Superharmonics and subharmonics in a phase lock loop F.M. detector.

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SUPERHARMONICS AND SUBHARMONICS IN
A PHASE LOCK LOOP F.M. DETECTOR

by

GURNAM SINGH GILL

A Thesis
Submitted to the Faculty of Graduate Studies through the Department of Electrical Engineering in Partial Fulfillment of the Requirements for the Degree of Master of Applied Science at the University of Windsor

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ABSTRACT

The response of a phase lock loop to FM signals when the centre frequencies of incoming signal and voltage controlled oscillator are the same, is described by a nonlinear differential equation of the form

$$\phi'' + A \phi' + B \sin \phi = C \cos (\omega_m t + \psi)$$

assuming that the loop filter is a simple R-C low pass filter. The restoring term in the equation is nonlinear. This equation is reduced to Duffing's equation by using the approximation $\sin \phi = \phi - \frac{\phi^3}{6}$ for the nonlinear term. This approximation holds good for phase error less than $\pi/2$ rad.

The equation is solved for fundamental and third superharmonic component using describing function techniques. Further, the equation is investigated for third subharmonic. A solution is assumed consisting of driving frequency component and third subharmonic component. A stability region is defined for the third subharmonic to exist.
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CHAPTER I

INTRODUCTION

The behaviour of a phase lock loop to frequency modulated signals is described by a differential equation of second order in which the restoring term is nonlinear and the forcing function is sinusoidal. This type of equation is quite common in physical systems where a mass is subjected to an elastic restoring force. It is also the equation of a pendulum when an external periodic force is applied and the amplitude of oscillations is not restricted to being small.

Quite a few solutions exist for this type of equation, periodic solutions with the same frequency as that of the impressed force and periodic solutions at subharmonic and superharmonic frequencies. Solutions depend upon the parameters of system, nature and amount of nonlinearity as well as on the initial conditions.

The nonlinearity in the phase lock loop is introduced by the phase detector stage where the two signals get multiplied. Jump phenomena and hysteresis exist in nonlinear systems. In linear systems the response curves are centred around the resonant frequency symmetrically whereas in nonlinear systems the response curves bend back on themselves in a direction depending upon the type of nonlinearity of restoring term. In the equation the nonlinearity is of "soft spring" type in which restoring force decreases as the displacement increases. The response curves shift towards the lower side of the resonant frequency for this type of nonlinearity, whereas the response curves bend back on themselves towards the higher side of the linear resonant frequency for the "hard spring" type nonlinearity.

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The original equation of the system is reduced to Duffing's equation by employing the approximation \( \sin \phi = \phi - \frac{\phi^3}{6} \) which limits the phase error to remain less than \( \pi/2 \) radians as at \( \pi/2 \) radians the error is 8%.

The equation is solved here for fundamental component and third superharmonics by using the describing function technique as described by Austin Blaquiere.

For \( \frac{1}{3} \) subharmonics, the solution is assumed and the conditions are found out for its existence by finding the stability region.
CHAPTER II

PHASE LOCK LOOP

2.1 General Description of Phase Lock Loop.

A Phase Lock Loop is fundamentally a feedback system and consists of a phase detector, filter and the voltage controlled oscillator (V.C.O.) as shown in Fig. 1. The V.C.O. is required to follow the reference signal

in phase. The phase detector produces a signal proportional to the instantaneous phase difference between the reference and V.C.O. signals. This is done by multiplying the reference signal and V.C.O. outputs giving the sum and difference of two phases. The phase detector is followed by a filter which eliminates the higher frequency components. The output of the filter, the error signal, which is the voltage proportional to phase difference is applied to V.C.O.

The V.C.O. is a voltage tuned oscillator which adjusts its frequency according to error signal voltage. There are many varieties of V.C.O. i.e. reactance tube, klystron, voltage controlled multivibrator. The essential

FIG. 1 BLOCK DIAGRAM OF PHASE LOCK LOOP
characteristic required of V.C.O. is that its displacement from centre frequency be linearly proportional to input voltage over the working frequency range.

The phase detector is basically a multiplier and in conjunction with the loop filter measures the phase difference between the reference signal and the V.C.O. signal. The output of the phase detector is proportional to a sine function of the phase difference between input and output. It thus introduces the inherent nonlinearity in the system. A phase lock system can be linearized by assuming that \( \sin \phi \approx \phi \) which holds good for small values of \( \phi \). For phase error of 30° the error is 5%. Nonlinearity should be taken into consideration for explanation of loop behaviour over a large range of phase error. The required characteristic of the phase detector is that it should not saturate over the amplitude and frequency range of interest.

The loop filter is used to filter out the unwanted harmonic output of the phase detector. Apart from filtering, it shapes the amplitude and phase response of the error signal, it determines the order of loop and thus the characteristics of the loop. A loop filter may be obtained by using passive components. An active device can also be used to provide extra loop gain.

When the reference signal and V.C.O. signal are synchronised in frequency and the phase error is \( \pi/2 \) rad., the two signals are said to be phase coherent. When the reference frequency shifts from the centre frequency, the phase error also changes around \( \pi/2 \) rad. provided the loop remains in lock. The range of phase variation can be decreased by increasing the loop gain. But loop gain cannot be increased greatly as it tends to make the system unstable.
For tracking FM signals modulated by a sinusoid the signal frequency varies around the carrier at a rate given by the modulating frequency and the maximum frequency change from the carrier is given by the deviation. The condition for the system to remain in lock is given by \[ \Delta \omega \leq \frac{K \Delta}{\sqrt{\alpha^2 + \omega_m^2}} \]

where \( \Delta \omega \) is frequency deviation
\( K \) is loop gain
\( \omega_m \) is frequency of modulating signal
\( \alpha \) is cut off frequency of low pass filter.

The error signal will be a combination of modulation frequency and the higher harmonics. If the system is detuned the d.c. signal will also be present.

The utility of the phase lock loop lies in its two remarkable properties i.e. tracking of signal and narrow bandwidth. As the loop is capable of following changes in phase, it may have a very narrow bandwidth as compared to signal frequency, thus acting as an effective filter of high Q. Due to these properties the phase lock loop behaves as an automatically adjustable tuned filter of very narrow bandwidth which can be as low as a fraction of a cycle per second. Wideband signals with low signal to noise ratio may be applied to a Phase Lock Loop for purification. It filters the noise because of its narrow bandwidth and there is considerable improvement of signal to noise ratio at the output. Thus it is very useful for tracking weak signals in a noisy environment.
2.2 Derivation of Loop Equation for FM Signal

Let the incoming signal to the phase lock loop be denoted by \( E_1 \sin \phi_1 \) and the V.C.O. output by \( E_2 \cos \phi_2 \). When the system is perfectly synchronised, the phase difference between the incoming signal and V.C.O. output is \( \pi/2 \) radians due to an inherent property of the phase detector. For convenience the two signals are taken as sine and cosine functions so that when the two signals are in perfect lock \( \phi = \phi_1 - \phi_2 \), the error signal may be zero.

![FIG. 2 FUNCTIONAL DIAGRAM OF PHASE LOCK LOOP](image)

Since the phase detector is acting as a multiplier, its output

\[
= E_1 E_2 \sin \phi_1 \cos \phi_2
\]

\[
= \frac{E_1 E_2}{2} \sin (\phi_1 - \phi_2) + \sin (\phi_1 + \phi_2) \quad (2-1)
\]

The term \( \sin (\phi_1 + \phi_2) \) can be dropped as the high frequency components are rejected by the filter. Thus the phase detector output

\[
= K_1 \sin (\phi_1 - \phi_2)
\]

\[
= K_1 \sin \phi \quad (2-2)
\]
which is proportional to a sine function of phase error.

where $K_1$ is phase detector constant in volts/rad.

Phase change caused by the error signal in the output of the V.C.O.
is $K_1 K_2 \frac{F(D)}{D} \sin \phi$

where $F(s)$ is the transfer function of the filter

\[ \frac{K_2}{D} \] is the transfer function of the V.C.O.

If $\omega_2$ is the centre frequency of V.C.O. then the output of the V.C.O. is

\[ E_2 \cos (\omega_2 t + \frac{K_1 K_2 F(D)}{D} \sin \phi) \] (2-3)

The phase lock loop is used for tracking a frequency modulated signal.

Thus the incoming signal is of the form

\[ E_1 \sin (\omega_1 t + \frac{\Delta \omega}{\omega_m} \sin \omega_m t) \] (2-4)

where $\omega_1$ is the carrier frequency

$\omega_m$ is frequency of modulating signal

$\Delta \omega$ is frequency deviation.

Substituting the values of input and output signal phases in the
error signal

\[ K_1 \sin \phi = K_1 \sin (\phi_1 - \phi_2) \]

\[ = K_1 \sin \left[ \omega_1 t + \frac{\Delta \omega}{\omega_m} \sin \omega_m t - \omega_2 t - \frac{K_1 K_2 F(s)}{D} \sin \phi \right] \] (2-5)
Comparing the angle on both sides

\[ \phi = (\omega_1 - \omega_2)t + \frac{\Delta \omega}{\omega_m} \sin \omega_m t - \frac{K_1 K_2}{D} F(D) \sin \phi \]  

(2-6)

A low pass filter is used in the system whose transfer function

\[ F(D) = \frac{\alpha}{D+\alpha} \]  

(2-7)

Substituting the expression for \( F(D) \) in eq. (2-6)

\[ \phi = (\omega_1 - \omega_2)t + \frac{\Delta \omega}{\omega_m} \sin \omega_m t - \frac{K_1 K_2}{(D+\alpha)} \sin \phi \]  

(2-8)

On differentiating eq. 2-8 twice we get

\[ \phi'' + \alpha \phi + K \alpha \sin \phi = \alpha(\omega_1 - \omega_2) + \Delta \omega \alpha \cos \omega_m t - \Delta \omega \omega_m \sin \omega_m t \]  

(2-9)

The actual system used is different from that of Fig. 1 and is shown in Fig. 3.

**FIG. 3 ACTUAL PHASE LOCK LOOP**

An I.F. amplifier is used to increase and control the loop gain. It provides effective amplification at the intermediate frequency. The same
thing could be done in Fig. 1 by using a d.c. amplifier by amplifying the filtered error signal. But that is somewhat inconvenient to materialize. It is simpler to mechanize I.F. amplifier. It necessitates the use of two multipliers instead of one and an oscillator at intermediate frequency.

Equation (2-9) is reduced to Duffing's equation by approximating the value of \( \sin \phi - \phi^3/6 \) which limits the phase error to remain less than \( \pi/2 \) rad. Eq. (2-9) becomes

\[
\ddot{\phi} + a\dot{\phi} + K\alpha \phi - K\alpha \phi^3/6
\]

\[
= a(\omega_1 - \omega_2) + \Delta\omega \cdot a \cos \omega_m t - \Delta\omega \cdot \omega_m \sin \omega_m t \quad (2-10)
\]

If the carrier frequency of the incoming signal and the centre frequency of the V.C.O. are the same to start with eq. (2-10 becomes

\[
\ddot{\phi} + a\dot{\phi} + K\alpha \phi - \frac{K\alpha}{6} \phi^3 = \Delta\omega \cdot a \cos \omega_m t - \Delta\omega \cdot \omega_m \sin \omega_m t \quad (2-11)
\]
CHAPTER III
SOLUTION FOR FUNDAMENTAL AND THIRD SUPERHARMONICS

\[
\frac{d^2 \phi}{dt^2} + \alpha \frac{d\phi}{dt} + K\alpha \phi - \frac{K\alpha}{6} \phi^3 = \Delta \omega \cdot \alpha \cos \omega_m t - \Delta \omega \cdot \omega_m \sin \omega_m t \quad (3-1)
\]

The equation (3-1) already derived in the Second Chapter represents the phase lock system with FM signal as the input with the restraints that the phase error is always less than \(\pi/2\) rad., the carrier frequency of FM signal and the centre frequency of the V.C.O. are the same. It relates phase error with the parameters of the system, modulating frequency and the deviation of incoming FM signal.

The equation is nonlinear because of the cubic term in \(\phi\). Forcing function is the sum of two sinusoids. It can be represented with magnitude of \(\Delta \omega \sqrt{\omega^2 + \omega_m^2}\) and phase angle \(\tan^{-1} \frac{\omega_m}{\alpha}\). In the present chapter equation (3-1) will be solved for fundamental and third superharmonic using Describing Function techniques\(^{[1]}\).

Assuming solution with fundamental and third harmonic component

\[
\phi = a_1 \cos (\omega_m t - \psi_1) + a_3 \cos (3 \omega_m t - \psi_3) \quad (3-2)
\]

\[
\phi' = -a_1 \omega_m \sin (\omega_m t - \psi_1) - 3a_3 \omega_m \sin (3 \omega_m t - \psi_3) \quad (3-3)
\]

\[
\phi'' = -a_1 \omega_m^2 \cos (\omega_m t - \psi_1) - 9a_3 \omega_m^2 \cos (3 \omega_m t - \psi_3) \quad (3-4)
\]

\[
y = \frac{d^2 \phi}{dt^2} + \alpha \frac{d\phi}{dt} + K\alpha \phi - \frac{K\alpha}{6} \phi^3
\]

\[
= -a_1 \omega_m^2 \cos (\omega_m t - \psi_1) - 9 \omega_m^2 a_3 \cos (3 \omega_m t - \psi_3)
\]

\[
- \alpha a_1 \omega_m \sin (\omega_m t - \psi_1) - 3 a_3 \omega_m \alpha \sin (3 \omega_m t - \psi_3)
\]

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\[ + K \alpha a_1 \cos(\omega m t - \Psi_1) + K \alpha a_3 \cos(3\omega m t - \Psi_3) \]

\[- \frac{K \alpha}{6} [a_1 \cos(\omega m t - \Psi_1) + a_3 \cos(3\omega m t - \Psi_3)]^3 \quad (3-5)\]

Expanding the last term of equation (3-5)

\[ [a_1 \cos(\omega m t - \Psi_1) + a_3 \cos(3\omega m t - \Psi_3)]^3 \]

\[ = \frac{3}{4} a_1^3 \cos(\omega m t - \Psi_1) + \frac{1}{4} a_3^3 \cos(3\omega m t - 3\Psi_3) \]

\[ + \frac{3}{4} a_3^3 \cos(3\omega m t - \Psi_3) + \frac{a_3}{4} \cos(9\omega m t - 3\Psi_3) \]

\[ + \frac{3}{2} a_1^2 a_3 \cos(3\omega m t - \Psi_3) + \frac{3}{4} a_1^2 a_3 \cos(5\omega m t - 2\Psi_1 - \Psi_3) \]

\[ + \frac{3}{4} a_1^2 a_3 \cos(\omega m t + 2\Psi_1 - \Psi_3) + \frac{3}{2} a_1^2 a_3 \cos(\omega m t - \Psi_1) \]

\[ + \frac{3}{4} a_1 a_3^2 \cos(7\omega m t - \Psi_1 - 2\Psi_3) + \frac{3}{4} a_1 a_3^2 \cos(5\omega m t + \Psi_1 - 2\Psi_3) \]

Taking terms upto third harmonic only, we have

\[ \left( \frac{3}{4} a_1^3 + \frac{3}{2} a_1^2 a_3 \right) \cos(\omega m t - \Psi_1) + \frac{3}{4} a_1^2 a_3 \cos(\omega m t + 2\Psi_1 - \Psi_3) \]

\[ + \left( \frac{3}{4} a_3^3 + \frac{3}{2} a_1^2 a_3 \right) \cos(3\omega m t - \Psi_3) + \frac{a_3}{4} \cos(3\omega m t - 3\Psi_1) \quad (3-6) \]

Writing eq. (3-5) in exponential form

\[ \gamma(t) = \dot{\Phi} + \alpha \dot{\Phi} + K \alpha \Phi - \frac{K \alpha}{6} [\frac{3}{4} a_1^2 + 2a_3^2] e^{-j\Psi_1} \]

\[ + \frac{3}{4} a_1^2 a_3 e^{-j(\Psi_3 - 2\Psi_1)} e^{j\omega m t} \]

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Writing equation (3-2) in exponential form

\[ \dot{\Phi}(t) = a_1 e^{j\omega t} + a_3 e^{j3\omega t} \]

Thus

\[ \dot{\Phi}_1(t) = a_1 e^{j\omega t} \]

\[ \dot{\Phi}_3(t) = a_3 e^{j3\omega t} \]

From equations (3-7) and (3-8)

\[ \ddot{y}(t) = \ddot{\Phi}_1 + \alpha \dot{\Phi}_1 + K_\alpha \dot{\Phi}_1 - \frac{K_\alpha}{6} \left[ \frac{3}{4} a_1^2 + 2a_3^2 \right] e^{j\omega t} + \frac{3}{4} a_1 a_3 e^{-j(\gamma_3 - 2\gamma_1)} e^{j\omega t} + \frac{a_3^3}{4} e^{j3\omega t} \]

From equations (3-9) and (3-10)

\[ \ddot{y}(t) = \ddot{\Phi}_1 + \alpha \ddot{\Phi}_1 - \frac{K_\alpha}{6} \left[ \frac{3}{4} a_1^2 + 2a_3^2 \right] e^{j\omega t} + \frac{3}{4} a_1 a_3 e^{-j(\gamma_3 - 3\gamma_1)} \ddot{\Phi}_1 \]
\[ \Phi'''' + \frac{1}{3} \Phi''' + \alpha \Phi'' - \frac{K\alpha}{6} \left[ \frac{3}{4} (a_1^2 + 2a_3^2) \right. \\
\left. + \frac{3}{4} a_1 a_3 e^{-j(3\Psi_1 - \Psi_3)} \right] \Phi' + \frac{a_1}{4a_3} e^{-j(3\Psi_1 - \Psi_3)} \Phi = 0 \] (3-11)

Defining
\[ \hat{y}(t) = \hat{y}_1(t) + \hat{y}_3(t) \] (3-12)

Comparing (3-11) and (3-12)

\[ \hat{y}_1(t) = \frac{\Phi''}{\Phi_1} + \alpha \frac{\Phi''}{\Phi_1} + \frac{K\alpha}{6} \left[ 6 - \frac{3}{4} (a_1^2 + 2a_3^2) \right. \\
\left. - \frac{3}{4} a_1 a_3 e^{-j(3\Psi_1 - \Psi_3)} \right] \frac{\Phi'}{\Phi_1} \] (3-13)

\[ \hat{y}_3(t) = \frac{\Phi''}{\Phi_3} + \alpha \frac{\Phi''}{\Phi_3} + \frac{K\alpha}{6} \left[ 6 - \frac{3}{4} (a_1^2 + 2a_3^2) \right. \\
\left. - \frac{1}{4a_3} e^{-j(3\Psi_1 - \Psi_3)} \right] \frac{\Phi'}{\Phi_3} \] (3-14)

Representing by a describing function,

\[ H_1(\omega_m, a_1, a_3) = -\omega_m^2 + \frac{K\alpha}{6} \left[ 6 - \frac{3}{4} (a_1^2 + 2a_3^2) - \frac{3}{4} a_1 a_3 e^{-j(3\Psi_1 - \Psi_3)} \right] + j\omega_m \] (3-15)

\[ H_3(3\omega_m, a_1, a_3) = - (3\omega_m)^2 + \frac{K\alpha}{6} \left[ 6 - \frac{3}{4} (a_1^2 + 2a_3^2) - \frac{a_1}{4a_3} e^{-j(3\Psi_1 - \Psi_3)} \right] + j3\omega_m \alpha' \] (3-16)

Rewriting the forcing function \( \Lambda_0 \cdot \alpha \cos \omega_m t - \Lambda_0 \omega_m \sin \omega_m t \)
as
\[ \Delta \omega \sqrt{\alpha^2 + \omega_m^2} \cos(\omega_m t + \tan^{-1} \frac{\omega_m}{\alpha}) \]

\[ = \Delta \omega \sqrt{\alpha^2 + \omega_m^2} \cos(\omega_m t + \gamma_o) \]

In exponential form it becomes

\[ \Delta \omega \sqrt{\alpha^2 + \omega_m^2} e^{j(\omega_m t + \gamma_o)} \]

\[ \frac{\Delta \omega \sqrt{\alpha^2 + \omega_m^2} e^{j(\omega_m t + \gamma_o)}}{a_1 e^{j(\omega_m t - \gamma_1)}} \]

\[ H_1(\omega_m, a_1, a_3) = \frac{\Delta \omega \sqrt{\alpha^2 + \omega_m^2} e^{j(\omega_m t + \gamma_o)}}{a_1 e^{j(\gamma_o + \gamma_1)}} \]

\[ (3-17) \]

\[ H_3(3\omega_m, a_1, a_3) = 0 \]  \[ (3-18) \]

Equating (3-15) and (3-17), (3-16) and (3-18)

\[ - \omega_m^2 a_1 + \frac{K\omega_m}{6} [6a_1 - \frac{3}{4}(a_1^2 + 2a_3^2)a_1 - \frac{3}{4}a_1^2a_3 e^{-j(\gamma_3 - 3\gamma_1)}] \]

\[ + j \omega_m a_1 = \Delta \omega \sqrt{\alpha^2 + \omega_m^2} e^{j(\gamma_1 + \gamma_o)} \]  \[ (3-19) \]

\[ - 9\omega_m^2 a_3 + \frac{K\omega_m}{6} [6a_3 - \frac{3}{4}(a_3^2 + 2a_1^2)a_3 - \frac{3}{4}a_1^3 e^{-j(3\gamma_1 - \gamma_3)}] \]

\[ + j3\omega_m \alpha a_3 = 0 \]  \[ (3-20) \]
Rewriting equations (3-19) and (3-20)

\[
(K_{\alpha} - \omega_m^2) a_1 - \frac{K_{\alpha}}{8} (a_1^2 + 2a_3^2) a_1 - \frac{K_{\alpha}}{8} a_1 a_3 e^{-j(\psi_3 - 3\psi_1)}
+ j \alpha \omega_m a_1 = \Delta \omega \sqrt{\alpha}^2 + \omega_m^2 e^{j(\psi_1 + \psi_o)}
\] (3-21)

\[
(K_{\alpha} - a_{w_m}^2) a_3 - \frac{K_{\alpha}}{8} (a_3^2 + 2a_1^2) a_3 - \frac{K_{\alpha}}{2} a_1 a_3 e^{-j(3\psi_1 - \psi_3)}
+ j 3 \omega_m a_3 = 0
\] (3-22)

Neglecting \(a_3^2\) as compared to \(a_1^2\) in equations (3-21) and (3-22)

\[
(K_{\alpha} - \omega_m^2) a_1 - \frac{K_{\alpha}}{8} a_1^2 - \frac{K_{\alpha}}{8} a_1 a_3 e^{-j(\psi_3 - 3\psi_1)}
+ j \alpha \omega_m a_1 = \Delta \omega \sqrt{\alpha}^2 + \omega_m^2 e^{j(\psi_1 + \psi_o)}
\] (3-23)

\[
(K_{\alpha} - \omega_m^2) a_3 - \frac{K_{\alpha}}{4} a_1^2 a_3 - \frac{K_{\alpha}}{24} a_1 a_3 e^{-j(3\psi_1 - \psi_3)}
+ j 3 \omega_m^2 a_3 = 0
\] (3-24)

Comparing real and imaginary parts of equations (3-23) and (3-24)

\[
(K_{\alpha} - \omega_m^2) a_1 - \frac{K_{\alpha}}{8} a_1^2 - \frac{K_{\alpha}}{8} a_1 a_3 \cos(\psi_3 - 3\psi_1)
\]
\[
= \Delta w \sqrt{\alpha^2 + \omega_m^2} \cos(\psi_1 + \psi_o) \quad (3-25)
\]

\[
K'o a_1^2 a_3 \sin(\psi_3 - 3\psi_1) + \alpha \omega_m a_1
\]

\[
= \Delta w \sqrt{\alpha^2 + \omega_m^2} \sin(\psi_1 + \psi_o) \quad (3-26)
\]

\[
(Ko - 9\omega_m^2)a_3 - \frac{Ko}{4} a_1^2 a_3 - \frac{Ko}{24} a_1^3 \cos(3\psi_1 - \psi_3) = 0 \quad (3-27)
\]

\[
\frac{Ko}{24} a_1^3 \sin(3\psi_1 - \psi_3) + 3\omega_m^2 a_3 = 0 \quad (3-28)
\]

Putting \( \psi_1 = 0 \) as reference phase in equations (3-25) to (3-28)

\[
(Ko - 9\omega_m^2)a_1 - \frac{Ko}{8} a_1^2 a_3 - \Delta w \sqrt{\alpha^2 + \omega_m^2} \cos \psi_o
\]

\[
= \frac{Ko}{8} a_1^2 a_3 \cos(\psi_3 - 3\psi_1) \quad (3-29)
\]

\[- \alpha \omega_m a_1 + \Delta w \sqrt{\alpha^2 + \omega_m^2} \sin \psi_o = \frac{Ko}{8} a_1^2 a_3 \sin(\psi_3 - 3\psi_1) \quad (3-30)
\]

\[
(Ko - 9\omega_m^2)a_3 - \frac{Ko}{4} a_1^2 a_3 = \frac{Ko}{24} a_1^3 \cos(3\psi_1 - \psi_3) \quad (3-31)
\]

\[- 3\omega_m^2 a_3 = \frac{Ko}{24} a_1^3 \sin(3\psi_1 - \psi_3) \quad (3-32)
\]

Equating the value of \( \sin(\psi_3 - 3\psi_1) \) from equations (3-30) and (3-32)
\[-\alpha \omega m a_1 + \Delta \omega \sqrt{\alpha^2 + w_m^2} \sin \varphi_o \]

\[= \frac{K\alpha}{8} a_1^2 a_3 \frac{3\omega m a_3}{24} \frac{K\alpha}{a_1^3} \]

or

\[-\alpha \omega m a_1^2 + a_1 \Delta \omega \sqrt{\alpha^2 + w_m^2} \sin \varphi_o = 3\omega m a_3^2 \]

(3-33)

From equations (3-29) and (3-31)

\[\left( K\alpha - \omega_m^2 \right) a_1 - \frac{K\alpha}{8} a_1^3 - \Delta \omega \sqrt{\alpha^2 + w_m^2} \cos \varphi_o \]

\[= \frac{K\alpha}{8} a_1^2 a_3 \left( K\alpha - \omega_m^2 \right) a_3 - \frac{K\alpha}{4} a_1^2 a_3 \]

Rewriting the above equation,

\[- \frac{K\alpha}{8} a_1^4 + \left( K\alpha - \omega_m^2 \right) a_1^2 - \Delta \omega \sqrt{\alpha^2 + w_m^2} \cos \varphi_o a_1 \]

\[= 3 \left( K\alpha - \omega_m^2 \right) a_3^2 - \frac{3}{4} K\alpha a_1^2 a_3^2 \]

(3-34)

Substituting the value of \( a_3^2 \) from equation (3-33) in equation (3-34) and dividing by \( a_1 \),

\[- a_1^3 \left( \frac{15}{8} \omega_m K\alpha^2 + \frac{3}{4} K\alpha \Delta \omega \sqrt{\alpha^2 + w_m^2} a_1^2 \sin \varphi_o \right) \]

\[+ a_1 \left( 12 K\omega_m^2 - 36\omega_m^3 \right) - 9\omega_m K\alpha \Delta \omega \sqrt{\alpha^2 + w_m^2} \cos \varphi_o \]
Substituting the values of $\sin \Psi_o$ and $\cos \Psi_o$ in equations (3-35), we have

\[-3(K_\alpha - g_m^2) \Delta_\omega \sqrt{1 + g_m^2} \sin \Psi_o = 0 \quad (3-35)\]

\[-\frac{15}{8} g_m K_\alpha a_1^3 + \frac{3}{4} K_\alpha g_m \Delta_\omega a_1^2\]

\[+ (12 K_\omega a_1^2 - 36 a_1^3) a_1 - 9 g_m a_1^2 \Delta_\omega\]

\[-3(K_\alpha - g_m^2) g_m \Delta_\omega = 0\]

Dividing both sides by $-\frac{15}{8} g_m K_\alpha^2$,

\[a_1^3 - \frac{2}{5} \frac{\Delta_\omega}{\alpha} a_1^2 + \left(\frac{96}{5} \frac{g_m^2}{K_\alpha} - \frac{32}{5} a_1^2 + \frac{24}{5} \frac{\Delta_\omega}{K_\alpha}\right) a_1\]

\[+ \frac{8}{5} \frac{\Delta_\omega}{\alpha} - \frac{72}{5} \frac{g_m^2 \Delta_\omega}{K_\alpha^2} = 0\quad (3-36)\]

Equation (3-36) can be written in the form

\[a_1^3 - \frac{2}{5} \frac{\Delta_\omega}{\alpha} a_1^2 - \frac{96(K_\alpha - g_m^2)}{15 g_m^2} a_1\]

\[+ \frac{24}{5} \frac{\Delta_\omega}{K_\alpha} + \frac{24 \Delta_\omega (g_m^2 - g_m^2)}{15 g_m^2 \alpha} = 0\quad (3-37)\]

Calculation of Fundamental Component.
Equation (3-36) can be solved for \( a_1 \), the amplitude of fundamental provided values of system parameters, modulating frequency and deviation are known. Frequency response can be determined by plotting the amplitude of fundamental as a function of modulating frequency.

**Calculation of Third Harmonic**

The third harmonic can be determined by solving for \( a_3 \).

From equation (3-33)

\[
9 \omega_m^2 a_3^2 = a_1 \Delta \omega \sqrt{\alpha^2 + \omega_m^2} \sin \gamma \omega_0 - a_1^2 \alpha \omega_m
\]

Substituting the value of \( \sin \gamma \omega_0 \) and dividing both sides by \( 9 \omega_m \),

\[
a_3^2 = \frac{1}{9} \frac{\Delta \omega}{\alpha} a_1 - \frac{a_1^2}{9}
\]

\[
a_3 = \sqrt{\frac{1}{9} \left( \frac{\Delta \omega}{\alpha} a_1 - \frac{a_1^2}{9} \right)}
\]

(3-38)

It is seen that for \( a_3 \) to be real \( \frac{\Delta \omega}{\alpha} > a_1 \). Substituting \( \frac{\Delta \omega}{\alpha} = a_1 \) in equation (3-37) gives the minimum value of \( \frac{\Delta \omega}{\alpha} \) for the third harmonic to exist.

\[
\left( \frac{\Delta \omega}{\alpha} \right)^2_{\text{min}} = 8 - 8 \frac{\omega_m}{\omega_n} - 8 \frac{\alpha}{K}
\]

(3-39)

Campbell observed a high third harmonic content under the following conditions [3]:

\[
\frac{\omega_m}{\omega_n} = 0.5,
\]

\[
\omega_n = 6.28 \times 10^3 \text{ radians/second}
\]
\[ \frac{\Delta n}{\alpha} = 6 \]

Under these conditions, the minimum value for \( \frac{\Delta n}{\alpha} \) as given by equation (3-39) is 6, which agrees with experimental observation.
CHAPTER IV

1/3 SUBHARMONIC OSCILLATIONS

When linear systems are excited, the output consists of free oscillations and forced oscillations. Free oscillations are transient and die out after sometime whereas forced oscillations sustain. In nonlinear systems this is not always the case. The frequency of system response depends not only on the frequency of applied force but also upon the amplitude of the input and the initial conditions.

Subharmonic oscillations can occur in which the smallest period of oscillations may be an integral multiple of the period of the applied force. The presence or absence of particular subharmonic depends upon the nature of nonlinearity also.

In our equation restoring term is nonlinear. In general the restoring nonlinearity can be

\[ f(\phi) = c_1 \phi + c_2 \phi^2 + c_3 \phi^3 + \ldots. \]

where \(c_1, c_2, c_3\) \ldots are constants.

When nonlinearity is given by

\[ f(\phi) = c_1 \phi + c_3 \phi^3 \]

in such a system, subharmonics of order \(\frac{1}{2}, \frac{1}{4}, \frac{1}{5}\) cannot occur [4]. In general it can be said that oscillations of order \(1/n\) may occur if \(c_n \phi^n\) term is present in nonlinearity.

In the equation of error signal the restoring nonlinearity is \(\sin \phi\). By approximation this becomes \(\phi - \frac{\phi^3}{6}\). There is the possibility of 1/3 subharmonic. It may exist under particular conditions.

4.1 Solution of 1/3 Subharmonics

The equation of error signal is
\[
\frac{d^2\phi}{dt^2} + \alpha \frac{d\phi}{dt} + K\phi - \frac{K\phi^3}{6} = \Delta \omega \cdot \alpha \cos \omega_m t - \Delta \omega \cdot \omega_m \sin \omega_m t \quad (4-1)
\]

Let \( \omega_m t = 3t \) \quad (4-2)

\[
\frac{d\phi}{dt} = \frac{\omega_m}{3} \frac{d\phi}{dt}
\]

\[
\frac{d^2\phi}{dt^2} = \frac{\omega_m^2}{9} \frac{d^2\phi}{dt^2}
\]

Substituting the above values in equation (4-1)

\[
\frac{\omega_m^2}{9} \frac{d^2\phi}{dt^2} + \frac{\omega_m \alpha}{3} \frac{d\phi}{dt} + K\phi - \frac{K\phi^3}{6} = \Delta \omega \cdot \alpha \cos 3t - \Delta \omega \cdot \omega_m \sin 3t \quad (4-3)
\]

Dividing both sides by \( \frac{\omega_m^2}{9} \)

\[
\frac{d^2\phi}{dt^2} + \frac{3\alpha}{\omega_m} \frac{d\phi}{dt} + \frac{9K\phi}{\omega_m^2} + \frac{3}{2} \frac{K\phi^3}{\omega_m^2} = \frac{9\Delta \omega \cdot \alpha}{\omega_m^2} \cos 3t - \frac{9\Delta \omega}{\omega_m} \sin 3t \quad (4-4)
\]

Assuming periodic solution for equation (4-4)

\[
\phi = z + x\sin t + y \cos T + W \cos 3t
\]

in which \( z \) is a constant term, \( x\sin t + y \cos T \) is subharmonic component and \( W \cos 3t \) is driving frequency component.

From Mandelstam and Papalexi, the amplitude \( W \) of the driving frequency component can be approximated by

\[
W = \frac{1}{1-3\gamma} \times \text{amplitude of forcing function}
\]
Nonlinearity being symmetrical, the constant term \( z \) in the solution can be dropped.

Therefore, the solution is

\[
\dot{\phi} = x \sin \tau + y \cos \tau + w \cos 3 \tau \tag{4-7}
\]

\[
\dot{\phi}' = -x \cos \tau - y \sin \tau - 3w \sin 3 \tau \tag{4-8}
\]

\[
\dot{\phi}'' = -x \sin \tau - y \cos \tau - 9w \cos 3 \tau \tag{4-9}
\]

Substituting the values of \( \dot{\phi}, \dot{\phi}', \dot{\phi}'' \) in equation (4-4)

\[
(-x \sin \tau - y \cos \tau - 9w \cos 3 \tau) + \frac{3x}{w_m} (x \cos \tau - y \sin \tau - 3w \sin 3 \tau) + \frac{9ky}{2} (x \sin \tau + y \cos \tau + w \cos 3 \tau)
\]

\[
- \frac{3}{2} \frac{K \alpha}{w_m} (x \sin \tau + y \cos \tau + w \cos 3 \tau)^3
\]

\[
= \frac{9A\alpha}{2w_m} \cos 3 \tau - \frac{9A\alpha}{w_m} \sin 3 \tau \tag{4-10}
\]

Coefficients of \( \sin \tau \) and \( \cos \tau \) in the expansion of cubic term in equation (4-10) are

\[
[\frac{3}{4} (x^3 - 3xy^2) + 3xy^2 - \frac{3}{2} xyw + \frac{3}{2} xw^2] \sin \tau
\]
+ \left[ \frac{3}{4} \left( \gamma^3 - 3x^2 \gamma \right) + 3x^2 \gamma + \frac{3}{4} \omega^2 \gamma - \frac{3}{4} wx^2 + \frac{3}{2} \gamma w^2 \right] \cos \tau \quad (4-11)

Equating Coefficients of \( \sin \tau \) equal to zero in equation (4-10)

\[- x - \frac{3x}{w_m} y + \frac{9K \gamma}{2} x - \frac{3}{2} \frac{K \gamma}{w_m} \left( x^3 - 3xy^2 \right) \frac{3}{4} + 3xy^2 \]

\[- \frac{3}{2} xyw + \frac{3}{2} xw^2 \right) = 0 \quad (4-12)

Rearranging equation (4-12)

\[- \frac{1 + \frac{9K \gamma}{2} - \frac{9}{8} \frac{K \gamma}{w_m} \left( x^2 + y^2 \right) - \frac{9}{4} \frac{K \gamma}{w_m} w^2 } \left( \gamma^3 - 3xy^2 \right) \frac{3}{4} + 3xy^2 \]

\[- \frac{3}{4} xyw + \frac{3}{2} xw^2 \right) = 0 \quad (4-13)

Equating coefficients of \( \cos \tau \) equal to zero in equation (4-10)

\[- y + \frac{3x}{w_m} x + \frac{9K \gamma}{2} y - \frac{3}{2} \frac{K \gamma}{w_m} \left( y^3 - 3x^2 y \right) \frac{3}{4} + 3x^2 y \]

\[+ \frac{3}{4} w^2 - \frac{3}{4} wy^2 + \frac{3}{2} yw^2 \right) = 0 \quad (4-14)

Rearranging equation (4-14)

\[\frac{3x}{w_m} x + y \left[ -1 + \frac{9K \gamma}{2} - \frac{9}{8} \frac{K \gamma}{w_m} \left( x^2 + y^2 \right) - \frac{9}{4} \frac{K \gamma}{w_m} w^2 \right] = \frac{9}{8} \frac{K \gamma}{w_m} w \left( y^2 - x^2 \right) \quad (4-15)\]
Defining \( A = \frac{9K\alpha}{w_m^2} - \frac{9}{8} \frac{K\alpha}{w_m} (x^2 + y^2) - \frac{9}{4} \frac{K\alpha}{w_m} w^2 \) (4-16)

Equations (4-13) and (4-15) can be rewritten as

\[
Ax - \frac{3\alpha}{w_m} y = -\frac{9}{4} \frac{K\alpha}{w_m} wxy
\]

(4-17)

\[
\frac{3\alpha}{w_m} x + Ay = \left(\frac{9}{8} \frac{K\alpha}{w_m} w\right) (y^2 - x^2)
\]

(4-18)

Squaring (4-17) and (4-18), and adding

\[
(A^2 + \frac{9\alpha^2}{w_m^2}) (x^2 + y^2) = \left(\frac{9}{8} \frac{K\alpha}{w_m} w\right)^2 (x^2 + y^2)^2
\]

(4-19)

or

\[
A^2 + \frac{9\alpha^2}{w_m^2} = \left(\frac{9}{8} \frac{K\alpha}{w_m} w\right)^2 (x^2 + y^2)
\]

(4-20)

Substituting value of \( A \) from (4-16) and putting \( x^2 + y^2 = R^2 \) in equation (4-20)

\[
\left(\frac{9K\alpha}{w_m^2} - \frac{9}{8} \frac{K\alpha}{w_m} \cdot \frac{R^2}{w_m} - \frac{9}{4} \frac{K\alpha}{w_m} \cdot \frac{w^2}{w_m}\right)^2 + \frac{9\alpha^2}{w_m^2}
\]

\[
= \left(\frac{9}{8} \frac{K\alpha}{w_m} w\right)^2 \cdot w^2 \cdot R^2
\]

(4-21)

Dividing both sides of equation (4-21) by \( \frac{81K\alpha^2}{4} \)

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Let
\[
\begin{align*}
R^2 &= S \\
\omega^2 &= T \\
\omega_m^2 &= k_1 \\
\frac{\omega_m^2}{9K} &= k_2
\end{align*}
\] (4-23)

Substituting the above values in equation (4-22)
\[
(1 - k_1 - \frac{S}{8} - \frac{T}{4})^2 + k_2 = \frac{ST}{64} 
\] (4-24)

or
\[
(8 - 8k_1 - S - 2T)^2 + 64k_2 = ST 
\] (4-25)

Expanding the above equation
\[
S^2 + 4T^2 + 3ST - 16(1 - k_1)S - 32(1 - k_1)T + 64\{(1 - k_1)^2 + k_2\} = 0 
\] (4-26)

The discriminant for equation (4-26) is negative. Therefore, it is an equation of ellipse. To facilitate the plotting of the above equation, it will be brought into standard form of ellipse i.e.
\[
\frac{(S - h)^2}{a^2} + \frac{(T - k)^2}{b^2} = 1 
\] (4-27)
by rotation of the coordinate axes.

Rotating the coordinate axes by an angle \( \theta \) will eliminate the product term \( ST \). The angle is given by

\[
\tan 2\theta = \frac{3}{1-4} = -1 \tag{4-28}
\]

or

\[
\frac{2 \tan \theta}{1 - \tan^2 \theta} = -1 \tag{4-29}
\]

Crossmultiplying and transposing,

\[
\tan^2 \theta - 2\tan \theta - 1 = 0 \tag{4-30}
\]

solution of quadratic equation (4-30) gives

\[
\tan \theta = 1 + \sqrt{2}, 1 - \sqrt{2}
\]

Taking positive value of \( \tan \theta \), we have

\[
\frac{\sin \theta}{\cos \theta} = 1 + \sqrt{2} \tag{4-31}
\]

or

\[
\sin \theta = \frac{1 + \sqrt{2}}{\sqrt{1 + (1 + \sqrt{2})^2}}
\]

\[
= \frac{1 + \sqrt{2}}{\sqrt{4 + 2\sqrt{2}}} \tag{4-32}
\]

\[
\cos \theta = \frac{1}{\sqrt{4 + 2\sqrt{2}}} \tag{4-33}
\]
Employing the transformation

\[
S = S' \cos \theta - T' \sin \theta
\]

\[
T = S' \sin \theta + T' \cos \theta
\]

we have

\[
S = \frac{S' - (1 + \sqrt{2})T'}{\sqrt{4 + 2\sqrt{2}}}
\]

\[
T = \frac{(1 + \sqrt{2})S' + T'}{\sqrt{4 + 2\sqrt{2}}}
\]

Substituting the value of \( S \) and \( T \) from equation (4-35) into equation (4-26)

\[
\left\{ \frac{S' - (1 + \sqrt{2})T'}{\sqrt{4 + 2\sqrt{2}}} \right\}^2 + 4\left\{ \frac{(1 + \sqrt{2})S' + T'}{\sqrt{4 + 2\sqrt{2}}} \right\}^2
\]

\[
+ 3 \frac{S' - (1 + \sqrt{2})T'}{\sqrt{4 + 2\sqrt{2}}} \cdot \frac{(1 + \sqrt{2})S' + T'}{\sqrt{4 + 2\sqrt{2}}}
\]

\[
- 16 (1 - k_1) \frac{S' - (1 + \sqrt{2})T'}{\sqrt{4 + 2\sqrt{2}}} - 32(1 - k_1) \frac{(1 + \sqrt{2})S' + T'}{\sqrt{4 + 2\sqrt{2}}}
\]

\[
+ 64(1 - k_1)^2 + k_2 \right) = 0
\]

Expanding and rearranging equation (4-36)

\[
\frac{16 + 11\sqrt{2}}{4 + 2\sqrt{2}} S'^2 + \frac{4 - \sqrt{2}}{4 + 2\sqrt{2}} T'^2 - \frac{16(3 + 2\sqrt{2})(1 - k_1)}{\sqrt{4 + 2\sqrt{2}}} S'
\]
\[ + \frac{16(\sqrt{2} - 1)(1 - k_1)}{\sqrt{4 + 2/2}} T' + 64\{ (1 - k_1)^2 + k^2 \} = 0 \]  \hspace{1cm} (4-37)

Rewriting equation (4-37) in \((S' - k)^2\) and \((T - k)^2\) form,

\[ \frac{16 + 11\sqrt{2}}{4 + 2\sqrt{2}} \left\{ S' - \frac{8(3 + 2/2) \sqrt{4 + 2/2} \cdot (1 - k_1)}{16 + 11\sqrt{2}} \right\}^2 \]

\[ + \frac{4 - \sqrt{2}}{4 + 2\sqrt{2}} \left\{ T' + \frac{8(\sqrt{2} - 1) \sqrt{4 + 2\sqrt{2}} \cdot (1 - k_1)}{4 - \sqrt{2}} \right\}^2 \]

\[ + 64\{ (1 - k_1)^2 + k^2 \} - \frac{16(3 + 2\sqrt{2})(1 - k_1)^2}{4(16 + 11\sqrt{2})} \]

\[ \frac{\{16(\sqrt{2} - 1)(1 - k_1)\}^2}{4(4 - \sqrt{2})} = 0 \]  \hspace{1cm} (4-38)

Simplifying equation (4-38)

\[ \frac{\{S' - 3.83(1 - k_1)\}^2}{0.216\{9.25(1 - k_1)^2 - 64k_2\}} + \frac{\{T' + 3.33(1 - k_1)\}^2}{2.64\{9.25(1 - k_1)^2 - 64k_2\}} = 1 \]

Equation (4-39) can be written as

\[ \frac{(S - h)^2}{a^2} + \frac{(T - k)^2}{b^2} = 1 \]  \hspace{1cm} (4-40)

where

\[ h = 3.83(1 - k_1) \]
\[
\begin{align*}
30 & = 3.83 \left(1 - \frac{w_m^2}{9K_\alpha}\right) \quad (4-41) \\
k & = -3.33 \left(1 - k_1\right) \\
= -3.33 \left(1 - \frac{w_m^2}{9K_\alpha}\right) \quad (4-42) \\
a & = \sqrt{0.216 \left[9.25(1 - \frac{w_m^2}{9K_\alpha})^2 - 64 \frac{w_m^2}{9K}\right]} \quad (4-43) \\
b & = \sqrt{2.64 \left[9.25(1 - \frac{w_m^2}{9K_\alpha})^2 - 64 \frac{w_m^2}{9K}\right]}
\end{align*}
\]

Equation (4-26) gives the relationship between \( S \) i.e. square of the subharmonic amplitude and \( T \) i.e. square of the amplitude of driving frequency component. Giving values to \( T \), the corresponding values of \( S \) can be determined.

\( T \) can be found out from Mandelstam and Papalexi approximation given by equations (4-6) and (4-23), which is

\[
T = \omega^2
\]

\[
= \left(\frac{-9}{8} \frac{A_\omega}{w_m^2} \sqrt{\alpha^2 + \omega_m^2}\right)^2
\]

Giving definite values to parameters of system, \( T \) can be determined which is further used to determine the subharmonic amplitude from the elliptical curves. \( T' \) and \( S' \) are the transformed co-ordinates and \((h, k)\) is the centre of ellipses.
4.2 Stability Check on 1/3 Subharmonic Solution

In linear system a single equilibrium condition exists. In nonlinear system more than a single equilibrium condition may appear. The stability is concerned with what will happen if a system is disturbed slightly near an equilibrium condition. If the oscillations die out, the equilibrium condition is stable and on the other hand if disturbance leads to larger and larger departure the equilibrium condition is unstable. A stable equilibrium condition exists actually whereas an unstable condition is not maintained.

The periodic states of equilibrium determined by equation are not always realized, but are able to exist only so long as they are stable. Now to check the stability; a small variation $\xi$ will be given to the equilibrium state.

Rewriting the normalized equation of the system

$$\frac{d^2 \phi}{d\tau^2} + \frac{3v}{w_m} \frac{d\phi}{d\tau} + \frac{9K\nu}{2w_m} \phi - \frac{3}{2} \frac{K\nu}{w_m} \phi^3 = 9 \frac{\nu m \omega}{w_m} \cos 3\tau - 9 \frac{\nu m}{w_m} \sin 3\tau \quad (4-44)$$

The assumed periodic solution is

$$\phi_o = x \sin \tau + y \cos \tau + w \cos 3\tau$$

$$= \sqrt{x^2 + y^2} \left( \frac{x}{\sqrt{x^2 + y^2}} \sin \tau + \frac{y}{\sqrt{x^2 + y^2}} \cos \tau \right) + w \cos 3\tau$$

$$= R \cos(\tau - \theta) + w \cos 3\tau$$
where \( R = \sqrt{x^2 + y^2} \)

\[ \theta = \tan^{-1} \frac{x}{y} \]

\( \xi_0 \) can be written as

\[ \xi_0 = \frac{R}{z} \left[ e^{i(\tau - \theta)} + e^{-i(\tau - \theta)} \right] + \frac{w}{2} (e^{j3\tau} + e^{-j3\tau}) \]

Considering small variation \( \xi \) from the equilibrium state, we have

\[ \ddot{\xi} = \ddot{\xi}_0 + \xi \quad (4-45) \]

Substituting (4-45) in equation (4-44)

\[ \frac{d^2}{d\tau^2} (\ddot{\xi}_0 + \xi) + \frac{3\gamma}{\omega_m} \frac{d}{d\tau} (\dot{\xi}_0 + \xi) + \frac{9K\gamma}{\omega_m^2} (\dot{\xi}_0 + \xi) \]

\[ - \frac{3}{2} \frac{K\gamma}{\omega_m^2} (\ddot{\xi}_0 + \xi)^3 = 9 \frac{\Delta \omega^2}{\omega_m^2} \cos 3\tau - \frac{9\Delta \omega}{\omega_m^2} \sin 3\tau \quad (4-46) \]

Expanding the nonlinear term

\[ \frac{3}{2} \frac{K\gamma}{\omega_m^2} (\ddot{\xi}_0 + \xi)^3 = \frac{3}{2} \frac{K\gamma}{\omega_m^2} (\ddot{\xi}_0 + \xi)^3 + 3\ddot{\xi}_0 \xi^2 + 3\xi_0^2 \xi) \]

Neglecting higher order terms of \( \xi \), the nonlinear term contributes

\[ \frac{9}{2} \frac{K\gamma}{\omega_m^2} \ddot{\xi}_0 \xi^2 \quad (4-47) \]

Taking variational components from equation (4-46) and (4-47)

\[ \frac{d^2\xi}{d\tau^2} + \frac{3\gamma}{\omega_m} \frac{d\xi}{d\tau} + \frac{9K\gamma}{\omega_m^2} \left(1 - \frac{\ddot{\xi}_0}{2}\right) \xi = 0 \quad (4-48) \]
Employing the transformation

\[ \xi = e^{\frac{-3\alpha}{2\omega_m} \tau} \eta \]  \hspace{1cm} (4-49)

to eliminate first derivative

\[ \frac{d\xi}{d\tau} = -\frac{3\alpha}{2\omega_m} e^{\frac{-3\alpha}{2\omega_m} \tau} \eta + e^{\frac{-3\alpha}{2\omega_m} \tau} \frac{d\eta}{d\tau} \]  \hspace{1cm} (4-50)

\[ \frac{d^2\xi}{d\tau^2} = \left(\frac{3\alpha}{2\omega_m}\right)^2 e^{\frac{-3\alpha}{2\omega_m} \tau} \eta - \frac{3\alpha}{2\omega_m} e^{\frac{-3\alpha}{2\omega_m} \tau} \frac{d\eta}{d\tau} - \frac{3\alpha}{2\omega_m} e^{\frac{-3\alpha}{2\omega_m} \tau} \frac{d^2\eta}{d\tau^2} \]  \hspace{1cm} (4-51)

Substituting (4-49), (4-50) and (4-51) in equation (4-48), and cancelling

\[ -\frac{3\alpha}{2\omega_m} \eta \]

\[ \left(\frac{3\alpha}{2\omega_m}\right)^2 \eta - \frac{3\alpha}{2\omega_m} \frac{d\eta}{d\tau} - \frac{3\alpha}{2\omega_m} \frac{d\eta}{d\tau} + \frac{d^2\eta}{d\tau^2} \]

\[ \frac{3\alpha}{\omega_m} \frac{d\eta}{d\tau} - \frac{3\alpha}{\omega_m} \frac{3\alpha}{2\omega_m} \eta + \frac{9K\gamma}{2} \left(1 - \frac{\phi}{2}\right) \frac{\eta}{\omega_m} = 0 \]  \hspace{1cm} (4-52)
Rewriting

\[
\frac{d^2 \eta}{dT^2} + \left\{ \frac{9K\alpha}{w_m^2} - \frac{9}{4} \frac{\alpha^2}{w_m^2} - \frac{9}{2} \frac{K\alpha}{w_m} \phi_o \right\} \eta = 0 \quad (4-53)
\]

Substituting the expression for \( \phi_o \) in equation (4-53)

\[
\frac{d^2 \eta}{dT^2} + \left[ K\alpha - \frac{\alpha^2}{4} - \frac{K\alpha}{2} \left\{ \frac{R}{2} e^{j(T - \theta)} + \frac{R}{2} e^{-j(T - \theta)} + \frac{w}{2} e^{j3T} + \frac{w}{2} e^{-j3T} \right\} \right] = 0 \quad (4-54)
\]

Expanding the square term,

\[
\frac{R^2}{4} e^{j2(T - \theta)} + \frac{R^2}{4} e^{-j2(T - \theta)} + \frac{w^2}{4} e^{j6T} + \frac{w^2}{4} e^{-j6T} + \frac{R^2}{2} + \frac{Rw}{2} e^{j(4T - \theta)} + \frac{Rw}{2} e^{-j(2T + \theta)} + \frac{Rw}{2} e^{j(2T + \theta)} + \frac{Rw}{2} e^{-j(4T - \theta)} + \frac{w^2}{2}
\]

Equation (4-54) becomes

\[
\frac{d^2 \eta}{dT^2} + \frac{9}{2} \left[ K\alpha - \frac{\alpha^2}{4} - \frac{K\alpha}{2} \left\{ \frac{R^2 + w^2}{2} + \frac{R^2}{2} e^{j2(T - \theta)} \right\} \right] = 0
\]

\[
\quad + \frac{R^2}{4} e^{-j2(T - \theta)} + \frac{Rw}{2} e^{-j(2T + \theta)} + \frac{Rw}{2} e^{-j(2T + \theta)} + \ldots \ldots \ldots \quad (4-55)
\]

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\[
\frac{d^2 \eta}{d\tau^2} + \frac{9}{2} \left[ K_\alpha - \frac{\alpha_1^2}{4} - \frac{K_\alpha}{2} \right] \left( \frac{R^2 + \omega^2}{2} \right) + \frac{R^2}{2} \cos 2(\tau - \theta) + Rw \cos (2\tau + \theta)] \eta = 0 \quad (4-56)
\]

\[
\frac{d^2 \eta}{d\tau^2} + \left[ \frac{9}{2} \left( K_\alpha - \frac{\alpha_1^2}{4} - \frac{K_\alpha}{2} \right) \cdot \frac{R^2 + \omega^2}{2} \right] - \frac{9K_\alpha}{2} \left\{ \frac{R^2}{2} \cos 2(\tau - \theta) + Rw \cos (2\tau + \theta) \right\} \eta = 0 \quad (4-57)
\]

Collecting together the terms in \( \cos 2\tau \) and \( \sin 2\tau \)

\[
\frac{d^2 \eta}{d\tau^2} + \left[ \frac{9}{2} \left( K_\alpha - \frac{\alpha_1^2}{4} - \frac{K_\alpha}{2} \right) \cdot \frac{R^2 + \omega^2}{2} \right] - 2\left( \frac{9}{4} \frac{K_\alpha}{\omega_m^2} \right) \cos 2\theta + Rw \cos \theta \cos 2\tau + \frac{9}{4} \frac{K_\alpha}{\omega_m^2} \left( \frac{R^2}{2} \sin 2\theta - Rw \sin \theta \right)
\]

\[
\sin 2\tau \ldots \ldots \ldots \eta = 0 \quad (4-58)
\]

Comparing with Hill's equation

\[
\frac{d^2 \eta}{d\tau^2} + [\theta_0 + 2 \theta_1 \cos(2\tau - \epsilon_1)] \eta = 0 \quad (4-59)
\]

\[
\theta_0 = \frac{9}{2} \left[ K_\alpha - \frac{\alpha_1^2}{4} - \frac{K_\alpha}{2} \right] \left( \frac{R^2 + \omega^2}{2} \right) \quad (4-60)
\]

\[
\theta_1c = \frac{9}{4} \frac{K_\alpha}{\omega_m^2} \left( \frac{R^2}{2} \cos 2\theta + Rw \cos \theta \right) \quad (4-61)
\]

\[
\theta_1s = -\frac{9}{4} \frac{K_\alpha}{\omega_m^2} \left( \frac{R^2}{2} \sin 2\theta - Rw \sin \theta \right) \quad (4-62)
\]
The stability condition for Hill's equation (4-59) is

\[
(\theta_o - 1)^2 + 2(\theta_o + 1)\delta^2 + \delta^4 > \theta_1^2
\]  

(4-63)

\(\delta\) is given by \(\frac{3\alpha}{2\omega_m}\), half the coefficient of first derivative in \(m\)-normalized equation (4-44)

\[
\theta_1^2 = \theta_1^2_c + \theta_1^2_s
\]

= \(\left(\frac{9}{4} \frac{K_\alpha}{\omega_m^2}\right)^2 \left(\frac{R^2}{2} \cos 2\theta + Rw \cos \theta\right)^2
\]

\[
+ \left(\frac{R^2}{2} \sin 2\theta - 12w \sin \theta\right)^2
\]

= \(\left(\frac{9}{4} \frac{K_\alpha}{\omega_m^2}\right)^2 \left[\frac{R^4}{4} + R^2 w^2 + R^3 w \cos 3\theta\right]
\]

(4-64)

From equation (4-63),

\[
\theta_o^2 + 2\theta_o (\delta^2 - 1) + 1 + 2\delta^2 + \delta^4 > \theta_1^2
\]  

(4-65)

Substituting the values of \(\theta_o\), \(\theta_1\), and \(\delta\),

\[
\left[\frac{9}{w_m^2} \left(K_\alpha - \frac{\alpha^2}{4} - \frac{K_\alpha}{2} \cdot \frac{R^2 + w^2}{2}\right)\right]^2
\]

\[
+ \frac{18}{w_m^2} \left(K_\alpha - \frac{\alpha^2}{4} - \frac{K_\alpha}{2} \cdot \frac{R^2 + w^2}{2}\right) \left(\frac{9}{w_m^2} \frac{\alpha^2}{2} - 1\right)
\]

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\[ + \frac{1}{2} \frac{\alpha^2}{w_m} + \frac{81}{16} \frac{\alpha^4}{w_m^4} > \frac{81}{16} \frac{K_{\alpha}^2}{w_m^4} \left( \frac{R^4}{4} + R_w^2 + R_{\alpha}^3 \cos \theta \right) \]  \hspace{1cm} (4-66) \]

Taking maximum value of Os 30 which gives sufficient condition for stability,

\[ \left\{ \frac{9}{2} \left( K_{\alpha} - \frac{\alpha^2}{4} - \frac{K_{\alpha}}{2} \cdot \frac{R^2 + w^2}{2} \right) \right\}^2 \]

\[ + \frac{18}{2} \left( K_{\alpha} - \frac{\alpha^2}{4} - \frac{K_{\alpha}}{2} \cdot \frac{R^2 + w^2}{2} \right) \left( \frac{9}{4} \frac{\alpha^2}{w_m^2} - 1 \right) \]

\[ + \left( \frac{9}{4} \frac{\alpha^2}{w_m^2} - 1 \right)^2 + 9 \frac{\alpha^2}{w_m^2} > \frac{81}{16} \frac{K_{\alpha}^2}{w_m^4} \left( \frac{R^4}{4} + R_w^2 + R_{\alpha}^3 \right) \]  \hspace{1cm} (4-67) \]

\[ \left\{ \frac{9}{2} \left( K_{\alpha} - \frac{\alpha^2}{4} - \frac{K_{\alpha}}{2} \cdot \frac{R^2 + w^2}{2} \right) + \frac{9}{4} \frac{\alpha^2}{w_m^2} - 1 \right\}^2 \]

\[ + \frac{9\alpha^2}{2} > \frac{81}{16} \frac{K_{\alpha}^2}{w_m^4} \left( \frac{R^4}{4} + R_w^2 + R_{\alpha}^3 \right) \]  \hspace{1cm} (4-68) \]

\[ \left\{ 9 \frac{K_{\alpha}}{2} \left( 1 - \frac{R^2}{4} - \frac{w^2}{4} - \frac{w_m^2}{9K_{\alpha}} \right) \right\}^2 + \frac{9\alpha^2}{2} > \frac{81}{16} \frac{K_{\alpha}^2}{w_m^4} \left( \frac{R^4}{4} + R_w^2 + R_{\alpha}^3 \right) \]

\[ + R_{\alpha}^3 \]  \hspace{1cm} (4-69) \]
\[
\frac{81 K}{\omega_m^2} \left[ \frac{2}{4} \left( 1 - \frac{R}{\omega_m^2} - \frac{w_2}{4} - \frac{w_m^2}{9K} \omega_m^2 \right) + \frac{w_m^2}{9K^2} \right] > \frac{81 K}{16 \omega_m^2}
\]

\[
\left( \frac{R}{4} + R^2 \omega^2 + R \omega \right)
\]

(4-70)

Cancelling \( \frac{81 K}{\omega_m^2} \) from both sides and expanding the square term,

\[
1 + \frac{R^4}{16} + \frac{w_4}{16} + \frac{\omega_m^4}{81 K} \omega_m^2 - \frac{R^2}{2} - \frac{w_2}{2} - \frac{w_m^2}{9K} + \frac{R^2 \omega^2}{8}
\]

\[
+ \frac{w_m^2}{18K} \omega^2 + \frac{w_m^2}{18K} \omega^2 > \frac{1}{16} \left( \frac{R^4}{4} + R^2 \omega^2 + R \omega \right)
\]

(4-71)

\[
\frac{3}{64} R^4 + \frac{w_4}{16} - \frac{1}{16} R^3 \omega + \frac{R^2 \omega^2}{16} + R \left( \frac{w_m^2}{18K} - \frac{1}{2} \right)
\]

\[
+ \omega^2 \left( \frac{w_m^2}{18K} - \frac{1}{2} \right) + 1 - \frac{2 w_m^2}{9K} + \frac{w_2}{9K^2} + \frac{w_m^4}{81 K^2} \omega_m^2 > 0
\]

(4-72)

The above inequality gives the stability region.

Under the following conditions Campbell \( [3] \) observed that the system is stable

\[
\Delta w = 94.1 \times 10^3 \text{ rad./sec.}
\]

\[
K = 6.28 \times 10^5 \text{ rad./sec.}
\]

\[
\alpha = 6.28 \times 10^3 \text{ rad./sec.}
\]

\[
\omega_m = 12.56 \times 10^4 \text{ rad./sec.}
\]
By substituting these values in equation (4-22) one obtains \( R = 0.816 \). When this value of \( R \) is substituted in equation (4-62), the stability inequality is satisfied.

Taking a point in the middle of region mapped by Campbell, e.g.

\[
\begin{align*}
\Delta \omega &= 120.6 \times 10^3 \text{ rad./sec.} \\
K &= 6.3 \times 10^5 \text{ rad./sec.} \\
\alpha &= 6.3 \times 10^3 \text{ rad./sec.} \\
\omega_m &= 144 \times 10^3 \text{ rad./sec.}
\end{align*}
\]

It is similarly found that 1/3 subharmonic can exist.
The equation for phase error of a phase lock loop for a FM signal is derived. The equation is nonlinear because of the restoring term. It is reduced to Duffing's equation by the approximation \( \sin \phi = \phi - \frac{\phi^3}{6} \). It is solved for fundamental and third superharmonic by the describing function method. The condition for minimum value of \( \frac{\Delta \omega}{\alpha} \) is established for the third superharmonic which agrees with the approximate values found by Campbell.

The subharmonic solution is also studied. A relation is derived which gives the square of the subharmonic amplitude as a function of the square of the amplitude of the forcing term. The condition for stability of the periodic solution of frequency \( \frac{\omega_0}{3} \) is obtained by determining whether any variation from this state, caused by a small change, attenuates or not as time increases. If the system is stable the perturbation will die out. Thus the equation is reduced to Hill's form and his stability criterion is applied. The inequality thus derived gives the stability region which satisfies Campbell's observation of the harmonic maps.
REFERENCES


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