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A new application of Routh-Hurwitz criterion.

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A NEW APPLICATION OF
ROUTH-HURWITZ CRITERION

by

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ABSTRACT

The object of this thesis is to provide a new field and technique for the application of Routh-Hurwitz criterion. Originally, the criterion provides a way to detect the system's absolute stability. However, by transforming the boundary of the complex s- plane, the Routh-Hurwitz criterion can also be used to detect the existence of natural frequencies of a system in a specified region.

This thesis consists of four chapters:

1. The first chapter is devoted to summarize briefly some important effects of natural frequencies on a system's performance which include the effect on speed of system response, bandwidth, resonance frequency, hidden oscillation in sampled-data system and time scaling in analog computer.
2. The second chapter is dedicated to review the Routh-Hurwitz criterion. By reducing the Hurwitz determinant to a triangular determinant, the non-zero array of the triangular determinant is equivalent to the Routhian Array, which provides a proof for the relationship between the Routhian Array and Hurwitz criterion.
3. The third chapter presents a technique of using the Routh-Hurwitz criterion to detect the existence of roots with positive real parts of a polynomial with complex coefficients.
4. The fourth chapter introduces the w-transformation, which transforms the boundary for the constant jw in complex s-plane to the imaginary axis in complex w-plane, so that the Routh-Hurwitz criterion can be applied to detect the existence of natural frequency of a system

which is greater than the given value w.

From the result of the study in this thesis it is of no doubt to conclude that the Routh-Hurwitz criterion is capable to detect the existence of roots of a polynomial with real or complex coefficients in a specified region by using a proper transformation.

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CHAPTER I

A SUMMARY OF SOME IMPORTANT EFFECTS OF NATURAL FREQUENCY ON A SYSTEM'S PERFORMANCE

The performance of any linear system can be described by its characteristic polynomial P(s) where s is the complex frequency.

$$P(s) = \sum_{k=0}^{n+1} a_k s^k \qquad (1\text{-}1)$$

or

$$P(s) = \prod_{k=1}^{n} (s + a_k - jw_k) \qquad (1\text{-}2)$$

For an absolutely stable system, one of the important factors which define the system performance in the characteristic polynomial is the imaginary part of the roots of P(s) which appears as w_k in equation (1-2) w_k is usually called the natural frequency of the system, and can have many values if N is of high order. The effects of w_k on a system's performance may be grouped into several categories.

1. Speed of System Response

In continuous linear system, especially when the system can be approximated by its dominant poles, the speed of the system performance in response to a certain excitation is determined by the natural frequency of the dominant poles, an approximate relationship has been derived by Chu[1] as follows:

$$T_p \propto \frac{1}{w_d} \tag{1-3}$$

where T_p is the time required for the system response to reach its first overshoot, and w_d is the natural frequency.

The smaller the value of T_p the faster will be the system response due to an excitation. Therefore, if it is desired to increase the speed of the system response, the natural frequency should be increased.

2. Bandwidth

The linear control system usually is operated at a comparatively low frequency. In order to eliminate the effect of noise, which is relatively of high frequency, it is necessary to keep the bandwidth sufficiently low. Assume the transfer function of the system is:

$$\frac{C(s)}{R(s)} = \frac{Q(s)}{P(s)} = \frac{\prod_{t=1}^{k} (s + a_t - jw_t)}{\prod_{i=1}^{n} (s + a_i - jw_i)} \tag{1-4}$$

where k is smaller than n.

The respone of the system in frequency domain can be obtained simply by replacing s in equation (1-4) by jw

$$\frac{C(jw)}{R(jw)} = \frac{\prod_{t=1}^{k} [j(w-w_t) + a_t]}{\prod_{i=1}^{n} [j(w-w_i) + a_i]} \tag{1-5}$$

The bandwidth of the system is defined as the largest value of w for which the ratio $\frac{C(jw)}{R(jw)}$ is not smaller than a specified value. From equation (1-5), it is not difficult to realize that the relatively high value of $\frac{C(jw)}{R(jw)}$ occured at the neighborhood of $w=w_i$. Therefore, in order to make the ratio $\frac{C(jw)}{R(jw)}$ small, the value of w should be larger than w_i. Consequently the larger the value of w_i, the larger the value of w is needed to make $\frac{C(jw)}{R(jw)}$ small enough, which means the larger the natural frequency, the larger the bandwidth will be, which will undesirably increase the effect of noise on the system response.

3. Resonance Frequency

The resonance frequency of a system is defined as the frequency at which the value of $\frac{C(jw)}{R(jw)}$ is at its maximum. As has been stated in the case of bandwidth, the maximum value of $\frac{C(jw)}{R(jw)}$ will occur at the neighborhood of w_i, and for the system of high order it may have several resonance frequencies due to several natural frequencies w_i. The effect of such resonance frequency should not be overlooked especially in the control system of aircraft and missiles, since in those systems the mechanical or structural vibrations can be coupled into the control system and act as a forcing function.If the resonance frequency of the system is high enough to resonate with the vibrations, then the performance of the system will be seriously affected.

4. Hidden Oscillation in Sampled-data System

In error control sampled-data systems, if the natural frequency of the open loop system is higher than half of the sampling frequency, then hidden oscillation will occur [2] and the method of z-transform analysis is no more applicable. Therefore in designing a sampled-data system, it is necessary to make sure that the sampling frequency is higher than twice the highest natural frequency of the open-loop system. If there is no minor loop in the open-loop system, the satisfaction of the above condition is very clear from the open-loop transfer function. However, if there is a minor loop in the open-loop system, some detailed investigation is needed to make sure that there is no natural frequency higher than half the sampling frequency in the open-loop system.

5. Scaling in Analog Computer

The analog computer has been widely used as a powerful tool in system simulation and in finding the solution of equations. One important drawback of the analog computer comes from the need of time scaling in the computer program. Since the ideal duration for a single dynamic solution obtained on general electronic analog computer is about a few seconds, it is necessary to adjust the speed of the system response to be within such ideal range. Since the speed of the system response is determined from the natural frequency of the system, therefore, before speeding up or slowing down the system response, a knowledge of the highest natural frequency of the system is one of the most important factors in the technique of time scaling.

The preceding discussion, shows that for a given system it is important to know the value or the highest value of the natural frequency of the system, or to check whether all the natural frequencies of the system lie inside a specified allowable range or region.

In order to obtain all such information, the best way is to solve the characteristic polynomial P(s) so that it appears as in equation (1-2), from which all the natural frequencies are known. However, if only the existence of the natural frequency of a system in a specified range or region is to be determined, solving the characteristic polynomial P(s) will be unnecessarily laborious. Therefore, a simple and straight forward approach has been developed for the same purpose.

CHAPTER II

ROUTH-HURWITZ CRITERION

It is known that the absolute stability of a system is determined merely by the existence of any root of the characteristic polynomial P(s) in the right-half of the complex s-plane. In other words, if P(s) has no root with positive real part then the system is defined as absolutely stable, otherwise, it is defined as absolutely unstable. Therefore, with the imaginary axis of the complex s-plane as the boundary of the region of stability, any root lying to the right of this boundary indicates the system is absolutely unstable, otherwise it is absolutely stable.

One way to detect the existence of any root in the unstable region is by solving the characteristic polynomial P(s). After finding all the roots of P(s), it will immediately tell whether the system is absolutely stable. However, it is rather laborious to solve the polynomial, especially when the system is of high order. To overcome this difficulty both Routh and Hurwitz have developed a simple and efficient method which is now named as the Routh-Hurwitz criterion. This criterion provides a simple method to detect the existence of roots of P(s) with positive real parts, i.e. which lie in the unstable region, without solving for any of the roots.

Routh-Hurwitz Criterion

Consider a polynomial P(s)

$$P(s) = \sum_{k=o}^{n} a_k s^k \tag{2-1}$$

where all the coefficients a_k are real and positive and non-zero. If any one of them is negative or zero, there will be at least one root of P(s) that has positive real part, and the following procedure becomes necessary.

On condition that all the coefficients are real and positive, the following determinant is constructed.

$$D = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \text{------} & \\ a_n & a_{n-2} & a_{n-4} & \text{------} & \\ 0 & a_{n-1} & a_{n-3} & \text{------} & \\ 0 & a_n & a_{n-2} & \text{------} & \\ 0 & 0 & a_{n-1} & a_{n-3} & \text{------} \\ 0 & 0 & a_n & a_{n-2} & \text{------} \\ \text{------} & & & & \end{vmatrix} \tag{2-2}$$

The dimension of the determinant D is equal to the order of the polynomial n.

Using the determinant reduction method the determinant D can be reduced to a triangular determinant.

$$D = \begin{vmatrix} a_{11} & a_{12} & - & - & - & - & - & - & - \\ 0 & a_{22} & a_{23} & - & - & - & - & - & - \\ 0 & 0 & a_{33} & a_{34} & - & - & - & - & - \\ 0 & 0 & 0 & a_{44} & a_{45} & - & - & - & - \\ - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & --- & a_{nn} \end{vmatrix} \qquad (2\text{-}3)$$

From equation (2-3) the following sequence is formed with the diagonal elements of the determinant D.

$$[a_{ii}]_{i = 1 \cdots n} = a_{11}, a_{22}, a_{33}, \ldots a_{nn} \qquad (2\text{-}4)$$

According to the Routh-Hurwitz criterion the number of sign changes in the sequence $[a_{ii}]$ is equal to the number of roots of P(s) with positive real parts. Therefore, if all the diagonal elements a_{ii} are positive, it indicates P(s) has no root with positive real part.

Further-more, during the determinant reduction procedure, if any one row of determinant D become an all-zero row, it indicates that there are one or more pairs of roots of P(s) in opposite signs. They can be a pair of conjugate roots with imaginary parts only, or a pair of real roots in opposite signs, or two pairs of conjugate complex roots in opposite signs, etc. All these roots can be determined from the auxiliary polynomial which is constructed with the elements of the preceding row, and the procedure can be continued after the zero row is replaced by the elements obtained from the coefficients after differentiating the auxiliary polynomial [3]. The following examples are devoted to illustrate the procedure described above.

Example 2-1

$$P(s) = s^5 + 6s^4 + 27s^3 + 68s^2 + 110s + 100 \qquad (2\text{-}5)$$

The determinant D is constructed according to equation (2-2)

$$D = \begin{vmatrix} 6 & 68 & 100 & 0 & 0 \\ 1 & 27 & 110 & 0 & 0 \\ 0 & 6 & 68 & 100 & 0 \\ 0 & 1 & 27 & 110 & 0 \\ 0 & 0 & 6 & 68 & 100 \end{vmatrix} \qquad (2\text{-}6)$$

Which can be reduced to :

$$D = \begin{vmatrix} 6 & 68 & 100 & 0 & 0 \\ 0 & 15.66 & 93.33 & 0 & 0 \\ 0 & 0 & 32.25 & 100 & 0 \\ 0 & 0 & 0 & 44.76 & 0 \\ 0 & 0 & 0 & 0 & 100 \end{vmatrix} \qquad (2\text{-}7)$$

From the diagonal elements of determinant D in equation (2-7), the following sequence is formed.

$$[a_{ii}]_{i=1...5} = 6,\ 15.66,\ 32.25,\ 44.76,\ 100 \qquad (2\text{-}8)$$

In the sequence $[a_{ii}]$, since there is no sign change, P(s) has no root with positive real part. The actual roots of P(s) are found to be:

$$\begin{aligned} s_1 &= -2 \\ s_{2,3} &= -1 \pm j2 \\ s_{4,5} &= -1 \pm j3 \end{aligned} \qquad (2\text{-}9)$$

In (2-9), there is no root with positive real part, so it is in agreement with the conclusion obtained from the sequence $[a_{ii}]$.

<u>Example 2-2</u>

$$P(s) = s^5 + 2s^4 + 11s^3 + 32s^2 + 70s + 100 \qquad (2\text{-}10)$$

Following the same procedure as in Example 2-1, the resulting sequence is:

$$[a_{ii}]_{i=1...5} = 2\ ,\ -5\ ,\ 40,\ 32.49,\ 100 \qquad (2\text{-}11)$$

Since there are two sign changes in the sequence $[a_{ii}]$, P(s) has two roots with positive real parts. The roots of P(s) are found to be:

$$s_1 = -2$$
$$s_{2,3} = -1 \pm j2 \qquad (2\text{-}12)$$
$$s_{4,5} = 1 \pm j3$$

In (2-12) it is clear that P(s) has two roots with positive real parts which are checked with the conclusion.

Example 2-3

$$P(s) = s^5 + 4s^4 + 18s^3 + 46s^2 + 81s + 90 \qquad (2\text{-}13)$$

During the reducing procedure, an all-zero row is found .

$$D = \begin{vmatrix} 4 & 46 & 90 & 0 & 0 \\ 0 & 6.5 & 58.5 & 0 & 0 \\ 0 & 0 & 10 & 90 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 90 \end{vmatrix} \qquad (2\text{-}14)$$

In equation (2-14) all the elements of the 4th row are equal to zero. Therefore the auxiliary polynomial is constructed from the 3rd row as follows:

$$P'(s) = 10s^2 + 90 \qquad (2\text{-}15)$$

Equation (2-15) results in :

$$s_{1,2} = \pm j3 \qquad (2\text{-}16)$$

which are some of the roots of P(s).

To continue the reduction process, P'(s) is differentiated with respect to s, thus:

$$\frac{dP'(s)}{ds} = 20s + 0 \qquad (2\text{-}17)$$

Substituting the coefficient of the polynomial in equation (2-17) into the determinant D in equation (2-14), we have:

$$D = \begin{vmatrix} 4 & 46 & 90 & 0 & 0 \\ 0 & 65 & 58.5 & 0 & 0 \\ 0 & 0 & 10 & 90 & 0 \\ 0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 10 & 90 \end{vmatrix} \qquad (2\text{-}18)$$

Then the reduction process of determinant D is continued until it becomes a triangular determinant. These diagonal elements of the triangular determinant form the following sequence:

$$[a_{ii}] = 4,\ 6.5,\ 10,\ 90,\ 90 \qquad (2\text{-}19)$$

In (2-19), since there in no sign change, P(s) has no root with positive real part. The roots of P(s) are found to be:

$$\begin{aligned} s_1 &= -\ 2 \\ s_{2,3} &= \pm\ j3 \\ s_{4,5} &= -\ 1 \pm j2 \end{aligned} \qquad (2\text{-}20)$$

In equation (2-20), it is clear that two roots of P(s), s_2 and s_3, are in opposite signs which are identical to the results obtained from the auxiliary polynomial P'(s) shown in equation (2-16). Moreover, in (2-20) it is obvious that P(s) has no root with positive real part.

Example 2-4

$$P(s) = s^5 + 8s^4 + 20s^3 + 12s^2 - 21s - 20 \qquad (2\text{-}21)$$

In equation (2-21), P(s) evidently has roots with positive real parts because some of the coefficients are negative. If the Routh-Hurwitz criterion is applied, an all-zero row will appear at the 4th row during the reduction process of the determinant D, and the auxiliary polynomial becomes:

$$P'(s) = 20s^2 - 20 \qquad (2\text{-}22)$$

From equation (2-22) it is easy to found that

$$s_{1,2} = \pm 1 \qquad (2\text{-}23)$$

s_1 and s_2 are roots of P(s) which occur in opposite signs.

Further in the reduction process of the determinant D, the following sequence is formed:

$$[a_{ii}] = 8,\ 20,\ -20,\ -20 \qquad (2\text{-}24)$$

Since the sign changes only once, P(s) should have one root with a positive real part, which is s =+1 as obtained from the auxiliary polynomial.

The roots of P(s) are found to be

$$s_{1,2} = \pm 1$$
$$s_3 = - 4 \qquad (2\text{-}25)$$
$$s_{4,5} = -2 \pm j1$$

The roots of P(s) in equation (2-23) are identical with the corresponding roots shown in equation (2-25), and with the conclusion obtained from sequence (2-24).

Example 2-5

$$P(s) = s^{10} + 20s^9 + 175s^8 + 880s^7 + 2823s^6 + 6100s^5 + 9225s^4 + 10120s^3 + 8276s^2 + 4880s + 1700 \qquad (2\text{-}26)$$

The 5th row is an all-zero row and auxiliary polynomial is:

$$P'(s) = 50s^6 + 50s^4 + 200s^2 + 200 \qquad (2\text{-}27)$$

Equation (2-27) can be factored into:

$$P'(s) = (s^2 + 1)\ (s^4 + 4) \qquad (2\text{-}28)$$

In equation (2-28), it is not difficult to find the roots of P'(s) which are also the roots of P(s):

$$s_{1,2} = \pm j1$$
$$s_{3,4,5,6} = \pm 1 \pm j1 \qquad (2\text{-}29)$$

In (2-29) s_1 and s_2 are complex roots in opposite signs, s_3 and s_4, s_5 and s_6 are two pairs of conjugate complex roots in opposite signs.

Differentiating P'(s) with respect to s, we have:

$$\frac{dP'(s)}{ds} = 300s^5 + 200s^3 + 400s \qquad (2\text{-}30)$$

The reduction process of Determinant D can be continued by replacing the elements in the zero row by the coefficients of $\frac{dP'(s)}{ds}$ in equation (2-30). Finally, the following sequence is formed from the diagonal elements of the triangular determinant.

$$[a_{ii}] = 10,\ 32,\ 54.38,\ 50,\ 250,\ -110,\ 240,\ 147,\ 200,\ 200 \qquad (2\text{-}31)$$

Since there are two sign changes in the sequence $[a_{ii}]$, it indicates P(s) has two roots with positive real parts, which are:

$$s_{3,4} = +1 \pm j1 \qquad (2\text{-}32)$$

as obtained in (2-29).

The actual roots of P(s) are found to be:

$$
\begin{aligned}
s_{1,2} &= \pm j \\
s_{3,4} &= 1 \pm j1 \\
s_{5,6} &= -1 \pm j1 \\
s_{7,8} &= -2 \pm j1 \\
s_{9,10} &= -3 \pm j1
\end{aligned}
\qquad (2\text{-}33)
$$

The roots in (2-29) are equal to the corresponding roots in (2-33). P(s) has only two roots with positive real parts, as shown in (2-33), and is in agreement with the Routh-Hurwitz criterion.

The above examples have clearly illustrated how to use the Routh-Hurwitz criterion to detéct the existence of any positive root of a polynomial P(s).

CHAPTER III

GENERALIZED ROUTH-HURWITZ CRITERION

If the coefficients of the polynomial P(s) are not real numbers but are complex numbers, then the original Routh-Hurwitz criterion is not applicable. Marden [4] and others have modified the Routh-Hurwitz criterion so that it can be applied to the case of a polynomial with complex coefficients. However, instead of modifying the criterion, there is another way to attain the desired result, that is by modifying the polynomial P(s) so that all the coefficients of the polynomial become real numbers without altering the real parts of the roots of P(s).

Suppose we have:

$$P(s) = \prod (s + d_i + jw_i)\ (s + d_i - jw_i)$$
$$= \prod (s^2 + 2a_i s + a_i^2 + w_i^2) \qquad (3\text{-}1)$$

In equation (3-1), since there is no complex number involved, the product of all the terms will result in a polynomial with real coefficients. However, if any one or some of the complex roots of P(s) are not in conjugate pairs, say if $s_1 = -b - jw_o$ and the other roots are inconjugate pairs, then P(s) may be expressed as:

$$P(s) = P'(s)\ (s + b + jw_o) \qquad (3\text{-}2)$$

where all the complex roots of P'(s) are in conjugate pairs or, in other words P'(s) is a polynomial with real coefficients. Expanding the product in equation (3-2), we have:

$$P(s) = P'(s)\ (s + a) + jw_o P'(s) \qquad (3\text{-}3)$$

Equation (3-3) shows P(s) contains two parts: a real part and an imaginary part. Performing the multiplication in equation (3-4) and collecting the terms with the same order of s, we have a polynomial with complex coefficients. Therefore, it is definite to conclude that the complex coefficients in a polynomial are introduced by the unpaired complex roots of the polynomials. Consequently, in order to eliminate the complex coefficients in a polynomial, it is necessary to introduce the conjugate pairs of the unpaired complex roots. However, for a given polynomial P(s), it is not easy to find out exactly the unpaired complex root of P(s), the only alternative is to intorduce the conjugate pairs of all the roots of P(s). In this way, the original real roots and the conjugate paired complex roots of P(s) become double roots and the unpaired complex root becomes a paired conjugate complex roots.

Consider the polynomial:

$$\begin{aligned} P(s) = {} & (s + a_i) \cdots\cdots\cdots \\ & \cdot (s + b_i + jw_{bi})\ (s + b_i - jw_{bi}) \\ & \cdot (s + c_i + jw_{ci})\ (s + d_i - jw_{di}) \qquad (3\text{-}4) \end{aligned}$$

Let $P^*(s)$ be defined as the conjugate polynomial of P(s)

$$\begin{aligned} P^*(s) = (s + a_i) & \quad \cdots\cdots\cdot \\ & \cdot (s + b_i - jw_{bi})\ (s + b_i + jw_{bi}) \\ & \cdot (s + c_i - jw_{ci})\ (s + d_i + jw_{di}) \end{aligned} \qquad (3\text{-}5)$$

By multiplying P(s) and $P^*(s)$, we have:

$$\begin{aligned} F(s) &= P(s) \cdot P^*(s) \\ &= (s + a_i)^2 \quad \cdots\cdots\cdots \\ &\quad \cdot (s + b_i + jw_{bi})^2\ (s + b_i - jw_{bi})^2 \\ &\quad \cdot (s + c_i + jw_{ci})\ (s + c_i - jw_{ci}) \\ &\quad \cdot (s + d_i + jw_{di})\ (s + d_i - jw_{di}) \end{aligned} \qquad (3\text{-}6)$$

In equation (3-6), all roots of F(s) are in conjugate pairs, therefore, by expanding equation (3-6) by multiplication we have a polynomial F(s) with real coefficients.

Four important points should be pointed out in the above result.

1. The order of polynomial F(s) is twice of that of the original P(s).
2. The real parts of all the roots of F(s) are the same as those of the original P(s). Therefore, if the original P(s) has no root with positive real part, then the new polynomial F(s) will also has no root with positive real part, and vice versa.
3. Since F(s) is a polynomial of real coefficients, and the real parts of all the roots of F(s) are identical with those of P(s), then the Routh-Hurwitz criterion can be applied to F(s) and the

conclusion obtained is applicable to P(s). In other words, instead of directly testing P(s), the polynomial with complex coefficients, we test F(s), the polynomial with real coefficients.

4. The number of roots of F(s) is twice of that of the original P(s). Therefore if F(s) has two roots with positive real parts, it indicates that P(s) has one root with positive real part.

In general, the modification of P(s) into F(s) can be done in the following two ways:

1. Suppose we have the polynomial:

$$P(s) = \sum_{i=0}^{n} (a_i + jb_i)\, s^i \tag{3-7}$$

From equation (3-7), the conjugate of P(s) is obtained as:

$$P^*(s) = \sum_{i=0}^{n} (a_i - jb_i)\, s^i \tag{3-8}$$

Multiplying P(s) with $P^*(s)$, we have:

$$\begin{aligned} F(s) &= P(s) \cdot P^*(s) \\ &= \sum_{i=0}^{n} (a_i + jb_i)\, s^i \cdot \sum_{k=0}^{n} (a_k - jb_k)\, s^k \end{aligned} \tag{3-9}$$

Expanding equation (3-9) and collecting similar terms, we have the following result:

$$F(s) = \sum_{i=0}^{2n} c_i s^i \tag{3-10}$$

where

$$c_i = \sum_{j=0}^{i} (a_j a_{i-j} + b_j b_{i-j}) \qquad (3\text{-}11)$$

In equation (3-11) and (3-10), it is obvious that all the coefficients of c_i are real numbers and they can be found easily by equation (3-11).

2. In equation (3-7), by separating all the terms with real coefficients from those with imaginary coefficients, we devide P(s) into two parts:

$$P(s) = \sum_{i=0}^{n} a_i s^i + j \sum_{i}^{n} b_i s^i \qquad (3\text{-}12)$$

The conjugate of P(s) obviously is:

$$P^*(s) = \sum_{i=0}^{n} a_i s^i - j \sum_{i=0}^{n} b_i s^i \qquad (3\text{-}13)$$

From equations (3-12) and (3-13) the following equations are formed

$$F(s) = (\sum_{i=0}^{n} a_i s^i)(\sum_{i=0}^{n} a_k s^k) + (\sum_{i=0}^{n} b_i s^i)(\sum_{i=0}^{n} b_k s^k) \qquad (3\text{-}14)$$

or

$$F(s) = (\sum_{i=0}^{n} a_i s^i)^2 + (\sum_{i=0}^{n} b_i s^i)^2 \qquad (3\text{-}15)$$

By expanding equation (3-15) and collecting terms of the same order of s, the following expression is obtained:

$$F(s) = \sum_{i=0}^{2n} c_i s^i \qquad (3\text{-}16)$$

where

$$c_i = \sum_{j=0}^{i} (a_j a_{i-j} + b_j b_{i-j}) \qquad (3\text{-}17)$$

(3-17) and (3-18) are identical with (3-11), (3-12) as obtained by method 1.

If any one or some of the coefficients c_i are zero or negative, it indicates that F(s) has roots with positive real parts, which is equivalent to saying that P(s) has roots with positive real parts. If only the existence of roots with positive real parts is to be detected, then further testing by the Routh-Hurwitz criterion is not necessary. Therefore, the Routh-Hurwitz criterion is applied only when all the coefficients of F(s) are positive and non-zeros.

The following examples are devoted to illustrate how to detect the existence of roots with positive real parts of a given polynomial P(s) with complex coefficients.

Example 3-1

$$P(s) = s^3 + (2 + j2)\cdot s^2 + (-2 + j4)\cdot s - 4 \qquad (3\text{-}18)$$

By the procedure described above, we have:

$$F(s) = P(s) \cdot P^*(s)$$
$$= s^6 + 4s^5 + 4s^4 + 0 + 12s^2 + 8s + 16 \qquad (3\text{-}19)$$

In equation (3-19), the term with s^3 is missing, which indicates that there is at least two roots of F(s) or one root of P(s) which has positive real part. The actual roots of P(s) are found to be

$$s_1 = -2$$
$$s_2 = -1 - j \qquad (3\text{-}20)$$
$$s_3 = 1 - j$$

In equation (3-20), it is clear that the third root s_3 of P(s) has positive real part. Consequently, the conclusion is justified.

Example 3-2

$$P(s) = s^3 + (1 + j2)\cdot s^2 + (-5 + j3)\cdot s - 6 - j2 \qquad (3\text{-}21)$$

From equation (3-21), F(s) is obtained as:

$$F(s) = s^6 + 2s^5 - 5s^4 - 10s^3 + 5s^2 + 48s + 40 \qquad (3\text{-}22)$$

In equation (3-22), since the coefficients of s^4 and s^3 are negative, F(s) and P(s) have roots with positive real parts and the Routh-Hurwitz test is not necessary to proceed. The roots of P(s) are found to be:

$$s_1 = -2$$
$$s_2 = -1 - j1 \qquad (3\text{-}23)$$
$$s_3 = 2 - j1$$

The third root, s_3, is shown to be with positive real part.

Example 3-3

$$P(s) = s^3 + (2 + j3)s^2 + (-3 + j7)s - 6 + j2 \qquad (3\text{-}24)$$

From equation (3-24) F(s) is obtained as :

$$F(s) = s^6 + 4s^5 + 7s^4 + 18s^3 + 46s^2 + 64s + 40 \qquad (3\text{-}25)$$

Since all the coefficients of F(s) in equation (3-25) are real and positive and non-zero, the Routh-Hurwitz criterion is applied to check the existence of any root of F(s) or P(s) which has positive real part. According to the procedure described in chapter 2, the following determinant is constructed.

$$D = \begin{vmatrix} 4 & 18 & 64 & 0 & 0 & 0 \\ 1 & 7 & 46 & 40 & 0 & 0 \\ 0 & 4 & 18 & 64 & 0 & 0 \\ 0 & 1 & 7 & 46 & 40 & 0 \\ 0 & 0 & 4 & 18 & 64 & 0 \\ 0 & 0 & 1 & 7 & 46 & 40 \end{vmatrix} \qquad (3\text{-}26)$$

After the determinant D is reduced to a triangular determinant, the following sequence is formed with the diagonal elements of the triangular determinant.

$$[a_{ii}] = 4, 2.5, -30, 30, 40, 40 \quad (3\text{-}27)$$

There are two sign changes in the sequence $[a_{ii}]$, so F(s) has two roots with positive real parts which in turn signifies that P(s) has one root with positive real part. The roots of P(s) are found to be:

$$\begin{aligned} s_1 &= -2 \\ s_2 &= -1 - j \\ s_3 &= 1 - j2 \end{aligned} \quad (2\text{-}28)$$

The third root, s_3, is obviously a root with positive real part.

<u>Example 3-4</u>

$$P(s) = s^3 + (5 + j2)s^2 + (7 + j7)s + 2 + j6 \quad (3\text{-}29)$$

The corresponding F(s) is found to be:

$$F(s) = s^6 + 10s^5 + 43s^4 + 102s^3 + 142s^2 + 102s + 40 \quad (3\text{-}30)$$

In equation (3-30), since all the coefficients are real and positive and non-zero, the Routh-Hurwitz criterion is applied. Finally, the following sequence is obtained:

$$[a_{ii}] = 10, 32.8, 61.82, 84.15, 60.42, 40 \quad (3\text{-}31)$$

There is no sign change in the sequence $[a_{ii}]$, indicating F(s) has no root with positive real part which in turn shows that P(s) has no root with positive real part. The roots of P(s) are found to be:

$$s_1 = -2$$
$$s_2 = -1 - j \qquad (3\text{-}32)$$
$$s_3 = -2 - j$$

The above examples clearly show that the existence of any root with positive real part of a polynomial with complex coefficients can be detected by Routh-Hurwitz criterion after a slight modification of the original polynomial.

In order to detect the existence of the natural frequency of a system in a certain region, the characteristic polynomial of the system is transformed from complex s-plane to complex w-plane by a transformation. After the transformation, the characteristic polynomial becomes a polynomial with complex coefficients. Therefore, the approach introduced in this chapter is called for.

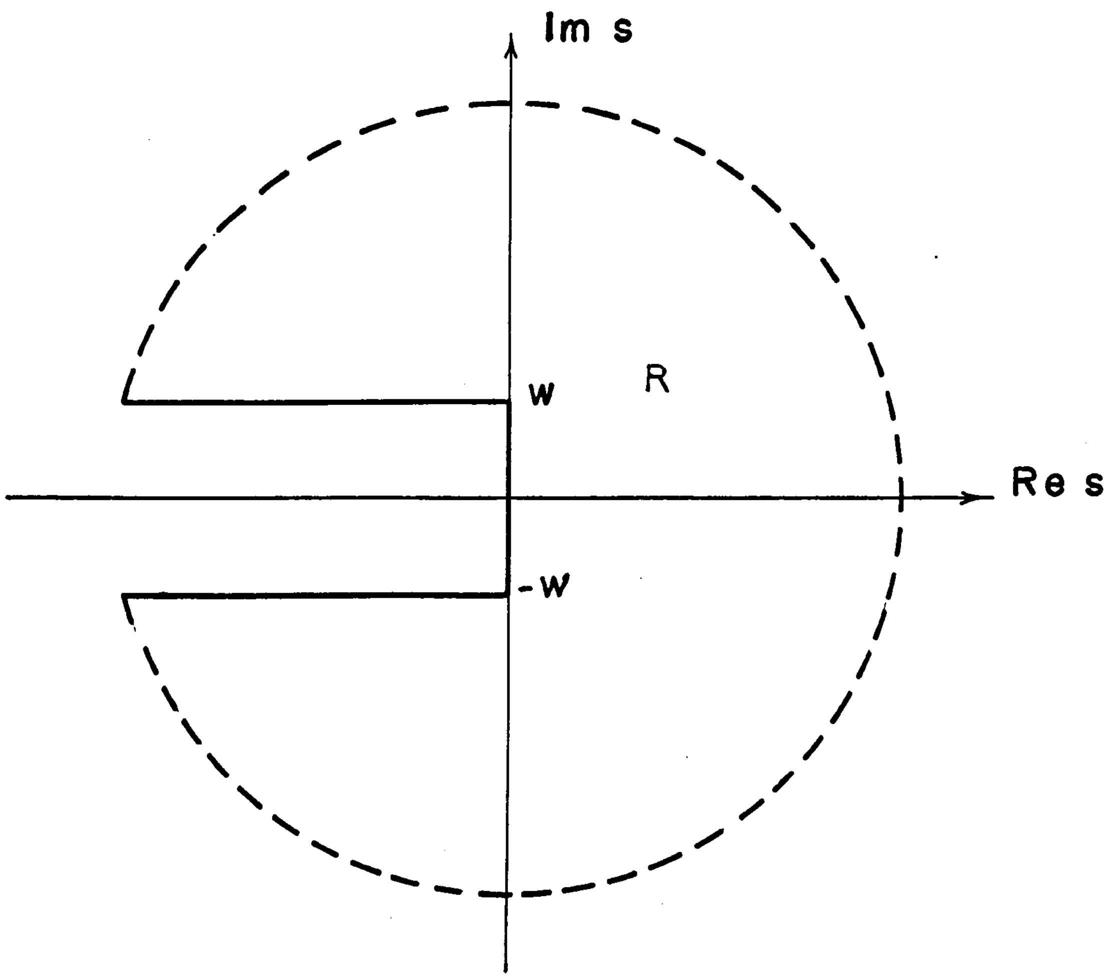
CHAPTER IV

W-TRANSFORMATION

For the same reason as states in Chapter I, it is desired that all the natural frequencies of a system be smaller than a certain value, say w, while the system is stable. In other words, on the complex s-plane a region R is specified, as shown in fig. 4-1, so that no root of the characteristic polynomial of the system is allowed to lie within this region. This region R can be divided into several sub-regions as shown in fig. 4-2 and fig. 4-3. In fig. 4-2 the region R_I is defined for the requirement of the absolute stability of the system and regions R_{II} and R_{III} are defined for the requirement of greatest allowable natural frequency of the system.

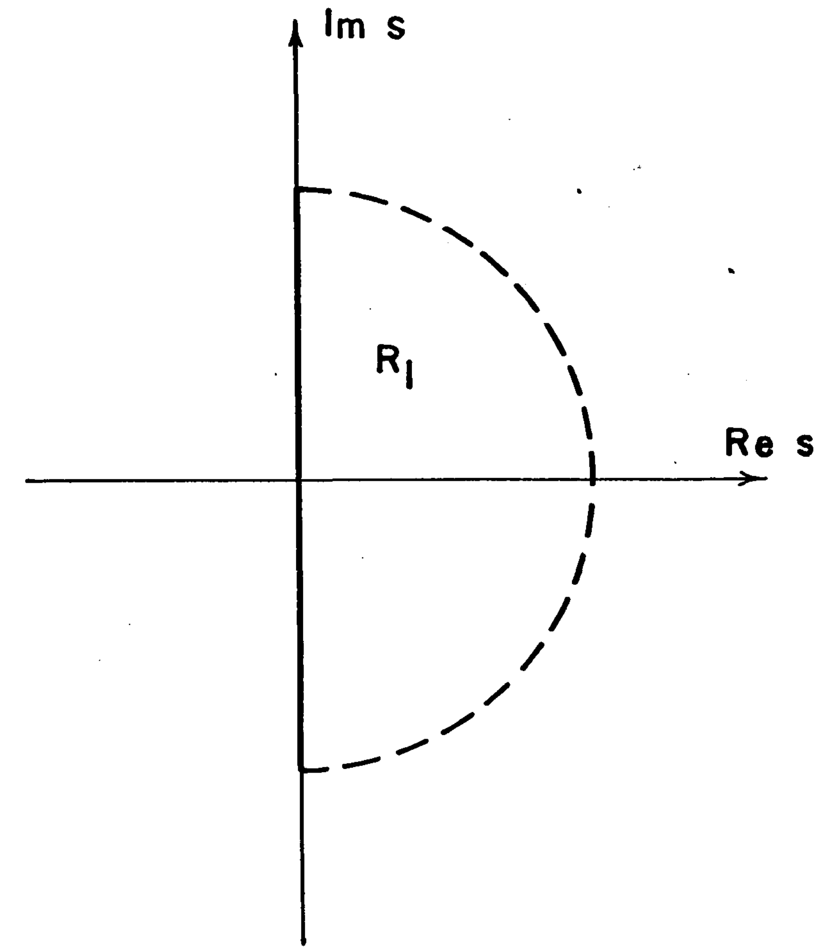
For a linear physical system, the characteristic polynomial of the system is a polynomial with real coefficients, which implies that all the complex roots of the characteristic polynomial should be in conjugate pairs if they exist. Therefore, if the system satisfies the requirement for region R_{II}, it will automatically satisfies the requirement for region R_{III}. Consequently, the problem is reduced to mere detecting the existence of any root of the characteristic polynomial which may lie in the regions R_I and R_{II}.

Since the boundary of region R_I is the imaginary axis of the complex s-plane, the Routh-Hurwitz criterion can be directly applied. However, for region R_{II}, Routh-Hurwitz criterion cannot be directly applied to detect the existence of roots of the characteristic



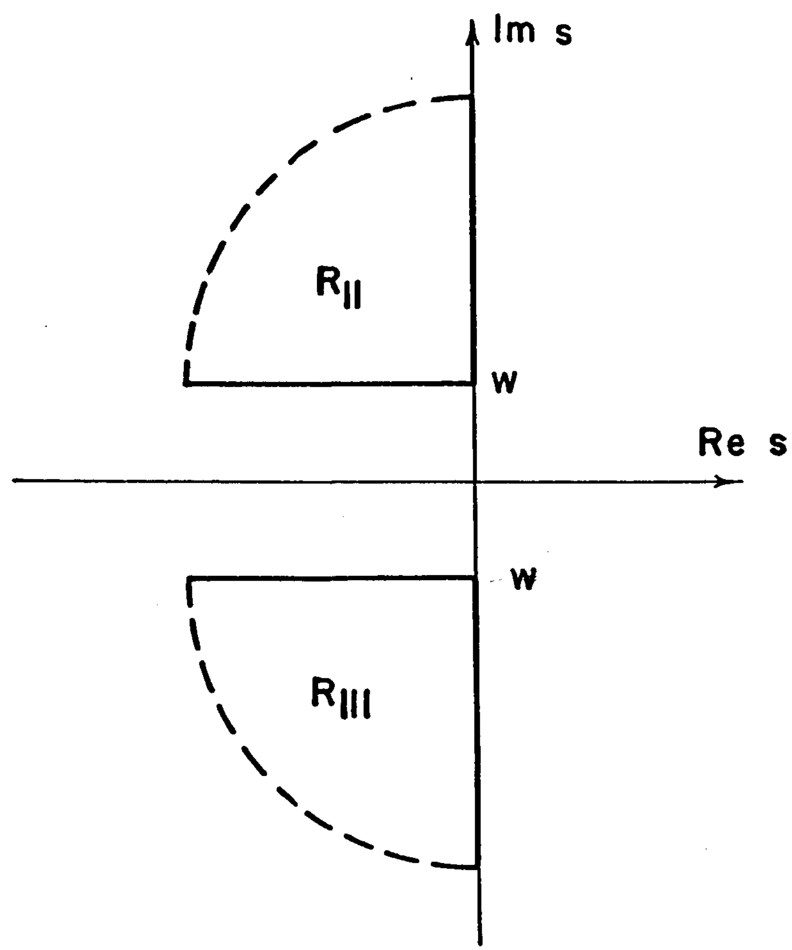
s - plane

Fig. 4 - 1



s - plane

Fig. 4 - 2



s - plane

Fig. 4 - 3

polynomial in this region, therefore, a transformation is introduced so that the boundary of region R_{II} becomes the imaginary axis of the complex plane.

By the following transformation:

$$S = j(W + w) \tag{4-1}$$

or

$$W = -j(s - jw) \tag{4-2}$$

the region R_{II} in complex s-plane is transformed into region R_{IV} in complex W-plane, as shown in fig. 4-4. The boundary of region R_{II} in s-plane is now transformed to the imaginary axis of W-plane, so that Routh-Hurwitz criterion is applicable to this region. However, after such a transformation as in equation (4-1) or (4-2), the characteristic polynomial becomes a polynomial with complex coefficients, this can be seen in the following:

Assume

$$P(s) = \sum_{i=1}^{n+1} a_i s^{n+1-i} \tag{4-3}$$

By the transformation equation (4-1), (4-3) becomes:

$$P(W)_w = \sum_{i=1}^{n+1} a_i (j)^{n+1-i} (W + w)^{n+1-i} \tag{4-4}$$

or

$$P(W)_w = \sum_{i=1}^{n+1} r_i W^{n-i+1} \tag{4-5}$$

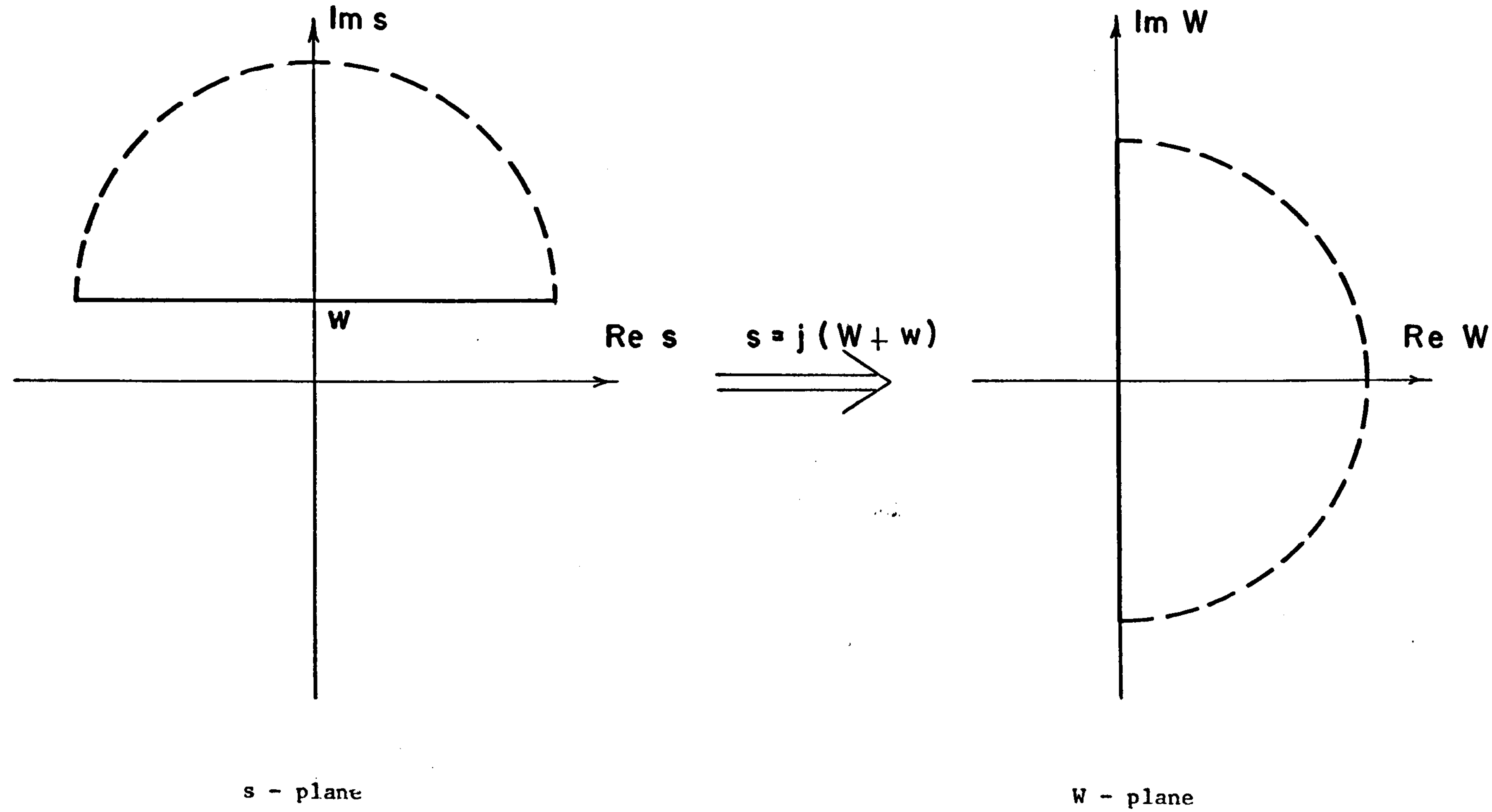


Fig. 4 - 4

$$r_i = \sum_{k=1}^{i} (a_k)\ (j)^{n-k+1} \binom{n-k+1}{i-k} (w)^{i-k} \qquad (4-6)$$

Where $\binom{x}{y}$ is the coefficient of the binomial expansion.

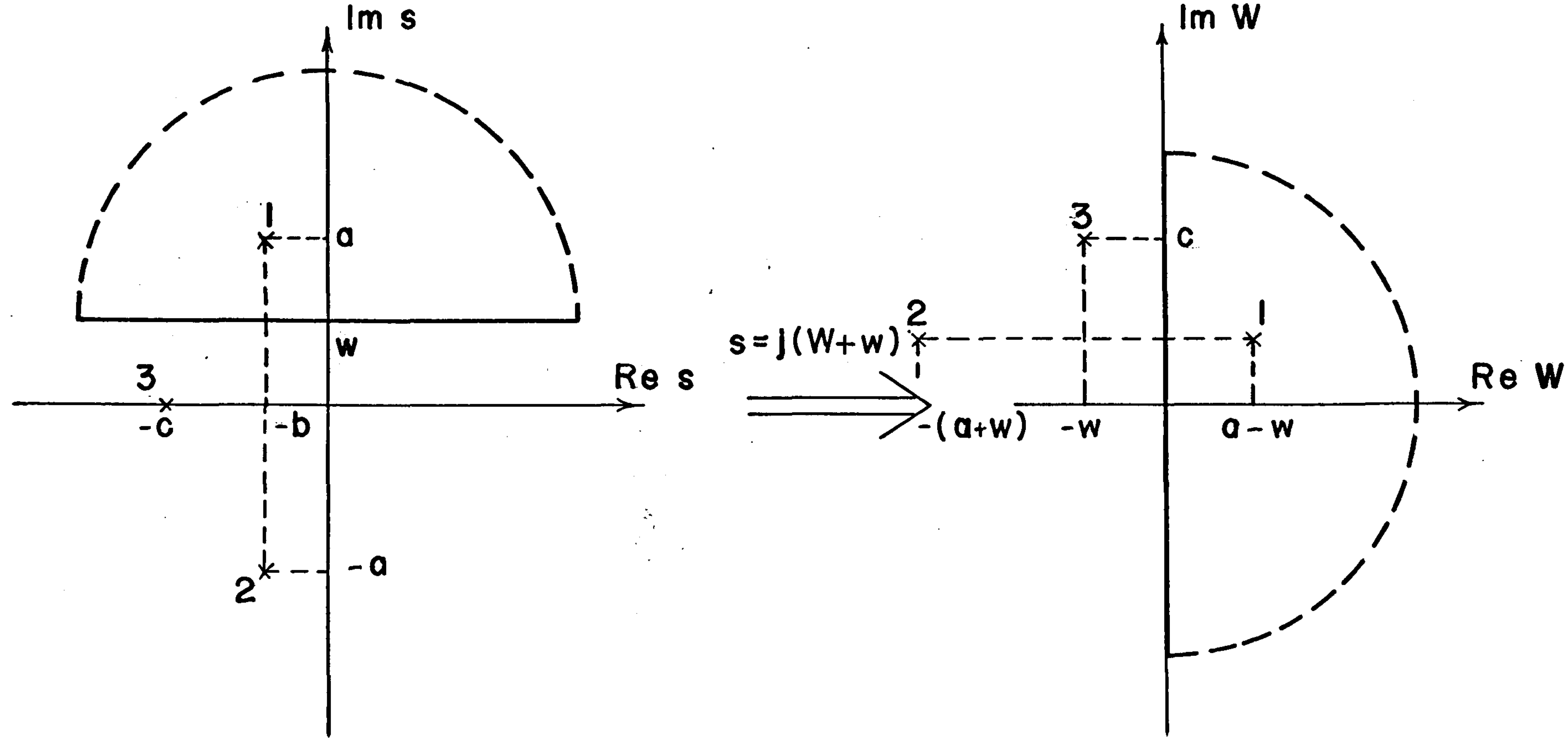
In equation (4-6), obviously r_i is a complex number, because it involves different powers of j. The problem of the existence of roots of P(s) in the region R_{II} in s-plane has now been changed to that of the existence of roots of $P(W)_w$ in the region R_{IV} in W-plane, that is, the problem now is to detect whether $P(W)_w$ has any root with positive real part.

Since $P(W)_w$ is a polynomial with complex coefficients, according to the technique introduced in Chapter III, it is necessary to construct another polynomial by multiplying $P(W)_w$ by its conjugate $P^*(W)_w$.

$$F(W) = P(W)_w \cdot P^*(W)_w \qquad (4-7)$$

As a result, the polynomial F(W) becomes a polynomial of real coefficients so that the Routh-Hurwitz criterion can be applied. However, in this special case of transformation, the following way is simplier:

In fig. 4-4, the transformation involves translation and rotation of the coordinates, so the roots of P(s) are transformed into W-plane as the roots of $P(W)_w$, as shown in fig. 4-5. The conjugate paired complex roots of P(s) become unpaired complex roots of $P(W)_w$. These unpaired complex roots of $P(W)_w$ make $P(W)_w$ to be a polynomial with complex coefficients.



s - plane

W - plane

Fig. 4-5

In order to make Routh-Hurwitz criterion to be applicable to $P(W)_w$, it is necessary to introduce the conjugate pair of the unpaired complex roots of $P(W)_w$ as shown in fig. 4-6(a), that is to introduce the roots in opposite signs to the roots of P(s) in s-plane as shown in fig. 4-6(b), this is proved as follows:

Since $F(W) = P(W) \cdot P^*(W)$ (4-8)

where $P(W) = P(s) \big|_{s = j(W + w)}$ (4-9)

and $P^*(W) = P(s) \big|_{s = -j(W + w)}$ (4-10)

equation (4-10) can be written as:

$$P^*(W) = P(-s) \big|_{s = j(W + w)} \qquad (4\text{-}11)$$

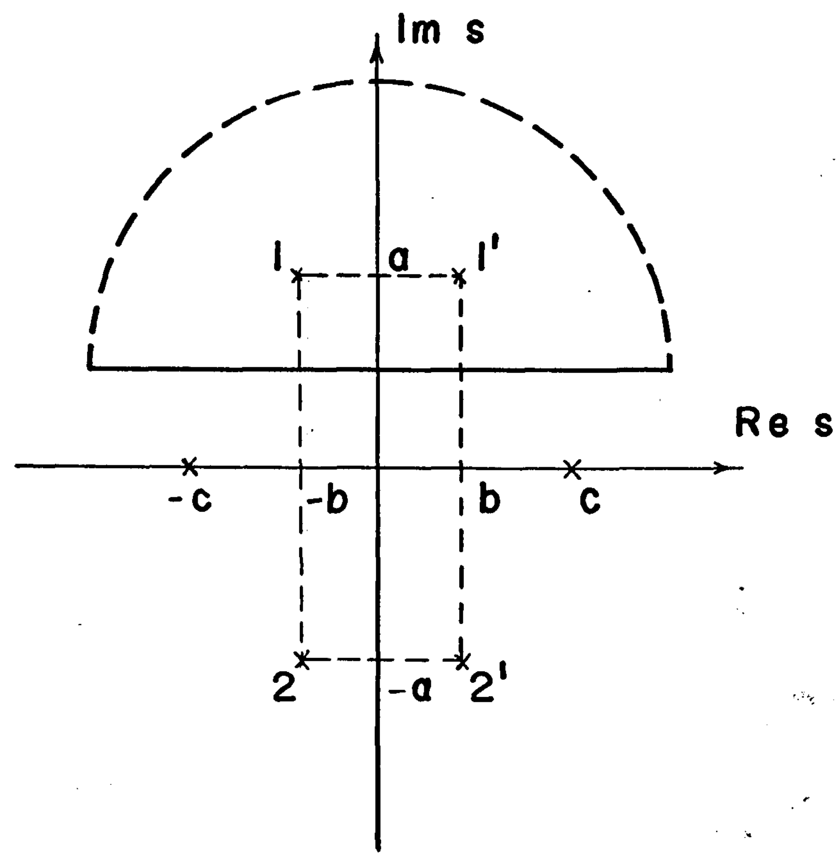
From equations (4-8), (4-9) and (4-11), F(W) can be expressed as:

$$F(W) = P(s) \cdot P(-s) \big|_{s = j(W + w)} \qquad (4\text{-}12)$$

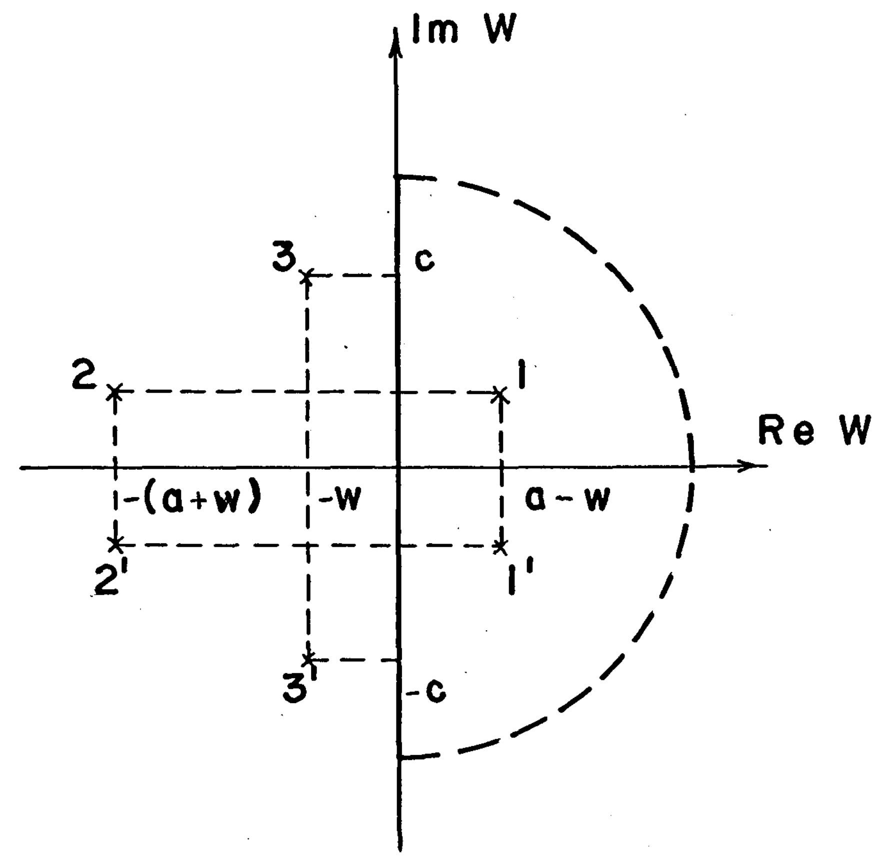
By the definition, the roots of P(-s) are equal to the minus of roots of P(s). Therefore the roots of F(s) are the roots of P(s) plus their negatives.

The characteristic polynomial P(s) can be divided into two parts:

$$P(s) = M(s) + N(s) \qquad (4\text{-}13)$$



(b) s - plane



(a) W - plane

Fig. 4 - 6

Where $M(s)$ = collection of terms with even powers of s

$N(s)$ = collection of terms with odd powers of s

Then

$$P(-s) = M(s) - N(s) \tag{4-14}$$

From equations (4-13) and (4-14),

$$F(s) = P(s) \cdot P(-s)$$
$$= M^2(s) - N^2(s) \tag{4-15}$$

In equation (4-15), since F(s) involves only the square of even terms and the square of odd terms, F(s) must be a polynomial with even powers of s only, and the total number of terms remains to be n+1.

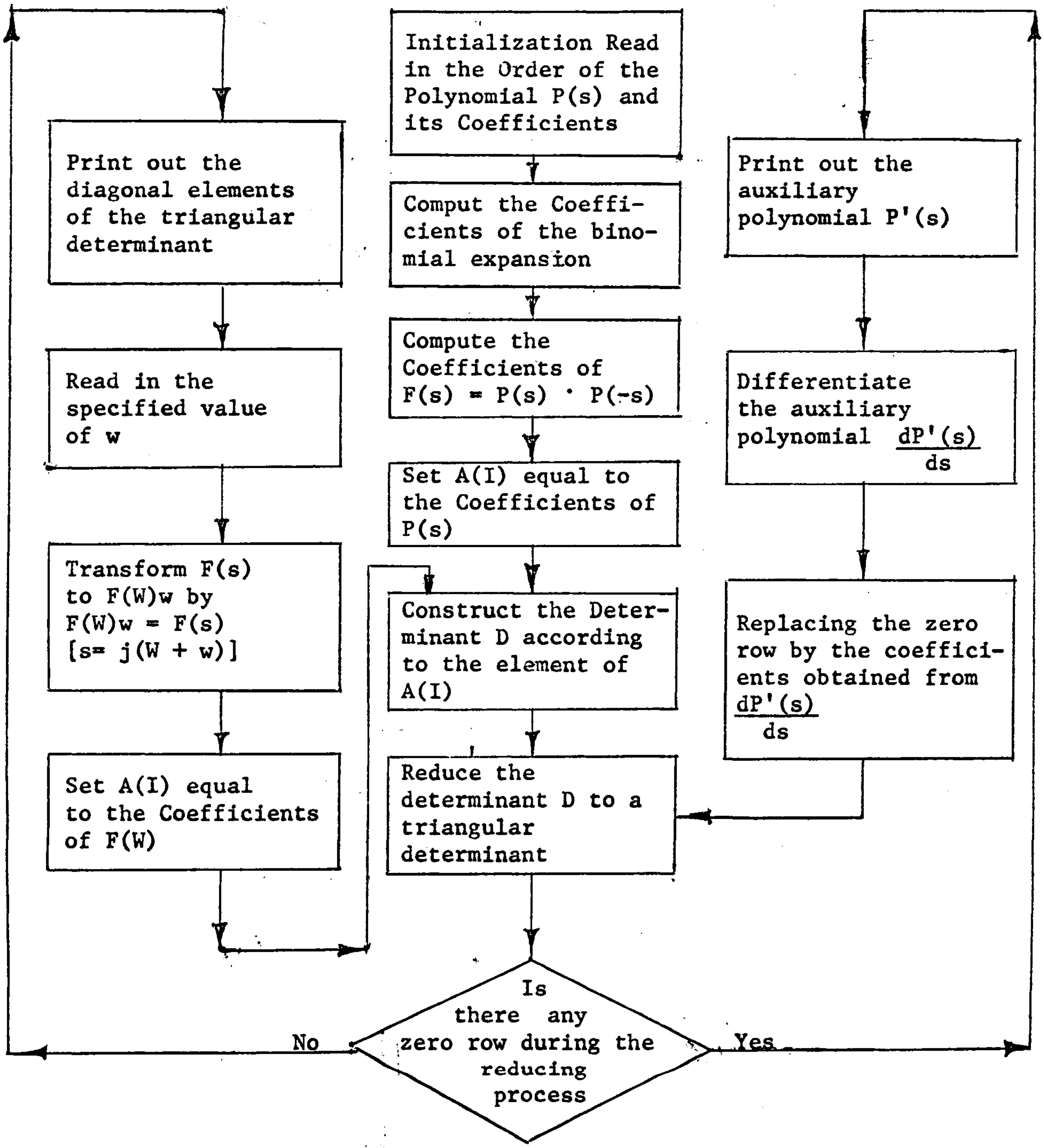
In equation (4-15), by the transformation, F(w) can be obtained without difficulty:

$$F(W) = F(s) \Big|_{s = j(W + w)} \tag{4-16}$$

To facilitate the transformation, a Fortran Computer program has been written, which includes the operation of transformation, modifying the polynomial, constructing the determinant and finally reducing it to a triangular determinant, all the examples given in this thesis have been done by using this program.

The following example has been done in order to illustrate the whole idea of the approach introduced in this thesis.

FLOW CHART OF COMPUTER PROGRAM



Example 4-1

Suppose the characteristic polynomial of a system is:

$$P(s) = s^5 + 5s^4 + 15s^3 + 25s^2 + 24s + 10 \qquad (4\text{-}17)$$

It is necessary to know:

1. If the system is absolutely stable.
2. If there is any natural frequency greater than 0.5, 1, 1.5, 2, 2.5 rad/sec.

First of all, the system's absolute stability is discernible by applying the Routh-Hurwitz criterion directly to P(s), and the following sequence results:

$$[a_{ii}] = 5,\ 10,\ 14,\ 14.86,\ 10 \qquad (4\text{-}18)$$

Since there is no sign change in the sequence $[a_{ii}]$, the system is absolutely stable.

Secondly, to determine the natural frequencies, more steps are required

$$\begin{aligned} F(s) &= M^2(s) - N^2(s) \\ &= -s^{10} - 5s^8 - 23s^6 + 5s^4 - 76s^2 + 100 \end{aligned} \qquad (4\text{-}19)$$

Case 1, w = 0.5 rad/sec.

using $$F(W)_{0.5} = F(s)\Big|_{s = j(W + 0.5)} \qquad (4\text{-}20)$$

F(W) is obtained as

$$F(W) = W^{10} + 5W^9 + 6.25W^8 - 5W^7 + 1.125W^6 + 41.875W^5 + 72.656W^4 + 59.687W^3 + 103.05W^2 + 82.519W + 119.653 \quad (4\text{-}21)$$

In equation (4-21) the negative coefficient of the term W^7, reveals that F(W) has roots with positive real parts or P(s) has natural frequencies greater than 0.5 rad/sec.

If the Routh-Hurwitz criterion is applied, the following sequence is formed with the diagonal elements of the triangular determinant.

$$[a_{ii}] = 82.5,\ 16.5,\ -5\text{x}10^{-6},\ -21,\ -3\text{x}10^{-5},\ 13.6\ \ 5\text{x}10^{-7},\ 52,\ -55\text{x}10^{-8},\ 1 \quad (4\text{-}22)$$

In the sequence (4-22), the four sign changes indicate that F(W) has 4 roots with positive real parts or P(s) has two natural frequencies greater than 0.5 rad/sec.

Case 2. w = 1 rad/sec.

$$F(W)_1 = F(s) \Big|\ s = j(W + 1) \quad (4\text{-}23)$$

From (4-19) and (4-23)

$$F(W) = W^{10} + 10W^9 + 40W^8 + 80W^7 + 93W^6 + 110W^5 + 210W^4 + 320W^3 + 356W^2 + 280W + 200 \quad (4\text{-}24)$$

By Routh-Hurwitz criterion, an all-zero row is produced during the determinant reduction process, and the auxiliary polynomial is found to be:

$$F'(W) = 50W^6 + 50W^4 + 200W^2 + 200 \quad (4\text{-}25)$$

From the auxiliary polynomial, it is easy to find the roots of F(W) which occur in pairs with opposite signs, as:

$$\begin{aligned} W_{1,2} &= \pm j1 \\ W_{3,4,5,6} &= \pm 1 \pm j1 \end{aligned} \quad (4\text{-}26)$$

Differentiating F'(W) with respect to W, there results:

$$\frac{dF(W)}{dW} = 300W^5 + 200W^3 + 400W \quad (4\text{-}27)$$

by replacing the element in the all-zero row by the coefficient of $\frac{d\,F(W)}{dW}$ which is obtained in (4-27), and continuing the reduction process, the following sequence can be obtained:

$$[a_{ii}] = 10, 32, 54.375, 50, 250, -110, 240, 147, 200, 200 \quad (4\text{-}28)$$

Since there are two sign changes in sequence $[a_{ii}]$, F(W) must has two roots with positive real parts and so P(s) has one natural frequency greater than 1 rad/sec.

Case 3. w = 1.5 rad/sec.

Following the same procedure, F(W) is found to be:

$$F(W) = W^{10} + 15W^9 + 96.25W^8 + 345W^7 + 771.125W^6 + 1175.625W^5 + 1401.406W^4 + 1506.562W^3 + 1448.675W^2 + 1044.433W + 487.817 \quad (4\text{-}29)$$

And finally the following sequence is formed

$$[a_{ii}] = 15,\ 1098.75,\ 14880,\ 62559,\ 96969,\ 55152,\ 0.066,\ 0.042,\ -7\times10^{-8},\ 1.148\times10^{-7} \quad (4\text{-}30)$$

Since there are still two sign changes in the sequence, F(W) must still has two roots with positive real parts which means that P(s) still has one natural frequency greater than 1.5 rad/sec.

Case 4. w = 2 rad/sec.

Following the same procedure F(W) is found to be:

$$F(W) = W^{10} + 20W^9 + 175W^8 + 880W^7 + 2823W^6 + 6100W^5 + 9225W^4 + 10120W^3 + 8276W^2 + 4880W + 1700 \quad (4\text{-}31)$$

During the reduction procedure an all-zero row is found. The auxiliary polynomial is:

$$F'(W) = 17W^2 + 17 \qquad (4\text{-}32)$$

From equation (2-32) it is easy to find the roots of F(W)

$$w = \pm j1 \qquad (4\text{-}33)$$

Differentiating F'(W) with respect to W

$$\frac{d\,F(W)}{dW} = 34W \qquad (4\text{-}34)$$

By replacing the element of the zero row by the coefficient of $\frac{d\,F(W)}{dW}$, and continuing the reduction process, the following sequence is obtained:

$$[a_{ii}] = 2,\ 131,\ 495.57,\ 1257,\ 2259,\ 2912,\ 2631,\ 1700,\ 1700,\ 1700 \qquad (4\text{-}35)$$

Now finally there is no sign change in the sequence $[a_{ii}]$, therefore, F(W) has no root with positive real part, meaning that P(s) has no natural frequency greater than 2 rad/sec.

Case 5. w = 2.5 rad/sec.

From the result in Case 4, since P(s) has no natural frequency greater than 2 rad/sec, it implies that P(s) has no natural frequency greater than 2.5 rad/sec.

The roots of P(s) are found to be:

$$s_1 = -1$$
$$s_{2,3} = -1 \pm j1 \qquad (4\text{-}36)$$
$$s_{4,5} = -1 \pm j2$$

In (4-36) P(s) has two natural frequencies:

$$W_1 = 1 \text{ rad/sec.}$$
$$W_2 = 2 \text{ rad/sec.} \qquad (4\text{-}37)$$

Therefore P(s) has two natural frequencies greater than 0.5 rad/sec. and one natural frequency greater than 1.5 rad/sec. and there is no natural frequency greater than 2 rad/sec. which are in agreement with the conclusion obtained by applying Routh-Hurwitz criterion.

Moreover, when w = 1, it is found that:

$$W_{1,2} = \pm j1 \qquad (4\text{-}38)$$
$$W_{3,4,5,6} = \pm 1 \pm j1 \qquad (4\text{-}39)$$

(4-38) shows that:

$$s = j(W + w)$$
$$= j(\pm j1 + 1)$$
$$= \mp 1 + j \qquad (4\text{-}40)$$

(4-40) are the roots of F(s), since

$$F(s) = P(s) \cdot P(-s) \tag{4-41}$$

It has just been proved that P(s) is absolutely stable, which means P(s) has no root with positive real part. From the above reasoning, it is clear that

$$s = - 1 + j \tag{4-42}$$

is one of the roots of P(s), which are equal to the roots in equation (4-36). From (4-39), it is obtained that:

$$\begin{aligned} s_{4,5,6,7} &= j(\pm 1 \pm j1 + 1) \\ &= \mp 1 + j2, \mp 1 + jo \end{aligned} \tag{4-43}$$

For the same reason as stated above, it is clear that:

$$\begin{aligned} s_4 &= -1 \pm j2 \\ s_5 &= - 1 \end{aligned} \tag{4-44}$$

are roots of P(s) which agree with those given in (4-36).

From the above discussion, it is shown that this method not only can determine the natural frequencies of the system, it also can find the corresponding damping coefficients of the system. Since the damping factor and natural frequency of the system are the only factors which determine the performance of the system, if the value of all these factors are known, then the total exact performance of the system is very easy to be described.

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