Analysis of gridwork in skew bridges.

Pijush Kanti Chowdhury
University of Windsor

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ANALYSIS OF
GRIDWORK IN SKEW BRIDGES

A THESIS
submitted to the Faculty of Graduate Studies
in Partial Fulfilment of the Requirement
for the Degree of Master of Applied Science in Civil Engineering from
the University of Windsor.

by
Pijush Kanti Chowdhury, B.Sc. Engg.
(Dacca, E. Pakistan, 1963)

Windsor, Ontario, Canada
1970
ABSTRACT

An elastic analysis of gridwork in skew bridges by the method of finite differences has been presented through this investigation. The analysis is based on the theory of equivalent orthotropic plate which is considered to be a substitute of gridwork and slab system of skew bridges.

By using appropriate boundary conditions finite difference equations have been derived for different typical network points covering the entire bridge which is simply supported on the two opposite sides and free at the other two. Simple formulae have been presented for computing bending moments in longitudinal and transverse girders of the grillage skew bridge. Several factors such as number of girders and diaphragms, their spacing and stiffness ratio, aspect ratio of the bridge and skew angle have been studied. A study of the influence of Poisson's ratio on the stress distribution has also been made.

An experimental study was performed on a model skew bridge under three different types of loadings. The results obtained from the tests are found to be in satisfactory agreement with the theoretical solutions.
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NOMENCLATURE

Bx, By Orthotropic flexural rigidities per unit width in x and y directions

Bxy, Byx Orthotropic torsional rigidities per unit width in x and y directions

b0, l0 Spacing of longitudinal girders and cross beams

C Torsional rigidity constant

C1 Shape factor involved in the torsion constant of rectangular section

E Modulus of elasticity

F_L, F_P Torsion constants of plane areas in longitudinal and transverse directions

G Shear modulus of rigidity

H Apparent torsional rigidity of the equivalent orthotropic plate

I_T, I_P Moment of inertia of plane areas with respect to longitudinal and transverse directions

Lx, Ly Span length and width of the bridge

Mx, My Bending moments per unit width, acting on sections, normal to x and y axes, respectively

Mxy, Myx Twisting moments per unit length acting on sections normal to x and y axes, respectively

P_o Concentrated load

P_0 Uniformly distributed load

q(x,y) Load intensity at point (x,y)

q_0 Line load per unit length in x direction

\tilde{q}_0 Equivalent combined load acting at node point x
Qx, Qy  Shear force per unit length perpendicular to x and y axes

Vx, Vy  Support reactions per unit length on edges perpendicular to x and y axes

w  Displacement component in z direction
w is called 'deflection'

x, y  Horizontal rectangular co-ordinates

u, v  Oblique co-ordinate axes as shown in Fig. 5

Ω  Torsional parameter

τx, τy  Distances between node points in x and y directions

τu, τv  Distances between node points in u and v directions

νx, νy  Poisson's ratio associated with x and y directions

σx, σy  Unit normal stresses in x and y directions

τxy, τyx  Unit shearing stresses on planes perpendicular to z axis but parallel to y and x axes

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]  Laplace's operator in two variables

U  =  \nabla^2 w

ϕ  Angle of skew

xi
INTRODUCTION

A grillage or gridwork is a structure composed of two systems of intersecting flexural members, the members in each system being parallel to one another and continuous through the point of intersections.

In the field of reinforced concrete the study of gridwork in skew bridges is of considerable interest and practical importance when a highway bridge is to cross streams, railways or other highways below at an oblique angle. Because of the present practice of transporting heavy loads, an accurate method of analyzing the behaviour of main girder and cross beam is essential. Owing to the high degree of statical indeterminancy, the actual stress distribution imposed on such a grid system by an external load is a problem in itself. The number of redundant components is generally considerable which complicates the numerical calculations so that analytical investigations become highly involved.

To reduce the size of the problem, Hendry and Jaeger [1]* assumed that the transverse members of a skew grid may be replaced by a continuous torsion-free spread medium of equivalent elastic rigidity. Further, the whole grid is taken as a simply supported

* Numbers in brackets refer to the number of reference in the Bibliography of this thesis.
beam carrying all the applied loads. The solution is then obtained in the form of first harmonic distribution coefficients, in terms of two dimensionless parameters. The method is restricted to a limited number of longitudinal girders and the engineer is to be satisfied with determining only the approximate critical moments because the complete solution of even a simple grid including the evaluation of all the stress resultants and deformations at every point of the structure is very impractical. Furthermore any change in loading data entail a separate series of calculations.

Langendonck [2] presented a method to analyse gridworks of skew bridges consisting of only two simply supported equal longitudinal girders connected by equal and equidistant transverse cross beams. With the assumption that the loads are applied at the intersections of the cross beams with the girders it was possible to yield an exact solution to the problem in terms of trigonometric polynomials, by satisfying conditions of static equilibrium and geometric compatibility. But the extension of this method to the case of bridges with a greater number of longitudinal girders involves cumbersome arithmetical computations.

Recently a remarkable change has taken place in the manner of the approach applied in structural analysis to the solution of gridwork problem. This is
the new concept of 'Equivalent orthotropy' [14, 3, 11, 13]. For the purpose of estimating overall deflections and stresses, the skew bridge stiffened with longitudinal and cross beams may be conceived to be replaced by a substitute 'Equivalent orthotropic plate' of a uniform spread longitudinal stiffness and a uniform spread transverse stiffness.

R. Bares and C. Massonet[3] have presented tables and diagrams for the distribution coefficients based on the theory of 'equivalent orthotropic plates' which are very effective in the analysis and design calculations of right girder bridges. Extending the theory of equivalent orthotropic plate to the analysis of skew grillage, Naruoka and Ohmura [4] derived skew network finite difference equations using Marcus' finite difference approach to calculate the influence coefficients for deflections and bending moments for simply supported orthotropic parallelogrammic plate. They employed a network proposed by Favre, dividing the plate into a 6x6 skew mesh.

Based on this analysis, Fujio, Ohmura and Naruoka [5] proposed formulae to calculate the longitudinal bending moment at mid-span of interior girder in grillage skew girder bridges. In formulating the finite difference equations they neglected the Poisson's ratio effect on deflections and moments of the equivalent orthotropic skew slab. But Kennedy and Tamberg [6] in their broad
and critical discussions against the background of available analytical and experimental method of solution of skew bridges have given special considerations on the influence of Poisson's ratio on the stress distribution of a skew slab.

Based on the same network as suggested by Favre, Basar and Yuksul [7] have also developed the finite difference equations for an orthotropic skew slab. No numerical results were given. But some difficulty was experienced by Basar and Yuksul (and presumably Naruoka and Ohmura) in satisfying the condition $U=0$ (Eq. 4.2) along the simply supported boundary of the orthotropic plates and a constraint is imposed near these edges, since they used six equations to eliminate five unknowns.

A similar problem occurs if Jensen's [17] network is used.

Fawcett [8] has suggested that the external points to the simply supported edges be left initially in the finite difference equations to be eliminated later on when the final set of simultaneous equations is being formed. Situations arising from the suggestions of Fawcett have been examined in this investigation.

Coull [18] has published an approximate method for the analysis of simply supported uniformly loaded orthotropic skew bridge slabs with two opposite edges...
free. He used the principle of least work in conjunction with the assumption that the load and stress components may be represented by a power series in the chordwise co-ordinates, the coefficients of this series being functions of spanwise position only. He found, on comparison with model tests on isotropic slabs, that agreement between theoretical and experimental results deteriorated with increasing skew.

Cheung, King and Zienkiewicz [19] applied the finite element method for the solution of isotropic skew plate problems. Recently Powell and Ogden [20] have published a paper on finite element method of analysis of orthotropic steel plate bridge decks in orthogonal configuration. This work can be extended to include the effects of skew for the analysis of gridwork in skew bridges. But the choice of a proper displacement function, satisfying not only the curvature criterion along the interface of the elements but also slope compatibility, appears to be a problem for an idealized orthotropic equivalent skew slab [19].

When the two systems of intersecting beams forming the gridwork of a skew bridge are not orthogonal, skew anisotropic plate theory as proposed by Lie [24] can be applied to the solution of gridwork in skew bridges. Since the governing differential equation of an anisotropic plate in skew configuration is very involved, a transformation of
the flexural and torsional rigidities from a skew aniso-
tropic plate to an equivalent orthotropic parallelogrammic
plate may be made and the analysis of grillage skew bridge
can be based on the theory of equivalent orthotropic skew
plate. However, the additional work taking the anisotropic
form of the system into account as well as an experimental
test on a plate with orthogonal beams will be carried out
in near future and will be reported in the literature.

The present investigation stems from the need to study
by means of an elastic theory several factors that enter
into the analysis and design of gridwork in skew bridges.
Such factors include number of girders and diaphragms, their
spacing and stiffness ratio in flexure and torsion, aspect
ratio of the bridge and skew angle.

Based on the theory of equivalent orthotropic parallelo-
grammic plate which is assumed to be a substitute of grid-
work and slab system of a skew bridge, finite difference
equations have been developed and compared with those of
available solutions [4,7]. A comparison of present analysis
with a numerical solution of a skew grillage based on the
theory of anisotropic plate [24,25] has also been made.
Favre's skew network has been used for deriving the finite
difference equations for deflections, bending and twisting
moments.

Theoretical solutions have also been verified with
experimental results.
THE THEORY OF ORTHOTROPIC PLATE AS APPLIED TO THE ANALYSIS OF A GRIDWORK AND SLAB SYSTEM IN ORTHOGONAL CONFIGURATION.

The study of the composite action of grid and slab system may be arranged to form a sequence of structural forms, the sequence beginning with an ideal orthotropic plate, a simple gridwork and ultimately ending with a slab and grid pattern.

1.1 Orthotropic plate

The analytical approach to the problem of an ideal orthotropic plate, which is composed of materials exhibiting elastic symmetry with respect to three mutually perpendicular planes (i.e. materials which are orthogonally anisotropic), is based on the classical Poisson-Kirchoff's simplifying assumptions [3] relating to the form of the material of the plate and to the state of strains induced by external loading. These are the same usual assumptions as used in the small deflection theory of isotropic plate.

The differential equation giving the relationship between the deflection and the loading of an ideal orthotropic plate, often referred to as Huber's equation [10] is:

\[ B_{x_1} \frac{\partial^4 w}{\partial x^4} + 2H_{x_1} \frac{\partial^4 w}{\partial x^2 \partial y^2} + B_{y_1} \frac{\partial^4 w}{\partial y^4} = q(x, y) \]  

(1.1)
where \( w \) is the deflection of the middle surface of the plate at any point \((x,y)\) and \( q(x,y) \) is the loading intensity.

The rigidities are defined as:

\[
\begin{align*}
B_{x_i} &= \frac{E_x h^3}{12 (1-\nu_x \nu_y)} \\
B_{y_i} &= \frac{E_y h^3}{12 (1-\nu_x \nu_y)} \\
2H_i &= 2 \left( 2\gamma' + B_{x_i} \right) \\
&= 2 \left( 2 \frac{G h^2}{12} + \nu_x B_{y_i} \right) \\
B_{x'y_i} &= \nu_x B_{y_i} = \nu_y B_{x_i} \\
\gamma' &= \frac{G h^2}{12}.
\end{align*}
\]

(1.2)

Following Huber we define the shear modulus as:

\[
G = \frac{E}{2 (1+\nu_x \nu_y)} \quad \text{where } E_x = E_y = E \quad (1.3)
\]

1.2 Simple Gridwork

Mathematical similarity which exists between the behaviour of plates and grillage seems to have been realised first by Timoshenko [14]. This basic concept has been used by Guyon and Massonet [3] who used Huber's solution of orthotropic plates and applied it to the analysis of right girder grillage.

Fig. 1 shows a system consisting of \( n \) simply supported longitudinal beams of span \( L_x \) running in the \( x \)-direction and \( m \) cross beams of length \( L_y \) running in the \( y \)-direction and free at the ends \( y = \pm \frac{L_y}{2} \).

For the purpose of analysis, the concept of 'Equivalent orthotropic plate' as a substitute of original gridwork is utilized where the elastic stiffness in both flexure and torsion of discrete beams are assumed

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to be continuously distributed to have a uniform spread longitudinal and transverse stiffness. The system is now submitted to a virtual deformation defined by the elastic surface \( w = w(x, y) \) which yields the governing equation for the equivalent substitute of a simple gridwork in the form:

\[
B_x' \frac{\partial^4 w}{\partial x^4} + 2H' \frac{\partial^4 w}{\partial x^2 \partial y^2} + B_y' \frac{\partial^4 w}{\partial y^4} = q(x, y)
\]

where

\[
B_x' = \frac{B_T}{b_0}, \quad B_y' = \frac{B_p}{b_0}
\]

\[
2H' = \gamma_T' + \gamma_p'
\]

\[
\gamma_T' = \frac{C_T}{b_0}, \quad \gamma_p' = \frac{C_p}{b_0}
\]

\( B_T = E_x I_T \) and \( B_p = E_y I_p \) are the flexural rigidities of the longitudinal and cross beams, respectively.

\( C_T \) and \( C_p \) are the torsional rigidities of the longitudinal and cross beams, respectively.

1.3 Slab with grillage in two mutually perpendicular direction.

The slab stiffened by longitudinal and cross beams may also be replaced by an 'Equivalent orthotropic plate' provided the ratio of gridwork spacing to slab boundary dimensions are small enough \( \frac{l_0}{l_x}, \frac{l_0}{l_y} \ll 1 \) to ensure approximate homogeneity of stiffness [11]. The elastic parameters relating to the substitute system are assumed to be continuously distributed in the two mutually perpendicular directions.
The theory of the 'Equivalent orthotropic plate' presupposes that both the longitudinal and transverse beams of a composite system are symmetrically placed with respect to the middle surface of the equivalent slab so that the true system possesses a horizontal plane of symmetry. But in a bridge deck system, both longitudinal and transverse beams are placed asymmetrically with respect to the slab portion of the cross section. An eighth order partial differential equation \[ [3] \] is obtained as a more rigorous solution based on the consideration of displacement components \( u, v, \) and \( w \) in all the three \( x, y \) and \( z \) directions. Bares \([3]\) has shown, from the analysis of the three dimensional problem mentioned, the important fact that the shear distribution is considerably dependent on support condition and loading intensity.

For the plane stress analysis, in order to minimise the error due to eccentric position of the beams with respect to the middle surface of the plate and hence the problems entailed with the torsional rigidity have been investigated by many authors \([3, 11, 13, 23]\). Huber's fourth order differential equation for the equivalent orthotropic plate which must satisfy both equation (1.1) governing the problem of orthotropic slab, and equation (1.4) corresponding to the problem of simple grid of beams was finally obtained in the form:
$B x \frac{\partial^2 w}{\partial x^4} + 2H \frac{\partial^3 w}{\partial x^2 \partial y^2} + B y \frac{\partial^4 w}{\partial y^4} = g(x,y) \quad (1.5)$

where

\[ B x = \frac{E_x h^3}{12 (1-\nu_{xx} \nu_{yy})} + \frac{E_x I_{sx}}{b_0} + \frac{E_x Z_1^2 h}{1-\nu_{xx} \nu_{yy}} \quad (a) \]

\[ B y = \frac{E_y h^3}{12 (1-\nu_{yy} \nu_{xy})} + \frac{E_y I_{sy}}{l_0} + \frac{E_y Z_2^2 h}{1-\nu_{yy} \nu_{xy}} \quad (b) \]

\[ 2H = B_x \lambda_y + B_y \lambda_x + 4C \quad (c) \]

and

- $h$ = thickness of the slab
- $b_0$ = spacing of the main girder
- $l_0$ = spacing of the cross beam

$Z_1$ and $Z_2$ are the distances of the neutral surface of the repeating section from the middle plane of the slab in longitudinal and transverse directions, respectively.

$I_{sx}$ = Moment of inertia of the longitudinal beam about the neutral surface i.e. $Z_1$ below the middle plane of the slab.

$I_{sy}$ = Moment of inertia of the cross beam about the neutral surface i.e. $Z_2$ below the middle plane of the slab.

$I_{sx}$ and $I_{sy}$ are calculated for the beam-sections without regard to the slab.

$4C = \tilde{\gamma}_T + \tilde{\gamma}_P$

The values of $\tilde{\gamma}_T$ and $\tilde{\gamma}_P$ are the torsional rigidities determined by means of torsion constants $F_T$ and $F_P$ of the sectional areas corresponding to the different

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elements constituting a section.

In case of an open slab and beam section (T-beam) as in Fig. 2, the torsion constant of the entire repeating cross-sectional area is given by the formula [3]

\[
\begin{align*}
F_T &= \frac{1}{2} b_0 h^3 C_i + b_i^3 (h_i - h) C_f \left( \frac{h_i}{b_i} < h_i / h \right), \\
F_P &= \frac{1}{2} b_o h^3 C_i + b_o^3 (h_i - h) C_f \left( \frac{h_o}{b_o} < \frac{h_o}{h} \right)
\end{align*}
\]

where the first term of the right hand side refer to the slab portion and the second term to the beam portion of the section. The value of the factor $C_1$ depends on the shape (side ratio) and is called the shape factor involved in the torsion constant of a rectangular section.

The value of $C_1$ is given in the following table.

<table>
<thead>
<tr>
<th>$\frac{h_i}{b_i}$</th>
<th>1</th>
<th>1.5</th>
<th>1.75</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>0.141</td>
<td>0.196</td>
<td>0.214</td>
<td>0.229</td>
<td>0.249</td>
<td>0.263</td>
<td>0.281</td>
<td>0.299</td>
<td>0.307</td>
<td>0.333</td>
</tr>
</tbody>
</table>

Now Equation (1.6C) of the apparent torsional rigidity of the equivalent system:

\[ 2H = B_x \lambda_y + B_y \lambda_x + 4C \]

is defined such that

\[ 4C = \gamma_T + \gamma_P = Gx \frac{Fr}{b_o} + Gy \frac{Fe}{l_o} \]

$G_x$ and $G_y$ are the shear modulus in $x$ and $y$ direction, respectively and are defined as:

\[ G_x = \frac{E_x}{2(1 + \sigma_{xy})} \quad \text{and} \quad G_y = \frac{E_y}{2(1 + \sigma_{xy})} \]
The stress couples, shear resultants and the vertical reactions are expressed as:

\[
\begin{align*}
N_x &= -B_x \left( \frac{\partial^2 w}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2} \right) \tag{a} \\
N_y &= -B_y \left( \frac{\partial^2 w}{\partial y^2} + \mu_x \frac{\partial^2 w}{\partial x^2} \right) \tag{b} \\
M_{xy} &= -2c \frac{\partial^2 w}{\partial x \partial y} \tag{c} \\
Q_x &= -B_x \frac{\partial^3 w}{\partial x^3} - \left( B_x \mu_y + 2c \right) \frac{\partial^2 w}{\partial x \partial y^2} \tag{d} \\
Q_y &= -B_y \frac{\partial^3 w}{\partial y^3} - \left( B_y \mu_x + 2c \right) \frac{\partial^2 w}{\partial x^2 \partial y} \tag{e} \\
V_x &= -B_x \left[ \frac{\partial^3 w}{\partial x^3} + \left( \frac{4c}{B_x} + \mu_y \right) \frac{\partial^2 w}{\partial x \partial y^2} \right] \tag{f} \\
V_y &= -B_y \left[ \frac{\partial^3 w}{\partial y^3} + \left( \frac{4c}{B_y} + \mu_x \right) \frac{\partial^2 w}{\partial x^2 \partial y} \right] \tag{g}
\end{align*}
\]

Positive directions of stresses, stress couples and the shear resultants of an orthotropic plate elements are shown in Fig. 3 and Fig. 4, respectively.

1.4 Dimensionless parameter:

A dimensionless parameter \( \Omega \) is introduced which is characteristic for the resistance in torsion of the structural pattern and limited by the values 0 and 1. This interval covers all the structural systems. For the simple grid of beams of weak torsional resistance \( \Omega = 0 \), while \( \Omega = 1 \) relates to the true slab. \( \Omega \) is expressed in the form:

\[
\Omega = \frac{\mu_y B_x + \mu_x B_y + \gamma_T + \gamma_P}{2 \sqrt{B_x B_y}} \tag{1.10}
\]

and is evaluated by employing the theorem of Betti.
which proves that:

\[ B_x \lambda_y = B_y \lambda_x \]

or \[ \lambda_y = \frac{B_y}{B_x} \lambda_x \quad (1.11) \]

1.5 Orthotropy of form:

In the analysis of gridwork, it is recognised at this state that the factors \( \lambda_x \) and \( \lambda_y \) although represent the relationship between the stress \( \sigma \) and the transverse strain \( \varepsilon \), are not material constants as 'Poisson's ratio' proper but are elastic constants corresponding to the form of the system. Hence the name 'orthotropy of form' as distinct from the 'orthotropy of material.'

The value of \( \lambda_y \) may be evaluated from the relations of Eq. (1.6) and (1.11). In the present analysis of gridwork in skew bridge, the influence of Poisson's ratio on the stress distribution will be studied for different values of \( \lambda_x \) and its relative importance will be discussed in details.

For the case where \( E_x = E_y = E \) and

\[ G_x = \frac{E}{2(1+\mu_x \mu_y)} = G_y \]

Eq. (1.10) reduces to:

\[ \Omega = \frac{\lambda_x B_y + \frac{E}{4 (1+\sqrt{\lambda_x \lambda_y})} \left[ \frac{F_r}{b_o} + \frac{F_p}{l_o} \right]}{B_y \sqrt{\frac{\lambda_x}{\lambda_y}}} \quad (1.12) \]

Ref. [3] has given several values of \( \lambda_x \) corresponding to the different types of gridwork with regard to the
different material of construction. For simple beam grids without slab the value of \( \mu_x = 0 \) is acceptable. For reinforced and prestressed concrete box section \( \mu_x = 0.10 \) and for open section comprising a single slab (e.g. T-beam section) \( \mu_x = 0.15 \) are sufficiently accurate. In case of orthotropic steel deck bridges the value of \( \mu_x \) may be taken as 0.3.

The flexural and torsional rigidities as defined in Eq. (1.6) for orthogonal equivalent gridwork will be assumed to be valid for an equivalent orthotropic skew plate. The differential equation governing the problem of the equivalent orthotropic skew plate which is a substitute of grillage in skew bridges will be derived in the next chapter.
FIG. 1 SIMPLE GRIDWORK

FIG. 2 GRIDWORK WITH SLAB
FIG. 3 POSITIVE DIRECTIONS OF STRESSES

FIG. 4 ORTHOTROPIC PLATE ELEMENT SHOWING
POSITIVE DIRECTIONS OF STRESS
COUPLES AND SHEAR RESULTANTS
II

APPLICATION OF ORTHOTROPIC PLATE THEORY
IN THE ANALYSIS OF GRIDWORK IN SKEW BRIDGES

Theoretical Background:

The plane stress solution obtained for orthogonal grid systems by orthotropic plate theory proved to be well in accordance with numerous experimental data and comparative analytical investigations gave further justification to this new method of solution.

Naruka and Ohmura [4] were the first to assume that the theory of orthotropic parallelogrammic plates will be effective to the same degree in the analysis of skew girder bridges as the orthotropic rectangular plate in the analysis of right girder bridges. Applying this basic concept the governing differential equation (1.5) of equivalent orthotropic plate may be transformed to skew co-ordinates parallel to the edges (u, v) for a skew gridwork system as follows:

If (x, y) are the rectangular co-ordinates of a point (Fig. 5) in the middle surface of the plate, with (u, v) the corresponding oblique co-ordinates and \( \phi \) the skew angle

\[
\begin{align*}
    u &= x + y \tan \phi \\
    v &= y \sec \phi
\end{align*}
\]

(Fig. 5)
Differentiating Eq. (2.1) with respect to \( x \) and \( y \) one obtains:

\[
\begin{align*}
\frac{\partial u}{\partial x} &= 1, \quad \frac{\partial u}{\partial y} = \tan \phi \\
\frac{\partial v}{\partial x} &= 0, \quad \frac{\partial v}{\partial y} = \sec \phi
\end{align*}
\] (2.2)

Since \( w \) is a function of both \( u \) and \( v \)

\[
\begin{align*}
\frac{\partial w}{\partial x} &= \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial w}{\partial u} \\
\frac{\partial w}{\partial y} &= \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial y} \\
&= \frac{\partial w}{\partial u} \cdot \tan \phi + \frac{\partial w}{\partial v} \cdot \sec \phi
\end{align*}
\] (2.3)

After successive partial differentiation of Eq. (2.3) the following relations are obtained:

\[
\begin{align*}
\frac{\partial^2 w}{\partial x^2} &= \frac{\partial^2 w}{\partial u^2} \\
\frac{\partial^2 w}{\partial x \partial y} &= \frac{\partial^2 w}{\partial u^2} \tan \phi + \frac{\partial^2 w}{\partial u \partial v} \sec \phi \\
\frac{\partial^2 w}{\partial y^2} &= \frac{\partial^2 w}{\partial u^2} \tan^2 \phi + 2 \frac{\partial^2 w}{\partial u \partial v} \tan \phi \sec \phi \\
&\quad + \frac{\partial^2 w}{\partial v^2} \sec^2 \phi \\
\frac{\partial^3 w}{\partial x^3} &= \frac{\partial^3 w}{\partial u^3} \\
\frac{\partial^3 w}{\partial x^2 \partial y} &= \frac{\partial^3 w}{\partial u^3} \tan \phi + \frac{\partial^3 w}{\partial u^2 \partial v} \sec \phi \\
\frac{\partial^3 w}{\partial x \partial y^2} &= \frac{\partial^3 w}{\partial u^3} \tan^2 \phi + 2 \frac{\partial^3 w}{\partial u^2 \partial v} \tan \phi \sec \phi \\
&\quad + \frac{\partial^3 w}{\partial u \partial v^2} \sec^2 \phi \\
\frac{\partial^3 w}{\partial y^3} &= \frac{\partial^3 w}{\partial u^3} \tan^3 \phi + 3 \frac{\partial^3 w}{\partial u \partial v^2} \tan^2 \phi \sec \phi \\
&\quad + 3 \frac{\partial^3 w}{\partial u^2 \partial v} \tan \phi \sec^2 \phi + \frac{\partial^3 w}{\partial v^3} \sec^3 \phi
\end{align*}
\] (2.4) (2.5)
\[
\frac{\partial^4 w}{\partial x^4} = \frac{\partial^4 w}{\partial y^4} \quad (a)
\]
\[
\frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{\partial^4 w}{\partial u^2 \partial v^2} + 2 \frac{\partial^4 w}{\partial u^2 \partial v} \tan \phi \sec \phi \quad (b)
\]
\[
\frac{\partial^4 w}{\partial y^4} = \frac{\partial^4 w}{\partial u^4} + 4 \frac{\partial^4 w}{\partial y^3 \partial v} \tan^3 \phi \sec \phi + 6 \frac{\partial^4 w}{\partial u^2 \partial v^2} \tan^2 \phi \sec^2 \phi + 4 \frac{\partial^4 w}{\partial u \partial v^3} \tan \phi \sec^3 \phi + \frac{\partial^4 w}{\partial v^4} \sec^4 \phi \quad (c)
\]

Putting the values of \( \frac{\partial^4 w}{\partial x^4} \), \( \frac{\partial^4 w}{\partial x^2 \partial y^2} \), \( \frac{\partial^4 w}{\partial y^4} \) into Equation (1.5), one obtains the fourth order partial differential equation of the equivalent orthotropic skew plate which is a substitute of gridwork in skew bridge in the following form:

\[
\frac{\partial^4 w}{\partial u^4} (B_x + 2H \tan^2 \phi + B_y \tan^3 \phi) + \frac{\partial^4 w}{\partial u^2 \partial v} (4H + 6B \tan^2 \phi) + \frac{\partial^4 w}{\partial v^4} = 0 \quad (2.6)
\]

The stress couples, shear resultants and vertical reactions of Eq. (1.9) may be expressed in skew co-ordinates as:

\[
M_x = -B_x \left[ \frac{\partial^3 w}{\partial u^2} + \frac{\partial w}{\partial u} \left( \frac{\partial^3 w}{\partial u^2} + \tan^2 \phi + 2 \frac{\partial^2 w}{\partial u \partial v} \tan \phi \sec \phi + \frac{\partial w}{\partial v^2} \sec^2 \phi \right) \right] \quad (a)
\]

\[
M_y = -B_y \left[ \frac{\partial^3 w}{\partial u^2} \tan^2 \phi + 2 \frac{\partial^2 w}{\partial u \partial v} \tan \phi \sec \phi + \frac{\partial w}{\partial v^2} \sec^2 \phi \right] \quad (b)
\]

\[
M_{xy} = -2 \left[ \frac{\partial^3 w}{\partial u^2} \tan \phi + \frac{\partial^2 w}{\partial u \partial v} \sec \phi \right] \quad (c)
\]

\[
Q_x = -B_x \frac{\partial^3 w}{\partial u^3} - (B_x B_y + 2) \left[ \frac{\partial^2 w}{\partial u^2} \tan^2 \phi + 2 \frac{\partial w}{\partial u^2 \partial v} \tan \phi \sec \phi \right] + \frac{\partial w}{\partial u^2 \partial v^2} \sec^2 \phi \quad (d)
\]

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Recently Patterson and Cusens [21] have presented a solution of orthotropic skew plate under uniform load. The solution is an extension of an early work by Kennedy and Huggins [22] who developed a method using a single infinite series representation of the deflection of an isotropic plate with edge stiffening beam. Results for the case of a slab simply supported on the two opposite sides and free at the other two were found unreliable due to the use of Kirchoff's two boundary equations instead of three. Moreover the method presented by Patterson and Cusen [21] does not consider the application of the concentrated loadings on the skew orthotropic slab.

Because of the complicated boundary conditions, an exact solution of the fourth order differential equation governing the behaviour of simply supported orthotropic skew slab with two opposite edges free has so far proved impossible. Since in a bridge structure a loaded vehicle would act more as a point load distributed only over a
small portion of the deck slab, finite difference method seems to be the best analytical method [8] available in dealing with such point loads. In the present investigation, the same approach which Basar and Yuksul [7] have utilized in their formulation of finite difference equations for a simply supported orthotropic skew slab has been followed in a somewhat different manner that leads to a fairly simple solution incorporating all the essential parameters of a gridwork in skew bridge which is simply supported on the two opposite edges and free at the other two.
Due to rapid development of the computing machines and the good convergence properties of the method of finite difference which replaces functions and their derivatives by algebraic expressions involving only the values of the functions at a finite number of points in or near the region or interval of interest, the complicated boundary value problems involved in this present investigation will be solved by this method. Replacement of functions reduces the problem to a set of simultaneous algebraic equations.

The method permits the immediate writing of the force-displacement relations in the form \[ [A]\{w\} = \{q\} \]

\[(3.1)\]

where \(\{q\}\) is a column of static loads acting at a predetermined set of points (called node points of a certain network) and \(\{w\}\) is the column corresponding to vertical displacements. \([A]\) is the conventional stiffness matrix obtained by a few algebraic operations.

The method of central finite differences will be applied in the solution of the problem.

3.1 Computation of Equivalent Plate Moments:

Solution of Eq. (3.1) yields numerical values of
deflections at the nodal points of the equivalent plate. By substitution of these values in appropriate moment equations (Eq. 2.8), numerical values of moments are found. Combining the sets of influence coefficients for $M_x$, $M_y$ and $M_{xy}$, influence coefficients for the equivalent principal moments with their directions at all the network points may be computed according to the equations below:

$$
\begin{align*}
M_{\text{max}} &= \frac{M_x + M_y}{2} \pm \sqrt{\left(\frac{M_x - M_y}{2}\right)^2 + M_{xy}^2} \\
\theta &= \frac{1}{2} \tan^{-1} \left( \frac{2M_{xy}}{M_x - M_y} \right)
\end{align*}
$$

3.2 Computation of Beam Moments:

The moments acting on the equivalent orthotropic plate may now be integrated over the whole flange width of the beam to compute the bending moments which are of greatest importance for design purposes. For example, the moment in longitudinal beam $B$ in Fig. 6A in $x$-direction is given by:

$$
M_{B(x)} = \int_{0.5a}^{1.5b_o} M_x dy
$$

Similarly moment in transverse beam $C$ in the $y$-direction is given by:

$$
M_{C(y)} = \int_{0.5c_o}^{1.5c_o} M_y dx
$$

(3.6a)

If the cross beams are in a direction at an angle $\theta$
with the y-axis, the bending and twisting moment of the transverse beam in the oblique direction \( v \), for example at point A, in Fig. 6A are given by \( [12] \)

\[
M_v = M_x' \sin^2 \phi + M_y' \cos^2 \phi - 2M_{xy}' \sin \phi \cos \phi \quad (3.6b)
\]

\[
M_{v\eta} = (M_x' - M_y') \sin \phi \cos \phi + M_{xy}' (\sin^2 \phi - \cos^2 \phi) \quad (3.6c)
\]

where \( M_x' \), \( M_y' \) and \( M_{xy}' \) are the integrated moments over the spacing of the gridwork at the section under considerations.

Equation (3.6b) and (3.6c) can be derived from the equilibrium of an element of the plate as shown in Fig. 6.

![Diagram](image.png)

**Fig. 6.** Moments acting on different planes

The integration of the equivalent plate moment for the beam may be carried out by Simpson's Rule \([9]\). Thus
for a mesh size of n division, the integral of a function $f(y)$ may be evaluated by:

$$\int_0^{y_n} f(y) \, dy = \frac{y_n - y_0}{2n} \left[ f(y_0) + 4f(y_1) + 2f(y_2) + \ldots + 4f(y_{n-1}) + f(y_n) \right]$$

where $(y_n - y_0)$ represents the spacing of the gridwork in either x or y direction and $f(y_0), f(y_1), f(y_2), \ldots$ etc., corresponds to the magnitude of functions of moments within the region $y_n$ and $y_0$. If the spacing of the longitudinal and cross beam is small and coincides with the mesh point layout, equivalent plate moment can be considered to be the average value within the flange-width of the gridwork.

3.3 Limitation and Accuracy of the Method of Finite Differences

The approximations by the finite difference method can be limited to a minor interference which breaks the deflection curve at discrete points and join them in straight lines. Hence the finite difference equations representing the original differential equations are valid as long as the finite number of points which have been reduced from an infinite number of points on the deflection surface and arranged to form a certain network are close enough for straight line approximations. Of course finite difference equations do not exactly represent the original governing equations and hence it is not an exact mathematical method. However, when properly applied, using finite number of network points it is a sufficiently accurate
tool for analysis of complicated structural systems.

3.4 Selection of Mesh Sizes:

In the present investigations since seven number of longitudinal and cross beams at different spacings have been provided for the model skew bridge, numerical solutions were obtained by dividing the equivalent bridge system into 6x6 skew panels so that nodal points of skew meshes coincided with the point of intersections of gridwork in both longitudinal and transverse directions to facilitate direct comparison between experimental and theoretical results at these points. However, the effect of different network spacings on the accuracy of the results has been examined by dividing the plate into 4x4, 6x6 and 8x8 meshes which resulted into 8, 18 and 32 simultaneous equations, respectively. It was observed that finer mesh sizes produced about 3% and 6% more accurate deflection and girder moment Mx, respectively, at the point of maximum stress intensity than the coarser mesh size. This is expected because of the inherent limitations of finite difference approximations.

Though the number of simultaneous equations increases considerably with the choice of finer mesh sizes, it does not impose a problem to the solution of the structural system when an electronic computer can be conveniently used for this purpose.
FIG. 6A GRIDWORK WITH DECK SLAB IN SKEW BRIDGE
IV
OUTLINE OF THE METHOD OF FINITE DIFFERENCES
AS APPLIED TO THE SOLUTION OF SKEW GRIDWORK

4.1 General Approach

Adopting Marcus' method, the fourth order differential equations of equivalent orthotropic plate can be split up into two equations of second order as given in reference [7] in the following form:

\[ \begin{align*}
B_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + B_y \frac{\partial^4 w}{\partial y^4} &= \varrho (x, y) \\
&= \left( \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \right) \left( B_x \frac{\partial^2 w}{\partial x^2} + B_y \frac{\partial^2 w}{\partial y^2} \right) \\
&\quad + (2H - B_x - B_y) \frac{\partial^2 w}{\partial y^2} \left( \frac{\partial^2 w}{\partial x^2} \right)
\end{align*} \quad (4.1) \]

Writing

\[ \begin{align*}
B_x \frac{\partial^2 w}{\partial x^2} + B_y \frac{\partial^2 w}{\partial y^2} &= U \\
\frac{\partial^3 w}{\partial x^2} &= X, \quad \frac{\partial^3 w}{\partial y^2} = Y
\end{align*} \quad (4.2, 4.3) \]

and

\[ 2H - B_x - B_y = D \]

Eq. (4.1) reduces to

\[ \begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + D \frac{\partial^2 x}{\partial y^2} &= \varrho \\
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + D \frac{\partial^2 y}{\partial x^2} &= \varrho
\end{align*} \quad (4.4a, 4.4b) \]

Eq. (4.4a) can now be easily put into finite difference
form using Favre's skew network for the gridwork in skew bridges.

Naruoka and Ohmura split up the fourth order differential equation in the form:

\[
\frac{\partial^4 u}{\partial y^4} + n \frac{\partial^3 u}{\partial x^3} = 0
\]

\[
\frac{\partial^2 u}{\partial y^2} + m \frac{\partial^3 u}{\partial x^3} = \frac{p}{By}
\]

where \( m = \frac{1}{By} \left( H + i \sqrt{Bx By - H^2} \right) \)

\( n = \frac{1}{By} \left( H - i \sqrt{Bx By - H^2} \right) \)

which are further abbreviations for the complicated terms that occur in the derivations of the equations.

Since the approach followed by Basar and Yuksul is relatively simple compared to that of Naruoka and Ohmura, derivation of finite difference equations in the present investigation is primarily based on Ref. [7].

The final expressions of the equations derived here, have been compared with those of Naruoka and Ohmura [4] which appear to agree when \( u_x \) and \( u_y \) in the present solution are put equal to zero. Comparison of the present solution with those of Basar and Yuksul [7] revealed minor discrepancies between the two solutions which may be attributed to the arithmetical computations for (1) interior point near the acute corner, (2) interior point near the obtuse corner, (3) general edge point and edge points near the acute and obtuse corners, respectively.
A comparison of the present solution by equivalent orthotropic plate theory for gridwork in skew bridges with the solution based on the theory of skew anisotropic slab formulated by Lie [24] and used by Naruoka [25] for analysis of grillage by Lie [24] and used by Naruoka [25] skew bridge system revealed very close agreement between the two solutions near the central portion. But a discrepancy in the value of deflection and moment at the free edge was observed which may be attributed to the unsatisfied boundary conditions as discussed in Section (b) of Chapter VIII of this thesis.

The simply supported edge boundary conditions and the boundary conditions of the free edge have been discussed and mathematically formulated when deriving the finite difference equations at different typical network points. Since six equations are to be used to eliminate five unknowns at the bridge boundaries, a constraint \(( w_{q} + w_{q} - 2w_{0} = 0 )\) Eq. (5.4e & 5.4f) is imposed near the simple support. However, it is felt reasonable to assume that this constraint imposed near the simply supported edges is valid, since the deflections at these points of questions are sufficiently small so that the value of \(( w_{q} + w_{q} - 2w_{0} )\) tends to zero in the limit. The suggestion by Faucett [8] of leaving the external points to be eliminated later when final set of equations is being formed, has been examined. Since it does not solve the boundary value problem, the external points for the typical network points near the boundary have been expressed in terms of deflections of internal points while formulating the finite difference operators.
4.2 **Finite Difference Approximations:**

The deformation of the equivalent skew slab defined by the elastic surface \( w = w(x, y) \) is also a function of \( w = w(u, v) \).

Recalling the finite difference approximations for the partial derivatives at a point \((x, y)\) or \((u, v)\) ranging over the domain of definition of the function, derivatives can be approximated in terms of deflections of nodal points of Favre's skew network as follows:

\[
\begin{align*}
\left( \frac{\partial^2 w}{\partial u^2} \right)_o &= \frac{1}{\gamma_u^2} \left( w_{11} - 2w_0 + w_1 \right) \\
\left( \frac{\partial^2 w}{\partial u \partial v} \right)_o &= \frac{1}{4\gamma_u \gamma_v} \left( w_{12} - w_{12} + w_{12'} - w_{12'} \right) \\
\left( \frac{\partial^2 w}{\partial v^2} \right)_o &= \frac{1}{\gamma_v^2} \left( w_{22'} - 2w_0 + w_2 \right)
\end{align*}
\]

Eq. (2.4) can now be written in finite difference approximations as:
\[
\begin{align*}
\left( \frac{\partial^2 w}{\partial x^2} \right)_0 &= \frac{1}{\gamma u^2} \left( w_{x'x'} - 2w_0 + w_i \right) \\
&= \frac{\chi^2}{\gamma y^2} \left( w_{x'x'} - 2w_0 + w_i \right) \quad (4.6a)
\end{align*}
\]

where \( \chi = \frac{\gamma y}{\gamma x} \) and \( \gamma_x = \gamma_u \)

\[
\begin{align*}
\left( \frac{\partial^2 w}{\partial y^2} \right)_0 &= \frac{1}{\gamma y^2} \left[ \tan^2 \phi \left( \frac{\chi^2}{\gamma y^2} (w_{x'x'} - 2w_0 + w_i) \right) + \frac{1}{2\gamma y^2} (w_{y'y'} - w_{y'y'} + w_{x'x'}) \tan \phi \sec \phi \\
&\quad + \frac{1}{\gamma y^2} (w_{x'x'} - 2w_0 + w_i) \sec^2 \phi \right]
\end{align*}
\]

but \( \gamma_y = \gamma_y \sec \phi \) and putting \( \beta = \chi \tan \phi \)

one can write

\[
\begin{align*}
\left( \frac{\partial^2 w}{\partial y^2} \right)_0 &= \frac{1}{\gamma y^2} \left[ \beta^2 (w_{x'x'} + w_i) - (2 + 2\beta^2) w_0 + \beta \left( -w_{y'y'} + w_{y'y'} + w_{x'x'} + w_{x'x'} \right) \\
&\quad + w_{y'^2} + w_{y'^2} \right] \quad (4.6b)
\end{align*}
\]

Hence

\[
\begin{align*}
\left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)_0 &= \frac{1}{\gamma y^2} \left[ (\chi^2 + \beta^2) (w_{x'x'} + w_i) - (2 + 2\beta^2 + 2\chi^2) w_0 \\
&\quad + \beta \left( -w_{y'y'} + w_{y'y'} + w_{x'x'} + w_{x'x'} \right) + w_{y'^2} + w_{y'^2} \right]
\end{align*}
\]

\[
= \frac{1}{\gamma y^2} \left[ \alpha (w_{x'x'} + w_i) - (2 + 2\alpha) w_0 \\
&\quad + \beta \left( -w_{y'y'} + w_{y'y'} + w_{x'x'} + w_{x'x'} \right) + w_{y'^2} + w_{y'^2} \right] \quad (4.6c)
\]

where \( \alpha = \beta^2 + \chi^2 \)

\[
\begin{align*}
U_0 &= B_x \left( \frac{\partial^2 w}{\partial x^2} \right)_0 + B_y \left( \frac{\partial^2 w}{\partial y^2} \right)_0 \\
&= \frac{1}{\gamma y^2} \left[ (B_x \chi^2 + B_y \beta^2) (w_{x'x'} + w_i) - (2B_x \chi^2 + 2B_y + 2\beta^3 \beta) w_0 \\
&\quad + \beta \left( -w_{y'y'} + w_{y'y'} + w_{x'x'} + w_{x'x'} \right) + B_y \left( w_{y'^2} + w_{y'^2} \right) \right]
\end{align*}
\]

\[
= \frac{1}{\gamma y^2} \left[ A (w_{x'x'} + w_i) - (2A + 2B_y) w_0 \\
&\quad + \beta \left( -w_{y'y'} + w_{y'y'} + w_{x'x'} + w_{x'x'} \right) + B_y \left( w_{y'^2} + w_{y'^2} \right) \right] \quad (4.7)
\]

where \( A = B_x \chi^2 + B_y \beta^2 \)

Following equations (4.6b) and (4.6c) the governing equation of the equivalent orthotropic skew plate

Eq. (4.4a) for general interior point (Point 0) can
be written as:

\[
\nabla \cdot \mathbf{w} = \frac{1}{\gamma^2} \left[ \alpha \left( \mathbf{u}_1' + \mathbf{u}_1 \right) - (2 + 2\alpha) \mathbf{u}_0 + \frac{\beta}{2} \left( -\mathbf{u}_1' + \mathbf{u}_1 - \mathbf{u}_1' - \mathbf{u}_1' \right) \right] + \frac{\partial}{\partial t} \left[ \beta \left( \mathbf{x}_1' + \mathbf{x}_1 \right) \right] - \left( 2 + 2\beta^2 \right) \mathbf{x}_0 \\
+ \frac{\beta}{2} \left( -\mathbf{x}_1' + \mathbf{x}_1 \right) + \frac{\beta}{2} \left( \mathbf{x}_1' - \mathbf{x}_1 \right) + \frac{\beta}{2} \left( \mathbf{x}_2' + \mathbf{x}_2 \right) = \frac{\dot{\mathbf{g}}}{\mathbf{g}_0} 
\]

(4.8)

where, from Equation (4.7) and (4.3) it follows:

\[
\mathbf{u}_1' = \frac{1}{\gamma^2} \left[ \mathbf{A} \left( \mathbf{w}_3' + \mathbf{w}_0 \right) - (2\mathbf{A} + 2\mathbf{B}_y) \mathbf{w}_1' \right] \\
+ \frac{\beta}{2} \mathbf{B}_y \left( -\mathbf{w}_3' + \mathbf{w}_2' + \mathbf{w}_3' - \mathbf{w}_2' \right) + \mathbf{B}_y \left( \mathbf{w}_{i2}' + \mathbf{w}_{i2} \right) \tag{4.9a} 
\]

\[
\mathbf{u}_1 = \frac{1}{\gamma^2} \left[ \mathbf{A} \left( \mathbf{w}_3 + \mathbf{w}_0 \right) - (2\mathbf{A} + 2\mathbf{B}_y) \mathbf{w}_1 \right] \\
+ \frac{\beta}{2} \mathbf{B}_y \left( -\mathbf{w}_3 + \mathbf{w}_2 + \mathbf{w}_3 - \mathbf{w}_2 \right) + \mathbf{B}_y \left( \mathbf{w}_{i2}' + \mathbf{w}_{i2} \right) \tag{4.9b} 
\]

\[
\mathbf{u}_{i2}' = \frac{1}{\gamma^2} \left[ \mathbf{A} \left( \mathbf{w}_3' + \mathbf{w}_2 \right) - (2\mathbf{A} + 2\mathbf{B}_y) \mathbf{w}_{i2}' \right] \\
+ \frac{\beta}{2} \mathbf{B}_y \left( -\mathbf{w}_3' + \mathbf{w}_4 + \mathbf{w}_3' - \mathbf{w}_4 \right) + \mathbf{B}_y \left( \mathbf{w}_{i4}' + \mathbf{w}_{i4} \right) \tag{4.9c} 
\]

\[
\mathbf{u}_{i2} = \frac{1}{\gamma^2} \left[ \mathbf{A} \left( \mathbf{w}_2 + \mathbf{w}_3 \right) - (2\mathbf{A} + 2\mathbf{B}_y) \mathbf{w}_{i2} \right] \\
+ \frac{\beta}{2} \mathbf{B}_y \left( -\mathbf{w}_4 + \mathbf{w}_3 + \mathbf{w}_4 - \mathbf{w}_3 \right) + \mathbf{B}_y \left( \mathbf{w}_i + \mathbf{w}_i \right) \tag{4.9d} 
\]

\[
\mathbf{u}_{i2}' = \frac{1}{\gamma^2} \left[ \mathbf{A} \left( \mathbf{w}_3' + \mathbf{w}_2' \right) - (2\mathbf{A} + 2\mathbf{B}_y) \mathbf{w}_{i2}' \right] \\
+ \frac{\beta}{2} \mathbf{B}_y \left( -\mathbf{w}_3' + \mathbf{w}_0 + \mathbf{w}_3' - \mathbf{w}_0 \right) + \mathbf{B}_y \left( \mathbf{w}_{i4}' + \mathbf{w}_{i4} \right) \tag{4.9e} 
\]

\[
\mathbf{u}_{i2} = \frac{1}{\gamma^2} \left[ \mathbf{A} \left( \mathbf{w}_2' + \mathbf{w}_3' \right) - (2\mathbf{A} + 2\mathbf{B}_y) \mathbf{w}_{i2} \right] \\
+ \frac{\beta}{2} \mathbf{B}_y \left( -\mathbf{w}_0 + \mathbf{w}_3 + \mathbf{w}_4' - \mathbf{w}_3 \right) + \mathbf{B}_y \left( \mathbf{w}_{i4} + \mathbf{w}_i \right) \tag{4.9f} 
\]

\[
\mathbf{u}_{i2}' = \frac{1}{\gamma^2} \left[ \mathbf{A} \left( \mathbf{w}_{i2}' + \mathbf{w}_{i2} \right) - (2\mathbf{A} + 2\mathbf{B}_y) \mathbf{w}_{i2}' \right] \\
+ \frac{\beta}{2} \mathbf{B}_y \left( -\mathbf{w}_{i2}' + \mathbf{w}_{i2} + \mathbf{w}_{i4} - \mathbf{w}_{i4} \right) + \mathbf{B}_y \left( \mathbf{w}_{i4} + \mathbf{w}_0 \right) \tag{4.9g} 
\]
\[ U_2 = \frac{1}{7y^2} \left[ A (w_i' + w_{12}) - (2A + 2By) w_2 \right. \\
\left. + \beta \frac{By}{2} (-w_{14} + w_{14} + w_{12} - w_1) + By (w_0 + w_4) \right] \quad (4.9h) \]

and \[ X_i' = \left( \frac{\partial w}{\partial x^2} \right)' = \frac{X^2}{7y^2} (w_{2y} - 2w_i + w_0) \quad (a) \]

\[ X_1 = \frac{X^2}{7y^2} (w_0 - 2w_1 + w_3) \quad (b) \]

\[ X_0 = \frac{X^2}{7y^2} (w_{12} - 2w_0 + w_1) \quad (c) \]

\[ X_{12} = \frac{X^2}{7y^2} (w_{32} - 2w_{12} + w_2) \quad (d) \]

\[ X_{12}' = \frac{X^2}{7y^2} (w_{32}' - 2w_{12}' + w_2') \quad (e) \]

\[ X_{12}' = \frac{X^2}{7y^2} (w_{22}' - 2w_{12}' + w_32') \quad (f) \]

\[ X_2' = \frac{X^2}{7y^2} (w_{12}' - 2w_2' + w_{12}') \quad (g) \]

\[ X_2 = \frac{X^2}{7y^2} (w_{12} - 2w_2 + w_1) \quad (h) \]

These finite difference approximations which have been used successfully in complicated boundary value problems can be utilized to formulate difference operators to be applied to different types of skew girder bridges, e.g. single span simply supported bridge, continuous bridge over several spans etc., by incorporating suitable boundary conditions. In what follows, the solution for a simply supported bridge grillage, pertaining to this investigation, will be obtained in terms of finite difference equations.
DERIVATIONS OF FINITE DIFFERENCE EQUATIONS

In this Chapter finite difference equations for a simply supported skew grillage bridge with two opposite edges free at y = ± V 2 will be derived. To cover the entire equivalent plate it is necessary to formulate difference equations for nine typical network points. These are: 1) General Interior Point, 2) Interior Point near the left simple support, 3) Interior Point near the right simple support, 4) Interior Point near the edge girder, 5) Interior Point near the acute corner, 6) Interior Point near the obtuse corner, 7) General edge Point, 8) Edge Point near the acute corner, 9) Edge Point near the obtuse corner, respectively.

5.1 General Interior Point:

Putting the values of \( U \) and \( \chi \) in terms of displacements from Eq. (4.7), (4.9) and (4.10) into equation (4.8) one can deduce the governing equation for general interior point as:

\[
\nabla^2 w = \left[ \alpha A (w_3' + 2w_0 + w_3) - \alpha (2A + 2B\gamma) (w_1' + w_1) \\
+ \frac{\alpha C}{2} B\gamma (w_3' + w_3') \right] + \alpha C B\gamma (w_1' + w_1) + \alpha C B\gamma (w_1' + w_1) \\
- (2 + 2\alpha) \left[ \beta A (w_1' + w_1) - (2A + 2B\gamma) w_0 + \frac{\beta}{2} B\gamma (w_1' + w_1) \right] \\
+ B\gamma (w_2' + w_2)
\]

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From equation (5.1), coefficients associated with deflection of different nodal points can be separated as shown below:

Coefft. of $w_0$: $2A \left(3 \alpha + 2\right) + By \left(\beta^3 + 4\alpha + 6\right) + DX^2 \left(6\beta^3 + 4\right)$

Coefft. of $w_1$: $-2 \alpha \left(2A + By\right) -2DX^2 \left(2\beta^3 + 1\right) -2A$

Coefft. of $w_i$: $-2 \alpha \left(2A + By\right) -2DX^2 \left(2\beta^3 + 1\right) -2A$

Coefft. of $w_2$: $-2 By \left(\alpha + 2\right) -2 \left(A + DX^2\right)$

Coefft. of $w_2'$: $-2 By \left(\alpha + 2\right) -2 \left(A + DX^2\right)$

Coefft. of $w_3$: $\alpha A + \beta^2 \left(DX^2 - \frac{By}{2}\right)$

Coefft. of $w_3'$: $\alpha A + \beta^2 \left(DX^2 - \frac{By}{2}\right)$

Coefft. of $w_4$: $By \left(1 - \frac{\beta^2}{2}\right)$
Coefft. of $w_A^i$ : $By (1 - \beta^2)$

\[
\begin{align*}
\text{"w\_2" : } & (1-\beta) (\alpha By + A + D x^2) - 2\beta By \\
\text{"w\_2\'" : } & (1+\beta) (\alpha By + A + D x^2) + 2\beta By \\
\text{"w\_4" : } & (1+\beta) (\alpha C By + A + D x^2) + 2\beta By \\
\text{"w\_4\'" : } & (1-\beta) (\alpha C By + A + D x^2) - 2\beta By \\
\text{"w\_4\" : } & \beta By \\
\text{"w\_4\'" : } & -\beta By \\
\text{"w\_4\" : } & -\beta By \\
\text{"w\_4\'" : } & \beta By \\
\text{"w\_34\" : } & \beta^2 By \\
\text{"w\_34\'" : } & \beta^2 By \\
\text{"w\_34\" : } & \beta^2 By \\
\text{"w\_34\'" : } & \beta^2 By \\
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\text{"w\_34\'" : } & \beta^2 By \\
\text{"w\_34\" : } & \beta^2 By \\
\text{"w\_34\'" : } & \beta^2 By \\
\text{"w\_32\" : } & \frac{\beta}{2} (\alpha C By + A + D x^2) \\
\text{"w\_32\'" : } & \frac{\beta}{2} (\alpha C By + A + D x^2) \\
\text{"w\_32\" : } & -\frac{\beta}{2} (\alpha C By + A + D x^2) \\
\text{"w\_32\'" : } & -\frac{\beta}{2} (\alpha C By + A + D x^2)
\end{align*}
\]
Using Favre's skew network, Equation (5.1) can now be conveniently presented as in Fig. 8(b).

\[
\begin{align*}
\beta^2 B/w_4 & \quad -\beta \cdot By \\
\beta (1-\beta^2) & \quad By (1-\beta^2) \\
-\beta (2A + By) & \quad -2\beta (2A + By) \\
\alpha A & \quad 2\alpha (2A + By) \\
\gamma A & \quad \eta B^2 (dx^2 - By/2) \\
\beta / (\alpha By) & \quad (1-\beta) (\alpha By) \\
\alpha A & \quad -2\beta (2A + By) \\
A + dx^2 & \quad -2A \\
A + dx^2 & \quad +2A + 2\beta By \\
A + dx^2 & \quad -2(\alpha + 2) \\
A + dx^2 & \quad +2A \\
A + dx^2 & \quad +2A + 2\beta By \\
A + dx^2 & \quad -2A \\
A + dx^2 & \quad +2A + 2\beta By
\end{align*}
\]

\[W = \frac{-q}{\rho_0 g} \]

Fig. 8(a). General Interior Point

Fig. 8(b). Finite difference equation for general Interior Point

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where \[ x = \frac{\lambda v}{T_x}, \]
\[ \beta = \kappa + \tan \phi, \]
\[ \alpha = \beta^2 + \kappa^2, \]
\[ A = B_x \kappa^2 + B_y \rho^2, \]
\[ D = 2H - B_x - B_y. \]

\[ (5.2) \]

5.2 **Interior Point near the left simple support**

**Boundary Conditions:**

The conditions on the simple support are as follows:

(I) Deflections along the edge are zero, i.e. \( w = 0 \)

Hence \( w|_{C} = 0 \) and \( w'|_{C} = 0 \)

(II) Since the condition of zero slope along the edge gives \( \frac{\partial^2 w}{\partial y^2} = 0 \) and moment perpendicular to the edge are zero \( (M_{n} = 0) \), these lead to the boundary condition that sum of the curvatures in two mutually perpendicular direction is zero along the edge.

Hence in terms of moments one can write:

\[ (M_{n} + M_{v})_{\text{support}} = (M_{x} + M_{y})_{\text{support}} = 0 \]

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From Eq. (1.9) it follows:

\[-B_x \left( \frac{\partial^2 w}{\partial x^2} + \lambda_y \frac{\partial^2 w}{\partial y^2} \right) - B_y \left( \frac{\partial^2 w}{\partial y^2} + \lambda_x \frac{\partial^2 w}{\partial x^2} \right) = 0 \]

or

\[ \frac{\partial^2 w}{\partial x^2} (B_x + B y \lambda_x) + \frac{\partial^2 w}{\partial y^2} (B_y + B x \lambda_y) = 0 \quad (5.3a) \]

Following Eq. (4.6a) and (4.6b)

\[ \left( \frac{\partial^2 w}{\partial x^2} \right)_1 = \frac{2}{\gamma y^2} \left( w_{2y} - 2 w_{1y} + w_0 \right) \quad (5.3b) \]

and

\[ \left( \frac{\partial^2 w}{\partial y^2} \right)_1 = \frac{1}{\gamma y^2} \left[ \frac{2}{\delta^2} \left( w_{2y} + w_0 \right) - (2 + 2 \beta^2) w_{1y} + \beta_2 \left( -w_{2y}^2 + w_2 + w_{2y}^2 - w_2' \right) + w_{1y}^2 + w_{1y} \right] \quad (5.3c) \]

Let

\[ B_x + B y \lambda_x = K_x \]

and

\[ B_y + B x \lambda_y = K_y \]

Combining condition (I) into Eq. (5.3a) with the values of curvature from Eq. (5.3b) and (5.3c) one can write:

\[ (K_x x^2 + K_y \beta^2) (w_{2y} + w_0) + K_2 \beta_2 \left( -w_{2y}^2 + w_2 + w_{2y}^2 - w_2' \right) = 0 \quad (5.4a) \]

From the condition of zero slope along the edge and by the method of interpolation one can deduce:

\[ \left( \frac{\partial w}{\partial v} \right)_0 = \frac{1}{2} \left[ \left( \frac{\partial w}{\partial v} \right)_{0y} + \left( \frac{\partial w}{\partial v} \right)_{0x} \right] = 0 \]

or

\[ \frac{1}{2} \left[ \frac{w_{2y} - w_{2y}'}{2 \lambda y} + \frac{w_2' - w_2}{2 \lambda y} \right] = 0 \]

or

\[ w_{2y} - w_{2y}' = w_2' - w_2 \quad (5.4b) \]
Let \( k_x x^2 + k_y \beta^2 = \delta \)

From Eq. (5.4a) and (5.4b) it follows:

\[
w_y = \frac{\beta k_y}{\delta} (w_2' - w_2) - w_0 \tag{5.4c}
\]

Following the same procedure one can obtain:

\[
w_{y_2} = \frac{\beta k_y}{\delta} (w_0 - w_x) - w_2 \tag{5.4d}
\]
\[
w_y' = \frac{\beta k_y}{\delta} (w_4' - w_0) - w_2' \tag{5.4e}
\]

From Eq. (5.4b) it follows:

\[
w_{y_2}' = \frac{\beta k_y}{\delta} (w_0 - w_x) - w_2' \tag{5.4f}
\]

Since

\[
w_{y_2}' - w_2' = w_0 - w_x \tag{5.4g}
\]

and

\[
w_y' - w_{y_2}' = w_0' - w_0 \tag{5.4h}
\]

Hence

\[
w_{y_2}' = \frac{\beta k_y}{\delta} (w_2' - w_2) - w_0 \tag{5.5a}
\]

and

\[
w_{y_2}' = \frac{\beta k_y}{\delta} (w_2' - w_2) - w_0' \tag{5.5b}
\]

The set of two equations (5.4e) and (5.4f) which have different values for the same external point, imposes a constraint \( w_4 + w_4' - 2w_0 = 0 \), near the simple support. It is, however, assumed in the present investigation that this constraint is valid near the simple support.

Now, putting the values of exterior mesh points outside the boundary of the plate in terms of deflections
of interior mesh points into Eq. (1.5) and maintaining the condition of zero deflections along the edge, the finite difference equation for interior point near the left simple support is obtained as follows:

\[ \nabla^2 w = \left[ \alpha A \left\{ \frac{\beta}{\delta} (w_2 - w_2') - w_0 + 2w_0 + W_3 \right\} - \alpha C (2A + 2By) w_1 \right. \\
+ \alpha C (By) \left\{ -\frac{\beta}{\delta} (w_0 - w_4) + w_2 + \frac{\beta}{\delta} (w_0 - w_4) - w_2' + w_3 - w_3' \right\} \\
+ \alpha C (By) (w_12' + w_12) - (2 + 2\alpha C) \left\{ A w_1 - (2A + 2By) w_0 \right. \\
\left. + \frac{\beta}{\delta} By (w_12 - w_12') + By (w_2' + w_2) \right\} \right) \\
+ \frac{\beta}{\delta} \left\{ A \left( -\frac{\beta}{\delta} (w_0 - w_4) + w_2 + w_32 + \frac{\beta}{\delta} (w_4 - w_0) - w_2' + w_32' \right) \\
+ (2A + 2By) (w_12 - w_12') \right) \\
+ \frac{\beta}{\delta} \left( 4w_0 - 2w_3 - 2\frac{\beta}{\delta} (w_2 - w_2') + 2w_0 - 2w_4 - 2w_4' + w_34 \right) \\
+ \frac{\beta}{\delta} (w_2 - w_2') - w_4 + \frac{\beta}{\delta} (w_2 - w_2') - w_4' + w_34' \right) \\
+ By (w_14' - w_14) \right) + A (w_12' + w_12) - (2A + 2By) (w_2 + w_2') \\
+ \frac{\beta}{\delta} (w_4 - w_4') + By (w_4' + 2w_0 + w_4) \right) \\
+ DX^2 \left[ \frac{\beta}{\delta} \left( \frac{\beta}{\delta} (w_2 - w_2') - w_0 + 2w_0 - 2w_1 + w_3 \right) - \left( 2 + 2\beta^2 \right) (2w_0 + w_1) \right. \\
\left. + \frac{\beta}{\delta} \left( -\frac{\beta}{\delta} (w_0 - w_4) + w_2 - 2w_12 + w_32 + \frac{\beta}{\delta} (w_0 - w_4) - w_2' \right. \\
\left. + 2w_12' - w_32' \right) + (2w_2' + w_12' - 2w_2 + w_12) \right] = \frac{Q}{\rho t} \gamma^4 \tag{5.6} \right] \\

Separating the coefficients associated with the different nodal point deflections, in the same way as in Section 5.1, Eq. (5.6) can be conveniently presented as in Fig. 10(b).
Fig. 10(a)

*Interior Point near the left simple support.*

![Diagram of interior point near a simple support](image)

Fig. 10(b) Finite difference equation for Interior Point near the left simple support.

<table>
<thead>
<tr>
<th>$\beta^2: A_k/\delta + B_y(1)$</th>
<th>$\beta: B_y$</th>
<th>$\beta^2: B_y/4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\frac{3}{4}\beta^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_y(4+2\alpha-\frac{\beta\delta}{2})$</td>
<td>$(1-\frac{3}{4})\beta(\alpha B_y + A) + \frac{\beta}{2}(\alpha B_y + A + Dx^2)$</td>
<td>$\frac{\beta^2}{2}(\alpha B_y + A + Dx^2)^2$</td>
</tr>
<tr>
<td>$+ A\left(\frac{\beta^2}{2}-2+4\alpha\frac{\beta}{2}\right)$</td>
<td>$+ Dx^2\left(4+4\alpha\frac{\beta}{2}\right)$</td>
<td>$-2A -2Dx^2\left(\frac{3}{2}\beta^2\right)$</td>
</tr>
<tr>
<td>$+ B_y\left(6+4\alpha+\frac{3}{2}\beta^2\right)$</td>
<td>$-2A -2Dx^2\left(\frac{3}{2}\beta^2\right)$</td>
<td>$+2\alpha(\alpha B_y + A + Dx^2)$</td>
</tr>
<tr>
<td>$- B_y(4+2\alpha+\frac{\beta^2}{2})$</td>
<td>$(1+\frac{3}{4})\beta(\alpha B_y + A) + \frac{\beta}{2}(\alpha B_y + A + Dx^2)$</td>
<td>$+ A + Dx^2$</td>
</tr>
<tr>
<td>$+ A\left(-\frac{\beta^2}{2}-2+\alpha&lt;\beta, \frac{K_y}{\delta}\right)$</td>
<td>$+ Dx^2\left(-\frac{\beta^2}{2}-2+\alpha&lt;\beta, \frac{K_y}{\delta}\right)$</td>
<td>$+ A + Dx^2$</td>
</tr>
<tr>
<td>$+ B_y\left(-\frac{\beta^2}{4}\right)$</td>
<td>$-\beta B_y$</td>
<td>$+ A + Dx^2$</td>
</tr>
<tr>
<td>$\beta^2: A_k/\delta + B_y\left(1-\frac{3}{4}\beta^2\right)$</td>
<td>$-\beta B_y$</td>
<td>$-\frac{\beta^2}{2}(\alpha B_y + A + Dx^2)^2$</td>
</tr>
</tbody>
</table>

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5.3 Interior Point near the Right Simple Support:

From the same boundary condition as in Sec. 5.2, external mesh points outside the boundary can be expressed as:

\[
\begin{align*}
W_3 & = \frac{K_y}{\delta} (w_2' - w_2) - w_0 \quad (5.7a) \\
W_{32} & = \frac{K_y}{\delta} (w_{41} - w_0) - w_2 \quad (5.7b) \\
W_{32}' & = \frac{K_y}{\delta} (w_0 - w_4) - w_2' \quad (5.7c) \\
W_{32}'' & = \frac{K_y}{\delta} (w_4 - w_0) - w_2'' \quad (5.7d) \\
W_{34} & = \frac{K_y}{\delta} (w_2 - w_2') - w_4 \quad (5.7e)
\end{align*}
\]

and condition of zero deflection along the edge yields:

\[
W_{14} = W_{12} = W_{12}' = W_{14}' = 0
\]

Putting these values in Equation (5.1) one can deduce the finite difference Equation near right simple support as: Fig. 11(a)

\[
\nabla \nabla w = \left[ \alpha A (w_3' + 2 w_0 + \frac{K_y}{\delta} (w_2 - w_2')) - w_0 \right] - \alpha (2 A + 2 B y) (w')
\]

\[
\quad + \frac{\alpha B y}{2} \left[ -w_3' + w_3' + \frac{K_y}{\delta} (w_4 - w_0) - w_2 - \frac{K_y}{\delta} (w_4 - w_0) + w_2' \right]
\]

\[
\quad + \alpha B y \left( w_1' + w_1' \right) - (2 + 2 \alpha) \left[ A (w_1') - (2 A + 2 B y) w_0 \right]
\]

\[
\quad + \beta B y \left( -w_1' + w_1' \right) + B y \left( w_2' + w_2 \right) \right] + \beta \left[ -w_2' + w_2' - \frac{K_y}{\delta} (w_0 - w_4') + w_2' \right]
\]

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Separating the coefficients associated with the different nodal deflections in the same way as in Section 5.1, Equation (5.8) can be presented as in Fig. 11(b).

\[
\ddot{w} + \frac{\beta}{2} \left( -w_{1''} + w_{1'} \right) + \beta \left( w_{2'} - 2 w_{2} + w_{0} \right) - 2 w_{3'} + \frac{\beta}{2} \left( w_{2'} - 2 w_{2} \right) - \left( 2A + 2By \right) \left( w_{2} + w_{2}' \right)
\]

\[
\begin{align*}
\frac{\beta^2}{2} \left( \lambda^2 - 2 w_{2} + 2 w_{0} + \frac{\beta}{2} \left( w_{2} - 2 w_{2} \right) \right) - w_{1'} + \frac{\beta}{2} \left( w_{1'} + w_{1}' \right) + \beta \left( w_{1'} + w_{1}' \right) \left( w_{1'} + w_{1}' \right) - \left( 2A + 2By \right) \left( w_{2} + w_{2}' \right)
\end{align*}
\]

Fig. 11(b)

Finite difference equation for Interior Point near the right simple support.
5.4 Interior Point near the free edge:

On the free edge at nodal points 3'2, 1'2, 2, 12 and 32, 
\( M_y = 0 \)

\[
(M_y)_{edge} = -B_y \left( \frac{\partial^2 w}{\partial y^2} + \mu_x \frac{\partial^2 w}{\partial x^2} \right) = 0
\]

\[ B_y \left( \frac{\partial^2 w}{\partial y^2} + \mu_x \frac{\partial^2 w}{\partial x^2} \right) = 0 \]

Adding and subtracting \( B_x \frac{\partial^2 w}{\partial x^2} \) to the above expression:

\[ B_x \frac{\partial^2 w}{\partial x^2} + B_y \frac{\partial^2 w}{\partial y^2} - B_x \frac{\partial^2 w}{\partial x^2} + B_y \mu_x \frac{\partial^2 w}{\partial x^2} = 0 \]

but \( U = B_x \frac{\partial^2 w}{\partial x^2} + B_y \frac{\partial^2 w}{\partial y^2} \)

Hence \( U_{edge} = (B_x - B_y \mu_x) \left( \frac{\partial^2 w}{\partial x^2} \right)_{edge} \) (5.9)

For points 1'2, 2, and 12, from Eq. (5.9) one obtains

\[ U_{1'2} = (B_x - B_y \mu_x) \frac{\gamma^2}{\lambda y^2} (w_{3'2} - 2w_{1'2} + w_2) \]

\[ = \frac{\gamma}{\lambda y^2} (w_{3'2} - 2w_{1'2} + w_2) \] (a)

Similarly \( U_2 = \frac{\gamma}{\lambda y^2} (w_{1'2} - 2w_2 + w_2) \) (b) \( \) (5.10)

\[ U_{12} = \frac{\gamma}{\lambda y^2} (w_2 - 2w_{12} + w_{3'2}) \] (c)

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where \( \gamma = (Bx - By/\Delta) \times ^2 \)

Comparing Equation (4.9c), (4.9d), (4.9h) with Eq. (5.10) one obtains the following relations:

\[
\frac{\beta^2}{2} By (-w_{34} + w_4) + By w_4 = -By w_1' - \frac{\beta^2}{2} By (w_{3'-w_0}) \\
+ (2By + 2A - 2 \gamma) w_{12} + (\gamma - A) (w_{3'2} + w_2) \tag{5.11a}
\]

\[
\frac{\beta^2}{2} By (-w_1' + w_{14}) + By w_4 = -By w_0 - \frac{\beta^2}{2} By (w_{1'-w_3}) \\
+ (2By + 2A - 2 \gamma) w_2 + (\gamma - A) (w_{1'2} + w_{12}) \tag{5.11b}
\]

\[
\frac{\beta^2}{2} By (-w_4 + w_{34}) + By w_{14} = -By w_1 - \frac{\beta^2}{2} By (w_0 + w_3) \\
+ (2By + 2A - 2 \gamma) w_2 + (\gamma - A) (w_{2} + w_{32}) \tag{5.11c}
\]

From Eq. (5.11) one can deduce:

\[
\frac{\beta^2}{4} By (w_{34} + w_{34} - 2w_4) + \frac{\beta^2}{4} By (-w_{14} + w_{14}) + By w_4 \\
= -By w_0 + (\gamma - A) (w_{12} + w_{12}) + (2By + 2A - 2 \gamma) w_2 \\
+ \frac{\beta^2}{4} By (w_{3'2} + w_0 + w_3) + \frac{\beta^2}{2} (2By + 2A - 2 \gamma) (w_{2} - w_{12}) \\
+ \frac{\beta^2}{2} (\gamma - A) (w_{3}' - w_{3}'2) \tag{5.12}
\]

With these values of nodal deflections outside the boundary Eq. (5.1) reduces to:

\[
\nabla^2 w = \left[ -\Delta A (w_{3'} + 2w_0 + w_3) - \Delta (2A + 2By) (w_{1'} + w_1)
\right. \\
+ \frac{\beta^2}{2} By (-w_{32} + w_{32}' + w_{32} - w_{32}') + \Delta By (w_{12} + w_{12} - w_{12} + w_{12}') \\
- (2 + 2 \Delta) \left\{ A (w_{1'} + w_1) - (2A + 2By) w_0 + \frac{\beta^2}{2} By (-w_{12} + w_{12} + w_{12}' - w_{12}') \right\}
\]
+ By \left( w_y' + w_0 \right) + \frac{\beta^2}{2} A \left( -w_3'^2 + w_3^2 + w_3^{'2} - w_3^2 \right)

- \frac{\beta^2}{2} \left( 2By + 2A \right) \left( -w_1'^2 + w_1^2 + w_1'^2 - w_1^2 \right) + \frac{\beta^2By}{4} \left( -2w_3^{'2} + 4w_0 + w_3^{'2} + w_3^2 - 2w_3^{'2} \right)

- 2w_4' + \frac{\beta^2}{2} By \left( w_1'^2 - w_1'^2 \right) + A \left( w_1'^2 + w_1^2 + w_1'^2 + w_1^2 \right) - \left( 2By + 2A \right) \left( w_0 + w_2 \right) + \frac{\beta^2}{2} By \left( w_4'^2 - w_4^{'2} \right) + By \left( w_4^{'2} + 2w_0 \right) - By \left( w_0 \right)

+ \left( \gamma^2 - A \right) \left( w_1'^2 + w_1^2 \right) + \left( 2By + 2A - 2A \right) w_2 + \frac{\beta^2By}{4} \left( w_3^{'2} - 2w_0 + w_3 \right)

+ \frac{\beta^2}{2} \left( 2By + 2A - 2A \right) \left( w_1^2 - w_1'^2 \right) + \frac{\beta^2}{2} \left( \gamma^2 - A \right) \left( w_3^2 - w_3'^2 \right)

+ Dx^2 \left[ \beta^2 \left( w_3^{'2} - 2w_1^2 + 2w_0 - 2w_1 + w_3 \right) - \left( 2 + 2\beta^2 \right) \left( w_1^2 - 2w_0 + w_1 \right) \right.

\left. + \frac{\beta^2}{2} \left( -w_3^{'2} + 2w_1^2 - 2w_1^2 + w_3^2 + w_3^{'2} - 2w_1'' + w_3'' + 2w_1'' \right) \right]

\left. + w_1'' - w_2^2 + w_1'' + w_1 - 2w_2 + w_1 \right] = \frac{\beta^2}{8} \lambda y^4 \quad (5.13)

Separating the coefficients of deflections; Eq. (5.13) can be presented as in Fig. 12(b).

![Fig. 12(b) Finite difference equation for Interior Point near the free edge.](Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.)
5.5 Interior Point near the Acute Corner:

From the boundary conditions of Section 5.2 and 5.4, it follows that when the exterior points (Eq. 5.4 \((b, c, d, g, h)\)) outside the boundary and (Eq. 5.11(a, b, c), (5.12)) are put into Eq. (5.1), finite difference equation for interior point near the acute corner is obtained as shown below:

\[
\begin{align*}
\nabla \nabla w &= \alpha A A \left\{ \left[ \frac{\beta \cdot k_y}{\delta} (w_2 - w_2') + w_0 + w_3 \right] - \alpha (2A + 2By) w_1 \right. \\
&+ \frac{\beta}{2} \alpha By \left\{ \left( -\frac{\beta \cdot k_y}{\delta} (w_4' - w_0) + w_2 + \frac{\beta \cdot k_y}{\delta} (w_4' - w_0) - w_2' + w_3 e - w_3 e' \right) \\
&+ \alpha By (w_2 + w_2) \right\} + \frac{\beta}{2} \left\{ A \left[ -\frac{\beta \cdot k_y}{\delta} (w_4' - w_0) + w_2 + w_3 e + \frac{\beta \cdot k_y}{\delta} (w_4 - w_0) \\
-w_2' + w_3 e' \right] - (2By + 2A) (w_12 - w_12') + \beta \cdot By \left[ (w_0 - 2w_3 - 2\beta \cdot k_y) (w_2' - w_2) \\
+ 2w_0 + \frac{\beta \cdot k_y}{\delta} (w_2' - w_2 - w_4' + w_3) \right] - By w_14' \right\} \\
&+ A (w_12 + w_12) - (2By + 2A) (w_2' + w_2) - \frac{\beta}{2} By w_14' + By (w_4' + 2w_0) \\
- By w_0 + (y-A) w_12 + (2By + 2A - 2y) w_2 + \frac{\beta \cdot By}{4} \left\{ \frac{\beta \cdot k_y}{\delta} (w_2' - w_2) - w_0 - 2w_0 + w_3 \right\}
\end{align*}
\]

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\[
\frac{\beta^2}{2} \left( 2BY + 2A - 2A' \right) \frac{w_{12}}{w_{12} - \beta \frac{K_Y}{\delta} \left( w_{4}-w_0 \right) + w_2} \]
\[
+ \frac{\beta^2}{2} \left( \frac{K_Y}{\delta} \left( w_{2'} - w_2 \right) - w_0 + 2w_0 - 2w_1 + w_3 \right) \left( 2 + 2 \beta^2 \right) (-2w_0 + w_1)
\]
\[
+ \frac{\beta^2}{2} \left[ \frac{-K_Y}{\delta} \left( w_{4'} - w_0 \right) + w_2 - 2w_{12} + w_{32} + \frac{K_Y}{\delta} \left( w_{4'} - w_0 \right) - w_{2'} \right]
\]
\[
= \frac{\ddot{q}}{\delta} A^4
\]

(5.14)

Separating the coefficients of deflections, Eq. 5.14 can be conveniently presented as in Fig. 13(b).

**Fig. 13(b)**

Finite difference equation for Interior Point near the Acute Corner.

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### 5.6 Interior Point near the Obtuse Corner:

Fig. 14(a)

Interior Point near the Obtuse Corner

Putting the values of nodal points outside the boundary (Eq. 5.7(a, b, c, d, e), Eq. 5.11 (a, b, c) and Eq. (5.12)) in Eq. (5.1) one can deduce the finite difference equation for Interior Point near the obtuse corner as:

\[
\nabla \nabla w = \left[ \alpha CA \left( w_{3'2} + 2w_0 + \frac{Ky}{d} (w_2 - w_{2'}) - w_0 \right) - \infty \left( 2By + 2A \right) w_{1'} \right.
\]
\[
+ \left( \frac{\beta}{2} \right) By \left( -w_{3'2} + w_{3'0} + \frac{Ky}{d} (w_0 - w_{4'}) - w_2 - \frac{Ky}{d} (w_0 - w_{4'}) + w_{2'} \right)
\]
\[
+ \left( \alpha \right) By \left( w_{1'2} + w_{1'i'2} \right) - (2 + 2 \infty) \left[ A w_{1'} - (2 BY + 2A) w_0 + \frac{\beta}{2} By \left( -w_{1'2} + w_{1'i'2} \right) \right.
\]
\[
+ By \left( w_{1'2'} + w_{2} \right) + \left( \frac{\beta}{2} \right) A \left( -w_{3'2} + \frac{Ky}{d} (w_0 - w_{4'}) - w_2 + w_{3'2'} - \frac{Ky}{d} (w_0 - w_{4'}) + w_{2'} \right)
\]
\[
- (2By + 2A) \left( -w_{1'2} + w_{1'i'2} \right) \right] + \left( \frac{\beta}{2} \right) By \left( w_{1'2'} + A \left( w_{1'2} + w_{1'i'2} \right) \right.
\]
\[
- (2BY + 2A) \left( w_{2'} + w_2 \right) + \left( \frac{\beta}{2} \right) By \left( w_{1'2}' + By \left( w_{4'} + 2w_0 \right) \right.
\]
\[
- By w_0 + (\gamma - A) w_{1'2} + (2A + 2By - 2Y) w_{2} + \left( \frac{\beta}{2} \right) By \left( w_{3'} - 2w_0 + \frac{Ky}{d} (w_{2} - w_{2'}) - w_0 \right)
\]
\[
+ \left( \frac{\beta}{2} \right) (Y - A) \left( \frac{Ky}{d} (w_{0} - w_{4'}) - w_{2} - w_{3'2} \right) + \left( \frac{\beta}{2} \right) (2A + 2By - 2Y) (\gamma - w_{1'2})
\]

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Separating the coefficients of deflections associated with different nodal points Eq. (5.15) can be conveniently presented as in Fig. 14(b).

![Finite difference Equation for Interior Point near the Obtuse Corner](image-url)
5.7 General Free Edge Point:

For the General Point on the free edge the second order differential equation (4.4b) is taken as the governing equation, i.e.

$$\nabla \nabla w = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + D \frac{\partial^2 u}{\partial x^2} = g(x,y)$$

Following Eq. (4.6a) and (4.6c) one can deduce:

$$\frac{1}{\gamma y^2} \left[ \alpha \left( U_{1} + U_{11} \right) - (2+2\alpha) U_{10} + \beta \left( -U_{12} + U_{12} + U_{12}' - U_{12}' \right) \right] + U_{2} + U_{2}' + U_{2} \right] + \frac{D x^2}{\gamma y^2} \left[ \gamma y - 2 \gamma y + \gamma y \right] = \frac{\bar{g}}{\beta}$$

From Eq. (5.9) for Points 1,0,1 one obtains the following relations:

$$U_{1} = \frac{\gamma y^2}{\gamma y^2} \left( w_{3}' - 2 w_{1}' + w_{0} \right)$$
$$U_{0} = \frac{\gamma y^2}{\gamma y^2} \left( w_{1}' - 2 w_{0} + w_{1} \right)$$
$$U_{1} = \frac{\gamma y^2}{\gamma y^2} \left( w_{0} - 2 w_{1} + w_{3} \right)$$
Since $M_y = 0$ on the edge

$$-B_y \left( \frac{\partial^2 w}{\partial y^2} + \lambda_x \frac{\partial^w}{\partial x^2} \right) = 0$$

or

$$\left( \frac{\partial^3 w}{\partial y^3} \right)_{\text{edge}} = \lambda_x \left( \frac{\partial^2 w}{\partial x^2} \right)_{\text{edge}}$$

Following equation (4.3) one can now deduce:

$$Y_i' = -\lambda_x x_i'$$

$$= -\lambda_x \frac{\partial^2}{\partial y^2} (w_3' - 2w_1' + w_0)$$

$$\gamma_0 = -\lambda_x \frac{\partial^2}{\partial y^2} (w_1' - 2w_0 + w_1) \quad (5.18)$$

$$\gamma_1 = -\lambda_x \frac{\partial^2}{\partial y^2} (w_0 - 2w_1 + w_3)$$

Again, from the boundary condition of free edge, i.e. vertical reaction

$$\left( \gamma_j \right)_{\text{edge}} = 0 \quad \text{Eq. (1.9g) and Eq. (1.8)} \quad \text{lead to:}$$

$$-B_y \left[ \frac{\partial^2 w}{\partial y^2} + \left( \frac{4}{B_y} + \lambda_x \right) \frac{\partial^2 w}{\partial x^2 \partial y} \right] = 0$$

or

$$B_y \frac{\partial^3 w}{\partial y^3} + (2H - B_x \lambda_y) \frac{\partial^2 w}{\partial x^2 \partial y} = 0 \quad (5.19)$$

Since

$$U = B_x \frac{\partial^2 w}{\partial x^2} + B_y \frac{\partial^2 w}{\partial y^2}$$

$$\frac{\partial U}{\partial y} = B_x \frac{\partial^3 w}{\partial x^2 \partial y} + B_y \frac{\partial^3 w}{\partial y^3}$$

Eq. (5.19) can now be written as:

$$\frac{\partial U}{\partial y} + (2H - B_x \lambda_y - B_x) \frac{\partial^3 w}{\partial x^2 \partial y} = 0 \quad (5.20)$$

Let $F(x,y)$ be a function of $x$ and $y$. From general edge point 0, (Fig. 15(a)) a perpendicular is drawn parallel
to y axis touching the network line at t and s respectively. Ref. [7].

Hence \( b = \gamma y \tan \phi \)

At Point \( 0 \), \( \left( \frac{\partial F}{\partial y} \right)_0 = \frac{F_t - F_s}{2\gamma y} \)

By interpolation:

\[
F_t = F_2 + \frac{F_{12} - F_{12}'}{2\gamma x} b
\]

\[
= F_2 + \frac{1}{2} \beta (F_{12} - F_{12}')
\]

Similarly

\[
F_s = F_2' - \frac{1}{2} \beta (F_{12}' - F_{12}')
\]

Hence \( \left( \frac{\partial F}{\partial y} \right)_0 = \frac{1}{2\gamma y} \left[ F_2 + \frac{1}{2} \beta (F_{12} - F_{12}') - \left( F_2' - \frac{1}{2} \beta (F_{12}' - F_{12}') \right) \right] \) (5.21)

Proceeding in the same way, for the function \( U \), one can deduce:

\[
\left( \frac{\partial U}{\partial y} \right)_0 = \frac{1}{2\gamma y} \left[ \left( U_2 + \frac{1}{2} \beta (U_{12} - U_{12}') \right) - \left( U_{12}' - \frac{1}{2} \beta (U_{12}' - U_{12}') \right) \right] \) (5.22)

Similar to Eq. (5.22)

\[
\left\{ \frac{\partial^2 U}{\partial x \partial y} \right\}_0 = \left\{ \frac{\partial^2 W}{\partial x^2 \partial y} \right\}_0
\]

\[
= \frac{\partial}{\partial y} \left( \frac{\partial^2 W}{\partial x^2} \right)_0
\]

\[
= \frac{\partial^2}{\partial y^2} \left( \frac{\partial}{\partial x} \left( \frac{\partial W}{\partial x} \right) \right)_0
\]

\[
= \frac{\partial^2}{\partial y^2} \left( \frac{\theta^2}{2\gamma^2} \left( w_{12}' - 2w_0 + w_1 \right) \right)
\]

\[
= \frac{\theta^2}{2\gamma^3} \left[ \left( w_{12}' + \frac{1}{2} \beta (w_2 - w_{32}') \right) - \left( w_{12}' - \frac{1}{2} \beta (w_2' - w_{32}') \right) \right]
\]

\[
- 2 \left( w_2 + \frac{1}{2} \beta (w_3 - w_{32}' - w_2') \right) + 2 \left( w_2' - \frac{1}{2} \beta (w_{12}' - w_{12}) \right)
\]

\[
+ \left[ w_{12} + \beta \left( w_{32}' - w_2 \right) \right]
\]

\[
- \left[ w_{12}' - \frac{1}{2} \beta (w_{32}' - w_2') \right] \) (5.23)
Putting the values of Eq. (5.22) and (5.23) into Eq. (5.20) one obtains:

\[ U_2 + \frac{1}{2} \beta (U_2' - U_1') - U_2' + \frac{1}{2} \beta (U_2' - U_1') = \]

\[ - \frac{\gamma}{\lambda y^2} \left[ \left\{ w_{1/2} + \frac{1}{2} \beta (w_2 - w_{3/2}) \right\} - \left\{ w_{1/2}' - \frac{1}{2} \beta (w_2' - w_{3/2}') \right\} \right] \]

\[ - 2 \left\{ w_2 + \frac{1}{2} \beta (w_{12} - w_{12}') \right\} + 2 \left\{ w_2' - \frac{1}{2} \beta (w_{12}' - w_{12}') \right\} \]

\[ + \left\{ w_{12} + \frac{1}{2} \beta (w_{32} - w_2) \right\} \]

where \( \gamma = \left( \frac{\varepsilon H - B_z \nu - B_x}{\nu} \right) \frac{1}{\lambda y^2} \)

Putting the value of \( U_2 \) from Eq. (5.24) into Eq. (5.16), the governing equation for general edge point is obtained as follows:

\[ \nabla \nabla w = \frac{1}{2} \gamma \left[ 2 U_2' - \beta (U_2' - U_1') \right] + \lambda y^2 \frac{\partial}{\partial x} \left( U_1' + U_1 \right) - \gamma y^2 (1 + \alpha) U_0 \]

\[ - \frac{\gamma}{2} \left[ \left\{ w_{1/2} + \frac{1}{2} \beta (w_2 - w_{3/2}) \right\} - \left\{ w_{1/2}' - \frac{1}{2} \beta (w_2' - w_{3/2}') \right\} \right] \]

\[ - 2 \left\{ w_2 + \frac{1}{2} \beta (w_{12} - w_{12}') \right\} + 2 \left\{ w_2' - \frac{1}{2} \beta (w_{12}' - w_{12}') \right\} \]

\[ + \left\{ w_{12} + \frac{1}{2} \beta (w_{32} - w_2) \right\} \]

\[ + D \gamma^2 \frac{\partial}{\partial x} (Y_0 - 2 Y_0 + Y_1) = \frac{\gamma y^2 \gamma^2}{2} \]

Comparing Eq. (5.17) with (4.9a), (4.7) and (4.9b) one can deduce the following relations:

\[ \frac{\beta}{2} By \left( -w_{32} + w_2 \right) + By w_1' = -By w_{12}' - \frac{\beta}{2} By \left( w_{32}' - w_2 \right) + (2 A + 2 B - 2 B) w_1' \]

\[ + (2 - A) \left( w_{32} + w_0 \right) \quad (a) \]
\[ \begin{align*}
\frac{\partial^2}{\partial x^2} & \left( -w_1' + w_2' \right) + B_y w_2 = -B_y w_2' - \frac{\partial}{\partial x} (w_{12}' - w_{12}) + (2A + 2B_y - 2\gamma) w_0 + (\gamma - A) (w_1' + w_1) \\
\frac{\partial^2}{\partial x^2} & \left( -w_2 + w_{32} \right) + B_y w_{12} = -B_y w_{12}' - \frac{\partial}{\partial x} (w_{23}' - w_{32}) + (2A + 2B_y - 2\gamma) w_1 + (\gamma - A) (w_0 + w_3)
\end{align*} \] (b) (5.26)

and
\[ -\nu \left[ w_{12}' + \frac{\nu}{\beta} (w_2 - w_{32}) - 2w_2 - \frac{\partial}{\partial x} (w_{12}' - w_{12}) + w_{12} + \frac{\nu}{\beta} (w_{32} - w_2) \right] = \frac{\nu}{B_y} \left[ -B_y w_{12}' - \frac{\partial}{\partial x} (w_{32}' - w_{2}') + (2A + 2B_y - 2\gamma) w_1' + (\gamma - A) (w_{1}' + w_1) \right] + (\gamma - A) (w_0 + w_3) \]

From the relations of Eq. (5.17), (5.18), (5.26) and (4.9) finite difference equations (5.25) for general edge point reduces to:

\[ \nabla \nabla w = A (w_{12}' + w_2') - (2A + 2B_y) w_2' + \frac{\partial}{\partial x} (w_1' + w_1 + w_{12}' - w_{12}) + B_y (w_0 + w_{4'}) - \frac{\partial}{\partial x} \left\{ A (w_{2}' + w_{32}') - (2A + 2B_y) w_{12}' \right\} + \frac{\partial}{\partial x} \left\{ B_y (w_0 + w_3 + w_{4'} - w_{32}') + B_y (w_1 + w_{34'}) - A (w_{32}' + w_{2}') \right\} + (2A + 2B_y) w_{12}' - \frac{\partial}{\partial x} \left\{ \alpha \frac{\nu}{2} w_1 + w_0 \right\} + \alpha \frac{\nu}{2} (w_0 - 2w_1 + w_3) - \frac{\nu}{\beta} \left\{ w_{12}' - \frac{\nu}{\beta} (w_{12}' - w_{12}) \right\} + \frac{\nu}{\beta} \left\{ w_{12}' - \frac{\nu}{\beta} (w_{32}' - w_2') \right\} - \frac{\nu}{2B_y} \left\{ -B_y w_{12}' - \frac{\partial}{\partial x} (w_{32}' - w_{2}') + (2A + 2B_y - 2\gamma) w_1' \right\} \]
Separating the coefficient of deflections associated with different nodal points Eq. (5.27) can be presented as in Fig. 15(d).

### Fig. 15(d)

Finite difference Equation for general free edge point.

#### 5.8 Edge point near the Acute Corner:

![Figure 16(a)](image-url)
From the boundary conditions of Sec. 5.2 and 5.7 it follows:

\[ w_1' = w_2' = w_3' = 0 \]

\[ w_2' = \frac{K_y \beta}{\delta} (w_2' - w_0) - \frac{K_y \omega_i^2}{\delta} - w_0 \]
\[ w_3' = \frac{K_y \beta}{\delta} (w_2' - w_0) - \frac{K_y \omega_i^2}{\delta} - w_4' \]
\[ w_3'' = \frac{K_y \beta}{\delta} (w_4' - w_0) - w_2' \]
\[ w_2'' = \frac{K_y \beta}{\delta} (w_4' - w_0) - w_2 \]

and from Eq. (5.17a) and the relation \( w_3' = -w_2 + w_3' + w_2' \) one can deduce:

\[ \beta w_2' + w_2' - \left( \frac{\gamma - A}{B_y} \right) w_3' = \beta w_2' + \frac{\gamma - A}{B_y} w_0 \]

(5.29)

Comparing (5.28a) and (5.29) it can be shown that

\[ w_3' = -w_0 \]

Putting these values into Equation (5.27) for general edge point, equation for edge point near the Acute Corner is obtained as follows:

\[ \nabla \nabla w = A (w_1') - (2A + 2B_y) w_2' + \frac{\beta}{2} B_y (w_1 - w_1') + B_y (w_0 + w_4') \]

\[ + \left[ \beta \left( w_2' + w_3' \right) - (2A + 2B_y) w_1' + \frac{\beta}{2} B_y (-w_0 + w_3 + w_4' - w_3') \right] \]

\[ + \left[ \frac{\beta}{2} \left( w_1 + w_4' \right) - A \left\{ \frac{K_y \beta}{\delta} (w_4' - w_0) \right\} - \frac{\beta}{2} B_y (2w_0 - 2w_4') \right] \]

\[ + \frac{\beta}{2} \left( w_0 - 2w_1 + w_3 \right) - A (1 + \omega) (-2w_0 + w_1) \]

\[ + \left( \frac{K_y \beta}{\delta} \right) \left\{ w_2' - \frac{K_y \beta}{\delta} (w_4' - w_0) + w_2' \right\} - \frac{\gamma}{2B_y} \left[ \frac{\beta}{2} B_y \left\{ \frac{K_y \beta}{\delta} (w_4' - w_0) - w_4' \right\} \right] \]

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Separating the coefficients associated with different nodal points Eq. (5.30) can be presented as in Fig. 16(b)

Fig. 16(b)

Finite difference equation
for edge Point near the Acute Corner
5.9 Edge Point near the Obtuse Corner:

From the boundary conditions of Sec. 5.2 and 5.7 it follows:

\[
\begin{align*}
W_1 &= w_{12}' - w_{14}' = 0 \\
W_3 &= \frac{K_y \beta}{\delta} (w_2 - w_2') - \frac{K_y}{\delta} w_{12} - w_0 \quad (a) \\
W_{34}' &= \frac{K_y \beta}{\delta} (w_o - w_4') - \frac{K_y}{\delta} w_{12} - w_0' \quad (b) \\
W_{32}' &= \frac{K_y \beta}{\delta} (w_o - w_4') - w_2' \quad (c) \\
W_{32} &= \frac{K_y \beta}{\delta} (w_0 - w_4') - w_2 \quad (d)
\end{align*}
\]

and from the Eq. (5.17c) and the relation \( W_{32} = -w_2 + w_{32}' + w_2' \) one can deduce:

\[
-\beta w_2 + w_{12} = \left( \frac{y - A}{By} \right) w_3 = -\beta w_2' + \left( \frac{y - A}{By} \right) w_0 \quad (5.32)
\]

Comparing (5.31a) and (5.32) it can be shown that

\[
W_3 = -w_0
\]
Putting these values into Eq. (5.27) for general edge point equation for edge point near the obtuse corner is obtained as follows:

$$\nabla \nabla w = A(w'_2) - (2A + 2By)w'_2 + \frac{\beta}{2} By (-w'_1 + w'_4) + By (w_0 + w_4')$$

$$- \frac{\beta}{2} \left[ A \left( w'_2 + \frac{\beta}{2} By (w_0 - w_4') - w'_2 \right) + \frac{\beta}{2} \left( -2w'_0 + 2w_4' \right) - A \left( w'_3 + w'_4 \right) \right] + (2A + 2By)w'_2 - \frac{\beta}{2} By (-w'_3 + w_0 + w'_4' - w'_4) - By (w'_1 + w'_4)$$

$$+ \frac{C}{2} \left( w'_3 - 2w'_1 + w_0 \right) - \gamma (1 + \alpha) (w'_1 - 2w_0)$$

$$+ \frac{C}{2} \left( w'_1 - \frac{\beta}{2} \left( w'_2 - w'_3 \right) \right) - \gamma \left( w'_2 - \frac{1}{2} \beta (-w'_1) \right)$$

$$+ \frac{C}{2} \left( w'_2 - \frac{1}{2} \beta (w'_0 - w'_4') - 2w_2 \right)$$

$$- \frac{C}{2} By \left[ -By w'_1 + \frac{\beta}{2} By (w_0 - w'_4) + (2A + 2By - 2\gamma)w'_1 + (2A - 2By - 2\gamma)w_0 + (2A - 2By)w'_2 \right]$$

$$- 2 \gamma \left( -By (w'_2) - \frac{\beta}{2} By (w'_1) \right) - (2A + 2By - 2\gamma)w'_3 + (2A - 2By)w'_4$$

$$- \frac{\beta}{2} By \left( w'_2 - \frac{\beta}{2} By (w'_0 - w'_4) + w'_1 \right)$$

$$- \frac{1}{2} \gamma w'_3 (w'_3 - 4w'_1 + 6w_0 - w_0) = \frac{C}{2} \gamma (A - \gamma)$$

Separating the coefficients associated with different nodal points Eq. (5.31) can be presented as in Fig. 17(b)

![Fig. 17(b)](image)

Finite difference equation for edge point near the obtuse corners.

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In these equations the quantity \( \tilde{g}_0 \) is the equivalent combined effects in terms of load per unit area for all the loads which act upon at point 0.

If a uniformly distributed load of \( q_0 \) per unit area, a line load of \( q_0 \) per unit of the length in \( x \) direction and a concentrated load \( P_0 \) act at point 0, \( \tilde{g}_0 \) is given by:

\[
\tilde{g}_0 = q_0 + q_0 \frac{P_0}{\gamma y} + \frac{P_0}{\gamma x \gamma y}
\]

If point 0 lies on an exterior edge of the plate, \( \tilde{g}_0 \) is given by

\[
\tilde{g}_0 = q_0 + \frac{q_0}{\gamma y} + \frac{P_0}{\gamma x \gamma y}
\]

5.10 Finite Difference Equations for Moments:

Finite difference equations for moments \( M_x, M_y \) and \( M_{xy} \) can be derived by substituting the finite difference approximations Eq. (4.6a), (4.6b) into appropriate moment equations (1.9). Their final expressions are the same as in Ref. [7] except for equation (5.38). However, complete derivations of moment equations are presented in this Section.

(a) General Interior Point:

\[
M_x = -\beta_x \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)_0 + \beta_x \frac{\partial^2 w}{\partial y^2} \right]
\]

\[
= -\frac{\beta_x}{\gamma y^2} \left[ \left( w_{1''} - 2w_0 + w_1 \right) + \beta_x \left( \beta_x^2 (w_{1'} + w_1) - \left( 2 + \beta_x^2 \right) w_0 \right) \right. \\
\left. + \frac{\beta_x}{2} \left( -w_{1''} + w_{1''} + w_{1'2''} - w_{12''} + w_{12'2} + w_{1'} + w_{2'} \right) \right] (5.32)
\]
Similarly:

\[ M_y = -\frac{E_y}{\gamma y^2} \left\{ \beta^2 (w_i' + w_i) - (z^2 + 2\beta^2) w_0 + \frac{\beta}{2} (-w_{i2} + w_{i2} + w_{i2}' - w_{i2}') \right\} + w_{i2}' + w_{i2} \right\} + \mu_x x^2 (w_i' - w_0 + w_i) \right\} 
\tag{5.33} \]

and

\[ M_{xy} = -2C \left[ \frac{\partial w}{\partial u} \tan \phi + \frac{\partial w}{\partial u} \sec \phi \right] 
= -\frac{2Cx^2}{\gamma y^2} \left[ \beta (w_i' - 2w_0 + w_i) + \frac{1}{4} (w_{i2} - w_{i2} + w_{i2}') \right. \left. \left( w_0 - w_{i2}' \right) \right] \tag{5.34} \]

These moment equations can be presented as in Fig. 18(a), 18(b) and (18c), respectively.

---

**Fig. 18(a)**

Finite difference equation

for \( M_x \) at general interior Point

---

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Fig. 18(b)

Finite difference equation

for $M_y$ at general interior point

$$
M_y = -\frac{B_y}{\gamma_y^2} x \left( \beta^2 + \mu x x^2 \right) \frac{w(i')}{w(0)} - 2 \left( \beta \mu x + \lambda x x^2 \right) \frac{w(0)}{w(i)} \frac{w(i)}{w(1)}
$$

Fig. 18(c)

Finite difference equation

for $M_{xy}$ at general interior point

$$
M_{xy} = -\frac{2 c x}{\gamma_y^2} x \left( \beta \frac{w(i')}{w(0)} - 2 \beta \frac{w(0)}{w(i)} \right)
$$
(b) Points on the left simple support:

From the boundary conditions as derived in Sec. 5.2 it follows that:

\[ w_2 = w_0 = w'_2 = 0 \]  
\[ M_n + M_v = M_x + M_y = 0 \]  

and

\[ \left( \frac{\partial w}{\partial v} \right)_o = \frac{1}{2} \left[ \left( \frac{\partial w}{\partial v} \right)_r + \left( \frac{\partial w}{\partial v} \right)_l \right] = 0 \]

or

\[ \frac{1}{2} \left[ \frac{w_{12} - w_{12}'}{2\gamma_v} + \frac{w_{12} - w_{12}'}{2\gamma_v} \right] = 0 \]

or

\[ w_{12} - w_{12}' = w_{12}' - w_{12} \]

From the relation:

\[ K_x \left( \frac{\partial^2 w}{\partial x^2} \right)_o + K_y \left( \frac{\partial^2 w}{\partial y^2} \right)_o = 0 \]

it follows:

\[ w'_r = -\beta \cdot \frac{K_y}{\delta} (w_{12} - w_{12}') - w_1 \]
Substituting these values of exterior nodal points into moment equations (5.32) and (5.34) one obtains:

\[
M_x = -\frac{B_x}{\lambda_y^2} \left[ \{ \beta \lambda_y - \frac{\beta_y k_y}{\delta} (x^2 + \lambda_y \beta_y^2) \} w_{12} - \{ \beta \lambda_y \} w_{12} \right] \tag{5.36}
\]

\[
M_y = -M_x \tag{5.37}
\]

\[
M_{xy} = -\frac{2c_x}{\lambda_y^2} \left[ (\beta_y k_y + \frac{1}{2}) w_{12} + (\beta_y k_y - \frac{1}{2}) w_{12} \right] \tag{5.38}
\]

Equations (5.36), (5.37) and (5.38) may be presented as in Fig. 19(b), (c) and (d), respectively.

---

**Fig. 19(b)**

Finite difference equation for \(M_x\) on simple support

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Fig. 19(c)

Finite difference equation
for $M_y$ on simple support

Fig. 19(d)

Finite difference equation
for $M_{xy}$ on simple support
(c) **Points on the right simple support:**

In a similar manner equations for moments on Point of right simple support may be derived from the relations:

\[
\begin{align*}
\omega_4 &= \omega_4' = \omega_2 = \omega_0 = \omega_2' = 0 \\
\omega_{i2} - \omega_{i2}' &= \omega_{i2}' - \omega_{i2} \\
\omega_i &= -\frac{Ky^3}{d} (\omega_{i2}' - \omega_{i2}) - \omega_i'
\end{align*}
\]  

\[(5.39)\]

Fig. 20(a)

Point on Right Simple Support
These moment equations can be presented as in Fig. 20(b), and Fig. 20(c), respectively:

\[ M_x = -\frac{B_x}{\nu^2} x \quad w(x') \]

\[ M_y = -\frac{B_y}{\nu^2} y \quad w(y') \]

**Fig. 20(b)**

Finite difference equation

for \( M_x \) and \( M_y \) on right simple support

\[ M_{xy} = -\frac{2c}{\nu^2} x \quad w(x') \]

\[ M_{xy} = -\frac{2c}{\nu^2} y \quad w(y') \]

**Fig. 20(c)**

Finite difference equation

for \( M_{xy} \) on right simple support

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(d) Points on the free edge:

on the free edge $M_y = 0$

Hence $-B_y \left( \frac{\partial^2 w}{\partial y^2} + \mu x \frac{\partial^2 w}{\partial x^2} \right) = 0$

or $\frac{\partial^2 w}{\partial y^2} = -\mu x \frac{\partial^2 w}{\partial x^2}$

but $M_x = -B_x \left( \frac{\partial^2 w}{\partial x^2} + \mu y \frac{\partial^2 w}{\partial y^2} \right)$

$= -B_x (1 - \mu x \mu y) \frac{\partial^2 w}{\partial x^2}$

Hence $(M_x)_o = -B_x \left( 1 - \mu x \mu y \right) \frac{x^2}{\gamma y^2} (w_i + 2w_o + w_1)$ \hspace{1cm} (5.40)

Eq. (5.40) can be presented as in Fig. 21(a)

\[ M_x = -B_x \left( 1 - \mu x \mu y \right) \frac{x^2}{\gamma y^2} \times \frac{1}{w(i')} \frac{-2}{w(o')} \frac{1}{w(i)} \]

Fig. 21(a)

Finite difference equation for $M_x$ on free edge.

For deriving the equation for $M_{xy}$ on free edge the nodal points outside the boundary can be expressed in terms of interior points as follows:

\[ \begin{array}{cccc}
\hline & 1' & 2 & 12 \\
\hline 1' & \hline
\end{array} \]

Fig. 21(b)

Point on free edge.
Since 
\[ \left( \frac{\partial w}{\partial x} \right)_0 = \frac{1}{2} \left[ \left( \frac{\partial w}{\partial x} \right)_2 + \left( \frac{\partial w}{\partial x} \right)_{2'} \right] \]

hence 
\[ \frac{w_i - w_i'}{2\gamma_x} = \frac{1}{2} \left[ \frac{w_{i2} - w_{i2}'}{2\gamma_x} + \frac{w_{i2'} - w_{i2}'}{2\gamma_x} \right] \]

or 
\[ \frac{1}{2} (w_{i2} - w_{i2'}) = w_i - w_i' - \frac{1}{2} (w_{i2'} - w_{i2'}) \quad (5.41) \]

Putting the value of \( w_{i2} \) and \( w_{i2'} \) into equation (5.34) one can deduce:

\[ M_{xy} = -\frac{2 \nu \xi}{\gamma_y^2} \left[ (\beta + \frac{1}{2}) w_i + (\beta - \frac{1}{2}) w_i' - 2\beta w_0 + \frac{1}{2} (w_{i2'} - w_{i2}) \right] \quad (5.42) \]

Eq. (5.42) can be presented as in Fig. 21(c)

\[ M_{xy} = -2 \frac{\nu \xi}{\gamma_y^2} w(i) \quad \frac{1}{2} \quad w(i') \quad \frac{1}{2} \]

\[ w(1) \quad -2\beta \quad w(0) \quad w(2) \]

\[ (\beta + \frac{1}{2}) \quad - \frac{1}{2} \quad (\beta - \frac{1}{2}) \]

Fig. 21(c)

Finite difference equation for \( M_{xy} \) on free edge.

5.11 Application of the method of finite difference:

By superposing the skew network on the equivalent orthotropic skew plate and applying the typical finite difference equations for each network points yield a set of simultaneous equations, the solution of which yields the numerical values of deflections at all network points. By substituting the values of
Deflection in proper moment equations as derived in Sec. 5.10 of this chapter, numerical values of moments acting on the equivalent orthotropic skew plate can be obtained.

Bending moments acting at different sections of longitudinal girder and transverse beams can be computed by integrating the equivalent plate moment over the flange width, according to the formula presented in Chapter Three.
VI

GENERAL SOLUTION AND STUDY OF THE FACTORS
THAT ENTER INTO THE ANALYSIS AND DESIGN
OF GRIDWORK IN SKEW BRIDGES

General Solution:

Finite difference equations as derived in Chapter V can be applied for each of network points (Fig. 22) to yield the force displacement relations in the form:

\[
[A][w] = \{\bar{q}\} \gamma^4
\]

where \(\{\bar{q}\}\) is the column of static load acting at the predetermined set of points and \([w]\) is the column corresponding to the vertical displacement. \([A]\) is the conventional stiffness matrix and is obtained by a few algebraic operations in the following form:

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & - & - & - & A_{1,18} \\
A_{21} & A_{22} & A_{23} & A_{24} & - & - & - & A_{2,18} \\
A_{31} & A_{32} & A_{33} & A_{34} & - & - & - & A_{3,18} \\
A_{41} & A_{42} & A_{43} & A_{44} & - & - & - & A_{4,18} \\
& & & & & & & \\
& & & & & & & \\
A_{18,1} & A_{18,2} & A_{18,3} & A_{18,4} & - & - & - & A_{18,18}
\end{bmatrix}
\]
To clarify the application, some of the equations for two typical network points are shown in the appendix.

By inverting the stiffness matrix \([A]\) and carrying out the multiplication with the load vector \([\mathbf{f}]\), deflections at different network points can be obtained. By substituting these values in proper moment equation as described in Sec. 5.11, numerical values of moments acting on the equivalent plate can be easily computed.

Computation of longitudinal and cross beam moments can be performed as outlined in Chapter III.

The finite difference solution for the gridwork with slab representing the modal skew bridge has been obtained. The deck slab is assumed to be an isotropic plate having dimension 36" x 30" x \(\frac{3}{16}\)" (Fig. 22) and stiffened by longitudinal and cross beams of different stiffnesses in both flexure and torsion. The material of the plate was hot rolled structural steel having Young's modulus of elasticity \(E = 30 \times 10^6\) psi and Poisson's ratio \(\nu = 0.3\).

The following are the loading conditions for which the solution has been obtained.

(a) A single concentrated load \(P\), located at centre.

(b) Two equal concentrated loads each of magnitude \(P/2\), located symmetrically along the longitudinal axis \((-\frac{L_x}{6},0)\) and \((\frac{L_x}{6},0)\).
(c) Two equal concentrated load each of magnitude $P/2$ located symmetrically along the transverse axis $(O_x-L/6)$ and $(O_x+L/6)$

Since the loading conditions and plate geometry are symmetrical about the central axis, only one half of the plate has been considered.

Factors that enter into the analysis and design of gridwork in skew bridges are: (a) Number of girders and diaphragms, their spacing and stiffness ratio, in both flexure and torsion; (b) Aspect ratio $(L_x/L_y)$ of the bridge where $L_x$ is the skew span between the supports parallel to the roadway and $L_y$ is the width of the bridge and (c) skew angle.

Theoretical solutions for the different factors influencing the analysis and design of gridwork in skew bridges have been presented graphically (Figs. 3.1 to 3.29). In varying the number of longitudinal and cross beams the total cross-sectional area in the two directions has been kept constant so that the cost per linear distance of the span length does not change appreciably, (Figs. 3.1 to 3.4).

Aspect ratio which is treated as one of the variables was varied from 1, 1.25, 1.5, 1.75 to 2.0 and its effects on deflections and intensity of moments are presented in Figs. 3.5 to 3.15.
The change of skew angle from 0, 15, 30, 45 to 60 degrees is treated as another variable and their effects on deflections and moments are presented in Figs. 3.16 to 3.23.

A disputable question in the value of Poisson's ratio which is not a material constant as Poisson's ratio proper but an elastic constant depending on the orthotropic form of the system has been treated as another variable. The value of \( \nu \) was varied from 0, 0.10, 0.15, 0.20, 0.25, 0.30 and 0.33 and their effects are shown in Figs. 3.24 to 3.29.

Graphical representation showing the influence of these different factors upon deflections and moment intensity \( M_x, M_y, M_{xy} \) and equivalent principal moments, have been presented for some typical node points, e.g. central point of the bridge, interior point near the obtuse corner, interior point near the acute corner, and central point on the free edge. In all cases the solutions have been obtained in the form of influence coefficients of deflections and moments for a unit central point loading. Influence coefficients have also been obtained for uniformly distributed loading, two point loadings and for aspect ratio at a different skew angle. The plate geometry is the same as for the model skew bridge. (Fig. 22).
VII

EXPERIMENTAL VERIFICATION OF THE THEORY

7.1 Description of the Model Bridge:

The gridwork and the deck plate of the model skew bridge were fabricated from a 36" (skew length) x 30" x 3/16" thick plate, 7 nos. of longitudinal beams (3/16" x 2") and 7 nos. of cross beams (3/16" x 1½") - all made of hot rolled structural steel. The two sets of intersecting flexural members forming the gridwork were welded intermittently to one side of the deck plate to form a skew mesh as shown in Fig. 22. In spite of cooling the model with cold water during the welding process, a considerable local warping resulted from the intense heat of welding. In order to have a flattened surface and to reduce the locked-in stresses the bridge model was annealed. Four flat bars of the same material were also subjected to the same heat treatment. They were tested in the universal testing machine to evaluate the modulus of elasticity E and Poisson's ratio under uniaxial tensile test, the average value of which were found to be 30 x 10^6 psi and 0.3, respectively. In order to have a simple line support along the two edges, the gap between the longitudinal beam and cross beam was filled up with pieces of ½" x ½" x 6½" long square bars by spot welding with the cross beams (Fig. 1.1).
7.2 Abutment Frame:

The model bridge was simply supported on the two specially machined 1" diameter steel rods resting on the two opposite edges of the abutment frame. The frame was fabricated from two \( \frac{1}{2} '' \times 3 \frac{1}{2} '' \times 15 '' \) deep channels which were rigidly connected to each other by two \( \frac{1}{4} '' \times 3 '' \times 8 '' \) deep I-beams welded to the frame -6" below the bridge deck level. The frame was in turn supported on six-standard steel blocks resting on a flat steel base (Fig. 1.2).

7.3 Loading Device:

The model skew grillage bridge was tested within the elastic range under the following types of loadings:

1. Concentrated load at the center:

The load was simulated concentrically on the central point of the bridge deck plate through a Thawing-Albert load cell which was attached to a hydraulic ram mounted under the beam of the testing structure. The load cell was calibrated by recording increments of strains corresponding to the direct load increments with a Budd portable type strain indicator (Model P-350) and a PCA - 300,000 lb. testing machine. (Calibration curve Fig. 2.24). A skew steel block (3\( \frac{1}{2} '' \) x 3\( \frac{1}{2} '' \) - 1" thick) with a groove underneath to accomodate the strain gage was placed below the load cell for transferring the load to the deck plate. (Fig. 1.4).
2. **Two equal concentrated loads applied on the longitudinal axis** $(\ell L, 0)$ and $(L, 0)$.

For this type of loading a Strainsert 100,000 lb. flat load cell was used which was calibrated with the same Budd portable type strain indicator and the PCA - 300,000 lb. testing machine. (Calibration curve Fig. 2.25). The load was simulated by using the same hydraulic ram as in Case 1 on a 4" x 4" - 2' ft. long solid bar resting symmetrically on two skew steel blocks dividing the central load into two equal concentrated loads. (Fig. 1.5).

3. **Two equal concentrated loads applied on the transverse axis** $(\ell L, 0)$ and $(0, -\ell L)$.

For this type of loading the same loading device as in Case 2 was used except that the two skew steel blocks were placed symmetrically on the transverse axis of the bridge model along the V-direction. (Fig. 1.6).

7.4 **Testing Procedure and Recording of Data:**

The PCA - 300,000 lbs. testing machine was used to simulate the load through the hydraulic ram in all the three cases.

Deflections at different points of intersections of gridwork were recorded by dial gauges (Mercer Dial gauges, accuracy $10^{-3}$ in.). To measure the strains, electrical resistance strain gauges were used. Strain rosettes (Type EA-06-125 RA - 120) were mounted on the
top face of the deck plate and linear gauges (Type EA - 06-062 AK - 120) were mounted on the bottom face of the longitudinal beams. Fig. 23 shows the location of the strain gages on both loaded and unloaded side of the model bridge. A Datron Digital strain indicator together with a switch and balance unit, a Datron polarity transposer and printer control unit was used to record the strains. (Fig. 1.7).

The results of the experimental tests under different types of loading conditions are presented in the form of figures, comparing the experimental and theoretical values at various points of gridwork of the skew model bridge. (Figs. 2.1 to 2.23).
BOTTOM VIEW

LONGITUDINAL RIBS (2" x 3/16")

LAYOUT OF NETWORK POINTS

1/4 fillet welding
1/2 x 1/2 square bar for simple line support along the edge

SEC. A - A

FIG. 22 GRIDWORK IN MODEL SKEW BRIDGE

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LAYOUT OF STRAIN ROSETTES ON LOADED SIDE

FIG. 23 LAYOUT OF LINEAR STRAIN GAGES ON THE UNLOADED SIDE
8.1 Discussions of Experimental and Theoretical Results:

For a single point loading on the midspan of central longitudinal girder, comparison of Figs. 2.1 to 2.4 shows that experimental values of deflections are higher (14% maximum) than the theoretical values. For the same type of loading comparison of girder bending moments calculated from the measured strain gauge readings (Figs. 2.5 to 2.11) shows that the moments are at some point 12% higher (Fig. 2.5) than the theoretical values and at some points 11% (Fig. 2.10) lower than the theoretical ones.

For two point loadings on the longitudinal and transverse axes, the experimental values of deflections are observed to be 13% (maximum) and 14% (maximum) higher than the theoretical ones (Figs. 2.12 and 2.19). For the same type of loadings experimental values of girder moments were within ±12% (maximum) of the theoretical values (Fig. 2.21).

A comparison of the normal stress distribution along the depth of the longitudinal beam (Fig. 2.23) shows that experimental values are 13% (maximum) higher than the theoretical ones.

Both experimental and theoretical values of deflections and moments are observed to have higher magnitude towards the
obtuse corner than those at the acute corner (Figs. 2.2 and 2.6). On the average, experimental values are observed to be higher than the theoretical solutions.

The deviation of the theoretical solutions from the experimental results may be attributed to the following:

(a) Effect of discontinuity on deflections and stresses:
Since the spacing of gridworks, the rigidity of which is assumed to be continuously distributed for the substitute orthotropic plate, is larger than the dimension of the applied load [23], which is usually the case in bridge deck design using concentrated wheel load, the discontinuity of steel plate deck system is of consequence in determinations of deflections of the system and bending moments and stresses of the individual members. The effect of the actual discontinuity could be considered by taking an effective width in computing rigidities of the equivalent system. Hence, it may be inferred that higher values of rigidities in the stiffness matrix, equation (3.1), result in lower theoretical deflections and consequently lower values of moment than the experimental results.

In order to examine the effect of effective width on the theoretical solutions of the problem, 90% of the rib spacing in the longitudinal direction of the bridge was considered instead of the full flange width of the original programme [23]. The results for deflections and moments were found to be only 2% higher than the original solutions. This, however, justifies the introduction of the concept of
an effective width in the solution of gridwork problems, the value of which depends on the span of ribs and load distribution on the deck floor. Ref. [23] gives an excellent treatment of the effective width of deck plate in orthotropic steel deck bridges. The same values of effective width as applied in the orthogonal bridge structures may be assumed to be valid for the skewed configuration of this structural system.

(b) Effect of unsatisfied boundary conditions at the free edge and imposition of constraints near the simple support:

The boundary conditions at a free edge were first expressed by Poisson as:

\[ M_y = 0, \quad M_{yx} = 0, \quad Q_y = 0 \]

But due to the nature of the fourth order differential equation governing the behaviour of the plate system, which is based on the small deflection theory, only two boundary conditions are possible at the free edge, and later on Kirchoff proved that the last two conditions concerning the twisting moment and shear force could be combined into one single condition in the form of an edge force expression as in equation (1.9g). These two boundary conditions, \( M_y = 0 \) and \( V_y = 0 \), which have been utilized in the formulation of finite difference equations for typical network points on the free edge, give rise to a value of \( M_{xy} \) at the free edge (Eq. 5.42). Similar expressions for \( M_{xy} \) at the free edge have been presented in
Ref. [4, 7, 17]. This violates the Poisson's original boundary conditions at the free edge and will have some effect in the deviation of experimental values from the theoretical solutions. Similarly, the constraints imposed near the simple support as discussed in Section 5.2 may be of some consequence in the determination of deflections and bending moments of the structural system.

(c) Effect of experimental and constructional inaccuracies of the model bridge:

Instead of cooling the model bridge with cold water, the substantial warping resulting from the intense heat of welding, could not be totally balanced by subsequent annealing of the system. The model had an uneven initial curvature in its neutral plane in both longitudinal and transverse directions. During the welding process and fitting of the ribs with deck plates, a great deal of residual stresses might have also been induced in the structure. The strain gage readings on which bending moments and stresses have been computed could give lower values than the theoretical ones because of these unbalanced locked-in stresses at some points (Figs. 2.5, 2.16, 2.21).

Due to the warping and non uniformity of the simple support condition along the edge because of shimming, the slightly unsymmetrical deflections of the model bridge under the symmetrical loading is one of the major causes of deviation between the experimental and theoretical results.
8.2 Discussions of Factors that influence the Analysis and Design of the Gridwork in Skew Bridges:

(a) Number of girders and diaphragms with different spacing and stiffness ratio:

Figs. 3.1 to 3.4 show the effects of variation of the number of longitudinal and transverse beams on deflections and moments of the gridwork system under a unit concentrated load at centre. As would be expected, larger errors can occur for decks with smaller number of girders (Fig. 3.1), since the assumption of the uniform spread medium will not be satisfied in that case. A system with seven number of girders seems to approach an optimum design as far as deflections and bending moments are concerned (Figs. 3.2, 3.3, and 3.4). Negative moment $M_x$ (Fig. 3.2) at the simple support indicates the effect of skew and the restraint imposed by the cross diaphragms on the free deflection surface of the central longitudinal girder.

In actual practice the road width, the skew span of the bridge and the type of highway loadings to which the bridge is anticipated to be subjected should determine the spacing of the girders and diaphragms to obtain an economical design. For this purpose several trial programmes can be run on a computer with different number of girders and diaphragms having different spacings and stiffness ratios in both flexure and torsion.

(b) Aspect ratio of the bridge:

The effects of aspect ratio on deflections and moments
of the gridwork in skew bridges are shown in Figs. 3.5 to 3.15. Figs. 3.5 to 3.9 show the variation of deflections and moments with aspect ratio for a skew angle of 45°, under a central concentrated load. Deflections of network points sharply decrease with decrease in aspect ratio (Fig. 3.5). While at this angle of skew, longitudinal girder moment $M_x$ decreases slowly with decrease of aspect ratio, transverse moment $M_y$ at a point near the acute corner tends to increase sharply with decrease of aspect ratio (Fig. 3.7). The twisting moment $M_{xy}$ and the principal moment of the equivalent system decrease at all points with decrease of aspect ratio (Fig. 3.8 and 3.9).

Examination of Figs. 3.10 and 3.11 shows that with a large skew angle of 60°, the variation of deflections and moments $M_x$ with aspect ratio are more rapid than the case with a 45° angle of skew under the same central concentrated load. Variations of deflections and moment $M_x$ for two point loading on the transverse axis and uniformly distributed load over the whole bridge deck are shown in Figs. 3.12 to 3.15. The variation appears to be similar in the two cases for the same angle of skew.

An aspect ratio in between 1 to 1.5 appears to be desirable, although in practice, the anticipated traffic density will determine the width of the skew bridge and hence the aspect ratio.
(c) **Skew Angle:**

Figures 3.16 to 3.23 show the effect of variations of skew angle on deflections and moments of the skew grillage system. With increase of skew angle, under a concentrated load at the centre, deflections of central nodal point decrease sharply after the skew angle has exceeded 30°. Up to 30 degrees, variations of deflections with the angle of skew is not very appreciable. Similar is the variation of longitudinal moment Mx with the angle of skew (Fig. 3.17). A slight increase in the value of deflections and moment (Figs. 3.16 and 3.17) is observed at the mid point of free edge girder with an increase of skew angle up to 30°.

Fig. 3.18 shows that after the skew angle has exceeded 30°, the transverse moment My at central interior point increases slowly up to a skew angle of 45° and tends to decrease slowly beyond this value. While My at the interior acute corner decreases appreciably, it increases sharply at the interior obtuse corner with further increase of skew angle. This points to stress concentration near the obtuse corner with increase of skew angle.

The influence of increasing angle of skew for two point loadings and uniformly distributed load on the deflections and moment Mx are shown in Figs. 3.20 to 3.23.

From this study it can be justified that up to a skew angle of 22°, the grillage in skew bridges can be analyzed in the same way as the right girder bridge by the method of lateral distribution and distribution coefficients [3].
(d) Poisson's Ratio:

The value of $\mu x$ was varied from 0.0 to 0.33 to show its influence on deflections and moments of the structural system (Figs. 3.24 to 3.29). Under a concentrated load acting at the centre, a change in the value of $\mu x$ from 0.0 to 0.33 resulted in a decrease of deflection up to 15% at central point, 3% at the mid point of free edge, 11% at the interior point near the obtuse corner and 8% at the interior point near the acute corner, respectively (Fig. 3.24). The influence of Poisson's ratio on moment $Mx$ (Fig. 3.25) for a concentrated load at centre appears to be appreciable. An increase of 8% and a decrease of 7% in the value of $Mx$ were observed for central interior point and mid point on the free edge for an increase in the value of $\mu x$ from 0.0 to 0.33 (Fig. 3.25). While a change of Poisson's ratio does not have an appreciable effect on moment $My$ at interior central point (Fig. 3.26), a sharp decrease of $My$ was observed at the interior point near the acute corner.

Figures 3.28 and 3.29 show the influence of Poisson's ratio on deflections and moments $Mx$ at different points on the bridge under uniformly distributed load. At the mid point of the free edge deflections increased by 4% for an increase of Poisson's ratio from 0 to 0.33 whereas deflections at the central point decreased by 10%. For the same variation of $\mu x$, the interior acute corner deflection increased by 4% and the interior obtuse
corner deflections decreased by 3%. Under the same uniform loading, though the moment Mx does not change appreciably at the central point and interior point near the acute corner, an increase of 5% in the value of Mx at the mid point of edge girder was observed for the same variation of dx (Fig. 3.29).

From this study it is clear that a proper value of dx which is not a material constant as Poisson's ratio proper but an elastic constant corresponding to the form of the system (since the value of dy is determined from the relation dy = (E/dx) dx should be incorporated in the analysis of gridwork with deck slab to yield a more accurate solution.

8.3 Conclusions:

From the investigation of this problem it has been shown that the theory of orthotropic plate can be effectively used in the analysis of gridwork in skew bridges.

The difficulties encountered in satisfying the boundary conditions along the simply supported edge and free edge of the structural system (the imposition of constraint w_4 + w_4 - 2w_0 = 0 near the simple support may not always be true depending on the loading condition) are not of a serious nature, since in actual practice extra reinforcement would be provided along the simply supported edge to take the support reactions and the free edge would normally have some form of footpath which would prevent heavy loads to
be applied at the free edge of the skew bridge.

By simply changing the data cards in the general computer programme, all the variable parameters such as number of girders and diaphragms, their spacing and stiffness ratios and aspect ratio for a particular alignment of a skew bridge can be examined to obtain an optimum design.

Due to simplicity in application and quite a good degree of accuracy in results, orthotropic plate analysis of gridwork in skew bridges by the method of finite differences, may be used as a powerful design tool.

8.4 Suggestions for further Research:

Though the elastic analysis dominates the field of bridge engineering, it is now generally acceptable that the understanding of any structure is incomplete unless its behaviour beyond the elastic range is fully investigated. Hence the elasto-plastic and plastic behaviour of gridwork in skew bridges may be of considerable interest and practical value for economical design purposes.

Since the ultimate strength design presupposes that the governing forces acting in the structure due to dead load and live load are calculated by elastic theory [6], it is recognized at this point that further research beyond the elastic range should be carried out to have a clear insight of the structural behaviour of gridworks in skew bridges.


FIG. 1.1 MODEL BRIDGE

FIG. 1.2 ABUTMENT FRAME
Fig. 1.3 Calibration of Load Cell

Fig. 1.4 Loading Device for a Concentrated Load at Centre.
Fig. 1.5 Loading Device for Two Point Loading on the Longitudinal Axis

Fig. 1.6 Loading Device for Two Point Loading on the Transverse Axis
Fig. 1.7 Digital Strain Indicator
FIGURES

COMPARISON OF EXPERIMENTAL
AND THEORETICAL RESULTS
Concentrated Load
4354 lbs. at Point 18
Ref. Fig. 22

Variations of Longitudinal Central Girder Deflections with Span Length.

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Concentrated Load
4354 lbs. at Point 18

Ref. Fig. 22

Variations of Longitudinal Edge Girder Deflections with Span Length.

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Fig. 2.3

Deflection at point 15

For concentrated load at centre

See Fig. 22 for nodal point locations

$\text{DEFLECTION} \times 10^3 \text{ in inches}$

$\text{LOAD IN LBS.}$

Experiment

Theory

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FIG. 2.4
DEFLECTIONS OF INTERIOR OBTUSE AND ACUTE CORNERS FOR CONCENTRATED LOAD AT CENTRE.

Experimental Value
▼ Point 10
○ Point 6
— Theory (Pt. 10)
— Theory (Pt. 6)

See Fig. 22 for nodal point locations.
Concentrated Load
4354 lbs. at Centre.
Ref. Fig. 22

Variations of longitudinal central girder moment $M_x$ with span length.

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Concentrated Load
4354 lbs. at Centre.
Ref. Fig. 22

FIG. 2.6

VARIATIONS OF LONGITUDINAL EDGE GIRDER MOMENT $M_x$ WITH SPAN LENGTH.
FIG. 2.7

VARIATION OF TRANSVERSE CROSS BEAM MOMENT $M_y$ WITH SKEW WIDTH.

Concentrated Load
4354 lbs. at Centre.
Ref. Fig. 22

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See Fig. 22 for nodal point location.

---

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See Fig. 22 for nodal point locations.

Fig. 2.9

$M_x \times 10^5$ in. $\cdot$ lbs.

$M_x \times 10^5$ at Point 3 for Concentrated Load at Centre.
FIG. 2.10

Mx AT POINT 10 FOR CONCENTRATED
LOAD AT CENTRE.

See Fig. 22 for
nodal point location.

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FIG. 2.11

See Fig. 22 for nodal point location.

Mx AT POINT 16 FOR CONCENTRATED LOAD AT CENTRE.
Two Equal Concentrated
Load each of 4495 lbs.
at Point 17'-17'
Ref. Fig. 22

FIG. 2.12

VARIATIONS OF LONGITUDINAL CENTRAL GIRDEN DEFLECTIONS WITH SPAN LENGTH.
Two equal concentrated load each of 4495 lbs. at Point 17'-17'.' Ref. Fig. 2

VARIATIONS OF LONGITUDINAL EDGE GIRDER DEFLECTIONS WITH SPAN LENGTH.
Two equal concentrated loads at Point 17-17'.
Ref. Fig. 22
Two equal concentrated loads at Point 17-17'
Ref. Fig. 22

FIG. 2.15
DEFLECTIONS OF INTERIOR OBTUSE AND ACUTE CORNER POINTS.

Experimental Value
- Point 6
- Point 10
- Theory (Point 10)
- Theory (Point 6)
Two equal concentrated loads each of 5005 lbs.
at Point 17-17'
Ref. Fig. 22
Two equal concentrated loads at Point 17-17'

Ref. Fig. 22

FIG. 2.17

Mx AT POINT 18 FOR TWO POINT LOADING ON LONGITUDINAL AXIS
Two equal concentrated loads at Point 17-17'
Ref. Fig. 22

FIG. 2.18

Mx AT POINT 3 FOR TWO POINT LOADING ON LONGITUDINAL AXIS
FIG. 2.19

VARIATIONS OF LONGITUDINAL CENTRAL GIRDER DEFLECTIONS WITH SPAN LENGTH.

Two equal concentrated load each of 4495 lbs. at Point 13'-13'
Ref. Fig. 22

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Two equal concentrated load each of 4495 lbs.
at Point 13'-13'
Ref. Fig. 22
Two equal concentrated load each of 5005 lbs.
at Point 13'-13'
Ref. Fig. 22
Two equal concentrated
doal each of 5005 lbs.
at Point 13-13'

**FIG. 2.22.**

**VARIATION OF LONGITUDINAL EDGE GIRDER MOMENT M_x WITH SPAN LENGTH.**

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Concentrated load 4354 lbs. at Centre

Two equal concentrated load each of 5005 lbs. at Point 17-17'

Two equal concentrated load of 5005 lbs. at Point 13-13'

FIG. 2.23
NORMAL STRESS DISTRIBUTION ALONG DEPTH
AT POINT 18

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CALIBRATION CURVE FOR A
STRAININSERT LOAD CELL.
CAPACITY 100,000 LBS.

1 \mu INCH CORRESPONDS TO
25.5 LBS.

STRAIN INDICATOR: BUDD
PORTABLE TYPE.
GAGE FACTOR 2.0

LOAD IN LBS. \times 10^3

Fig. 2.25
FIGURES

THEORETICAL RESULTS
<table>
<thead>
<tr>
<th>No. of Girders</th>
<th>Spacing</th>
<th>$B_{x/y}$</th>
<th>$A_t/A_p$</th>
<th>$\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$L_x/2$</td>
<td>2.021</td>
<td>0.758</td>
<td>0.221</td>
</tr>
<tr>
<td>5</td>
<td>$L_x/4$</td>
<td>1.612</td>
<td>0.715</td>
<td>0.246</td>
</tr>
<tr>
<td>6</td>
<td>$L_x/5$</td>
<td>1.875</td>
<td>0.785</td>
<td>0.235</td>
</tr>
<tr>
<td>7</td>
<td>$L_x/6$</td>
<td>1.880</td>
<td>0.740</td>
<td>0.232</td>
</tr>
<tr>
<td>9</td>
<td>$L_x/8$</td>
<td>1.641</td>
<td>0.556</td>
<td>0.246</td>
</tr>
</tbody>
</table>

Unit Load at Point 18

Ref. Fig. 22

**Fig. 3.1**

Variations of Deflections with No. of Girders
Unit concentrated load at Centre

Ref. Fig. 22

FIG. 3.2 VARIATIONS OF LONGITUDINAL CENTRAL GIRDER MOMENT $M_x$ WITH SPAN LENGTH

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Unit Load at Centre

Ref. Fig. 22.

FIG. 3.3 EDGE GIRDER MOMENT Mx VS. SPAN LENGTH
Fig. 3.4 Variations of My with Skew Width
Unit concentrated load at Centre.
Skew Angle $45^\circ$
Ref. Fig. 22

**Fig. 3.5 Deflection vs. Aspect Ratio**

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Unit concentrated Load at Centre
Skew Angle 45°
Ref. Fig. 22

$\text{M}_x \times 10^2$ in inch-lbs.

\begin{align*}
\text{ASPECT RATIO} & \quad \frac{L_x}{L_y} \\
2.0 & \quad 1.75 \\
1.5 & \quad 1.25 \\
1.0 & \\
\end{align*}

FIG. 5.6 $M_x$ VS. ASPECT RATIO
Unit concentrated
Load at Centre
Skew Angle 45°
Ref. Fig. 22
FIG. 3.8 $M_{xy} \times 10^4$ in. lbs. vs. ASPECT RATIO $L_x/L_y$

- Point 18
- Point 3
- Point 10
- Point 6

Unit concentrated Load at Centre Skew Angle 45° Ref. Fig. 22

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Unit concentrated Load at Centre
Skew Angle 45° Ref. Fig. 22
Point 18
Point 3
Point 10
Point 6

FIG. 3.9 MAXIMUM PRINCIPAL MOMENT B.M. max
VS. ASPECT RATIO

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Unit concentrated Load
at Centre
Skew Angle $60^\circ$
For nodal point location
See Fig. 22

FIG. 3.10 DEFLECTIONS VS. ASPECT RATIO
Unit concentrated Load at Centre
Skew Angle 60°

For nodal point locations
See Fig. 22

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Two point loading on
Transverse Axis at Point 13-13'
Skew Angle 45°
Ref. Fig. 22
Two point Loading on
Transverse Axis at Point 13-13'
Skew Angle 45°
Ref. Fig. 22

FIG. 3.13 Mx VS. ASPECT RATIO

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Unit uniformly distributed load
Skew Angle $45^\circ$
Ref. Fig. 22
FIG. 3.15 Mx VS. ASPECT RATIO

Unit uniformly distributed Load
Skew Angle 45°
Ref. Fig. 22

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FIG. 3.16 DEFLECTIONS VS. ANGLE OF SKEW

Unit concentrated Load at Centre
Ref. Fig. 22

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FIG. 3.17 Mx \times 10^2 \text{ in.-lbf.} vs. Angle of Skew

Unit Load at Centre
Ref. Fig. 22

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Unit concentrated
Load at Centre
Ref. Fig. 22
Fig. 3.19 B.Mmax vs. Angle of Skew

Unit concentrated Load at Centre
Ref. Fig. 22
Two Point Loading on
Transverse Axis at Point 13-13'
Ref. Fig. 22

Fig. 3.20  DEFLECTIONS VS. ANGLE OF SKEW
FIG. 3.21 Mx VS. ANGLE OF SKEW

Two Point Loading at Point 13-13'
Ref. Fig. 22
Unit uniformly distributed
Load
Ref. Fig. 22

FIG. 3.22 DEFLECTIONS VS. ANGLE OF SKEW
FIG. 3.23 $M_x$ VS. ANGLE OF SKEW

Unit uniformly distributed Load

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Unit concentrated load at Centre
Skew Angle 45°
Ref. Fig. 22

FIG. 3.24 DEFLECTIONS VS. POISSON'S RATIO
FIG. 3.25 Mx vs. Poisson's Ratio

- Unit concentrated Load at Centre
- Skew Angle 45°

Mx x 10^2 in. lbs.

Poisson's Ratio

37.5, 35.0, 32.5, 30.0, 17.5, 15.0, 12.5, 10.0, 7.5, 5.0

0.00, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.33

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Unit concentrated
Load at Centre
Skew Angle 45°
Ref. Fig. 22

FIG. 3.26 My VS. POISSON'S RATIO
Unit concentrated Load at Centre
Skew Angle 45°
Ref. Fig. 22

FIG. 3.27 B.Mmax VS. POISSON'S RATIO
Unit uniformly distributed Load
Skew Angle 45°
Ref. Fig. 22

FIG. 3.28  DEFLECTION VS. POISSON'S RATIO
FIG. 3.29 \( M_x \) VS. POISSON'S RATIO

Uniformly distributed Load
Skew Angle 45°
Ref. Fig. 22
A. APPLICATION OF FINITE DIFFERENCE EQUATIONS AT DIFFERENT MESH POINTS

1. At point 18, Eq. (5.1) represented by Fig. 8(b) can be applied as follows:

\[
\begin{align*}
\{2A(3\alpha+2) + B_y(\beta^2 + 4\alpha + 6) + Dx^2(6\beta^2 + 4)\}w_{18} + \{-2B_y(\alpha+2)-2(A + 2px^2)\}w_{13} + \{B_y(1-\beta/2)\}w_{12} + \{-\beta.B_y\}w_{11} + \{\beta^2.B_y/4\}w_6 \\
+ \{(1+\beta)(\alpha.B_y + A + Dx^2) + 2\beta.B_y\}w_{12} + \{-2\alpha(2A+B_y) - 2Dx^2(2\beta^2 + 1) - 2A\}w_{17} + \{(1-\beta)(\alpha.B_y + A + Dx^2) - 2\beta.B_y\}w_{14} + \{-\beta.B_y\}w_{9'} \\
+ \{-\beta/2(\alpha.B_y + A + Dx^2)\}w_{11} + \{\alpha.A + \beta^2(Dx^2 - B_y/2)\}w_{16} \\
+ \{\beta/2(\alpha.B_y + A + Dx^2)\}w_{15'} + \{\beta^2.B_y/4\}w_{10'}
\end{align*}
\]

Model Bridge
ref. fig. 22

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\[
\begin{align*}
+ \{ \beta \, B_y \} \, W_9 + \{ (1-\beta) \, (c \, B_y + A + D \, x^2) \} \, W_{14} \\
+ \{ -2 \, \alpha \, (2 \, A + B_y) \} \, -2 \, D \, x^2 \, (2 \, \beta^2 + 1) \, -2 \, A^2 \} \, W_{17} \\
+ \{ \beta \, B_y \} \, W_{12} + \{ -\beta \, B_y \} \, W_7 + \{ \beta^2 \, B_y \} \, W_{10} \\
+ \{ \beta \, A + D \, x^2 \} \, W_{15} + \{ \alpha \, A + \beta^2 \, (D \, x^2 - B_y) \} \, W_{16} \\
+ \{ -\beta^2 \, A + D \, x^2 \} \, W_{13} + \{ B_y \} \, (1-\beta^2) \} \, W_8 = \{ \frac{P}{\lambda \times A} \} \, \gamma^4 \\
\end{align*}
\]

2. At Point 3, Eq. (5.27) represented by Fig. 15(d)
can be applied to obtain the following deflection equation:

\[
\begin{align*}
\{ \alpha \, y + \frac{y'}{2B_y} (A - y) - \beta \, \frac{B_y}{4} - D \, x^2 \, y \} \, W_1 \\
+ \{ -2 \, \alpha \, x - y - \psi - 2 \, \frac{B_y}{A - y} + 2 \, D \, x \, y \} \, W_2 \\
+ \{ (1+\beta) \, (c \, B_y + A + D \, x^2) \} \, W_7 + \{ -\beta \, B_y \} \, W_7 + \{ \beta^2 \, B_y \} \, W_{10} \\
+ \{ -\beta^2 \, A + D \, x^2 \} \, W_{13} + \{ B_y \} \, (1-\beta^2) \} \, W_8 = \{ \frac{P}{\lambda \times A} \} \, \gamma^4 \\
\end{align*}
\]

The formulation of the stiffness matrix [A] can now be easily performed when the appropriate finite difference equations are applied to each of the mesh points in a similar manner.
### Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>Lx</td>
<td>No. of cross beams</td>
</tr>
<tr>
<td>Ly</td>
<td>No. of longitudinal beams</td>
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<tr>
<td>m</td>
<td>Spacing of the cross beams</td>
</tr>
<tr>
<td>n</td>
<td>Spacing of the longitudinal beams</td>
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<tr>
<td>( \phi )</td>
<td>Thickness of slab</td>
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<tr>
<td>E (_x)</td>
<td>Width of longitudinal beams</td>
</tr>
<tr>
<td>E (_y)</td>
<td>Clear depth of longitudinal beams</td>
</tr>
<tr>
<td>((h_2 - h))</td>
<td>Clear depth of cross beam</td>
</tr>
<tr>
<td>H (_y)</td>
<td>Repeating width of slab in y-direction</td>
</tr>
<tr>
<td>Z (_1)</td>
<td></td>
</tr>
<tr>
<td>b (_2)</td>
<td>Width of cross beam</td>
</tr>
<tr>
<td>F (_x)</td>
<td>Skewed width</td>
</tr>
<tr>
<td>H (_x)</td>
<td>Repeating width of slab in x-direction</td>
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<tr>
<td>Z (_2)</td>
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</tr>
<tr>
<td>((h_2 - h))</td>
<td>Clear depth of cross beam</td>
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<tr>
<td>I (_x)</td>
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<tr>
<td>I (_y)</td>
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<tr>
<td>XNU</td>
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<td>YNU</td>
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### Computer Language

(In Order of Program)

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<th>Symbol</th>
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<tr>
<td>FX</td>
<td></td>
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<tr>
<td>HY</td>
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</tr>
<tr>
<td>Z1</td>
<td></td>
</tr>
<tr>
<td>FV</td>
<td></td>
</tr>
<tr>
<td>FX = (FV/\cos(F))</td>
<td></td>
</tr>
<tr>
<td>HX</td>
<td></td>
</tr>
<tr>
<td>Z2</td>
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<td>ZY</td>
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<td>YNU</td>
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</table>
NOMENCLATURE

\begin{align*}
B_x & \\
B_y & \\
\gamma_y & \\
\gamma_x & \\
\chi & = \gamma_y / \gamma_x \\
\beta & = \chi + \tan \phi \\
F_T & \\
F_P & \\
G_x & \\
G_y & \\
A & = B_y / B_x + B_x \chi^2 \\
H & = (B_x / B_y + B_y / B_x + 4) / 2 \\
\alpha & = B^2 + \chi^2 \\
\gamma & = (B_x - B_y / B_x) \chi^2 \\
D & = 2H - B_x - B_y \\
K_x & = B_x + B_y B_x \\
K_y & = B_y + B_x B_y \\
\delta & = \chi^2 K_x + B^2 K_y \\
\psi & = (2H - B_x B_y - B_x) \chi^2 \\
\Omega & \\
M_x & \\
M_y & \\
M_{xy} & \\
B.M. \ max. & \text{Principal moment of the equivalent plate} \\
B.M. \ min. & \\
\theta & - \text{Angle, the maxm. Principal moment make with x-axis}
\end{align*}
C. FLOW DIAGRAM OF GENERAL COMPUTER PROGRAM

READ INPUT DATA

COMPUTE PARAMETERS
\(B_x, B_y, A_T, A_p, I_x, I_y\)
\(\chi, \beta, \gamma, H, D, C, \text{ etc.}\)

PRINT SECTIONAL PROPERTIES \(Z_1, Z_2, I_x, I_y\)

PRINT RIGIDITY CONSTANTS \(B_x, B_y, A_T, A_p\) AND DIMENSIONLESS PARAMETER \(\Omega\)

COMPUTE ELEMENTS OF STIFFNESS MATRIX

PRINT STIFFNESS MATRIX

CALL SUBROUTINE FOR MATRIX INVERSION

COMPUTE DEFLECTIONS

PRINT DEFLECTIONS

COMPUTE EQUVALENT PLATE PRINCIPAL MOMENTS AND DIRECTIONS

PRINT OUTPUT FOR \(M_x, M_y, M_{xy}, B.M.\max, B.M.\min, \theta\).

ARE RESULTS FOR ANOTHER GRIDWORK IN SKEW BRIDGE REQUIRED?

YES

END

NO
LEVEL 1, 59 54

1. MONDAY, 8:00 A.M. TO 4:00 P.M.

2. TUESDAY, 8:00 A.M. TO 4:00 P.M.

3. WEDNESDAY, 8:00 A.M. TO 4:00 P.M.

4. THURSDAY, 8:00 A.M. TO 4:00 P.M.

5. FRIDAY, 8:00 A.M. TO 4:00 P.M.

6. LEVEL 2, 59 54

7. MONDAY, 8:00 A.M. TO 4:00 P.M.

8. TUESDAY, 8:00 A.M. TO 4:00 P.M.

9. WEDNESDAY, 8:00 A.M. TO 4:00 P.M.

10. THURSDAY, 8:00 A.M. TO 4:00 P.M.

11. FRIDAY, 8:00 A.M. TO 4:00 P.M.

12. LEVEL 3, 59 54

13. MONDAY, 8:00 A.M. TO 4:00 P.M.

14. TUESDAY, 8:00 A.M. TO 4:00 P.M.

15. WEDNESDAY, 8:00 A.M. TO 4:00 P.M.

16. THURSDAY, 8:00 A.M. TO 4:00 P.M.

17. FRIDAY, 8:00 A.M. TO 4:00 P.M.

DATE = 7/112

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LEVEL 1, DAY 1

A(1, 1) = 0
A[1, 4] = 367*2*8/3
A[1, 10] = 2, * 8
A(1, 11) = 367*2*8
A(1, 12) = (2, * 8, * 367*2*8
A[1, 13] = 2, * 8
A(1, 14) = 2, * 8
A[1, 15] = 2, * 8
A(1, 16) = 2, * 8
A[1, 17] = 2, * 8
A(1, 18) = 2, * 8

1 + 2*8

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A(1,2) = A(1,1)

A(1,1) = A(2,1)

A(2,2) = A(2,1)

A(2,1) = A(1,2)

A(1,3) = A(3,1)

A(2,3) = A(3,2)

A(3,1) = A(0,1)

A(3,2) = A(0,2)

A(3,3) = A(1,3)

A(3,4) = A(2,3)

A(3,5) = A(3,4)

A(3,6) = A(3,5)

A(4,1) = A(0,1)

A(4,2) = A(0,2)

A(4,3) = A(0,3)

A(4,4) = A(0,4)

A(4,5) = A(0,5)

A(4,6) = A(0,6)

A(5,1) = A(0,1)

A(5,2) = A(0,2)

A(5,3) = A(0,3)

A(5,4) = A(0,4)

A(5,5) = A(0,5)

A(5,6) = A(0,6)

A(6,1) = A(0,1)

A(6,2) = A(0,2)

A(6,3) = A(0,3)

A(6,4) = A(0,4)

A(6,5) = A(0,5)

A(6,6) = A(0,6)

A(7,1) = A(0,1)

A(7,2) = A(0,2)

A(7,3) = A(0,3)

A(7,4) = A(0,4)

A(7,5) = A(0,5)

A(7,6) = A(0,6)

A(7,7) = A(0,7)

A(8,1) = A(0,1)

A(8,2) = A(0,2)

A(8,3) = A(0,3)

A(8,4) = A(0,4)

A(8,5) = A(0,5)

A(8,6) = A(0,6)

A(8,7) = A(0,7)

A(8,8) = A(0,8)

A(9,1) = A(0,1)

A(9,2) = A(0,2)

A(9,3) = A(0,3)

A(9,4) = A(0,4)

A(9,5) = A(0,5)

A(9,6) = A(0,6)

A(9,7) = A(0,7)

A(9,8) = A(0,8)

A(9,9) = A(0,9)

A(10,1) = A(0,1)

A(10,2) = A(0,2)

A(10,3) = A(0,3)

A(10,4) = A(0,4)

A(10,5) = A(0,5)

A(10,6) = A(0,6)

A(10,7) = A(0,7)

A(10,8) = A(0,8)

A(10,9) = A(0,9)

A(10,10) = A(0,10)

A(11,1) = A(0,1)

A(11,2) = A(0,2)

A(11,3) = A(0,3)

A(11,4) = A(0,4)

A(11,5) = A(0,5)

A(11,6) = A(0,6)

A(11,7) = A(0,7)

A(11,8) = A(0,8)

A(11,9) = A(0,9)

A(11,10) = A(0,10)
21 PRINT 7I, I, W(I)
. 71 FORMAT (1HO, 30X, 2HW, I2, 2H<#E13.6)
PRINT 16, DETERM
16 FORMAT (1HO, 1HDETERMINANT IS/E13.6)
B1=-R X/(AH**2)
A20=(BET*YMU-BET*YK)*(AS**2+BET**2*YMU)/DEL)*B1
A21=(-BET*YMU-BET*YK)*(AS**2+BET**2*YMU)/DEL)*B1

RMX(25)=0.
RMX(24)=A20*W(15)+A21*W(5)
RMX(23)=A20*W(16)+A21*W(10)
RMX(22)=A20*W(11)+A21*W(15)
RMX(21)=A20*W(6)+A21*W(16)
RMX(20)=A20*W(1)+A21*W(11)
RMX(19)=0.
RMX(18)=(-YMU*BET*W(12))/2.*YMU*W(13)+YMU*BET*W(14)
1+2.*((AS**2+BET**2*YMU)*W(17))/2-2.*((AS**2+YMU+BET**2*YMU)*W(18))
RMX(17)=(-YMU*W(11))/2-2.*YMU*W(12)+((AS**2+BET**2*YMU)*W(16)
1+2.*((AS**2+BET**2*YMU)*W(17))/2+2.*((AS**2+BET**2*YMU)*W(18)
2+YMU*W(15)/2.+YMU*W(14))
RMX(16)=(YMU*BET*W(11))/2-2.*((AS**2+YMU+BET**2*YMU)*W(16)
1+2.*((AS**2+YMU+BET**2*YMU)*W(17))
2+YMU*BET*W(18)/2.
RMX(15)=(YMU*BET*W(17))/2.
RMX(14)=(YMU*BET*W(17))/2.
RMX(13)=(-BET*YMU*W(7))/2.+YMU*W(8)+YMU*BET*W(9)/2.
1+((AS**2+BET**2*YMU)*W(12))/2-2.*((AS**2+YMU+BET**2*YMU)*W(13)
2+YMU*BET*W(14)/2.+YMU*W(18)
3-YMU*W(17)
RMX(12)=(-BET*YMU*W(6))/2.+YMU*W(7)+YMU*BET*W(8)/2.
1+(AS**2+BET**2*YMU)*W(11)/2-2.*((AS**2+YMU+BET**2*YMU)*W(12)
2+((AS**2+BET**2*YMU)*W(13))+YMU*BET*W(16)/2.+YMU*W(17)
3-YMU*BET*W(13)/2.)
RMX(11)=(YMU*W(6))/2.+(YMU*BET*W(7)/2.2.*((AS**2+YMU+BET**2*YMU)*W(11)
1+(AS**2+BET**2*YMU)*W(12))/2.+YMU*W(16)
1+((AS**2+BET**2*YMU)*W(17))/2.2.*((AS**2+BET**2*YMU)*W(9)
2+YMU*BET*W(14)/2.+YMU*W(15))
1+((AS**2+BET**2*YMU)*W(15))/2.2.*((AS**2+BET**2*YMU)*W(9)
2+YMU*W(14)/2.-YMU*BET*W(13)/2.+YMU*W(17)
3/2.1*
RMX(9)=(-BET*YMU*W(3))/2.+YMU*W(4)/2.+YMU*W(5)+YMU*BET*W(4)/2.
1+((AS**2+BET**2*YMU)*W(7))/2-2.*((AS**2+YMU+BET**2*YMU)*W(8)
2+((AS**2+BET**2*YMU)*W(9))+YMU*BET*W(12)/2.+YMU*W(13)
3-YMU*BET*W(14)/2.)
RMX(8)=(-BET*YMU*W(1))/2.+YMU*W(2)/2.+YMU*W(3)/2.
1+((AS**2+BET**2*YMU)*W(6))/2-2.*((AS**2+YMU+BET**2*YMU)*W(7)
2+((AS**2+BET**2*YMU)*W(8))+YMU*BET*W(11)/2.+YMU*W(12)
3-YMU*BET*W(13)/2.1)
RMX(7)=(YMU*W(1))/2.+YMU*BET*W(2)/2.-2.*((AS**2+YMU+BET**2*YMU)*W(6)
1+((AS**2+BET**2*YMU)*W(7))+YMU*BET*W(11)/2.+YMU*W(12)
3-YMU*BET*W(13)/2.1)\nRMX(6)=(YMU*W(1))/2.+YMU*BET*W(2)/2.-2.*((AS**2+YMU+BET**2*YMU)*W(6)
1+((AS**2+BET**2*YMU)*W(7))+YMU*BET*W(11)/2.+YMU*W(12)
3-YMU*BET*W(13)/2.1)\nRMX(5)=(YMU*W(1))/2.+YMU*BET*W(2)/2.-2.*((AS**2+YMU+BET**2*YMU)*W(6)
1+((AS**2+BET**2*YMU)*W(7))+YMU*BET*W(11)/2.+YMU*W(12)
3-YMX*W(13)/2.)
RMX(5) = A11 * (W(4) - 2 * W(5))
RMX(4) = A11 * (W(3) - 2 * W(4) + W(5))
RMX(3) = A11 * (W(2) - 2 * W(3) + W(4))
RMX(2) = A11 * (W(1) - 2 * W(2) + W(3))

RMX(1) = A11 * W(1) - 2 * W(2) + W(3))

DO 22 I = 1, 25

22 FORMAT (IHO, 30X, 4HRMX*, I2, 2H<#E13.6)

B2 = -BY/(AH**2)

PMX(25) = 0.

RMY(24) = -RMX(24)
RMY(23) = -RMX(23)
RMY(22) = -RMX(22)
RMY(21) = -RMX(21)
RMY(20) = -RMX(20)
RMY(19) = 0.

RMY(18) = (-BET*W(12) + 2 * W(13) + BET*W(14) + 2 * (BET**2 + AS**2 * XMU) * W(17)
1 - 2 * (1 + BET**2 + AS**2 * XMU) * W(18)) * B2

RMY(17) = (-BET*W(11) / 2 + W(12) + (BET**2 + AS**2 * XMU) * W(16)
1 - 2 * (1 + BET**2 + AS**2 * XMU) * W(17) + (BET**2 + AS**2 * XMU) * W(18)
2 + BET*W(15) / 2 + W(14)) * B2

RMY(16) = (W(11) + BET*W(12) / 2 - 2 * (1 + BET**2 + AS**2 * XMU) * W(16)
1 + (BET**2 + AS**2 * XMU) * W(17) + W(15) - BET*W(14) / 2) * B2

RMY(15) = (-BET*W(9) / 2 + W(10) + (BET**2 + AS**2 * XMU) * W(14)
1 - 2 * (1 + BET**2 + AS**2 * XMU) * W(15) + BET*W(13) / 2 + W(16)) * B2

RMY(14) = (-BET*W(8) / 2 + W(9) + BET*W(10) / 2 + (BET**2 + AS**2 * XMU) * W(13)
1 - 2 * (1 + BET**2 + AS**2 * XMU) * W(14) + BET*W(12) / 2 + W(16)) * B2

RMY(13) = (-BET*W(7) / 2 + W(8) + BET*W(9) / 2 + (BET**2 + AS**2 * XMU) * W(12)
1 - 2 * (1 + BET**2 + AS**2 * XMU) * W(13) + (BET**2 + AS**2 * XMU) * W(14)
2 + BET*W(11) / 2 + W(13) - BET*W(12) / 2) * B2

RMY(12) = (-BET*W(6) / 2 + W(7) + BET*W(8) / 2 + (BET**2 + AS**2 * XMU) * W(11)
1 - 2 * (1 + BET**2 + AS**2 * XMU) * W(12) + (BET**2 + AS**2 * XMU) * W(13)
2 + BET*W(10) / 2 + W(12) - BET*W(11) / 2) * B2

RMY(11) = (W(6) + BET*W(7) / 2 - 2 * (1 + BET**2 + AS**2 * XMU) * W(11)

RMY(10) = (-BET*W(4) / 2 + W(5) + BET*W(6) / 2 + (BET**2 + AS**2 * XMU) * W(9)
1 - 2 * (1 + BET**2 + AS**2 * XMU) * W(10) + BET*W(14) / 2 + W(15)) * B2

RMY(9) = (-BET*W(3) / 2 + W(4) + BET*W(5) / 2 + (BET**2 + AS**2 * XMU) * W(8)
1 - 2 * (1 + BET**2 + AS**2 * XMU) * W(9) + (BET**2 + AS**2 * XMU) * W(10)
2 + BET*W(13) / 2 + W(14) - BET*W(15) / 2) * B2

RMY(8) = (-BET*W(2) / 2 + W(3) + BET*W(4) / 2 + (BET**2 + AS**2 * XMU) * W(7)
1 - 2 * (1 + BET**2 + AS**2 * XMU) * W(8) + (BET**2 + AS**2 * XMU) * W(9)
2 + BET*W(12) / 2 + W(13) - BET*W(14) / 2) * B2

RMY(7) = (-BET*W(1) / 2 + W(2) + BET*W(3) / 2 + (BET**2 + AS**2 * XMU) * W(6)
1 - 2 * (1 + BET**2 + AS**2 * XMU) * W(7) + (BET**2 + AS**2 * XMU) * W(8)
2 + BET*W(11) / 2 + W(12) - BET*W(13) / 2) * B2

RMY(6) = (W(1) + BET*W(2) / 2 - 2 * (1 + BET**2 + AS**2 * XMU) * W(6)
1 + (BET**2 + AS**2 * XMU) * W(7) + W(11) - BET*W(12) / 2) * B2

RMY(5) = 0.
RMY(4) = 0.
RMY(3) = 0.
RMY(2) = 0.
RMY(1) = 0.

DO 23 I = 1, 25

23 FORMAT (IHO, 30X, 4HRMY*, I2, 2H<#E13.6)

A12 = -2 * C * AS / (AH**2)
DO 57 I=1,6
57 FORMAT (30,1,THETA(I))
503 ADD V2 (VX, VY, VXTHETA*Y, 12, 2H<=513.6)
GO TO 706
500 STOP
END
VITA AUCTORIS

1940
Born at Rangamati, Chittagong Hill Tracts, East Pakistan on January 12.

1957
Completed Secondary School from Rangamati Gov't. H.E. School and received first grade Scholarship.

1959
Stood first class second in order of merit in Intermediate Science Examination from Sir A.T. College, Kanungopara, under Dacca University.

1963
Stood First class Second in B.Sc. Engg. (Civil) final examination from East Pakistan University of Engg. and Technology, Dacca.

1964
Joined the Associated British Consultant Ltd., Chittagong, E.P. as an Assistant Engineer.

1965
Appointed as a Resident Engineer in 'The Engineers Ltd.' Dacca, E. Pakistan.

1967
Promoted to the Post of Senior Engineer (Design) and held this position until left for Canada for higher studies.

1968-70
Enrolled as a Graduate (Teaching and Research) Assistant in the University of Windsor, while pursuing a Master's Program in the Department of Civil Engineering.