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Comparison of performance indices in the optimal control of a second order system.

Bosko Cirjanic

University of Windsor

COMPARISON OF PERFORMANCE INDICES IN THE OPTIMAL CONTROL OF A SECOND ORDER SYSTEM

by

Boško Ćirjanić

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C. St. Pierre

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ABSTRACT

In the design of control systems, the selection of a performance index is frequently a difficult one. In some applications, such as, time optimal control, the index is predetermined, but in other cases the decision is not as clear cut. The selection of a performance index is an important one, since it determines the nature of the system transient response. Therefore, some guidance is required in selecting a suitable performance index.

The work described in this thesis makes an attempt to simplify the selection of a suitable performance index. This is done by designing the system so as to minimize a certain performance index during the transient period of a second order system. Additional indices are evaluated during the transient period and the results are tabulated for each index. This was carried out for six performance indices, and each time all the indices are evaluated. The results for each transient response were tabulated in order to provide a quick reference for the selection of a suitable index.

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NOMENCLATURE

T	system time constant
K	gain of the system (= 1.0)
y	K/T
ζ	damping ratio of a second order system
G(s)	open loop transfer function in the Laplace variable
E(s)	closed loop error in the Laplace variable s
e(t)	closed loop error in the time domain
$F(\bar{x},\bar{m},t)$ and $\phi(\bar{x},\bar{m},t)$	performance index which may be a function of $\bar{x},\bar{m}$, and t
ISE	$\int_0^{\infty} e(t)^2 dt$
ITSE	$\int_0^{\infty} t\, e(t)^2 dt$
R(s)	Laplace transform of the time function step input r(t)
c(s)	Laplace transform of the time function c(t)
$m^o(t)$	optimum input to the system
$\bar{x}(t)$	System state variables
$H(\bar{x},\bar{m},\bar{p},t)$ or H	Hamiltonian function
p(t)	Adjoint system state variable
E	The desired value of the system output

$$\mathrm{sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

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I. INTRODUCTION

The design of control systems by the classical control techniques is rarely used at the present time, due mainly to the better methods now available with the modern control theory. The amount of information that can be obtained about the system using classical control is limited to the system frequency response, stability, transient response, etc. Most of the design techniques in classical control are by graphical means. This makes the design somewhat laborious, since trial and error methods have to be used in the design.

Optimization techniques can be used in classical control as well as in modern control theory. System optimization is concerned with making some performance index or criteria take on a extremum value, in which case we have optimum control. A performance index is used to uniquely determine the optimum operating conditions of the system. When a system is optimum it is usually only optimum with respect to the performance index or criteria used.

There are basically two optimizing techniques:

i) The type of controller and the nature of the system are predetermined and the parameters of both are selected so as to optimize some criteria.

ii) The controller is designed so as to optimize some chosen performance criteria of the system.

The first method falls into the class of parameter optimization. Parameter optimization has limitations in that the system response is usually oscillatory with overshoots. The performance indices normally used with parameter optimization are ISE and ITSE. Other indices are not so amenable to parameter optimization and are seldom used.

The second method uses either the dynamic programming or the maximum principle. Dynamic programming always results in a feedback controller with time varying gains if optimized for the finite time interval. The feedback loops make the overall system stable during the optimizing interval and the effect of any disturbances at the output are reduced. Dynamic programming does however require a knowledge of all the state variables, which makes it unsuitable for certain systems. This limitation can in some cases be overcome by using state variable estimation techniques. Another serious disadvantage with dynamic programming is that it is not very easily applicable to discontinuous control.

The maximum principle involves extremization of the Hamiltonian function. If the performance index is to be minimized the Hamiltonian is maximized. Extremization of the Hamiltonian provides an adjoint system, the output of which is fed through a controller to the process or plant. The controller can be continuous or discontinuous, depending upon the performance index used. If the plant to be controlled is stable, then the adjoint system is unstable. This is not a serious limitation since the adjoint system can be made stable over the optimizing interval.

One important point that should be realized is that the performance index dictates the type of controller that will be used. This is more apparent by observing the Hamiltonian function. If the control signal m, appears in the performance index to a power greater than unity, then this will result in analog or continuous control, otherwise the control will be discontinuous (may be bang-bang, or on-off). Also, the performance index dictates whether the resultant controller is open or closed loop. For a closed loop controller the output state variable must appear in the performance index to a power greater than unity, otherwise the controller will be open loop.

The aim of the work in this thesis was to study the role of the performance indices in optimal control of a second order system. This problem is of interest in industry where it is desirable to get the plant or process up to its operating condition and at the same time extremize some performance or cost criteria. The possibility of using either dynamic programming or the maximum principle will be considered.

II. PERFORMANCE INDICES IN OPTIMAL CONTROL

In optimal control, it is necessary to have some means of assessing the performance or the quality of control of the system. A performance index is introduced to fulfill this requirement. The performance index depends entirely on the type of system being controlled. Minimum fuel indices are used in applications such as space vehicles and sattelite control systems. In these situations the amount of fuel that can be carried is severely limited and any manuevre must be performed using the least amount of fuel. Other indices, such as minimum time, require that the manuevre be carried out in the minimum possible time. This index could be used in the dive or surface control system section of a submarine. No one index could possibly be used in a complex system to define the optimum performance, and usually large systems are broken down into small sections where we can apply the appropriate indices.

A performance index is used as a means of determining uniquely the optimum operating conditions of a control system or process. One possible performance index is to minimize the system error.

We would like the system to respond to the command without error. This is not generally possible and our only alternative is to operate the system in the best possible way subject to any imposed constraints. The controller is designed incorporating the imposed constraints and a performance index is used to check the system performance. The choice of the performance index is an important one since it determines the nature of system response. In some cases undesirable results can be obtained by the wrong choice of a performance index. Often a compromise has to be reached in the selection of a suitable index, especially if the most suitable one is difficult to evaluate or impossible to optimise. Listed below is a brief summary of some of the most common performance indices.

The system described in this thesis has the open loop transfer function,

$$G(s) = \frac{K/T}{s(s + 1/T)} \quad \text{where } K = 1.0 \qquad (2.1)$$

and is shown in a closed loop configuration in Fig. 1.

i) $\int_{o}^{t_f} \left(e^2 + \lambda m^2\right) dt$

This is one of the most widely used indices, involving quadratic terms of error and the system control signal m. The λ in the index is the Lagrange multiplier if there are constraints in the system, otherwise it is only a weighting factor. This index attempts to minimize both the system error and also the input energy. It is easily applicable in either the maximum principle or the dynamic programming techniques. In both cases it gives continuous and closed loop control.

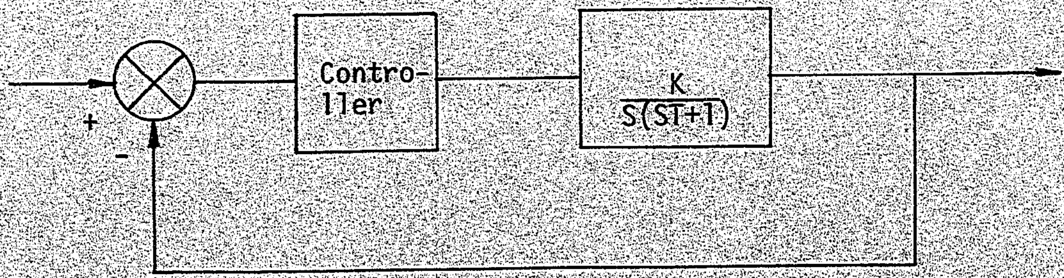


FIGURE 1

General Closed Loop System with Controller

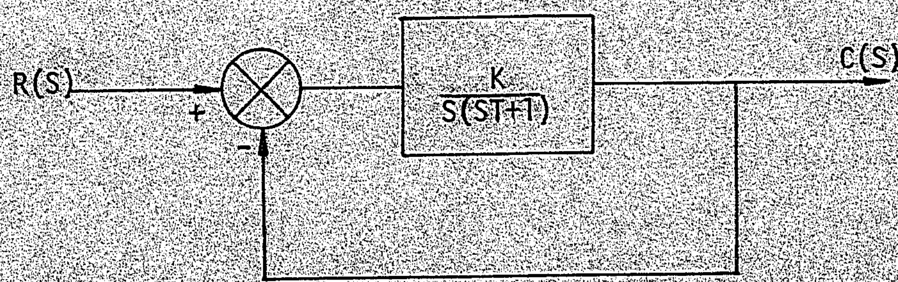


FIGURE 2

Closed Loop Second Order System

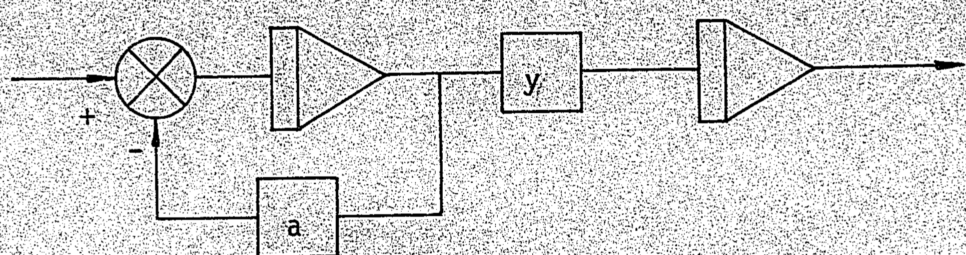


FIGURE 4

Schematic Diagram of aSecond Order System

ii) $\int_{o}^{t_f} e^2 dt$

This index penalizes both positive and negative errors equally, since the product is always positive. It has been widely used in the past, especially in parameter optimization techniques. Minimization of this index using parameter optimization, produces lightly damped systems having poor relative stability[1,5]. It is also insensitive to small errors or disturbances at the output, which will go undetected. The integral could be made zero by applying an infinite input to the system. This is, however, impractical since it is difficult to generate such high inputs and there would invariably be saturation in the system. In practice this integral would have to be minimized with fixed constraints on various quantities in the system.

iii) $\int_{o}^{t_f} dt$

This index is commonly called the minimum time or time optimal performance index. It is used when it is desired to transfer the system from some fixed initial state to some fixed final state in the minimum possible time. For a second order system with constraints on the control signal, it has been found that maximum available power should be used at all times to either accelerate or to brake the system[6]. This form of control is commonly called "bang-bang" control.

iv) $\int_{o}^{t_f} \lambda m^2 dt$

This is called the minimum energy performance index. It is a measure of the energy required to transfer the system from a fixed initial state to some prescribed final state. The time required for this manuevre, t_f, may or may not be specified. The λ in the index is the Lagrange multiplier if there are constraints on the input m. This index is used when the input energy to the system is limited and m is chosen so as to minimize the power over the optimizing interval.

v) $\int_{o}^{t_f} te^2 dt$

This index is similar to the ISE, except that it does not penalize as severely large initial errors. It does, however, penalize errors that persist for a long time. Because of this it is more sensitive to disturbances at the output, than the ISE index. Minimization of this index using parameter optimization produces lightly damped systems having poor relative stability. The final time t_f may or may not be specified. If it is, then only the error up to the time t_f is of interest.

vi) $\int_{o}^{t_f} |m| \, dt$

This is the minimum fuel index, and is particularly useful in applications where the amount of available fuel is limited. In these cases the controller is designed so that the system consumes the minimum amount of fuel in transferring the system from some fixed initial state to some fixed final state. Usually other measures of the system are sacrificed, e.g. settling time, etc., in order to achieve the minimum fuel requirement. For type 0 and type 1 systems only one sign of the control signal is required, that is either +M or -M and zero. This requires ON-OFF control. However, for a type 2 or higher systems, a change of sign of the control signal is required, with possibly a zero input in between the controller switchings. If the control signal to a type 0 or type 1, system changes sign, this will result in sub-optimal control.

III. PARAMETER OPTIMIZATION

Parameter optimization involves the selection of controller or system parameters in such a manner that the optimum operating conditions are achieved. This form of optimization is used when the type of controller and system have been chosen, but their parameters can be selected almost at will. This form of optimization is usually the cheapest since it requires very little change to the existing system. There is a considerable amount of literature available on this subject[1,2,3], only a brief summary will be given here.

The most used performance criterion with parameter optimization with step type inputs to the system is the ISE. The ISE is defined as,

$$ISE = \int_0^\infty e(t)^2 dt \qquad (3.1)$$

The parameters of the controller and the system are chosen so as to minimize this integral over the period of integration. The integral (3.1) can be transformed from the time into the frequency domain as shown below.

$$ISE = \int_0^\infty e(t)^2 dt = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} E(s) \cdot E(-s)\, ds \qquad (3.2)$$

where E(s) is the Laplace transform of the time error function e(t).

The value of the right hand side of (3.2) can be found from published tables providing E(s) can be written in the form

$$E(s) = \frac{c(s)}{d(s)} \tag{3.3}$$

where $$c(s) \triangleq c_o + c_1 s + \ldots + c_{n-1} s^{n-1} \tag{3.4}$$

and $$d(s) \triangleq c_o + d_1 s + \ldots + d_n s^n \tag{3.5}$$

and where d(s) has zeros in the left half plane only. This manipulation of the ISE is due to Parseval and is referred to as Parseval's theorem. Thus the evaluation of the ISE is simplified and the results are available as published tables.

The minimization of the ISE on a second order system, we have the choice of two parameters which can be optimized. Consider the second order system whose transfer function is given by

$$G(s) = \frac{K}{s(sT+1)} \tag{3.6}$$

and we need to determine K and T to make ISE take on the minimum value. To make use of Parseval's theorem we need the closed loop system error of Fig. 2, and the error is given by

$$E(s) = \frac{R(ST + 1)}{TS^2+S+K} = \frac{R(s + 1/T)}{s^2+1/T\ S + K/T} \tag{3.7}$$

$$E(s) = \frac{R(s+a)}{s^2+as+Ka} \tag{3.8}$$

where R is the step input to the system

and a is 1/T

Using Parseval's theorem on (3.8), we have

$$ISE = R\left[\frac{Ka + a^2}{2Ka^2}\right] = R\left[\frac{1}{2a} + \frac{1}{2K}\right] \qquad (3.9)$$

The minimization of the ISE with respect to K and T gives a trivial result, since it requires that $K = \infty$ and $T = 0$. If K is very high the resultant system response is oscillatory and the relative stability would be very poor.

A more meaningful result will be obtained if the optimization is carried out with respect to the system damping ratio ζ. For the system of Fig. 2 it can be shown using Parseval's theorem that the damping which minimizes the ISE is

$$\zeta = 0.5 \qquad (3.10)$$

With this value of ζ, it can be shown that

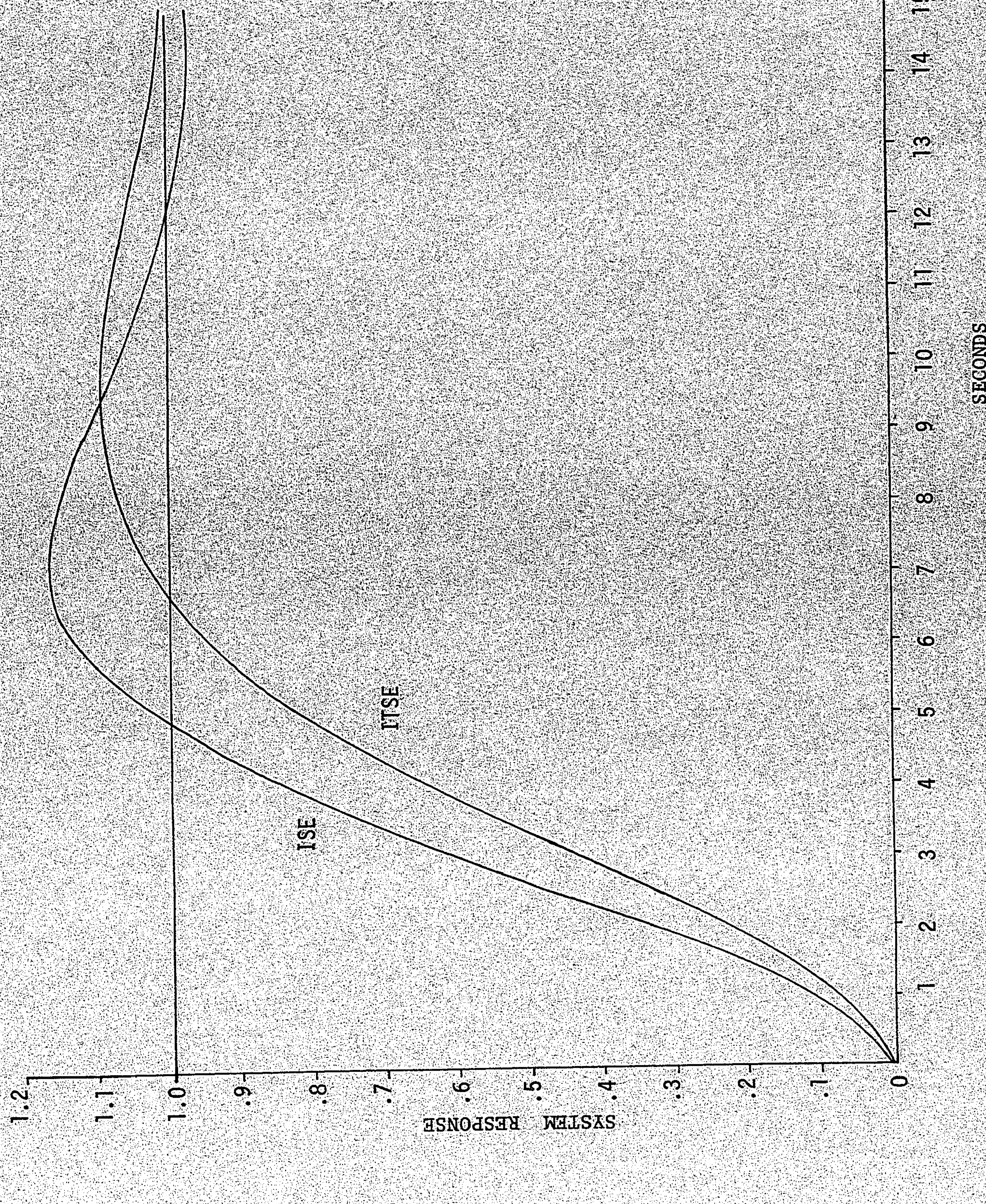
$$ISE_{min} = T \qquad (3.11)$$

and the optimum $K = 1/T$ (3.12)

Parseval's theorem has been extended by Westcott[4], for the use of the ITSE performance index. If a similar optimization procedure is carried out as above, but this time using the ITSE index, the following results will be obtained.

FIGURE 3

Step Response of a Second Order System
for the ISE and ITSE Index



The damping which minimizes ITSE is $\zeta = 0.595$ (3.13)

and

$$ITSE_{min} = T^2 \tag{3.14}$$

and the optimum

$$K = 1.0/1.44\ T^2 \tag{3.15}$$

Comparing the values of ζ obtained to make both indices take on a minimum value, both results agree with that obtained by Graham and Lathrop[1]. The system of Fig. 2 was simulated on a digital computer, the simulation results are displayed in Fig. 3.

The simulation results of Fig. 3 show that as the damping ratio is decreased this results in a more oscillatory response. The values of ISE and ITSE agree with the calculated, the simulated values are

$$ISE = 1.999 \tag{3.16}$$

$$ITSE = 4.006 \tag{3.17}$$

Both the results were taken for a 15 second simulation interval. In parameter optimization, we know that the best system response will be obtained if $\zeta = 0.7$, and what we are doing in effect, is trying to find an index which gives this result. The simulation results confirm that the index which gives a damping ratio of around 0.7 gives the best overall results, in this case ITSE would seem to be superior. Its settling time is smaller and also the overshoot is less than for the ISE index.

IV. OPTIMAL CONTROL

A control system can be optimal in a sense that a performance index or criteria is extremized. Ideally, we would like the control system to excecute the commands with no error at all. This is almost impossible in practice and the next best solution is to try and minimize the system error. We need not take the system error as the criteria, we could just as well minimize the fuel or the energy to the system. Whatever performance index we use, we must ensure that the system is operating optimally. It should be remembered that usually a system is only optimal with respect to one performance index. It is impossible to make a system optimal with respect to all our indices. The two most powerful optimizing techniques available at the present time are dynamic programming and the maximum principle. Both methods will be described and their advantages and disadvantages will be discussed in the next two sections.

4.1 DYNAMIC PROGRAMMING

Dynamic programming has been found to be very useful with certain types of optimal control problems. Its main advantage is that it provides a closed loop controller with time varying gains that approach zero at the end of the optimizing interval. This is in contrast with the maximum principle where the adjoint vectors tend to infinity. The theory behind the dynamic programming will be stated without proof.

Consider an nth order system characterized by the differential equation.

$$\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t) + \bar{D}(t)\bar{m}(t) \tag{4.1}$$

where $\bar{x}$ is an n x 1 vector representing the state of the process

$\bar{m}$ is an r x 1 control vector

$\bar{A}$ is the coefficient matrix

$\bar{D}$ is the driving matrix

The problem now is to determine the optimum control signal m(t) which will extremize an integral performance index of the type

$$I(m) = \int_{t}^{t_f} F(\bar{x},\bar{m},t)dt \tag{4.2}$$

over the interval of time t to t_f. Let the minimum of the integral (4.2) be

$$f(\bar{x},t) = \min_{m} \int_{t}^{t_f} F(\bar{x},\bar{m},t)dt \tag{4.3}$$

Bellman's principle of optimality states that if we have the optimal trajectory, then any portion of this trajectory must necessarily be optimal itself. Applying the principle of optimality to (4.3) yields

$$f(\bar{x},t) = \min_{m}\int_{t}^{t+\Delta} F(\bar{x},\bar{m},t)dt + \min_{m}\int_{t+\Delta}^{t_f} F(\bar{x},\bar{m},t)dt \tag{4.4}$$

$$f(\bar{x},t) = \min_{m} \int_{t}^{t+\Delta} F(\bar{x},\bar{m},t)dt + f(\bar{x} + \dot{\bar{x}}\Delta\ ,\ t+\Delta\ ) \tag{4.5}$$

Assuming Δ to be very small and expanding the second term on the right hand side of (4.5) using the Taylor series, we have,

$$f(\bar{x},t) = \min_{m} \left[F(\bar{x},\bar{m},t)\Delta + f(\bar{x},t) + \dot{\bar{x}}\frac{\partial f}{\partial \bar{x}}\Delta + \frac{\partial f}{\partial t}\Delta + E(\Delta) \right] \tag{4.6}$$

where $E(\Delta)$ is the error incurred by the truncation of the Taylor series. Taking the limit as Δ tends to zero, we have that

$$\min_{m} \left[F(\bar{x},\bar{m},t) + \dot{\bar{x}}\frac{\partial f}{\partial \bar{x}} + \frac{\partial f}{\partial t} \right] = 0 \tag{4.7}$$

rearranging (4.7), we have

$$-\frac{\partial f}{\partial t} = \min_{m} \left[F(\bar{x},\bar{m},t) + \dot{\bar{x}}\frac{\partial f}{\partial \bar{x}} \right] \tag{4.8}$$

Equation (4.8) is known as Bellmann's functional equation. The optimization problem has been reduced to one in which we have to solve the partial differential equation (4.8) for the function $f(\bar{x},t)$. The optimum input is obtained from (4.8) and is given by,

$$\frac{\partial}{\partial m} \left[F(x,m,t) + \dot{\bar{x}}\frac{\partial f}{\partial \bar{x}} \right] \Bigg|_{m=m^{o}} = 0 \tag{4.9}$$

but $\quad \dot{\bar{x}} = \bar{A}(t)\ \bar{x}(t) + \bar{D}(t)\ m(t)$

hence (4.9) is simplified to

$$\frac{\partial}{\partial m}\left[F(\bar{x},\bar{m},t) + \left(\bar{A}(t)\ \bar{x}(t) + \bar{D}(t)\ \bar{m}(t)\right)\cdot\frac{\partial f}{\partial \bar{x}}\right]_{m=m^o} = 0 \quad (4.10)$$

The optimum input obtained from (4.10) is substituted into (4.8) and the resulting partial differential equation is solved for the function $f(\bar{x},t)$.

If the function $F(\bar{x},\bar{m},t)$ is quadratic in the system error $e(t)$ and the control signal $m(t)$, then by using Merriam's parametric expansion[9] the function $f(\bar{x},t)$ can be approximated by

$$f(\bar{x},t) = bo - \sum_{j=1}^{n} b_j x_j + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j \quad (4.11)$$

where n is the order of the system, and the b's are time varying gains that will appear as parameters of the controller.

Substituting (4.11), (4.10) into (4.8) and simplifying the resulting equation will give $1 + N + N(N+1)/2$ first order differential equations. The final values of the gains $b(t)$ are zero and hence the first order differential equations are integrated backwards in time until the $b(t)$ gains reach steady state. The values of the parameter $b(t)$ are stored and fed into the system backwards which will yield the optimal system performance.

The above procedure is useful if the performance index is quadratic in the error $e(t)$ and the input $m(t)$. For other indices Merriam's parametric equation cannot be used since it has been found by the author using $\int_0^{t_f} e^2 dt$ and $\int_0^{t_f} \lambda m^2 dt$ indices to give undesirable results. Thus dynamic programming is not very useful for other indices than the one described above, since we have to solve for the function $f(\bar{x},t)$ and then Bellmann's functional equation. This may be very difficult and the maximum principle seems to offer more hope.

4.2 THE MAXIMUM PRINCIPLE

The maximum principle provides a method of obtaining an optimal solution for control systems. It is capable of handling optimization problems of extremizing a functional subject to certain constraints. This is very important, since the optimizing procedure using Variational Calculus often results in the system having unbounded control signals. In practice we have constraints on the control signal and also on some of the system state variables. It is for this reason that the maximum principle is particularly useful as an optimizing technique. The maximum principle will be stated here without proof.

Consider an nth order system which is characterised by

$$\dot{\bar{x}} = \bar{f}(\bar{x},\bar{m},t) \qquad (4.12)$$

where $\bar{x}$ is the n x 1 state vector

and $\bar{m}$ is the r x 1 control vector

It is assumed that the control vector is to be confined to a region M of the space $m_1 . . . m_r$. The allowable region for m could be defined without any loss of generality as,

$$m_j \leqslant 1 \qquad j = 1,2,. . . r \qquad (4.13)$$

The functions $m_1,. . . m_r$ are assumed to be piecewise continuous at the end points $t = t_o$ and $t = t_f$, and if there are

are any discontinuities in m over the interval, (t_o, t_f) the right and left hand limits must exist at these discontinuities[7]. Thus for the left hand limit we must have

$$m_i(\tau) = \lim_{\substack{t \to \tau \\ t < \tau}} m_i(t) \qquad (4.14)$$

It is more convenient to deal with the left hand limit otherwise we are forced to work in forward time. For $\overline{m}$ to be an admissible input the properties of (4.13) and (4.14) have to be satisfied. Let the system be described by a set of differential equations

$$\dot{x}_i = f_i(\overline{x}, \overline{m}, t) \qquad i = 1, 2, \ldots n \qquad (4.15)$$

We are to find an admissible control vector $\overline{m}(t)$, such that the system of (4.14) is taken from some initial state to some fixed final state in such a manner so as to optimize the performance criterion. Let the performance index be of the type

$$I(m) = \int_{t_o}^{t_f} \phi(\overline{x}, \overline{m}, t)\,dt \qquad (4.16)$$

The system performance will be judged to be optimum when (4.16) takes on a minimum value with respect to all admissible $\overline{m}(t)$. Let us introduce a Hamiltonian function

$$H(\overline{x}, \overline{P}, \overline{m}, t) = \sum_{i=1}^{n} P_i f_i - \phi(\overline{x}, \overline{m}, t) \qquad (4.17)$$

where the functions $P_i(t)$ are given by

$$\dot{P}_i(t) = -\frac{\partial H}{\partial x_i} \qquad i = 1,2,\ldots n \tag{4.18}$$

From the Hamiltonian (4.17), we have

$$\bar{x}_i = \frac{\partial H}{\partial P_i} \qquad i = i,2,\ldots n \tag{4.19}$$

with the boundary conditions $x(t_o) = x^o$, $x(t_f) = x'$

If $\bar{m}^*(t)$ is the optimal control, then there exists[8] a vector $\bar{P}^*(t)$ which satisfies (4.18) and at every instant of time $t_o < t < t_f$

$$H(\bar{x}^*,\bar{P}^*,\bar{m}^*,t) \geqslant H(\bar{x},\bar{P},\bar{m},t) \tag{4.20}$$

or

$$H(\bar{x}^*,\bar{P}^*,\bar{m},t) = \max_{\bar{m}\in M} H(\bar{x},\bar{P},\bar{m},t) \tag{4.21}$$

The above procedure has been carried out minimizing the performance index. If on the other hand we wanted to maximize the index, then we need to minimize the Hamiltonian and the negative sign of (4.17) would be changed. Thus the design of an optimal control system has been reduced to that of maximizing or minimizing the Hamiltonian function (4.17). The following section will deal with maximizing the Hamiltonian function for various performance indices.

4.2 i) $\int_o^{t_f}(e^2 + \lambda m^2)dt$

Consider the second order system which is described by the following state equations.

$$\dot{x}_1 = yx_2 \tag{4.22}$$

$$\dot{x}_2 = -ax_2 + m \tag{4.23}$$

The system described by (4.22) and (4.23) is shown in Fig. 4. Using (4.17) the Hamiltonian becomes

$$H = -(\,(E - x_1)^2 + \lambda m^2) + P_1 y x_2 + P_2\,(-ax_2 + m) \tag{4.24}$$

hence

$$\dot{P}_1 = -\frac{\partial H}{\partial x_1} = -\,2(E - x_1) \tag{4.25}$$

$$\dot{P}_2 = -\frac{\partial H}{\partial x_2} = aP_2 - yP_1 \tag{4.26}$$

$$\frac{\partial H}{\partial m} = -\,2\lambda m + P_2 \tag{4.27}$$

the optimum input $m^o = \dfrac{P_2}{2\lambda}$ (4.28)

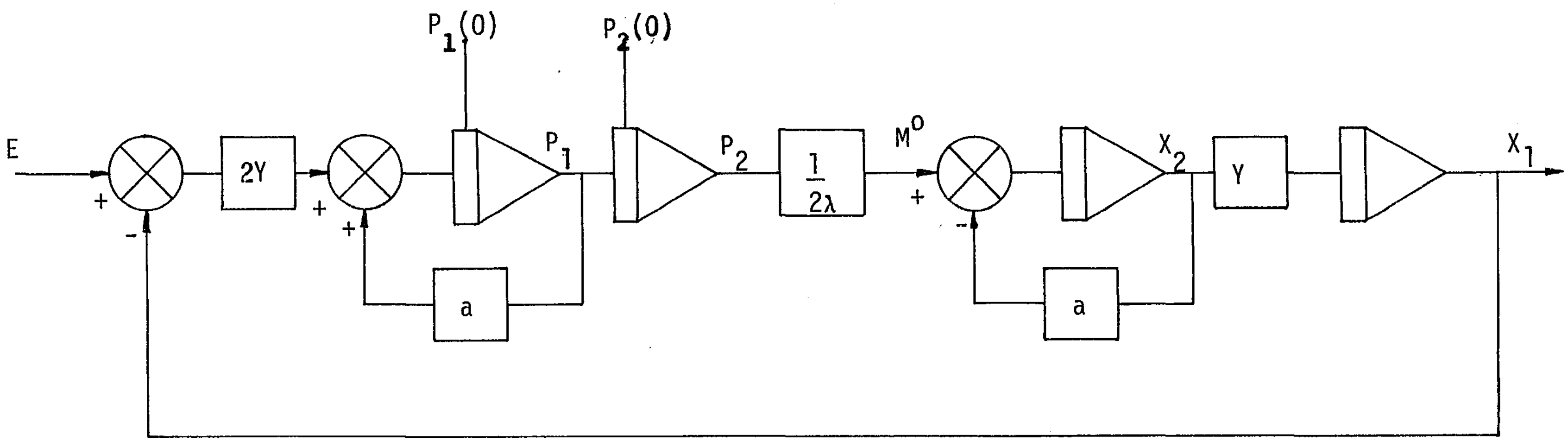
From (4.25) through (4.28) the adjoint system can be determined and is shown in Fig. 5.

Equations (4.27) and (4.28) can only be applied if the control signal m is not on the boundary of the permissible region. This is due to the fact that $\partial H/\partial m$ is not defined on the boundary. In the case of continuous control signals, equation (4.27) and (4.28) are valid. However, for bang-bang control the control signal is on the boundary of the permissible region at all times and hence equations (4.27) and (4.28) are invalid.

It is evident from Fig. 5 that the plant or process is reflected in the adjoint system. For a stable plant the adjoint

FIGURE 5

Adjoint System for the $\int_0^{t_f} e^2 + \lambda m^2 dt$ Index



system will always turn out to be unstable. This is inherent in the maximum principle and makes the control slightly complex. With a step input of magnitude E applied to the adjoint system, the initial conditions $P_1(0)$ and $P_2(0)$ have to be determined, which will drive the system to the desired state optimally. One desirable feature of the adjoint system of Fig. 5 is that it is closed loop and the effect of any disturbances in the system will be reduced due to the negative feedback.

If the disturbances within the system are large, it may be possible for the system to go unstable due to these disturbances. If the system is to remain optimal with any disturbance, this would necessitate new initial conditions on the adjoint vectors. This would not be possible in practice since the disturbance would have to be detected and the initial conditions on the adjoint system vectors would have to be altered without the system straying from the optimal trajectory.

4.2 ii) $\int_o^{t_f} e^2 dt$

The Hamiltonian for this index and the system shown in Fig. 4 is

$$H = -(E-x_1)^2 + P_1 y x_2 + P_2(-ax_2 + m) \qquad (4.29)$$

hence

$$\dot{P}_1 = -\frac{\partial H}{\partial x_1} = -2(E-x_1) \qquad (4.30)$$

$$\dot{P}_2 = -\frac{\partial H}{\partial x_2} = aP_2 - yP_1 \qquad (4.31)$$

the optimum input $m^o = M\text{sgn}\left[P_2\right]$ (4.32)

The adjoint system is obtained using (4.30) through (4.32) and is shown in Fig. 6.

Again the adjoint system is unstable and also there is "bang-bang" control. Bang-bang control is when the input is at its maximum value and is either accelerating or braking the system. The initial conditions $P_1(0)$ and $P_2(0)$ have to be determined to provide optimum control. The system of Fig. 6 is closed loop and the effect of any disturbances will be reduced.

4.2 iii) $\int_o^{t_F} dt$

This is the minimum time or time optimal performance index. The Hamiltonian for this index and the system of Fig. 4 is

$$H = -1 + P_1 yx_2 + P_2(-ax_2 + m) \qquad (4.33)$$

hence

$$\dot{P}_1 = -\frac{\partial H}{\partial x_1} = 0 \qquad (4.34)$$

$$\dot{P}_2 = -\frac{\partial H}{\partial x_2} = aP_2 - yP_1 \qquad (4.35)$$

the optimum input $m^o = M\text{sgn}\left[P_2\right]$ (4.36)

The adjoint system is obtained using (4.34) through (4.36) and is shown in Fig. 7.

Since the performance index does not include m at all, we can expect bang-bang control. An undesirable feature of the

FIGURE 6

Adjoint System for the $\int_{0}^{t} te^{2}\,dt$ Index

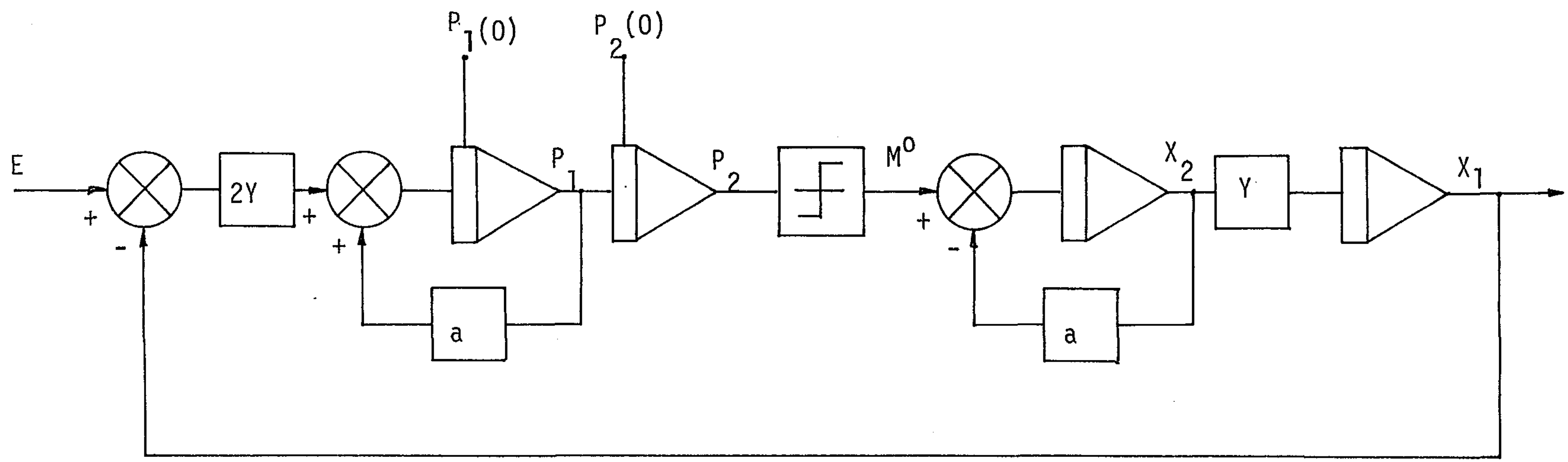
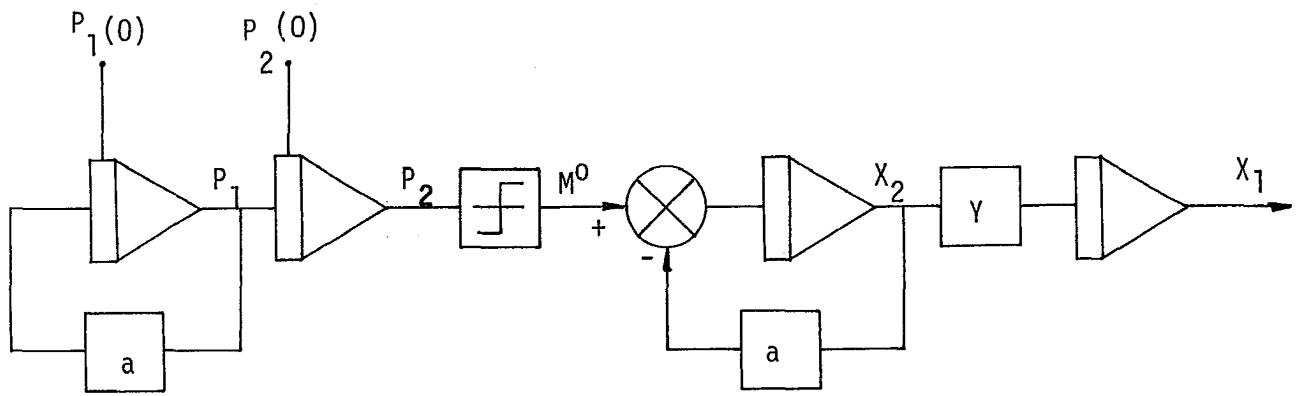


FIGURE 7

Adjoint System for the $\int_0^{t_f} dt$ Index



adjoint system of Fig. 7 is that it is open loop. There is no control over any disturbances within the system, and because of this it may be difficult to implement in practice. The initial conditions $P_1(0)$ and $P_2(0)$ have to be determined for optimum control.

4.2 iv) $\int_o^{t_f} \lambda m^2 dt$ - Minimum energy

The Hamiltonian for this performance index and the system of Fig. 4 is,

$$H = -\lambda m^2 + P_1 y x_2 + P_2(-ax_2 + m) \qquad (4.37)$$

hence

$$\dot{P}_1 = -\frac{\partial H}{\partial x_1} = 0 \qquad (4.38)$$

$$\dot{P}_2 = -\frac{\partial H}{\partial x_2} = aP_2 - yP_1 \qquad (4.39)$$

$$\frac{\partial H}{\partial m} = -2\lambda m + P_2 \qquad (4.40)$$

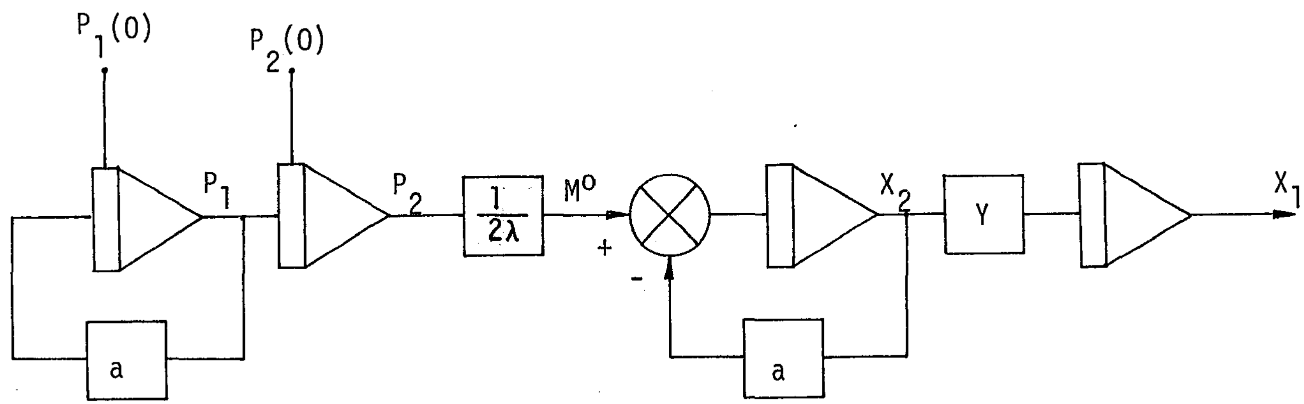
the optimum input $m^o = \dfrac{P_2}{2\lambda}$ (4.41)

The adjoint system is obtained using (4.38) through (4.51) and is shown in Fig. 8.

The performance index is quadratic in m and because of this we have continuous control. This system is open loop and the input to the system is non-dependent on the system variables. The initial conditions $P_1(0)$ and $P_2(0)$ have to be determined

FIGURE 8

Adjoint System for the $\int_0^t f\lambda m^2 dt$ Index



for optimum control.

4.2 v) $\int_{o}^{t_f} te^2 dt$

The Hamiltonian for this performance index and the system of Fig. 4 is

$$H = -t(E-x_1)^2 + P_1 y x_2 + P_2(-ax_2 + m) \qquad (4.42)$$

hence

$$\dot{P}_1 = -\frac{\partial H}{\partial x_1} = -2t(E-x_1) \qquad (4.43)$$

$$\dot{P}_2 = -\frac{\partial H}{\partial x_2} = aP_2 - yP_1 \qquad (4.44)$$

the optimum input $m^o = M\text{sgn}\left[P_2 \right]$ (4.45)

The adjoint system is obtained using (4.43) through (4.45) and is shown in Fig. 9.

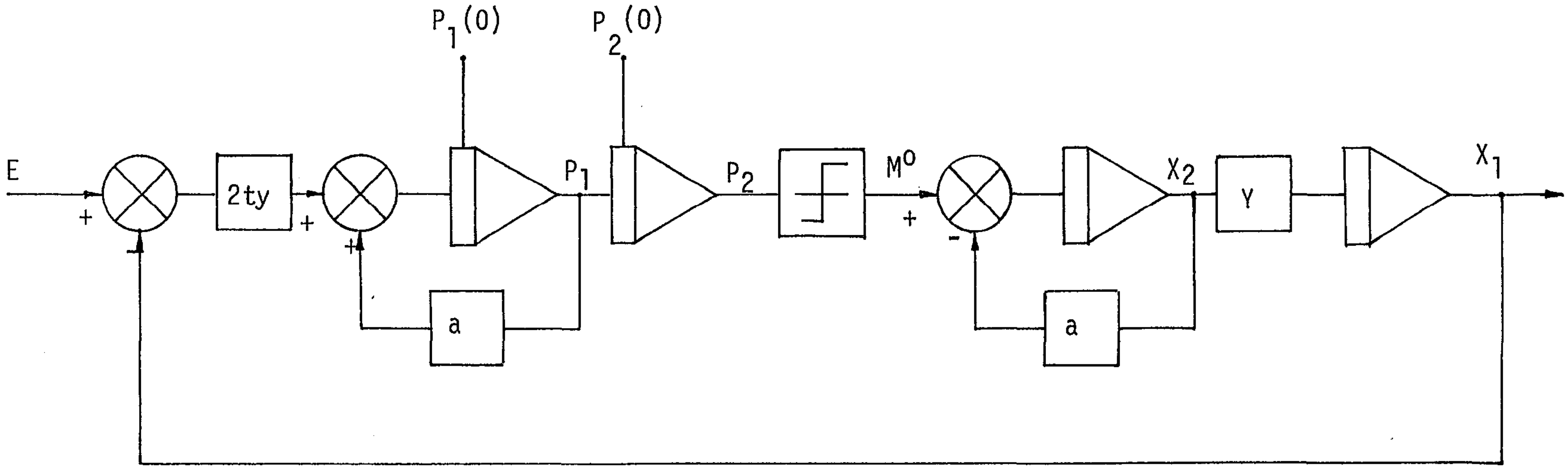
Since the performance index is quadratic in the system error e, this will result in a closed loop controller. Because m does not appear in the index, the control is bang-bang. The only difference between this and the ISE adjoint system is that time appears in this system as a multiplying factor otherwise everything is the same.

4.2 vi) $\int_{o}^{t_f} |m|\, dt$ - Minimum fuel

The Hamiltonian for the minimum fuel index and the system of Fig. 4 is,

FIGURE 9

Adjoint System for the $\int_0^{t_f} te^2 dt$ Index



$$H = -|m| + P_1 y x_2 + P_2(-a x_2 + m) \quad (4.46)$$

hence

$$\dot{P}_1 = -\frac{\partial H}{\partial x_1} = 0 \quad (4.47)$$

$$\dot{P}_2 = -\frac{\partial H}{\partial x_2} = aP_2 - yP_1 \quad (4.48)$$

the optimum input $m^o = \begin{cases} +1.0 & \text{if} \quad P_2 \geqslant 1.0 \\ 0 & \text{if} \quad |P_2| < 1.0 \\ -1.0 & \text{if} \quad P_2 \leqslant -1.0 \end{cases}$ (4.49)
(4.49)

The adjoint system is obtained from (4.47) through (4.49) and is shown in Fig. 10. The controller is different from any of the others in that we have ON-OFF control. This shows that for optimum system performance the control has to be +1.0 and zero.

The controller for each index was simulated together with the second order system and the results of the simulation are shown in Fig. 11 through Fig. 14. A general computer program which was used for the simulation is shown in the appendix. Only minor modifications are required to the program to optimize with respect to some other index.

The results of the simulation of the second order system have been shown graphically. The system response was judged to be acceptable if the output was within one percent of the desired value. One unforseen result is that all the bang-bang controllers gave the same result. This was not apparent at the beginning and to confirm this result a third order system was simulated with two indices that gave bang-bang control.

FIGURE 10

Adjoint System for the $\int_0^t f|m|dt$ Index

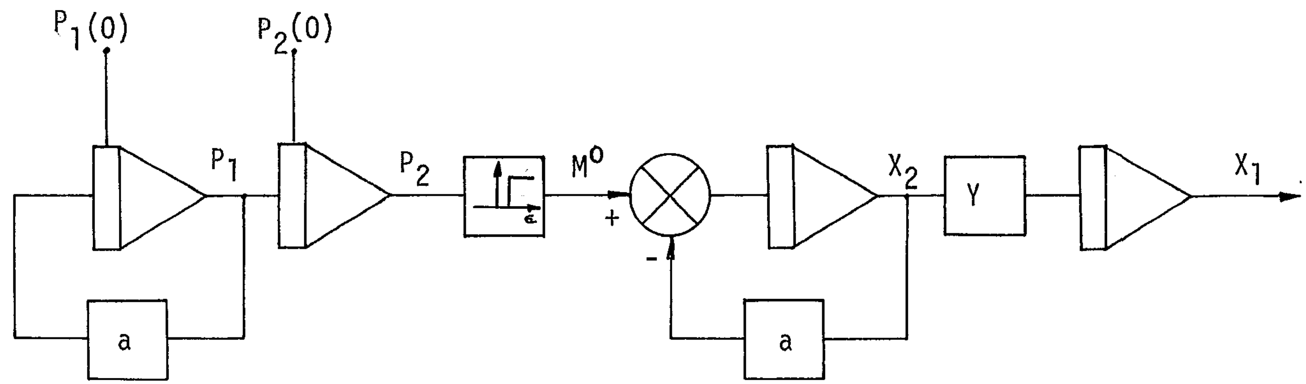


FIGURE 11a

Transient Response of a Second Order System with $a\int_0^{t_f} e^2 + \lambda m^2 dt$ Controller

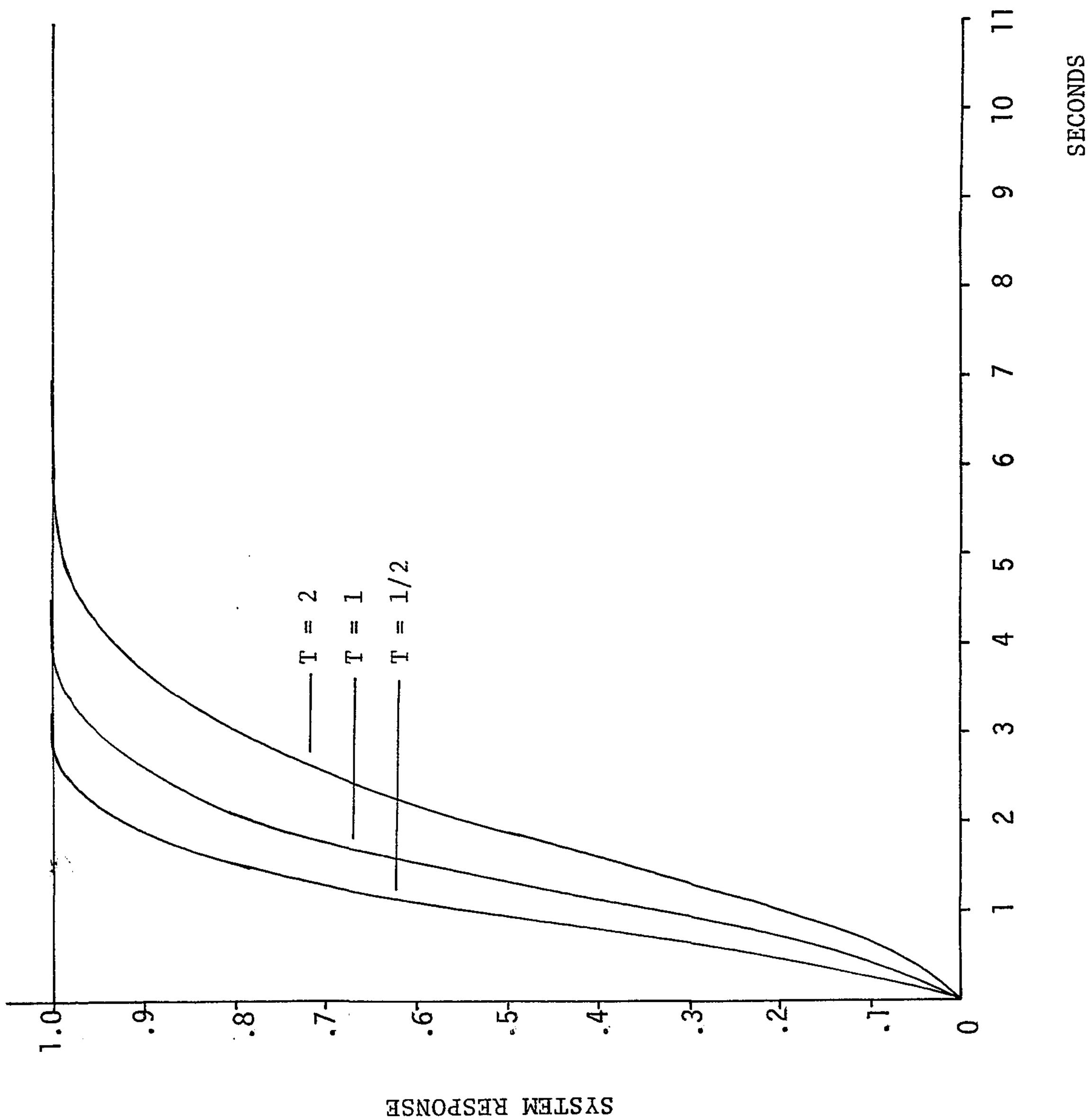


FIGURE 11b

Optimum Input for Controller of Fig. 11a

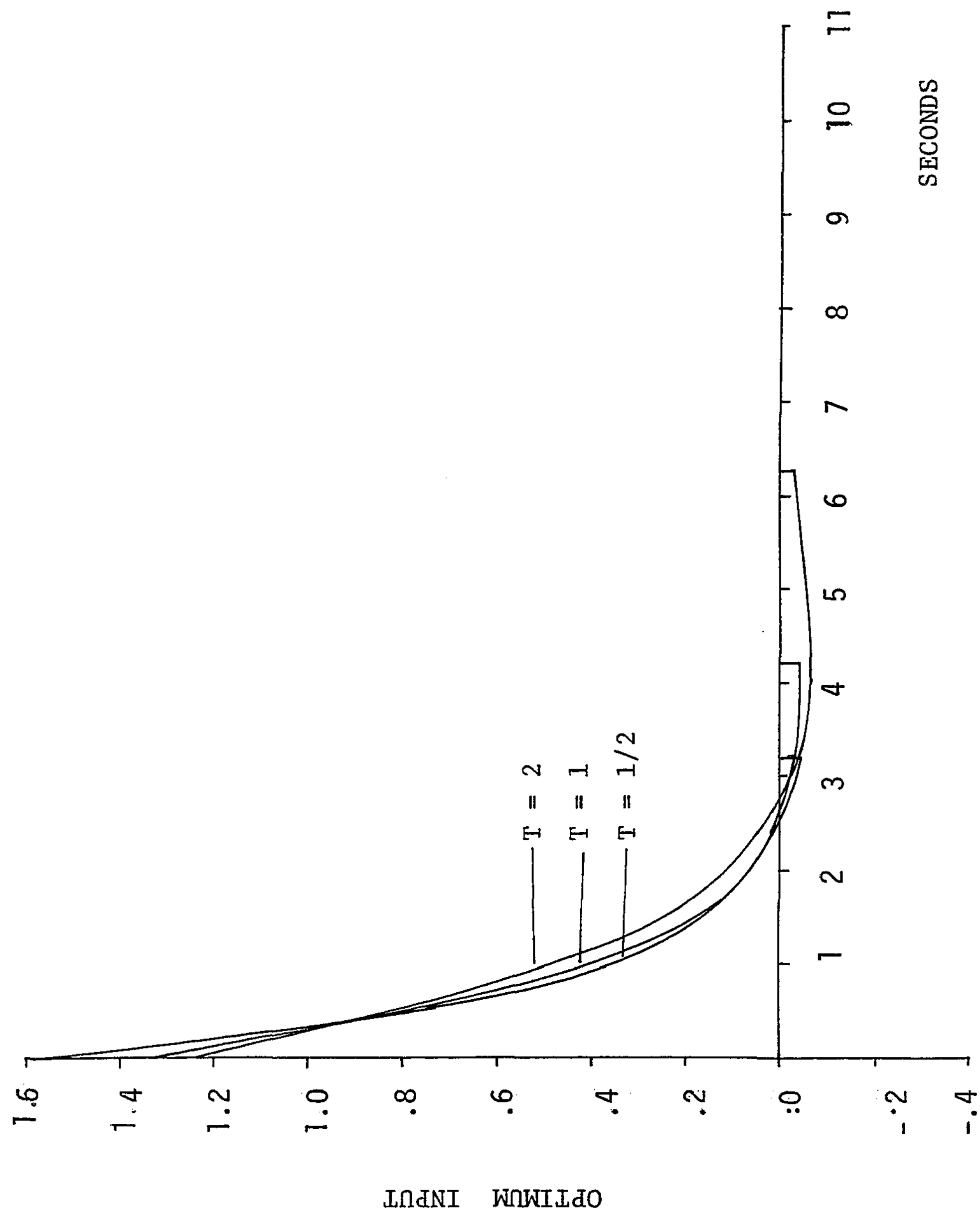


FIGURE 12a

Transient Response of a Second Order System

$\int_0^t f e^2 dt$, $\int_0^t f dt$, $\int_0^t f t e^2 dt$ Controllers.

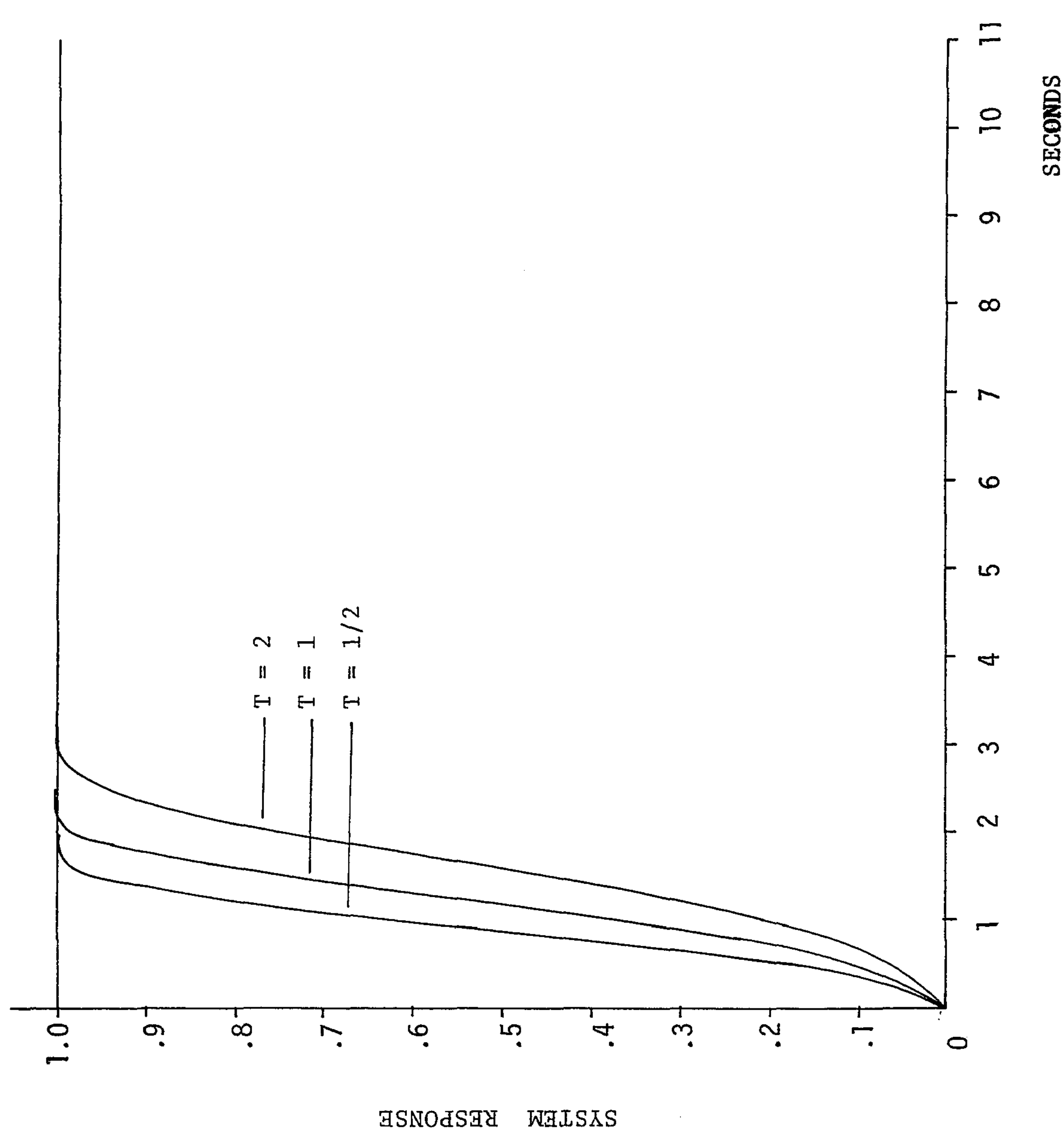


FIGURE 12b

Optimum Input for the Controllers of Fig. 12a

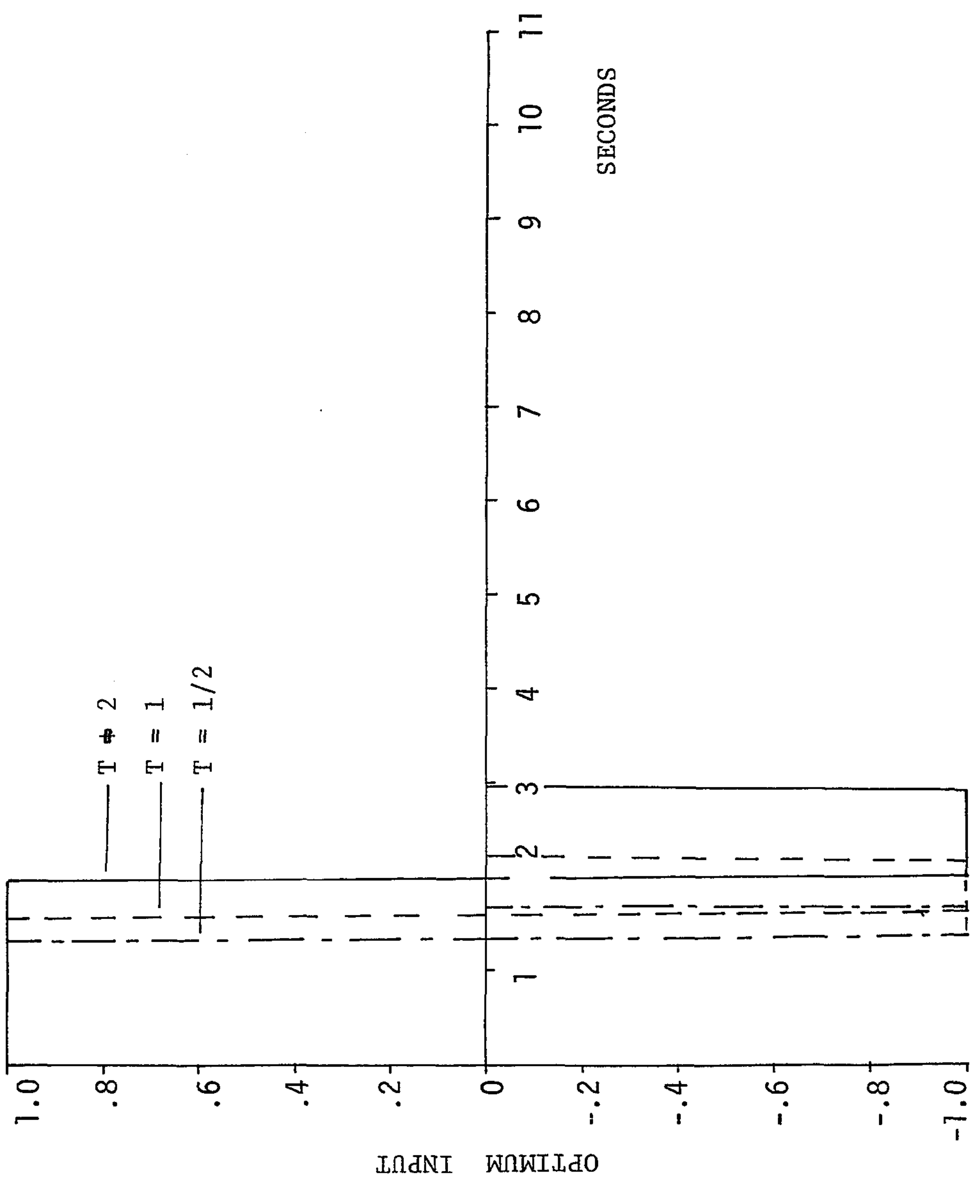


FIGURE 13a

Transient Response of a Second Order System
with a $\int_0^{t_f} \lambda m^2 dt$ Controller

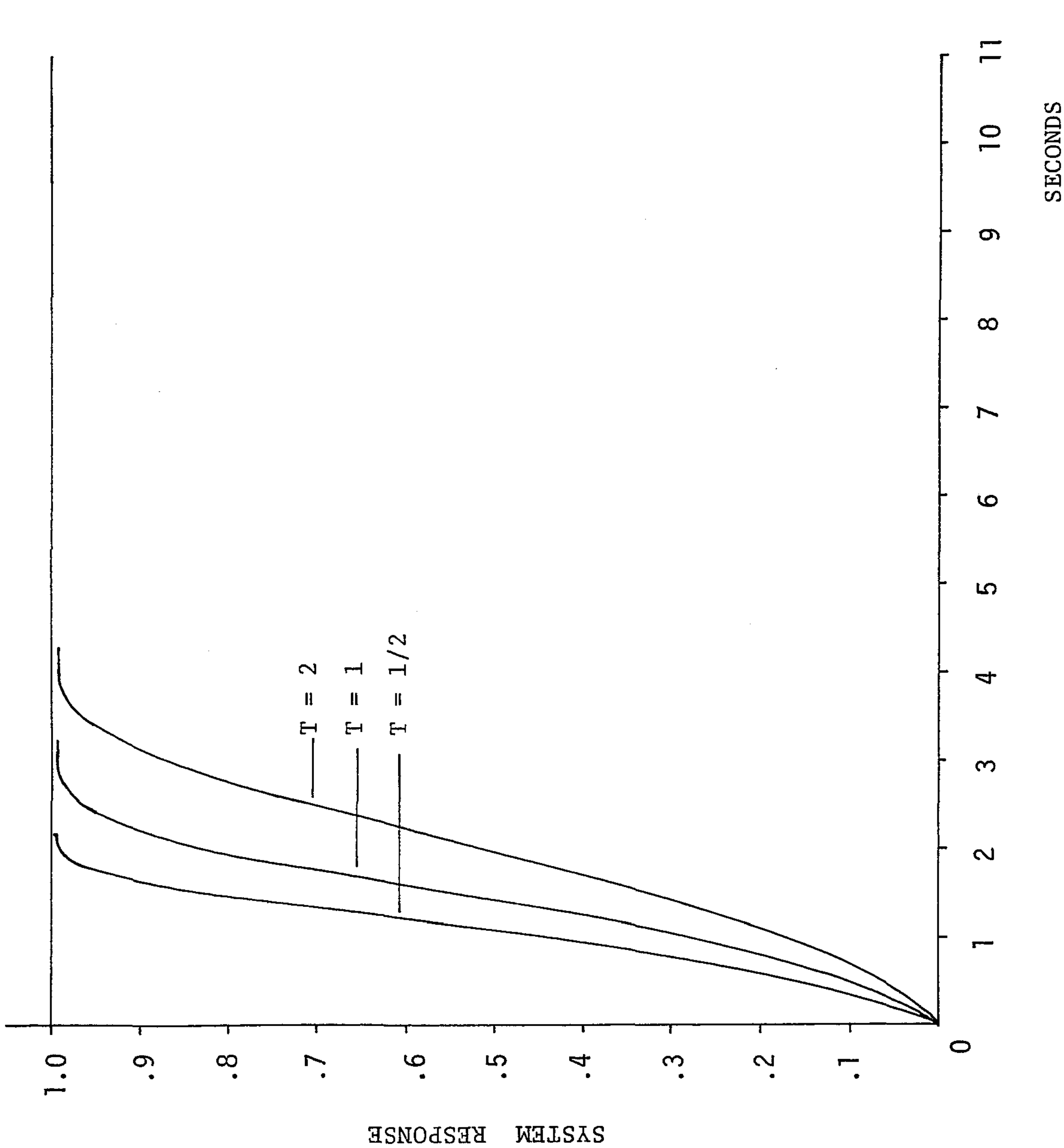


FIGURE 13b

Optimum Input for the Controller of Fig. 13a.

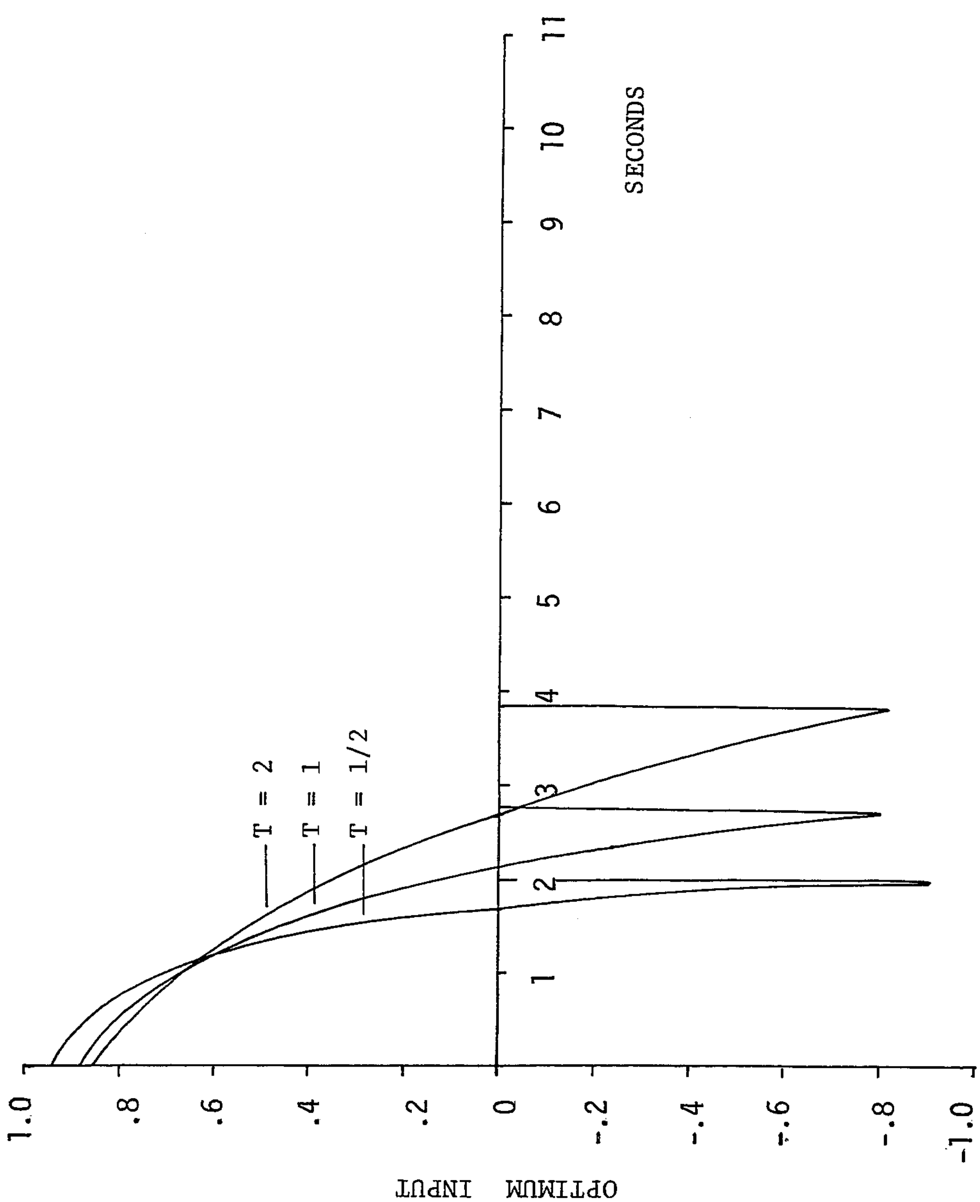


FIGURE 14a

Transient Response of a Second Order System with a $\int_{o}^{t_f} |m| dt$ Controller.

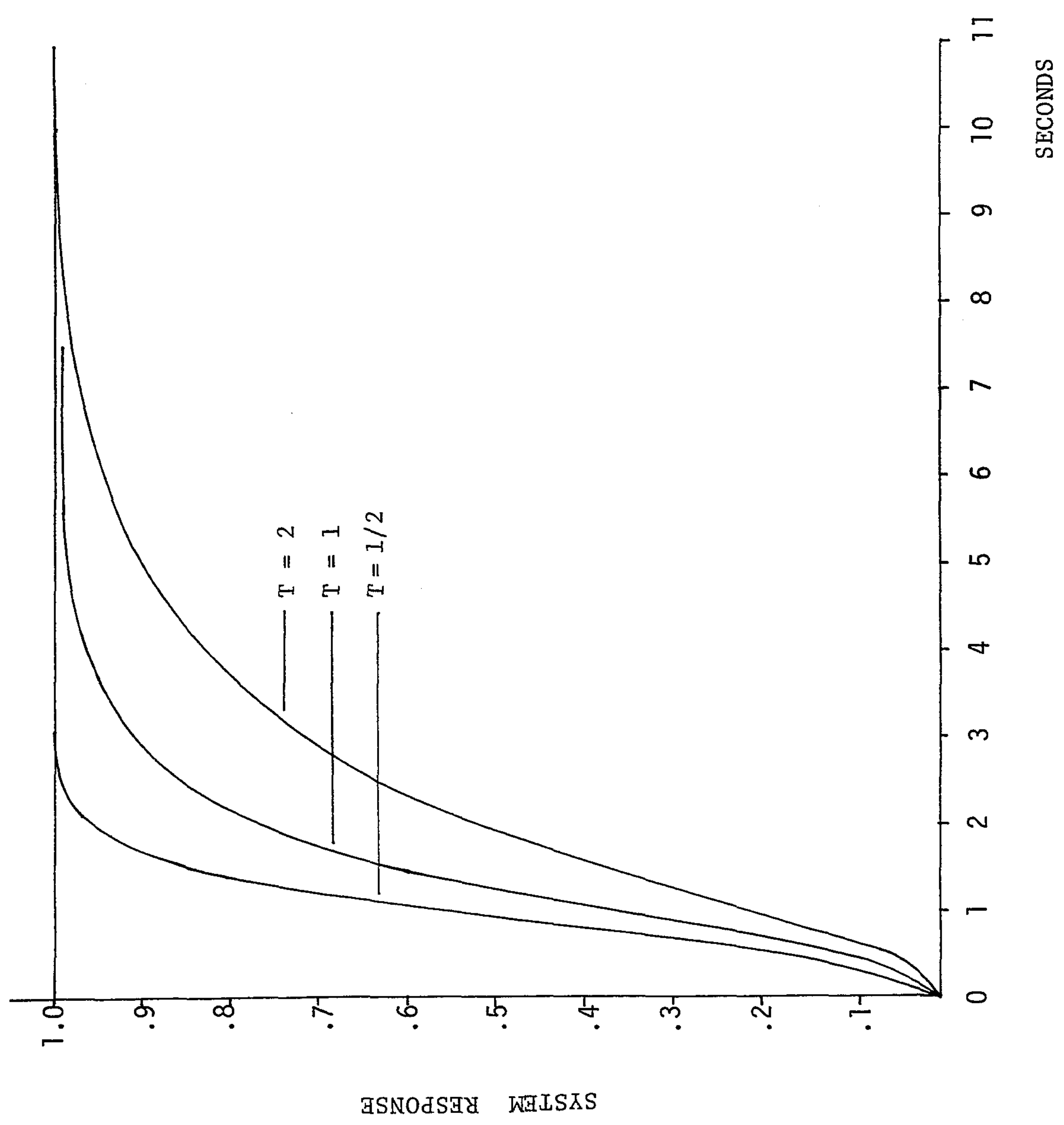
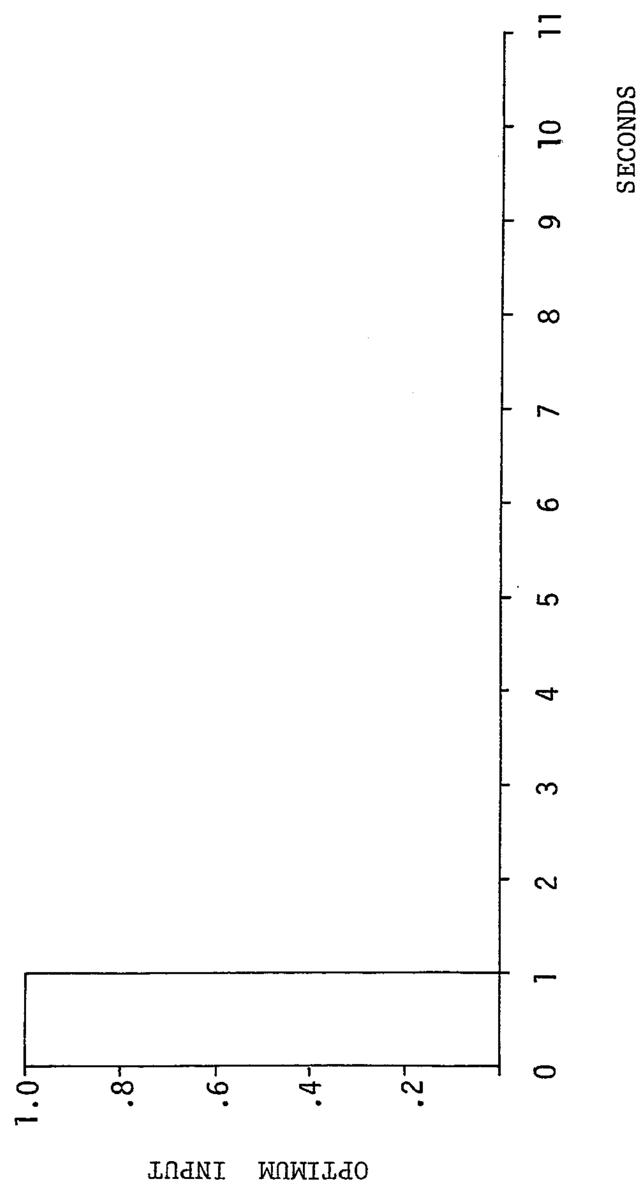


FIGURE 14b

Optimum Input for the Controller of Fig. 14a.



4.3 i) $\int_{o}^{t_f} e^2 dt$

Consider the third order system whose transfer function is given by

$$G(s) = \frac{K}{s^2(sT+1)} \quad (4.50)$$

or

$$G(s) = \frac{K/T}{s^2(s+a)} = \frac{y}{s^2(s+a)} \quad (4.51)$$

where

$$y = 1/T = a$$

The system of (4.51) is represented schematically in Fig. 15, from which we have the following state equations,

$$\dot{x}_1 = x_2 \quad (4.52$$

$$\dot{x}_2 = yx_3 \quad (4.53)$$

$$\dot{x}_3 = -ax_3 + m \quad (4.54)$$

The Hamiltonian for this index and the system of (4.51) is

$$H = -(E-x_1)^2 + P_1x_2 + P_2yx_3 + P_3(-ax_3 + m) \quad (4.55)$$

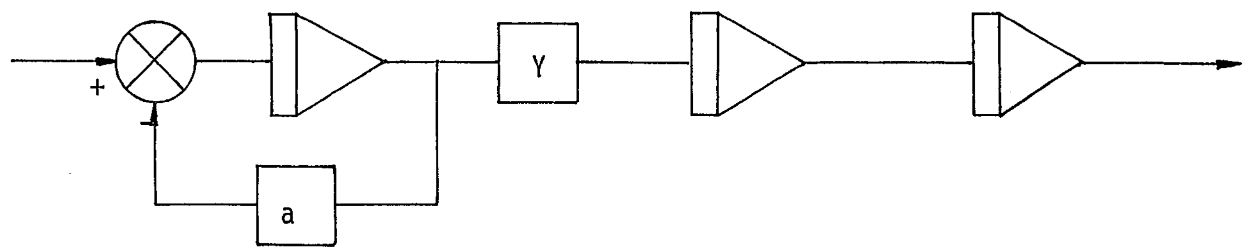
hence

$$\dot{P}_1 = -\frac{\partial H}{\partial x_1} = -2(E-x_1) \quad (4.56)$$

$$\dot{P}_2 = -\frac{\partial H}{\partial x_2} = -P_1 \quad (4.57)$$

FIGURE 15

Schematic Diagram of a Third Order System



$$\dot{P}_3 = -\frac{\partial H}{\partial x_3} = aP_3 - yP_2 \qquad (4.58)$$

the optimum input $m^o = M\text{sgn}\left[P_3\right]$ (4.59)

The adjoint system is obtained using (4.56) through (4.59) and is shown in Fig. 16. The simulation results are shown graphically in Fig. 18 and are also tabulated in Table 2.

4.3 ii) $\int_o^{t_f} dt$

The Hamiltonian for this index and the system of (4.51) is

$$H = -1 + P_1x_2 + P_2yx_3 + P_3(-ax_3 + m) \qquad (4.60)$$

hence

$$\dot{P}_1 = -\frac{\partial H}{\partial x_1} = 0 \qquad (4.61)$$

$$\dot{P}_2 = -\frac{\partial H}{\partial x_2} = P_1 \qquad (4.62)$$

$$\dot{P}_3 = -\frac{\partial H}{\partial x_3} = aP_3 - yP_2 \qquad (4.63)$$

the optimum input $m^o = M\text{sgn}\left[P_3\right]$ (4.64)

The adjoint system is obtained using (4.61) through (4.64) and is shown in Fig. 17. The simulation results are shown graphically in Fig. 18 and are also tabulated in Table 2.

FIGURE 16

Adjoint System for the $\int_o^{t_f} e^2 dt$ Performance Index

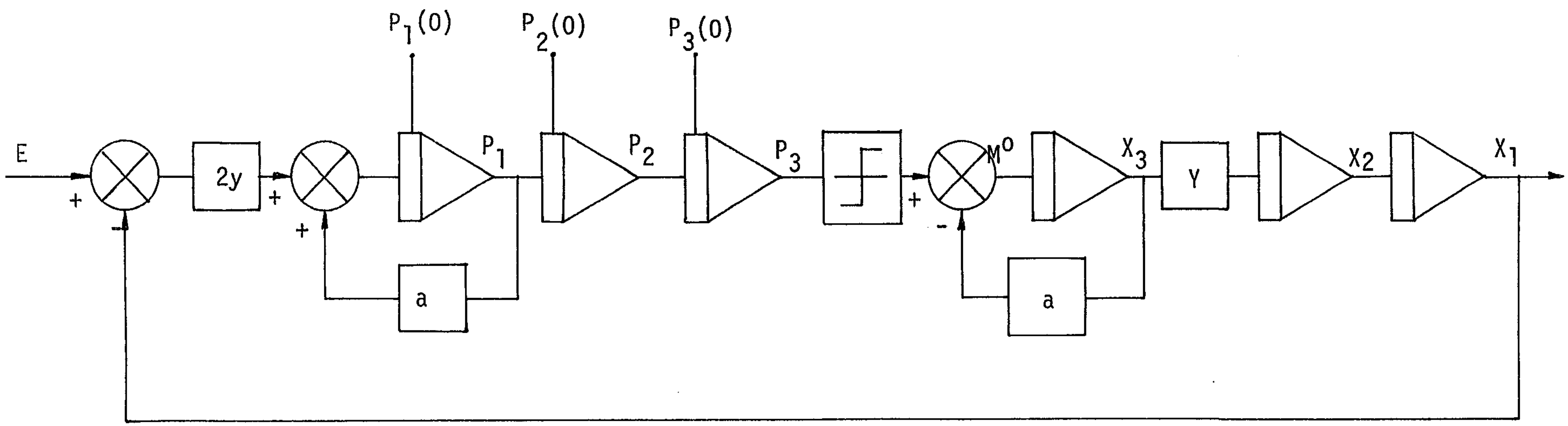


FIGURE 17

Adjoint System for the $\int_0^{t_f} f\,dt$ Performance Index

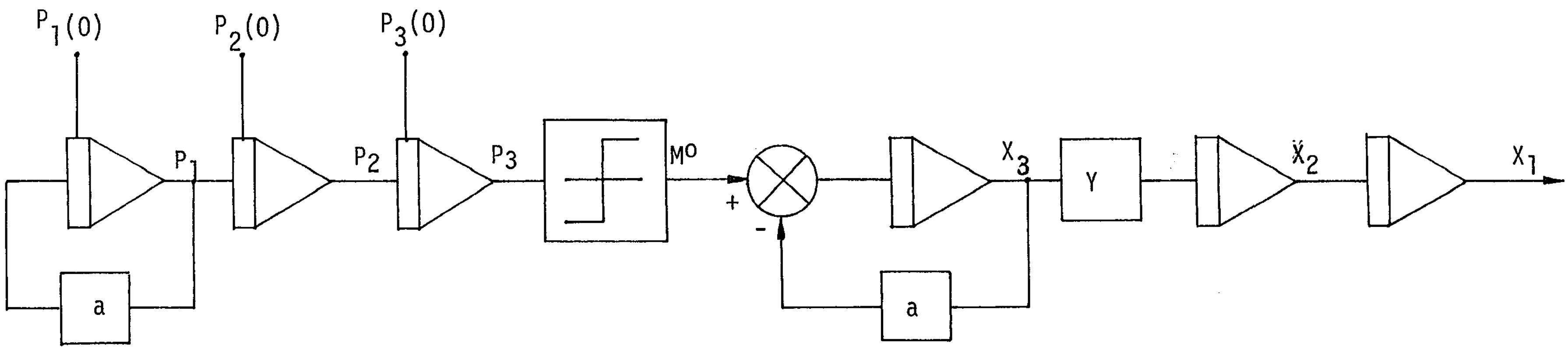


FIGURE 18a

Transient Response of the Third Order System
For The Controllers of Fig. 16 and Fig. 17

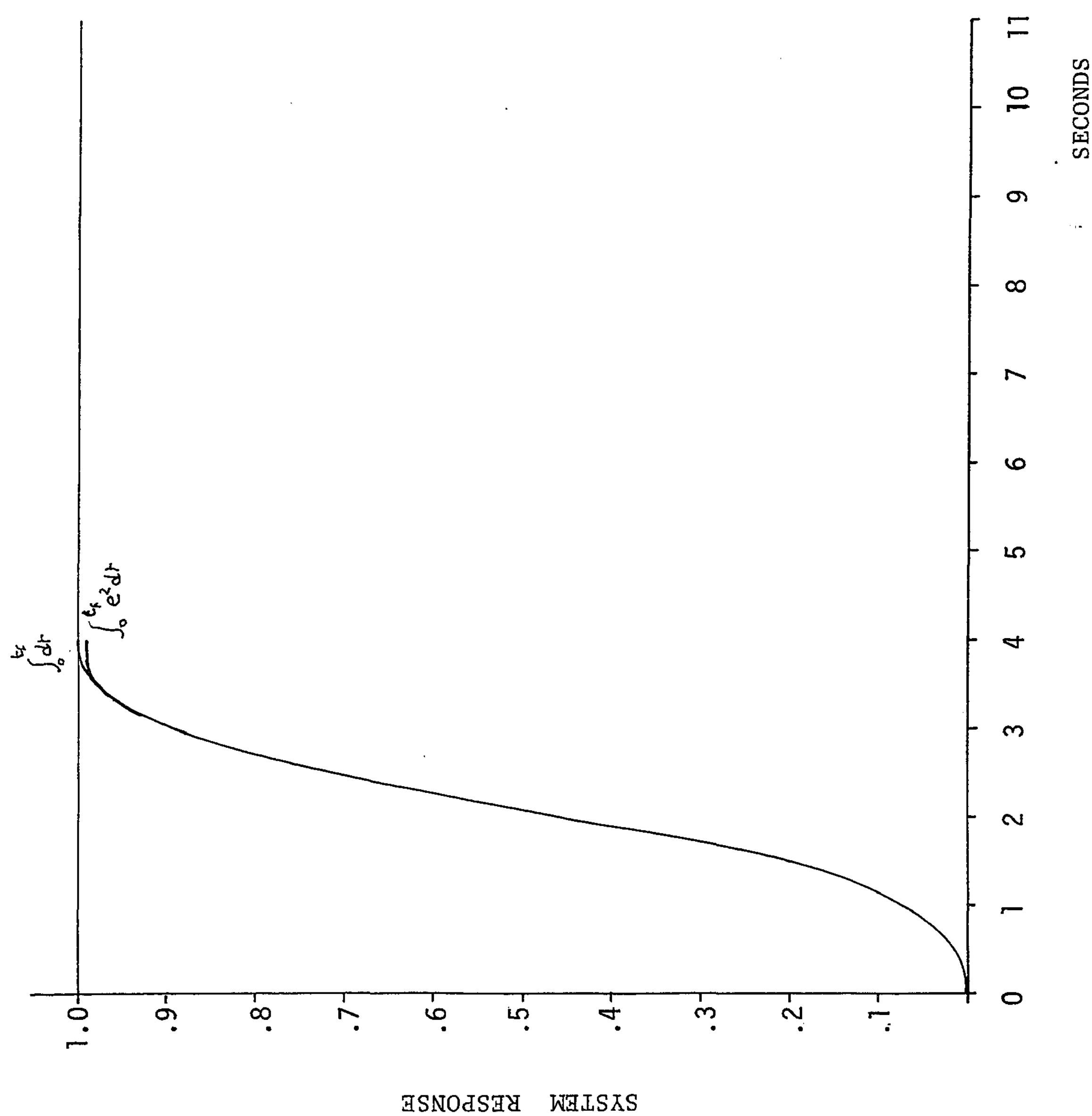
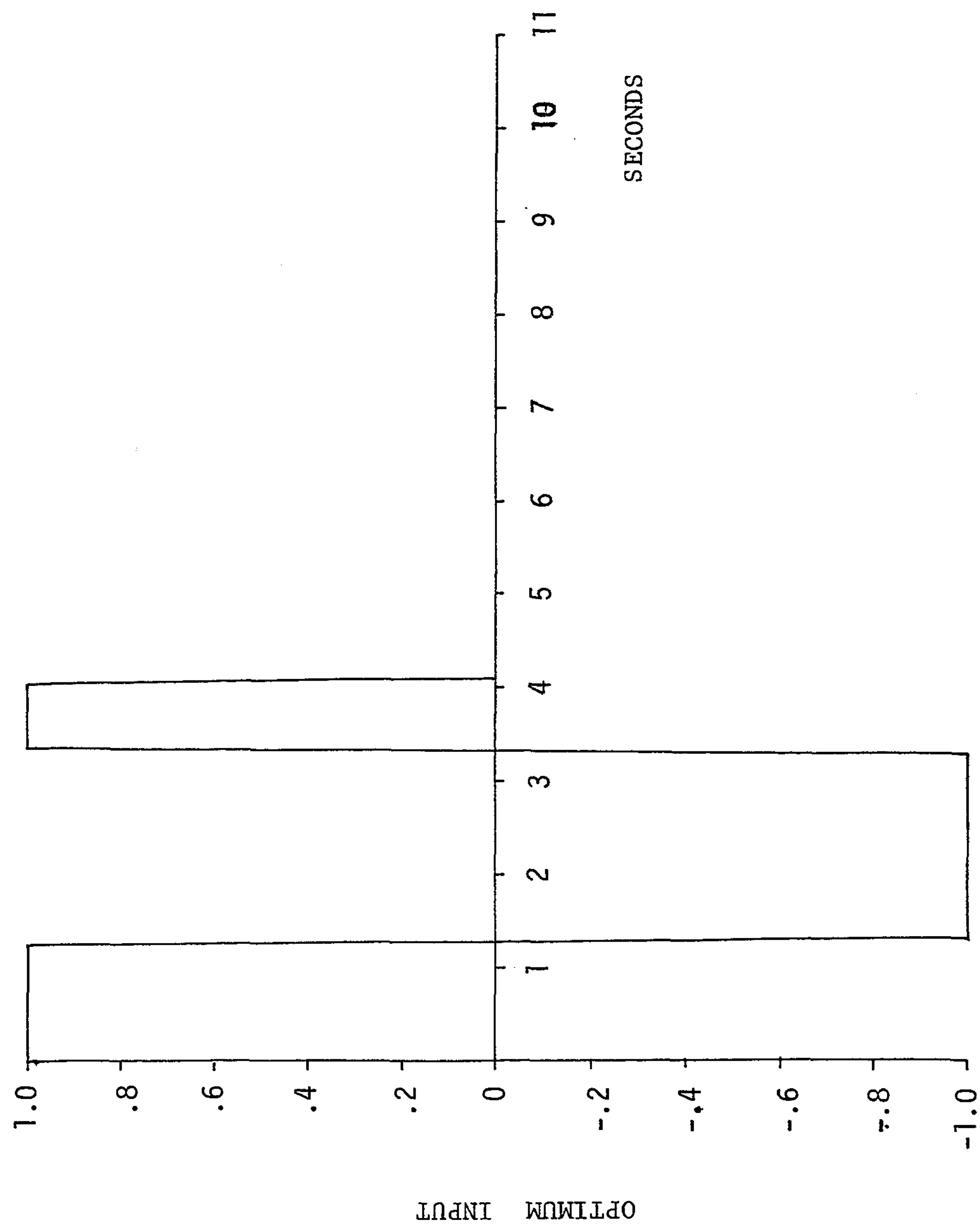


FIGURE 18b

Optimum Input for the Controllers of Fig. 16 and Fig. 17



V. EVALUATION OF INDICES

The simulation results, Table 1, showed that the indices which provide bang-bang control have the best settling time. If controllers are designed to minimize these indices, then, although the individual adjoint systems are different, the system performance for each index is identical. This result is a useful one since it allows us to interchange the time optimal, ISE and ITSE indices, and we can be certain that the system response will remain optimal with respect to any one of these indices. Although the controller designed for minimum time gives the same system response as the ISE and ITSE, both ISE and ITSE controllers are closed loop whereas the time optimal controller is open loop. In most cases a closed loop controller is desirable because the effect of any disturbances within the system will be reduced with closed loop controllers. Also the input to the system with a closed loop controller is dependent upon the system variables. Of the three indices which give bang-bang control the ISE index is probably the best. It gives the same system response as the others and it gives closed loop control. It is also simpler to implement than the ITSE controller. Another important point is that it is impossible to have a time optimal and also a minimum fuel or energy system. These two requirements are contradictory and one has to be foresaken for the other one. If both minimum time and fuel requirements are important, then it is best to fix the amount of fuel available

Table 1a

Minimum Value of Each Performance Index for a Second Order System. Time Constant T=2.0. Simulation Time 10 seconds. λ=1

Controller Designed For The Index	Minimum Value of Each P.I.					
	$\int(e^2+\lambda m^2)dt$	$\int e^2dt$	$\int dt$	$\int\lambda m^2dt$	$\int te^2dt$	$\int\lvert m\rvert dt$
$\int(e^2+\lambda m^2)dt$	2.2775	1.3973	6.2999	0.8802	1.2882	1.3384
$\int e^2dt$	4.1615	1.1948	2.9500	2.9666	0.8290	2.9666
$\int dt$	4.1615	1.1948	2.9500	2.9666	0.8290	2.9666
$\int\lambda m^2dt$	2.5740	1.4301	3.8500	1.1440	1.2423	1.8594
$\int te^2dt$	4.1615	1.1948	2.9500	2.9666	0.8290	2.9666
$\int\lvert m\rvert dt$	2.4769	1.4685	9.9999	1.0083	1.5529	1.0083

Table 1b

Minimum Value of Each Performance Index for a Second Order System. Time Constant T=1.0. Simulation Time 10 seconds. $\lambda=1$

	Minimum Value of Each P.I.					
	$\int e^2+\lambda m^2 dt$	$\int e^2 dt$	$\int dt$	$\int \lambda m^2 dt$	$\int te^2 dt$	$\int \lvert m \rvert dt$
$\int (e^2+\lambda m^2) dt$	1.8032	0.9655	4.4999	0.8377	0.6122	1.1098
$\int e^2 dt$	3.0784	0.8868	2.2000	2.1916	0.4555	2.1916
$\int dt$	3.0784	0.8868	2.2000	2.1916	0.4555	2.1916
$\int \lambda m^2 dt$	2.0328	1.0208	2.8000	1.0120	0.6286	1.5085
$\int te^2 dt$	3.0784	0.8868	2.2000	2.1916	0.4555	2.1916
$\int \lvert m \rvert dt$	1.9627	0.9710	7.4499	0.9917	0.6197	0.9917

Table 1c

Minimum Value of Each Performance Index for a Second Order System. Time Constant T=0.5 Simulation Time 10 seconds. $\lambda=1$

Controller Designed For The Index	Maximum Value of Each P.I.					
	$\int(e^2+\lambda m^2)dt$	$\int e^2dt$	$\int dt$	$\int\lambda m^2dt$	$\int te^2dt$	$\int\lvert m\rvert dt$
$\int(e^2+\lambda m^2)dt$	1.5414	0.6669	3.2199	0.8746	0.3021	1.0404
$\int e^2dt$	2.3318	0.6752	1.6600	1.6566	0.2748	1.6566
$\int dt$	2.3318	0.6752	1.6600	1.6566	0.2748	1.6566
$\int\lambda m^2dt$	1.6960	0.7347	2.0000	0.9614	0.3341	1.2728
$\int te^2dt$	2.3318	0.6752	1.6600	1.6566	0.2748	1.6566
$\int\lvert m\rvert dt$	1.7050	0.6883	3.0999	1.0167	0.2938	1.0167

and make the system time optimal with this constraint. This would provide sub-optimal control, but there is no other way around this problem.

Of the three indices that include the control signal m, probably the $\int_0^{t_f}(e^2+\lambda m^2)dt$ is the best all round index. It is better in every respect than the $\int_0^{t_f}|m|\,dt$, except in the amount of fuel consumed to transfer the system from its initial to its final state. The difference in the amount of fuel consumed is of the order of 33% more for the $\int_0^{t_f}(e^2+\lambda m^2)dt$, than for the minimum fuel index.

The $\int_0^{t_f}\lambda m^2 dt$ performance index gave somewhat similar result as the $\int_0^{t_f}(e^2+\lambda m^2)dt$, in each case giving a slightly higher value for each index except for the minimum time and the ITSE. It was thought that if the system was designed to minimise one particular index, then no other controller could possibly give a smaller value of the index than the controller designed for this index. This was shown to be not the case. The controllers designed to minimize $\int_0^{t_f}(e^2+\lambda m^2)dt$ and $\int_0^{t_f}|m|\,dt$ gave a smaller value for the index $\int_0^{t_f}\lambda m^2 dt$ than did the controller designed for the $\int_0^{t_f}\lambda m^2 dt$ index. This would indicate that if the design criteria is to be the conservation of fuel or energy, then the $\int_0^{t_f}(e^2+\lambda m^2)dt$ and $\int_0^{t_f}|m|\,dt$ indices give better results than $\int_0^{t_f}\lambda m^2 dt$. However, the $\int_0^{t_f}\lambda m^2 dt$ index gives the best settling time of the indices that involve the control signal m. This again shows that it is not possible to have minimum time and minimum fuel or energy control.

Some of the indices provide open loop controllers, which

Table 2.

Minimum Value of the Two Performance Indices for a Third Order System. Time Constant T=2.0 seconds. Simulation time 10 seconds.

Controller Designed for the Index	Minimum Value of Each Performance Index	
	$\int e^2 dt$	$\int dt$
$\int e^2 dt$	1.6558	4.1000
$\int dt$	1.6557	4.0000

is an undesirable feature. This is not a very serious drawback since once the optimum trajectories have been determined they can be stored using function generators, and the resultant open loop controller can be made closed loop. This may be a better method of controlling the system since it would obligate the need for the adjoint system. This has been tried by Roots and Lees[8] with success, for the $\int_0^{t_f}(e^2 + \lambda m^2)dt$ performance index.

It is difficult to compare directly the two indices that gave continuous control due to the fact that the optimum input m^o is different for each index. In the simulation described in this thesis λ was the same for both performance indices. This may not be the best comparison and perhaps a more realistic comparison would be to choose λ so that the amount of fuel consumed is the same for both indices. Another possibility is to impose identical constraints on the magnitude of the control signal m^o.

VI. CONCLUSION

6.1 $\int_0^{t_f} (e^2 + \lambda m^2) dt$

The simulation results showed this to be a very useful performance index. Most optimal control literature uses this index more frequently than any of the others. This is due to the fact that it gives a satisfactory system response and it can be handled either using dynamic programming or the maximum principle. Its only disadvantage is that the settling time is inferior to the other indices, except the minimum fuel index. If the settling time is of secondary importance, this is probably the most useful all-round index.

6.2 $\int_0^{t_f} e^2 dt$

Of the indices that give bang-bang control this is probably the best one. It has a closed loop controller which is desirable but not essential. As with the other bang-bang controllers, the fuel and energy consumptions are relatively high with respect to the minimum fuel index.

6.3 $\int_0^{t_f} dt$

This index has been extensively used in optimal control, especially in the phase-plane analysis. It has been shown in this thesis that identical results can be ob tained using an $\int_0^{t_f} e^2 dt$ performance index with the added advantage that the latter provides a closed loop controller. If the optimization is carried out using the maximum principle this index is inferior to $\int_0^{t_f} e^2 dt$. It has the same disadvantages as $\int_0^{t_f} e^2 dt$ with an additional one, that is, the controller is open loop and is therefore independent of the system variables.

6.4 $\int_0^{t_f} \lambda m^2 dt$

This index was found to be inferior in most respects to $\int_0^{t_f} (e^2 + \lambda m^2) dt$ index, and also to the indices giving bang-bang control. For most applications if continuous control is desirable the $\int_0^{t_f} (e^2 + \lambda m^2)\, dt$ index would give better results than the $\int_0^{t_f} \lambda m^2 dt$ index.

6.5 $\int_0^{t_f} te^2 dt$

This index gave the same system response as the other bang-bang indices. Its only disadvantage is that it is slightly more complex to implement than the $\int_0^{t_f} e^2 dt$ index. Since these two indices give the same system response, the $\int_0^{t_f} e^2 dt$ index would in most cases be the more useful.

6.6 $\int_0^{t_f} |m| dt$

The minimum fuel index gives very poor settling times. This is a result of the input being on for only a short time. The controller required to achieve this requirement is of the

ON-OFF type for a type 0 or 1 system. For a type 2 system a 3 position relay is required.

Of the six performance indices considered the $\int_0^{t_f} (e^2 + \lambda m^2) dt$ and $\int_0^{t_f} e^2 dt$ are the best. The first index can be used very easily using either dynamic programming or the maximum principle. However, the dynamic programming technique is better suited for this index than the maximum principle especially for third order systems or higher.

The $\int_0^{t_f} e^2 dt$ index is better suited to the maximum principle since it requires discontinuous control. For systems of higher order than two it becomes very difficult to solve for the initial conditions on the adjoint vectors.

The findings in this thesis can be summed up briefly as,

i) Optimization using dynamic programming is not easily applicable to indices other than the $\int_0^{t_f} (e^2 + \lambda m^2) dt$.

ii) The performance index dictates the type of controller, i.e. continuous or discontinuous. This has also been reported by Roots and Lees[10].

iii) The system response is the same for any performance index that gives bang-bang control.

APPENDIX

The program used to evaluate the initial conditions for the $\int_{o}^{t} fe^2 dt$ performance index is shown for both the second and third order systems. Minor modifications are required to accommodate the other performance indices.

```
      C     PONTRYAGINS MAXIMUM PRINCIPLE
      C     SECOND ORDER SYSTEM WITH ONE TIME CONSTANT
      C     INTEGRAL ERROR SQUARED PERFORMANCE INDEX
      C     FOURTH ORDER RUNGE-KUTTA INTEGRATION
      C     DATA CARDS CONTAIN THE NUMBER OF EQUATIONS
      C     THE SIZE OF THE INCREMENT
      C     THE INITIAL VALUE OF X
      C     AND THE INITIAL VALUES FOR EACH F(X)
      C     THE FIRST NUMBER IS IN I FORMAT
      C     THE REST ARE IN F FORMAT
            DIMENSION XN(10),X(10),Q(10,4),FX(10),A(200,10)
            DIMENSION AB(3),EMK(200)
            READ 100 ,N,H,TN,(XN(K),K=1,N)
        100 FORMAT(I2,12F4.2)
            WRITE(6,103)N,H,TN,(XN(K),K=1,N)
        103 FORMAT(1H0,I2,12(2X(F4.2)))
            KK=0
            T1=2.0
            AA=1.0/T1
            Y=AA*4.0
            F=1.0
            ALAMDA=1.0
            XN(3)=1.0
            XN(4)=-2.0
            XO=XN(3)
            XODOT=XN(4)
            X(1)=0.0
            X(2)=0.0
            DXO=0.02
            DXODOT=-0.02
            JK=0
         12 JK=JK+1
            IF(JK.GT.90)GO TO 500
            J2=1
            NN=0
            WRITE(6,14)JK,XO,XODOT,X(1),X(2)
         14 FORMAT(5X,I5,4(2XF10.5))
          3 DO 15 M=1,N
            X(M)=0.0
         15 FX(M)=0.0
            NN=NN+1
          1 L=1
            T=TN
            DO 777 K=1,N
        777 X(K)=XN(K)
            GO TO 101
         10 DO 151 K=1,N
        151 Q(K,L)=H*FX(K)
            T=TN+H/2.
            DO 252 K=1,N
        252 X(K)=XN(K)+Q(K,L)/2.
            L=2
            GO TO 101
         20 DO 251 K=1,N
        251 Q(K,L)=H*FX(K)
            T=TN+H/2.
            DO 352 K=1,N
        352 X(K)=XN(K)+Q(K,L)/2.
```

```
      L=3
      GO TO 101
   30 DO 351 K=1,N
  351 Q(K,L)=H*FX(K)
      T=TN+H
      DO 452 K=1,N
  452 X(K)=XN(K)+Q(K,L)
      L=4
      GO TO 101
   40 DO 451 K=1,N
  451 Q(K,L)=H*FX(K)
      GO TO 7
  101 CONTINUE
      IF(X(2).LT.0.0)GO TO 114
      IF(X(3))110,111,112
  110 EMK(J2)=-1.0
      GO TO 115
  114 X(2)=0.0
  111 EMK(J2)=0.0
      GO TO 115
  112 EMK(J2)=1.0
  115 CONTINUE
      FX(1)=Y*X(2)
      FX(2)=-AA*X(2)+EMK(J2)
      FX(3)=X(4)
      FX(4)=2.0*Y*(E-X(1))+AA*X(4)
      FX(5)=(E-X(1))**2+ALAMDA*EMK(J2)**2
      FX(6)=(E-X(1))**2
      FX(7)=1.0
      FX(8)=ALAMDA*EMK(J2)**2
      FX(9)=TN*(E-X(1))**2
      FX(10)=ABS(EMK(J2))
    4 GO TO (10,20,30,40),L
    7 TN=TN+H
      DO 8 K=1,N
    8 XN(K)=XN(K)+(1./6.)*(Q(K,1)+2.*Q(K,2)+2.*Q(K,3)+Q(K,4))
      DO 2 INT=1,N
    2 A(J2,INT)=XN(INT)
      IF(J2.LT.2)GO TO 66
      IF(A(J2,2).LE.0.0) A(J2,7)=A(J2-1,7)
      IF(ABS(A(J2,1)-E).LE.0.01.AND.ABS(A(J2-1,1)-E).LE.0.01.AND
     1ABS(A(J2,2)).LE.0.02)GO TO 25
      IF(A(J2,2).LT.0.0) GO TO 55
      GO TO 66
   55 IF(KK.EQ.0) GO TO 13
      EMK(J2)=0.0
      A(J2,2)=0.0
      IF(KK.GT.0) GO TO 25
   66 CONTINUE
      J2=J2+1
      IF(J2-200)1,1,13
   13 CONTINUE
      IF(KK.GT.0)GO TO 25
      WRITE(6,21) A(J2,1),A(J2,2),J2,NN,A(J2,6),XO,DXO,XODOT,DXO
   21 FORMAT(5X,2F10.4,2I5,5F10.4)
      IF(NN.GT.1)GO TO 6
      AB(NN)=A(J2,1)
      DXO=0.02*(ABS(E-AB(NN)))+0.0001
```

```
      IF(ABS(E-AB(NN)).LE.0.05) DXO=DXO/2.0
      XO=XO+DXO
      DO 32 J=1,N
   32 XN(J)=0.0
      DO 33 K=1,N
      DO 33 J2=1,200
   33 A(J2,K)=0.0
      XN(3)=XO
      XN(4)=XODOT
      J2=1
      TN=0.0
      GO TO 3
    6 IF(NN.GT.2)GO TO 9
      AB(NN)=A(J2,1)
      AK=AB(NN-1)-AB(NN)
      IF(AK)41,42,43
   41 AK=-1.0
      GO TO 44
   42 AK=0.0
      GO TO 44
   43 AK=1.0
   44 CONTINUE
      IF(AB(NN).GT.E) GO TO 45
      XO=XO-2.0*AK*DXO
      GO TO 46
   45 XO=XO+2.0*AK*DXO
   46 CONTINUE
      DXODOT=0.02*(ABS(E-AB(NN)))+0.0001
      IF(ABS(E-AB(NN)).LE.0.05) DXODOT=DXODOT/2.0
      XODOT=XODOT+DXODOT
      DO 34 J=1,N
   34 XN(J)=0.0
      DO 47 K=1,N
      DO 47 J2=1,200
   47 A(J2,K)=0.0
      XN(3)=XO
      XN(4)=XODOT
      J2=1
      TN=0.0
      GO TO 3
    9 AB(NN)=A(J2,1)
      BK=AB(NN-1)-AB(NN)
      IF(BK)51,52,53
   51 BK=-1.0
      GO TO 54
   52 BK=0.0
      GO TO 54
   53 BK=1.0
   54 CONTINUE
      IF(AB(NN).GT.E)GO TO 56
      XODOT=XODOT-2.0*BK*DXODOT
      GO TO 57
   56 XODOT=XODOT+2.0*BK*DXODOT
   57 DO 58 J=1,N
   58 XN(J)=0.0
      DO 59 K=1,N
      DO 59 J2=1,200
   59 A(J2,K)=0.0
```

```
          XN(3)=XO
          XN(4)=XODOT
          TN=0.0
          GO TO 12
      500 WRITE(6,505)
      505 FORMAT( ' CONVERGENCE HAS NOT BEEN OBTAINED')
          GO TO 700
       25 CONTINUE
          IF(A(J2,2).LE.0.0) GO TO 211
          GO TO 212
      211 A(J2,2)=0.0
          EMK(J2)=0.0
      212 J2=J2+1
          KK=KK+1
          IF(KK.LT.10) GO TO 1
          TN=0.0
          L=1
          DO 26 K=1,N
       26 A(L,K)=0.0
          A(1,3)=XO
          A(1,4)=XODOT
          J=1
          WRITE(6,75)J,TN,EMK(1),(A(L,K),K=1,N)
       75 FORMAT(1H1,4X,I5,12F10.5)
          I=J2-1
          TN=H
          DO 62 J=1,I
          WRITE(6,61) J,TN,EMK(J ),(A(J,K),K=1,N)
       61 FORMAT(5X,I5,12F10.4)
       62 TN=TN+H
      700 STOP
          END
```

```
C       PONTRYAGINS MAXIMUM PRINCIPLE
C       THIRD ORDER SYSTEM WITH ONE TIME CONSTANT
C       INTEGRAL ERROR SQUARED PERFORMANCE INDEX
C       FOURTH ORDER RUNGE-KUTTA INTEGRATION
C       DATA CARDS CONTAIN THE NUMBER OF EQUATIONS
C       THE SIZE OF THE INCREMENT
C       THE INITIAL VALUE OF X
C       AND THE INITIAL VALUES FOR EACH F(X)
C       THE FIRST NUMBER IS IN I FORMAT
C       THE REST ARE IN F FORMAT
        DIMENSION XN(12),X(12),Q(12,4),FX(12),A(200,12)
        DIMENSION AB(4),EMK(200)
        READ 100 ,N,H,TN,(XN(K),K=1,N)
    100 FORMAT(I2,14F4.2)
        WRITE(6,103)N,H,TN,(XN(K),K=1,N)
    103 FORMAT(1H0,I2,14(2X(F4.2)))
        KK=0
        T1=2.0
        AA=1.0/T1
        Y=AA
        E=1.0
        ALAMDA=1.0
        XN(4)=1.75
        XN(5)=-2.75
        XN(6)=1.35
        XA=XN(4)
        XB=XN(5)
        XC=XN(6)
        X(1)=0.0
        X(2)=0.0
        X(3)=0.0
        DXA=0.05
        DXB=-0.05
        DXC=-0.05
        JK=0
     12 JK=JK+1
        IF(JK.GT.100)GO TO 500
        J2=1
        NN=0
        WRITE(6,14)JK,XA,DXA,XB,DXB,XC,DXC,X(1),X(2),X(3)
     14 FORMAT(5X,I5,9(2XF10.5))
      3 DO 15 M=1,N
        X(M)=0.0
     15 FX(M)=0.0
        NN=NN+1
      1 L=1
        T=TN
        DO 777 K=1,N
    777 X(K)=XN(K)
        GO TO 101
     10 DO 151 K=1,N
    151 Q(K,L)=H*FX(K)
        T=TN+H/2.
        DO 252 K=1,N
    252 X(K)=XN(K)+Q(K,L)/2.
        L=2
        GO TO 101
     20 DO 251 K=1,N
```

```
  251 Q(K,L)=H*FX(K)
      T=TN+H/2.
      DO 352 K=1,N
  352 X(K)=XN(K)+Q(K,L)/2.
      L=3
      GO TO 101
   30 DO 351 K=1,N
  351 Q(K,L)=H*FX(K)
      T=TN+H
      DO 452 K=1,N
  452 X(K)=XN(K)+Q(K,L)
      L=4
      GO TO 101
   40 DO 451 K=1,N
  451 Q(K,L)=H*FX(K)
      GO TO 7
  101 CONTINUE
      IF(J2.LT.2)GO TO 102
      IF(EMK(J2-1).EQ.0.0)GO TO 114
  102 CONTINUE
      IF(X(4))110,111,112
  110 EMK(J2)=-1.0
      GO TO 115
  114 A(J2,2)=0.0
      A(J2,3)=0.0
  111 EMK(J2)=0.0
      GO TO 115
  112 EMK(J2)=1.0
  115 CONTINUE
      FX(1)=X(2)
      FX(2)=Y*X(3)
      FX(3)=-AA*X(3)+EMK(J2)
      FX(4)=X(5)
      FX(5)=X(6)
      FX(6)=AA*X(6)-2.0*Y*(E-X(1))
      FX(7)=(E-X(1))**2+ALAMDA*EMK(J2)**2
      FX(8)=(E-X(1))**2
      FX(9)=1.0
      FX(10)=ALAMDA*EMK(J2)**2
      FX(11)=TN*(E-X(1))**2
      FX(12)=ABS(EMK(J2))
    4 GO TO (10,20,30,40),L
    7 TN=TN+H
      DO 8 K=1,N
    8 XN(K)=XN(K)+(1./6.)*(Q(K,1)+2.*Q(K,2)+2.*Q(K,3)+Q(K,4))
      DO 2 INT=1,N
    2 A(J2,INT)=XN(INT)
      IF(J2.LT.2)GO TO 66
      IF(A(J2,2).LE.0.0) A(J2,7)=A(J2-1,7)
      IF(ABS(A(J2,1)-E).LE.0.02.AND.ABS(A(J2-1,1)-E).LE.0.02.AN
     1ABS(A(J2,2)).LE.0.03.AND.ABS(A(J2,3)).LE.0.03)GO TO 25
      IF(A(J2,2).LE.-0.05.OR.ABS(A(J2,1)-E).GE.E)GO TO 55
      GO TO 66
   55 IF(KK.EQ.0) GO TO 13
      EMK(J2)=0.0
      A(J2,2)=0.0
      A(J2,3)=0.0
      IF(KK.GT.0) GO TO 25
```

```
   66 CONTINUE
      J2=J2+1
      IF(J2-200)1,1,13
   13 CONTINUE
      IF(KK.GT.0)GO TO 25
      WRITE(6,21) A(J2,1),A(J2,2),A(J2,3),J2,NN,A(J2,9),XA,DXA,
     1XC,DXC
   21 FORMAT(5X,3F10.4,2I5,7F10.4)
      IF(NN.GT.1)GO TO 6
      AB(NN)=A(J2,1)
      DXA=0.02*(ABS(E-AB(NN)))
      IF(ABS(E-AB(NN)).LE.0.02) DXA=DXA/2.0
      XA=XA+DXA
      DO 32 J=1,N
   32 XN(J)=0.0
      DO 33 K=1,N
      DO 33 J2=1,200
   33 A(J2,K)=0.0
      XN(4)=XA
      XN(5)=XB
      XN(6)=XC
      J2=1
      TN=0.0
      GO TO 3
    6 IF(NN.GT.2)GO TO 9
      AB(NN)=A(J2,1)
      AK=AB(NN-1)-AB(NN)
      IF(AK)41,42,43
   41 AK=-1.0
      GO TO 44
   42 AK=0.0
      GO TO 44
   43 AK=1.0
   44 CONTINUE
      IF(AB(NN).GT.E) GO TO 45
      XA=XA-2.0*AK*DXA
      GO TO 46
   45 XA=XA+2.0*AK*DXA
   46 CONTINUE
      DXB=0.02*(ABS(E-AB(NN)))
      IF(ABS(E-AB(NN)).LE.0.02) DXB=DXB/2.0
      XB=XB+DXB
      DO 34 J=1,N
   34 XN(J)=0.0
      DO 47 K=1,N
      DO 47 J2=1,200
   47 A(J2,K)=0.0
      XN(4)=XA
      XN(5)=XB
      XN(6)=XC
      J2=1
      TN=0.0
      GO TO 3
    9 IF(NN.GT.3)GO TO 11
      AB(NN)=A(J2,1)
      BK=AB(NN-1)-AB(NN)
      IF(BK)51,52,53
   51 BK=-1.0
```

```
      GO TO 54
   52 BK=0.0
      GO TO 54
   53 BK=1.0
   54 CONTINUE
      IF(AB(NN).GT.E)GO TO 56
      XB=XB-2.0*BK*DXB
      GO TO 57
   56 XB=XB+2.0*BK*DXB
   57 CONTINUE
      DXC=0.01*(ABS(E-AB(NN)))
      IF(ABS(E-AB(NN)).LE.0.02) DXC=DXC/2.0
      XC=XC+DXC
      DO 58 J=1,N
   58 XN(J)=0.0
      DO 59 K=1,N
      DO 59 J2=1,200
   59 A(J2,K)=0.0
      XN(4)=XA
      XN(5)=XB
      XN(6)=XC
      J2=1
      TN=0.0
      GO TO 3
   11 AB(NN)=A(J2,1)
      CK=AB(NN-1)-AB(NN)
      IF(CK)91,92,93
   91 CK=-1.0
      GO TO 94
   92 CK=0.0
      GO TO 94
   93 CK=+1.0
   94 CONTINUE
      IF(AB(NN).GT.E)GO TO 95
      XC=XC-2.0*CK*DXC
      GO TO 96
   95 XC=XC+2.0*CK*DXC
   96 DO 97 J=1,N
   97 XN(J)=0.0
      DO 98 K=1,N
      DO 98 J2=1,200
   98 A(J2,K)=0.0
      XN(4)=XA
      XN(5)=XB
      XN(6)=XC
      TN=0.0
      GO TO 12
  500 WRITE(6,505)
  505 FORMAT( ' CONVERGENCE HAS NOT BEEN OBTAINED')
      GO TO 700
   25 CONTINUE
      IF(A(J2,2).LE.0.0) GO TO 211
      GO TO 212
  211 A(J2,2)=0.0
      A(J2,3)=0.0
      EMK(J2)=0.0
  212 J2=J2+1
      KK=KK+1
```

```
      IF(KK.LT.10) GO TO 1
      TN=0.0
      L=1
      DO 26 K=1,N
   26 A(L,K)=0.0
      A(1,4)=XA
      A(1,5)=XB
      A(1,6)=XC
      J=1
      WRITE(6,75)J,TN,EMK(1),(A(L,K),K=1,N)
   75 FORMAT(1H1,4X,I5,14F8.4)
      I=J2-1
      TN=H
      DO 62 J=1,I
      WRITE(6,61) J,TN,EMK(J ),(A(J,K),K=1,N)
   61 FORMAT(5X,I5,14F8.4)
   62 TN=TN+H
  700 STOP
      END
```

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VITA AUCTORIS

1943	Born on November 11, in Kadina Luka, Ljig, Yugoslavia.
1959	Completed secondary education at Turves Green Boys School, Birmingham, England.
1965	Graduated from South Birmingham Technical College, Birmingham, England with the Higher National Certificate in Electrical Engineering.
1969	Graduated from the University of Aston in Birmingham England with the degree of B.Sc. in Electrical Engineering.
1970	Candidate for the degree of M.A.Sc. in Electrical Engineering at the University of Windsor.