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Buckling strength of frames under primary bending.

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BUCKLING STRENGTH OF FRAMES
UNDER PRIMARY BENDING

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ABSTRACT

A systematic method of analyzing the elastic stability of frames against buckling is presented. Two possible means of attack are developed: (1) a flexibility method based on campatibility considerations; and (2) a stiffness method based on equilibrium considerations. In both cases primary bending moments occur before buckling takes place.

In the special case of sidesway buckling it is possible to use the principle of superposition because of the presence of bifurcation of deflected shapes. The sidesway mode is assumed to consist of two parts; (1) a symmetrically deformed frame, and (2) an infinitesimally small antisymmetrical configuration. The superposition principle can be employed automatically by differentiating the equations set up from the original sideways deflected frame. The advantage to this method lies in the fact that the original equations are very difficult to solve explicitly. On the other hand the differentiated equations can be simplified to a determinant which defines the criterion of stability when it vanishes.

ACKNOWLEDGMENTS

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TABLE OF CONTENTS

LIST OF FIGURES

NOTATION

c	stability factor (carry-over factor of a member)
Δ c	change in carry-over factor
C	stability factor
Δ C	change in stability factor
D	displacement
Δ D	infinitesimal sidesway displacement
E	Young's modulus of elasticity
F	flexibility of a member ($F = \frac{L}{EI}$)
H	transverse shear force
Δ H	change of transverse shear force
i,j	subscripts indicating the ends of a member
I	moment of inertia of a member
k	$=\sqrt{\frac{P}{EI}}$
K	stiffness of a member ($K= \frac{EI}{L}$)
L	length of a member
M	bending moment
Δ M	change of bending moment
N	loading parameter
p	axial forces at ends of a member

Δp	change of axial force at ends of a member
P	external force acting at a joint ($P = (1+N)\frac{wL_2}{2}$)
ΔP	change of external force acting at a joint
$\overline{P}$	external force acting at a joint ($\overline{P} = N\frac{wL_2}{2}$)
R	bar rotation
ΔR	change in bar rotation
s	stability factor (non-dimensional stiffness)
Δs	change in non-dimensional stiffness
S	stability factor
ΔS	change in stability factor
v	subscript indicating member with variable flexibility
V	vertical reaction
ΔV	change in vertical reaction
w	intensity of uniformly distributed load
x,y	displacements at ends of a member in the x and y directions
θi, θj	angular rotation of member at joints i and j
θab	angular rotation of joint a of member ab
θo	angular rotation due to span load w.
ϕ	$= \sqrt{\frac{P}{EI}}\,L$

CHAPTER I

INTRODUCTION

The importance, in the design of frameworks, whether pin connected trusses or rigidly jointed frames, of the action of compressive members is well known. In trusses each member is divorced from its neighbour except with regard to axial forces. However, because of the rigidly connected joints of frames, deflection of one member causes distortion of every other member in the framework. Thus, in rigid frames stability depends on the buckling strength of the whole structure.

The theory of buckling strength of framework subject to axial forces only, has been well established for several decades. Until the advent of electronic computers, facillitating numerical calculations, little work had been done on frames subject to primary and secondary moments at the instance of buckling. A notable exception to this statement is the work of Chwalla and Jokisch[3] who in 1941 applied a slope deflection method of analysis to an example frame including the effect of primary bending moment. More recently Masur, Donnell, Chang[12] and Lu[10] have also contributed to the study of buckling in the presence of primary bending.

The work of this thesis is intended to present a direct analytical solution of elastic instability by setting up in matrix form a system of linear homogeneous equations, expressing the interrelations between forces and displacements. A determinant can then be developed expressing the criterion of stability. These equations include terms allowing variation of the end fixity of supports by the addition of a member ab with variable stiffness as illustrated in Fig. 1. Lu[10] introduced a similar framework, but used a convergence method to arrive at the final solution. In this writer's opinion the direct solution seems more straightforward.

A consistent sign convention is used throughout, although two modes of analysis are presented. These are: (1) flexibility method - a compatibility analysis arranged in tabular form which leads to a set of "equations of compatibility", and (2) stiffness method-an equilibrium analysis leading to a set of "slope-deflection equations" employing stability functions. In both avenues of approach the equations that result are differentiated to simulate a condition of superposition and the matrices are then simplified to the well-known determinant criterion.

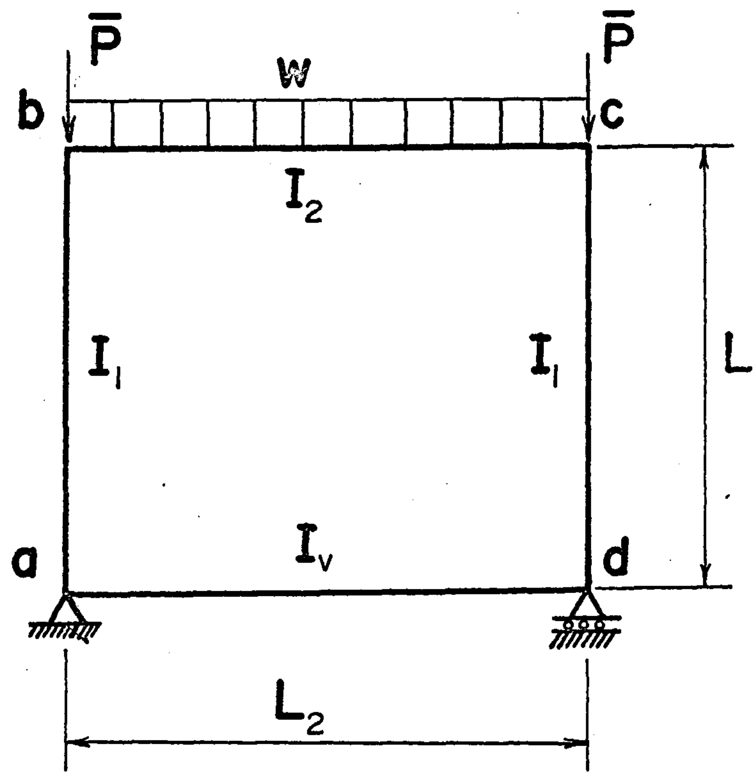

Fig. 1 FRAME DIMENSIONS AND LOADING

CHAPTER II

HISTORICAL DISCUSSION

When an investigation of elastic buckling is undertaken-especially from a chronological point of view-the work of Euler must always be placed in a position of primary importance. His classic formula for the critical load of a pinned end strut

$$P_{cr} = \pi^2 \frac{EI}{L^2}$$

is still a basic guide for any work connected with slender compression members.

In 1919 BLEICH presented the method of four moment equations in which a systematic analysis of the stability of plane frameworks with rigid joints is derived. His method depends on the condition of continuity at a point where two or more members are rigidly joined. Each equation expresses a relation between the four terminal moments of two adjacent connected members and the bar rotation.

As mentioned before, Chwalla and Jokisch[3] first derived the slope-deflection equations for stability. In this method the angular rotation of the joints and the bar rotation are considered as variables in the stability equations.

Both of these methods may be termed analytical solutions to the stability problem in which the vanishing determinant of the coefficients of the equations defines the buckling load of the frame.

Various convergence methods have also been applied to the problem of elastic instability. James[6] in 1935 converted the moment distribution method as developed by Hardy Cross to a form including the effect of axial load in the members. Shortly after, Lundquist[6] also presented stability criteria based on the Hardy Cross method. However, these criteria require the use of trial and error procedures to solve, instead of leading to a direct solution.

Until the mid 1950's Chwalla[3] was the only author known to this writer who considered the effect of bending of a frame before buckling occurred. In 1952 Bleich[3] reviewed Chwalla's work and suggested that future work be carried out. Subsequently with the introduction of competent electronic computers the problem was attacked by Livesley[8], Masur, Chang and Donnell[12], and Lu[10]. By introducing stability factors Masur, et al., presented a systematic approach using the slope-deflection and the moment distribution method. The structure analyzed was a pin connected frame under the action of two concentrated vertical loads placed symmetrically on the horizontal beam so that they produced primary bending moments. Then, in 1963 Lu[10] extended the slope-defection analysis to include the effects of a uniformly distributed span load.

CHAPTER III

FLEXIBILITY METHOD

Actual collapse of a frame such as that shown in Fig. 1 is caused by a number of factors although there are two primary ones, namely inelasticity and the second order effects of "buckling" instability. This thesis considers only the effects of elastic instability and as a result the usual assumptions made in a study of elasticity are adhered to.

In Fig. 1 the interrelation of axial load $\overline{P}$ to uniform load w as developed by Lu[10] is adopted.

$$\overline{P} = N \frac{wL_2}{2} \qquad (3\text{-}1)$$

in which N is a numerical parameter. Therefore the total axial force in the columns is

$$P = (1+N) \frac{wL_2}{2} \qquad (3\text{-}2)$$

The beam-column shown in Fig. 3 represents a typical member connecting joints i and j. M_i and M_j are moments applied at joints i and j respectively, and the axial force is signified by p. The angles of rotation at the joints i and j are θ_i and θ_j respectively, and the bar rotation is R. Y_i and Y_j are the support reactions

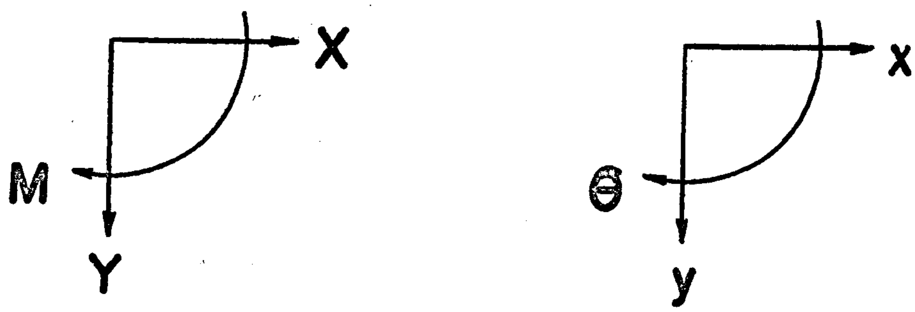

Fig. 2 SIGN CONVENTION

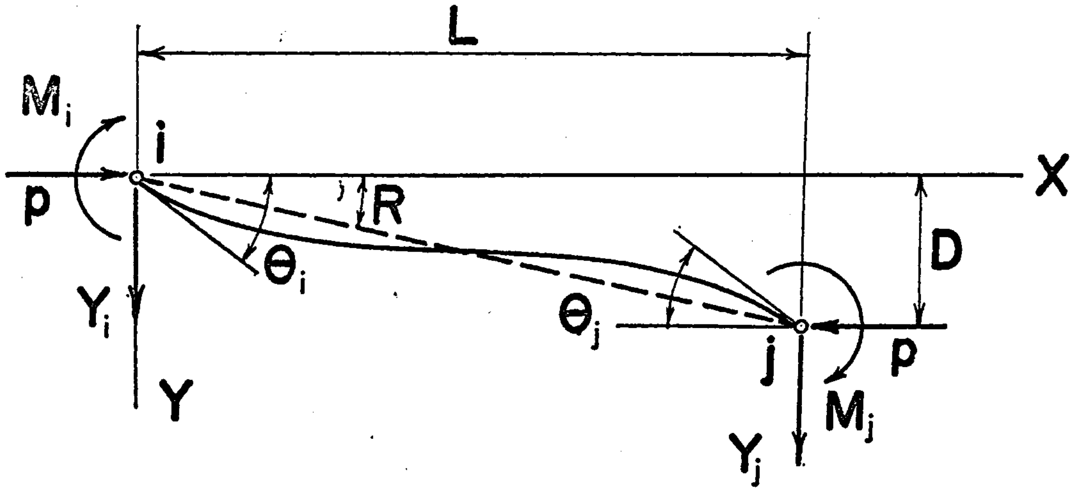

Fig. 3 TYPICAL MEMBER

at i and j respectively.

By means of the general differential equation for beams under axial load

$$\left[\frac{d^2y}{dx^2} + \frac{P}{EI}\; y\right]'' = 0 \tag{3-3}$$

the following interrelations between forces and displacements can be found as derived in Appendix A:

$$\theta_i = M_i\, F\left(C-\frac{1}{\phi^2}\right) - M_j\, F\left(S+\frac{1}{\phi^2}\right) + Y_i L\, \frac{F}{\phi^2} \tag{3-4a}$$

$$\theta_j = -M_i F\left(S + \frac{1}{\phi^2}\right) + M_j F\left(C-\frac{1}{\phi^2}\right) + Y_i L\, \frac{F}{\phi^2} \tag{3-4b}$$

$$R = -\, M_i\, \frac{F}{\phi^2} - M_j\, \frac{F}{\phi^2} + Y_i\, L\, \frac{F}{\phi^2} \tag{3-4c}$$

in which

$$F = \frac{L}{EI} \tag{3-5a}$$

$$\phi^2 = \frac{pL^2}{EI} \tag{3-5b}$$

C and S are stability factors denoted by the following expressions:

$$C = \frac{1}{\phi^2}\,(1-\phi \cot \phi) \tag{3-6a}$$

$$S = \frac{1}{\phi^2}\left(\frac{\phi}{\sin \phi} - 1\right) \tag{3-6b}$$

The angle of rotation caused by a uniform load w on a beam column is, from Timoshenko[16]

$$\theta_o = (C + S) \left[1 - \frac{\phi\ (1 + \cos\phi)}{2 \quad \sin} \right] \frac{wL}{P}$$

By the law of superposition, θ_o can be algebraically added to the joint rotations, θ_i and θ_j .

Equations (3-4), expressed in matrix form, become

$$\begin{bmatrix} \theta_i \\ \\ \theta_j \\ \\ R \end{bmatrix} = \begin{bmatrix} F\left(C - \frac{1}{\phi^2}\right) & -F\left(S + \frac{1}{\phi^2}\right) & \frac{1}{pL} \\ \\ -F\left(S + \frac{1}{\phi^2}\right) & F\left(C - \frac{1}{\phi^2}\right) & \frac{1}{pL} \\ \\ -\frac{1}{pL} & -\frac{1}{pL} & \frac{1}{pL} \end{bmatrix} \begin{bmatrix} M_i \\ \\ M_j \\ \\ Y_i L_1 \end{bmatrix} \qquad (3\text{-}8)$$

If joints i and j are considered to be on the x axis, i.e. D = R = 0 then the matrix equation becomes

$$\begin{bmatrix} \theta_i \\ \\ \theta_j \end{bmatrix} = \begin{bmatrix} FC & -FS \\ \\ -FS & FC \end{bmatrix} \begin{bmatrix} M_i \\ \\ M_j \end{bmatrix} \qquad (3\text{-}9)$$

Symmetrical Deformation

The frame in Fig. 1 is now analyzed for its symmetrical mode of instability. The deflected shape at the point of buckling is as shown in Fig. 4. By applying

TABLE 3 - 1

TABULAR PRESENTATION OF INTERRELATIONS BETWEEN FORCES AND DISPLACEMENTS

	θ_{ab}	θ_{ba}	θ_{bc}	θ_{cb}	θ_{cd}	θ_{dc}	θ_{da}	θ_{ad}
APPLY MOMENT AT a	$M_{ab}\ F_1C_1$	$-M_{ab}\ F_1S_1$	0	0	0	0	$-M_{ad}\ F_vS_v$	$M_{ad}\ F_vC_v$
APPLY MOMENT AT b	$-M_{ba}\ F_1S_1$	$M_{ba}\ F_1C_1$	$M_{bc}\ F_2C_2$	$-M_{bc}\ F_2S_2$	0	0	0	0
APPLY MOMENT AT c	0	0	$-M_{cb}\ F_2S_2$	$M_{cb}\ F_2C_2$	$M_{cd}\ F_1C_1$	$-M_{cd}\ F_1S_1$	0	0
APPLY MOMENT AT d	0	0	0	0	$-M_{dc}\ F_1S_1$	$M_{dc}\ F_1C_1$	$+M_{da}\ F_vC_v$	$-M_{da}\ F_vS_v$
INITIAL ROTATION FROM SPAN LOAD	0	0	θ_o	$-\theta_o$	0	0	0	0

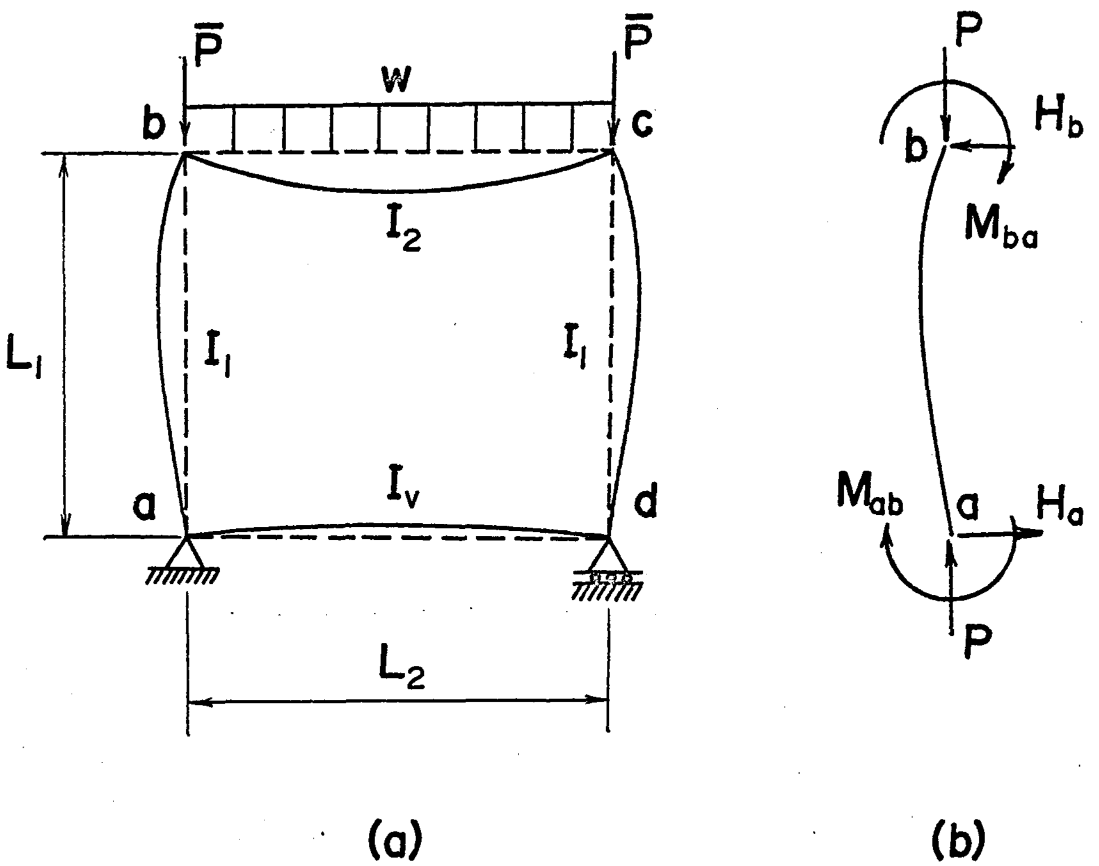

Fig. 4 SYMMETRICALLY DEFORMED FRAME

the matrix equation (3-9) to each member in turn and arranging in tabular form the analysis can be very easily presented.

Continuity of the structure requires that at joint a

$$\theta_{ad} - \theta_{ab} = 0 \qquad (3\text{-}10)$$

Similarly at joint b

$$\theta_{ba} - \theta_{bc} = 0 \qquad (3\text{-}11)$$

From Table 3-1, equations (3-10) and (3-11) become

$$M_{ad}\ F_vC_v - M_{da}\ F_vS_v - M_{ab}\ F_1C_1 + M_{ba}\ F_1S_1 = 0 \qquad (3\text{-}12)$$

$$-M_{ab}\ F_1S_1 + M_{ba}\ F_1C_1 - M_{bc}\ F_2C_2 + M_{cb}\ F_2S_2 - \theta_o = o \quad (3\text{-}13)$$

Now remove column ab from the structure and consider the forces acting on it. By summing moments about point a the following equation results:

$$M_{ab} + M_{ba} - H_bL_1 = 0 \qquad (3\text{-}14)$$

The frame is then disengaged as illustrated in Fig. 5. Only the moments which produce the assumed deflection of the members in Fig. 4 are indicated. Since, in this method, equilibrium is everywhere assumed to be satisfied the moments on the members can be related through the use of equilibrium at the joints.

Let $M_{ba} = M_1$

and $M_{ab} = M_2$

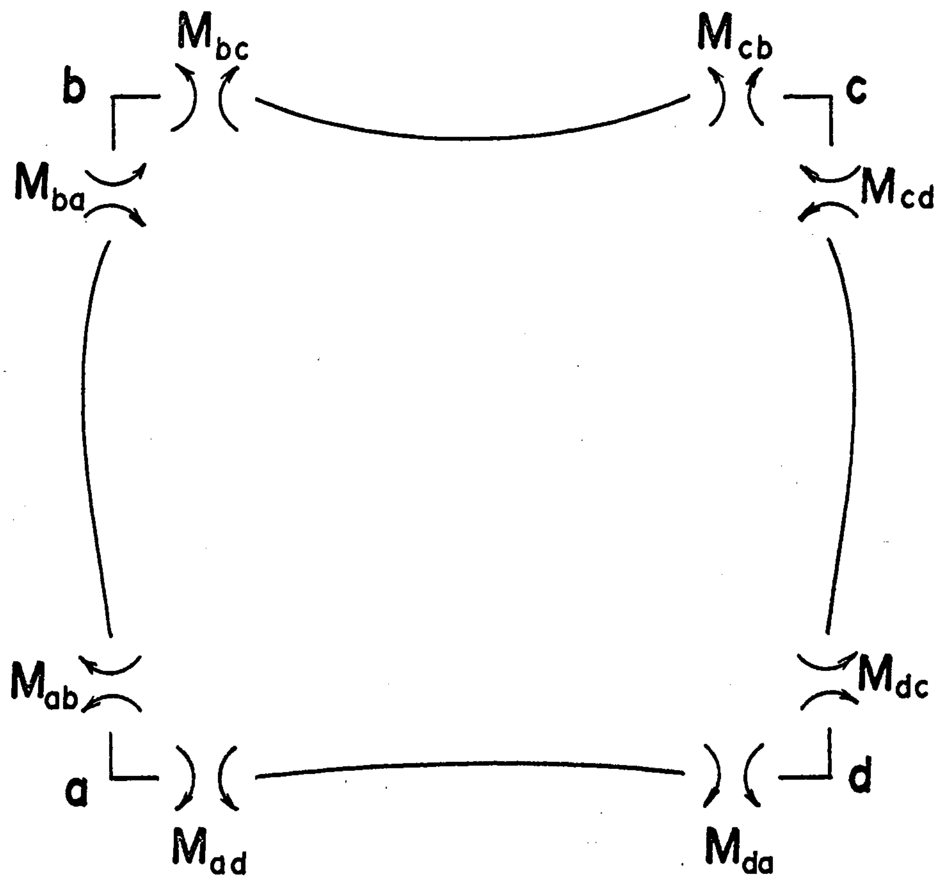

Fig. 5 DISENGAGED SYMMETRICAL FRAME

By following the above procedure

$$M_1 = M_{ba} = -M_{bc} = M_{cb} = -M_{cd} \tag{3-15}$$

and $M_2 = M_{ab} = M_{ad} = -M_{da} = -M_{dc}$

By using equations (3-12), (3-13), (3-14), and (3-15) the following matrix equation results.

$$\begin{bmatrix} F_1S_1 & F_v(C_v + S_v) - F_1C_1 & 0 \\ F_1C_1 + F_2(C_2 + S_2) & -F_1S_1 & -\theta_o \\ 1 & 1 & -H_hL_1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ 1 \end{bmatrix} = 0 \tag{3-16}$$

However, the column matrix $\begin{bmatrix} M_1 \\ M_2 \\ 1 \end{bmatrix}$ is not equal to zero.

Therefore the determinant of the coefficients must be zero and the criterion of stability has been obtained.

$$\begin{vmatrix} F_1S_1 & F_v(C_v + S_v) - F_1C_1 & 0 \\ F_1C_1 + F_2(C_2 + S_2) & -F_1S_1 & -\theta_o \\ 1 & 1 & -H_bL_1 \end{vmatrix} = 0 \tag{3-17}$$

By varying the values of F_v (the flexibility of member ad) from zero to infinity the end conditions of a three member frame abcd can be modified from entirely fixed to completely free to rotate.

For example, the result for a frame with pinned ends, ie., F_v is infinity, is

$$H L_1 \left[F_1 C_1 + F_2 (C_2 + S_2) \right] - \theta_o = 0 \qquad (3\text{-}18)$$

Similarly the equation for a frame with fixed ends, ie., F_v is zero, is

$$HL_1 \left[\frac{C_1}{C_1 + S_1} \quad F_1 \frac{(C_1^{\,2} - S_1^{\,2})}{C_1} + F_2 (C_2 + S_2) \right] - \theta_o = 0 \qquad (3\text{-}19)$$

Sidesway Deformation

Unless the frame of Fig. 1 is braced against sidesway, it will always buckle in an antisymmetrical or sidesway mode before the value of the critical load for symmetrical deformation is reached. This means that for some loading stage there exists two possible stable modes of deformation and the point at which this phenomenon occurs on a load-deformation diagram is called the bifurcation point. A proof of the existence of such a phenomenon was given by Chwalla[3] for a simple portal frame.

In order to obtain a stability criterion for the sidesway mode of deformation, the frame must be allowed to deflect an infinitesimal amount into the mode of failure as illustrated in Fig. 6 (a).

Using equation (3-8) a table similar to table (3-1) is constructed which includes the effect of sidesway.

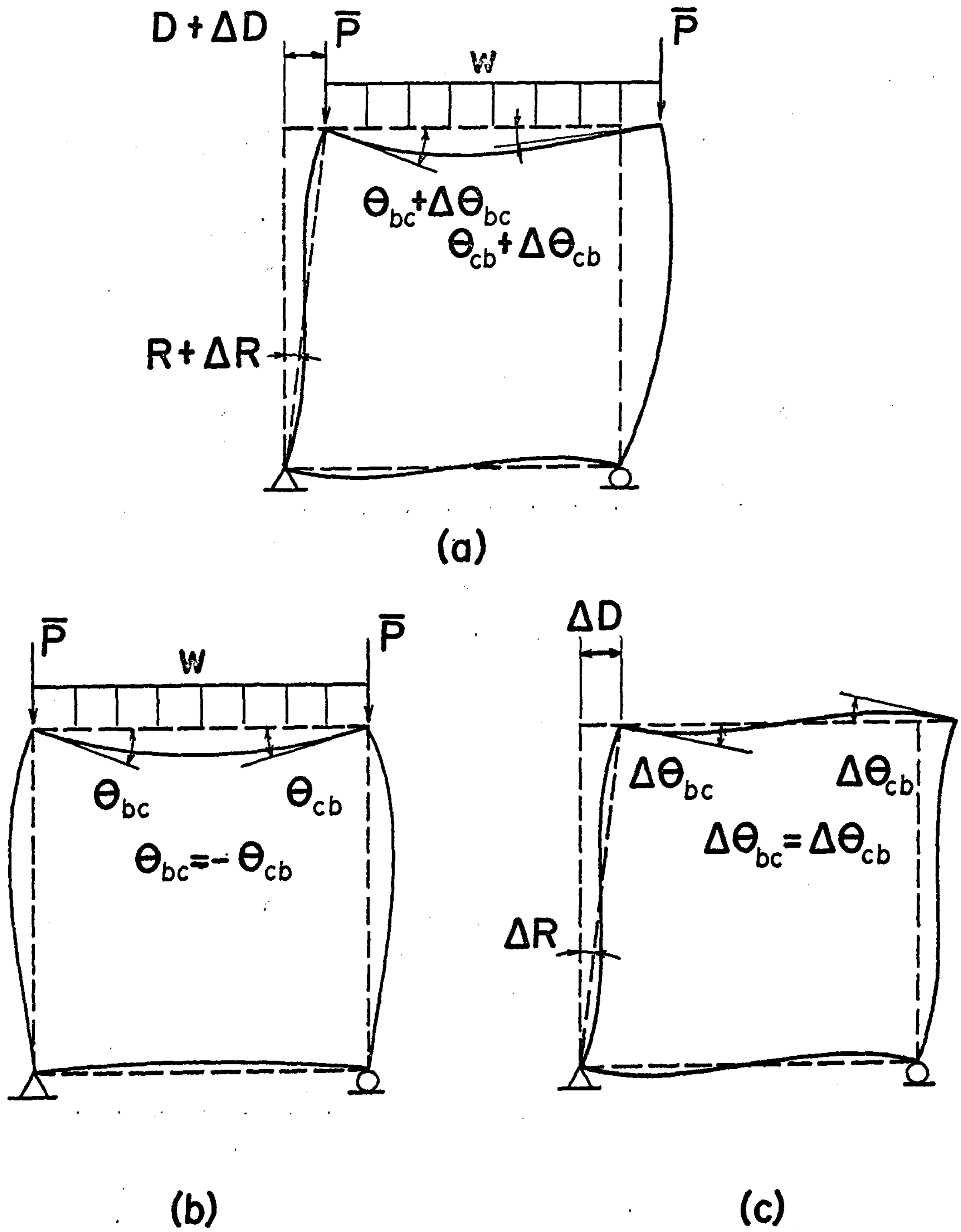

Fig. 6 FRAME UNDER SIDESWAY DEFORMATION

The same conditions of continuity still apply, ie,

$$\theta_{ad} - \theta_{ab} = 0 \qquad (3\text{-}10)$$

$$\theta_{ba} - \theta_{bc} = 0 \qquad (3\text{-}11)$$

However another condition of compatibility is evident from Fig. 6, i.e.,

$$R_b L_1 - D = 0 \qquad (3\text{-}20)$$

From Table 3-2, equations (3-10), (3-11) and (3-20) become

$$- M_{ab} F_1 (S_1 + \frac{1}{\phi^2_1}) + M_{ba} F_1 (C_1 - \frac{1}{\phi_1^2}) + \frac{H_b L_1}{PL_1} - M_{bc} F_2 C_2$$

$$+ M_{cd} F_2 S_2 - \theta_o = 0 \qquad (3\text{-}21)$$

$$M_{ad} F_v C_v - M_{da} F_v S_v - M_{ab} F_1 (C_1 - \frac{1}{\phi^2_1}) + M_{ba} F_1$$

$$(S_1 + \frac{1}{\phi_1^2}) - \frac{H_b L_1}{PL_1} = 0 \qquad (3\text{-}22)$$

$$\left[- M_{ab} \frac{1}{PL_1} - M_{ba} \frac{1}{PL_1} + \frac{H_b L_1}{PL_1} \right] L_1 - D = 0 \qquad (3\text{-}23)$$

These equations alone represent the solution to the sidesway problem, but it is very difficult to obtain an explicit solution from them. However, by means of the principle of superposition the frame can be segregated

TABLE 3 - 2

TABULAR PRESENTATION OF INTERRELATIONS BETWEEN FORCES AND DISPLACEMENTS

	θ_{ab}	θ_{ba}	θ_{bc}	θ_{cb}	θ_{cd}	θ_{dc}	θ_{da}	θ_{ad}	R_b	R_c
APPLY MOMENT AT a	$M_{ab}\ F_1C_1$	$-M_{ab}\ F_1S_1$	0	0	0	0	$-M_{ad}\ F_vS_v$	$M_{ad}\ F_vC_v$	0	0
APPLY MOMENT AT b	$-M_{ba}\ F_1S_1$	$M_{ba}\ F_1C_1$	$M_{bc}\ F_2C_2$	$-M_{bc}F_2S_2$	0	0	0	0	0	0
APPLY MOMENT AT c	0	0	$-M_{cb}\ F_2S_2$	$M_{cb}F_2C_2$	$M_{cd}F_1C_1$	$-M_{cd}\ F_1S_1$	0	0	0	0
APPLY MOMENT AT d	0	0	0	0	$-M_{dc}F_1S_1$	$M_{dc}\ F_1C_1$	$M_{da}\ F_vC_v$	$-M_{da}F_vS_v$	0	0
APPLY LATERAL FORCE	$-M_{ab}F_1/\phi_1^2$ $-M_{ba}F_1/\phi_1^2$ $+H_bL_1/PL_1$	$-M_{ab}F_1/\phi_1^2$ $-M_{ba}F_1/\phi_1^2$ $+H_bL_1/PL_1$	0	0	$-M_{cd}F_1/\phi_1^2$ $-M_{dc}F_1/\phi_1^2$ $+H_cL_1/PL_1$	$-M_{cd}F_1/\phi_1^2$ $-M_{dc}F_1/\phi_1^2$ $+H_cL_1/PL_1$	0	0	$-\frac{M_{ab}+M_{ba}}{PL_1}$ $+\frac{H_bL_1}{PL_1}$	$-\frac{M_{cd}+M_{dc}}{PL_1}$ $+\frac{H_cL_1}{PL_1}$
INITIAL SPAN LOAD ROTATION	0	0	θ_o	$-\theta_o$	0	0	0	0	0	0

into two parts consisting of; (1) a symmetrically deformed frame and (2) a frame with an infinitesimal sidesway. This can be performed automatically by differentiating equations (3-21), (3-22) and (3-23).

$$- \Delta M_{ab} F_1(S_1 + \frac{1}{\phi_1^2}) - M_{ab} F_1 \left[\Delta S_1 + \Delta(\frac{1}{\phi_1^2})\right] +$$

$$\Delta M_{ba} F_1(C_1 - \frac{1}{\phi_1^2}) + M_{ba} F_1 \left[\Delta C_1 - \Delta(\frac{1}{\phi_1^2})\right] +$$

$$\frac{\Delta H_b L_1}{PL_1} + H_b L_1 \Delta(\frac{1}{PL_1}) - \Delta M_{bc} F_2 C_2 - M_{bc} F_2 \Delta C_2$$

$$+ \Delta M_{cb} F_2 S_2 + M_{cb} F_{22} \Delta S_2 - \Delta \theta_o = 0 \qquad (3\text{-}24)$$

$$\Delta M_{ad} K_v C_v + M_{ad} K_v \Delta C_v - \Delta M_{da} K_v S_v - M_{da} K_v \Delta S_v$$

$$- \Delta M_{ab} F_1 (C_1 - \frac{1}{\phi_1^2}) - M_{ab} F_1 \left[\Delta C_1 - \Delta(\frac{1}{\phi_1^2})\right]$$

$$+ \Delta M_{ba} F_1(S_1 + \frac{1}{\phi_1^2}) + M_{ba} F_1 \left[\Delta S_1 + \Delta(\frac{1}{\phi_1^2})\right] - \Delta \frac{H_b L_1}{PL_1}$$

$$- H_b L_1 \Delta(\frac{1}{PL_1}) = 0 \qquad (3\text{-}25)$$

$$\left[- \frac{\Delta M_{ab}}{PL_1} - M_{ab} \Delta(\frac{1}{PL_1}) - \frac{\Delta M_{ba}}{PL_1} - M_{ba} \Delta(\frac{1}{PL_1}) +\right.$$

$$\frac{\Delta H_b L_1}{P L_1} + H_b L_1 \, \Delta \left(\frac{1}{P L_1} \right) \Big] \; L_1 - \Delta D = 0 \qquad (3\text{-}26)$$

In each of equations (3-24), (3-25) and (3-26) there are two basic groups of terms. One group consists of terms in which the moment and lateral force H are differentiated; the other group consists of terms in which the moment and lateral force are not differentiated. These groups represent respectively the infinitesimal sidesway and the symmetrical modes of deformation. The following expressions apply to equation (3-24), (3-25), and (3-26):

$$\Delta S_2 = \frac{d\,S_2}{d\,H} \; \Delta H_2 = S_2' \; \Delta H_2$$

$$\Delta C_2 = C_2' \; \Delta H_2$$

$$\Delta \theta_o = \theta_o' \; \Delta H_2 \qquad (3\text{-}27)$$

$$\Delta S_v = S_v' \; \Delta H_v$$

$$\Delta C_v = C_v' \; \Delta H_v \qquad (3\text{-}28)$$

$$\Delta S_1 = S_1' \; \Delta p_1$$

$$\Delta C_1 = C_1' \; \Delta p_1 \qquad (3\text{-}29)$$

in which $S' = \frac{S}{2p} \left[\frac{C}{S} + C\frac{2S}{C-S} - \frac{C}{C^2-S^2} \right]$

$$\text{and} \quad C' = -\frac{C}{2p}\left[\frac{C - 3S}{C + S} + \frac{S^2}{C(C^2 - S^2)}\right] \qquad (3\text{-}30)$$

For the derivation of S' and C' see Appendix C.
However in the perfectly antisymmetrical infinitesimal sidesway deformation, it is evident that the axial force in both horizontal members is zero, i.e., $\Delta H_2 = \Delta H_v = 0$. Therefore

$$\Delta S_2 = \Delta C_2 = \Delta\theta_o = \Delta S_v = \Delta C_v = 0 \qquad (3\text{-}31)$$

For the symmetrically deflected frame the relation between shear force, H_b and moments, M_{ab} and M_{ba} is the same as in equation (3-14).

$$M_{ab} + M_{ba} - H_b L_1 = 0 \qquad (3\text{-}14)$$

Both of the frames of Fig. 6 can be disengaged at the joints as was done in the previous section concerned with symmetrical buckling only. In fact the symmetrically deformed portion of this section has the same result as that of equations (3-15)

$$\begin{aligned} & M_1 = M_{ba} = -M_{bc} = M_{cb} = -M_{cd} \\ \text{and} \quad & M_2 = M_{ab} = M_{ad} = -M_{da} = -M_{dc} \end{aligned} \qquad (3\text{-}15)$$

When the antisymmetrical frame is disengaged as shown in Fig. 7, the following equations result:

$$\begin{aligned} & \Delta M_1 = \Delta M_{ba} = -\Delta M_{bc} = -\Delta M_{cb} = \Delta M_{cd} \\ \text{and} \quad & \Delta M_2 = \Delta M_{ab} = -\Delta M_{ad} = -\Delta M_{da} = \Delta M_{dc} \end{aligned} \qquad (3\text{-}32)$$

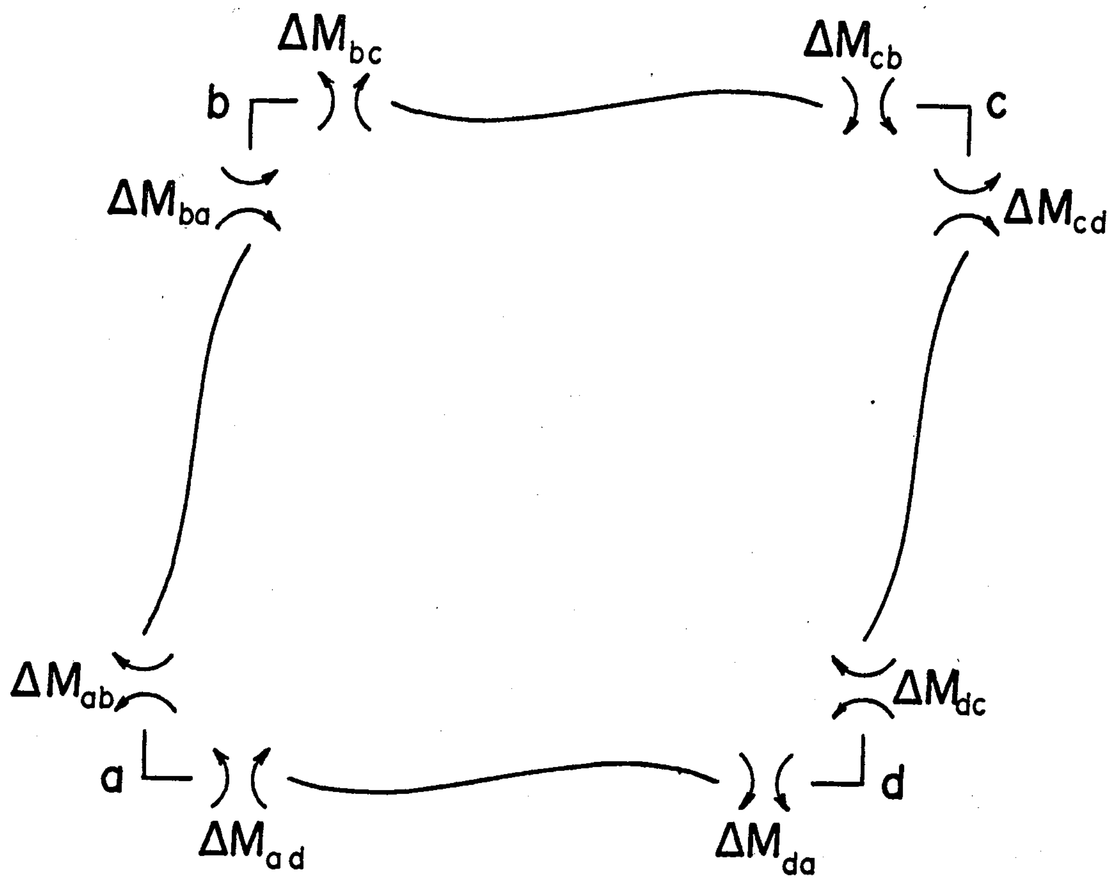

Fig. 7 DISENGAGED ANTISYMMETRICAL FRAME

in which ΔM_1 and ΔM_2 are assumed arbitrarily to be equal to ΔM_{ba} and ΔM_{ab} respectively.

By employing the relations (3-14), (3-15), (3-29), (3-31), and (3-32) equations (3-24), (3-25) and (3-26) simplify to:

$$\Delta M_1 \left[F_1(C_1 - \frac{1}{\phi_1^2}) + F_2(C_2 - S_2)\right] - \Delta M_2 F_1 (S_1 + \frac{1}{\phi_1^2})$$

$$+ \left[M_1 F_1 C_1' - M_2 F_1 S_1'\right] \Delta P_1 = 0 \tag{3-33}$$

$$\Delta M_1 F_1 (S_1 + \frac{1}{\phi_1^2}) + \Delta M_2 \left[F_v(C_v - S_v) + F_1(C_1 - \frac{1}{\phi_1^2})\right]$$

$$- \left[M_1 F_1 S_1' - M_2 F_1 C_1'\right] \Delta P_1 = 0 \tag{3-34}$$

$$\Delta M_1 (\frac{1}{P}) + \Delta M_2 (\frac{1}{P}) + \Delta D = 0 \tag{3-35}$$

in which $M_1 = - \dfrac{\left[F_v(C_v + S_v) - F_1 C_1\right]}{F_1^2 S_1^2 + \left[F_v(C_v + S_v) - F_1 C_1\right]\left[F_1 C_1 + F_2(C_2 + S_2)\right]} \theta_o$

and $M_2 = - \dfrac{\theta_o}{F_1 S_1 + \frac{1}{F_1 S_1}\left[F_v(C_v - S_v) - F_1 C_1\right]\left[F_1 C_1 + F_2(C_2 + S_2)\right]}$

The relations for M_1 and M_2 are derived directly from equation (3-21)and (3-22). Now consider the equilibrium of the structure shown in Fig. 8 (a). If the summation of moments about d equals zero, then

$$0 = (V_1 + \Delta V_1) L_2 - P(L_2 - \Delta D) + P\,\Delta D + M_{ab} + M_{cd} + \Delta M_{ab} + \Delta M_{cd}$$

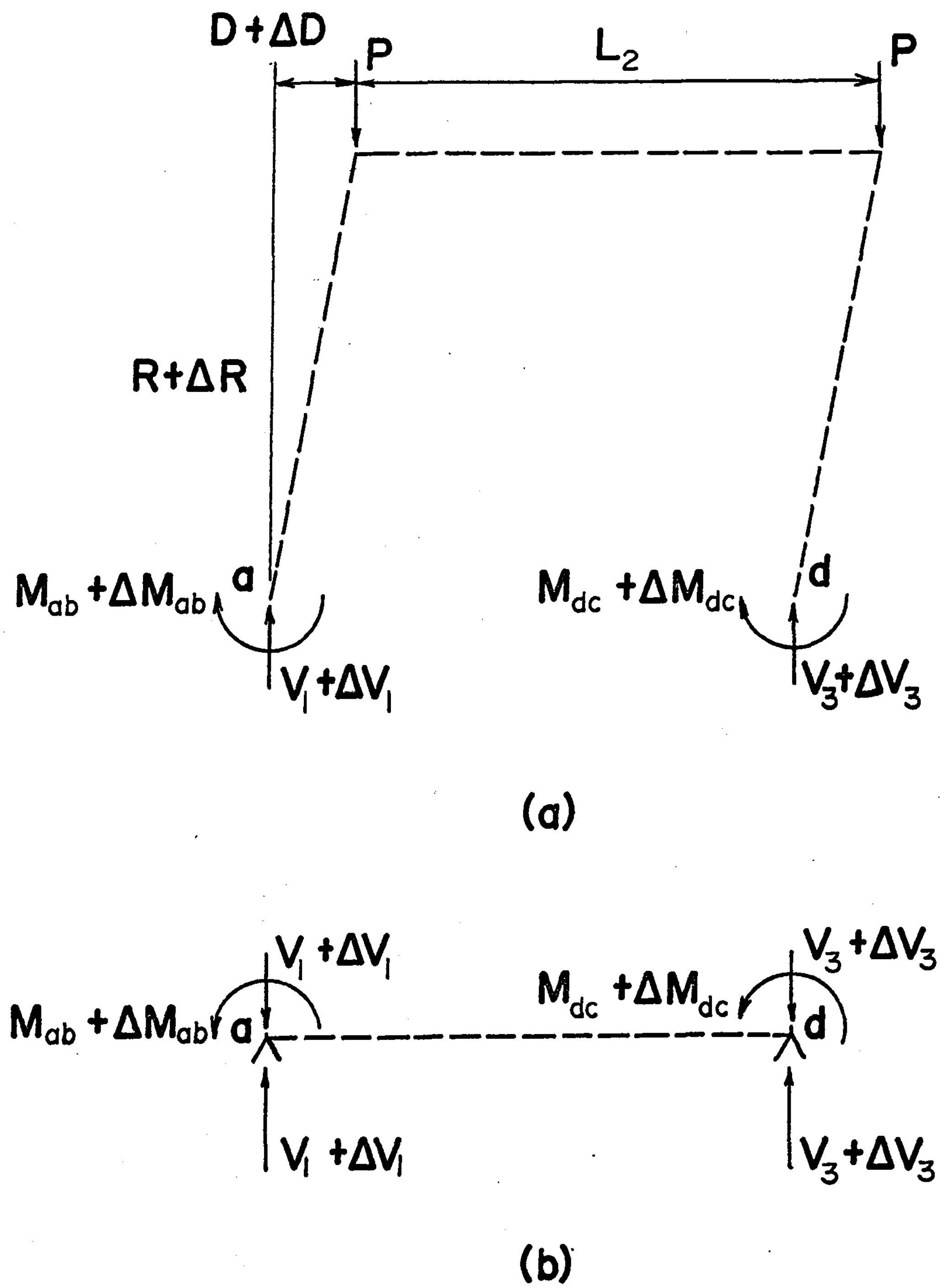

Fig. 8 FORCE EQUILIBRIUM AFTER SIDESWAY

96367

But from the symmetrical configuration it is obvious that $V_1 = P$.

Then after employing equations (3-15) and (3-32),

$$\Delta D = -\frac{\Delta P_1 L_2}{2P} - \frac{\Delta M_2}{P} \tag{3-36}$$

Substitute equation (3-36) into equation (3-35) and arrange in matrix form.

$$C = \begin{bmatrix} F_1\left(C_1 - \frac{1}{\phi_1^2}\right) + F_2(C_2 - S_2) & -F_1\left(S_1 + \frac{1}{\phi_1^2}\right) & F_1(M_1C_1' - M_2S_1') \\ F_1\left(S_1 + \frac{1}{\phi_1^2}\right) & -F_1\left(C_1 - \frac{1}{\phi_1^2}\right) + F_v(C_v - S_v) & F_1(M_1S_1' - M_2C_1') \\ 2 & 0 & -L_2 \end{bmatrix} \begin{bmatrix} \Delta M_1 \\ \Delta M_2 \\ \Delta P_1 \end{bmatrix}$$

But the column matrix $\begin{bmatrix} \Delta M_1 \\ \Delta M_2 \\ \Delta P_1 \end{bmatrix}$ is not equal to zero; hence the determinant of its coefficients must be zero, i.e,

$$\begin{vmatrix} F_1\left(C_1 - \dfrac{1}{\phi_1^{\,2}}\right) + F_2(C_2 - S_2) & -F_1\left(S_1 + \dfrac{1}{\phi_1^{\,2}}\right) & F_1(M_1C_1{'} - M_2S_1{'}) \\ F_1\left(S_1 + \dfrac{1}{\phi_1^{\,2}}\right) & -F_1\left(C_1 - \dfrac{1}{\phi_1^{\,2}}\right) + F_v(C_v - S_v) & F_1(M_1S_1{'} - M_2C_1{'}) \\ 2 & 0 & -L_2 \end{vmatrix} = 0 \qquad (3\text{-}37)$$

This determinant represents the criterion of stability for a frame which buckles in a sidesway mode. By expanding the determinant a theoretical solution can be found for the critical load at which the frame will become unstable if sidesway is allowed to occur.

CHAPTER IV

STIFFNESS METHOD

The stiffness or slope-deflection method of analysis has been fully treated by Lu[10] except with regard to the automatic development of the superposition principle as mentioned before in Chapter III. Also Lu[10] does not introduce the variation of support fixity directly into his analysis. For these reasons the author feels that it is of interest to analyze the sidesway mode of deformation by the stiffness method.

This method of analysis differs from the flexibility method in that the conditions of compatibility are replaced by equations of equilibrium as the requirements of analysis, i.e, compatibility of the structure is everywhere assumed to be satisfied in the stiffness method while equilibrium is assumed satisfied in the flexibility method. Thus summation of moments at joints a and b, and the equilibrium of column ab are the conditions which must be fulfilled for a proper analysis. Now consider the frame shown in Fig. 6.

$$M_{bc} = K_2 s_2 \theta_b + K_2 s_2 c_2 \theta_c - K_2 s_2 (1 + c_2) R_2 - M_{Fbc}$$

$$M_{ba} = K_1 s_1 \theta_b + K_1 s_1 c_1 \theta_a - K_1 s_1 (1 + c_1) R_1$$

$$M_{ab} = K_1 s_1 \theta_a + K_1 s_1 c_1 \theta_b - K_1 s_1 (1 + c_1) R_1 \tag{4-1}$$

$$M_{ad} = K_v s_v \theta_a + K_v s_v c_v \theta_d - K_v s_v (1 + c_v) R_v$$

$$M_{da} = K_v s_v c_v \theta_a + K_v s_v \theta_d - K_v s_v (1 + c_v) R_v$$

$$M_{dc} = K_1 s_1 \theta_d + K_1 s_1 c_1 \theta_c - K_1 s_1 (1 + c_1) R_1$$

in which $K = EI / L$

$$s = \frac{\phi (\sin \phi - \phi \cos \phi)}{2 - 2 \cos \phi - \phi \sin \phi} \tag{4-2}$$

$$c = \frac{\phi - \sin \phi}{\sin \phi - \phi \cos \phi}$$

Also from Lu

$$M_{Fbc} = K_2 \left[1 - \frac{\phi}{2} \left(\frac{1 + \cos \phi}{\sin \phi} \right) \quad \frac{wL_2}{H_b} \right] \tag{4-3}$$

which is the fixed end moment for a uniformly distributed load w. Also in equations (4-1) both R_v and R_2 and any change in R_v and R_2 are considered to be zero.

In matrix form the equations of equilibrium are

$$
\begin{bmatrix}
K_1s_1 + K_vs_v & K_1s_1c_1 & 0 & K_vs_vc_v & -K_1s_1(1+c_1) \\
K_1s_1c_1 & K_1s_1 + K_2s_2 & K_2s_2c_2 & 0 & -K_1s_1(1+c_1) \\
K_1s_1(1+c_1) & K_1s_1(1+c_1) & 0 & 0 & PL_1 - 2K_1s_1(1+c_1)
\end{bmatrix}
\begin{bmatrix}
\theta_a \\ \theta_b \\ \theta_c \\ \theta_d \\ R_1
\end{bmatrix}
+ \begin{bmatrix} 0 \\ -M_{Fbc} \\ -HL_1 \end{bmatrix} = 0 \qquad (4\text{-}4)
$$

Now equation (4-4) is differentiated.

$$\begin{bmatrix} K_1\Delta s_1 + K_v\Delta s_v & K_1[s_1\Delta c_1 + \Delta s_1 C_1] & 0 & K_v[\Delta s_v c_v + s_v\Delta c_v] & -K_1[\Delta s_1(1 + c_1) + s_1\Delta c_1] \\ K_1[s_1\Delta c_1 + \Delta s_1 c_1] & K_1\Delta s_1 + K_2\Delta s_2 & K_2[\Delta s_2 c_2 + s_2\Delta c_2] & 0 & -K_1[\Delta s_1(1 + c_1) + s_1\Delta c_1] \\ K_1[\Delta s_1(1 + c_1) + s_1\Delta c_1] & K_1[\Delta s_1(1 + c_1) + s_1\Delta c_1] & 0 & 0 & \Delta P.L_1 -2K_1[\Delta s_1(1+c_1)+ s_1\Delta c_1] \end{bmatrix}$$

$$\begin{bmatrix} \theta_a \\ \theta_b \\ \theta_c \\ \theta_d \\ R_1 \end{bmatrix} + \begin{bmatrix} K_1 s_1 + K_v s_v & K_1 s_1 c_1 & 0 & K_v s_v c_v & -K_1 s_1(1 + c_1) \\ K_1 s_1 c_1 & K_1 s_1 + K_2 s_2 & K_2 s_2 c_2 & 0 & -K_1 s_1(1 + c_1) \\ K_1 s_1(1 + c_1) & K_1 s_1(1 + c_1) & 0 & 0 & PL_1 \ -2k_1 s_1(1 + c_1) \end{bmatrix} \begin{bmatrix} \Delta\theta_a \\ \Delta\theta_b \\ \Delta\theta_c \\ \Delta\theta_d \\ \Delta R_1 \end{bmatrix} + \begin{bmatrix} 0 \\ - \Delta M_{Fbc} \\ - \Delta H_2 L_1 \end{bmatrix} = 0$$

After equation (4-4) is differentiated it can be seen from Fig. 6 (b) and 6 (c) that

$$\theta_c = -\theta_b$$

$$\theta_d = -\theta_a \tag{4-6a}$$

and

$$\Delta\theta_c = \Delta\theta_b$$

$$\Delta\theta_d = \Delta\theta_a \tag{4-6b}$$

Also in equation (4-5)

$$\Delta s_2 = \frac{ds_2}{dH_2} \Delta H_2 = s_2' \Delta H_2$$

$$\Delta c_2 = c_2' \Delta H_2 \tag{4-7}$$

$$\Delta s_v = s_v' \Delta H_v$$

$$\Delta c_v = c_v' \Delta H_v \tag{4-8}$$

$$\Delta s_1 = s_1' \Delta p_1$$

$$\Delta c_1 = c_1' \Delta p_1 \tag{4-9}$$

However for the perfectly antisymmetrically deflected frame in Fig. 6 (c) it is obvious that $\Delta H_2 = \Delta H_v = 0$

Therefore

$$\Delta s_2 = \Delta c_2 = \Delta s_v = \Delta c_v = 0$$

Then equation (4-5) becomes after substituting and simplifying

$$\begin{bmatrix} K_1 s_1{}' \theta_a + K_1(s_1 c_1{}' + s_1{}' c_1)\ \theta_b \\ K_1(s_1 c_1{}' + s_1{}' c_1)\ \theta_a + K_1 s_1{}' \theta_b \\ K_1 s_1{}'(1 + c_1) - s_1 c_1{}' \quad (\theta_a + \theta_b) \end{bmatrix} \begin{bmatrix} \Delta p_1 \end{bmatrix} +$$

$$\begin{bmatrix} K_1 s_1 + K_v s_v(1 + c_v) & K_1 s_1 c_1 & - K_1 s_1(1 + c_1) \\ K_1 s_1 c_1 & K_1 s_1 + K_2 s_2(1 + c_2) & - K_1 s_1(1 + c_1) \\ K_1 s_1(1 + c_1) & K_1 s_1(1 + c_1) & PL_1 - 2K_1 s_1(1 + c_1) \end{bmatrix} \begin{bmatrix} \Delta \theta_a \\ \Delta \theta_b \\ \Delta R_1 \end{bmatrix} = 0 \quad (4\text{-}10)$$

in which θ_a and θ_b can be found from equations (4-1)

$$\theta_a = \frac{(K_1 s_1 c_1)\ M_{Fbc}}{(K_1 s_1 c_1)^2 - \left[K_1 s_1 + K_2 s_2(1-c_2)\right]\left[K_1 s_1 + K_v s_v(1-c_v)\right]} \quad (4\text{-}11a)$$

and

$$\theta_b = - \frac{\left[K_1 s_1 + K_v s_v (1-c_v)\right] \quad M_{Fbc}}{(K_1 s_1 c_1)^2 - \left[K_1 s_1 + K_2 s_2 (1-c_2)\right]\left[K_1 s_1 + K_v s_v (1-c_v)\right]}$$

(4-11b)

The factors s' and c' were found by Masur, Chang and Donnell[12]

$$s' = \frac{s}{2p}\,(1-c^2 s) \qquad (4\text{-}12a)$$

$$c' = \frac{1+c}{2p}\left[1 - c\, s\, (1-c)\right] \qquad (4\text{-}12b)$$

To simplify equation (4-10) further it is necessary to consider Fig. 8 (a). Summation of moments about point d is zero.

Therefore

$$0 = (V_1 + \Delta V_1)\, L_2 - P\left[L_2 - (D + \Delta D)\right] + P\,(D + \Delta D)$$

$$+ M_{ab} + \Delta M_{ab} + M_{dc} + \Delta M_{dc} \qquad (4\text{-}13)$$

By noting that $D = 0$ and substituting appropriate values for M_{ab} , M_{dc} , ΔM_{ab} and$_d$ ΔM_{dc} it is found that

$$0 = \Delta V_1 L_2 + 2P\,\Delta D + 2K_1 s_1\left[\Delta\theta_a + c_1\ \Delta\theta_b - (1+c_1)\ \Delta R_1\right]$$

Since $\Delta D = \Delta R_1 L_1$

$$\Delta V_1 = \Delta p_1 = -\frac{2K_1 s_1}{L_2}\Delta\theta_a - \frac{2K_1 s_1 c_1}{L_2}\Delta\theta_b - \frac{2}{L_2}\left[PL_1 - K_1 s_1 (1+C_1)\right]$$

$$\Delta R_1 \qquad (4\text{-}14)$$

By substituting equation (4-14) into equation (4-10) an equation will be found such that

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} \Delta\theta_a \\ \Delta\theta_b \\ \Delta R_1 \end{bmatrix} = 0 \qquad (4\text{-}15)$$

But the column matrix $\begin{bmatrix} \Delta\theta_a \\ \Delta\theta_b \\ \Delta R_1 \end{bmatrix}$ can not be zero; so the determinant of its coefficients must be zero, ie,

$$\begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = 0 \qquad (4\text{-}16)$$

in which

$$T_{11} = K_1 s_1 + K_v s_v (1+c_v) - \frac{2K_1{}^2 s_1}{L_2}\left[s_1{}'\theta_a + (s_1 c_1{}' + s_1' c_1)\,\theta_b\right]$$

$$T_{12} = K_1 s_1 c_1 - \frac{2K_1^2 s_1 c_1}{L_2}\left[s_1'\theta_a + (s_1 c_1' + s_1' c_1)\,\theta_b\right]$$

$$T_{13} = -K_1 s_1(1 + c_1) - \frac{2K_1}{L_2}\left[PL_1 - K_1 s_1(1+c_1)\right]\left[s_1'\theta_a + (s_1 c_1' + s_1' c_1)\theta_b\right]$$

$$T_{21} = K_1 s_1 c_1 - \frac{2K_1^2 s_1}{L_2}\left[(s_1 c_1' + s_1' c_1)\,\theta_a + s_1'\theta_b\right]$$

$$T_{22} = K_1 s_1 + K_2 s_2(1 + c_2) - \frac{2K_1^2 s_1 c_1}{L_2}\left[(s_1 c_1' + s_1' c_1)\,\theta_a + s_1'\theta_b\right]$$

$$T_{23} = -K_1 s_1(1+c_1) - \frac{2K_1}{L_2}\left[PL_1 - K_1 s_1(1+c_1)\right]\left[(s_1 c_1' + s_1' c_1)\theta_a + s_1'\theta_b\right]$$

$$T_{31} = K_1 s_1(1 + c_1) - \frac{2K_1^2 s_1}{L_2}\left[s_1'(1+c_1) + s_1 c_1'\right](\theta_a + \theta_b)$$

$$T_{32} = K_1 s_1(1+c_1) - \frac{2K_1^2 s_1 c_1}{L_2}\left[s_1'(1+c_1) + s_1 c_1'\right](\theta_a + \theta_b)$$

$$T_{33} = PL_1 - 2K_1 s_1(1 + c_1) - \frac{2K_1}{L_2}\left[PL_1 - K_1 s_1(1+c_1)\right]\left[s_1'(1+c_1)\right.$$

$$\left. + s_1 c_1'\right](\theta_a + \theta_b)$$

CHAPTER V

CONCLUSIONS

Two orderly, systematic methods of analyzing the buckling stability of frames were presented. In both methods the phenomenon of bifurcation was employed to aid in simplifying the analysis of the sidesway mode of instability. Since bifurcation occurs only for a frame which is symmetrical with regard to geometry, physical properties and loading, this would appear to be a very restricted case for analysis. However in actual construction this situation is often encountered.

APPENDIX A

If equation (3-3) is integrated twice it becomes

$$\frac{d^2y}{d\,x^2} = -\frac{M}{EI}$$

in which

$$M = M_i + py - Y_i x$$

Thus

$$\frac{d^2y}{dx^2} = -\frac{1}{EI}\left[M_i + py - Y_i\, x\right]$$

Let $k = \sqrt{P/EI}$

$$\frac{d^2y}{dx^2} + k\,y = \frac{1}{EI}\left[Y_i x - M_i\right]$$

The well-known solution for this linear differential equation is

$$y = A \sin k\,x + B \cos k\,x + \frac{Y_i x}{P} - \frac{M_i}{P}$$

The following boundary conditions are evident:

1) at $x = 0$
   a) $y = 0$
   b) $y' = \theta_i$

2) at $x = L$
   a) $y = D$
   b) $y' = \theta_j$

Substituting boundary condition 1 (a) it is found that

$$B = \frac{M_i}{P}$$

Now using boundary condition 2 (a) it is noted that

$$A = \frac{1}{\sin kL}\left[D + \frac{M_i}{P} - \frac{Y_i L}{P} - \frac{M_i}{P}\cos kL\right]$$

But from consideration of the equilibrium of the beam column of Fig. 3

$$D = \frac{Y_i L}{P} - \frac{M_i}{P} - \frac{M_j}{P}$$

Then

$$A = -\frac{1}{\sin kL}\quad\frac{M_i}{P}\cos kL + \frac{M_j}{P}$$

Hence

$$y' = -\frac{1}{\sin kL}\left[\frac{M_i}{P}\cos kL + \frac{M_j}{P}\right]k\cos k\,x - \frac{M_i}{P}k\sin kx + \frac{Y_i}{P}$$

At x = 0 the angular rotation is

$$\theta_i = -\frac{k}{\sin kL}\left[M_i\frac{\cos kL}{P} + \frac{M_j}{P}\right] + \frac{Y_i}{P}$$

$$\theta_i = -M_i\frac{L}{EI}\frac{k}{P}\frac{EI}{L}\cot kL - M_j\frac{L}{EI}\frac{k}{P}\frac{EI}{L}\frac{1}{\sin kL} + \frac{Y_i}{P}$$

$$= M_i\frac{L}{EI}\left[\frac{EI}{PL^2} - \frac{\cot kL}{kL} - \frac{EI}{PL^2}\right] - M_j\frac{L}{EI}\left[\frac{1}{kL\sin kL} - \frac{EI}{PL^2} + \frac{EI}{PL^2}\right] + \frac{Y_i}{P}$$

But $kL = \phi = \sqrt{\frac{PL^2}{EI}}$

and $F = \frac{L}{EI}$

Therefore

$$\Theta_i = M_i F\left[C - \frac{1}{\phi^2}\right] - M_j F\left[S + \frac{1}{\phi^2}\right] + \frac{Y_i}{P}$$

Similarly if $x = L$ then

$$\Theta_j = - M F\left[S + \frac{1}{\phi^2}\right] + M_j F\left[C - \frac{1}{\phi^2}\right] + \frac{Y_i}{P}$$

APPENDIX B

STABILITY FACTORS

(1) BLEICH

$$C = \frac{1}{\phi^2} \left(1 - \phi \cot \phi\right)$$

$$S = \frac{1}{\phi^2} \left(\frac{\phi}{\sin \phi} - 1\right)$$

$$\text{in which } \phi = \sqrt{\frac{P}{EI}}\, L$$

(2) LIVESLEY

$$c = \frac{\phi - \sin \phi}{\sin \phi - \phi \cos \phi}$$

$$s = \frac{\phi \left(\sin \phi - \phi \cos \phi\right)}{2 - 2 \cos \phi - \phi \sin \phi}$$

$$m = \frac{1}{1 - \dfrac{\phi^2}{s\,(1+c)}}$$

$$\text{in which } \phi = L\sqrt{\frac{P}{EI}}$$

(3) TIMOSHENKO

$$\phi(u) = \frac{3}{u} \left(\frac{1}{\sin 2u} - \frac{1}{2u}\right)$$

$$\Psi(u) = \frac{3}{2u} \left(\frac{1}{2u} - \frac{1}{\tan 2u}\right)$$

$$\text{in which } u = \frac{L}{2}\sqrt{\frac{P}{EI}}$$

INTERRELATIONS

a) BLEICH and LIVESLEY

$$C = \frac{1}{s\ (1-c^2)}$$

$$S = \frac{c}{s\ (1-c^2)}$$

b) TIMOSHENKO and BLEICH

$$\phi(u) = 6\ S$$

$$\Psi(u) = 3\ C$$

c) TIMOSHENKO and LIVESLEY

$$\phi\ (u) = \frac{3\ c\ m}{1 + c}$$

$$\Psi\ (u) = \frac{3m}{2\ (1+c)}$$

APPENDIX C

DERIVATION OF S' AND C'

From the work of Masur, Chang and Donnell,[12] the expressions for s', $\bar{s}'$ and c' are

$$s' = \frac{s}{2p}\,(1-c^2 s)$$

$$\bar{s}' = \frac{\bar{s}}{2p}\left[\frac{1-3c}{1-c} + c^2 s\right] \qquad \text{(a)}$$

and

$$c' = \frac{1+c}{2p}\left[1 - c\,s\,(1-c)\right]$$

From Appendix B

$$S = \frac{c}{\bar{s}}$$

Now

$$S' = \frac{dS}{dp} = \frac{dS}{d(\frac{c}{\bar{s}})} \times \frac{d(\frac{c}{\bar{s}})}{dp}$$

But

$$\frac{dS}{d(\frac{c}{\bar{s}})} = 1$$

and

$$\frac{d(\frac{c}{\bar{s}})}{dp} = \frac{\bar{s}\,\frac{dc}{dp} - c\,\frac{d\bar{s}}{dp}}{(\bar{s})^2}$$

Then after substituting

$$\frac{dS}{dp} = \frac{\frac{\bar{s}(1+c)}{2P}\quad 1-cs(1-c) \; - \; c\,\frac{\bar{s}}{2P}\left[\frac{1-3C}{1-c} + c^2 s\right]}{\bar{s}^2}$$

which simplifies to

$$\frac{dS}{dp} = \frac{1}{2P\bar{s}}\left[1 + \frac{2c^2}{1-c} - cs\right]$$

Again from appendix B

$$\bar{s} = \frac{1}{C} \;\; ; \; c = \frac{S}{C} \text{ and } s = \frac{C}{C^2 - S^2} \qquad \text{(b)}$$

After substituting and simplifying

$$S' = \frac{S}{2p}\left[\frac{C}{S} + \frac{2S}{C-S} - \frac{C}{C^2 - S^2}\right]$$

By employing the same method as above along with the relation $C = \frac{1}{\bar{s}}$ it can be noted that

$$C' = \frac{dC}{dp} = \frac{dC}{d\bar{s}} \times \frac{d\bar{s}}{dp}$$

in which $\frac{dC}{d\bar{s}} = -\frac{1}{(\bar{s})^2}$

Now

$$C' = -\frac{1}{(\bar{s})^2}\ \frac{\bar{s}}{2p}\left[\frac{1-3c}{1-c} + c^2 s\right]$$

which simplifies to

$$C' = -\frac{1}{2P\bar{s}}\left[\frac{1-3c}{1-c} + c^2 s\right]$$

By substituting equations (b) into the expression for C' and simplifying the following equation can be obtained:

$$C' = -\frac{C}{2p}\left[\frac{C-3S}{C-S} + \frac{S^2}{C(C^2-S^2)}\right]$$

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VITA AUCTORIS

1941 James Mervin Douglas was born in Leamington, Ontario, Canada on December 3, 1941.

1947 In September, 1947, he entered S.S.No. 8 Public School in the township of Gosfield North, Essex County, Ontario.

1954 In September, 1954, he enrolled at Essex District High School, Essex, Ontario where he obtained his secondary education.

1959 In September, 1959, he entered Essex College of Assumption University of Windsor in the Applied Science course.

1963 In June, 1963, he graduated from Assumption University of Windsor, with a B.A.Sc. in civil engineering and in September he entered the course leading to M.A.Sc. in Civil Engineering, at University of Windsor.