1-1-1962

Matrix analysis of frames with semi-rigid connections.

Gerard R. Monforton
University of Windsor

MATRIX ANALYSIS OF FRAMES WITH SEMI-RIGID CONNECTIONS

A THESIS
Submitted to the Faculty of Graduate Studies through the Department of Civil Engineering in Partial Fullfillment of the Requirements for the Degree of Master of Applied Science at Assumption University of Windsor

by

GERARD R. MONFORTON
B.A.Sc., Assumption University of Windsor, 1961

Windsor, Ontario, Canada
1962

APPROVED BY:

T. S. Wu

A. A. Duquette

J. C. Campbell

ABSTRACT

In this thesis frames with semi-rigid connections are investigated by employing matrix algebra which is especially suited to operations by high speed digital computers. The displacement method is the basis for the proposed analysis. Stiffness matrices for members with semi-rigid connections are presented in the form of the stiffness matrices for members with rigid connections modified by a correction matrix whose elements are functions of two parameters designated as "fixity factors" of the member. A formula,

$$P = p - S\,C\,U\,(\,U^T S\,C\,U\,)^{-1}\,(\,U^T p + p_o\,)$$

is established for determining the force components on the members of a frame.

ACKNOWLEDGMENTS

The writer expresses gratitude to Dr. T. S. Wu for his stimulating encouragement and guidance in the preparation of this work, and to the National Research Council of Canada for sponsoring this project as well as for its financial assistance.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

In the conventional analysis of steel structures the beam to column connections are considered as either perfectly hinged or rigidly fixed. These assumptions are not entirely consistent with conditions often encountered in practice; however they have been adopted because of the simplicity in analysis and design. Beam or girder to column connections have partial restraint depending upon the type of connection used. Although by including the effect of this restraint in analysing frames a more economical design would result, it has been almost entirely neglected because of the more time required and because of a lack of a systematic method of performing the analysis on more complex types of structures.

The slope deflection and moment distribution methods, modified for members with partial end restraint, have been proposed by J. Charles Rathburn (12), John F. Baker (2) and Bruce Johnston and Edward H. Mount (9). These methods are sufficient for simple frames; however they are too cumbersome to be used directly in the investigation of highly redundant frames.

The advent of high speed electronic computers has stimulated the analysis of complex frames by matrix methods. The force and displacement methods were developed by J. H. Argyris (1) in order to solve structural problems using matrix algebra. The displacement method, a matrix formulation of the joint equilibrium equations in which linear and angular displacements are chosen as redundants, is ideally suited to electronic digital computers and it is particularly systematic for the analysis of frameworks, however complex.

The work embodied in this thesis consisted of developing a method of analysis which is readily applicable to digital computation for frames with semi-rigid connections. Relationships between forces and displacements at the ends of a member with partial restraint were found by using the conjugate beam method. The force components, in terms of the end displacements, were arranged in matrix form. The modified stiffness were then expressed as the stiffness matrices for rigid joints multiplied by a correction matrix whose elements are functions of the "fixity factors" of the members. When the connections of the structure are defined, these factors can be evaluated by formulae or determined by experimental investigation.

A formula for the force components at the ends of the members of the frame was established, based on the joint equilibrium equations. The evaluation of the force components involves multiplication, inversion and subtraction of known matrices. These matrix operations are conveniently performed by any digital computer.

The adaptibility of digital computers to solve problems by the derived method was demonstrated. The general procedure is illustrated by a flow chart and several numerical examples are worked out by employing the LGP-30 digital computer.

A consistent sign convention is used throughout the thesis.

CHAPTER II

HISTORICAL DISCUSSION

In 1915 Wilbur M. Wilson and George A. Maney (16) first applied the slope deflection equation to frames with rigid joints. This was followed by Hardy Cross's (5) developments of the moment distribution method in 1926 and the column analogy in 1930. These original methods remain as the outstanding systems for analyzing rigid frames, but in recent years electronic digital computers have motivated the analysis of structures by matrix methods. These methods have been advocated by Stanley U. Benscoter (4), J. H. Argyris (1), Fernando Venancio Filho (6) and others, and they are now accepted as significant advancements in structural analysis.

The slope deflection and moment distribution methods were both applied to frames with semi-rigid connections in the 1930's by John F. Baker (2) in England and J. Charles Rathbrun (12) in the United States. In 1942 Bruce Johnston and Edward H. Mount (9) refined Baker's analysis by considering the effect due to the width of the members. From 1950 to 1955 research projects dealing with semi-rigid connections were carried out at Oklahoma State University (10).

Extensive experimental research has been sponsored, especially in Great Britain and the United States, to enhance the possibility of including the partial restraint of connections in the design of structural frameworks. In 1931 to 1936, a group of men, headed by Cyril Batho (3) conducted tests at the University of Birmingham to find a relationship between the moment applied at a riveted connection and the corresponding rotation. Their findings were in close agreement with those of Rathburn (12) of the College of the City of New York. A series of tests to compare the rigidity of welded and riveted connections were conducted at the University of Toronto by C.R.Young and K.B.Jackson (17) in 1934. The most recent report was published in 1947 by the American Institute of Steel Construction Committee on Steel Structures Research (8). This body recommended a dependable percentage of restraint for various types of semi-rigid connections.

The findings of the above research groups indicate that for practical purposes a linear relationship exists between the applied moment and the relative rotation of the beam and column. This linear relationship, shown in Fig. 1, is valid only for a specific design region; but for practical purposes, it may be considered the acceptable relationship in the design of frames with semi-rigid joints.

A possible tests setup for determining the moment-rotation curve of the interior joint of a frame with beam to column flange connections is shown in Fig. 2. An applied load induces a moment M and a rotation ϕ at the joint. The relative rotation for various values of moment is measured by means of sensitive level bars, and a moment-rotation curve is plotted as in Fig. 1. The inverse slope of the straight line portion of the graph is calculated and is called the "semi-rigid connection factor" for that connection. Thus,

$$Z = \phi/M \qquad (2\text{-}1)$$

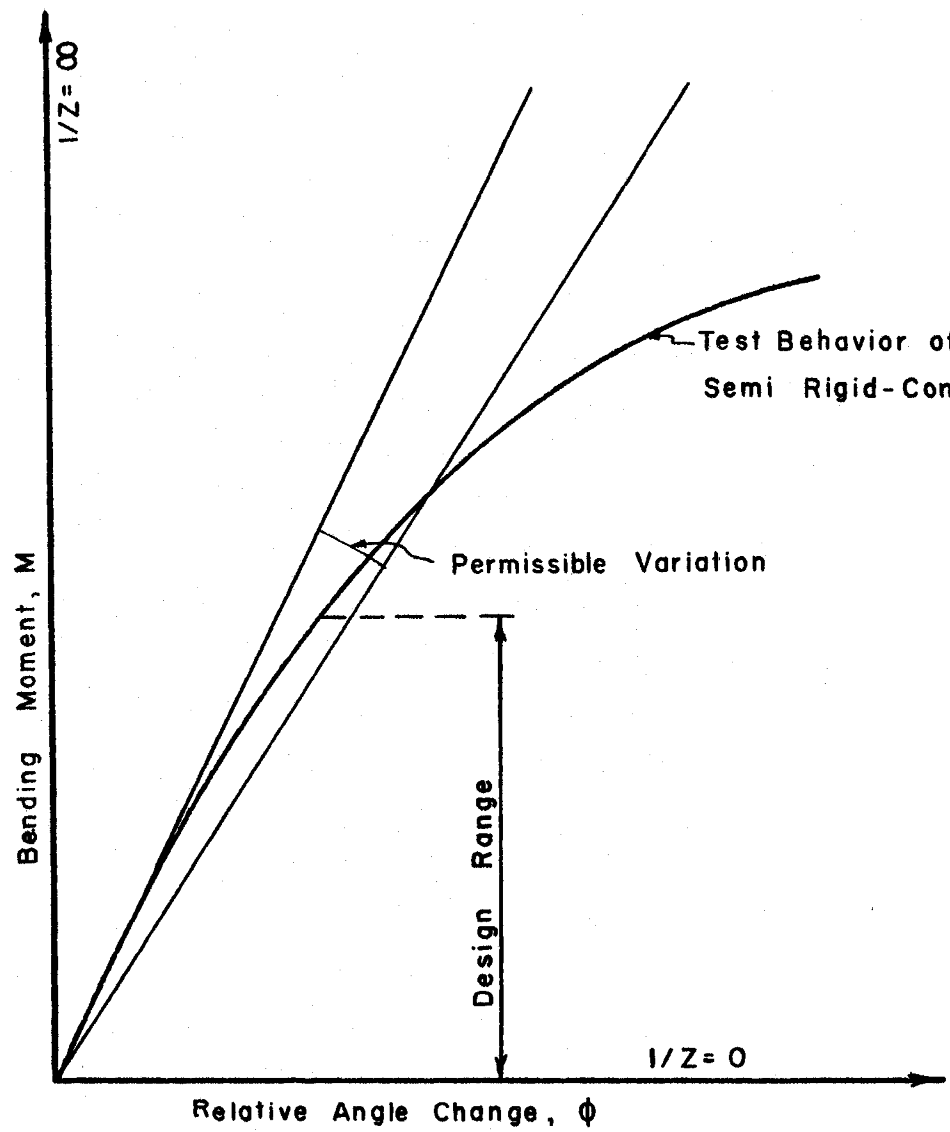


FIG. 1. MOMENT-ROTATION CURVE FOR SEMI-RIGID CONNECTION

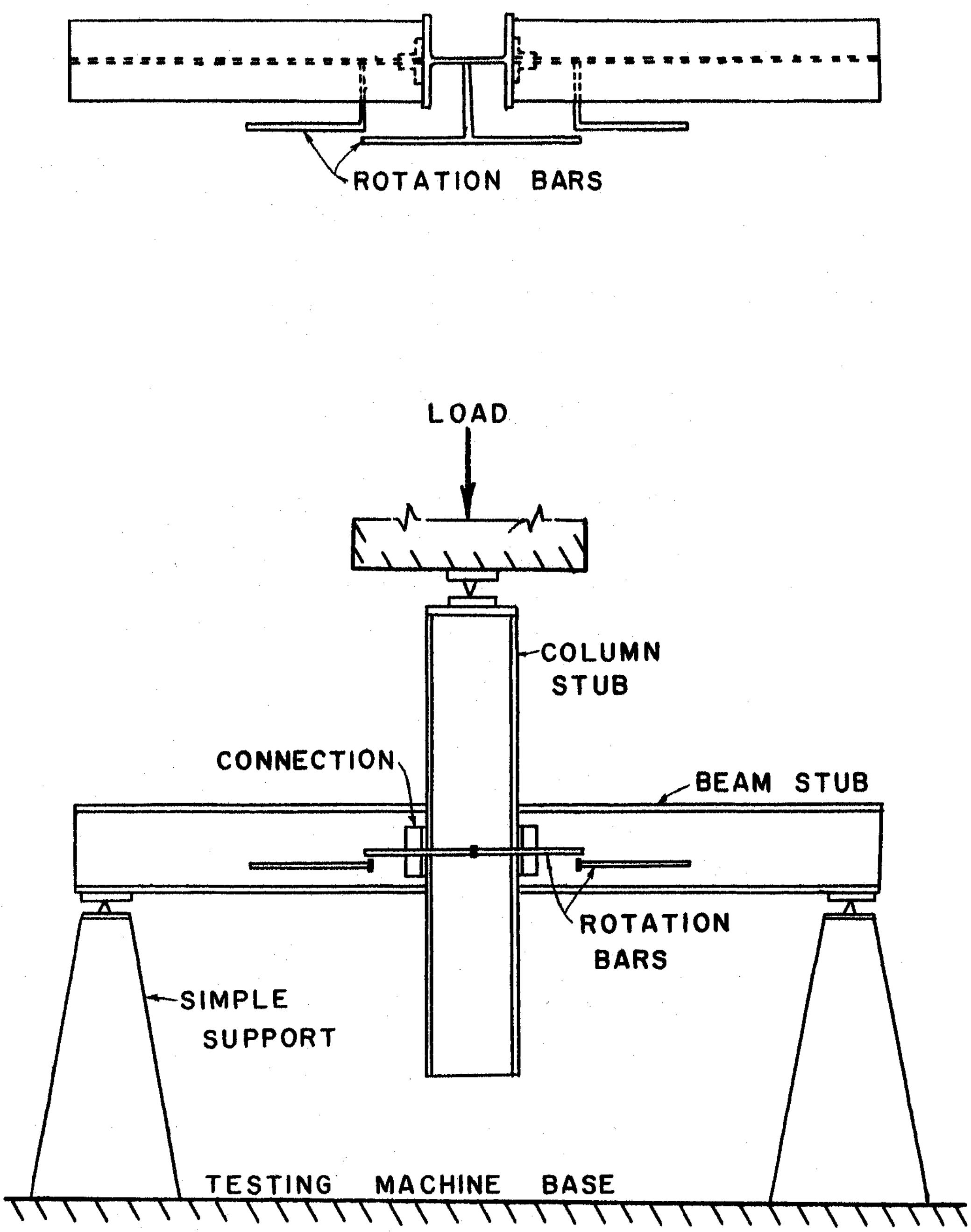


FIG. 2 TEST SETUP FOR M-ϕ CURVE

The semi-rigid connection factor may be defined as an angle change per unit moment and its magnitude depends upon the rigidity of the connection. The extreme values of Z are infinity for a frictionless pin connection and zero for a perfectly rigid connection. The Z factor for most standard connections has been found by experimental and theoretical investigations, and the results of both methods are in close agreement. Appendix A contains Z factors which have been collected from published tables.

CHAPTER III

STIFFNESS MATRIX

Modified Slope Deflection Equations

The moments at the restrained ends of a loaded member consist of three contributing sources:

(1) that due to the fixed end moments

(2) that due to the rotation of the ends of the member

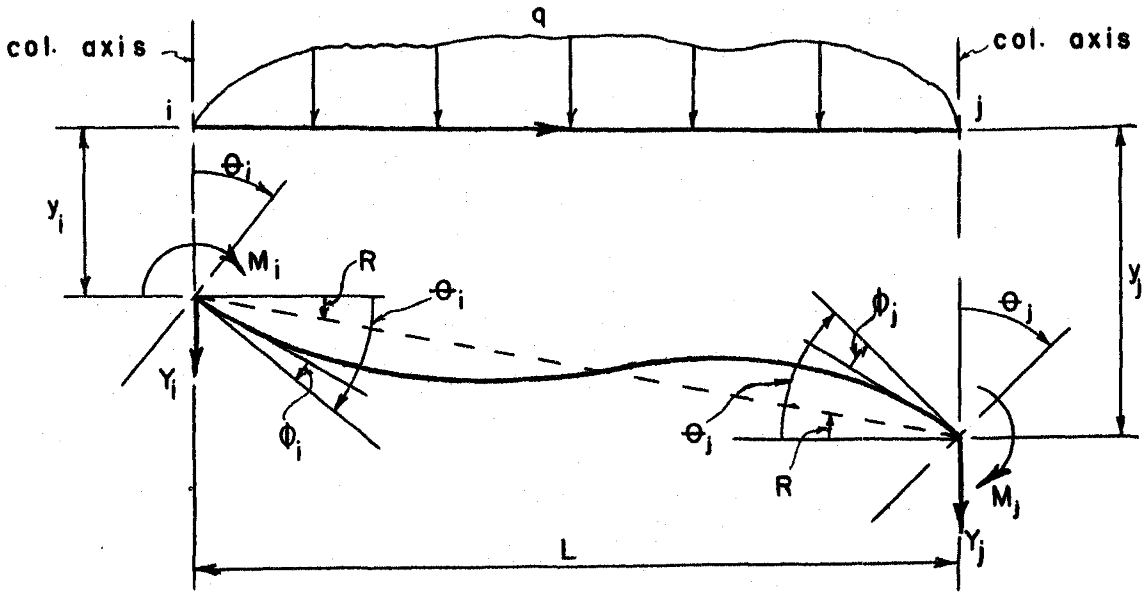
(3) that due to the relative displacement of the ends of the member

An additional element, namely the semi-rigid connection factor contributes to the moments of members with semi-rigid connections; this element is interdependent on the above three sources. For this reason it is not possible to separate the derivation of the slope deflection equation into three parts, which is the customary procedure for rigid connections (14).

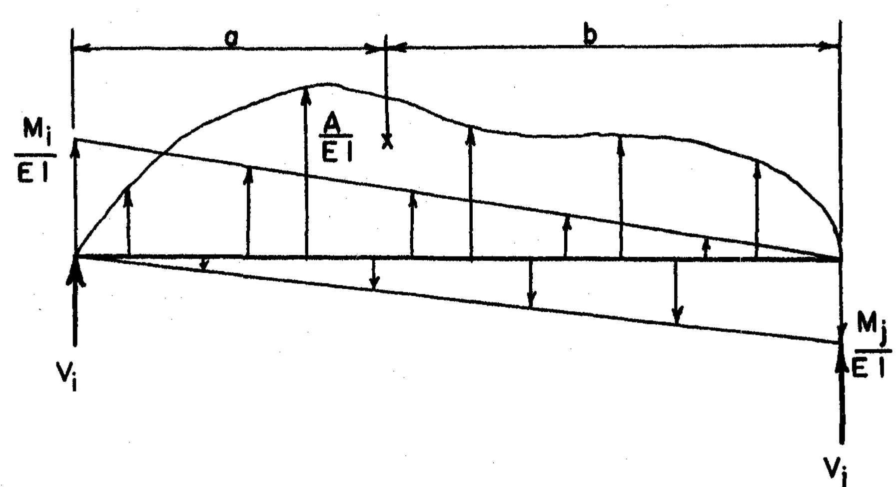
The following derivation of the relationship between the forces and displacements at the ends of a member with semi-rigid connections is based upon the conjugate beam method. The adopted sign convention is shown in Fig. 3(a) and is referred to the longitudinal axis of the member.

Fig. 3(b) represents a typical member of a frame with any external loading q. M_i and M_j are the final moments acting on the ends of the beam (and through the semi-rigid connection, on the column) at i and j respectively. θ_i and θ_j are the rotation of the column axes at their

(a) Sign Convention



(b) Member Under Loading



(c) Conjugate Beam and Loading

FIG. 3 CONJUGATE BEAM FOR SEMI-RIGID CONNECTIONS intersection with the beam axis. Y_i and Y_j are the shearing forces at i and j respectively. ϕ_i and ϕ_j represent the angle change at the ends of the loaded member and they are due to the flexibility of the connections; as previously described experimental tests indicate that for practical purposes $\phi = MZ$. The net rotation of the ends of the member itself are $\theta_i - \phi_i - R$ for end i and $\theta_j - \phi_j - R$ for end j, where R is the angle of rotation of the member due to the relative displacements of the two ends. All directions shown in Fig. 3(b) are positive.

Fig. 3(c) shows the loading on the conjugate beam. A represents the area under the simple beam moment diagram due to the external loading only, and it is easily calculated for various loading conditions. Since the shear forces, V_i and V_j, at the ends of the conjugate beam equal the net angle change at the ends of the actual beam, then we have

$$V_i = \theta_i - \phi_i - R = \frac{M_i L}{3EI} - \frac{M_j L}{6EI} + \frac{Ab}{EIL} \qquad (3\text{-}1)$$

$$V_j = \theta_j - \phi_j - R = \frac{M_j L}{3EI} - \frac{M_i L}{6EI} - \frac{Aa}{EIL} \qquad (3\text{-}2)$$

Substituting $\phi_i = M_i Z_i$ and $\phi_j = M_j Z_j$ into equations (3-1) and (3-2) gives

$$\theta_i - R = \frac{M_i L}{3EI} - \frac{M_j L}{6EI} + \frac{Ab}{EIL} + M_i Z_i \qquad (3\text{-}3)$$

$$\theta_j - R = \frac{M_j L}{3EI} - \frac{M_i L}{6EI} - \frac{Aa}{EIL} + M_j Z_j \qquad (3\text{-}4)$$

Rearranging the terms, we have

$$M_j \left(\frac{L + 3EIZ_i}{L}\right) = \frac{3EI}{L} (\theta_i - R) + \frac{M_j}{2} - \frac{3Ab}{L^2} \qquad (3\text{-}5)$$

$$M_j \left(\frac{L + 3EIZ_j}{L}\right) = \frac{3EI}{L} (\theta_j - R) + \frac{M_i}{2} - \frac{3Aa}{L^2} \qquad (3\text{-}6)$$

Introducing the dimensionless parameters,

$$\gamma_i = \frac{L}{L + 3EIZ_i} \qquad (3\text{-}7)$$

$$\gamma_j = \frac{L}{L + 3EIZ_j} \qquad (3\text{-}8)$$

into equations (3-5) and (3-6) and solving for M_i and M_j, we have

$$M_i = \frac{6EI}{L} \gamma_i \cdot \frac{2\theta_i + \gamma_i\theta_j - (2 + \gamma_j)R}{4 - \gamma_i \gamma_j} - \frac{6A}{L^2} \cdot \frac{\gamma_i(2b - \gamma_j a)}{4 - \gamma_i \gamma_j} \qquad (3\text{-}9)$$

$$M_j = \frac{6EI}{L} \gamma_j \cdot \frac{2\theta_j + \gamma_j\theta_i - (2 + \gamma_i)R}{4 - \gamma_i \gamma_j} + \frac{6A}{L^2} \cdot \frac{\gamma_j (2a - \gamma_i b)}{4 - \gamma_i \gamma_j} \qquad (3\text{-}10)$$

Equations (3-9) and (3-10) are the slope deflection equations for members with semi-rigid connections. The last term on the right hand side of both equations is defined as the "modified" fixed end moments. The sign preceeding these terms follows the adopted sign convention; formulae have been worked out for the modified fixed end moments for various loading conditions and appear in Appendix B.

The dimensionless parameters, γ_i and γ_j , depend upon the Z-factor of the connections and the geometry of the member and are designated as "fixity factors" of the member. The value of γ varies between zero for a frictionless pin connection and unity for a perfectly rigid connection. From equations (3-9) and (3-10), if the fixity factor is unity for both ends of a member, the equations reduce to the standard form of the slope deflection equations.

Modified Stiffness Matrices

The equations derived in the preceeding section can be used to solve frames with semi-rigid connections. However, the operations involved for complex structures would be excessive for hand computation, and the form of the equations is not well suited for computer operations. Relationships for the internal forces acting at the ends of a member, in terms of the end displacements, can be found by applying equations (3-9) and (3-10).

Planar Frames

Equations (3-9) and (3-10) in expanded form give

$$M_i = \frac{4EI}{L}\cdot\frac{3\gamma_i}{4-\gamma_i\gamma_j}\,\theta_i + \frac{6EI}{L^2}\cdot\frac{\gamma_i(2+\gamma_j)}{4-\gamma_i\gamma_j}\,y_i + \frac{2EI}{L}\cdot\frac{3\gamma_i\gamma_j}{4-\gamma_i\gamma_j}\,\theta_j - \frac{6EI}{L^2}\cdot\frac{\gamma_i(2+\gamma_j)}{4-\gamma_i\gamma_j}\,y_j \qquad (3\text{-}11)$$

$$M_j = \frac{2EI}{L}\cdot\frac{3\gamma_i\gamma_j}{4-\gamma_i\gamma_j}\,\theta_i + \frac{6EI}{L^2}\cdot\frac{\gamma_j(2+\gamma_i)}{4-\gamma_i\gamma_j}\,y_i + \frac{4EI}{L}\cdot\frac{3\gamma_j}{4-\gamma_i\gamma_j}\,\theta_j - \frac{6EI}{L^2}\cdot\frac{\gamma_j(2+\gamma_i)}{4-\gamma_i\gamma_j}\,y_j \qquad (3\text{-}12)$$

The shearing forces at the ends are

$$Y_i = \frac{6EI}{L^2}\cdot\frac{\gamma_i(2+\gamma_j)}{4-\gamma_i\gamma_j}\,\theta_i + \frac{12EI}{L^3}\cdot\frac{\gamma_i+\gamma_j+\gamma_i\gamma_j}{4-\gamma_i\gamma_j}\,y_i + \frac{6EI}{L^2}\cdot\frac{\gamma_j(2+\gamma_i)}{4-\gamma_i\gamma_j}\,\theta_j - \frac{12EI}{L^3}\cdot\frac{\gamma_i+\gamma_j+\gamma_i\gamma_j}{4-\gamma_i\gamma_j}\,y_j \qquad (3\text{-}13)$$

$$Y_j = -\frac{6EI}{L^2}\cdot\frac{\gamma_i(2+\gamma_j)}{4-\gamma_i\gamma_j}\,\theta_i - \frac{12EI}{L^3}\cdot\frac{\gamma_i+\gamma_j+\gamma_i\gamma_j}{4-\gamma_i\gamma_j}\,y_i - \frac{6EI}{L^2}\cdot\frac{\gamma_j(2+\gamma_i)}{4-\gamma_i\gamma_j}\,\theta_j + \frac{12EI}{L^2}\cdot\frac{\gamma_i+\gamma_j+\gamma_i\gamma_j}{4-\gamma_i\gamma_j}\,y_j \qquad (3\text{-}14)$$

The semi-rigidity of the connections does not affect the axial deformation of the member and therefore the axial loads, in terms of the axial displacements, are given by

$$X_i = \frac{EA}{L} (x_i - x_j) \tag{3-15}$$

$$X_j = - \frac{EA}{L} (x_i - x_j) \tag{3-16}$$

Equations (3-11) to (3-16) may be written in matrix form as

$$F_i = K_{ii} D_i + K_{ij} D_j \tag{3-17}$$

and $$F_j = K_{ji} D_i + K_{jj} D_j \tag{3-18}$$

where the components of these equations are shown in Fig. 4. Equations (3-17) and (3-18) represent the internal forces due to the end displacements of a member in a planar frame. The K matrices of equations (3-17) and (3-18) are called modified stiffness matrices for planar frames.

Space Frames

Before electronic computers were available for general engineering operations the study of more complex framed structures was of academic interest only, since the arithmetic work involved was excessive. Now, however, frames of a three dimensional nature, even with partial restraint, can be analyzed readily by the extension of the ideas previously illustrated for simpler planar frames.

The sign convention adopted is that of the right hand rule illustrated in Fig. 5. In the stiffness matrices shown in Fig. 6, α, β and γ represent the fixity factors for rotation about the x, y and z axes respectively. The value of γ is defined by equations (3-7)

$$F_i = \begin{bmatrix} X_i \\ Y_i \\ M_i \end{bmatrix} \qquad F_j = \begin{bmatrix} X_j \\ Y_j \\ M_j \end{bmatrix} \qquad D_i = \begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix} \qquad D_j = \begin{bmatrix} x_j \\ y_i \\ \theta_j \end{bmatrix}$$

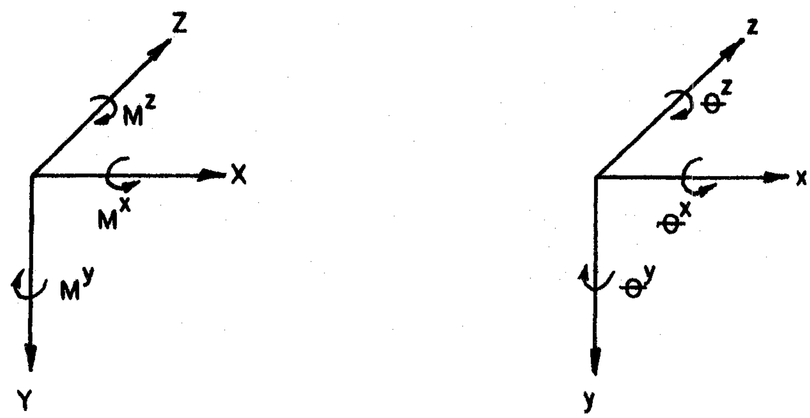
$$K_{ii} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3}\,\frac{\gamma_i+\gamma_j+\gamma_i\gamma_j}{4-\gamma_i\gamma_j} & \frac{6EI}{L^2}\,\frac{\gamma_i(2+\gamma_j)}{4-\gamma_i\gamma_j} \\ 0 & \frac{6EI}{L^2}\,\frac{\gamma_i(2+\gamma_j)}{4-\gamma_i\gamma_j} & \frac{4EI}{L}\,\frac{3\gamma_i}{4-\gamma_i\gamma_j} \end{bmatrix}$$

$$K_{ij} = \begin{bmatrix} \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3}\,\frac{\gamma_i+\gamma_j+\gamma_i\gamma_j}{4-\gamma_i\gamma_j} & \frac{6EI}{L^2}\,\frac{\gamma_j(2+\gamma_i)}{4-\gamma_i\gamma_j} \\ 0 & \frac{-6EI}{L^2}\,\frac{\gamma_i(2+\gamma_j)}{4-\gamma_i\gamma_j} & \frac{2EI}{L}\,\frac{3\gamma_i\gamma_j}{4-\gamma_i\gamma_j} \end{bmatrix}$$

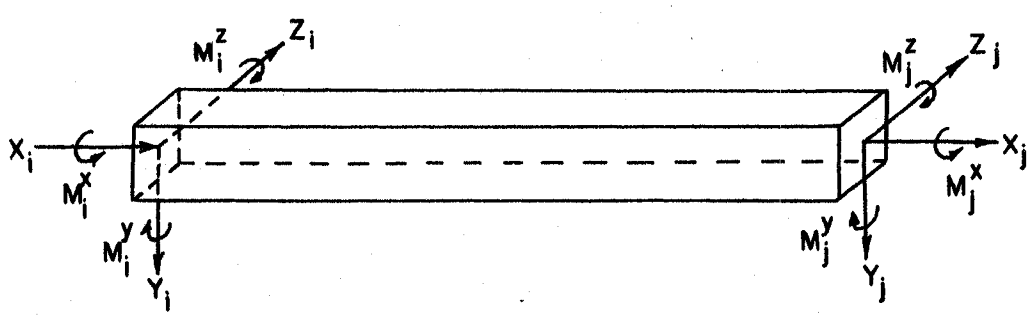
$$K_{ji} = \begin{bmatrix} \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3}\,\frac{\gamma_i+\gamma_j+\gamma_i\gamma_j}{4-\gamma_i\gamma_j} & \frac{-6EI}{L^2}\,\frac{\gamma_i(2+\gamma_i)}{4-\gamma_i\gamma_j} \\ 0 & \frac{6EI}{L^2}\,\frac{\gamma_j(2+\gamma_i)}{4-\gamma_i\gamma_j} & \frac{2EI}{L}\,\frac{3\gamma_i\gamma_j}{4-\gamma_i\gamma_j} \end{bmatrix}$$

$$K_{jj} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3}\,\frac{\gamma_i+\gamma_j+\gamma_i\gamma_j}{4-\gamma_i\gamma_j} & \frac{-6EI}{L^2}\,\frac{\gamma_j(2+\gamma_i)}{4-\gamma_i\gamma_j} \\ 0 & \frac{-6EI}{L^2}\,\frac{\gamma_j(2+\gamma_i)}{4-\gamma_i\gamma_j} & \frac{4EI}{L}\,\frac{3\gamma_j}{4-\gamma_i\gamma_j} \end{bmatrix}$$

FIG. 4 COMPONENTS OF EQUATIONS (3-17) and (3-18)



(a) Sign Convention



(b) Forces Acting at Ends of Member in a Space Frame

FIG. 5. SIGN CONVENTION FOR SPACE FRAMES

and (3-8). However, to the writer's knowledge the Z factors for the x and y axes have not been determined experimentally or theoretically, and therefore the fixity factors, α and β, cannot be explicitly defined at this time. Once the Z factors for these two axes are determined it will be possible to find the values of α and β within the same limits as γ. Standard connections have little resisting capacity about the y-axis and thus β will approach zero. For other than standard connections, both β and γ must be determined by experimental investigations. Twisting about the x-axis of a member will be very slight in most connections and therefore the value of α will be close to unity.

Applying equations (3-9) and (3-10) for bending about the y and z axes and considering torsion about the x-axis yields

$$F_i = K_{ii} D_i + K_{ij} D_j \tag{3-19}$$

and

$$F_j = K_{ji} D_i + K_{jj} D_j \tag{3-20}$$

Equations (3-19) and (3-20) represent the internal force components due to the end displacements of a member in a space frame, and the elements of these equations are shown in Fig. 6. The K matrices are designated as modified stiffness matrices for space frames.

Correction Matrices

The similarity between the modified stiffness matrices K of Figs. 4 and 6 and the stiffness matrices S for rigid joints, Figs. 7(a) and 8(a) for planar and space frames respectively, is readily apparent: each element of the K matrices is equal to the corresponding element of the S matrix multiplied by a term which is a function of the fixity factors

$$F_i = \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ M_i^x \\ M_i^y \\ M_i^z \end{bmatrix} \qquad F_j = \begin{bmatrix} X_j \\ Y_j \\ Z_j \\ M_j^x \\ M_j^y \\ M_j^z \end{bmatrix} \qquad D_i = \begin{bmatrix} x_i \\ y_i \\ z_i \\ \theta_i^x \\ \theta_i^y \\ \theta_i^z \end{bmatrix} \qquad D_j = \begin{bmatrix} x_j \\ y_j \\ z_j \\ \theta_j^x \\ \theta_j^y \\ \theta_j^z \end{bmatrix}$$

$$K_{ii} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3}\cdot\frac{\gamma_i+\gamma_j+\gamma_i\gamma_j}{4-\gamma_i\gamma_j} & 0 & 0 & 0 & \frac{6EI_z}{L^2}\cdot\frac{\gamma_i(2+\gamma_j)}{4-\gamma_i\gamma_j} \\ 0 & 0 & \frac{12EI_y}{L^3}\cdot\frac{\beta_i+\beta_j+\beta_i\beta_j}{4-\beta_i\beta_j} & 0 & \frac{-6EI_y}{L^2}\cdot\frac{\beta_i(2+\beta_j)}{4-\beta_i\beta_j} & 0 \\ 0 & 0 & 0 & \frac{GJ}{L}\alpha & 0 & 0 \\ 0 & 0 & \frac{-6EI_y}{L^2}\cdot\frac{\beta_i(2+\beta_j)}{4-\beta_i\beta_j} & 0 & \frac{4EI_y}{L}\cdot\frac{3\beta_i}{4-\beta_i\beta_j} & 0 \\ 0 & \frac{6EI_z}{L^2}\cdot\frac{\gamma_i(2+\gamma_j)}{4-\gamma_i\gamma_j} & 0 & 0 & 0 & \frac{4EI_z}{L}\cdot\frac{3\gamma_i}{4-\gamma_i\gamma_j} \end{bmatrix}$$

$$K_{ij} = \begin{bmatrix} \frac{-EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-12EI_z}{L^3}\cdot\frac{\gamma_i+\gamma_j+\gamma_i\gamma_j}{4-\gamma_i\gamma_j} & 0 & 0 & 0 & \frac{6EI_z}{L^2}\cdot\frac{\gamma_j(2+\gamma_i)}{4-\gamma_i\gamma_j} \\ 0 & 0 & \frac{-12EI_y}{L^3}\cdot\frac{\beta_i+\beta_j+\beta_i\beta_j}{4-\beta_i\beta_j} & 0 & \frac{-6EI_y}{L^2}\cdot\frac{\beta_j(2+\beta_i)}{4-\beta_i\beta_j} & 0 \\ 0 & 0 & 0 & \frac{-GJ}{L}\alpha & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{L^2}\cdot\frac{\beta_i(2+\beta_j)}{4-\beta_i\beta_j} & 0 & \frac{2EI_y}{L}\cdot\frac{3\beta_i\beta_j}{4-\beta_i\beta_j} & 0 \\ 0 & \frac{-6EI_z}{L^2}\cdot\frac{\gamma_i(2+\gamma_j)}{4-\gamma_i\gamma_j} & 0 & 0 & 0 & \frac{2EI_z}{L}\cdot\frac{3\gamma_i\gamma_j}{4-\gamma_i\gamma_j} \end{bmatrix}$$

FIG. 6 COMPONENTS OF EQUATIONS (3-19) AND (3-20)

$$K_{ji} = \begin{bmatrix} -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-12EI_z}{L^3}\cdot\frac{\gamma_i+\gamma_j+\gamma_i\gamma_j}{4-\gamma_i\gamma_j} & 0 & 0 & 0 & \frac{-6EI_z}{L^2}\cdot\frac{\gamma_i(2+\gamma_j)}{4-\gamma_i\gamma_j} \\ 0 & 0 & \frac{-12EI_y}{L^3}\cdot\frac{\beta_i+\beta_j+\beta_i\beta_j}{4-\beta_i\beta_j} & 0 & \frac{6EI_y}{L^2}\cdot\frac{\beta_i(2+\beta_j)}{4-\beta_i\beta_j} & 0 \\ 0 & 0 & 0 & \frac{-GJ}{L}\alpha & 0 & 0 \\ 0 & 0 & \frac{-6EI_y}{L^2}\cdot\frac{\beta_j(2+\beta_i)}{4-\beta_i\beta_j} & 0 & \frac{2EI_y}{L}\cdot\frac{3\beta_i\beta_j}{4-\beta_i\beta_j} & 0 \\ 0 & \frac{6EI_z}{L^2}\cdot\frac{\gamma_j(2+\gamma_i)}{4-\gamma_i\gamma_j} & 0 & 0 & 0 & \frac{2EI_z}{L}\cdot\frac{3\gamma_i\gamma_j}{4-\gamma_i\gamma_j} \end{bmatrix}$$

$$K_{jj} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3}\cdot\frac{\gamma_i+\gamma_j+\gamma_i\gamma_j}{4-\gamma_i\gamma_j} & 0 & 0 & 0 & \frac{-6EI_z}{L^2}\cdot\frac{\gamma_j(2+\gamma_i)}{4-\gamma_i\gamma_j} \\ 0 & 0 & \frac{12EI_y}{L^3}\cdot\frac{\beta_i+\beta_j+\beta_i\beta_j}{4-\beta_i\beta_j} & 0 & \frac{6EI_y}{L^2}\cdot\frac{\beta_j(2+\beta_i)}{4-\beta_i\beta_j} & 0 \\ 0 & 0 & 0 & \frac{GJ}{L}\alpha & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{L^2}\cdot\frac{\beta_j(2+\beta_i)}{4-\beta_i\beta_j} & 0 & \frac{4EI_y}{L}\cdot\frac{3\beta_j}{4-\beta_i\beta_j} & 0 \\ 0 & \frac{-6EI_z}{L^2}\cdot\frac{\gamma_j(2+\gamma_i)}{4-\gamma_i\gamma_j} & 0 & 0 & 0 & \frac{4EI_z}{L}\cdot\frac{3\gamma_j}{4-\gamma_i\gamma_j} \end{bmatrix}$$

FIG. 6 (cont'd) COMPONENTS OF EQUATIONS (3-19) AND (3-20)

$$S_{ii} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^3} & \frac{4EI}{L} \end{bmatrix} \qquad S_{ij} = \begin{bmatrix} \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{-6EI}{L^3} & \frac{2EI}{L} \end{bmatrix}$$

$$S_{ij} = \begin{bmatrix} \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} \end{bmatrix} \qquad S_{jj} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

(a) Stiffness Matrices for Rigid Joints

$$C_{ii} = C_{ji} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{4\gamma_j - 2\gamma_i + \gamma_i\gamma_j}{4-\gamma_i\gamma_j} & -2L\frac{\gamma_i(1-\gamma_j)}{4-\gamma_i\gamma_j} \\ 0 & \frac{6}{L}\frac{\gamma_i-\gamma_j}{4-\gamma_i\gamma_j} & 3\frac{\gamma_i(2-\gamma_j)}{4-\gamma_i\gamma_j} \end{bmatrix}$$

$$C_{ij} = C_{jj} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{4\gamma_i - 2\gamma_j + \gamma_i\gamma_j}{4-\gamma_i\gamma_j} & 2L\frac{\gamma_j(1-\gamma_i)}{4-\gamma_i\gamma_j} \\ 0 & \frac{6}{L}\frac{\gamma_i-\gamma_j}{4-\gamma_i\gamma_j} & 3\frac{\gamma_j(2-\gamma_i)}{4-\gamma_i\gamma_j} \end{bmatrix}$$

(b) Correction Matrices

FIG. 7. S AND C MATRICES FOR PLANAR FRAMES

$$
S_{ii} = \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix}
$$

$$
S_{ij} = \begin{bmatrix}
\frac{-EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{-GJ}{L} & 0 & 0 \\
0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L}
\end{bmatrix}
$$

(a) Stiffness Matrices

FIG. 8. S AND C MATRICES FOR SPACE FRAMES

$$S_{ji} = \begin{bmatrix} -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} \\ 0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{-GJ}{L} & 0 & 0 \\ 0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\ 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \end{bmatrix}$$

$$S_{jj} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} \\ 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\ 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix}$$

FIG. 8 (cont'd) S AND C MATRICES FOR SPACE FRAMES

$$
C_{ii} = C_{ji} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{4\gamma_j - 2\gamma_i + \gamma_i\gamma_j}{4-\gamma_i\gamma_j} & 0 & 0 & 0 & -2L\frac{\gamma_i(1-\gamma_j)}{4-\gamma_i\gamma_j} \\
0 & 0 & \frac{4\beta_j - 2\beta_i + \beta_i\beta_j}{4-\beta_i\beta_j} & 0 & -2L\frac{\beta_i(1-\beta_j)}{4-\beta_i\beta_j} & 0 \\
0 & 0 & 0 & \alpha & 0 & 0 \\
0 & 0 & \frac{6}{L}\frac{\beta_i-\beta_j}{4-\beta_i\beta_j} & 0 & 3\frac{\beta_i(2-\beta_j)}{4-\beta_i\beta_j} & 0 \\
0 & \frac{6}{L}\frac{\gamma_i-\gamma_j}{4-\gamma_i\gamma_j} & 0 & 0 & 0 & 3\frac{\gamma_i(2-\gamma_j)}{4-\gamma_i\gamma_j}
\end{bmatrix}
$$

$$
C_{ij} = C_{jj} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{4\gamma_i - 2\gamma_j + \gamma_i\gamma_j}{4-\gamma_i\gamma_j} & 0 & 0 & 0 & 2L\frac{\gamma_j(1-\gamma_i)}{4-\gamma_i\gamma_j} \\
0 & 0 & \frac{4\beta_i - 2\beta_j + \beta_i\beta_j}{4-\beta_i\beta_j} & 0 & 2L\frac{\beta_j(1-\beta_i)}{4-\beta_i\beta_j} & 0 \\
0 & 0 & 0 & \alpha & 0 & 0 \\
0 & 0 & \frac{6}{L}\frac{\beta_i-\beta_j}{4-\beta_i\beta_j} & 0 & 3\frac{\beta_j(2-\beta_i)}{4-\beta_i\beta_j} & 0 \\
0 & \frac{6}{L}\frac{\gamma_i-\gamma_j}{4-\gamma_i\gamma_j} & 0 & 0 & 0 & 3\frac{\gamma_j(2-\gamma_i)}{4-\gamma_i\gamma_j}
\end{bmatrix}
$$

(b) Correction Matrices

FIG. 8 S AND C MATRICES FOR SPACE FRAMES (cont'd) of the member. Therefore, a correction matrix, C, by which the S matrix can be post-multiplied to produce the corresponding K matrix can be found.

Let

$$K = SC \quad (3\text{-}21)$$

from which

$$C = S^{-1}K \quad (3\text{-}22)$$

The resulting C matrices are defined as correction matrices and are shown in Fig. 7(b) for planar frames and in Fig. 8(b) for space frames.

Substituting into equations (3-17) to (3-20) we have:

$$F_i = S_{ii} \; C_{ii} \; D_i + S_{ij} \; C_{ij} \; D_j \quad (3\text{-}23)$$

$$F_j = S_{ji} \; C_{ji} \; D_i + S_{jj} \; C_{jj} \; D_j \quad (3\text{-}24)$$

Equations (3-23) and (-24) represent the internal forces on a member due to its end displacements.

CHAPTER IV

METHOD OF ANALYSIS

Problems in structural analysis can be solved using matrix algebra by two methods. The force, or flexibility method takes forces and moments as unknowns. It is generally better suited for investigation of complex aircraft and shell structures. The strain energy method is an example of the force method. The displacement, or stiffness method considers linear and angular displacements as unknowns. This method is especially suited for the analysis of structural frameworks. An example is the slope deflection method.

Both the force and displacement methods have advantages and disadvantages and therefore the choice of one method over the other is not necessarily obvious. In structural frameworks the member displacements coincide with the joint displacements and therefore the unknown displacements of the ends of all the members meeting at a joint can be found in terms of the joint displacements. Invariably the number of unknowns in the displacement method is smaller than the corresponding number in the force method.

The total internal forces at the ends of a member consist of those due to the end displacement and those due to the external span loading. Thus

$$P_i = F_i + p_i \qquad (4\text{-}1)$$

$$P_j = F_j + p_j \qquad (4\text{-}2)$$

Substituting equations (3-21) and (3-22) for F_i and F_j, we have

$$P_i = S_{ii} \; C_{ii} \; D_i + S_{ij} \; C_{ij} \; D_j + p_i \qquad (4\text{-}3)$$

$$P_j = S_{ji} \; C_{ji} \; D_i + S_{jj} \; C_{jj} \; D_j + p_j \qquad (4\text{-}4)$$

Equations (4-3) and (4-4) may be written in matrix form:

$$\begin{bmatrix} P_i \\ P_j \end{bmatrix} = \begin{bmatrix} S_{ii} & 0 \\ 0 & S_{jj} \end{bmatrix} \begin{bmatrix} C_{ii} & 0 \\ 0 & C_{jj} \end{bmatrix} \begin{bmatrix} D_i \\ D_j \end{bmatrix} + \begin{bmatrix} 0 & S_{ij} \\ S_{ji} & 0 \end{bmatrix} \begin{bmatrix} C_{ji} & 0 \\ 0 & C_{ij} \end{bmatrix} \begin{bmatrix} D_i \\ D_j \end{bmatrix} + \begin{bmatrix} p_i \\ p_j \end{bmatrix} \qquad (4\text{-}5)$$

Since $C_{ii} = C_{ji}$ and $C_{jj} = C_{ij}$ (Figs. 7 and 8), equation (4-5) may be written:

$$\begin{bmatrix} P_i \\ P_j \end{bmatrix} = \begin{bmatrix} S_{ii} & S_{ij} \\ S_{ji} & S_{jj} \end{bmatrix} \begin{bmatrix} C_{ii} & 0 \\ 0 & C_{jj} \end{bmatrix} \begin{bmatrix} D_i \\ D_j \end{bmatrix} + \begin{bmatrix} p_i \\ p_j \end{bmatrix} \qquad (4\text{-}6)$$

or

$$P_m = S_m \; C_m \; D_m + p_m \qquad (4\text{-}7)$$

Equation (4-7) represents the total internal forces at ends i and j of any member m.

If we consider a frame consisting of n members, the forces at the ends of all the members referred to their own axis may be written as:

$$\begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ P_m \\ \cdot \\ \cdot \\ P_n \end{bmatrix} = \begin{bmatrix} S_1 & 0 & \cdot\,\cdot & 0 & \cdot\,\cdot & 0 \\ 0 & S_2 & \cdot\,\cdot & 0 & \cdot\,\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot\,\cdot & & & \cdot \\ 0 & 0 & \cdot\,\cdot & S_m & \cdot\,\cdot & 0 \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot\,\cdot \\ 0 & 0 & \cdot\,\cdot & 0 & \cdot\,\cdot & S_n \end{bmatrix} \begin{bmatrix} C_1 & 0 & \cdot\,\cdot & 0 & \cdot\,\cdot & 0 \\ 0 & C_2 & \cdot\,\cdot & 0 & \cdot\,\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot\,\cdot & & \cdot \\ 0 & 0 & \cdot\,\cdot & C_m & \cdot\,\cdot & 0 \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot\,\cdot \\ 0 & 0 & \cdot\,\cdot & 0 & \cdot\,\cdot & C_n \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ \cdot \\ \cdot \\ D_m \\ \cdot \\ \cdot \\ D_n \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ p_m \\ \cdot \\ \cdot \\ p_n \end{bmatrix} \qquad (4\text{-}8)$$

or

$$P = SCD + p \tag{4-9}$$

for the whole frame.

Let D' represent k unknown joint displacements referred to the fixed co-ordinate system shown in Fig. 9(a). The k components of D' may easily be determined by inspection of the frame, and the member displacements D may be expressed in terms of D'.

Let U be a 6n.k matrix composed of the transformation matrices T_m, Fig. 9(b), of each member such that

$$D = UD' \tag{4-10}$$

Substituting equation (4-10) into equation (4-9) we have

$$P = SCUD' + p \tag{4-11}$$

Pre-multiplying both sides by the transpose of U we have

$$U^T P = U^T SCUD' + U^T p \tag{4-12}$$

$U^T P$ represents k sets of the sum of the internal force components at the joints where D' is induced, in the direction of D'. Since the structure is in equilibrium, the sum of these force components and the external force components acting on these joints must be equal to zero. Thus

$$U^T P + p_o = 0 \tag{4-13}$$

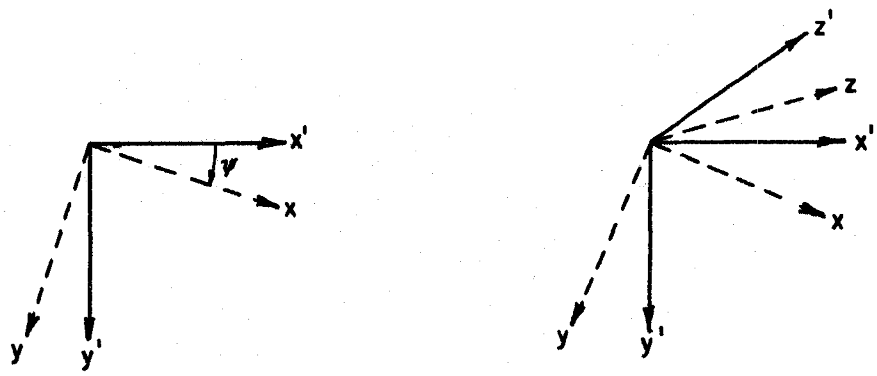
or

$$U^T SCUD' + U^T p + p_o = 0 \tag{4-14}$$

Equation (4-14) represents, in fact, k equilibrium equations in terms of k unknowns.

Solving for D', we have

$$D' = - (U^T SCU)^{-1} (U^T p + p_o) \tag{4-15}$$



(a) Fixed Co-ordinate Systems

$$T_m = \begin{bmatrix} 1 & m & 0 \\ -m & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_m = \begin{bmatrix} l_x & m_x & n_x & 0 & 0 & 0 \\ l_y & m_y & n_y & 0 & 0 & 0 \\ l_z & m_z & n_z & 0 & 0 & 0 \\ 0 & 0 & 0 & l_x & m_x & n_x \\ 0 & 0 & 0 & l_y & m_y & n_y \\ 0 & 0 & 0 & l_z & m_z & n_z \end{bmatrix}$$

(b) Transformation Matrices

FIG. 9. FIXED CO-ORDINATE SYSTEMS AND TRANSFORMATION MATRICES

Equation (4-15) is a formula for the k unknown joint displacements referred to the fixed co-ordinate system.

Replacing D' in equation (4-11) by equation (4-15) we have

$$P = p - SCU\,(U^T SCU)^{-1}\,(U^T p + p_o) \qquad (4\text{-}16)$$

Equation (4-16) is a formula for the internal forces on the members of a frame referred to their own longitudinal axis.

From equation (4-16), the following remarks can be made:

1. The 6n.1 matrix, P, is the solution of the problem and is obtained by multiplication, inversion, addition and subtraction of matrices S, C, U, U^T, p and p_o.

2. The 6n.1 matrix, S, depends only upon the geometric and elastic properties of the members.

3. The 6n.1 matrix, C, depends only upon the fixity factors and length of the members.

4. The 6n.k matrix U and k.6n matrix U^T depend only upon the orientation of the members.

5. The 6n.1 matrix p depends upon the external loads on the span of the members and upon their fixity factors.

6. The k.1 matrix p_o depends only upon the external loads on the joints of the frame.

7. The order of the matrix to be inverted, $(U^T SCU)$, is k.k.

8. All the matrix operations mentioned are easily performed by any digital computer.

The order of the matrices included in equation (4-16) can easily be decreased by neglecting the effect due to the axial deformation of the members. In arranging the stiffness matrix it is not necessary to include the forces of all the members of the structure; by considering only the forces necessary to establish the equilibrium equations, the order of the matrices can be further reduced.

Because the correction matrices were derived based upon the presence of both shear and moment at the ends of a member, if the moment at the end of a member is desired, the shearing force at that end must also be included, and vice versa.

CHAPTER V

SAMPLE CALCULATIONS

General Calculations

The matrix operations involved in the formula, given by equation (4-16), for the total internal forces at the ends of the members of a frame can be performed by using various techniques. As previously mentioned, hand computation is possible but would prove to be tedious and time consuming for all but the simplest problems. The use of a desk calculator would hasten the calculations, but the inversion of U^TSCU would require considerable time. An electronic computer can handle large order matrices with phenomenal speed and accuracy and it is, therefore, the most effective tool available. The problems that can be solved using the derived expressions are limitless; the storage capacity of the available computer is the only restriction that limits the size of the problem.

When a problem is first presented, it is necessary to study it and break it down into a number of smaller problems. Whether or not an electronic computer is at one's disposal, it is helpful to specify the sequence in which the matrix operations are to be performed. In this respect a flow chart, which outlines the optimum number of steps which will lead to the desired solutions, can be prepared. Fig. 10 shows the flow chart for the solution of the internal forces and displacements given by equations (4-15) and (4-16).

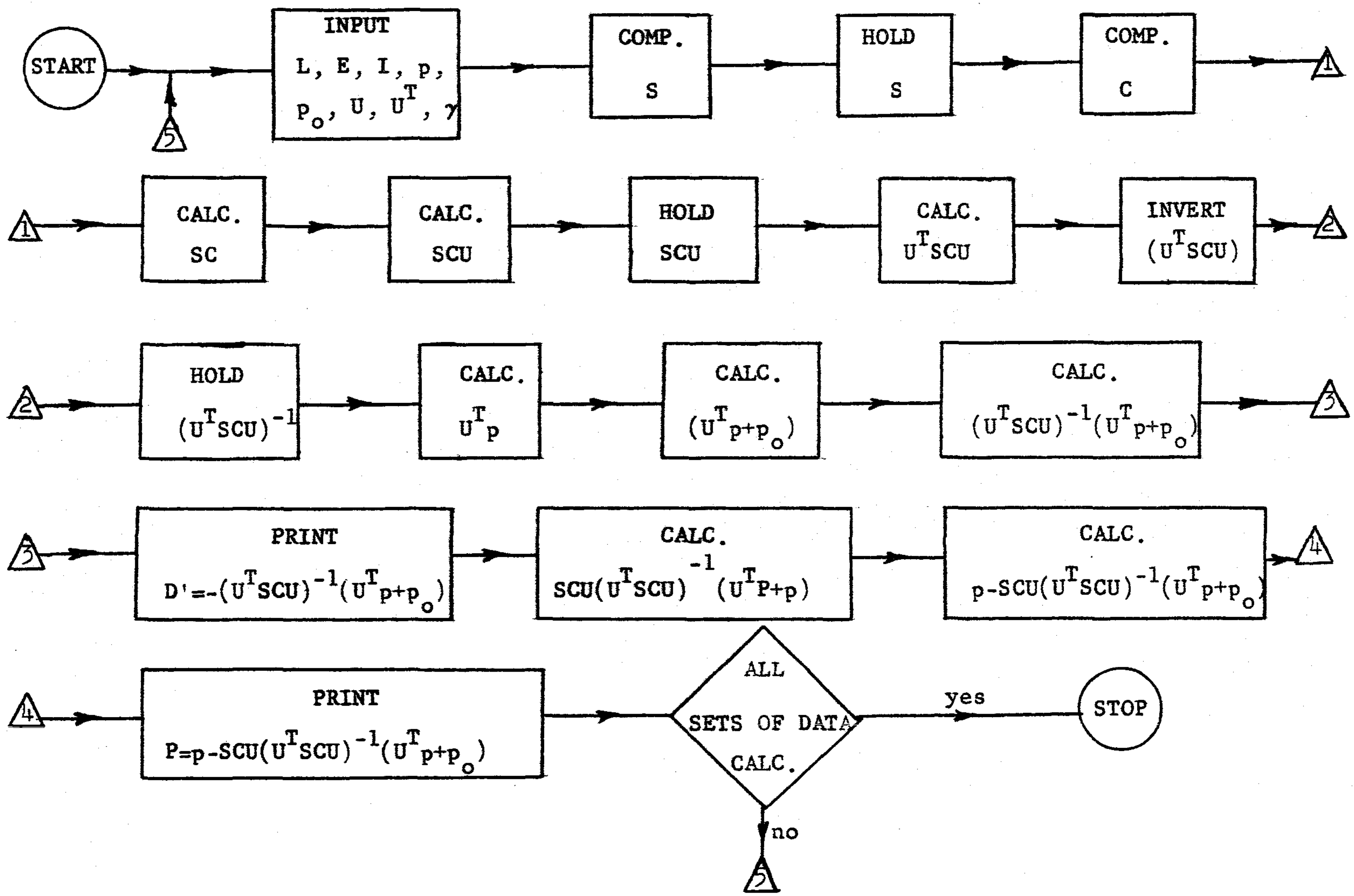


FIG. 10 FLOW CHART

It is debatable whether the S and C matrices should be computed by the computer or if they should be computed using a desk calculator and then entered directly into the machine as data. This choice will ordinarily depend upon the analyst's judgement and the complexity of the problem.

Numerical Examples

Example 1 - Single Bay, One Story Frame (10)

In Fig. 11 is shown a portal frame in which the beam is connected to the columns by means of standard beam connections. The feet of the columns are perfectly fixed.

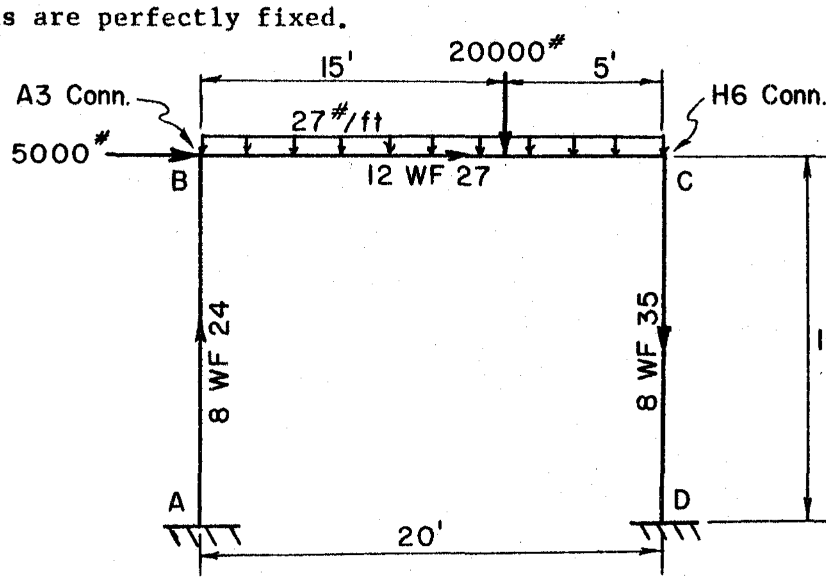


FIG. 11. SINGLE BAY, ONE STORY PORTAL FRAME

From Table A-1, Appendix A, $1/Z_{BC} = .319 \times 10^8$ and $1/Z_{CB} = 5.499 \times 10^8$. From Steel Construction (13), $I_{AB} = 82.5$ in^3, $I_{BC} = 204.1$ in^3, $I_{CD} = 126.5$ in^3 and $E = 29 \times 10^6$ p.s.i. The fixity factors are found to be $\gamma_{AB} = \gamma_{BA} = \gamma_{CD} = \gamma_{DC} = 1$; $\gamma_{BC} = .301$ and $\gamma_{CB} = .882$. The S and C matrices for the frame, neglecting axial deformation are:

$$S=\begin{bmatrix} 4.055\times10^{3} & -3.889\times10^{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ -3.889\times10^{5} & 4.980\times10^{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.135\times10^{3} & 6.162\times10^{5} & -5.135\times10^{3} & 6.162\times10^{5} & 0 & 0 \\ 0 & 0 & 6.162\times10^{5} & 9.864\times10^{7} & -6.162\times10^{5} & 4.932\times10^{7} & 0 & 0 \\ 0 & 0 & -5.135\times10^{3} & -6.162\times10^{5} & 5.135\times10^{3} & -6.162\times10^{5} & 0 & 0 \\ 0 & 0 & 6.162\times10^{5} & 4.932\times10^{7} & -6.162\times10^{5} & 9.864\times10^{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6.218\times10^{3} & 5.971\times10^{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 5.971\times10^{5} & 7.640\times10^{7} \end{bmatrix}$$

$$C=\begin{bmatrix} 1.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .8546 & -4.565 & 0 & 0 & 0 & 0 \\ 0 & 0 & -.0039 & .2700 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -.0790 & 79.24 & 0 & 0 \\ 0 & 0 & 0 & 0 & -.0039 & 1.204 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.000 \end{bmatrix}$$

The unknown joint displacements D' are $x_B = x_C$, θ_B and θ_C. Thus,

$$\begin{bmatrix} y_{BA} \\ \theta_{BA} \\ y_{BC} \\ \theta_{BC} \\ y_{CB} \\ \theta_{CB} \\ y_{CD} \\ \theta_{CD} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_B \\ \theta_B \\ \theta_C \end{bmatrix} \text{, or } D = UD' \text{ .}$$

The internal force components at the ends of the members due to the external span loading, p, and the external forces acting on the joints, p_o, are:

$$p = \begin{bmatrix} Y_{BA} \\ M_{BA} \\ Y_{BC} \\ M_{BC} \\ Y_{CB} \\ M_{CB} \\ Y_{CD} \\ M_{CD} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -5270 \\ -72448 \\ -15270 \\ 676131 \\ 0 \\ 0 \end{bmatrix} \text{ , } p_o = \begin{bmatrix} X_B \\ M_B \\ M_C \end{bmatrix} = \begin{bmatrix} -5000 \\ 0 \\ 0 \end{bmatrix}$$

Substituting S, C, U, U^T, p and p_o into equations (4-15) and (4-16), and performing the necessary matrix operations, we obtain:

$$P = \begin{bmatrix} Y_{BA} \\ M_{BA} \\ Y_{BC} \\ M_{BC} \\ Y_{CB} \\ M_{CB} \\ Y_{CD} \\ M_{CD} \end{bmatrix} = \begin{bmatrix} 357.2 \\ 13452.3 \\ -5471.6 \\ -13452.3 \\ -14528.4 \\ 503761.8 \\ -4642.8 \\ -503761.8 \end{bmatrix}, \text{ and } D' = \begin{bmatrix} X_B \\ \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} .4548 \\ .003825 \\ -.003039 \end{bmatrix}$$

Table 5-1 compares the end moments and maximum positive moments of the frame with semi-rigid connections and those for the same frame with perfectly rigid joints.

Table 5-1

Maximum Positive and Negative Moments for Rigid and Semi-Rigid Cases, Example 1.

MEMBER	MOMENTS (ft-k)					
	SEMI-RIGID			RIGID		
	M_i	M_j	M_s	M_i	M_j	M_s
AB	-82035.0	13452.3		-69642.3	27498.3	
BC	-13452.3	503761.8	971475.8	-27498.3	526849.4	835471.2
CD	503761.8	-387656.0		-526849.4	-391006.8	

Example 2 - Two Bay Frame

Fig.12 illustrates a frame of two bays having different heights. The fixity factors for the members are indicated in the diagram and EI is taken as 100 k-ft^2. for purposes of computation.

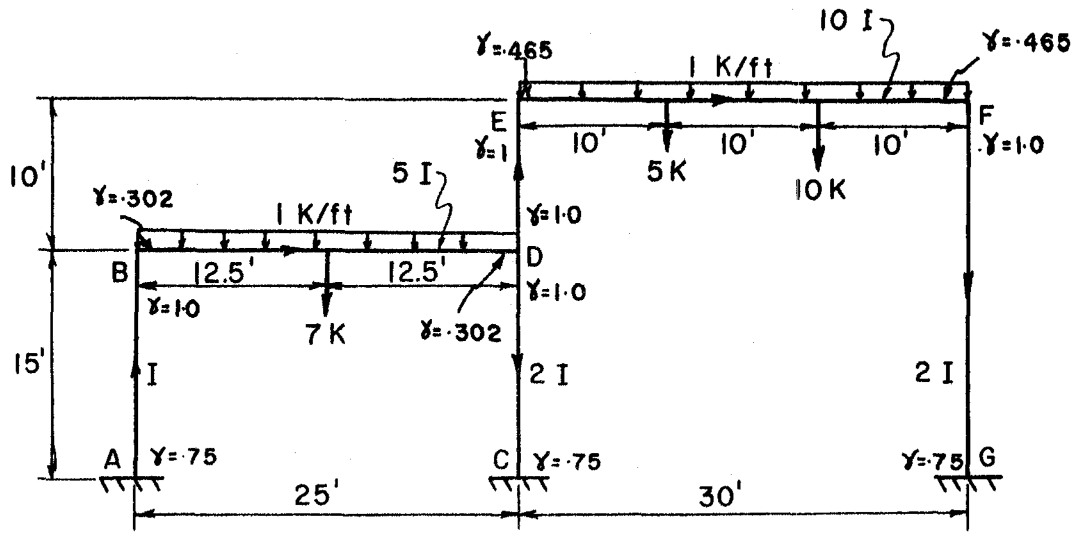


FIG. 12. TWO BAY FRAME

$$
\begin{bmatrix} Y_{BA} \\ \theta_{BA} \\ Y_{BD} \\ \theta_{BD} \\ Y_{DB} \\ \theta_{DB} \\ Y_{DC} \\ \theta_{DC} \\ Y_{DE} \\ \theta_{DE} \\ Y_{ED} \\ \theta_{ED} \\ Y_{EF} \\ \theta_{EF} \\ Y_{FE} \\ \theta_{FE} \\ Y_{FG} \\ \theta_{FG} \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} X_B \\ \theta_B \\ \theta_D \\ X_E \\ \theta_E \\ \theta_F \end{bmatrix}, \text{ or } D = UD'
$$

$$S=\begin{bmatrix}
.3556 & -2.667 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2.667 & 26.67 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & .3840 & 4.800 & -.3840 & 4.800 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4.800 & 80.00 & -4.800 & 40.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -.3840 & -4.800 & .3840 & -4.800 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4.800 & 40.00 & -4.800 & 80.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .7111 & 5.333 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5.333 & 53.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.400 & 12.00 & -2.400 & 12.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.00 & 80.00 & -12.00 & 40.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.400 & -12.00 & 2.400 & -12.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.00 & 40.00 & -12.00 & 80.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .4444 & 6.667 & -.4444 & 6.667 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.667 & 133.3 & -6.667 & 66.67 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.4444 & -6.667 & .4444 & -6.667 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.667 & 66.67 & -6.667 & 133.3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .1536 & 1.920 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.920 & 32.00
\end{bmatrix}$$

$$
C = \begin{bmatrix}
.5385 & 2.308 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-.0308 & 1.154 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & .1779 & -2.696 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .3936 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & .1779 & 2.696 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & .3936 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .5385 & -2.308 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .0308 & 1.154 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .3029 & -3.945 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .5659 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .3029 & 3.945 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .5659 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .5385 & -3.846 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .0185 & 1.154
\end{bmatrix}
$$

The p and p_o matrices are shown below along with the final results for P.

$$p = \begin{bmatrix} 0 \\ 0 \\ -16.00 \\ -29.11 \\ -16.00 \\ +29.11 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -21.67 \\ -69.05 \\ -23.33 \\ +72.42 \\ 0 \\ 0 \end{bmatrix} \qquad p_o = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \quad P = \begin{bmatrix} -1.960 \\ 19.58 \\ -15.82 \\ -19.58 \\ -16.18 \\ 24.06 \\ -0.260 \\ -6.442 \\ 1.700 \\ -17.62 \\ -1.700 \\ 34.62 \\ -21.90 \\ -34.62 \\ -23.10 \\ 31.02 \\ -1.700 \\ -31.02 \end{bmatrix}$$

Table 5-2 compares the moments for the rigid and semi-rigid cases. The beams of a frame as that of Fig. 12 would ordinarily be designed as beams with simple supports. For these end conditions the maximum span moments, M_s, for BD and EF are 121.9 ft-k and 188.9 ft-k respectively. Comparing these values of M_s with those in Table 5-2 it can be seen that a significant saving in steel is possible by considering the semi-rigidity of connections.

TABLE 5-2

Maximum Positive and Negative Moments for Rigid and Semi-Rigid Cases

Example 2

MEMBER	MOMENTS (ft-K)					
	SEMI-RIGID			RIGID		
	M_i	M_j	M_s	M_i	M_j	M_s
AB	9.82	19.58		21.26	33.09	
BD	-19.58	24.06	100.04	-33.09	49.22	80.72
DC	-6.44	2.54		-21.31	-1.25	
DE	-17.62	34.62		-27.91	49.11	
EF	-34.62	31.02	158.19	-49.11	36.82	153.14
FG	-31.02	-11.48		-36.82	-16.19	

Example 3 - Two Bay Frame with Gable Roof

Fig. 13 shows a two bay frame with a gable roof on one of the bays. All of the joints are assumed to be rigidly fixed except that at E. The value of the fixity factor at E is varied between 0 and 1 and the resulting moemnts are compared. EI is taken as 1000 K-ft^2.

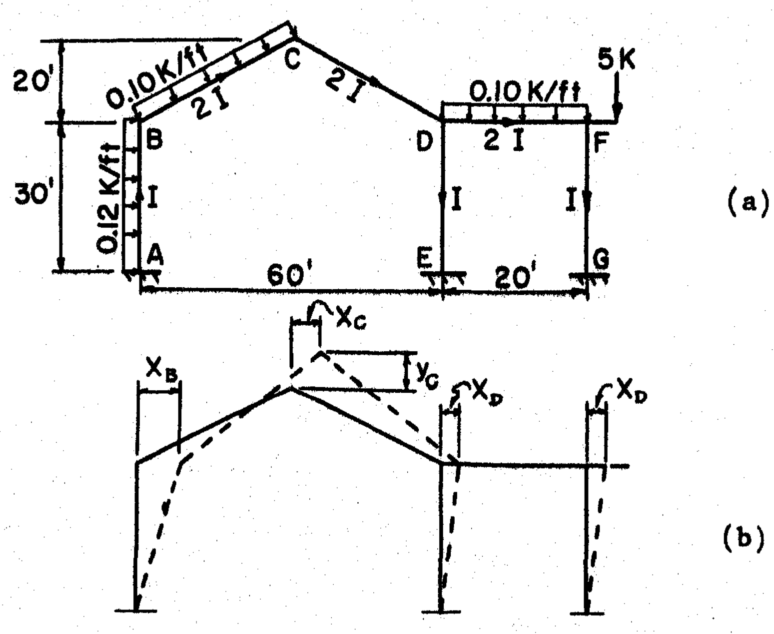


FIG. 13. TWO BAY FRAME WITH GABLE ROOF

From Fig. 13(b), $x_C = 1/2\ (x_D + x_B)$

$$y_C = 3/4\ (x_B - x_C)$$

$$\begin{bmatrix} y_{BA} \\ \theta_{BA} \\ y_{BC} \\ \theta_{BC} \\ y_{CB} \\ \theta_{CB} \\ y_{CD} \\ \theta_{CD} \\ y_{DC} \\ \theta_{DC} \\ y_{DE} \\ \theta_{DE} \\ y_{DF} \\ \theta_{DF} \\ y_{FD} \\ \theta_{FD} \\ y_{FG} \\ \theta_{FG} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ .555 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -.347 & 0 & 0 & .901 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -.901 & 0 & 0 & .347 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.555 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_B \\ \theta_B \\ \theta_C \\ x_D \\ \theta_D \\ \theta_F \end{bmatrix}$$

or $D = UD'$

$$
s = \begin{bmatrix}
.4444 & -6.667 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6.667 & 133.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & .5120 & 9.231 & -.5120 & 9.231 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 9.231 & 221.9 & -9.231 & 110.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -.5120 & -9.231 & .5120 & -9.231 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 9.231 & 110.9 & -9.231 & 221.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .5120 & 9.231 & -.5120 & 9.231 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 9.231 & 221.9 & -9.231 & 110.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -.5120 & -9.231 & .5120 & -9.231 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 9.231 & 110.9 & -9.231 & 221.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .4444 & 6.667 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.667 & 133.3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.000 & 30.00 & -3.000 & 30.00 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 30.00 & 400.0 & -30.00 & 200.0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.000 & -30.00 & 3.000 & -30.00 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 30.00 & 200.0 & -30.00 & 400.0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .4444 & 6.667 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.667 & 133.3
\end{bmatrix}
$$

$$
C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{11-11} & c_{11-12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{12-11} & c_{12-12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$p = \begin{bmatrix} -1.800 \\ +9.000 \\ -1.80 \\ -10.83 \\ -1.800 \\ +10.83 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1.000 \\ -3.333 \\ -1.000 \\ +3.333 \\ 0 \\ 0 \end{bmatrix} \qquad p_o = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -25 \end{bmatrix}$$

Table 5-3 gives the values of c_{11-11}, c_{11-12}, c_{12-11} and c_{12-12}, (shown in the C matrix) for various values of γ_E.

TABLE 5-3

Values of c_{11-11}, c_{11-12}, c_{12-11}, and c_{12-12}

γ_E	c_{11-11}	c_{11-12}	c_{12-11}	c_{12-12}
0	-.5000	-15.00	.0500	1.500
.25	-.2000	-12.00	.0400	1.400
.50	.1429	-8.571	.0286	1.286
.75	.5385	-4.615	.0154	1.154
1.00	1.000	0	0	1.000

The resulting moments at the ends of the members are given in Table 5-4.

TABLE 5-4

Moments at Ends of Members for Various Values of γ_E, Example 3

Member	Moments (ft-k)				
	γ_E=0	γ_E=.25	γ_E=.50	γ_E=.75	γ_E=1.0
AB	-34.94	-33.74	-32.66	-31.69	-29.20
BA	-8.35	-7.96	-7.60	-7.28	-6.99
BC	+8.35	+7.96	+7.60	+7.28	+6.99
CB	+9.23	+10.31	+11.27	+12.13	+12.92
CD	-9.23	-10.31	-11.27	-12.13	-12.92
DC	-2.24	-2.62	-2.97	-3.27	-3.55
DE	-18.02	-18.66	-19.22	-19.73	-20.18
ED	0	-6.48	-12.28	-17.48	-22.21
DF	+20.26	+21.28	+22.18	+23.00	+23.73
FD	+47.59	+45.16	+42.99	+41.03	+39.27
FG	-22.59	-20.16	-17.99	-16.03	-14.27
GF	-29.79	-26.70	-24.01	-21.47	-19.24

Example 4 - Grid Framework

In Fig. 14 is shown a grid frame which is part of a floor system. It is assumed that sideway is prevented and that EI = 100 and GJ = .2 EI, E = constant for all members. The fixity factors are given in the figure.

$$\begin{bmatrix} y_{BA} \\ \theta^x_{BA} \\ \theta^z_{BA} \\ y_{BC} \\ \theta^x_{BC} \\ \theta^z_{BC} \\ y_{BE} \\ \theta^x_{BE} \\ \theta^z_{BE} \\ y_{EB} \\ \theta^x_{EB} \\ \theta^z_{EB} \\ y_{ED} \\ \theta^x_{ED} \\ \theta^z_{ED} \\ y_{FE} \\ \theta^x_{FE} \\ \theta^z_{FE} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_B \\ \theta^x_B \\ \theta^z_B \\ y_E \\ \theta^x_E \\ \theta^z_E \end{bmatrix}$$

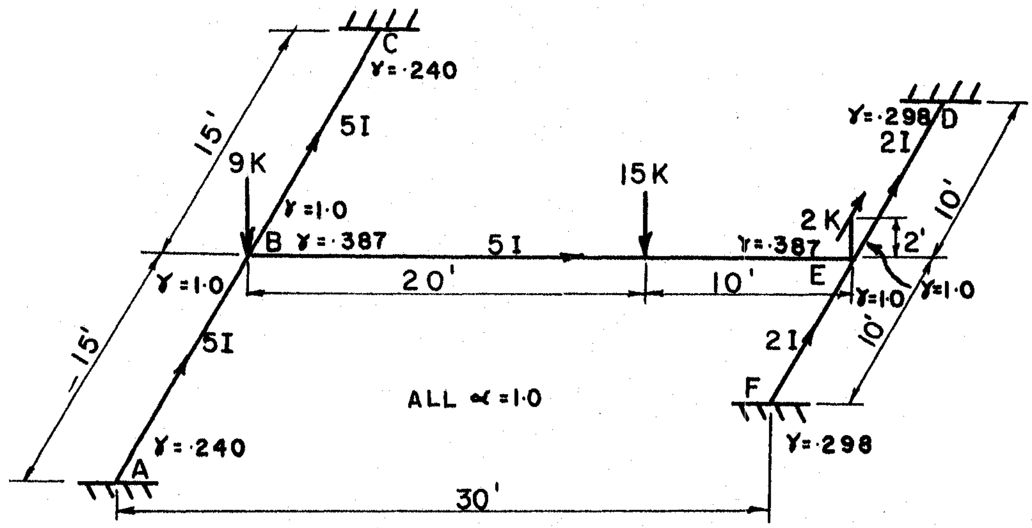


FIG. 14. GRID FRAMEWORK

The p and p_o matrices are shown below along with the resulting P matrix which is the solution to the problem.

$$p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -5.00 \\ 0 \\ -20.32 \\ -10.00 \\ 0 \\ 28.32 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad p_o = \begin{bmatrix} -9.00 \\ 0 \\ 0 \\ 0 \\ 4.00 \\ 0 \end{bmatrix} \qquad P = \begin{bmatrix} 7.102 \\ 4.154 \\ -80.56 \\ 7.109 \\ 4.154 \\ 80.65 \\ -5.211 \\ .0988 \\ -8.308 \\ -9.789 \\ 9.983 \\ 5.119 \\ -4.992 \\ 37.17 \\ 4.670 \\ -4.992 \\ -33.27 \\ .4944 \end{bmatrix}$$

$$
S=\begin{bmatrix}
1.778 & 0 & -13.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6.667 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-13.33 & 0 & 133.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.778 & 0 & 13.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6.667 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 13.33 & 0 & 133.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .2222 & 0 & 3.333 & -.2222 & 0 & 3.333 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.333 & 0 & 0 & -3.333 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3.333 & 0 & 66.67 & -3.333 & 0 & 33.33 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -.2222 & 0 & -3.333 & .2222 & 0 & -3.333 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.333 & 0 & 0 & 3.333 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3.333 & 0 & 33.33 & -3.333 & 0 & 66.67 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.400 & 0 & 12.00 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.00 & 0 & 80.00 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.400 & 0 & -12.00 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.000 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12.00 & 0 & 80.00
\end{bmatrix}
$$

$$C=\begin{bmatrix}
-.2128 & & 6.064 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-.0809 & 0 & 1.404 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -.2128 & & -6.064 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .0809 & 0 & 1.404 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .2399 & 0 & -5.697 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .4864 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2399 & 0 & 3.697 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .4864 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.1378 & 0 & -3.793 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .1138 & 0 & 1.379 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.1378 & 0 & 3.793 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.1138 & 0 & 1.379
\end{bmatrix}$$

Table 5-5 compares the moments on the ends of the members for the semi-rigid and rigid cases.

TABLE 5-5

Moments on Ends of Members for Rigid and Semi-Rigid Cases

Example 4

Mem.	Moments (ft-k)							
	Semi-Rigid				Rigid			
	M_i^x	M_j^x	M_i^z	M_j^z	M_i^x	M_j^x	M_i^z	M_j^z
AB	-4.15	4.15	-25.97	-80.56	-6.00	6.00	-56.78	-56.70
BC	4.15	-4.15	80.65	25.98	6.00	-6.00	56.78	56.82
BE	.0988	-.0988	-8.31	9.98	.0806	-.0806	-12.01	11.22
ED	-4.99	4.99	37.17	14.02	-5.61	5.61	24.12	23.13
EF	4.99	-4.99	-33.27	-13.43	5.61	-5.61	-21.17	-20.20

The values of M_s^z for the simply supported case are 105 ft-k for AC and 50 ft-k for DF. The values of M_s^z for the semi-rigid case are 80.65 ft-k and 37.17 ft-k, respectively.

CHAPTER VI

CONCLUSIONS

A method of investigating planar and space frames with semi-rigid connections was derived using the displacement method of matrix analysis. Correction matrices were found which depend upon the "fixity factors" of the members. These factors are defined such that the analyst is able to solve frames having joints which vary from hinged to rigid connections. Formulae, given by equations (4-15) and (4-16) were established for the displacements of the joints of a frame and for the forces acting at the ends of the members. These formulae are composed of known matrices and their solutions were shown to be readily adaptible to electronic digital computers. The method derived is limitless; the capacity of the available computer is the only restriction which limits the size of the problem that can be solved.

APPENDIX A

SEMI-RIGID CONNECTION FACTOR

Table A-1

Z-Factors for Standard AISC Beam Connections (10)

Web Thickness (in.)	Connection Number	Series A-B-H-K $\frac{1}{z \times 10^8}$	Series HH-KK $\frac{1}{z \times 10^8}$
5/8	3	0.402	0.411
1/2	3	0.371	0.379
3/8	3	0.343	0.351
1/4	3	0.319	0.326
3/4	4	1.079	1.105
5/8	4	0.994	1.018
1/2	4	0.917	0.884
3/8	4	0.850	0.870
1/4	4	0.789	0.808
3/4	5	3.692	3.798
5/8	5	3.392	3.489
1/2	5	3.126	3.215
3/8	5	2.888	2.970
3/4	6	6.490	6.676
5/8	6	5.961	6.131
1/2	6	5.499	5.650
3/8	6	5.071	5.216

Table continued on next page

Table A-1 (continued)

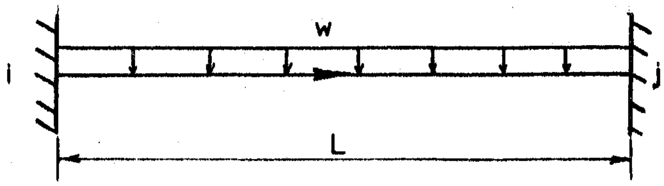
Web Thickness (in.)	Connection Number	Series A-B-H-K $\frac{1}{z \times 10^8}$	Series HH-KK $\frac{1}{z \times 10^8}$
3/4	7	10.433	10.733
5/8	7	9.590	9.854
1/2	7	8.835	9.088
3/4	8	15.745	16.168
5/8	8	14.455	14.843
1/2	8	13.316	13.686
7/8	9	24.80	25.236
3/4	9	24.82	23.184
5/8	9	24.83	21.283
1/2	9	24.86	19.635
1	10	37.058	38.129
7/8	10	33.793	34.784
3/4	10	31.042	31.984
5/8	10	28.535	29.358

Dependable percentages of rigidity for semi-rigid connections are tabulated in Reference (8) and a formula is given relating this percentage of rigidity to Z.

APPENDIX B

MODIFIED FIXED END MOMENTS

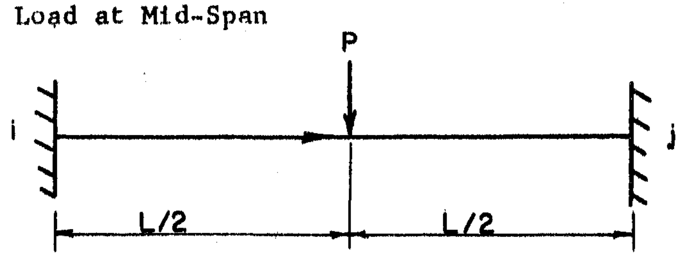
Uniformly Distributed Load



$$M_i = \frac{wL^2}{12} \cdot \frac{3\gamma_i(2-\gamma_j)}{4-\gamma_i\gamma_j}$$

$$M_j = \frac{wL^2}{12} \cdot \frac{3\gamma_j(2-\gamma_i)}{4-\gamma_i\gamma_j}$$

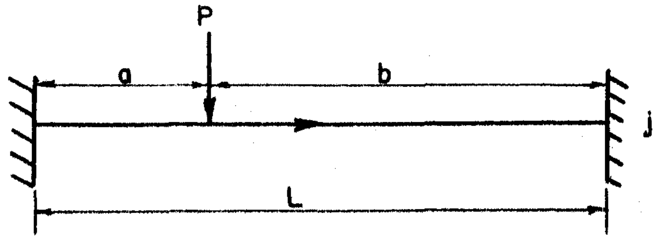
Concentrated Load at Mid-Span



$$M_i = \frac{PL}{8} \quad \frac{3\gamma_i(2-\gamma_j)}{4-\gamma_i\gamma_j}$$

$$M_j = \frac{PL}{8} \cdot \frac{3\gamma_j(2-\gamma_i)}{4-\gamma_i\gamma_j}$$

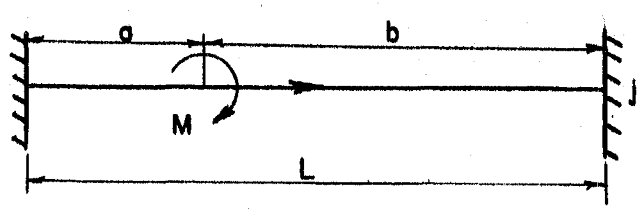
Concentrated Load at Any Point



$$M_i = \frac{Pab^2}{L^2}\left[\frac{\gamma_i L(4-\gamma_j) - \gamma_i a(2+\gamma_j)}{b(4-\gamma_i\gamma_j)}\right]$$

$$M_j = \frac{Pa^2b}{L^2}\left[\frac{2\gamma_j L(1-\gamma_i) + \gamma_j a(2+\gamma_i)}{a(4-\gamma_i\gamma_j)}\right]$$

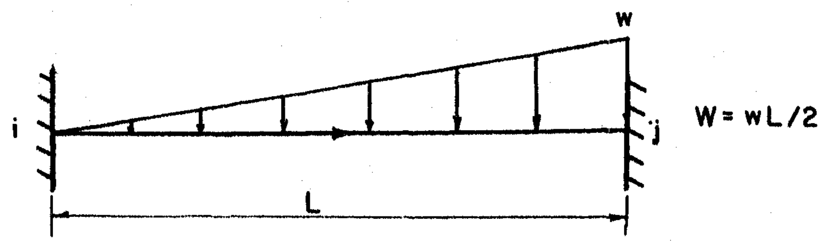
Moment on Span



$$M_i = \frac{M}{L^3}\cdot\frac{\gamma_i}{4-\gamma_i\gamma_j}\left[2a^3(1-\gamma_j)+3ab(2a-\gamma_j b)+b^3(4-\gamma_j)\right]$$

$$M_j = \frac{M}{L^3}\cdot\frac{\gamma_j}{4-\gamma_i\gamma_j}\left[2b^3(1-\gamma_i)+3ab(2b-\gamma_i a)+a^3(4-\gamma_i)\right]$$

Triangular Distributed Load



$$M_i = \frac{WL}{15} \cdot \frac{\gamma_i(7-4\gamma_j)}{4-\gamma_i\gamma_j}$$

$$M_j = \frac{WL}{30} \cdot \frac{\gamma_j(16-7\gamma_i)}{4-\gamma_i\gamma_j}$$

APPENDIX C

LGP-30 COMPUTATION OF EXAMPLE 1

The problems contained in Chapter V were all solved using the LGP-30 (Royal McBee) digital computer. The floating point interpretive system (24.4) was employed throughout. In addition, matrix inversion (D1-P30.0) and matrix multiplication (D1-P-30.2) subroutines, compiled by Purdue University School of Electrical Engineering, were stored in the computer. The following is the detailed program for Example 1; it follows exactly the flow chart outlined in Chapter V.

Storage

Subroutine Storage:

Matrix inversion (D1-P30.0) stored at 4000

Matrix multiplication (D1-P30.2) stored at 4300

Data Storage

Stiffness matrix, S(8x8), stored at 4600

Correction matrix, C(8x8) stored at 4800

Transformation matrix, U(8x3), stored at 5000

Transpose of U(3x8), stored at 5100

p(8x1) matrix stored at 5200

p_o(3x1) matrix stored at 5220

Program Storage

Program stored at 5300

Program

Program input codes:

	;0005300	start fill
	/0000000	set modifier

Location	Instruction	Contents of Address	Notes
5300	r6300	Floating point	24.2
01	u0400		
02	i0000	data input	
03	e0000	matrix mult.	
04	r4315	calling sequence	
05	u4300		
06	z4600	S	(8x8)
07	z0808		
08	z4800	C	(8x8)
09	z0808		
10	z5500	SC	(8x8)
11	r4315		
12	u4300		
13	z5500	SC	(8x8)
14	z0808		
15	z5000	U	(8x3)
16	z0803		
17	z5700	SCU	(8x3)
18	r4315		
19	u4300		
20	z5100	U^T	(3x8)
21	z0308		
22	z5700	SCU	(8x3)
23	z0803		
24	z5800	U^TSCU	(3x3)
25	r4003	matrix inv.	
26	u4000	calling sequence	

Location	Instruction	Contents of Address	Notes
27	35800	$(U^TSCU)^{-1}$	(3x3)
28	r4315		
29	u4300		
30	z5100	U^T	(3x8)
31	z0308		
32	z5200	p	(8x1)
33	z0801		
34	z5900	U^Tp	(3x1)
35	r6300		
36	u0400		
37	1c0003	set ind. reg.	
38	1e0000	no. 1	
39	1i0002		
40	1b5900	U^Tp	(3x1)
41	1a5220	p_o	(3x1)
42	1h5240	U^Tp+p_o	(3x1)
43	1z5340		
44	e0000		
45	r4315		
46	u4300		
47	z5800	$(U^TSCU)^{-1}$	(3x3)
48	z0303		
49	z5240	(U^Tp+p_o)	(3x1)
50	z0301		
51	z5250	$(U^TSCU)^{-1}(U^Tp+p_o) = -D'$	(3x1)
52	r6300		
53	u0400		
54	2c0003	set index reg.	
55	2e0000	no. 2	
56	2i0002		
57	2b5250	-D'	(3x1)
58	y0000	D'	(3x1)

Location	Instruction	Contents of Address	Notes
59	p0000	print D'	
60	m0000		car.ret.
61	2z5357		
62	e0000		
63	r4315		
5400	u4300		
01	z5700	SCU	(8x3)
02	z0803		
03	z5250	$(U^T SCU)^{-1}(U^T p+p_o)$	(3x1)
04	z0301		
05	z5900	$SCU(U^T SCU)^{-1}(U^T p+p_o)$	(8x1)
06	r6300		
07	u0400		
08	3c0008	set ind. reg. no. 3	
09	3e0000		
10	3i0002		
11	3b5200	p	(8x1)
12	3s5900	$SCU(U^T SCU)^{-1}(U^T p+p_o)$	(8x1)
13	p0000	P	print P
14	m0000		car.ret.
15	3z5411		
16	z0000		stop
17	.0005300		

LITERATURE CITED

(1) Argyris, J. H., "Energy Theorems and Structural Analysis", Aircraft Engineering, Vol. 27, Nos. 2 and 3, February and March, 1955.

(2) Baker, John F., "Methods of Stress Analysis", First Report of the Steel Structures Research Committee, Dept. of Scientific and Industrial Research of Great Britain.

(3) Batho, Cyril, First, Second and Final Reports, Steel Structures Research Committee, Department of Scientific and Industrial Research of Great Britain, H. M. Stationery Office, London, 1931-36.

(4) Benscoter, Stanley V., "Matrix Analysis of Continuous Beams", Trans. A.S.C.E., Vol. 112, 1947, p. 1109.

(5) Cross, Hardy, and N. D. Morgan, "Continuous Frames of Reinforced Concrete," John Wiley + Sons, Inc., New York, N.Y., 1932.

(6) Filko, Fernando Venancio, "Matrix Analysis of Plane Rigid Frames", Proc. A.S.C.E., Vol. 86, Part 2, 1960, ST7.

(7) Fowler, Keith T., "Slope Deflection Equations for Curved Members", Trans. A.S.C.E., Vol. 116, 1951, p.125.

(8) Hechtman, Robert A., and Bruce G. Johnston, "Riveted Semi-Rigid Beam-to-Column Building Connections." Progress Report Number 1, American Institute of Steel Construction, 1947.

(9) Johnson, Bruce, and Edward H. Mount, "Analysis of Building Frames with Semi-Rigid Connections," Trans. A.S.C.E., Vol. 107, 1942, pp. 993-1033.

(10) Lothers, John E., "Advanced Design in Structural Steel", Prentice Hall, Inc., Englewoods Cliffs, N.J., 1960, pp. 367-405.

(11) Parcel, John I., and Robert B. B. Moorman, "Analysis of Statically Indeterminate Structures", John Wiley + Sons, Inc., New York N.Y., 1959.

(12) Rathburn, J. Charles, "Elastic Properties of Riveted Connection", Trans. A.S.C.E. Vol. 101, 1936, pp. 524-595.

(13) Steel Construction, American Institute of Steel Construction, New York, N.Y., Fifth Edition, 1959.

(14) Sutherland, Hale, and Harry Lake Bowman, "Structual Theory," 4th Ed., John Wiley + Sons, 1950, pp. 239-242.

(15) Tuma, Jan J., Kerry S. Havner, and Frank Hedges, "Analysis of Frames with Curved and Bent Members", Proc. A.S.C.E., Vol. 84, Part 2, 1958, ST5.

(16) Wilson, Wilbur M., and George A. Maney, "Wind Stresses in the Steel Frames of Office Buildings", Bulletin No. 80, Eng. Experiment Station, Univ. of Illinois, 1915.

(17) Young, C. R., and K. B. Jackson, "The Relative Rigidity of Welded and Riveted Connections", Canadian Journal of Research, July and August, 1934.

NOMENCLATURE

A	area under the simple beam moment diagram due to external lateral loads.
C	correction matrix
D	displacements at ends of members, in matrix form, referred to their longitudinal axis
D'	unknown joint displacements, in matrix form, referred to a fixed set of axes
E	Young's modulus
F	internal forces at the ends of a member, due to its end displacements, in matrix form
G	shear modulus
i, j	subscripts indicating the ends of a member
I^y, I^z	moment of inertia about the y and z axes of a member
J	polar moment of inertia of a member
K	modified stiffness matrix
l,m,n	direction cosines of the axis of a member, relative to a fixed axis
L	length of a member
M^x,M^y,M^z	bending moments about the x, y and z axes of a member
M_S	maximum positive moment in the span of a member
p	forces at the ends of a member due to the external span loads, in matrix form
P_o	external forces acting on the joints of a frame, in matrix form
P	total internal forces at the ends of the members of a frame, in matrix form
R	angle of rotation of a member, due to the relative displacements of its ends
S	stiffness matrix

T_m	transformation matrix for a single member
U	transformation matrix for the whole frame
x,y,z	displacements at the ends of a member in the x, y and z directions
Z,Y,Z	forces at the ends of a member, in the x, y and z directions
Z	semi-rigid connection factor
α,β,γ	fixity factors for rotation about the x, y and z axes of a member
$\theta^x,\theta^y,\theta^z$	rotation of the column axes at their interesection with the beam axes, about the x, y and z axes
ϕ	relative rotation of the beam and column axis
ψ	angle of rotation of a member, measured clockwise from the x-axis of the fixed co-ordinate system to the x-axis of the member

VITA AUCTORIS

1938 Gerard Roland Monforton was born in Windsor, Ontario, Canada on July 21, 1938.

1944 In September, 1944, he entered Sacred Heart School, La Salle, Ontario where he obtained his elementary education.

1952 In September, 1952, he enrolled at Assumption High School, Windsor Ontario, where he obtained his secondary education.

1957 In September, 1957, he enrolled at Assumption University of Windsor Windsor, Ontario, Canada, to study civil engineering.

1961 In May, 1961, he was graduated from Assumption University of Windsor with a Bachelor of Applied Science degree. He was awarded the Board of Governors Gold Medal in civil engineering. In September, 1961, he continued his studies at the same university in order to obtain the degree of Master of Applied Science in civil engineering.