^m$-seminorm is defined as

$$\|E\|_m = \left( \int_a^b |(E^{(m)})^2 dx \right)^{1/2} \quad (1.2.1)$$
where $E(x) = W(x) - \overline{W}(x)$ is the global error of the interpolation problem, $E^{(m)} = \frac{d^m E}{dx^m}$.

Carey and Dinh developed an adaptive algorithm for the selection of optimal $\xi(x)$ in the interpolation problem. The mesh on which the global error of the interpolation problem is minimized in the $H^m$-seminorm is obtained as the solution of (1.1.5), i.e. find a grading function such that an upper bound of $\|E\|_m^2$ is minimized with respect to the variations of the mesh coordinates. Their derivation is presented below.

Expanding the error $E(x)$ and its derivatives on the subinterval $[x_{i-1}, x_i]$ in a Fourier sine series,

$$E(x) = \sum_{n=1}^{\infty} a_n \sin \frac{nn(x-x_i)}{h_i},$$

then

$$E^{(m)}(x) = \begin{cases} \displaystyle (-1)^{m/2} \sum_{n=1}^{\infty} a_n \left( \frac{nn}{h_i} \right)^m \sin \frac{nn(x-x_{i-1})}{h_i}, & m \text{ even} \\ \displaystyle (-1)^{(m-1)/2} \sum_{n=1}^{\infty} a_n \left( \frac{nn}{h_i} \right)^m \cos \frac{nn(x-x_{i-1})}{h_i}, & m \text{ odd} \end{cases}$$

We then have Parseval's identity for the $H^m$-seminorms of the error

$$\int_{x_{i-1}}^{x_i} [E^{(m)}]^2 dx = \frac{h_i}{2} \sum_{n=1}^{\infty} a_n^2 \left( \frac{nn}{h_i} \right)^{2m}$$

and similarly for $k \geq m$,
\[
\int_{x_{i-1}}^{x_i} [E^{(k+1)}]^2 \, dx = \frac{h_i}{2} \sum_{n=1}^{\infty} a_n^2 \left( \frac{nn}{h_i} \right)^{2(k+1)}.
\] (1.2.5)

Multiplying both sides of equation (1.2.5) by \( \left( \frac{h_i}{\pi} \right)^{2(k+1-m)} \) gives

\[
\left( \frac{h_i}{\pi} \right)^{2(k+1-m)} \int_{x_{i-1}}^{x_i} [E^{(k+1)}]^2 \, dx = \frac{h_i n^{2(k-m)}}{2} \sum_{n=1}^{\infty} a_n^2 \left( \frac{nn}{h_i} \right)^{2m}.
\] (1.2.6)

Comparing equations (1.2.4) and (1.2.6) term-by-term leads to the inequality

\[
\int_{x_{i-1}}^{x_i} [E^{(m)}]^2 \, dx \leq \left( \frac{h_i}{\pi} \right)^{2(k+1-m)} \int_{x_{i-1}}^{x_i} [E^{(k+1)}]^2 \, dx.
\] (1.2.7)

Since \( W(x) \) is assumed to be a piecewise polynomial of degree \( k \), we have

\[
E^{(k+1)}(x) = W^{(k+1)}(x).
\]

Hence, the inequality (1.2.7) reduces to

\[
\int_{x_{i-1}}^{x_i} [E^{(m)}]^2 \, dx \leq \left( \frac{h_i}{\pi} \right)^{2(k+1-m)} \int_{x_{i-1}}^{x_i} [W^{(k+1)}(x)]^2 \, dx.
\] (1.2.8)

The global error of the interpolation in \( H^m \)-seminorm is given by
\[ |E|_m^2 \leq \sum_{n=1}^{\infty} \left( \frac{h_i}{\pi} \right)^{2(k+1-m)} \int_{x_{i-1}}^{x_i} [W^{(k+1)}(x)]^2 \, dx \]

i.e.

\[ |E|_m^2 \leq \sum_{n=1}^{\infty} \left( \frac{h_i}{\pi} \right)^{2(k+1-m)} \left( W(\overline{x}_i) \right)^2 (1 + O(h_i)) \]  \hspace{1cm} (1.2.9)

where \( \overline{x}_i \) is an intermediate point of subinterval \( [x_{i-1}, x_i] \).

By the definition of grading function,

\[ \int_{x_{i-1}}^{x_i} \xi'(x) \, dx = \frac{1}{N} \]  \hspace{1cm} (1.2.10)

Furthermore

\[ \int_{x_{i-1}}^{x_i} \xi'(x) \, dx = h_i \xi'(\overline{x}_i) (1 + O(h_i)) \]  \hspace{1cm} (1.2.11)

From (1.2.10) and (1.2.11), \( h_i \) can be obtained,

\[ h_i = \frac{1}{N \xi'(\overline{x}_i)} (1 + O(h_i)) \]  \hspace{1cm} (1.2.12)

Substituting (1.2.12) into (1.2.9) gives

\[ |E|_m^2 \leq \frac{1}{C_{\pi N}} \sum_{n=1}^{\infty} \left( \frac{h_i}{\pi} \right)^{2(k+1-m)} \frac{[W^{(k+1)}(\overline{x}_i)]^2}{[\xi'(\overline{x}_i)]^2 (1 + O(h_i))} \]  \hspace{1cm} (1.2.13)

The sum in expression (1.2.13) can be interpreted as the Riemann sum. Therefore

\[ |E|_m^2 \leq \frac{1}{C_{\pi N}} \int_a^b \frac{[W^{(k+1)}(x)]^2}{[\xi'(x)]^2 (1 + O(h))} \, dx (1 + O(h_c)) \]  \hspace{1cm} (1.2.14)
where \( h = \max h_i \).

Now, let the right hand term in (1.2.14) be the objective function \( R \) in (1.1.5), i.e. we consider an optimization problem as following

\[
\min_{\xi} \frac{1}{(\pi N)^{(2(k+1)-m)}} \int_a^b \frac{[W^{(k+1)}(x)]^2}{[\xi'(x)]^{2(k+1)-m}} \, dx \quad (1.2.15)
\]

where \( W^{(k+1)}(x) \) is assumed to be known. Then (1.2.15) is a typical problem in calculus of variations with respect to the grading function \( \xi(x) \).

The grading function \( \xi(x) \) satisfies the Euler-Lagrange equation

\[
\frac{d}{dx} \frac{[W^{(k+1)}(x)]^2}{[\xi'(x)]^{2(k+1)-m+1}} = 0 \quad (1.2.16)
\]

Equation (1.2.16) can be solved for \( \xi'(x) \), i.e.

\[
\xi(x) = \text{constant} \int_a^b \frac{[W^{(k+1)}(x)]^{2/(2(k+1)-m+1)}}{dx} \, dx \quad (1.2.17)
\]

Finally, because of the boundary condition, \( \xi(b)=1 \), the constant in (1.2.17) can be determined leading to

\[
\xi(x) = \frac{\int_a^x [W^{(k+1)}(x)]^{2/(2(k+1)-m+1)} \, dx}{\int_a^b [W^{(k+1)}(x)]^{2/(2(k+1)-m+1)} \, dx} \quad (1.2.18)
\]

By the definition of the grading function, for a given
number of grid points, the optimal mesh \( \{x_i\} \) can be
determined by solving the following system of equations

\[
\int_a^b \frac{\left[W^{(k+1)}(x)\right]^{2/3(k+1)+1}}{[W^{(k+1)}(x)]^{2/3(k+1)+1}} dx = \frac{1}{N}, \quad i=1,2,...,N.
\]

(1.2.19)

As an example, if piecewise linear interpolation is
used \( (k=1) \), the grading function in the \( L^2 \)-norm \( (m=0) \) is

\[
\xi(x) = \frac{\int_a^b [W^{(')}(x)]^{2/3} dx}{\int_a^b [W^{(')}(x)]^{2/3} dx}.
\]

(1.2.20)

In Carey and Dinh's paper, it was proved that the error
measure in (1.2.15) is equivalent to the energy functional
in finite elements for a two-point boundary value problem.

1.3 The outline of this thesis

Carey and Dinh's optimal mesh algorithm has been
applied for solving partial differential equations in \[13\]
where the global error of interpolation is used as an error
indicator and an example from potential theory is
considered. Theoretically, it can be proven that the global
error of interpolation is equivalent to the global error of
the numerical solution only in special cases. In chapter
two, a systematic simulation of the algorithm in potential
problems is presented. Several examples from fluid dynamics
and classical mechanics with and without singularities are

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
considered where constant, linear and quadratic boundary elements are employed. Carey and Dinh's method is also extended to solving the linear elasticity equations. Similarly, a systematic simulation of the linear elasticity problems with and without singularities is presented. In addition, the algorithm in this chapter is based on the minimization of the asymptotic global error of interpolation rather than the upper bound of the global error as used in Carey and Dinh's work.

In chapter three, a combined adaptive h-r algorithm is developed. This idea is based on the minimization of a truncation error which results in finding an asymptotic optimal mesh and the optimal number of grid points. Finally, numerical examples with and without singularities for potential problems are presented where the adaptive h-r algorithm and linear boundary elements are used.

In chapter four, we extend the adaptive h-r algorithm to the linear elasticity equations. Due to a fundamental difference in the nature of the singularity in the potential and linear elasticity equations, an iteration process has to be used to find the optimal grading function in the latter case.

The numerical results in chapters three and four indicate that the combined adaptive h-r boundary element method seems to be more effective in improving the solution quality than the h- or r-methods individually.

In chapter five, some results on the stability of the optimal mesh algorithm are discussed. First, it will be
proven that the optimal mesh distribution is continuously dependent on the distribution function and the optimal error indicator. Secondly, we extend the analysis into a class of singular problems which arise in fluid dynamics and linear elasticity. It is obvious that the optimal mesh corresponding to the singular distribution function cannot satisfy the condition of the quasi-uniform mesh presented by Pereyra and Sewell [37] where the discussion was restricted to smooth problems. Hence, the concept of the k-regular mesh will be introduced. We also prove that the optimal mesh corresponding to some singular problems is k-regular on which the error indicator has an optimal order. Similar results on stability are also presented. If the distribution function has roots in its domain, it results in an unreasonable mesh (too sparse near the roots). For such cases, a modified distribution function is presented. The optimal mesh corresponding to the modified distribution function is still asymptotically optimal and the error indicator has an asymptotic minimum value.

Chapter six contains some conclusions, suggestions and possible extensions of the basic ideas in this thesis.
Chapter Two

The Optimal Mesh Algorithm (r-method)

2.1 Introduction

The optimal mesh algorithm was originally applied in the context of approximation theory. Only recently it has been extended to the finite element solution of differential equations. Generally, all optimal mesh algorithms are based on the concept of equidistributing a certain error indicator, i.e. the error indicator is uniformly distributed on each element under some norm. As will become clear, the optimal mesh algorithm for finite element and finite difference methods is applied primarily to one dimensional problems. Two-dimensional extensions, if possible, are not straightforward.

Due to the nature of the boundary element method, which reduces a two-dimensional boundary value problem into an integral equation along the one-dimensional boundary, it is possible to employ the optimal mesh algorithm for the computational purposes. By the maximum principle, the minimization on the boundary will result in the minimization in the whole domain [27]. Carey and Kennon [13] were the first to extend some of the results from the one-dimensional interpolation theory to the two-dimensional boundary element
solution of Laplace's equation. Carey and Kennon's approach is based on the grading function described in chapter one and the error analysis of an upper bound for the global error of interpolation where the function values are implicitly given by the boundary integral equations.

In this chapter, a stricter error analysis of the one-dimensional interpolation problem is presented. An asymptotic global error estimate is given which is equivalent to the upper bound of the global error derived by Carey and Kennon. This asymptotic global error will be used as an error indicator in the optimal mesh algorithm. Next, a systematic simulation of the optimal mesh algorithm is presented in the case of Laplace's equation, where the constant, linear and quadratic direct boundary element formulations are used. Finally, we extend the algorithm to Navier's equations for linear elasticity.

2.2 Error Indicator

Carey and Dinh's adaptive optimal mesh algorithm is based on the minimization of an error indicator which is an upper bound on the global error of interpolation. In this section, we prove that their minimization is equivalent to minimizing the asymptotic global error of interpolation.

Let $W(x)$ be a function defined on $[a,b]$. For a given partition

$$\Pi: x_0 < x_1 < \ldots < x_n$$
on \([a,b]\), let \(W_h(x)\) be a piecewise interpolation polynomial of \(W(x)\). Let \(x_i^j\) \(j=0,1,...,k\), be nodal points in the subinterval \([x_{i-1},x_i]\) with

\[
x_i^c = x_{i-1}, \quad x_i^k = x_i, \quad i=0,1,...,n.
\]

Then in \([x_{i-1},x_i]\), \(W_h(x)\) is a polynomial of degree \(k\) which interpolates \(W(x)\) in \(x_i^j\), \(j=0,1,...,k\). The global error of the interpolation is defined as

\[
E(x) = W(x) - W_h(x)
\]

By standard results from interpolation theory, \(E(x)\) can be expressed as

\[
E(x) = W[x_i^0,x_i^1,...,x_i^k,x] \prod_{j=0}^k (x - x_i^j)
\]

where \(W[x_i^0,x_i^1,...,x_i^k,x]\) denotes the Newton divided difference of \(W(x)\) in \(x_i^j\).

**Lemma 2.1** (see reference [30] pp.252) Let \(y_i\in[a,b]\) and \(f(y)\) have a continuous \(n\)th derivative in \([a,b]\). Then if the points \(y_0,y_1,...,y_n\) are distinct,

\[
f[y_0,y_1,...,y_n] = \int_0^{t_1} \int_0^{t_2} ... \int_0^{t_{n-1}} f^{(n)}(y_n-y_{n-1}) + \ldots + t_1 [y_i-y_0] + y_0) dt_n
\]

**Lemma 2.2** If \(f(y)\) has continuous derivatives of order \(m+k+1\) in \([a,b]\) and \(y_i\in[a,b]\) and \(y_i\neq y_j\) for \(i\neq j\), then there exists
\( \eta \in \min(y_i), \max(y_i) \) such that

\[
\frac{d^m}{dy^m} f[y_0, y_1, \ldots, y_k, y] = \frac{m!}{(m+k+1)!} \frac{d^{(m+k+1)}}{dy^{m+k+1}} f(\eta)
\]

(2.2.3)

Proof: By Lemma 2.1,

\[
f[y_0, y_1, \ldots, y_k, y] = \int_0^t \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_k} y_k \cdot y_3 = C_{m+k+1} d\tau d\rho
\]

(2.2.4)

Differentiating both sides of (2.2.4), we obtain

\[
\frac{d^m}{dy^m} f[y_0, y_1, \ldots, y_k, y] = \int_0^t \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_k} y_k \cdot y_3 = C_{m+k+1} d\tau d\rho
\]

(2.2.5)

By the mean value theorem, (2.2.5) can be expressed as

\[
\frac{d^m}{dy^m} f[y_0, y_1, \ldots, y_k, y] = f^{(k+m)}(\eta) \int_0^t \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_k} t^m d\tau d\rho
\]

(2.2.6)

where \( \eta \in \min(y_i), \max(y_i) \). Then

\[
\frac{d^m}{dy^m} f[y_0, y_1, \ldots, y_k, y] = \frac{m!}{(m+k+1)!} \frac{d^{(m+k+1)}}{dy^{m+k+1}} f(\eta)
\]

(2.2.7)

We now consider an asymptotic error analysis. Similar
to Carey and Dinh's analysis in section 1.2, we use the $H^m$-seminorm defined in $[a,b]$ as follows:

$$\|E\|_m^2 = \int_a^b \left( E^{(m)}(x) \right)^2 dx .$$

**Theorem 2.1** With the above notation, if $W(x) \in C^{m+k+1}[a,b]$, and function $T(x) = W^{(k+1)}(x)^2$ is strictly positive ($T(x) > 0$), the global error (with the $H^m$-seminorm, $m \leq k$) can be expressed as

$$\|E\|_m^2 = C \int_a^b \frac{W^{(k+1)}(x)^2}{(\xi')^2(k+1-m)} dx + o(C)$$

(2.2.7)

where $C$ is a constant, $\xi(x)$ is the grading function defined in chapter one and $o(C)$ denotes an infinitesimal quantity of the term ahead of it as $N \to \infty$.

**Proof:** (see Appendix I, 7.1).

By Theorem 2.1, the optimal mesh can be constructed by minimizing the asymptotic global error rather than its upper bound used in Carey and Dinh's algorithm. Because of the similarity of (2.2.7) and (1.2.15), it is obvious that the optimal mesh with respect to the asymptotic global error analysis is the same as that in Carey and Dinh's algorithm.

Clearly, Theorem 2.1 is true under some other norms and seminorms such as $L_1$-norm and a weaker continuity condition.

Similar to the minimization described in chapter one, the optimal mesh can be considered as the following optimization problem.
Then the optimal mesh can be determined by

$$ \min \ E = C \int_{a}^{b} \left[ W^{(k+1)}(\xi) \right]^{2} \left( \frac{1}{\xi} \right)^{2(k+1-m)} \ dx . $$

Then the optimal mesh can be determined by

$$ \frac{k(i-1)+j}{nk} = \frac{\int_{a}^{x_{i}} |W^{(k+1)}(\xi)|^{2/2(k+1-m)+1} \ dx}{\int_{a}^{b} |W^{(k+1)}(\xi)|^{2/2(k+1-m)+1} \ dx} $$

$$ i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, k \quad (2.2.8) $$

and on the optimal mesh, the error indicator has its minimum value as follows

$$ E_{0} = C \int_{a}^{b} |W^{(k+1)}(\xi)|^{2/q} \ dx \quad (2.2.9) $$

where $q=2(k+1-m)+1$.

2.3. Adaptive Optimal Mesh Algorithm

Equation (2.2.8) in section 2.2 defines an asymptotic optimal mesh. In the adaptive optimal mesh algorithm, the function $T(\xi) = |W^{(k+1)}(\xi)|^{2/2(k+1-m)+1}$ involving the unknown solution must be replaced by its approximation $\tilde{T}(\xi)$ constructed by the discrete boundary element solution $\{w_i\}$. In this thesis, we only consider the cases of $m=0$ and $k=0, 1, 2$ and employ $k+1$ nodal values for $k=0, 2$ and $k+2$ nodal values for $k=1$ from the adjacent elements, i.e.
\[ W^{(k+1)}(s) = \begin{cases} 
W[s_j, s_{j+1}] & k=0 \\
W[s_{j-1}, s_j, s_{j+1}] + W[s_j, s_{j+1}, s_{j+2}] & k=1 \\
W[s_{j-1}, s_j, s_{j+1}, s_{j+2}] & k=2 
\end{cases} \]

in \([s_j, s_{j+1}]\). At the points which are fixed due to geometrical constraints, the algorithm uses one sided divided difference for the calculation of \(W^{(k+1)}(s)\). Therefore, the optimal mesh is determined by the following system of equations

\[
\frac{i}{N} = \frac{\int_{a}^{x_i} \mathcal{T} \mathcal{O} dx}{\int_{a}^{b} \mathcal{T} \mathcal{O} dx} \quad (2.3.1)
\]

where we still use notation \(\langle x_i \rangle, i=0,1,\ldots,N\), replacing \(\langle x^j_i \rangle, i=0,1,\ldots,n, j=0,1,\ldots,k\), and \(N=nk\). Hence, the optimal mesh algorithm is an iterative procedure. After the numerical solution on an initial mesh is obtained with the boundary element method, the new mesh can be determined by solving (2.3.1). Then, solving the boundary element equations again on the new mesh gives a further numerical solution which, in principle, is of higher accuracy. The general formulation for the boundary integral equation and boundary element method are presented in Appendix 7.2.

In general, the system of equations is nonlinear and therefore, a numerical method such as Newton iteration or secant method can be used to solve them iteratively.
However, when the distribution function \( T(x) \) is a piecewise constant polynomial, a direct algorithm for solving equation (2.3.1) has been presented by De Boor [17] and Carey and Dinh [12], where iteration is not necessary. For the case of piecewise linear polynomial, a direct algorithm is shown below.

Let \( \{x_i\} \) be a mesh on the boundary and \( \{\bar{x}_i\} \) the asymptotic optimal mesh determined by (2.3.1). \( T(x) \) is a piecewise linear function defined on \([a, b]\) and it is expressed as

\[
T(x) = b_i + a_i(x-x_i) \quad x \in [x_i, x_{i+1}].
\]  

(2.3.2)

Substituting (2.3.2) into (2.3.1) gives

\[
\frac{1}{N} = \frac{\int_{x_l}^{x_{l+1}} \left[ b_{p+1} + a_{p+1}(x-x_l) \right] dx + \sum_{l=1}^{p} \int_{x_{l-1}}^{x_l} \left[ b_l + a_l(x-x_l) \right] dx}{A}.
\]

i.e.

\[
\frac{1}{N} = \frac{\frac{1}{2} a_{p+1} (\bar{x}_l-x_l)^2 + b_{p+1} (\bar{x}_l-x_l) + \sum_{l=1}^{p} \left( \frac{1}{2} a_l h_l^2 + b_l h_l \right)}{A},
\]

leading to

\[
\frac{1}{N} \left( \frac{1}{2} a_{p+1} (\bar{x}_l-x_l)^2 + b_{p+1} (\bar{x}_l-x_l) + \sum_{l=1}^{p} \left( \frac{1}{2} a_l h_l^2 + b_l h_l \right) \right) - A \frac{1}{N} = 0
\]

(2.3.3)
where \( h_l = (x_l - x_{l-1}) \),

\[
A = \sum_{l=1}^{N} \left[ \frac{1}{2} a_l h_l^2 + b_l h_l \right]
\]

and \( p \) is chosen to satisfy

\[
\sum_{l=1}^{p+1} \left[ \frac{1}{2} a_l h_l^2 + b_l h_l \right] > \frac{1}{N} A \geq \sum_{l=1}^{P} \left[ \frac{1}{2} a_l h_l^2 + b_l h_l \right]
\]  \( (2.3.4) \)

Solving equation (2.3.3), we obtain

\[
\bar{x}_i - x_p = \frac{-b_{p+1} + \sqrt{b_{p+1}^2 + 2a_{p+1} c_p}}{a_{p+1}}
\]

which is a real root where

\[
c_p = A \frac{1}{N} - \sum_{l=1}^{P} \left[ \frac{1}{2} a_l h_l^2 + b_l h_l \right].
\]

Thus, the asymptotic optimal mesh can be obtained by the following formula

\[
\bar{x}_i - x_p = \frac{-b_{p+1} + \sqrt{b_{p+1}^2 + a_{p+1} c_p}}{a_{p+1}}, i=1,2, \ldots, N. \quad (2.3.5)
\]

In this thesis, we only employ the piecewise constant approximation to \( T(x) \). Now, the strategy of the optimal mesh algorithm is summarized as follows:

**STEP 1:** Decide on the important part of the boundary where the solution has to be accurately determined;

**STEP 2:** Decide on the choice of \( W(x) \);

**STEP 3:** Construct an initial mesh on the boundary;
STEP 4: Solve the original problem using the boundary
element method;

STEP 5: On the part of the boundary in STEP 1, construct a
piecewise constant distribution function and then
solve the equation (2.3.1) using the algorithm in
[13,32]. In the case of piecewise linear function,
equation (2.3.1) can be solved by the algorithm
presented in (2.3.3) through (2.3.6);

STEP 6: Check the stopping criterion. If the user tolerance
is met, stop. Otherwise, return to STEP 4.

There are two choices for the stopping criterion which
can be used in STEP 6 as follows:

1. Use the difference of the meshes between two successive
iterations, i.e.

\[ \varepsilon = \sum_{i=1}^{N} |x_i - \bar{x}_i| \]  \hspace{1cm} (2.3.7)

which was recommended by Carey and Dinh [12].

2. Use the difference of the minimum error indicator
between two successive iterations

\[ \varepsilon = |E_o(x_i) - E_o(\bar{x}_i)| \]  \hspace{1cm} (2.3.8)

where \( E_o \) is the minimum error indicator presented in
(2.2.9).

In the last chapter, we will prove that the two
stopping criteria are asymptotic equivalent. A stopping
criterion based on the physics of the problem has been
presented in [B1].

It is important to note that, in this chapter, we do not specify a stopping criterion as described in Step 6. All examples use a total of three iterations and the two quantities in (2.3.7) and (2.3.8) are evaluated.

2.4 Potential

In this section, the optimal mesh algorithm is tested against two benchmark problems with the constant, linear and quadratic direct boundary element formulation. Both problems involve the solution of Laplace's equation with singularities on the boundary.

Since there is only one unknown at each node for the potential problem, the natural choice for $W(s)$ is this unknown variable. For constant elements, the optimal mesh can be determined by solving

$$\frac{i}{N} = \frac{\int_a^b |W'(x)|^{2/(3-2m)} dx}{\int_a^b |W'(x)|^{2/(3-2m)} dx} \quad \text{(constant element)}$$

$i=1,2,\ldots,N$, where $m$ is dependent on the norm to be used. For the linear and quadratic elements the mesh is described by the following two equations respectively.
\[ \frac{1}{N} = \frac{\int_{a}^{b} |W''(x)|^{2/(5-2m)} \, dx}{\int_{a}^{b} |W''(x)|^{2/(5-2m)} \, dx} \quad \text{linear element} \]

\[ \frac{1}{N} = \frac{\int_{a}^{b} |W'''(x)|^{2/(7-2m)} \, dx}{\int_{a}^{b} |W'''(x)|^{2/(7-2m)} \, dx} \quad \text{quadratic element} \]

\( i=1,2,\ldots,N \). In this chapter, only \( m=0 \) corresponding to the \( L_2 \)-norm is employed.

**Example 2.1** Consider Laplace's equation in a rectangular region of dimensions 2 x 1, subject to the boundary conditions described in figure 2.1 (the well-known Motz I problem), where \( p=\frac{\partial u}{\partial n} \). The solution has a square root flux singularity at point O. In general, the adaptive optimal mesh algorithm deals with the entire boundary but it can equally be applied to portions of the boundary which require more attention. In the present problem, the algorithm is applied to the segment AB.

In order to test the adaptive algorithm, the initial mesh consisted of 48 nodes and uniform spacing for constant, linear and quadratic elements as shown in figure 2.2. After three iterations, all three meshes (CBEM = constant element, LBEM = linear element, QBEM = quadratic element, where linear and quadratic elements are conforming) resulted in a
higher concentration of grid points near point O as indicated in figures 2.3 through 2.5. The final mesh corresponding to the quadratic elements has a higher concentration of nodes near the point O, as compared to the constant and linear elements.

Tables 2.1 through 2.3 contain the quantitative information on the performance of the algorithm for all three types of boundary elements. In these tables, column 2 is a measure of the relative percentage error of the approximate solution on the segment OB (measured in the $L_2$ norm), column 3 is the minimum error indicator described by (2.2.9) and column 4 presents the difference of the meshes in two successive iterations. LR results were obtained by using linear elements with 204 grid points and a local refinement technique, and therefore are sufficiently
accurate to represent the exact solution for comparative purposes. The variable $u^h$ represents the numerical solution of the potential problem. Since the approximation to the second and third derivatives is not as accurate as that to the first derivatives near the singular point $O$, the convergence of algorithm with constant element is fastest as shown in column 4 of tables 2.1-2.3.

The numerical solutions on OB are shown respectively in figures 2.6, 2.7 and 2.8.

| No. of iteration | $\frac{|u^h-u_{Lk}|_o}{|u^h|_o}$ (OB) | $E_o$ (OB) | $\sum_{oB} |\tilde{x}_i-x_i|$ |
|------------------|-----------------------------------|------------|------------------|
| 0 initial        | 0.825 %                           | 0.331 E 00 | 0.363 E 00       |
| 1                | 0.604 %                           | 0.663 E-01 | 0.272 E-01       |
| 2                | 0.511 %                           | 0.261 E-01 | 0.250 E-02       |
| 3                | 0.498 %                           | 0.201 E-01 | 0.512 E-03       |
### TABLE 2.2 Motz I with linear element

| No. of iteration | \( \frac{|u^h-u^{Lx}|_{0}}{|u^h|_{0}} \) | \( E_0 \) (OB) | \( \Sigma |\tilde{x}_i-x_i| \) |
|------------------|----------------------------------|----------------|-------------------|
| 0 initial        | 0.622 %                          | 0.156 E 00     | 0.131 E 01        |
| 1                | 0.401 %                          | 0.412 E-01     | 0.281 E 00        |
| 2                | 0.342 %                          | 0.202 E-01     | 0.742 E-01        |
| 3                | 0.298 %                          | 0.152 E-01     | 0.112 E-01        |

### TABLE 2.3 Motz I with quadratic element

| No. of iteration | \( \frac{|u^h-u^{Lx}|_{0}}{|u^h|_{0}} \) | \( E_0 \) (OB) | \( \Sigma |\tilde{x}_i-x_i| \) |
|------------------|----------------------------------|----------------|-------------------|
| 0 initial        | 0.488 %                          | 0.939 E-01     | 0.114 E 00        |
| 1                | 0.271 %                          | 0.315 E-01     | 0.190 E 00        |
| 2                | 0.176 %                          | 0.178 E-01     | 0.645 E-01        |
| 3                | 0.136 %                          | 0.121 E-01     | 0.128 E-01        |

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 2.2 Initial Mesh

Figure 2.3 The Mesh in Iteration 3 for Motz I (CBEM)
Figure 2.4 The Mesh in Iteration 3 for Motz I (LBEM)

Figure 2.5 The Mesh in Iteration 3 for Motz I (QBEM)
Figure 2.6 The Potential on OB for Motz I (CBEM)

Figure 2.7 The Potential on OB for Motz I (LBEM)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 2.8: The Potential on O3 for Motz 1 (QBEM)
Example 2.2 The next boundary value problem to be solved is the well known Motz II problem which is depicted in figure 2.9.

\[ \nabla^2 u = 0 \]

Figure 2.9 Motz II

The initial mesh is again uniform as in the previous example and is shown in figure 2.2. This initial mesh represents the three cases of constant, linear and quadratic elements. The optimal meshes at the third iteration are presented in figures 2.10, 2.11 and 2.12 corresponding to constant, linear and quadratic elements, respectively. Because of the presence of flux singularities at the points O and D, it is natural that the nodes should move toward these points. Once again, the mesh corresponding to the quadratic elements has a higher density near the two
singularities than the constant and linear counterparts.

Tables 2.4 through 2.6 represent the error and convergence results on section OB. The corresponding results on the section CD are shown in tables 2.7 through 2.9 for all three types of elements. The LR results are obtained by using linear elements with 224 grid points and local refinement techniques which can in fact represent the exact solution needed for comparative purposes. The same comments concerning the faster convergence of constant elements as in example 2.1 can be applied here.

**TABLE 2.4 Motz II with constant element**

| No. of iteration | $\frac{|u^h - u^{LR}|}{|u^h|}$ | $E_o$ (OB) | $\sum |x_i^e - x_i|$ |
|------------------|-------------------------------|------------|-----------------|
| 0 initial        | 2.795 %                       | 0.344 E 00 | 0.163 E 01      |
| 1                | 2.041 %                       | 0.691 E-01 | 0.152 E-01      |
| 2                | 1.819 %                       | 0.217 E-01 | 0.873 E-02      |
| 3                | 1.814 %                       | 0.177 E-01 | 0.991 E-03      |

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
TABLE 2.5 Motz II with linear element

| No. of iteration | $\frac{\| u^h_u - u^L \|_{C^0(B)}}{\| u^h \|_\infty}$ | $E_o$ (C^0(B)) | $\sum_{oB} |x_i - x_i|$ |
|------------------|---------------------------------|----------------|-----------------|
| 0 initial        | 1.961 %                         | 0.184 E 00     | 0.133 E 01      |
| 1                | 1.481 %                         | 0.412 E-01     | 0.554 E-01      |
| 2                | 0.937 %                         | 0.187 E-01     | 0.102 E-02      |
| 3                | 0.872 %                         | 0.143 E-01     | 0.355 E-03      |

TABLE 2.6 Motz II with quadratic element

| No. of iteration | $\frac{\| u^h_u - u^L \|_{C^0(B)}}{\| u^h \|_\infty}$ | $E_o$ (C^0(B)) | $\sum_{oB} |x_i - x_i|$ |
|------------------|---------------------------------|----------------|-----------------|
| 0 initial        | 1.432 %                         | 0.965 E-01     | 0.107 E 01      |
| 1                | 0.741 %                         | 0.270 E-01     | 0.154 E 00      |
| 2                | 0.448 %                         | 0.129 E-01     | 0.573 E-01      |
| 3                | 0.322 %                         | 0.995 E-02     | 0.932 E-02      |
**TABLE 2.7 Motz II with constant element**

| No. of iteration | \( \frac{|\mathbf{u}^h - \mathbf{u}^{LR}|_{0}^{\text{CDD}}}{|\mathbf{u}^h|_0^{\text{CDD}}} \) | \( E_0 \) (CDD) | \( \sum_{\text{CD}} |\check{x}_i - x_i| \) |
|------------------|-----------------|----------------|-----------------|
| 0 initial        | 1.826 \%        | 0.372 E 00     | 0.158 E 01      |
| 1                | 0.982 \%        | 0.561 E-01     | 0.455 E-01      |
| 2                | 0.940 \%        | 0.211 E-01     | 0.121 E-02      |
| 3                | 0.935 \%        | 0.196 E-01     | 0.312 E-03      |

**TABLE 2.8 Motz II with linear element**

| No. of iteration | \( \frac{|\mathbf{u}^h - \mathbf{u}^{LR}|_{0}^{\text{CDD}}}{|\mathbf{u}^h|_0^{\text{CDD}}} \) | \( E_0 \) (CDD) | \( \sum_{\text{CD}} |\check{x}_i - x_i| \) |
|------------------|-----------------|----------------|-----------------|
| 0 initial        | 1.421 \%        | 0.134 E 00     | 0.146 E 00      |
| 1                | 0.686 \%        | 0.377 E-01     | 0.928 E-01      |
| 2                | 0.433 \%        | 0.162 E-01     | 0.454 E-02      |
| 3                | 0.399 \%        | 0.135 E-01     | 0.988 E-03      |

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
The approximate solutions on section OB are shown in figures 2.13 through 2.15 and those for section CD are presented in figures 2.16 through 2.18.
Figure 2.10 The Mesh in Iteration 3 for Motz II (CBEMD)

Figure 2.11 The Mesh in Iteration 3 for Motz II (CLBEMD)

Figure 2.12 The Mesh in Iteration 3 for Motz II (CQBEMD)
Figure 2.13 The Potential on OB for Motz II (CBEM)

Figure 2.14 The Potential on OB for Motz II (LBEM)
Figure 2.15 The Potential on OB for Motz II (QBEM)

Figure 2.16 The Potential on CD for Motz II (CBEM)
Figure 2.17 The Potential on CD for Motz II (LBEM)

Figure 2.18 The Potential on CD for Motz II (QBEM)
2.5 Linear elasticity

The numerical results presented in the previous section show that the optimal mesh algorithm is reliable for solving Laplace's equation. Here we extend the algorithm originally developed by Carey and Kennon [9] to treat the concept of asymptotic optimal mesh in the boundary element elastostatic problem. This extension has several important features. To begin with, the algorithm is simple enough to be incorporated into existing two-dimensional boundary element stress analysis codes. Furthermore, the algorithm leads to an asymptotic optimal discretization and therefore a more accurate BEM solution. In order to demonstrate the strategy, three benchmark problems are treated, one of which is a centered crack problem. All three benchmark problems involve boundaries consisting of straight line segments. The procedure is similar to that of the previous section, the major difference being in the choice of function $W(x)$. Both the linear and the quadratic boundary elements are experimented with. The reason for excluding the constant elements is that they are found to result in inaccurate predictions.

A word of caution is in order. Preferably, in the two-dimensional elasticity equations, the displacement variables are referred to the $(x_1, x_2)$ coordinate system. But for the sake of convenience, in this chapter we are referring to the $(x, y)$ system.
The elasticity computer program used in this thesis is designed for two-dimensional problems and thickness variable is not requested as an input.

Example 2.3 Here we consider a simple plane stress problem of two dimensional linear elasticity. The dimensions, boundary conditions and material properties are given in figure 2.19 where the imposed load is $P = p_y/c$ and $P = 50.0$.

![Figure 2.19 The Idealized Plane Stress, Ex. 2.3, $c=0.25, L=1.0$, Young's modulus $E=50.0$, Poisson's ratio $\nu=0.3$.](image)

The exact solution to the idealized plane stress problem can easily be derived from the equilibrium equation (see Appendix II, 8.2) where the displacements are given by

$$u_1(x,y) = \frac{P}{E} \frac{xy}{c} \quad (2.5.1)$$
\[ u_2(x,y) = -\frac{vP}{2E} \frac{y^2}{c} - \frac{P}{2Ec} x^2 \] (2.5.2)

In the above expression, \( u_1 \) and \( u_2 \) are the displacements in the \( x \) and \( y \) directions respectively.

The first step is to identify the portion (or portions) of the boundary where the mesh is to be redistributed. In this particular problem, the optimal mesh algorithm is to be applied on the top and bottom horizontal boundaries. An initial boundary element mesh is chosen and shown in figure 2.20. Note that the mesh is not uniform adjacent to the corners on the two horizontal segments.

Since an elasticity problem has two dependent variables, there can be different choices for the function \( WC(x) \). In this example the choice

\[ WC(s) = u_2(s) \]

was made, where \( s \) is the arc length on the boundary. We also note that \( u_1 \) cannot be chosen as \( WC(x) \) because \( u_1 = 0 \) on both top and bottom horizontal boundaries. For the above choice and linear elements, the optimal mesh is determined by (2.3.1) with the function \( TC(x) \) given by

\[ TC(s) = \left| u_2''(s) \right|^{2/3}. \] (2.5.3)

The closed form analytic solution (2.5.2) implies that on the top and bottom boundaries, \( u_2(s) \) is a quadratic function of \( x \). Therefore, the function in (2.5.3) is constant, i.e., an optimum uniform mesh is anticipated. Note
that the actual redistribution takes place only on the top
and bottom horizontal boundaries. Convergence to a uniform
mesh is expected.

At this point the algorithm described in section 2.3
was applied and the results after three iterations are
recorded in Table 2.10. The second and third columns of this
table reflect the relative error in the two displacement
components on the bottom boundary measured in the \( L^2 \)-norm.
Note that as the mesh is redistributed, the error is reduced
monotonically. The fourth column in Table 2.10 describes the
development from the uniform mesh on the bottom boundary. In
the last row, EOD denotes the boundary element solution on
the exact optimal mesh, i.e. in (2.3.1). \( \bar{\Omega}(\Omega) = \Omega(\Omega) \) is a
known analytic solution.

| No. of iteration | \( \frac{|u_1 - u_1^h|_0}{|u_1^h|_0} \) (AB) | \( \frac{|u_2 - u_2^h|_0}{|u_2^h|_0} \) (AB) | \( \sum_{AB} |x_i - x_i^E| \) |
|------------------|---------------------------------------------|---------------------------------------------|-----------------------------|
| 0 Initial        | 1.20 %                                       | 3.98 %                                       | 0.258 E 00                 |
| 1                | 1.01 %                                       | 2.52 %                                       | 0.296 E-01                 |
| 2                | 0.99 %                                       | 2.39 %                                       | 0.152 E-02                 |
| 3                | 0.98 %                                       | 2.38 %                                       | 0.118 E-03                 |
| EOD              | 0.98 %                                       | 2.38 %                                       | 0.0                        |
The optimal mesh at the third iteration is also shown in figure 2.21. The next two figures show the exact and the approximate boundary element solutions at different iteration stages. It is worth noting that examples 2.3 and 2.4 are dealing with a problem which is simple mathematically, therefore, the optimal mesh does not significantly change the quality of the solution. The effectiveness of the process will be more noticeable in example 2.5 which is a centered crack problem.

Note that the optimal mesh algorithm cannot be applied to example 2.3 if higher order elements are used. This is due to the fact that the exact distribution function vanishes.

In order to check the performance of a quadratic element in the optimal mesh algorithm, example 2.4, which does not suffer from the above deficiency, will be introduced.
Figure 2.20 The Initial Mesh for Ex. 2.3

Figure 2.21 The Mesh in Iteration 3 for Ex. 2.3
Figure 2.22 The Displacement in x Direction on AB for Ex. 2.3

Figure 2.23 The displacement in y Direction on AB for Ex. 2.3
Example 2.4  Once again a simple plane stress problem in two-dimensional linear elasticity is considered. The dimensions, boundary conditions and material properties are stated in figure 2.24. The imposed loads are described by

\[
\begin{align*}
\bar{P}_1 &= P_1 \\
\bar{P}_2 &= \frac{3P}{4c^3} (c^2 - y^2)L \\
\bar{P}_3 &= \frac{3P}{4c^3} (4L^2y - 2y^3/3) + P_2y/c
\end{align*}
\]

where \( P_1 = 0.625 \) and \( P_2 = 40.0 \). The purpose of imposing such a complicated system of boundary conditions is to ensure that the exact solution is a higher order polynomial.

\[\text{Figure 2.24 The Idealized Plane Stress, Ex. 2.4}\]

\( c=0.25, L=1.0, \) Young's modulus \( E=50.0, \) Poisson's ratio \( \nu=0.2 \)
The displacement components for this problem can be expressed as

\[ u_1(x, y) = -\frac{3P_1}{4Ec^3} \left( \frac{y^3}{3} + \frac{2}{3} xy^2 + \nu \left( \frac{y^3}{3} - yyc^3 - \frac{2}{3} yc^3 \right) \right) + \frac{xy\nu^2}{Ec} \]  

(2.5.4)

\[ u_2(x, y) = \frac{3P_1}{4Ec^3} \left( \frac{y^4}{12} - \frac{c^2y^2}{2} - \frac{2}{3} yc^3 + \nu \left[ \frac{y^4}{6} - \frac{x^2y^2}{2} \right] \right) - \nu \frac{P_2y^2}{2Ec} + \frac{3P_1}{4Ec^3} \left( -\frac{x^4}{12} + (2+\nu) \frac{c^2x^2}{2} \right) - \frac{P_2x^2}{2Ec} \]  

(2.5.5)

The detailed derivation is presented in Appendix II, 8.3.

As in example 2.3, the optimal mesh algorithm is to be applied to the top and bottom horizontal boundaries. The choice of function \( W(s) \) is somewhat different. For the comparative purposes, here it is assumed that

\[ W(s) = u_1(s) \]

In this case, the exact optimal mesh for the quadratic approximation on the top and bottom horizontal boundaries is uniform.

The initial mesh is the same as in example 2.3 (see figure 2.20) but the elements are quadratic, i.e. each
element has three nodes with the middle node located at the center.

Table 2.11 shows that the algorithm converges to a uniform mesh (exact optimal mesh). The relative errors measured in the $L_2$-norm also follow a consistent pattern.

| No. of iteration | $\frac{|u_1^h-u_1^0|}{|u_1^0|}$ (AB) | $\frac{|u_2^h-u_2^0|}{|u_2^0|}$ (AB) | $\sum_{AB} |x_i - x_i^E|$ |
|------------------|---------------------------------|---------------------------------|------------------|
| 0 initial        | 4.431%                          | 3.921%                          | 0.400 E 00       |
| 1                | 4.421%                          | 3.621%                          | 0.277 E-01       |
| 2                | 4.417%                          | 3.410%                          | 0.162 E-01       |
| 3                | 4.418%                          | 3.288%                          | 0.791 E-02       |
| EOD              | 4.418%                          | 3.205%                          |                   |

For the sake of completeness we have included figures 2.25, 2.26 and 2.27 which are the optimal mesh at the third iteration and the two displacement components at different iteration stages.
Figure 2.25: The mesh in Iteration 3 for Ex. 2.4
Figure 2.26 The Displacement in x Direction on AB for Ex. 2.4

Figure 2.27 The Displacement in y Direction on AB for Ex. 2.4
Example 2.5 The final example in this chapter is a rectangular plate, containing a central crack under mode I loading as shown in figure 2.28. Due to the symmetrical nature of the geometry and boundary conditions, only one quarter of the domain has to be considered. This is represented by the shaded region in figure 2.28. No closed form analytical solution is available for this problem. Therefore, in order to have a reference solution, the linear boundary element method with 180 nodes and local refinement technique near the crack tip was used (denoted by LR).

Figure 2.28 Central Crack $c=1.0$, $L=2.0$, $a=0.5$, $\sigma=1.0$

Young's modulus $E=20000.0$, Poisson ratio $\nu=0.3$
Both linear and quadratic boundary elements are employed for testing the optimal mesh algorithm for this problem. The function \( W(s) \) is chosen to include the COD information (Crack Opening Displacement) as follows:

\[
W(s) = \begin{cases} 
  u_1(s) & -1.0 < x < 0.5, \ y = 0 \\
  u_2(s) & 0.5 < x < 1.0, \ y = 0 
\end{cases}
\]

The initial mesh used for both types of discretizations is shown in figure 2.29. The bottom horizontal boundary was used for mesh redistribution purposes. This was based on the knowledge that a region of high stress gradient surrounds the crack tip. Figures 2.30 and 2.31 are the final grid configurations (third iteration) for the linear and the quadratic boundary elements, respectively. The high concentration of grid points in the vicinity of the crack tip is not surprising.

The error and convergence results are shown in Table 2.12 and 2.13 with the contents similar to those of example 2.3.

In a linear elastic fracture mechanics problem such as the present example, one of the most important parameters involved is the stress intensity factor defined by (see Appendix II, 8.4)

\[
K_I = \lim_{r \to 0} \frac{E}{4} \frac{\sqrt{2\pi}}{r} \frac{u_2}{\sqrt{r}} \quad (2.5.5)
\]

where \( r \) is the distance from the tip of the crack.
TABLE 2.12 Centered crack with linear element

| No. of iteration | \( \frac{|u_2^h-u_2^{LR}|}{|u_2^h|} \) (OB) | \( \sum |\bar{x}_i-x_i| \) |
|------------------|----------------------------------|-----------------|
| 0 initial        | 2.45 %                           | 0.448 'E 01     |
| 1                | 1.65 %                           | 0.251 E-02      |
| 2                | 1.24 %                           | 0.152 E-02      |
| 3                | 1.19 %                           | 0.811 E-03      |

TABLE 2.13 Centered crack with quadratic element

| No. of iteration | \( \frac{|u_2^h-u_2^{LR}|}{|u_2^h|} \) (OB) | \( \sum |\bar{x}_i-x_i| \) |
|------------------|----------------------------------|-----------------|
| 0 initial        | 1.44 %                           | 0.102 E 01      |
| 1                | 0.88 %                           | 0.105 E 00      |
| 2                | 0.69 %                           | 0.285 E-01      |
| 3                | 0.61 %                           | 0.126 E-01      |

The numerical stress intensity factor is obtained by the COD method used on the displacement based on the first three nodes adjacent to the crack tip and least squares technique.
The exact value of $K_I$ is approximately denoted by $K_{LR}^I$ and is taken to be 1.25. Tables 2.14 and 2.15 display the results of the optimal mesh algorithm for the linear and quadratic elements, respectively. Both tables indicate that the optimal mesh algorithm converges rapidly and produces accurate and consistent results only after three iterations.

### TABLE 2.14 Centered crack with linear element

| No. of iteration | $\frac{K_I^h}{\sigma \sqrt{aR}}$ | $\frac{|K_I^h-K_{LR}^I|}{K_{LR}^I}$ |
|------------------|----------------------------------|----------------------------------|
| 0 initial        | 1.159                            | 7.272 %                          |
| 1                | 1.184                            | 5.288 %                          |
| 2                | 1.193                            | 4.550 %                          |
| 3                | 1.197                            | 4.272 %                          |
| LR               | 1.250                            | 0.0                              |

Figures 2.32 and 2.33 are the COD plots for linear and quadratic elements after three iterations.
### TABLE 2.15 Centered crack with quadratic element

| No. of iteration | $\frac{K^h_I}{\sigma \sqrt{a\pi}}$ | $\frac{|K^h_I-K^{LR}_I|}{K^{LR}_I}$ |
|------------------|---------------------------------|-------------------------------------|
| 0 initial        | 1.175                           | 5.884 %                             |
| 1                | 1.197                           | 4.248 %                             |
| 2                | 1.202                           | 3.808 %                             |
| 3                | 1.205                           | 3.624 %                             |
| LR               | 1.250                           | 0.0                                 |
Figure 2.29 The Initial Mesh for the Crack

Figure 2.30 The Mesh in Iteration 3 for the Crack (LBEMD)

Figure 2.31 The Mesh in Iteration 3 for the Crack (QBEMD)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 2.32 The Displacement in y Direction on CB LBEM

Figure 2.33 The Displacement in y Direction on CB CQBEM
Example 2.6 We now consider an extension of example 2.5 where a different choice of the distribution function is made. Here, \( W(s) = u_1(s) \) on \( y=0 \). All the parameters and the initial mesh are the same as indicated in example 2.5. The results are shown in figures 2.34–2.37 and Tables 2.16–2.17 by the linear boundary elements. By the asymptotic expression of the solution (see Appendix II 8.4), \( u_1 \) is approximately a linear function near the crack tip. Because of the singularity at the tip, numerical solution is not very accurate in that region as seen in figure 2.35. Since the error in the numerical solution leads to a large second derivative near the crack tip, the nodal points still move toward the tip (see figure 2.34) and therefore, we obtain the numerical results similar to those in example 2.5.

### TABLE 2.16 The Error for Ex. 2.5

| No. | \( \frac{|u_1^{LR} - u_1^{ih}|}{|u_1^{LR}|} \) (OBD) | \( \frac{|u_1^{LR} - u_1^{ih}|}{|u_1^{LR}|} \) (OBD) | \( \frac{|u_2^{LR} - u_2^{ih}|}{|u_2^{LR}|} \) (OBD) |
|-----|---------------------------------|---------------------------------|---------------------------------|
| 0 Initial | 1.82 % | 2.45 % | 1.13 % |
| 1 | 1.50 % | 1.48 % | 0.69 % |
| 2 | 1.37 % | 1.25 % | 0.59 % |
| 3 | 1.35 % | 1.20 % | 0.55 % |
TABLE 2.17 The Stress Intensity Factor for Ex. 2.5

| No. | $\frac{K_I}{\sigma \sqrt{\pi a}}$ | $\frac{|K_I - K_{LR}^I|}{K_{LR}^I}$ |
|-----|-------------------------------|---------------------------------|
| 0   | 1.159                         | 7.27 %                          |
| Initial |                                |                                 |
| 1   | 1.183                         | 5.36 %                          |
| 2   | 1.190                         | 4.80 %                          |
| 3   | 1.194                         | 4.48 %                          |
| LR  | 1.25                          |                                 |
Figure 2.34 The Mesh in Iteration 3 for Ex. 2.6

Figure 2.35 The Displacement in x Direction on CB for Ex. 2.6
Figure 2.36 The Displacement in x Direction on AC for Ex. 2.6

Figure 2.37 The Displacement in y Direction on CB for Ex. 2.6

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
2.6 Conclusion

In this chapter, we have presented a systematic simulation of the optimal mesh algorithm. The numerical results indicate that the algorithm is effective in the boundary element solution of Laplace’s equation and Navier’s equations of linear elasticity. Some important comments are summarized below:

1. The optimal mesh algorithm converges quickly. The errors in all examples are reduced monotonically. The numerical calculations also show that the meshes corresponding to known polynomial solutions converge to the exact optimal distribution. The meshes corresponding to the singular problems have a high concentration near the singular points as expected. For both potential and linear elasticity equations, the mesh corresponding to quadratic boundary elements has a higher concentration near the singularities than the other two methods. This agrees with the theoretical analysis.

2. According to the numerical results, the improvement of the solution quality is evident for all three types of elements. This improvement is partially dependent on the choice of the degree of elements. The higher the degree of the shape functions (elements), the more dramatic the improvement in the approximate solution.

3. Although the optimal mesh algorithm with quadratic boundary elements produces better results, the computer CPU
time is also higher. The average CPU times (IBM 4381) for the simulation of both the potential and the elasticity problems are recorded in Table 2.16, showing that the quadratic boundary elements are more expensive than the constant and linear elements.

<table>
<thead>
<tr>
<th></th>
<th>Potential (sec.)</th>
<th>Elasticity (sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>5.52</td>
<td>N/A</td>
</tr>
<tr>
<td>Linear</td>
<td>6.50</td>
<td>43.84</td>
</tr>
<tr>
<td>Quadratic</td>
<td>8.09</td>
<td>109.01</td>
</tr>
</tbody>
</table>

4. In the r-method, the purpose is to solve the governing equation on the optimal mesh and the numerical results are expected to be better, which is verified numerically in this chapter.

In the neighbourhood of singularities, the improvement of solution quality is very evident. Since the optimal mesh algorithm is based on the concept of equidistribution of error, the improvement is restricted. Generally speaking, for a given number of grid points the error of the numerical solution only converges to the minimum error rather than to an arbitrarily small tolerance. Thus the optimal mesh algorithm only partially improves the quality of numerical
solutions. To see this, we have also tested the Motz I and Motz II problems with 60 grid points on the boundary by using constant, linear and quadratic elements. Clearly, better numerical solutions are obtained for the discretization with 60 grid points. This disadvantage of the optimal mesh algorithm will be considered and resolved in the next chapter where a new combined refinement/redistribution algorithm is developed.
Chapter Three
An Adaptive h-r Boundary Element Algorithm
For the Laplace Equation

3.1 Introduction

The optimal mesh algorithm described in the previous chapters is based on the concept of equidistribution of the error indicator. The complexity (degrees of freedom) of the algorithm is fixed and therefore it can only partially improve the quality of the numerical solution. In this algorithm, the error indicator does not fall below an arbitrary small tolerance.

In this chapter, a combined h-r algorithm (mesh refinement and redistribution) designed for the boundary element method is presented. The combined algorithm is based on a truncation error indicator. Here, the mesh distribution and the complexity of the algorithm are adapted simultaneously such that the truncation error indicator is minimized and is pushed to fall below a given user defined tolerance. Finally, in order to keep the algebra minimal and explain the ideas in a simple fashion, linear elements are employed in solving two benchmark problems. This element assumes a linear approximation to the potential, flux and geometry, respectively.
3.2 The Truncation Error Indicator

We restrict our attention to a two-dimensional domain $\Omega$ where Laplace's equation is to be solved. Using a standard argument (see Appendix 7.3), one can convert the involved boundary value problem into an integral equation with singular kernels represented by

$$
c(s)u(s) + \int_{\Omega} u(s')p(s, s')ds' - \int_{\Omega} p(s')u(s', s)ds' = 0
$$

(3.2.1)

where $s$ is the position of source defined on $\Omega$. Let an arbitrary point be fixed on the boundary from which both $s$ and $s'$, the arc lengths involved, are being measured.

![Arc length Coordinate System](image)

**Fig. 3.1 The Arc length Coordinate System**

The boundary element discretization can now be based on equation (3.2.1). The boundary is divided into elements
separated by the nodes \( \{s_i\}, i=1,2, \ldots, N \). On each segment \( \Gamma_i \), the unknown solution \( u \) and its normal derivative \( p \) are approximated by linear functions. Mathematically, this approximation is denoted by

\[
u(s) \approx u^h(s) = B_1(s) u^h_{j_1} + B_2(s) u^h_{j_2} \text{ se}_{\Gamma_j} \tag{3.2.2}\]

\[
p(s) \approx p^h(s) = B_1(s) p^h_{j_1} + B_2(s) p^h_{j_2} \text{ se}_{\Gamma_j} \tag{3.2.3}\]

The function \( B_1(s) \) and \( B_2(s) \) are linear shape functions defined in Appendix I, 7.3. The variables \( u^h_{j_1}, u^h_{j_2}, p^h_{j_1} \) and \( p^h_{j_2} \) in equations (3.2.2) and (3.2.3) are the nodal values corresponding to the potential solution and its normal derivative, respectively.

The approximations \( u^h(s) \) and \( p^h(s) \) are substituted in the integral equation (3.2.1) and the resulting expression is collected at the node points \( \{s_i\} \). This process leads to the following system of linear integral equations,

\[
c_i u^h + \sum_j \int_{\Omega_j} u^h(s') p^m(s_i, s') ds' - \sum_j \int_{\Omega_j} p^h(s') u^m(s_i, s') ds' = 0 \quad i=1,2, \ldots, N \tag{3.2.4}\]

Assuming that the exact solution and its normal derivative are twice continuously differentiable on the typical segment \( \Gamma_j \), polynomial interpolation results in

\[
u(s) = B_1(s) u_{j_1} + B_2(s) u_{j_2} + (s-s_{j-1})(s-s_j)u[s_{j-1}, s_j, s] \text{ se}_{\Gamma_j} \tag{3.2.5}\]
\[ p(s) = B_1(s) p_{j_1} + B_2(s) p_{j_2} + (s-s_{j-1})(s-s_j) p[s_{j-1}, s_j, s] \]
\[ \text{set } j \]  \hspace{1cm} \tag{3.2.8}

where

\[ u_{j_1} = u(s_{j-1}) \hspace{1cm} u_{j_2} = u(s_j) \]
\[ p_{j_1} = p(s_{j-1}) \hspace{1cm} p_{j_2} = p(s_j) \]

and \( u[. .] \) and \( p[. .] \) denote the second divided differences of functions \( u(s) \) and \( p(s) \), respectively. Note that there is a difference between the set of values \( (u_{j_1}^h, u_{j_2}^h, p_{j_1}^h, p_{j_2}^h) \) and \( (u_{j_1}, u_{j_2}, p_{j_1}, p_{j_2}) \). The former refers to the boundary element discretization process whereas the latter refers to the interpolation process.

Defining \( \tilde{u}(s) \) and \( \tilde{p}(s) \) to be the linear interpolants to \( u(s) \) and \( p(s) \) respectively, one can write

\[ \tilde{u}(s) = B_1(s) u_{j_1} + B_2(s) u_{j_2} \hspace{1cm} \text{set } j \]  \hspace{1cm} \tag{3.2.7}
\[ \tilde{p}(s) = B_1(s) p_{j_1} + B_2(s) p_{j_2} \hspace{1cm} \text{set } j \]  \hspace{1cm} \tag{3.2.8}

Using \( \tilde{u}(s) \) and \( \tilde{p}(s) \), equations (3.2.5) and (3.2.6) can be rewritten as

\[ u(s) = \tilde{u}(s) + (s-s_{j-1})(s-s_j) u[s_{j-1}, s_j, s] \]  \hspace{1cm} \tag{3.2.9}
Substituting (3.2.9) and (3.2.10) into the original integral equation (3.2.1) one obtains

\[ p(s) = \tilde{p}(s) + (s-s_{j-1})(s-s_j)p[s_{j-1},s_j,s'] \]  \hspace{1cm} (3.2.10)

Collocating equation (3.2.11) at the points \( s_i \), \( i=1,2,\ldots,N \), one arrives at the system of linear equations (3.2.12)

\[ c(s_i)u(s_i) + \]

\[ + \sum_j \int_{\Gamma_j} \left[ \tilde{u}(s') + (s'-s_{j-1})(s'-s_j)u[s_{j-1},s_j,s'] \right] p[s_i,s']ds' \]

\[ - \sum_j \int_{\Gamma_j} \left[ \tilde{p}(s') + (s'-s_{j-1})(s'-s_j)p[s_{j-1},s_j,s'] \right] u[s_i,s']ds' \]

\[ = 0 \quad i=1,2,\ldots,N, \]  \hspace{1cm} (3.2.11)

Rearranging the above equation gives
The right hand side represents truncation error \( R(s_i) \) which is obtained by substituting the interpolants \( \tilde{u}(s) \) and \( \tilde{p}(s) \) in the integral equation (3.2.1) and collocating at \( s_i \).

\[
\begin{align*}
\sum_{j} \left[ (s'-s_{j-1})(s'-s_j)u(s_{j-1},s_j,s') \right] p^*(s_i,s') ds'
\end{align*}
\]

\[ i=1,2,...,N \]  
(3.2.14)
the following two expressions.

\[
\begin{align*}
\int_{\Gamma_j} (s'-s_{j-1})(s'-s_j)u[s_{j-1}, s_j, s']p^*(s_i, s')ds' \\
&= -\frac{h_j^3}{\delta} u[s_{j-1}, s_j, \theta_j]p^*(s_i, \theta_j) \quad (3.2.15)
\end{align*}
\]

\[
\begin{align*}
\int_{\Gamma_j} (s'-s_{j-1})(s'-s_j)p[s_{j-1}, s_j, s']u^*(s_i, s')ds' \\
&= -\frac{h_j^3}{\delta} p[s_{j-1}, s_j, \gamma_j]u^*(s_i, \gamma_j) \quad (3.2.16)
\end{align*}
\]

where \( \theta_j, \gamma_j \in (s_{j-1}, s_j) \). Noting the property of the divided differences

\[
\begin{align*}
u[s_{j-1}, s_j, \theta_j] &= \frac{1}{2} u''(\alpha_j) \\
p[s_{j-1}, s_j, \gamma_j] &= \frac{1}{2} p''(\beta_j)
\end{align*}
\]

where \( \alpha_j, \beta_j \in (s_{j-1}, s_j) \), the truncation error in equation (3.2.14) can be therefore written as

\[
R(s_i) = -\frac{1}{12} \sum_{j=1}^{N} h_j^3 \left[ u''(\alpha_j)p^*(s_i, \theta_j) - p''(\beta_j)u^*(s_i, \gamma_j) \right]
\]

Now taking the absolute value and using triangle inequality give
If the $L_1$ norm of $R(s)$ is used as the error indicator, one can proceed as follows:

\[
|R(s_i)| \leq \frac{1}{12} \sum_{j=1}^{N} h_j \left( |u''(a_j)| |p^*(s_i, \theta_j)| + |p''(\beta_j)| |u^*(s_i, \gamma_j)| \right)
\]

(3.2.17)

where $h = \max h_i, i=1,2,\ldots, N$. Substituting (3.2.17) into (3.2.18) gives

\[
\|R\| \equiv \int |R(s)| ds = \sum_{i=1}^{N} h_i |R(s_i)| (1+O(h)) \]

(3.2.18)

Reversing the summation process in (3.2.19) yields

\[
|R| \leq \frac{1}{12} \sum_{i=1}^{N} h_i \left( \sum_{j=1}^{N} h_j \left( |u''(a_j)| |p^*(s_i, \theta_j)|
\right.
\right.

\[
\left. + |p''(\beta_j)| |u^*(s_i, \gamma_j)| \right) \right) (1+O(h)) \]

(3.2.19)

As we are interested in the asymptotic optimal mesh distribution, the inner sum in (3.2.20) can be regarded as a Reimann sum, i.e.
\[ |R| \leq \frac{1}{12} \sum_{j=1}^{N} h_j \left( |u''(c_j)| \int_{\Omega} |p^*(s,\theta_j)| ds + |p''(c_j)| \int_{\Omega} |u^*(s,\gamma_j)| ds \right) (1 + O(h)) \]  

For the purpose of asymptotic optimality and under the assumption of the continuity of the solution, (3.2.21) can be rewritten as

\[ |R| \leq \frac{1}{12} \sum_{j=1}^{N} h_j \left( |u''(c_j)| \int_{\Omega} |p^*(s,\xi_j)| ds + |p''(c_j)| \int_{\Omega} |u^*(s,\xi_j)| ds \right) (1 + O(h)) \]

where \( \xi_j \in (s_{j-1}, s_j) \).

Finally, defining the function \( W(s) \) by

\[ W(s') = |u''(s')| \int_{\Omega} |p^*(s,s')| ds + |p''(s')| \int_{\Omega} |u^*(s,s')| ds \]

(3.2.22)

the following upper bound of the truncation error is obtained

\[ |R| \leq \frac{1}{12} \sum_{j=1}^{N} h_j W(\xi_j) = E \]  

(3.2.23)

The right side of equation (3.2.23) will be used as the error indicator and it is minimized in the combined h-r
algorithm. Similar to the process in section 1.3, one can consider a minimization problem as following

$$\min E = \frac{1}{12} \sum_{j=1}^{N} h_j^3 W(\zeta_j)$$

(3.2.24)

Introducing the grading function $\xi(s)$, one has

$$h_j = \frac{1}{\xi'(\zeta_j)N}$$

and therefore the variable $h_j$ can be eliminated from (3.2.24). The optimization in (3.2.24) is a standard problem of the calculus of variations with respect to the grading function and can be summarized by

$$\min_{\xi(s)} E = \frac{1}{12 N^2} \int_{\partial\Omega} \frac{W(s)}{\xi'(s)^2} ds.$$  

The optimal mesh $(s_i)$ is determined by the following equations

$$\frac{1}{N} = \frac{\int_0^{s_i} [W(s)]^{1/3} ds}{\int_0^S [W(s)]^{1/3} ds}$$

(3.2.25)

where $S$ is the total perimeter of region $\Omega$. The mesh distribution determined by (3.2.25) minimizes the error indicator $E$ and the resulting optimal value is given by

$$E_0 = \frac{1}{12N^2} \left( \int_{\partial\Omega} [W(s)]^{1/3} ds \right)^3$$

(3.2.26)
3.3 The Adaptive h-r Algorithm

One starts with a discretization of the boundary which is denoted by \( \{s_i\} \). Using this, a boundary element solution \( u^h(s) \) and its normal derivative \( p^h(s) \) are obtained. Based on this approximate solution, the function \( W(s) \) in (3.2.22) is constructed (note that similar to the treatment in section 2.3, \( u''(s) \) and \( p''(s) \) should be replaced by an approximation extracted from the boundary element solution). Once the function \( W(s) \) is available, the upper bound on the truncation (i.e. \( E \)) can be evaluated.

If this truncation error in the relative sense is small enough, i.e.

\[
\frac{E}{\| u^h \| + \alpha \| p^h \|} \leq \varepsilon_{tol}
\]  

(3.3.1)

no refinement or redistribution is required. The variable \( \varepsilon_{tol} \) is a user specified tolerance and \( \| \cdot \| \) is the \( L_1 \) norm. On the other hand, if inequality (3.3.1) is not satisfied, the mesh refinement and redistribution are performed. The number of grid points is estimated as follows:

\[
\frac{E_0}{\| u^h \| + \alpha \| p^h \|} \leq \varepsilon_{tol}
\]  

(3.3.2)

or equivalently
\[
\frac{1}{12N^2} \left( \int_{\Omega} [W(s)]^{1/3} ds \right)^3 \leq \varepsilon_{tol} (\|u^h\| + a\|p^h\|). \quad (3.3.3)
\]

noting that the expression from \(E_o\) was taken from (3.2.26).

The coefficient \(a\) is included in order to make the two terms in the denominator have the same unit. The presence of \(a\) is extremely important for physical considerations. In this thesis, \(a\) is assumed to be one.

Solving the above inequality for \(\hat{N}\), the following estimate is obtained,

\[
\hat{N} = \text{Int} \left[ 1 + \frac{\left( \int_{\Omega} [W(s)]^{1/3} ds \right)^{2/3}}{\sqrt{12(\|u^h\| + a\|p^h\|)\varepsilon_{tol}}} \right] \quad (3.3.4)
\]

where \(\text{Int}[a]\) denotes the integer part of \(a\). The new mesh redistribution with respect to \(\hat{N}\) points is then obtained according to equation (3.2.25).

The steps involved in the algorithm are summarized below:

STEP 1 Specify an initial boundary element grid and truncation tolerance \(\varepsilon_{tol}\).

STEP 2 Solve the governing equation using linear boundary elements.

STEP 3 Calculate \(E\) in (3.2.23) and if the inequality (3.3.1) is satisfied, stop. Otherwise continue.

STEP 4 Calculate the number of nodes from (3.3.4),
STEP 5 Construct the optimal mesh by solving equation (3.2.25).

STEP 6 Go to STEP 2.

In general, the h-r adaptive algorithm proposed here deals with the entire boundary. However, it is possible and even more effective to apply the algorithm to the portions of boundary where the solution changes rapidly. In such cases, a rigorous mathematical analysis can be very difficult. Similarly, the second derivatives of $u$ and $p$ can be approximately obtained by the method in section 2.3.

Another issue is the apparent complexity of the function $W(s)$ in (3.2.22). On the surface, the evaluation of $W(s)$ seems to be complicated. In fact, since $u^*(s_i, s')$ and $p^*(s_i, s')$ for $i=1, 2, ..., M$ at Gaussian points of each element have been evaluated in the quadrature routine of the boundary element program, the integrals in expression (3.2.22) can easily be obtained numerically. Therefore, one can calculate this function efficiently and without an excessive number of function evaluations. In addition, the method for solving equation (3.2.25) in STEP 5 has been discussed in section 2.4.

3.4 Numerical Results

In this section, the proposed h-r algorithm is tested against two benchmark problems. The first problem involves the solution of Laplace's equation in an exterior domain but
involves no singularities. The second one is the classical Motz I problem which is well known for its square root singularity on the boundary. The details of the numerical experimentation are described below.

**Example 3.1** Consider the flow of an inviscid, irrotational and incompressible fluid of velocity \( V \) past a cylinder of diameter 1. If the cylinder is long enough compared with its diameter, the problem can be assumed to be two dimensional as shown in figure 3.1. Let \( \psi \) represent the stream function, i.e., the velocity components in the \( x \) and \( y \) directions are given by

\[
\begin{align*}
  u &= \frac{\partial \psi}{\partial y}, \\
  v &= -\frac{\partial \psi}{\partial x}
\end{align*}
\]

![Fig. 3.2 The Flow Past a Cylinder](image)
One can decompose the stream function $\Psi$ into two parts as follows

$$\Psi = \Psi y + \psi$$

where $\Psi y$ describes the uniform flow ignoring the cylinder and $\psi$ is a perturbation stream function. The flow field can now be described in terms of the perturbation stream function $\psi$ as

$$\nabla^2 \psi = 0 \quad \text{for}\quad |r| > 0.5$$

$$\psi = -\Psi y \quad \text{on}\quad |r| = 0.5$$

$$\psi = \propto \frac{1}{r} \quad \text{as}\quad |r| \to \infty$$

The initial discretization is uniform and consisting of 18 linear boundary elements as shown in figure 3.3. After two iterations, the refined and optimal mesh distribution takes the shape depicted in figure 3.4 (obviously, the cylinder is approximated by the collection of linear segments). Note the concentration of nodes at the poles of the cylinder as expected. The exact, initial and final solutions of $\frac{\partial \psi}{\partial n}$ are given in figure 3.5. Although one can see an improvement, it is not dramatic. This is not surprising since the solution does not involve a singularity and the initial discretization is reasonably accurate.

Table 3.1 contains quantitative information on the performance of the algorithm corresponding to $\varepsilon_{\text{tol}} = 2\%$. In this table, column 2 represents the number of nodes at each
iteration. Column 3 is a measure of the relative percentage error for $p = \frac{\partial \psi}{\partial n}$ on the boundary of the cylinder. Here $p$ represents the exact solution and $p^h$ its boundary element approximation. Finally, the last column is the relative measure of the truncation error indicator expressed in (3.2.24), which falls below $\epsilon_{\text{tol}}$.

### TABLE 3.1 The Flow Past a Cylinder

| No. of Iter. | No. of Nodes | $\frac{|p - p^h|}{|\psi| + \alpha |p|}$ | $\frac{E}{|\psi| + \alpha |p|}$ |
|--------------|--------------|----------------------------------|----------------------------------|
| 0, initial  | 18           | 1.51 %                           | 4.02 %                           |
| 1            | 24           | 0.81 %                           | 2.11 %                           |
| 2            | 26           | 0.59 %                           | 1.59 %                           |
| uniform      | 26           | 0.76 %                           |                                  |

Figure 3.3 The Initial Mesh for the Cylinder
Figure 3.4 The Mesh in Iteration 2 for the Cylinder

Figure 3.5 The Flux on the Cylinder
Example 3.2  The Motz I problem governed by the Laplace equation has been considered in chapter two by using the optimal mesh algorithm. The boundary conditions and geometry can be found in figure 2.1. In general, the h—r adaptive algorithm proposed in this chapter deals with the entire boundary, but one can also apply it to portions of the boundary. Similarly, for the present combined algorithm we still focus our attention on the segment AB on which the singularity lies.

The initial grid is a uniform mesh consisting of 12 linear boundary elements covering segment AB and a total of 36 elements covering the entire boundary as shown in figure 3.6. After three iterations, the number of elements on side AB increased to 23 with concentration at the crack tip as indicated in figure 3.7. Note that the nodes on the other three sides have remained fixed. The calculations were performed with tolerance value of \( \varepsilon_{\text{tol}} = 0.4\% \).

The computed solution on OA and the normal derivative on OB are shown in figure 3.8 and 3.9. The Motz problem does not have an exact closed form solution; therefore, the results of a boundary element analysis with 204 linear elements are assumed to be sufficiently accurate to represent the exact solution and are shown in figures 3.8 and 3.9 for comparative purposes. Table 3.2 contains quantitative information on the calculations performed. The recorded results are as in the previous example with the
exception that the $L_1$ norm refers to integration on the segment OB.

TABLE 3.2 Motz I (h-r Method)

| No. of Iter. | No. of Nodes (OB) | $\frac{|p - p^h|}{|\psi| + |\alpha p|}$ | $\frac{E}{|\psi| + |\alpha p|}$ |
|--------------|------------------|---------------------------------|---------------------------------|
| initial      | 7                | 2.11 %                          | 1.21 %                          |
| 1            | 10               | 1.03 %                          | 0.52 %                          |
| 2            | 11               | 0.85 %                          | 0.41 %                          |
| 3            | 11               | 0.53 %                          | 0.39 %                          |

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 3.6 The Initial Mesh for Motz I (h-r Method)

Figure 3.7 The Mesh in Iteration 3 for Motz I (h-r Method)
Figure 3.8 Numerical Solution $u$ on OB (h-r)

Figure 3.9 Numerical Solution $p=\sigma u/\sigma n$ on AO (h-r)
3.5 Conclusion

In this chapter, a combined adaptive h-r algorithm has been proposed. The algorithm is based on a sound formulation and is verified by the numerical experiments conducted. Although the algorithm is described for the Laplace equation, it will be extended to the equation of two dimensional elasticity in the next chapter. The extension is not straightforward because there is a fundamental difference between the two classes of integral equations representing the two sets of the boundary value problems.
Chapter Four

An Adaptive h–r Algorithm for the Equations of Linear Elasticity

4.1 Introduction

In the previous chapter, an adaptive h–r algorithm for the boundary element solution of Laplace's equation was formulated. Numerical results indicate that the solution quality is greatly improved for problems involving boundary singularities.

In this chapter, we extend the combined algorithm to the equations of linear elasticity. Since strongly singular kernels arise in the integral equations corresponding to elasticity problems, the treatment of the minimization of the truncation error indicator will be different from that in the Laplace equation. It turns out that the numerical results are still satisfactory, especially for singular problems, i.e. those arising in fracture mechanics. In the h–r algorithm, the advantages of both techniques are maintained and therefore are more promising than h or r algorithms separately.

4.2 Truncation Error

The Navier's equation of linear elasticity (neglecting
body forces) can be transformed by using Betti’s reciprocal theorem, to a boundary integral equation, written as (see Appendix 7.3):

\[ c(s)u_l(s) + \int_{\Gamma} u_k(s')p^*_l_k(s,s')ds' - \int_{\Gamma} p_k(s')u^*_l_k(s,s')ds' = 0 \quad l=1,2 \quad (4.2.1) \]

where \( s \) is the position of source defined on \( \Gamma=\partial\Omega \) and \( \int_{\Gamma} \) denotes the Cauchy principal value integral along \( \partial\Omega \). As defined in chapter 3, \( s \) and \( s' \) in (4.2.1) are arc length coordinates. \( c(s) \) is dependent on the smoothness of the boundary geometry at \( s \in \Gamma \). \( u^*_l_k \) and \( p^*_l_k \) are the fundamental solutions described in Appendix I, 7.3.

The discretization is achieved by the partitioning \( \{s_j\}, j=1,2,...,N \), on the boundary. On each segment \( \Gamma_j \) the unknown solution \( u(x) \) and its normal derivative \( p(x) \) are approximated by linear functions.

It is assumed that the exact solution and its normal derivative are three times and twice continuously differentiable on the boundary, respectively. Using linear interpolation theory,

\[ u_l(s) = B(s)u^1_{j1} + B(s)u^1_{j2} + (s-s_{j-1})(s-s_j)u_l [s_{j-1}, s_j, s] \quad \text{se} \Gamma_j \quad (4.2.2) \]
\[ p_l(s) = B_1(s)p_{j_1}^i + B_2(s)p_{j_2}^i + (s-s_{j-1})(s-s_j)p_l(s_{j-1}, s_j, s) \]

where \( u_l[...] \) and \( p_l[...] \) denote the divided differences of functions \( u_l(s) \) and \( p_l(s) \), \( l=1,2 \) respectively. We also define \( \tilde{u}_l(s) \) and \( \tilde{p}_l(s) \) by

\[ \tilde{u}_l(s) = B_1(s)u_{j_1}^i + B_2(s)u_{j_2}^i \quad \text{se} \Gamma_j \]  \hspace{1cm} (4.2.4)

\[ \tilde{p}_l(s) = B_1(s)p_{j_1}^i + B_2(s)p_{j_2}^i \quad \text{se} \Gamma_j . \]  \hspace{1cm} (4.2.5)

In solving an elasticity problem by linear boundary elements, one replaces \( u_l(s) \) and \( p_l(s) \) in equation (4.2.1) with their linear approximations \( \tilde{u}_l(s) \) and \( \tilde{p}_l(s) \), respectively. The discrete form can be expressed as

\[ c(s_i)\tilde{u}_l(s_i) + \sum_j \int_{\Gamma_j} \tilde{u}_k(s')\tilde{p}_l^{(k)}(s_i, s')ds' \]

\[ - \sum_j \int_{\Gamma_j} \tilde{p}_k(s')\tilde{u}_l^{(k)}(s_i, s')ds' = 0 \]

\[ i=1,2,\ldots, N . \]  \hspace{1cm} (4.2.6)

Note that \( \sum_j \int_{\Gamma_j} \) still denotes a Cauchy principal value integral. Comparing equations (4.2.1) and (4.2.6), the truncation error at \( s_i \) for the linear elasticity equation with linear boundary element approximation can be defined as
\( H_1(s_i) = \)

\[
= \sum_j \left( \int_{\Gamma_j} (s'-s_{j-1})(s'-s_j) u_k(s_{j-1}, s_j, s') \right) p^u_{l k}(s_i, s') ds'
\]

\[- \sum_j \left( \int_{\Gamma_j} (s'-s_{j-1})(s'-s_j) p_k(s_{j-1}, s_j, s') \right) u^u_{l k}(s_i, s') ds'
\]

\[
i = 1, 2, \ldots, N, \quad l = 1, 2.
\]

(4.2.7)

For the second integral in (4.2.7), the same approach as in chapter 3 can be used, i.e. we have

\[
\sum_j \left( \int_{\Gamma_j} (s'-s_{j-1})(s'-s_j) p_k(s_{j-1}, s_j, s') \right) u^u_{l k}(s_i, s') ds'
\]

\[
= \sum_j \frac{h_j}{12} p_k(s_{j-1}, s_j, \alpha_j) u^u_{l k}(s_i, \alpha_j)
\]

where \( \alpha_j \in (s_{j-1}, s_j) \). Using the grading function, we obtain

\[
\sum_j \left( \int_{\Gamma_j} (s'-s_{j-1})(s'-s_j) p_k(s_{j-1}, s_j, s') \right) u^u_{l k}(s_i, s') ds'
\]

\[
= -\frac{1}{12N^2} \int_{\Gamma} \frac{p''(s') u^u_{l k}(s_i, s')}{\xi'(s')^2} ds' + o(\infty)
\]

(4.2.8)

where \( \xi'(s) \) is a grading function. Since a strongly singular kernel arises from the first integral in (4.2.7), a different procedure must be employed for its evaluation. Now
we show that the first integral in (4.2.7) can be transformed into an integral with a weakly singular kernel. Let

\[ p_{ik}(s, s') = p_{ik}^*(s, s') - \delta_{ik} c(s) \quad (4.2.9) \]

where \( c(s) \) was defined in (4.2.1). Then, the first integral in (4.2.7) can be expressed as the summation of two integrals with kernels \( p_{ik}^*(s, s') \) and \( \delta_{ik} c(s) \), respectively, and we have

\[
\sum \int \left( (s'-s_{j-1})(s'-s_j) u_k(s_{j-1}, s_j, s') \right) p_{ik}^*(s_i, s') ds' \int_{G_j} \rho_{ik}^*(s_i, s') ds' + o(\infty) \quad (4.2.10)
\]

where \( s_j = (s_{j-1} + s_j) / 2 \). The proof of (4.2.10) is stated in Appendix I, 7.4. By noting that \( \int \rho_{ik}^*(s_i, s') ds' = 0 \) and using the Abel Partial Summation Formula in (4.2.10), we obtain

\[
\sum \int \left( (s'-s_{j-1})(s'-s_j) u_k(s_{j-1}, s_j, s') \right) p_{ik}^*(s_i, s') ds' \int_{G_j} \rho_{ik}^*(s_i, s') ds' = \frac{1}{12} \sum \left( h_{j-1}^2 u_k(s_{j-1}) - h_j^2 u_k(s_j) \right) a_{ik}(s_i, s') \]

\[ + o(\infty) \quad i=1,2,\ldots,N, \quad i,k=1,2 \]

where
Furthermore, by using the grading function, the above equation can be rewritten as

\[ q_{lk}(s_i, s') = \int_0^s p_{lk}^{*}(s_i, t) dt. \quad (4.2.11) \]

The right hand side of (4.2.12) is an integral with a weakly singular kernel. Now, substituting (4.2.9) and (4.2.12) into (4.2.7), the truncation error can be expressed as

\[
\sum_{j} \left[ (s'-s_{j-1})(s'-s_j) u_k[s_{j-1}, s_j, s'] \right] p_{lk}^{*}(s_i, s') ds' \\
= -\frac{1}{12N^2} \int_{\Gamma} \frac{d}{ds}\left( \frac{u''(s')}{\xi'(s')^2} \right) q_{lk}(s_i, s') ds' + o(\varepsilon^*). \quad (4.2.12)
\]

Letting \( \| \cdot \| \) denote the \( L_1 \) norm, a new norm \( \| \cdot \|_\infty \) is defined as follows,
\[ \| H \|_w = \| H_1(s) \| + \| H_2(s) \| \]

i.e.

\[ \| H \|_w = \int_\Gamma \left( \| H_1(s) \| + \| H_2(s) \| \right) ds \quad (4.2.14) \]

In the discrete form, (4.2.14) can be expressed as

\[ \| H \|_w = \sum_i h_i \left( \| H_1(s_i) \| + \| H_2(s_i) \| \right) (1 + O(h)) \quad (4.2.15) \]

where \( h = \max h_i \). Substituting (4.2.13) into (4.2.15), the truncation error under the new norm is

\[ \| H \|_w = \frac{1}{12N^2} \sum_{i=1}^2 \sum_i h_i \left[ \frac{1}{\Gamma} \int_{s'} \frac{u_k''(s')}{\xi'(s')^2} q_{kk}(s_i, s') \ ds' \right] \]

\[ + \frac{1}{12N^2} \sum_{i=1}^2 \sum_i h_i \left[ \frac{1}{\Gamma} \int_{s'} \frac{\delta_{kk}}{\xi'(s')^2} c(s_i) \ ds' \right] \]

\[ + \frac{1}{12N^2} \sum_{i=1}^2 \sum_i h_i \left[ \frac{1}{\Gamma} \int_{s'} \frac{p_{kk}''(s') u_k''(s_i, s')}{\xi'(s')^2} \ ds' \right] + O(h) \quad (4.2.16) \]

and furthermore,

\[ \| H \|_w \leq \frac{1}{12N^2} \sum_{i=1}^2 \sum_i h_i \left[ \frac{1}{\Gamma} \int_{s'} \left( \frac{u_k''(s')}{\xi'(s')^2} \right) q_{kk}(s_i, s') \ ds' \right] \]

97

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Reversing the summation and the integration in (4.2.17) and noting that the sum can be regarded as a Riemann sum, we obtain

\[ \| H \|_\infty \leq \frac{1}{12N^2} \sum_{l=1}^{2} \int_{\Gamma} \left| \frac{u''(s')}{\xi'(s')^2} \right| | q^*_l k(s') | ds' \]

\[ + \frac{1}{12N^2} \sum_{l=1}^{2} \int_{\Gamma} \left| \frac{p''(s')}{\xi'(s')^2} \right| | q^*_l k(s') | ds' \]

\[ + \frac{1}{12N^2} \sum_{l=1}^{2} \int_{\Gamma} \left| \frac{u''(s')}{\xi'(s')^2} \right| | g^*_l k(s') | ds' + o(\infty) \tag{4.2.18} \]

where

\[ q^*_l k(s') = \int_{\Gamma} |q_{l k}(s, s')| ds \tag{4.2.19} \]

\[ g^*_l k(s') = \int_{\Gamma} |u^*_l k(s, s')| ds \tag{4.2.20} \]

\[ c^*_s = \int_{\Gamma} c(s) ds = \frac{1}{2\delta} \tag{4.2.21} \]

Let

\[ \]
In the next section, the above expression will be used as an error indicator in the adaptive h-r algorithm and the grading function is chosen such that $E$ achieves its minimum value.

4.3 The Adaptive Algorithm

As described in the previous chapters, the optimal mesh problem can be considered as a minimization problem presented below

$$
\min E = \frac{1}{12N^2} \sum_{l=1}^{2} \left( \int_{\Gamma} \frac{d}{ds'} \left( \frac{u''(s')}{{\xi'}(s')^2} \right) q_{lk}^* ds' + \int_{\Gamma} \frac{u''(s')}{{\xi'}(s')^2} \delta_{lk} ds' + \int_{\Gamma} \frac{p''(s')}{{\xi'}(s')^2} g_{lk}^* ds' \right).
$$

In the next section, the above expression will be used as an error indicator in the adaptive h-r algorithm and the grading function is chosen such that $E$ achieves its minimum value.

4.3 The Adaptive Algorithm

As described in the previous chapters, the optimal mesh problem can be considered as a minimization problem presented below

$$
\min E = \frac{1}{12N^2} \sum_{l=1}^{2} \left( \int_{\Gamma} \frac{d}{ds'} \left( \frac{u''(s')}{{\xi'}(s')^2} \right) q_{lk}^* ds' + \int_{\Gamma} \frac{u''(s')}{{\xi'}(s')^2} \delta_{lk} ds' + \int_{\Gamma} \frac{p''(s')}{{\xi'}(s')^2} g_{lk}^* ds' \right).
$$

Obviously, the objective function in (4.3.1) is different from that discussed in chapter 3 for Laplace's equation. This is a very complicated optimization problem and it is difficult to find an analytic solution for it. Therefore, an iterative method has to be employed for solving (4.3.1).
Express (4.3.1) as

$$\min_{\xi} E = \frac{1}{12N^2} \int_{\Gamma} \frac{1}{\xi'(s')^2} WC(\xi', s') \, ds'$$  \hspace{1cm} (4.3.2)

where

$$WC(\xi, s') = \sum_{l=1}^{2} \left[ \frac{\xi''(s')}{\xi'(s')} - u_k'(s') \right] q_{lk}^*(s')$$

\hspace{1cm} + \frac{c^*}{12N^2} \sum_{l=1}^{2} \left[ \int_{\Gamma} \left( \frac{\xi''(s')}{\xi'(s')} \right) q_{lk}^*(s') \, ds' \right] \left( \delta_{lk} + \int_{\Gamma} \frac{p_k'(s')}{\xi'(s')} g_{lk}^*(s') \, ds' \right)$$  \hspace{1cm} (4.3.3)

Let $\xi'(s')$ and $\xi''(s')$ in $WC(\xi, s')$ be known, then the $\xi'(s')$

$$\min_{\xi} E = \frac{1}{12N^2} \int_{\Gamma} \left\{ \int_{\Gamma} \frac{\xi''(s')}{\xi'(s')} \, ds' \right\} q_{lk}^*(s') \, ds'$$

\hspace{1cm} + \frac{c^*}{12N^2} \int_{\Gamma} \frac{u_k''(s')}{\xi'(s')} \, ds' + \int_{\Gamma} \frac{p_k'(s')}{\xi'(s')} g_{lk}^*(s') \, ds' \right\}$$  \hspace{1cm} (4.3.1)

As indicated in chapter 3, (4.3.4) is a classical problem from the calculus of variations. The solution to the Euler-Lagrange equation associated with (4.3.4), $\xi(x)$, can be expressed as

$$\xi(s') = \frac{\int_{0}^{s'} \left[ WC(\xi', s') \right]^{1/3} \, ds'}{\int_{0}^{S} \left[ WC(\xi', s') \right]^{1/3} \, ds'}$$  \hspace{1cm} (4.3.5)

where $S$ is the perimeter of the boundary. By the definition of grading function, the equation (4.3.5) can be rewritten
The asymptotic optimal mesh \( \{s_i\} \), i.e., the solution of (4.3.2), is determined by iteratively solving equation (4.3.6). This iteration procedure is

\[
\text{Initial mesh } \rightarrow \bar{s} \rightarrow \text{new mesh } \{s_i\} \rightarrow \bar{s} \ldots
\]

In general, the iteration continues until the difference of the meshes between two successive iterations is small enough, i.e.

\[
\Sigma |s_j^* - s_j| \leq \delta_{\text{tol}}.
\]

In practice, for the sake of convenience we directly solve (4.3.6) for the optimal mesh \( \{s_j\} \) in terms of the mesh at the last iterative step of the h-r adaptive algorithm.

On the asymptotically optimal mesh, the objective function \( E \) in (4.3.1) achieves its minimum which can be expressed as

\[
E_o = \frac{1}{12N^2} \left( \int_{\Gamma} W^{4/3} \, ds \right)^3.
\]  

We now describe the adaptive algorithm more systematically. One starts with a discretization of the boundary which is denoted by \( \{s_j\} \). Using this mesh, the
approximate boundary element solution and the associated
normal derivative are obtained by linear boundary elements.
Based on the initial mesh (initial grading function) and the
approximate solution, W is evaluated, in which \( u''(s) \) and
\( p''(s) \) are replaced by their numerical approximations and
therefore, the truncation error function \( E \) is calculated.

If this truncation error in the relative sense is small, i.e.

\[
\frac{E}{\|u_h\|_H^2 + \alpha \|p_h\|_H^2} \leq \varepsilon_{\text{tol}},
\]

(4.3.8)

no mesh refinement and redistribution are required (\( u_h \) and
\( p_h \) are the numerical solutions). The variable \( \varepsilon_{\text{tol}} \) is a user
specified tolerance. For the explanation concerning \( \alpha \), see
section 3.3. On the other hand, if inequality (4.3.7) is
not satisfied, mesh refinement and redistribution are
required. The new number of grid points is estimated from
the inequality

\[
\frac{E_0}{\|u_h\|_H^2 + \alpha \|p_h\|_H^2} \leq \varepsilon_{\text{tol}}.
\]

(4.3.9)

Using the expression of \( E_0 \) in (4.3.7) and solving the above
inequality for \( \hat{N} \), the number of grid points on the new mesh
may be obtained from

\[
\hat{N} = \text{Int} \left[ 1 + \frac{\left( \int \frac{W^{1/3}}{ds} \right)^{2/3}}{\sqrt{12(\|u_h\|_H^2 + \alpha \|p_h\|_H^2)\varepsilon_{\text{tol}}}} \right]
\]

(4.3.10)
Mesh redistribution is then performed by solving equation (4.3.6).

The steps involved in the algorithm are summarized below:

STEP 1 Specify an initial boundary element grid and tolerance \( \varepsilon_{\text{tol}} \).

STEP 2 Solve the governing equation by linear boundary element method and evaluate the functions \( q_{lk}^*(s) \) and \( g_{lk}^*(s) \) defined in (4.2.19) and (4.2.20).

STEP 3 Calculate \( \tilde{\xi}' \) and evaluate the distribution function in (4.3.3) on the boundary.

STEP 4 Calculate \( E \) in (4.3.1) and if the inequality (4.3.7) is satisfied, stop. Otherwise continue.

STEP 5 Calculate the number of nodes from (4.3.10), i.e. refine.

STEP 6 Construct the optimal mesh by solving equation (4.3.6).

STEP 7 Go to STEP 2.

The calculation details are similar to those considered in section 3.3. Since function \( u_{lk}^*(s_i,s') \) and \( p_{lk}^*(s_i,s') \) for \( i = 1,2,\ldots,N \) and \( l,k=1,2 \) have been evaluated at Gaussian points of each element in the quadrature routine of the boundary element program, \( q_{lk}(s_i,s') \) in (4.2.11) can be evaluated using trapezoid rule and furthermore, \( q_{lk}^*(s') \) and \( g_{lk}^*(s') \) defined in (4.2.19) and (4.2.20) can be evaluated similarly. \( \tilde{\xi}'(s) \) and \( \tilde{\xi}''(s) \) can be constructed by using the initial
mesh. Therefore, one can calculate $W(\bar{z}', s')$ without excessive operation.

4.4 Numerical Results

In order to test the algorithm in the previous section, two examples from classical mechanics are discussed. The details of the numerical experimentation are described below.

**Example 4.1** We consider a plane stress problem where a circular region is under a uniform pressure of intensity $P$. The dimensions, boundary conditions and material properties are given in figure 4.1. This problem possesses a closed form analytical solution.

In order to test the adaptive algorithm, the initial mesh was chosen as the uniform mesh in figure 4.1. After two iterations the resulting mesh is shown in figure 4.2. The initial and final numerical solutions are given respectively in figures 4.3 and 4.4.

Table 4.1 presents the error and convergence information and the performance of the algorithm corresponding to $\varepsilon_{\text{tol}} = 0.3\%$. 

104

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Table 4.1  The Error for the Circle under Uniform Pressure

| i  | No. of Nodes | $\frac{E}{|u_h| + \alpha |P_h|}$ | $\frac{|u_1 - u_{1h}|}{|u_1|}$ | $\frac{|u_2 - u_{2h}|}{|u_2|}$ |
|----|--------------|---------------------------------|-------------------------------|-------------------------------|
| 0  | Initial      | 12                              | 0.84 %                        | 2.93 %                        | 3.14 %                        |
| 1  | 20           | 0.32 %                          | 2.11 %                        | 2.14 %                        |
| 2  | 22           | 0.26 %                          | 1.61 %                        | 2.12 %                        |
| uniform | 22     | 1.71 %                          | 2.16 %                        |

(For the definitions of $|\cdot|$ and $|\cdot|_\infty$, see section 4.2)
Figure 4.1 The Initial Mesh for Ex. 4.1, diameter = 6.0
P = 100.0, Young's modulus E = 10^3, Poisson ratio μ = 0.1

Figure 4.2 The Mesh in Iteration 2 for Ex. 4.1
Figure 4.3 The Displacement in x Direction on the Circle

Figure 4.4 The Displacement in y Direction on the Circle
Example 4.2 We now test a singular problem in spite of the fact that the theoretical analysis in section 4.2 is based on the hypothesis of a smooth solution. Consider the rectangular plate containing a centered crack under a mode I loading which was treated in chapter 2 using the optimal mesh algorithm. The plate is subjected to a uniform tension P. Due to symmetry, only one quarter of the plate is considered. Here the adaptive algorithm is still used along the section AB on which the crack lies.

The initial mesh distribution is shown in figure 4.5 with 40 linear boundary elements. The mesh after three iterations is shown in figure 4.6. It contains 59 linear boundary elements with a concentration at the crack tip. The displacement on OA and traction on OB are shown in figures 4.7 and 4.8 respectively (the LR solution is obtained by the same method as in chapter 2). The tolerance of $e_{tol} = 1\%$ is used. For this singular problem, the numerical results after three iterations are much better than those on the initial mesh.

As a mode I problem, the stress intensity factor can be calculated from the crack opening displacement (COD) (see Appendix II, 8.5), i.e.,

$$K_1 = \frac{\sqrt{2\pi}}{4} \frac{E u_2}{\gamma r} \ \ \ \ \ (4.4.1)$$

The comparison of the stress intensity factors for the initial, final and LR solutions are given in Table 4.3.

108

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Table 4.2 The Error for the Central Crack Problem

| No. | No. of Nodes | $\frac{E}{|u_h|^++|P_h|^+}$ | $\frac{|u_{1h}^{LR} - u_{1h}^1|}{|u_{1h}^1|}$ (AO) | $\frac{|u_{2h}^{LR} - u_{2h}^2|}{|u_{2h}^2|}$ (COB) |
|-----|--------------|---------------------------|--------------------------------|--------------------------------|
| 0  | Initial     | 42                        | 3.53 %                          | 20.11 %                          | 8.53 %                          |
| 1  |             | 52                        | 1.39 %                          | 13.24 %                          | 3.48 %                          |
| 2  |             | 57                        | 1.12 %                          | 7.49 %                           | 2.38 %                          |
| 3  |             | 59                        | 0.88 %                          | 3.89 %                           | 1.67 %                          |

Table 4.3 The Stress Intensity Factor for the Central Crack Problem

| No. | $\frac{K_I}{\sigma\sqrt{na}}$ | $\frac{|K_I - K_I^{LR}|}{K_I^{LR}}$ |
|-----|-----------------------------|---------------------------------|
| 0  | Initial         | 1.033                          | 17.35 %                         |
| 1  |                | 1.139                          | 8.88 %                          |
| 2  |                | 1.177                          | 5.84 %                          |
| 3  |                | 1.199                          | 4.08 %                          |
| LR |                | 1.25                           |                                 |
Figure 4.5 Initial Mesh for Centre Crack (h-r Method)

Figure 4.6 The Mesh in Iteration 3 for Centre Crack (h-r Method)
Figure 4.7 The Displacement in y Direction on CB (h-r)

Figure 4.8 The Displacement in y Direction on AC (h-r)
4.5 Conclusion

In this chapter, the combined adaptive h-r algorithm proposed in chapter 3 is extended to the Navier’s equation for linear elasticity. Although this is a parallel development to the Laplace equation, the theoretical analysis and the proposed algorithm are more complex due to the presence of a strongly singular kernel. The numerical results for the two benchmark problems are satisfactory. The effectiveness of the algorithm is however more pronounced in the second case.
Chapter Five
Stability and Singularity

5.1 Introduction

The discretization of a domain plays an important role in the quality of the numerical solution of differential equations. In both the r and h-r algorithms, an error function is minimized in order to find the optimal mesh and therefore, the optimal mesh is very much dependent on the nature of the error function. The general discussion on the stability of the optimal mesh algorithm is presented in this chapter. This discussion requires several concepts which are defined as follows.

Definition 5.1 \( \mathcal{T}(x) \) is a distribution function defined on \([a,b]\) if \( \mathcal{T}(x) \) is nonnegative and not identically zero in \([a,b]\) (i.e., \( \mathcal{T}(x) \geq 0 \) and \( \mathcal{T}(x) \neq 0 \)).

The error function associated with a distribution function \( \mathcal{T}(x) \) and a particular mesh can be defined as

\[
ECT(\xi) = \int_a^b \frac{\mathcal{T}(x)^{1+r}}{\xi' \xi} \, dx \tag{5.1.1}
\]

where \( r > 0 \), \( \xi \) is any grading function related to the above mesh and \( \xi' = d\xi/dx \).

Definition 5.2 Let \( \mathcal{T}(x) \) be a distribution function defined on \([a,b]\). The mesh \( \{x_i\} \) constructed by
\[ \frac{\int_a^i T(x) \, dx}{\int_a^b T(x) \, dx} = \frac{1}{N} \tag{5.1.2} \]

is said to be the optimal mesh with respect to \( T(x) \).

The grading function corresponding to an optimal mesh is expressed by

\[ \xi(x) = \frac{\int_a^x T(x) \, dx}{\int_a^b T(x) \, dx} \tag{5.1.3} \]

In Carey and Dinh's algorithm [12], \( r = 2(k+1-m) \) and

\[ T(x) = |w^{k+1}(x)| \]

Differentiating both sides of (5.1.3), we have

\[ \xi'(x) = \frac{T(x)}{\int_a^b T(x) \, dx} \tag{5.1.4} \]

The error function in (5.1.1) on the optimal mesh can be calculated by substituting (5.1.4) into (5.1.1) leading to

\[ E^o(T) = \left( \frac{1}{\int_a^b T(x) \, dx} \right)^{1+\tau} \tag{5.1.5} \]

Definition 5.3: \( E^o(T) \) in (5.1.5) is said to be the optimal error with respect to the distribution function \( T(x) \).

In practice, the distribution function is unknown and a numerical approximation is used to represent it. This causes
an error in the optimal error indicator $E_0(T)$, and the optimal mesh distribution. The effect of a small perturbation in the grading function has been considered in [12]. These results were obtained under two crucial hypotheses on the distribution function as follows:

(a) The distribution function $T(\mathbf{x})$ is strictly positive, i.e.,

$$T(\mathbf{x}) \geq c > 0$$  \hspace{1cm} (5.1.6)

where $c$ is a constant independent of $N$ (the number of nodes in the domain).

(b) The distribution function is continuous on $[a,b]$.

Clearly, not all problems satisfy these two conditions. Pereyra and Sewell [37] have given a discussion on the stability and the convergence of the optimal mesh algorithm under restriction (b) and the quasi-uniformness of the mesh (i.e., $(\max h_i)/(\min h_i) \leq \text{const.}$). In their discussion, the distribution functions are not required to satisfy condition (a). Loosely speaking, they introduced a class of modified distribution functions defined by

$$\hat{T}(\mathbf{x}) = \max \{T(\mathbf{x}), \varepsilon\}, \varepsilon > 0$$  \hspace{1cm} (5.1.7)

which are strictly positive. The optimal mesh with respect to this strictly positive modified distribution function is quasi-uniform. However, the error indicator is not asymptotically optimal, i.e.

$$\|E_0(\hat{T}) - E_0(T)\| \to 0 \quad \text{as } N \to \infty$$

115

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
In chapters 2, 3 and 4, we have shown that the optimal mesh algorithm works for problems involving no singularities but the improvement is not dramatic. The numerical results show that for problems with singularity in the domain, the improvement in the solution quality is more evident. However, in such examples restriction (b)' is not satisfied. Furthermore, the optimal mesh does not satisfy the condition of quasi-uniformness presented by Pereyra and Sewell.

In this chapter, a discussion on the stability of the optimal mesh and the optimal error indicator with respect to a strictly positive and continuous distribution function is presented. We later extend the discussion into a class of singular distribution functions. For the general case, a class of modified distribution functions \( \hat{T}(x) \) is used to replace the exact distribution functions. The optimal mesh with respect to the modified distribution function is \( k \)-regular (to be defined later). For this modified class, the error indicator and the mesh are asymptotically optimal, i.e.,

\[
\|E_0(\hat{T}) - E_0(T)\| \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad \text{(5.1.8)}
\]

and

\[
\|x_i - \hat{x}_i\| \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad \text{(5.1.9)}
\]

for some norm, where \( E_0(\hat{T}) \) and \( \{\hat{x}_i\} \) are the optimal error indicator and the optimal mesh with respect to the modified distribution function \( \hat{T}(x) \). The analysis presented is rather
theoretical in nature but of fundamental importance in the numerical algorithm used in the previous chapters.

5.2 Continuous Distribution Functions

For a continuous distribution function satisfying the condition (a) in section 5.1, the optimal error indicator \( E_0(T) \) is continuously dependent on the distribution function \( T(x) \) [12]. Mathematically, this means that for any two distribution functions \( T_1(x) \) and \( T_2(x) \) satisfying (a) and (b), there exists a constant \( c \) independent of \( N \) such that

\[
\frac{|E_0(T_1) - E_0(T_2)|}{E_0(T_2)} \leq c \|T_1 - T_2\|_1 \quad (5.2.1)
\]

In the following theorem, we will extend the result of (5.2.1) to show that the optimal mesh is also continuously dependent on the distribution function.

Theorem 5.1 Let \( T_1(x) \) and \( T_2(x) \in C^0[a,b] \) be strictly positive. If \( \{x_i^1\} \) and \( \{x_i^2\} \) are two optimal meshes with respect to \( T_1(x) \) and \( T_2(x) \), there exists a constant \( c > 0 \) such that

\[
\max_i |x_i^1 - x_i^2| \leq c \|T_1 - T_2\|_1 \quad (5.2.2)
\]

Proof: By the definition of the optimal mesh in Section 5.1, the optimal meshes \( \{x_i^1\} \) and \( \{x_i^2\} \) satisfy

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[
\frac{1}{N} = \frac{\int_a^b T_1(x) \, dx}{A_1} \quad (5.2.3)
\]

and

\[
\frac{1}{N} = \frac{\int_a^b T_2(x) \, dx}{A_2} \quad (5.2.4)
\]

where \( A_1 = \int_a^b T_1(x) \, dx \), and \( A_2 = \int_a^b T_2(x) \, dx \). Subtracting (5.2.4) from (5.2.3) gives

\[
\int_a^{x_i} T_1(x) \, dx - \int_a^{x_i} T_2(x) \, dx = \frac{1}{N} (A_1 - A_2).
\]

Furthermore

\[
\int_{x_i}^{x_1} T_1(x) \, dx = \frac{1}{N} (A_1 - A_2) - \int_{x_i}^{x_2} (T_1 - T_2) \, dx \quad (5.2.5)
\]

By the mean value theorem, (5.2.5) can be simplified to

\[
(x_1^i - x_2^i) = \frac{1}{T_2(x_i^* \{N (A_1 - A_2) - \int_{x_i}^{x_2} (T_1 - T_2) \, dx \}}
\]

where \( x_i^* \in (x_i^1, x_i^2) \). By (5.2.6), we obtain

\[
|x_1^i - x_2^i| \leq \frac{2}{T_2(x_i^*)} \int_a^b |T_1 - T_2| \, dx \quad (5.2.7)
\]

Since \( T_2(x) \) is strictly positive, (5.2.2) can be obtained.
directly from (5.2.7).

It has been proven [37] that the optimal mesh with respect to a strictly positive distribution function \( T(x) \in C^0 [a, b] \) is quasi-uniform. We will prove that the condition of strict positiveness is also necessary for the quasi-uniformness of the mesh as follows.

**Theorem 5.2** If \( T(x) \in C^0 [a, b] \), then the optimal mesh with respect to \( T(x) \) is quasi-uniform if and only if \( T(x) \) is strictly positive.

**Proof:** The sufficient condition has been proved in [37]. We only prove the necessary condition. Let \( \{x_i\}_n \) be the quasi-uniform optimal mesh with respect to \( T(x) \), i.e., for any \( N \) there exists a constant \( c > 0 \) such that

\[
\frac{\max h_i}{\min h_i} \leq c . \tag{5.2.8}
\]

By equation (5.1.2), we obtain

\[
\frac{1}{N} = \frac{\int_{x_{i-1}}^{x_i} T(x) dx}{A} = \frac{T(\bar{x}_i) h_i}{A} \tag{5.2.9}
\]

where \( \bar{x}_i \in (x_{i-1}, x_i) \) and \( A = \int_a^b T(x) dx \). Using (5.2.9) for index \( m \) on which \( h_i \) has its minimum and (5.2.8) gives

\[
\frac{h_i}{h_m} = \frac{T(\bar{x}_m)}{T(\bar{x}_i)} \leq c
\]

Since \( h_m \leq (b-a)/N \) and \( T(\bar{x}_m) = \max T(\bar{x}_i) \), we obtain
Since $T(x)$ is continuous in $[a,b]$, then for a large enough $N$, (5.2.10) leads to

$$T(x) \geq \frac{1}{c} \max T(x) > 0,$$

i.e., $T(x)$ is strictly positive in $[a,b]$.

5.3 Singular Distribution Functions

In Section 5.2 we restricted our discussion to a continuous and strictly positive distribution function. In this section we extend the results to a class of singular distribution functions of the type

$$T(x) = |x-x^*|^\alpha f(x).$$

which arise in those singular problems shown in chapters 3 and 4. In (5.3.1), one has $-1 < \alpha \leq 0$ and $f(x) \in C^0[a,b]$. Obviously, $T(x)$ in (5.3.1) does not satisfy condition (b) if $f(x)$ is strictly positive and $\alpha<0$.

Without loss of generality, it is assumed that $x^*=a$. Clearly, the optimal mesh with respect to the distribution function (5.3.1) is not quasi-uniform.

The concept of $k$-regularity which plays an important role in this case was used by Babuska and Rheinboldt in [5]. The definition is as follows:

**Definition 5.4** Let $\Pi_N$ be a set of partitions on $[a,b]$. Then $\Pi_N$ is said to be $k$-regular with $k > 0$, if there exists a constant $c>0$ independent of $N$ such that
\[
\frac{\left(\max h_i\right)^k}{\min h_i} \leq c. \quad \Box \quad (5.3.2)
\]

**Theorem 5.3** If \( f(x) \) is strictly positive, the optimal mesh with respect to \( T(x) \) in (5.3.1) is \( k \) regular with \( k = \frac{1}{1+c_1+\alpha} \).

**Proof:** By (5.2.9) and (5.3.1), we obtain

\[
\frac{A}{N} = \frac{1}{\alpha+1} f(x) h_i^{\alpha+1} \quad (5.3.3)
\]

\( h_i \) can now be obtained from (5.3.3), resulting in

\[
h_i = \left( \frac{A\alpha+1}{Nf(x)} \right)^{1/(\alpha+1)} \quad (5.3.4)
\]

On the other hand, by (5.2.9),

\[
\frac{A}{N} = \frac{1}{\alpha+1} f(x) h_i (\bar{x}_i - a) \alpha \leq \frac{1}{\alpha+1} f(x) h_i (a) \alpha \quad \text{for } i > 1
\]

\[
(5.3.5)
\]

where \( f(x) = \max f(x) \) and \( x \in (x_{i-1}, x_i) \). Substituting (5.3.4) into (5.3.5), \( h_i \) can be obtained from (5.3.5)

\[
h_i \geq \frac{A\alpha+1}{N} \left( \frac{A\alpha+1}{Nf(x)} \right)^{-\alpha/(\alpha+1)} \quad \text{for } i > 1. \quad (5.3.6)
\]

Since \( f(x) \) is strictly positive, combining (5.3.4) and (5.3.6) gives

\[
h_m = \min h_i \geq \frac{c}{N^{\alpha/(\alpha+1)}} \quad (5.3.7)
\]

where \( c \) is a positive constant. Similarly, using (5.2.9).
we obtain

$$\max h_i = \max \frac{A}{NT_m} \leq \frac{A}{NT_m}$$  \hspace{1cm} (5.3.8)

where \(T_m = \max T(x)\). Since \(T(x)\) is also strictly positive, combining (5.3.7) and (5.3.8) leads to

$$\left(\frac{\max h_i}{\min h_i}\right)^k = \left(\frac{\max h_i}{h_m}\right)^k \leq \text{const.} \frac{N^k}{N^{\alpha+1}}.$$  \hspace{1cm} (5.3.9)

The inequality (5.3.9) means that the optimal mesh is \(k\)-regular provided that \(k \geq 1/(\alpha+1)\).

If \(f(x)\) is strictly positive, for the class of distribution functions in (5.3.1) the continuous dependence results (5.2.1 and 5.2.2) are still true.

In theorem 5.3, condition \((b)\) was basically removed while condition \((a)\) still holds. We now try to weaken both conditions \((a)\) and \((b)\) by considering the following types of singular distribution functions

$$T(x) = (x-a)^\alpha |x^* - x|^{\beta} g(x)$$  \hspace{1cm} (5.3.10)

where \(\beta \geq 0\) and \(g(x) \in C^0[a,b]\). An example of this case was seen in chapter 3 (centre crack with QBEM). Obviously, \(T(x)\) above has its root in the domain if \(\alpha < 0\), \(\beta > 0\) and \(g(x)\) is strictly positive. Hence \(T(x)\) does not satisfy conditions \((a)\) and \((b)\).

Without loss of generality, we assume that there is only a unique root in \([a,b]\) and \(x^* = b\). We now define a modified distribution function \(\hat{T}(x)\) of \(T(x)\) as
\[ \hat{T}(x) = \max \{ T(x), \varepsilon \} \quad (5.3.11) \]

and

\[ \varepsilon = \frac{V}{N^\gamma} \]

where \( V \) is a positive constant and \( 0 \leq \gamma \leq \min \{ 1, \beta \} \) and \( \gamma \neq 1 \).

**Theorem 5.4** For \( \beta > 0 \), the optimal mesh with respect to the modified distribution function (5.3.11) is \( k \)-regular with \( k = 1/(1 - \gamma)(\alpha + 1) \) provided that \( g(x) \in C^0[a, b] \) is strictly positive.

**Proof:** Let \( \{ x_i \} \) be the optimal mesh with respect to \( \hat{T}(x) \). Since \( \alpha < 0 \), \( \beta > 0 \) and \( \gamma > 0 \), then

\[ \hat{T}(x) = \max \{ (x-a)^\alpha (b-x)^\beta g(x), \frac{V}{N^\gamma} \} \]

\[ \leq \max \{ (b-a)^\beta g_m(x-a)^\alpha, V \} \quad (5.3.12) \]

where \( g_m = \max g(x) \). Hence for a large enough \( N \), there exists a positive constant \( C_i \) such that

\[ \hat{T}(x) \leq C_i (x-a)^\alpha. \quad (5.3.13) \]

By (5.2.9) and (5.3.13),

\[ \frac{A}{N} \leq \frac{C_i}{\alpha + 1} [(x_i - a)^{\alpha + 1} - (x_{i-1} - a)^{\alpha + 1}]. \quad (5.3.14) \]

Since \( \alpha < 0 \), the inequality (5.3.14) reduces to
\[
\frac{A}{N} \leq \frac{C_i}{a+1} h_i^{\alpha+1}
\]  \hspace{1cm} (5.3.15)

and \( h_i \) obtained from (5.3.15) is expressed as

\[
h_i = \left( \frac{A C_i}{N C_1} \right)^{i/(\alpha+1)} \hspace{1cm} i=1,2,\ldots,N.
\]  \hspace{1cm} (5.3.16)

Again using (5.2.9) gives

\[
\frac{A}{N} = \int_{x_{i-1}}^{x_i} \hat{T}(x)dx \geq \frac{V}{N^\gamma} h_i \hspace{1cm} i=1,2,\ldots,N.
\]

i.e.

\[
h_i \leq \frac{A}{V} \frac{1}{N^{1-\gamma}}.
\]  \hspace{1cm} (5.3.17)

Combining (5.3.16) and (5.3.17) leads to

\[
\frac{\text{(max } h_i)^k}{\text{min } h_i} \leq \frac{C_2}{N^{k(1-\gamma)-1/(\alpha+1)}}
\]  \hspace{1cm} (5.3.18)

where \( C_2 \) is a positive constant. Obviously, when

\[
k \geq \frac{1}{(1+\alpha)(1-\gamma)}
\]

the right hand term of (5.3.18) is bounded, i.e. the mesh is \( k \)-regular.

On the other hand, since \( \alpha > 0 \), for a large enough \( N \) and \( x \in [a, x_i] \),
\[ \hat{T}(\kappa) = \max \left\{ (x-a)^{\alpha} \beta (b-x)^{\beta} g(x), \frac{V}{N'} \right\} = (x-a)^{\alpha} \beta (b-x)^{\beta} g(x). \]

Replacing \( f(x) \) in (5.3.4) with \( (b-x)^{\beta} g(x) \), we obtain

\[
h_i = \left( \frac{A \alpha + 1}{N (b-x_i)^{\beta} f(x_1)} \right)^{1/(\alpha+1)} \quad (5.3.10)
\]

For \( x \in [x_{\underline{n-1}}, b] \),

\[ \hat{T}(\kappa) = \max \left\{ (x-a)^{\alpha} \beta (b-x)^{\beta} g(x), \frac{V}{N'} \right\} \geq \max \left\{ C_3 (b-x)^{\beta}, \frac{V}{N'} \right\} \]

(5.3.20)

where \( C_3 = (b-a)^{\alpha} g_m \), \( g_m = \min g(x) \). Then by (5.2.9)

\[
\frac{A}{N} \leq \int_{x_{\underline{n-1}}}^{x_1} \max \left\{ C_3 (b-x)^{\beta}, \frac{V}{N'} \right\} dx. \quad (5.3.21)
\]

Since

\[
\max \left\{ C_3 (b-x)^{\beta}, \frac{V}{N'} \right\} = \begin{cases} \frac{V}{N'} & z \leq x \leq b \\ C_3 (b-x)^{\beta} & a \leq x \leq z \end{cases}
\]

where

\[
z = b - \left( \frac{V}{C_3 N'} \right)^{1/\beta} \quad (5.3.22)
\]

If \( z \leq x_{\underline{n-1}} \), the inequality (5.3.21) reduces to

\[
\frac{A}{N} \leq h_N \frac{V}{N'}
\]

\[ \text{i.e.} \]
If \( z \geq x_{N-1} \), then the right hand term in (5.3.21) can be divided into two integrations and (5.3.20) can be expressed as

\[
\frac{A}{N} \leq \int_{x_{N-1}}^{z} C_3 (b-x)^{\beta} \, dx + \int_{z}^{b} \frac{V}{N^\gamma} \, dx
\]

and furthermore

\[
\frac{A}{N} \leq \frac{C_3}{\beta+1} \left[ (b-z)^{1+\beta} - h_N^{1+\beta} \right] + \frac{V}{N^\gamma} (b-z)
\]

i.e.

\[
\frac{C_3}{\beta+1} h_N^{1+\beta} \leq \frac{C_3}{\beta+1} (b-z)^{1+\beta} + \frac{V}{N^\gamma} (b-z).
\]  

(5.3.24)

Substituting (5.3.22) into (5.3.24)

\[
h_N \leq \left( \frac{\beta+1}{C_3} \frac{V}{N^\gamma} \left[ \frac{V}{C_3 N^\gamma} \right]^{1/\beta} + \left[ \frac{V}{C_3 N^\gamma} \right]^{1/(\beta+1)} \right)^{1/(1+\beta)}.
\]

(5.3.25)

Simplifying (5.3.25) gives

\[
h_N \leq (2+\beta)^{1/(1+\beta)} \left( \frac{V}{C_3} \right)^{1/\beta} \frac{1}{N^{\gamma/\beta}}.
\]

(5.3.26)

Substituting (5.3.26) into (5.3.21) gives

\[
\frac{A}{N} \leq \int_{x_{N-1}}^{b} \max \left( (2+\beta)^{1/(1+\beta)} \left( \frac{V}{C_3} \right)^{1/\beta} \frac{1}{N^{\gamma/\beta}} \right.
\]

\[
\left. \frac{V}{N^\gamma} \right)
\]

126

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
where $C_4$ is a positive constant and the $h_n$ obtained from (5.3.27) can be expressed as

$$h_n \geq \frac{A}{C_4} \frac{1}{N^{1-\gamma}}.$$  \hfill (5.3.28)

Combining (5.3.23) and (5.3.28), for any case

$$h_n \geq \frac{C_5}{N^{1-\gamma}}.$$  \hfill (5.3.29)

Finally, by using (5.3.19) and (5.3.29), we obtain

$$\frac{(\max h_i)^k}{\min h_i} \geq \frac{h_n^k}{h_i^k} \geq \frac{\text{const.}}{N^{k(1-\gamma)-1/(\alpha+1)}}.$$  \hfill (5.3.30)

The combination of the inequalities (5.3.18) and (5.3.30) means that the optimal mesh is $k$-regular with $k=1/(1-\gamma)(\alpha+1)$. \hfill \square

It should be noted that theorem 5.3 is a special case of theorem 5.4 when $V=0$ and $\beta=\gamma \to 0$, in which case, $T(x) = \hat{T}(x)$. In general, the parameters $\gamma$ and $V$ can be used to adjust the density of mesh distribution.

**Theorem 5.5** Letting $\beta>0$ and $g(x)\in C^0[a,b]$ be strictly positive, one has

$$\|\hat{T} - T\|_{L^1} = O\left(\frac{1}{N^{\gamma+\gamma/\beta}}\right).$$  \hfill (5.3.31)
Proof: Let \( \tilde{x}_m \) be the smallest root of the following equation

\[
TC(x) = (x-a)^\alpha (b-x)^\beta g(x) = \frac{V}{N^{\gamma}}
\]  

(5.3.32)
i.e.,

\[
(b-\tilde{x}_m) = \left( \frac{V}{N^{\gamma}} \frac{1}{(\tilde{x}_m-a)^\alpha g(\tilde{x}_m)} \right)^{1/\beta}.
\]  

(5.3.33)

Since

\[
||T - \tilde{T}||_{L^1} \leq \int_{\tilde{x}_m}^b |TC(x) - TC(x)| dx
\]

then

\[
||\hat{T} - T||_{L^1} \leq \int_{\tilde{x}_m}^b \left[ \frac{V}{N^{\gamma}} + (x-a)^\alpha (b-x)^\beta g(x) \right] dx
\]
i.e.

\[
||\hat{T} - T||_{L^1} \leq \left[ \frac{V}{N^{\gamma}} (b-\tilde{x}_m) + \frac{(\tilde{x}_m-a)^\alpha g(\tilde{x}_m) (b-\tilde{x}_m)^{\beta+1}}{N^{\gamma+\gamma/\beta}} \right]
\]  

(5.3.34)

where \( \tilde{x} \in (\tilde{x}_m, b) \). Substituting (5.2.31) into the inequality (5.3.34) gives

\[
||\hat{T} - T||_{L^1} \leq \frac{b_1}{N^{\gamma+\gamma/\beta}}
\]  

(5.3.35)

where \( b_1 \) is a positive constant.

On the other hand, letting \( \bar{x}_M \) be the largest root of equation (5.3.32), i.e.
\[ (b - \bar{x}_M) = \left( \frac{V}{N^\gamma} \frac{1}{(\bar{x}_M - a)^\alpha g(\bar{x}_M)} \right)^{1/\beta} \]  

(5.3.36)

then

\[ TC(x) \leq \frac{V}{N^\gamma} \quad \text{for } \bar{x}_M \leq x \leq b \]

Hence

\[ \|T - \hat{T}\|_L^2 \geq \int_{\bar{x}_M}^{b} \left( \frac{V}{N^\gamma} - TC(x) \right) dx \quad (5.3.37) \]

Since

\[ \int_{\bar{x}_M}^{b} TC(x) dx = \frac{(\hat{x} - a)^\alpha g(\hat{x})}{(\beta + 1)} (b - \bar{x}_M)^{\beta + 1} \]  

(5.3.38)

where \( \hat{x} \in (\bar{x}_M, b) \), substituting (5.3.38) into (5.3.37) reduces it to

\[ \|T - \hat{T}\|_L^2 \geq \left( \frac{V}{N^\gamma} - \left[ \frac{(\hat{x} - a)^\alpha g(\hat{x})}{(\beta + 1)} (b - \bar{x}_M)^{\beta + 1} \right] (b - \bar{x}_M) \right). \]  

(5.3.39)

By substituting (5.3.38) into (5.3.39), we obtain

\[ \|T - \hat{T}\|_L^2 \geq \left( V - \frac{(\hat{x} - a)^\alpha g(\hat{x})}{(\beta + 1)(\bar{x}_M - a)^\alpha g(\bar{x}_M)} \right) \frac{1}{N^\gamma} (b - \bar{x}_M)^{\beta + 1} \]  

(5.3.40)

Note that \( \hat{x}, \bar{x}_M \rightarrow b \) as \( N \rightarrow \infty \). Hence for a large enough \( N \),

\[ V - \frac{(\hat{x} - a)^\alpha g(\hat{x})}{(\beta + 1)(\bar{x}_M - a)^\alpha g(\bar{x}_M)} \geq b_2 > 0 \]  

(5.3.41)
where $b_2$ is a positive constant. Finally, substituting (5.3.38) and (5.3.41) into (5.3.40) gives

$$\|\hat{T} - T\|_{L^1} \geq \left( \frac{V}{(\frac{1}{2} - a)^{\alpha g(x_m)}} \right)^{1/\beta} \frac{b_2}{N^{\gamma + \gamma/\beta}}. \quad (5.3.42)$$

Now, (5.3.31) can be obtained directly by combining (5.3.42) and (5.3.35).

We are also able to extend the continuous dependence results for the optimal error and the optimal mesh presented in section 5.2 into the class of singular distribution functions. By (5.2.1) and theorem 5.5, we can prove the following corollary:

**Corollary** With the notation and hypothesis in theorem 5.5,

$$\frac{|E_{o}(\hat{T}) - E_{o}(T)|}{E_{o}(T)} \leq O\left( \frac{1}{N^{\gamma + \gamma/\beta}} \right) \quad (5.3.43)$$

Similarly, the extension of theorem 5.1 into the class of singular distribution functions is described below.

**Theorem 5.6** Let $\beta > 0$, $g(x) \in C^0[a,b]$ be strictly positive and $\{x_i\}$ and $\{\hat{x}_i\}$ be the two optimal meshes with respect to $T(x)$ and $\hat{T}(x)$ respectively. Then, there exists a constant $c > 0$ independent of $N$ such that

$$\max |x_i - \hat{x}_i| \leq \frac{c}{N^{\gamma + \gamma/\beta}}. \quad (5.3.44)$$

**Proof:** Replacing $T_1(x)$, $T_2(x)$, $\{x_i^1\}$ and $\{x_i^2\}$ with $T(x)$, $\hat{T}(x)$, $\{x_i\}$ and $\{\hat{x}_i\}$ respectively in the proof of theorem 5.1.
inequality (5.2.7) can be expressed as

\[ |x_i - \hat{x}_i| \leq \frac{2}{T(x_i)} \| T - \hat{T} \|_{L_1} \]  \hspace{1cm} (5.3.45)

where \( \bar{x}_i \in \{ \min(x_i, \hat{x}_i), \max(x_i, \hat{x}_i) \} \). Since \( T(x) \geq 2V/N^\gamma \), then (5.3.45) reduces to

\[ |x_i - \hat{x}_i| \leq \frac{2N^\gamma}{V} \| T - \hat{T} \|_{L_1}. \]  \hspace{1cm} (5.3.46)

Finally using the theorem 5.5 gives

\[ |x_i - \hat{x}_i| \leq \frac{2}{V} \cdot O\left(\frac{1}{N^{\gamma/\beta}}\right) \]  \hspace{1cm} (5.3.47)

The corollary of theorem 5.5 and theorem 5.6 shows that the mesh and error with respect to the modified distribution function are asymptotically optimal.
Chapter Six

Conclusions

In this thesis, we have discussed the stability of the Carey and Dinh's optimal algorithm and have presented a systematic simulation of the adaptive mesh redistribution algorithm. A combined adaptive h-r algorithm for the boundary element method is developed which is applicable to the Laplace equation and the equation of linear elasticity. Some important comments are summarized below:

1. The adaptive algorithms shown in this thesis can easily be incorporated into the commercial boundary element codes for the potential and elasticity equations. Numerical results have indicated that the CPU time of the optimal algorithm for Laplace's equation and linear elasticity equation constitute only five percent and two percent of the total CPU time, respectively.

2. By using these adaptive algorithms, considerable improvement in the solution quality can be made. This is particularly true for problems involving singularities (cracks). In fracture mechanics it is very important to obtain accurate information in the vicinity of the crack.

3. The deficiency of the r-method is that the error indicator converges to a minimum value and cannot be made
arbitrarily small. Therefore, the combined adaptive h-r algorithm exhibits a better performance when compared to the r-method. In this combined adaptive algorithm, the mesh distribution and the complexity of the problem are altered simultaneously such that the error indicator is minimized to fall below a given tolerance.

4. The theoretical analysis of the effect associated with the numerical approximation to the exact distribution function has been less explored [50]. The numerical experiments in this thesis indicate that the interpolation approximation of the numerical solution is reasonable.

5. For almost all adaptive boundary element algorithms known, the main drawback is the computing cost. The adaptive algorithms presented in this thesis are still expensive, compared with the adaptive finite element methods. A preliminary iterative scheme for solving the global linear system of equations in the case of constant boundary elements was explored by Schipper [53]. Further research has not been pursued. Perhaps one should focus on the cost of constructing the global matrix which is the main part in a three dimensional boundary element method for an adaptive fashion.

6. Another deficiency of the optimal mesh algorithm is its restriction to two dimensions. Since these algorithms are based on the concept of equidistribution of a certain error indicator [12,17,58], for the error function \( R(x,y) \) in two variables, the mesh on a two-dimensional surface which
equidistributes the error function surface is not unique and unstable although in some cases, the mesh can be determined [15]. Hence, the interpretation of the optimal mesh algorithm in the present form is limited and needs further research for implementation in three-dimensional boundary elements.
Chapter Seven

Appendix I

7.1 The Proof of Theorem 2.1

Theorem 2.1 With the notation in chapter 2, if \( W(x) \in C^{m,k+1} [a,b] \) and the distribution function \( T(x) = [W^{(k+1)}(x)]^2 \) is strictly positive, the global error (on the \( H^m \)-seminorm, \( m \leq k \)) can be expressed as

\[
|e^{(m)}|^2 = C \int_a^b \frac{[W^{(k+1)}(x)]^2}{(T(x))^{2(k+1-m)}} \, dx + o(x) \tag{7.1.1}
\]

where \( C \) is a constant, \( T(x) \) is the grading function defined in chapter 1 and \( o(x) \) denotes an infinitesimal quantity of the term ahead it as \( N \to \infty \).

Proof: Differentiate (2.2.1)

\[
e^{(m)}(x) = \sum_{p=0}^{m} \left( \frac{d^p W^m}{dx^p} \right) \frac{d^{m-p}}{dx^{m-p}} \prod_{j=0}^{k} (x - x_j) \tag{7.1.2}
\]

where

\[
W^m(x) = W(x_0, x_1, \ldots, x_k, x)
\]

By the lemma 2.2.
\[
\frac{d^p W^*}{dx^p} = \frac{m!}{(m+k+1)!} \frac{d^{(p+k+1)}}{dx^{p+k+1}} W(\eta) \quad (7.1.3)
\]

where \( \eta \in (x_{i-1}, x_i) \). Since \( W(\cdot) \in C^{m+k+1}[a,b] \), equation (7.1.3) leads to
\[
\left| \frac{d^p W^*}{dx^p} \right| \leq M \quad \text{for } p \leq m, \quad (7.1.4)
\]

where \( M \) is a positive constant.

On the other hand,
\[
\frac{d^{m-p}}{dx^{m-p}} \prod_{j=0}^{k} (x - x_i^j) = (m-p)! \sum_{\alpha_l \in I_k^q} \prod_{l=1}^{q} (x - x_i^l)
\]
\[
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (7.1.5)
\]

where \( q = k - m + p + 1 \) and \( I_k^q \) is a set of all combinations of \( q \) entries in \( \{0, 1, \ldots, k\} \).

For any \( x \in (x_{i-1}, x_i) \) and \( q > k - m + 1 \), i.e. \( p > 0 \), obviously we have
\[
\sum_{\alpha_l \in I_k^q} \prod_{l=1}^{q} (x - x_i^l) = o \left( \sum_{\alpha_l \in I_k^{k-m+1}} \prod_{l=1}^{k-m+1} (x - x_i^l) \right).
\]

By (7.1.5), for \( p > 0 \)
\[
\left( \frac{d^{m-p}}{dx^{m-k}} \prod_{j=0}^{k} (x - x_i^j) \right) = o \left( \frac{d^{m}}{dx^{m}} \prod_{j=0}^{k} (x - x_i^j) \right). \quad (7.1.6)
\]

By the property of divided differences,
\[ W^*(x) = \frac{1}{(k+1)!} W^{(k+1)}(\tilde{\eta}) \]  
\[ \text{where } \tilde{\eta} \in [x_{i-1}, x_i). \]

Noting that \( W^*(x) \) is strictly positive in \([a, b]\) and combining (7.1.4) and (7.1.6) gives

\[ \left( \frac{d^p W}{dx^p} \right) \left( \frac{d^{m-p}}{dx^{m-p}} \prod_{j=0}^{k} (x - x_i^j) \right) = o \left( \frac{d^m}{dx^m} \prod_{j=1}^{k} (x - x_i^j) \right) \]

for \( p > 0 \).  
\[ (7.1.8) \]

Substituting (7.1.8) into (7.1.2), the global error can be expressed as

\[ e^{(m)}(x) = W^*(x) \left( \frac{d^m}{dx^m} \prod_{j=0}^{k} (x - x_i^j) \right) + o(x) \]

Furthermore, by (7.1.5) for \( p = 0 \), we obtain

\[ e^{(m)}(x) = W^*(x) \cdot k! \left( \sum_{\alpha \in k} \prod_{l=0}^{k-m+1} (x - x_i^l) \right) + o(x) \]

Therefore the error in subinterval \([x_{i-1}, x_i]\) is

\[ \left| e^{(m)} \right|_i^k = \left( \frac{m!}{k!} \right)^2 \left( W^*(x) \right)^2 \left( \sum_{\alpha \in k} \prod_{l=1}^{k-m+1} (x - x_i^l) \right)^2 dx \]

\[ + o(x). \]

By the mean value theorem,
\[ \left| e^{(m)} \right|_{i}^{2} = \left[ W^{k} \right] (\xi_{i})^{m!} \left( \sum_{\alpha_{l} \in \Gamma_{i}^{k}} \prod_{l=1}^{k-m+1} (x - x_{l}) \right) dx \]

where \( \xi_{i} \in (x_{i-1}, x_{i}) \). Using the transformation

\[ (x-x_{i}) = t_{i}, \]

where \( t_{i} = (x_{i} - x_{i-1}), \) and noting

\[ (x - x_{i}) = (x - x_{i-1}) - (x_{i} - x_{i-1}) \]

we obtain

\[ \prod_{l=1}^{k-m+1} (x - x_{i}) = \prod_{l=1}^{k-m+1} \left( t_{i} - (h_{i1} + h_{i2} + \ldots + h_{il}) \right) \]

i.e.

\[ \prod_{l=1}^{k-m+1} (x - x_{i}) = h_{i} \prod_{l=1}^{k-m+1} \left( t_{i} - \sum_{s=1}^{l} \frac{h_{is}}{h_{i}} \right) \]

where \( h_{is} = (x_{i}^{s} - x_{i-1}^{s-1}) \). Substituting (7.1.10) into (7.1.9) leads to

\[ \left| e^{(m)} \right|_{i}^{2} = \left[ W^{k} \right] (\xi_{i})^{m!} 2^{k-m+1} \left( t_{i} - \sum_{s=1}^{l} \frac{h_{is}}{h_{i}} \right) \]

\[ + \alpha(x) \]
Letting

\[ H(ch_i, h_i) = \int_0^1 \left( \sum_{(\alpha_k) \in k} \left( t - \sum_{s=1}^{a_k} \frac{h_{is}}{h_i} \right) \right)^2 dt \]

then

\[ \|e^{(m)}\|^2 = \left[ W(\xi_i) \right]^2 h_i^{2(k+1-m)+1} + o(\xi) \]  \hspace{1cm} (7.1.12)

By the definition of grading function

\[ h_{is} = \frac{1}{nk} \frac{1}{\xi'(\overline{x}_i)} \]  \hspace{1cm} (7.1.13)

where \( \overline{x}_i \in (x_i^{s-1}, x_i^s) \) and

\[ h_i = \sum_{s=1}^{k} h_{is} = \frac{1}{n} \frac{1}{\xi'(\overline{x}_i)} \]  \hspace{1cm} (7.1.14)

where \( \overline{x}_i \in (x_i, x_{i+1}) \). By using (7.1.14) and (7.1.13), \( H(ch_i, h_i) \) can be expressed as

\[ H = \int_0^1 \left( \sum_{(\alpha_k) \in k} \left( t - \sum_{s=1}^{a_k} \frac{\xi'(\overline{x}_i)}{k} \frac{1}{\xi'(\overline{x}_i)} \right) \right)^2 dt \]

and furthermore

\[ H = \int_0^1 \left( \sum_{(\alpha_k) \in k} \left( t - \frac{\xi'(\overline{x}_i)}{k} \sum_{s=1}^{a_k} \frac{1}{\xi'(\overline{x}_i)} \right) \right)^2 dt \]

i.e.

139
where \( \hat{x}_i \in \langle x_{i-1}, x_i \rangle \). Substituting (7.1.15) and (7.1.14) into (7.1.12), we obtain

\[
\|e^{(m)}\|^2 = \left( \frac{1}{\xi'(\hat{x}_i)} \right)^{2(k+1-m)} \sum_{i=1}^{N} \left( W^m(\zeta_i) \right)^2 \left( \frac{1}{\xi'(\hat{x}_i)} \right)^{2(k+1-m)} h_i \rho \left( x_i^2, \bar{x}_i, \xi' \right) + o(\infty) \cdot (7.1.16)
\]

Adding the two sides of equation (7.1.16) respectively for \( i=1, 2, \ldots, N \) lead to

\[
\|e^{(m)}\|^2 = \frac{1}{n^{2(k+1-m)}} \sum_{i=1}^{N} \left( W^m(\zeta_i) \right)^2 \left( \frac{1}{\xi'(\hat{x}_i)} \right)^{2(k+1-m)} h_i \\
\rho \left( x_i^2, \bar{x}_i, \xi' \right) + o(\infty). \cdot (7.1.17)
\]

By equality (7.1.17),

\[
W^m(\zeta_i) = \frac{1}{(k+1)!} W^{(k+1)}(\zeta_i)
\]

where \( \zeta_i \in \langle x_{i-1}, x_i \rangle \). Since \( x_i^2, \bar{x}_i, \hat{x}_i \) and \( \zeta_i \in \langle x_{i-1}, x_i \rangle \), as \( N \to \infty \).

\[
H = \int_0^1 \left( \sum_{\zeta_i \in \Gamma_k} \prod_{l=1}^{k-m+1} \left( t - \frac{a_l \xi'(\hat{x}_i)}{k \xi'(\hat{x}_i)} \right) \right)^2 dt
\]

140

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
where \( H^* \) is a constant independent of index \( i \). Hence the right hand of (3.1.19) can be expressed in an integral form, i.e.

\[
\|e^{(m)}\|^2 = \frac{H^*}{n^{2(k+1-m)}} \left( \frac{m!}{(k+1)!} \right)^2 \int_a^b [W^{(k+1)}(x)]^2 \left( \frac{1}{\xi',(x)} \right)^{2(k+1-m)} dx + o(n) \quad (7.1.18)
\]

Let

\[
C = \frac{H^*}{n^{2(k+1-m)}} \left( \frac{m!}{(k+1)!} \right)^2
\]

At this point, the theorem is immediately derived from (3.1.20).

7.2 The Extension of Mean Value Theorem

**Theorem 7.1**  Let \( f(x) \in C^0[a,b] \), \( g(x) \geq 0 \) and \( f(x)g(x) \) be integrable in interval \([a,b]\) and \( \lim_{x \to a} f(x) = +\infty \). Then there exists \( \xi \in (a,b) \) such that

\[
\int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx \quad (7.2.1)
\]

**Proof:** Let
\[ f(x_i) = \min_{x \in [a, b]} f(x) \]

Then
\[ \int_a^b f(x)g(x) \, dx \geq f(x_i) \int_a^b g(x) \, dx \]  \hspace{1cm} (7.2.2)

Since \( \lim_{x \to a} f(x) = +\infty \) and \( g(x) \geq 0 \), there exists \( x_2 \in (a, b) \) such that
\[ \int_a^b f(x)g(x) \, dx \leq f(x_2) \int_a^b g(x) \, dx \]  \hspace{1cm} (7.2.3)

By the continuity of \( f(x) \) in \([x_1, x_2]\) and the intermediate value theorem, combining (7.2.2) and (7.2.3) results in (7.2.1). □

7.3 The Basic Formulation of The Boundary Element Method

In this section we attempt to present a general formulation for the boundary integral equations arising from elliptic boundary value problems. As special cases, the boundary integral equations for the potential and the elasticity problems are considered.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n=2 \) or 3, with a sufficiently smooth boundary \( \partial \Omega \). The results for unbounded domains can be similarly obtained under some additional growth conditions at infinity.

Consider a second order linear partial differential equation with the appropriate boundary conditions

\[ Lu = 0 \quad \text{in} \quad \Omega \]
\[ B u = g \quad \text{on } \Omega \]  

where \( L \) is a second order linear differential operator, \( B \) is a boundary differential operator and \( u : \overline{\Omega} \subset \mathbb{R}^N \rightarrow H \) where \( H \) is a finite dimensional real Hilbert space.

Let \( \{ e_\nu \} \) be an orthonormal basis in the Hilbert space \( H \) and \( u_\nu^* : \overline{\Omega} \mapsto \langle \nabla, \nabla \rangle \vert \nabla \rightarrow H \) be the fundamental solutions of operator \( L \), i.e., \( u_\nu^* \) satisfies

\[ L^* u_\nu^* (x, y) = \delta(x-y) e_\nu \]  

where \( \delta \) is the Dirac distribution function. \( L^* \) is the adjoint operator of \( L \). Hence

\[ \langle u_\nu^*, Lu \rangle - \langle L^* u_\nu^*, u \rangle = \text{div} \, AC(u, u_\nu^*) \]  

where \( \langle , \rangle \) denotes the inner product and \( AC(u, u_\nu^*) \) is the bilinear concomitant.

Let \( x \in \Omega \). Integrating both sides of equation (7.3.3) over \( \Omega \) and using the divergence theorem,

\[ \int_{\Omega} \langle u_\nu^*(x, y), Lu(y) \rangle - \langle L^* u_\nu^*(x, y), u(y) \rangle \, dV(y) \]

\[ = \int_{\partial \Omega} AC(u(y), u_\nu^*(x, y)) \cdot n(y) \, ds(y) \]  

where \( n(y) \) is the unit outward normal to \( \Omega \) at the point \( y \) and the dot denotes scalar product in \( \mathbb{R}^N \). Using (7.3.1) and (7.3.2), (7.3.4) can be simplified as
\( \langle e_p, u \rangle = \int_{\partial \Omega} A(u, u^*) \cdot n \, ds \) \hspace{1cm} (7.3.5)

If the operator \( L \) has an explicit form

\[
L = \left( \sum_{i,j} \frac{\partial}{\partial x_i} (a_{i,j} \frac{\partial}{\partial x_j}) + \sum_i b_i \frac{\partial}{\partial x_i} + c \right)
\]

(7.3.6)

where \( a_{i,j}, b_i \) and \( c \) are linear operators on \( H \), then (7.3.5) can be replaced by

\[
\langle e_p, u \rangle = \int_{\partial \Omega} \langle u^*, D_u \rangle ds - \int_{\partial \Omega} \langle D_u^*, u \rangle ds + \int_{\partial \Omega} \langle u^*, B_u \rangle ds
\]

(7.3.7)

where

\[
D_n = \sum_{i,j} a_{i,j} \frac{\partial}{\partial y_j} n_i, \quad D_n = \sum_{i,j} a_{i,j}^* \frac{\partial}{\partial y_j} n_i
\]

and

\[
B_n = \sum_i b_i n_i
\]

The \( \langle e_p, u \rangle \) in (7.3.7) can be expressed as

\[
\langle e_p, u \rangle = \sum_{p,q} c_{pq} (x) u_q (x)
\]

(7.3.8)

where \( c_{pq} (x) \) is given by

\[
c_{pq} (x) = -\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \langle D_u^*, e \rangle ds
\]

where \( \Gamma_\varepsilon \) is the segment of sphere of radius \( \varepsilon \) around point \( x \). For \( x \in \text{Int} \Omega \) where \( \text{Int} \Omega \) is the interior of \( \Omega \),

\[
c_{pq} (x) = 0.
\]

For \( x \in \partial \Omega \), \( c_{pq} (x) \) is dependent upon the smoothness of the
boundary \( \partial \Omega \). If the boundary is smooth in a small neighbourhood of point \( x \), then

\[
c_{pq}(x) = \delta_{pq}.
\]

Hence, in this case we obtain a boundary integral equation as following

\[
\frac{1}{2} u(x) = \int_{\partial \Omega} \langle u^*(x, y), D u(y) \rangle ds(y)
\]

\[
- \int_{\partial \Omega} \langle \mathbf{D}_n u^*(x, y), u(y) \rangle ds(y)
\]

\[
+ \int_{\partial \Omega} \langle u^*(x, y), \mathbf{B} u(y) \rangle ds(y) \quad (7.3.9)
\]

The equation (7.3.9) is the fundamental equation in the boundary element method.

If \( u \) is prescribed on the boundary (Dirichlet condition), equation (7.3.9) is the well-known Fredholm integral equation of the first kind. If \( D_n u \) is prescribed on the boundary (Neumann condition), the equation (7.3.9) is the Fredholm integral equation of the second kind. A more common case is a mixed boundary condition in which on part of \( \partial \Omega \), \( u \) is prescribed and on the rest of \( \partial \Omega \), \( D_n u \) is given.

We now consider two special cases.

(a) Potential problem

If the operator \( L \) is the Laplace operator on the scalar valued function \( u: \bar{\Omega} \subset \mathbb{R}^N \rightarrow \mathbb{R}^t \), the equation (7.3.1) is a Laplace equation, namely
\[ \Delta u = 0 \quad \text{in } \Omega \]

\[ u|_{\partial \Omega_1} = f \]

\[ \frac{\partial u}{\partial n}|_{\partial \Omega_2} = g \]

where \( \partial \Omega_1 + \partial \Omega_2 = \partial \Omega \). In this case, the fundamental solution of operator \( \Delta \) can be expressed as

\[ u^*(x,y) = \frac{1}{2\pi} \ln \frac{1}{r_{xy}} \]  \( \text{(7.3.11)} \)

in two dimensions (\( N=2 \)) and

\[ u^*(x,y) = \frac{1}{(N-2)\omega_{N-1}} \ln \frac{1}{r_{xy}} \]  \( \text{(7.3.12)} \)

in \( N > 2 \) dimensions, where \( r_{xy} = |x-y| \) is the distance between the points \( x \) and \( y \) in \( \mathbb{R}^N \) and \( \omega_{N-1} \) is the surface area of the sphere in \( \mathbb{R}^N \).

(b) Linear elasticity

If \( L \) is the Lame operator, equation (7.3.1) is the Navier equation

\[ Lu = \mu \Delta u + (\lambda+\mu)\text{grad } \text{div}(u) = 0 \quad \text{in } \Omega \]  \( \text{(7.3.13)} \)

where \( \mu > 0 \) and \( \lambda > -2\mu/N \) are Lame constants and \( u(x) = (u_1, u_2, \ldots, u_N) \) is the displacement vector at \( x \in \Omega \) and the boundary conditions can be expressed as

\[ u|_{\partial \Omega_1} = f \]

\[ p|_{\partial \Omega_2} = g \]
where \( p \) is boundary traction

In this case, the fundamental solution of the Lame operator for two-dimensional plane strain is

\[
\mathbf{u}^*(x,y) = \frac{1}{8\pi \mu(1-\nu)} \left[(3-4\nu)\ln \frac{1}{r} \delta_{ij} + \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_j}\right]
\]

and

\[
p^*_{ik} = -\frac{1}{4\pi (1-\nu) r} \left[\frac{\partial r}{\partial n} \left((1-2\nu) \delta_{ik} + 2 \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_k}\right)
- (1-2\nu) \left[\frac{\partial r}{\partial x_i n_k} - \frac{\partial r}{\partial x_k n_i}\right]\right]
\]

where \( \delta_{ij} \) is the Kronecker delta, \( \nu \) and \( G \) are Poisson's ratio and shear modulus, respectively. For plane stress, \( \nu \) is replaced by \( \bar{\nu} = \nu/(1+\nu) \).

In this thesis, we only concentrate on the two dimensional problem. Without the body force and interior sources, the integral equations in the two cases can be expressed as

\[
c(x,\alpha) u(x) + \int_{\Gamma} p^*(x,y) u(y) \, dy = \int_{\Gamma} u^*(x,y) p(y) \, dy \quad (7.3.16)
\]

where \( x \in \Gamma \) is the arc length coordinate and \( p(x) \) is the normal derivative of function \( u(x) \). The variable \( p^*(x,y) \) is also the normal derivative of \( u^*(x,y) \) on the boundary. \( c(x) \) is a constant and only dependent upon the local geometrical smoothness of the boundary. If the boundary is smooth in a small neighbourhood of point \( x \), then \( c(x) = 1/2 \).
For a given partition \( \{x_i\}, i=1,2,\ldots,n \) on \( \Gamma \), let \( \{x_i^j\} 
\) be a further partition of \( \Gamma_i \) where \( \sum \Gamma_i = \Gamma \). Approximating \( u(x) \) and \( p(x) \) through piecewise interpolation polynomials of degree \( k \),

\[
    u(x) \approx \sum_j B_i^j(x)u_i^j = B_i^T u_i \quad (7.3.17)
\]

\[
    p(x) \approx \sum_j B_i^j(x)p_i^j = B_i^T p_i \quad (7.3.18)
\]

where \( B_i^j(x) \), \( j = 0, 1, \ldots, k \), are the basis functions of interpolation of order \( k \) in the subinterval \( [x_{i-1}, x_i] \) satisfying

\[
    B_i^j(x) = \delta_{jk}.
\]

Replacing \( u(x) \) and \( p(x) \) in \( (3.2.16) \) with \( (3.2.17) \) and \( (3.2.18) \), respectively, leads to

\[
    c(x)u(x) + \sum \int p^*(x,y)B_i^j(x)\tau u_i^j dy = \sum \int u^*(x,y)B_i^j(x)\tau p_i^j dy. \quad (7.3.19)
\]

Imposing \( (3.2.18) \) at \( x=x_i^j \),

\[
    c_i^j u_i^j + \sum_{i,j} \int p^*(x_i^j,y)B_i^j(y)\tau d y \ u_i^j = \sum_{i,j} \int u^*(x_i^j,y)B_i^j(y)\tau d y \ p_i^j \quad (7.3.20)
\]

Letting
\[ H_i^j = \int_{\Gamma_i} p(x_i, y) B^i(y^T) dy \]

\[ G_i^j = \int_{\Gamma_i} u(x_i, y) B^i(y^T) dy \]

Equation (7.3.20) can be expressed as

\[ c_i^j u_i^j + \sum_{i,j} H_i^j u_i^i = \sum_{i,j} G_i^j p_i \]  

(7.3.21)

Assembling equation (7.3.21) for \( i = 1, 2, \ldots, n \) and \( j = 0, 1, \ldots, k \)

(7.3.21) can be expressed in a matrix form as

\[ H U = G P. \]  

(7.3.22)

By imposing the prescribed boundary conditions of the problem, equation (7.3.22) can be rearranged in such a way that a final linear system of algebraic equations is easily obtained

\[ A X = b \]  

(7.3.23)

where \( A \) is, in general, a fully populated and unsymmetric matrix of order \( nk \) and \( X \) is a vector containing all the boundary unknowns. The discretized boundary element solution can be obtained by solving the system of linear algebraic equations (7.3.23) via direct or iterative methods.

7.4 An Interpolation Error Analysis of the Cauchy Mean Value Integral.
In this section, we will prove the formula (4.2.12) which is an important tool in section 4.2. Generally, we can consider the following Cauchy mean value integral

\[ I(\bar{x}) = \int_a^b F(x) \frac{G(x)}{x-\bar{x}} \, dx \]

where \( \bar{x} \in [a,b] \). The conditions of \( F(a) = F(b) \) and \( G(a) = G(b) \) are satisfied and therefore, without a loss of generality, we can assume that \( \bar{x} \in (a,b) \).

For the given partitions \( \Pi \{ x \} \) on \([a,b] \), we use a linear approximation to \( F(x) \) where \( G(x) / (x-\bar{x}) \) is considered as a weight function. Then the interpolation error can be expressed as

\[ \varepsilon(\bar{x}) = \int_a^b \omega(\bar{x}) \frac{G(x)}{x-\bar{x}} \, dx \quad (7.4.1) \]

where

\[ \omega(\bar{x}) = (x-x_{j-1})(x-x_j) F(x_{j-1},x_j) \quad x \in [x_{j-1},x_j] \]

\[ j=1,2,...,N. \]

In order to simplify the notation we rewrite \( \varepsilon(\bar{x}) \) as follows

\[ \varepsilon(\bar{x}) = \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} \omega(\bar{x}) \frac{G(x)}{x-\bar{x}} \, dx \]

The main theorem is described below.

**Theorem 7.2** With the above notation, if the following conditions are satisfied

1. \( G(x) \in C^4[a,b] \) and \( F(x) \in C^3[a,b] \).

2. The partition \( \Pi \{ x \} \) satisfies Lipschitz's condition as follows
where \( h = \text{max} h_i \) and \( C \) is a constant independent on the partition \( \Pi_N \langle x_j \rangle \).

3. \( x = x_p \) is a grid point, then

\[
E \Delta \equiv \varepsilon (\Delta) - \sum \frac{h^3}{12} F(C) \int_{x_{j-1}}^{x_j} \frac{G(\Delta)}{x-x} \, dx \equiv o(h^2) \quad (7.4.2)
\]

where \( \bar{x}_j = (x_{j-1} + x_j) / 2 \).

**Proof:** Let \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \) satisfy

\[
\bar{x} + \varepsilon = x_{p+q}, \quad \bar{x} - \varepsilon = x_{p-q}
\]

Then \( E \Delta \) can be expressed as

\[
E \Delta \equiv \left( \sum_{j \leq p-q} \int_{x_{j-1}}^{x_j} (x-x_{j-1}) (x-x_j) F(x_{j-1}, x_j, x) \frac{G(\Delta)}{x-x} \, dx \right)
\]

\[
+ \sum_{j \geq p+q} \frac{h^3}{12} F(C) \int_{x_{j-1}}^{x_j} \frac{G(\Delta)}{x-x} \, dx \right) \]

\[
+ \sum_{p-q < j \leq p+q} \int_{x_{j-1}}^{x_j} (x-x_{j-1}) (x-x_j) F(x_{j-1}, x_j, x) \frac{G(\Delta)}{x-x} \, dx
\]

\[
+ \sum_{p-q < j \leq p+q} \frac{h^3}{12} F(C) \int_{x_{j-1}}^{x_j} \frac{G(\Delta)}{x-x} \, dx \right) \]

\[
\equiv E_1 + E_2 + E_3 \quad (7.4.3)
\]

For the first integral in (7.4.3), using the Mean Value Theorem leads to
\[
E_1 = \left\{ \sum_{j=p+q}^{x_j} \int_{x_{j-1}}^{x_j} \left( \frac{G(x)}{x-x_j} \right) F[x_{j-1}, x, x_j] \frac{G(x)}{x-x_j} \right\} dx
\]

\[
+ \sum_{j=p+q}^{h^3 \frac{1}{12} \int_{x_j}^{x_{j-1}} \frac{G(x)}{x-x_j} \frac{G(x)}{x-x_j} \right\} dx
\]

\[
= \sum_{j=p+q}^{h^2 \frac{1}{12} \int_{x_j}^{x_{j-1}} \frac{1}{x-x_j} \frac{1}{x-x_j} \right\} dx
\]

\[
+ \sum_{j=p+q}^{h^2 \frac{1}{12} \left( \int_{x_j}^{x_{j-1}} \frac{1}{x-x_j} \frac{1}{x-x_j} \right)} \int_{x_j}^{x_{j-1}} \frac{1}{x-x_j} \frac{1}{x-x_j} \right\} \left(7.4.4\right)
\]

Since

\[
\int_{x_{j-1}}^{x_j} \frac{(x-x_{j-1})(x-x_j)}{x-x_j} dx = -h^2 \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \left( \frac{h_j}{x_j-x} \right)^n
\]

and

\[
\int_{x_{j-1}}^{x_j} \frac{1}{x-x_j} dx = \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{h_j}{x_j-x} \right)^n
\]

(7.4.4) can be simplified as

152
\[ E_1 = \sum_{j < p-q} \left( I_n^C \alpha_j \sum_{j \geq p+q} \left( \frac{h_j^2}{(x_j - \overline{x})^3} \right)^n \right) + \sum_{j < p-q} h_j \ln \left| 1 - \frac{h_j}{x_j - \overline{x}} \right| . \]  

Let \( \varepsilon_2 = C_1 h^T \) with \( T > 0 \) and \( C_1 > 0 \) dependent on the partition, chosen such that \( \overline{x} - \varepsilon_2 \) is a grid point. Clearly, \( T \geq C_1 > 0 \) for a quasi-uniform mesh. Since \( \overline{x} \in (a, b) \), we can assume that \( \overline{x} - \varepsilon_2 \geq a \) and \( \overline{x} - \varepsilon_2 = x_{p-q} \) (\( p > q \)). Let \( \varepsilon_1 = x_{p+q} - \overline{x} \). Without loss of generality, we can also assume that \( \varepsilon_2 \leq \varepsilon_1 \). Therefore, \( h_j / x_j - \overline{x} \geq h_j / \varepsilon_2 \) and there exists a constant \( C > 0 \) such that

\[ |E_1| \leq C_2 \sum_{j < p-q} \left( h_j^2 \left( \frac{h_j}{\varepsilon_2} \right)^3 + h_j^3 \frac{h_j}{\varepsilon_2} \right) = C_2 \sum_{j = 1}^{N} O(h_j^{3-T} + h_j^{4-T}). \]  

The second integral in (7.4.3) can be rewritten as

\[ E_2 = \sum_{j \geq p+q} \int_{x_{j-1}}^{x_j} (x - x_{j-1}) (x - x_j) F(x_{j-1}, x_j, x) \frac{G(x)}{x - \overline{x}} \, dx \]

By the Mean Value Theorem, we have

\[ E_2 = \sum_{j \geq p+q} \frac{h_j^3}{2} \int_{0}^{1} \frac{t(1-t)}{x_{j-1} + th_j - \overline{x}} f(x_{j+1} + th_j - \overline{x}_j + th_j - \overline{x}) \, dt \]

where \( f(x) = F(x_{j-1}, x_j, x) \) and therefore,

\[ E_2 = \sum_{i = 1}^{q} \frac{h_{p+i}^3}{2} \int_{0}^{1} \frac{t(1-t)}{x_{p+i-1} + th_{p+i} - \overline{x}} f(x_{p+i-1} + th_{p+i} - \overline{x}_{p+i-1} + th_{p+i} - \overline{x}) \, dt \]

153

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
where
\[ f^+ = f(x_{p+i-1} + th_{p+i} - \infty) \quad G^+ = G(x_{p+i-1} + th_{p+i} - \infty) \]
\[ f^- = f(x_{p+i+1} - th_{p+i} - \infty) \quad G^- = G(x_{p+i+1} - th_{p+i} - x_{p+i+1}) \].

By the condition 2, we have
\[ |h^3_{p+i} - h^3_{p+i+1}| \leq C_2 h^3 (x_{p+i} - x_{p+i+1}) \]
and therefore, there exists \( C_3 > 0 \) such that
\[ \left| \frac{h^3_{p+i} - h^3_{p+i+1}}{2} \right| \int_0^1 \frac{t(1-t)}{x_{p+i-1} + th_{p+i} - x} f^+ G^+ dt \]
\[ \leq -C_3 h^2 (x_{p+i} - x_{p+i+1}) \ln 1 \frac{h_{p+i}}{x_{p+i} - \infty} \right| < 2C_3 h^3. \]  \( (7.4.8) \)
By the continuity of $F(x)$ and $G(x)$,

$$\left| \int_0^t \frac{t(1-t)}{x_{p+i-1} + th_{p+i} - x} (f^+ - G^-) \right| \leq C_2 \int_0^t \frac{t(1-t)}{x_{p+i-1} + th_{p+i} - x} \left( x_{p+i-1} - x_{p-i} + th_{p+i} - h_{p-i+1} \right) \leq C_2$$  \hspace{1cm} (7.4.8)

In addition, the third integral in (7.4.7) can also be rewritten as

$$\int_0^t \left( \frac{t(1-t)}{x_{p+i-1} + th_{p+i} - x} - \frac{t(1-t)}{x_{p+i-1} + th_{p+i} - x} \right) f^+ G^- dt$$

$$= \int_0^t \frac{(x_{p+i-1} - x_p) + th_{p+i} - h_{p+i}}{(x_{p+i-1} + th_{p+i} - x_p) + th_{p+i} - h_{p+i}} dt. \hspace{1cm} (7.4.10)$$

By the condition 2, there exists $C_4 > 0$ such that

$$\sum_{k=1}^{i} h_{p-k+1} (x_{p+k} - x_{p-k+1}) \leq C_4 (x_{p+i-1} - x_{p-i+1}) (x_{p+i-1} - x_{p-i+1})$$

and therefore, (7.4.10) can be simplified as

$$\int_0^t \left( \frac{t(1-t)}{x_{p+i-1} + th_{p+i} - x} - \frac{t(1-t)}{x_{p+i-1} + th_{p+i} - x} \right) f^+ G^- dt$$

$$\leq \int_0^t \frac{C_3 (x_{p+i-1} - x_{p-i+1}) (x_{p+i-1} - x_{p-i+1}) + th_{p+i} - h_{p+i}}{(x_{p+i-1} + th_{p+i} - x) + th_{p+i} - h_{p+i}} f^+ G^-$$
\[ \leq \int_0^t (1-t) \left( 2C_g + \frac{2h}{h_{p-i+1}} \right) f'' G' \, dt \leq C_5 . \quad (7.4.11) \]

Substituting (7.4.8), (7.4.9) and (7.4.11) into (7.4.7), we obtain

\[ E_2 = -\sum_{i=1}^{q} 2C_g h^i + \sum_{i=1}^{q} \frac{h^i}{2} C_2 + \sum_{i=1}^{q} \frac{h^i}{2} C_5 \]

\[ \leq O(h^{2+T}) + O(h^{2+T}) + O(h^{2+T}) = O(h^{2+T}). \quad (7.4.12) \]

For the third integral \( E_3 \) in (7.4.3), the analysis is similar to \( E_2 \) such that

\[ |E_3| \leq O(h^{2+T}). \quad (7.4.13) \]

Now, substituting (7.4.8), (7.4.12) and (7.4.13) into (7.4.3) results in

\[ E(\tilde{\infty}) \leq O(h^{5-3T} + h^{4-T}) + O(h^{2+T}) + O(h^{2+T}) \]

If we choose \( \tau > 0 \), the above equation leads to the conclusion of the theorem, namely,

\[ E(\tilde{\infty}) = O(h^2) \quad \square \quad (7.4.14) \]
Chapter Eight
Appendix II

8.1 Fundamental Equations in Linear Elasticity

In this section, some important equations in linear elasticity are presented which can be found in [59].

The differential equations of equilibrium for two dimensions (neglecting body force) can be expressed by

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \]  
(8.1.1)

\[ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \]  
(8.1.2)

and the compatibility equation is

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left( \sigma_x + \sigma_y \right) = 0 \]  
(8.1.3)

Equations (8.1.1)-(8.1.3) are valid for both plane stress and plane strain. Let \( \phi \) be stress function which is related to the stress as follows

\[ \frac{\partial^2 \phi}{\partial y^2} = \sigma_x, \quad \frac{\partial^2 \phi}{\partial x^2} = \sigma_y, \quad -\frac{\partial^2 \phi}{\partial x \partial y} = \tau_{xy} \]  
(8.1.4)

and therefore, stress function satisfies

\[ \frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \]  
(8.1.5)

Let \( u \) and \( v \) be displacements in x-direction and y-direction
respectively. In the case of plane stress, we have

\[
\frac{\partial u}{\partial x} = -\frac{1}{E} (\sigma_x - \nu \sigma_y), \tag{8.1.6}
\]

\[
\frac{\partial v}{\partial y} = \frac{1}{E} (\sigma_y - \nu \sigma_x), \tag{8.1.7}
\]

\[
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{G} \tau_{xy}. \tag{8.1.8}
\]

Let \( p_x \) and \( p_y \) be traction on the boundary defined by

\[
p_x = \sigma_x n_x + \tau_{xy} n_y, \tag{8.1.9}
\]

\[
p_y = \sigma_y n_y + \tau_{xy} n_x. \tag{8.1.10}
\]

The equations in this section will be solved in following sections for some simple cases where all examples are plane stress problems.

8.2 The analytic solution for example 2.3

In the case of example 2.3, the solution of equation (8.1.5) is the following cubic polynomial

\[
\phi = Py^3/c. \tag{8.2.1}
\]

By (8.1.4), we have

\[
\sigma_x = Py/c, \quad \sigma_y = 0, \quad \tau_{xy} = 0 \tag{8.2.2}
\]

which satisfy the traction boundary condition. By (8.1.6) - (8.1.8),

\[
\frac{\partial u}{\partial x} = \frac{1}{E} (\sigma_x - \nu \sigma_y) = \frac{1}{Ec} Py, \tag{8.2.3}
\]

\[
\frac{\partial v}{\partial y} = \frac{1}{E} (\sigma_y - \nu \sigma_x) = \frac{\nu}{Ec} Py. \tag{8.2.4}
\]
\[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{G} \tau_{xy} = 0 \quad (8.2.5) \]

and therefore,
\[ u = \frac{1}{Ec} P_{xy} + f_1(y), \quad (8.2.6) \]
\[ v = -\frac{\nu}{2Ec} Py^2 + f_2(x). \quad (8.2.7) \]

By the symmetry, i.e., \( u(0,y) = 0 \) (see figure 2.195), we obtain \( f_1(y) = 0 \). Differentiating the two sides of (8.2.6) and (8.2.7), respectively,
\[ \frac{\partial u}{\partial y} = \frac{1}{Ec} P_x \quad \text{and} \quad \frac{\partial v}{\partial x} = f'_2(x). \quad (8.2.8) \]

Substituting (8.2.8) into (8.2.5) leads to
\[ \frac{1}{Ec} P_x + f'_2(x) = 0, \]
i.e.,
\[ f'_2(x) = -\frac{P}{2Ec} x^2 + \text{const.}. \]

By the boundary condition \( v(0,0) = f_2(0) = 0 \) (see figure 2.195), \( f_2(x) = -\frac{P x^2}{2Ec} \). Now, the displacements can be expressed as
\[ u = \frac{1}{Ec} P_{xy} \quad (8.2.9) \]
\[ v = -\frac{\nu}{2Ec} Py^2 - \frac{P}{2Ec} x^2 \quad (8.2.10) \]

8.3 The analytic solution of example 2.4

Similar to the discussion in last section, the solution of equation (8.1.5) is a polynomial of order 5. The stress can be expressed by
\[ \sigma_x = \frac{3P}{4c^3} (x^2 y - \frac{2}{3} y^3) + P_{y/c} \quad (8.3.1) \]
\[ \sigma_y = -\frac{3P}{4c^3} \left( yc^2 - \frac{1}{3} y^3 \right) - \frac{P_1}{2} \quad \text{(8.3.2)} \]
\[ \tau_{xy} = \frac{3P}{4c^3} \left( c^2 - y^2 \right) \quad \text{(8.3.3)} \]

which satisfy equations (8.1.1)-(8.1.3). By (8.1.8), (8.1.9), (8.3.1)-(8.3.3) and boundary condition for traction (see Fig.2.24), on the boundary \( x = L \) we have:

\[ P_x \bigg|_{x=L} = \sigma_x \bigg|_{x=L} = -\frac{3P}{4c^3} \left( L^2 y - \frac{2}{3} y^3 \right) + \frac{P_2 y}{c} \equiv F_3 \quad \text{(8.3.4)} \]
\[ P_y \bigg|_{x=L} = \tau_{xy} \bigg|_{x=L} = \frac{3P}{4c^3} \left( c^2 - y^2 \right) L \equiv F_2 \quad \text{(8.3.5)} \]

On the other boundary, the boundary conditions are satisfied because:

\[ P_x \bigg|_{x=-L} = -\sigma_x \bigg|_{x=-L} = -F_3 \]
\[ P_y \bigg|_{x=-L} = -\tau_{xy} \bigg|_{x=-L} = F_2 \]
\[ P_x \bigg|_{y=c} = \tau_{xy} \bigg|_{y=c} = 0 \]
\[ P_y \bigg|_{y=c} = \sigma_y \bigg|_{y=c} = -\frac{P_1}{2} \equiv F_4 \quad \text{(8.3.6)} \]
\[ P_x \bigg|_{y=-c} = -\tau_{xy} \bigg|_{y=-c} = 0 \]
\[ P_y \bigg|_{y=-c} = -\sigma_y \bigg|_{y=-c} = 0 \]

Now, we need to solve the equations (8.1.6)-(8.1.8) for the displacement. Substituting (8.3.1)-(8.3.3) into (8.1.6)-(8.1.8), we obtain

\[ \frac{\partial u}{\partial x} = \frac{1}{E} \left[ \frac{3P}{4c^3} \left( x^2 y - \frac{2}{3} y^3 \right) + \frac{P_2 y}{c} \right] - \frac{\nu}{E} \left[ \frac{P}{4c^3} y^3 - \frac{3P}{4c} \right] \frac{y}{y - \frac{P_1}{2}} \]
\[
\frac{\partial v}{\partial x} = \frac{1}{E} \left[ \frac{P}{4c^3} y^3 - \frac{3P}{4c} y - \frac{P}{2} x \right] - \frac{\nu}{E} \left[ \frac{3P}{4c^3} \left( x^2 y - \frac{2}{3} y^3 \right) + P \frac{y}{c} \right]
\]

and therefore,

\[
u = \frac{1}{E} \left[ \frac{P}{4c^3} (x^3 y - 2xy^2) + \frac{P}{2} y \right] - \frac{\nu}{E} \left[ \frac{3P}{4c^3} y^3 \right] - \frac{P}{2} y + f_1(y)
\]

\[
v = \frac{1}{E} \left[ \frac{P}{16c^3} - \frac{3P}{8c} y^2 - \frac{P}{2} y \right] - \frac{\nu}{E} \left[ \frac{3P}{8c^3} \left( x^2 y^2 - \frac{1}{3} y^4 \right) + \frac{P}{2c} y^2 \right] + f_2(x)
\]

(8.3.7)

By the symmetry, i.e., \(u(x,0,0) = 0\) (see figure 2.24), we have
\(f_1(y) = 0\). Substituting (8.3.7) and (8.3.3) into (8.1.8) results in

\[
u = \frac{1}{E} \left[ \frac{3P}{4c^3} (x^3 - 3xy^2) + \frac{P}{2} x \right] - \frac{\nu}{E} \left[ \frac{3P}{4c^3} y^2 \right] + f_2(x)
\]

\[
= \frac{1}{G} \frac{3P}{4c^3} (c^2 - y^2)x
\]

(8.3.8)

Noting condition \(v(x,0,0) = 0\) and solving (8.3.8) for \(f_2(x)\), we have

\[
f_2(x) = \frac{1}{E} \left[ \frac{3P}{4c^3} \left( \frac{x^4}{12} + (2 + \nu)x^2c^2/2 \right) - \frac{P}{2c} \right]
\]

(8.3.9)

and therefore, the displacement can be expressed as

\[
u = \frac{1}{E} \left[ \frac{P}{4c^3} (x^3 y - 2xy^2) + \frac{P}{2} y \right] - \frac{\nu}{E} \left[ \frac{3P}{4c^3} y^3 \right] - \frac{P}{2} y + f_1(y)
\]

(8.3.10)

\[
v = \frac{1}{E} \left[ \frac{P}{16c^3} - \frac{3P}{8c} y^2 - \frac{P}{2} y \right] - \frac{\nu}{E} \left[ \frac{3P}{8c^3} \left( x^2 y^2 - \frac{1}{3} y^4 \right) + \frac{P}{2c} y^2 \right]
\]

\[+ \frac{1}{E} \left[ \frac{3P}{4c^3} \left( \frac{x^4}{12} + (2 + \nu)x^2c^2/2 \right) - \frac{P}{2c} \right]
\]

(8.3.11)
8.4 The formula for stress intensity factor

For the crack under a mode I loading (see figure 8.1), the displacement can be expressed as [39]

\[
\begin{align*}
  u_1 &= \frac{2(1+\nu)}{E} (r/2\pi)^{1/2} \left[ K_1 \cos \frac{\theta}{2} \left( \frac{p-1}{2} + \sin^2 \frac{\theta}{2} \right) ight] \\
  &\quad - \frac{(1+\nu)}{4E} (1-k)(p+1)\sigma(r\cos\theta+a) \\
  u_2 &= \frac{2(1+\nu)}{E} (r/2\pi)^{1/2} \left[ K_1 \sin \frac{\theta}{2} \left( \frac{p-1}{2} - \cos^2 \frac{\theta}{2} \right) ight] \\
  &\quad + \frac{(1+\nu)}{4E} (1-k)(3-p)\sigma\sin\theta
\end{align*}
\]

(8.4.1) (8.4.2)

Figure 8.1

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
where \( p = \frac{(3-\nu)}{(1+\nu)} \) in plane stress, and \( (3-4\nu) \) in plane strain and \( k \) is a proportion of \( \sigma \) in the \( x \) direction. Therefore, the stress intensity factor for a plane stress problem can be calculated by

\[
K_I = \lim_{r \to 0} \frac{E(2\pi)^{1/2} u_r}{4 r^{1/2}}
\]

8.5 The analytic solution of example 4.1

For the case of the circle under uniform pressure, the displacement can be expressed as [54]

\[
\begin{align*}
  u &= \frac{P(x-1)}{4G} \cos \alpha \\
  v &= \frac{P(x-1)}{4G} \sin \alpha
\end{align*}
\]

where \( r \) is radius and \( x = \frac{(3-\nu)}{(1+\nu)} \) for plane stress.
References


[18] De Boor, C. Good approximation by spline with variable


[26] Hsiao, G.C., Kleinmman, R. E., Li, R-X. and Vandenberg,


Weiwei Sun was born in 1957 in Shanghai, China. He received his B.Sc. in Mathematics at Northwestern Polytechnical University in 1981. He received M.Sc. in Mathematics at Xian Jiaotong University in 1984 and University of Windsor in 1987.