1-1-1970

On quantum-mechanical zero-point energy and energy-momentum density.

Ronald W. Gable
University of Windsor

ON QUANTUM-MECHANICAL ZERO-POINT ENERGY

AND

ENERGY-MOMENTUM DENSITY

A Thesis

Submitted to the Faculty of Graduate Studies through

the Department of Physics in Partial Fulfillment

of the Requirements for the Degree of

Master of Science at the

University of Windsor

by

RONALD W. GABLE

B.Sc., University of Waterloo, I969

Windsor, Ontario, Canada

I970

ABA19895

APPROVED BY:

L. Halpern

Dr. L. Halpern (chairman)

M. Schlesinger

Dr. M. Schlesinger

G. Drake

Dr. G. Drake

301518

ABSTRACT

This thesis contains preparatory work for treating the problem of zero-point energy rigorously from the point of view of quantum mechanics. In order to realize this we have constructed and analyzed the Green's tensor of the electromagnetic field equations with homogeneous boundary conditions. The causal and retarded Green's tensors were considered and it is shown that the zero-point energy obtained from the causal tensor is the same as that obtained from the results of statistical mechanics by Casimir, Fierz and Boyer. The method used, employing the Green's tensor, allows a much more general treatment of the problem than that of statistical mechanics.

In the last section it is shown that the realistic case of a sphere which is partly penetrable by the radiation can be described by a model which is analytically identical to that of a fermion gas at finite temperature, so that the apparatus of statistical mechanics may be used to treat the problem. The resulting forces on the enclosing body in our model arise from the deviation of the energy density from Planck's law for limited extension of the enclosing body and this fact suggests a method by which the problem can be solved.

ACKNOWLEDGMENTS

I would like to express my appreciation for the assistance granted to me in the preparation of this thesis by Dr. L. Halpern, under whose supervision this investigation was conducted, and for many helpful discussions with Dr. M. Miketinac and Dr. K. Zischka. I gratefully acknowledge the financial assistance which was extended to me in the form of scholarships, by the National Research Council of Canada and by the University of Windsor.

Ron Gable

June, 1970.

CHAPTER I

INTRODUCTION

The idea that inertia is ultimately an electromagnetic phenomenon and that inertial mass is basically an inductive effect had its origin in the study of the electrodynamics of charges in motion. Although Maxwell's electromagnetic stress tensor, as the spatial part of the energy-momentum tensor of the electromagnetic field, contained implicit ideas conducive to this new conception - as is known today but was unknown before the rise of relativity - it was only in 1881 that Joseph John Thomson envisaged the possibility of reducing inertia to electromagnetism. Following his lead, men such as G. F. FitzGerald, Oliver Heaviside, L. Boltzmann, W. Wien, W. Kaufmann, Max Abraham, Fritz Hasenöhrl, and H. A. Lorentz tackled the problem.

These investigations were challenged by what was called the substantial concept of physical reality, and sparked a competition between the science of mechanics and the science of electromagnetism for primacy in physics. The era of mechanical interpretations of electromagnetic phenomenon initiated by William Thomson (Lord Kelvin) and Maxwell in their search for mechanical models of the ether was still at its peak and was the first attempt at a conceptual unification of physics. A physical body, according to the substantial concept of physical reality, is first of all what it is; only on the basis of its intrinsic, invariable, and permanent nature, of which mass was the physical expression and inertial mass the quantitative measure, did it act as it did. The electromagnetic concept proposed to deprive matter of its intrinsic nature.

Neither of the above two points of view is held as valid in modern physics. However, the electromagnetic concept of mass expressed fully a fundamental tenet of modern physics and of the modern philosophy of matter: "matter does not do what it does because it is what it is, but it is what it is because it does what it does".[1]

The various electromagnetic theories of the electron were beset by numerous difficulties. The most perplexing problem presents itself in the so-called Abraham-Lorentz model of the electron. Abraham regards the electron as a rigid sphere with a homogeneous distribution of charge. Such an assumption leads to the undesirable result of electromagnetic forces from the distinct parts of the charge distribution tending to expand the structure. Ad-hoc stresses were postulated by Poincaré in order to stabilize the particle's finite charge configuration. Since all the effects of classical electromagnetism seemed to have been already incorporated in the model, the Poincaré stresses

---

[1] M. Jammer, Concepts of Mass in Classical and Modern Physics, Harper and Row, New York, 1964, pp. 152-153.

were presumed to be mechanical, non-electromagnetic. Further investigations of this problem were not very convincing until quite recently. Quantum mechanics and notably quantum electrodynamics have succeeded in shedding new light on the problem of electron structure.

In 1953, Casimir, encouraged by his successful calculations of the attraction of two conducting parallel plates due to quantum electromagnetic zero-point energy,[2] suggested that Poincaré's stresses could be viewed as a quantum electromagnetic effect due to zero-point energy.[3] His idea is quite simple, and has the added virtue that now the electron model is filled out entirely by electromagnetic effects.

The Abraham-Lorentz model is extended by Casimir as follows. In its rest frame, a charged particle is regarded as a conducting spherical shell carrying a homogeneous surface charge of total magnitude e. Taking the radius of the shell as a in the intermediate calculations, we find the electrostatic energy of the configuration is, in Gaussian units,

$$E = \frac{e^2}{2a} \qquad (1\text{-}1)$$

with a corresponding tension $e^2/8\pi a^4$ tending to expand the sphere. On the other hand, the presence of the conducting boundary alters the zero-point energy of the universe.

---

[2] H. B. G. Casimir, Koninkl. Ned. Akad. Wetenschap, Proc. 51, 793 (1948).

[3] H. B. G. Casimir, Physica, 19, 846 (1956).

Arguing by analogy with the parallel plate calculations, Casimir suggested that the zero-point energy might tend to collapse the sphere giving an energy

$$E = -K\frac{\hbar c}{2a} \qquad (1\text{-}2)$$

where K is a constant, and hence a corresponding tension

$$-K\frac{\hbar c}{8\pi a^4}$$

This tension would supply the Poincare stress, making the configuration stable - independent of the value of the radius a provided that

$$\frac{e^2}{8\pi a^4} = K\frac{\hbar c}{8\pi a^4} \qquad (1\text{-}3)$$

ie., provided that the total charge on the sphere is such that

$$\frac{e^2}{\hbar c} = K \qquad (1\text{-}4)$$

The condition (1-4) is independent of the radius a of the configuration so that after calculating the constant K for a sphere of finite radius, we may allow a to go to zero so as to avoid any further questions of electron structure. This suggests that, if by some chance, the model did represent an approximation to nature, it might be possible to calculate the value of the fine structure constant α as the Casimir constant appearing in the zero-point energy of a conducting spherical shell. This constant K is dimensionless, following uniquely and unambiguously from the electromagnetic normal modes of a sphere.

Encouragement for the above suggestions can be obtained using the exact results of the parallel plate calculations. (Casimir's results for this case were confirmed in an independent calculation by Fierz[4] and have been confirmed experimentally by Sparnaay[5] and van Silfhout.[6]) If we take the parallel plate result for the zero-point energy of two conducting plates of area A and separation d

$$\Delta E = \frac{-\pi^2 \hbar c A}{720 d^3} \qquad (1\text{-}5)$$

and very roughly approximate a sphere of radius a as two parallel plates of area a a distance a apart, then we get from (1-5),

$$\Delta E_{sphere} = -0.09 \frac{\hbar c}{2a} \qquad (1\text{-}6)$$

giving a value for Casimir's constant K only about ten times as large as that of the fine structure constant. We might regard this as relatively good agreement for such a rough approximation.

Intrigued by the above considerations, Boyer carried out a detailed calculation for the force on a spherical conducting shell hoping to confirm Casimir's conjectures. A description of Boyer's method and results is presented below.

---

[4] M. Fierz, Helv. Phys. Acta., 33, 855, 1960.

[5] M. J. Sparnaay, Physica, 24, 751, 1958.

[6] A. van Silfhout, Dispersion Forces Between Macroscopic Objects, Drukkerij, Holland, N.V., Amsterdam, 1966.

The formulation of the problem of the zero-point energy of a conducting spherical shell can be carried out in direct analogy with Casimir's calculation for the force between two parallel plates.[7] As shown in Fig. I, a large conducting spherical shell of radius R is considered, which is the quantization universe. The problem is then to evaluate the difference in the zero-point energy of a spherical conducting shell when the shell is changed in radius from a size some fixed fraction $1/\eta$ (for example $\frac{1}{\eta}=\frac{1}{2}$) of the radius of the universe down to radius a, the size of physical interest. Divergent series occur, just as in Casimir's calculations, and Boyer, following Fierz, suggests that the physically appropriate cutoff parameter is the wave-length. The final physical situation corresponds to a universe of radius R, very large compared to the radius of the electron. Thus the quantum zero-point energy of a conducting spherical shell is given in terms of the nodes of standing waves.

Both Casimir and Boyer evaluate the zero-point energy of their respective configurations in the same way. The first step is to form the expression for the energy difference with the cutoff parameter included in the expression and expressed for a finite quantization universe. Next the limit of an infinitely extended quantization universe is taken, the subtraction of the series performed, and then

---

[7] T. H. Boyer, Physical Review, 174, No. 5, 1764, 1968.

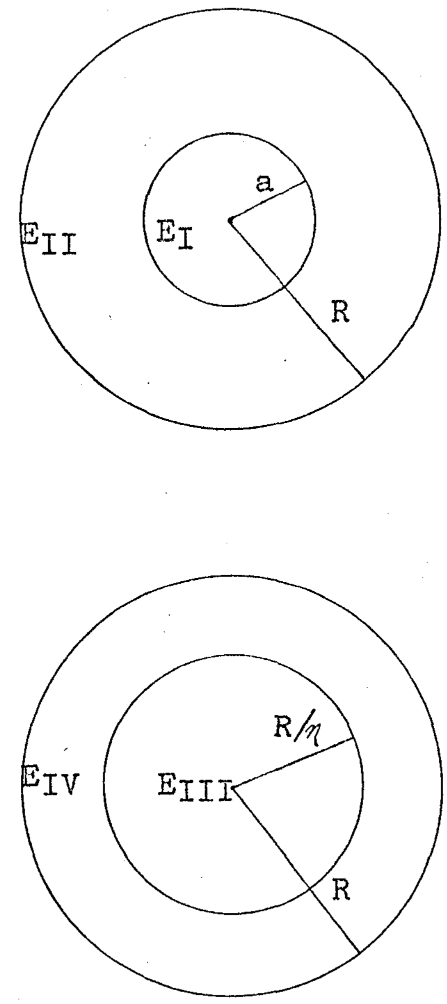


Fig. I. Spherical configurations for a finite quantization universe of radius R.

the cutoff allowed to go to zero. In both cases, finite results are obtained.

Boyer obtains the following expression for the potential due to the zero-point energy.

$$\Delta E(a) = \lim_{R\to\infty} \lim_{\lambda\to 0} \left\{ \left[E_I(a,\lambda) + E_{II}(a,R,\lambda) \right] - \left[E_{III}(\tfrac{R}{\eta},\lambda) + E_{IV}(\tfrac{R}{\eta},R,\lambda) \right] \right\} \qquad (1\text{-}7)$$

where

$$E_I(a,\lambda) = \frac{1}{2}\hbar c \sum_{\ell=1}^{\infty} (2\ell+1) \sum_{s=1}^{\infty} \left[k_{\ell s}(a) \exp(-\lambda k_{\ell s}(a)) + \bar{k}_{\ell s}(a) \exp(-\lambda \bar{k}_{\ell s}(a)) \right] \qquad (1\text{-}8)$$

with

$$j_\ell(a k_{\ell s}(a)) = 0, \text{ and } \frac{d}{dx}\left[x j_\ell(x \bar{k}_{\ell s}(a)) \right]_{x=a} = 0 \qquad (1\text{-}9)$$

$$E_{II}(a,R,\lambda) = \frac{1}{2}\hbar c \sum_{\ell=1}^{\infty} (2\ell+1) \sum_{s=1}^{\infty} \left[K_{\ell s}(a,R) \exp(-\lambda K_{\ell s}(a,R)) + \bar{K}_{\ell s} \exp(-\lambda \bar{K}_{\ell s}(a,R)) \right] \qquad (1\text{-}10)$$

with

$$j_\ell(a K_{\ell s}(a,R))/n_\ell(a K_{\ell s}(a,R)) - j_\ell(R K_{\ell s}(a,R))/n_\ell(R K_{\ell s}(a,R)) = 0$$

$$\frac{(d/dx)\left[x j_\ell(x \bar{K}_{\ell s}(a,R)) \right]_{x=a}}{(d/dx)\left[x n_\ell(x \bar{K}_{\ell s}(a,R)) \right]_{x=a}} - \frac{(d/dx)\left[x j_\ell(x \bar{K}_{\ell s}(a,R)) \right]_{x=R}}{(d/dx)\left[x n_\ell(x \bar{K}_{\ell s}(a,R)) \right]_{x=R}} = 0 \qquad (1\text{-}11)$$

and E_{III} and E_{IV} are found by setting $a \to R/\eta$ in E_I and E_{II} respectively. ($j_\ell(x)$ and $n_\ell(x)$ are respectively the spherical -Bessel and -Neumann functions, which are defined in terms of the ordinary Bessel functions as, $j_\ell(x) = \sqrt{\frac{\pi}{2x}}\, J_{\ell+\frac{1}{2}}(x)$, $n_\ell(x) = \sqrt{\frac{\pi}{2x}}\, N_{\ell+\frac{1}{2}}(x)$.)

After a complicated analysis of the roots of the equations (1-9) and (1-11), Boyer is able to transform his expression for the zero-point energy into,

$$\Delta E(a) = \lim_{\lambda \to 0} \frac{\hbar c}{2a} \sum_{\ell=1}^{\infty} (2\ell+1) \times$$
$$\times \int_0^\infty dx \left(S_\ell(x) - [S_\ell(x)] + \bar{S}_\ell(x) - [\bar{S}_\ell(x)] - 1 \right) \frac{d}{dx} \left(x F(\lambda/a, x) \right) \qquad (1\text{-}12)$$

with
$$S_\ell(x) = -\frac{1}{\pi} \arctan\left(\frac{j_\ell(x)}{n_\ell(x)} \right) \qquad (1\text{-}13)$$
$$\bar{S}_\ell(x) = 1 - \frac{1}{\pi} \arctan\left(\frac{(d/dx)(x j_\ell(x))}{(d/dx)(x n_\ell(x))} \right) \qquad (1\text{-}14)$$

and F(x) any suitable cutoff function, such as F(x)=exp(-x), and λ is the wavelength. The notation [ ] denotes a step function which occurs in the following form of the Euler summation formula.

$$\sum_{x=1}^{n} f(x) = \int_1^n f(x)dx + \int_1^n (x - [x] - \tfrac{1}{2}) f'(x)dx + \tfrac{1}{2}[f(1) + f(n)]$$

The expression on the right in (1-12) can be evaluated numerically. Boyer's evaluation of this expression led him to the final result

$$E(a) \doteq +0.093\, \hbar c/2a \qquad (1\text{-}15)$$

The corresponding forces are outward, tending to expand the shell.

Boyer's result has the opposite sign to the qualitative formula suggested by Casimir, and hence, seems to rule out the possibility of explaining Poincaré stresses by quantum zero-point energy. It should also be noted that the Casimir constant obtained by Boyer is about twelve times larger than the fine structure constant α .

Boyer has completed an analysis of certain other aspects of quantum zero-point energy connected with both microscopic and macroscopic phenomena.[8] He evaluates the zero-point energy of a long conducting cylinder with a piston and finds an attractive force in this case. His calculations are performed using the same formal procedure as in the case of the sphere.

Following a suggestion by Dr. Halpern, we view the Casimir problem (for the sphere) as one of quantum field theory. The fields considered are a quantized photon field and an external field (one whose values in all space-time points can be governed by macroscopic devices and is thus a given quantity) given in the local limit by the sphere. By such a consideration, one obtains an expression for the zero-point energy, which is of some interest as a source of gravitational fields. This procedure actually gives the energy-momentom density of the zero-point energy, which, as in most problems of quantum field theory, is divergent. It is shown that the divergent expression for the total energy (energy integral over all space) corresponds to that of the zero-point energy calculated by conventional means. (The removal of the divergence by conventional methods will not be treated in this thesis, but is left to a future paper.)

The configurations chosen for the calculations of the zero-point energy done in connection with this thesis differ

---

[8] T. H. Boyer, thesis, Harvard University, 1968, (unpublished).

from those of Boyer shown in Fig. 1. A differential change in the radius of the inner sphere was considered in the belief that such a change would give more dependable information than the finite but very large change in the radius of the inner sphere considered by Boyer. This procedure leads to expressions involving the zeros of the radial functions and the derivatives of these functions that do not, as in Boyer's work, vanish in the limit of a very large outer sphere. Numerical evaluation of the final expressions was not attempted since a suitable convergence factor was unavailable. The subtraction of divergences is a major problem in itself and beyond the scope of this thesis.

Following another suggestion by Dr. Halpern, a cutoff function is introduced which makes the total energy equivalent to that of a Fermi gas of unlimited particle number. This reduces the problem to that of a mathematical model of such a gas.

CHAPTER 2

STUDY OF THE GREEN'S TENSOR

The energy density discussed is expressed in terms of a Green's tensor of the electromagnetic field inside and outside the sphere. We are able to give the 4x4 Green's tensor for both cases with given boundary conditions on the surface of the sphere.

We begin by considering Maxwell's equations in the Coulomb gauge.

$$\nabla^2 A_0 = -4\pi J_0 \tag{2-1}$$

$$\nabla^2 \vec{A} - \partial_0^2 \vec{A} = -4\pi \vec{J} + \vec{\nabla}\partial_0 A_0 \tag{2-2}$$

and define

$$\begin{aligned} L_{\alpha\beta} &= \nabla^2 - \partial_0^2 , \quad \alpha,\beta = 1,2,3 . \\ L_{00} &= \nabla^2 \\ \vec{J}_{transversal} &= \vec{J} - \nabla\partial_0 A_0 / 4\pi \\ I_\alpha &= \vec{J}_{transversal} ; \quad I_0 = J_0 \end{aligned}$$

Then we can rewrite (2-1) and (2-2) as

$$L_{\alpha\beta} A_\beta = -4\pi I_\alpha , \quad \alpha\beta = 0,1,2,3. \tag{2-3}$$

We find the solution of (2-1) as[9]

$$A_0(\bar{x}, x_0) = \int \delta(x_0 - x_0') \left[\frac{1}{|\bar{x} - \bar{x}'|} - \frac{a/|\bar{x}'|}{\left|\bar{x} - \frac{a^2}{|\bar{x}'|^2}\bar{x}'\right|} \right] J_0(\bar{x}', x_0') d^4x' \tag{2-4}$$

---

9 J.D. Jackson, Classical Electrodynamics, John Wiley and Sons, Inc., New York, 1962, p.42.

where a is the radius of the sphere.

In determining (2-3) using (2-1) and (2-2) we have used the fact that a vector can always be separated into a longitudinal and a transverse part. Doing this in the Coulomb gauge, we obtain the familiar relations

$$\vec{J} = \vec{J}_\ell + \vec{J}_t \tag{2-5}$$

$$\vec{\nabla} \times \vec{J}_\ell = 0 \tag{2-6}$$

$$\vec{\nabla} \cdot \vec{J}_t = 0 \tag{2-7}$$

$$\vec{J}_\ell = -\frac{1}{4\pi} \vec{\nabla} \int \frac{\vec{\nabla}' \cdot \vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x}' \tag{2-8}$$

$$\vec{J}_t = \frac{1}{4\pi} \vec{\nabla} \times \vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x}' \tag{2-9}$$

$$\vec{\nabla} \partial_0 A_0(x) = -\vec{\nabla} \int \frac{\vec{\nabla}' \cdot \vec{J}(\vec{x}', x_0')}{|\vec{x} - \vec{x}'|} d^3\vec{x}' \tag{2-10}$$

The Green's tensor (spatial part) is constructed using an eigenfunction expansion[10] for each component of the tensor. Formally, this expansion has the form

$$G(\vec{x}, \vec{x}') = \sum \frac{A_n(\vec{x}) A_n^*(\vec{x}')}{\omega^2 - \omega_n^2} \tag{2-11}$$

All that is assumed is the existence of the complete system of eigenfunctions and eigenvalues for the vector Helmzholtz equation

$$\nabla^2 A_n(x, y, z; \omega) + \omega^2 A_n(x, y, z; \omega) = 0$$

for the spherical region considered.

---

[10] A. Sommerfeld, Lectures in Theoretical Physics, VI, Academic Press, New York, 1964, p.183.

Then, using the solutions of (2-3), we obtain, in the Coulomb gauge, the following 4x4 symmetric Green's tensor.

$$G_{00}(x,x') = \frac{\delta(x_0-x_0')}{|\bar{x}-\bar{x}'|} - \frac{\delta(x_0-x_0')a/|\bar{x}'|}{|\bar{x}-\frac{a^2}{|\bar{x}'|^2}\bar{x}'|} \tag{2-12}$$

$$G_{0\alpha}(x,x') = G_{\alpha 0}(x,x') = 0 \tag{2-13}$$

$$G_{\alpha\beta'}(x,x') = \int_C dk_0 \exp(-ik_0(x_0-x_0')) \times$$

$$\times\left[\sum_{L,M,n} \frac{\overset{m}{A}_{\alpha LM}(\bar{x},k_{Ln})\overset{m*}{A}_{\beta LM}(\bar{x}',k_{Ln})}{k_{Ln}^2-k_0^2} + \sum_{L,M,m} \frac{\overset{e}{A}_{\alpha LM}(\bar{x},k_{Lm})\overset{e}{A}_{\beta LM}(\bar{x}',k_{Lm})}{k_{Lm}^2-k_0^2}\right] \tag{2-14}$$

where $\overset{m}{A}_\alpha$ are the components of the magnetic multipole field solution to (2-3) and $\overset{e}{A}_\alpha$ are the components of the electric multipole field solution. The significance of the k_{Ln} and k_{Lm} will be discussed later in connection with the exact form of the multipole solutions.

It should be noted that in (2-14), both $\overset{m}{A}_\alpha$ and $\overset{e}{A}_\alpha$ are transverse vector components, and no longitudinal components enter the spatial portion of the Green's tensor. (Note that $G_{\alpha\beta'}(x,x')$ is written for $G_\alpha(x,x')_\beta$.)

The components of the Green's tensor given above satisfy

$$L_{\alpha\beta}G_{\beta\gamma'}(x,x') = -4\pi\left\{\delta_{\alpha\gamma'}\delta^4(x-x') - P_{\ell\alpha\gamma'}\delta(x_0-x_0')\right\}$$

$$= -4\pi\,\delta(x_0-x_0')P_{t\alpha\gamma'} \tag{2-15}$$

where P_ℓ is a projection operator for the longitudinal field, given by

$$P_{\ell\,\alpha\mu} = \frac{\nabla_\alpha}{4\pi} \int \frac{d^4x' \, \nabla'_\alpha \, \delta(x_0 - x_0')}{|\bar{x} - \bar{x}'|} \qquad (2\text{-}16)$$

and $G_{\alpha\beta'}(x,x')=0$ on the boundary of the sphere.

This is the complete Green's tensor which can be used to find solutions to Maxwell's equations in a sphere for any given 4-current density. In a later section the explicit form of the Green's tensor for the annular region contained between two concentric spherical shells will be given as well as that for the single spherical shell.

The Green's tensor as given contains an integral which must be evaluated along a specified contour in the complex k_0 plane. Depending on the contour chosen, one obtains either an advanced, retarded, or causal Green's tensor. The contours for the causal and retarded functions are shown in Fig. 2 and Fig. 3 respectively. The difference between the two functions will be shown explicitly later when the Green's tensor is written in terms of the normal modes of the sphere.

We now turn to the question of how the Green's tensor for the interior of the sphere behaves under gauge transformations, recalling that it is written in the Coulomb gauge above.

We note that any two-point function $F_{k\ell'}(x,x')$ of the form $\partial_k \varphi_{\ell'}(x,x')$ when added to $G_{k\ell'}(x,x')$ (written in the

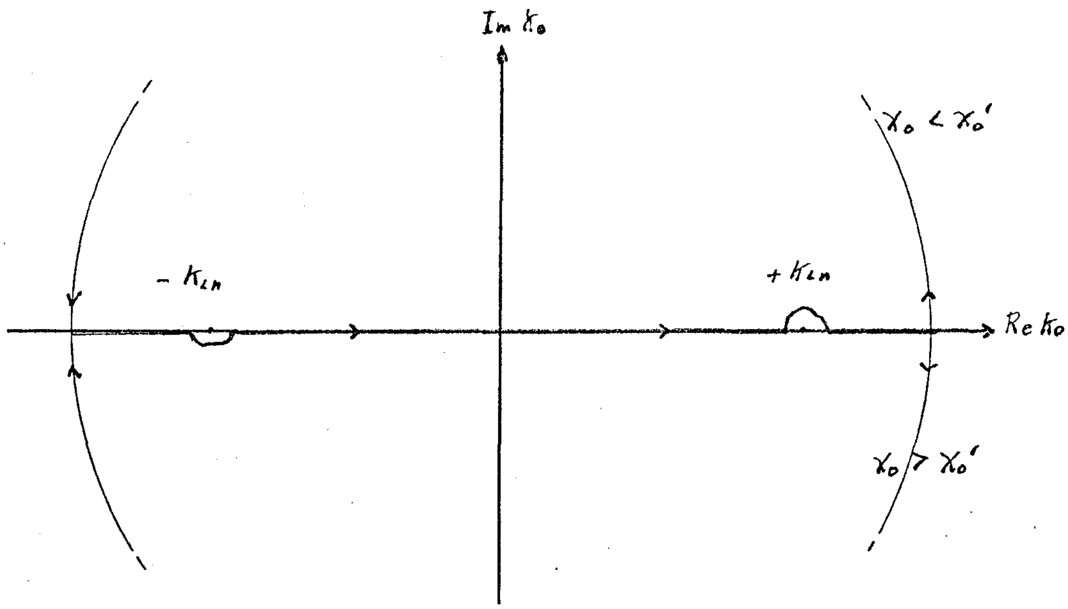


Fig. 2. Path of integration for the function G_{causal} in the complex k_o plane.

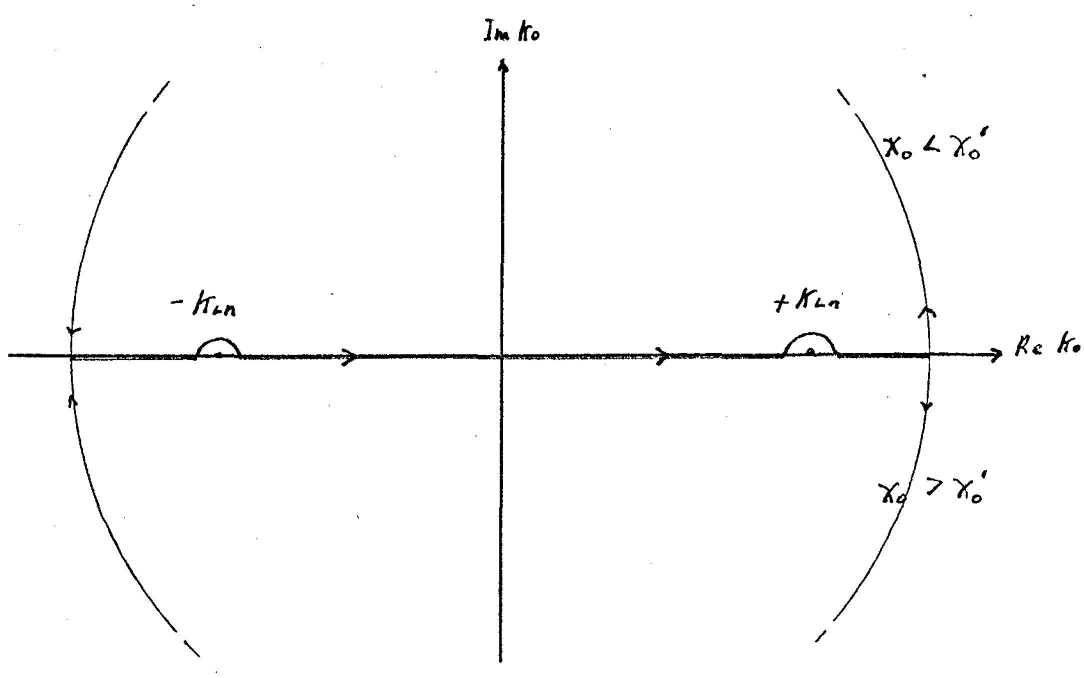


Fig. 3. Path of integration for the function $G_{retarded}$ in the complex k_o plane.

Coulomb gauge) will give rise to the same electric and magnetic fields, but will, in general, no longer satisfy the equations for the vector potential in the Coulomb gauge.

Adding such a two-point function to the Green's tensor, we get, in order to satisfy the condition $\partial^{\kappa} G_{\kappa \ell'} = 0$, the requirement

$$\partial^{\kappa} F_{\kappa \ell'}(x, x') = -\delta_{o\ell'} \partial_o G_{oo}(x, x') \qquad (2\text{-}17)$$

where $G_{oo}(x, x')$ is given by (2-12). Thus

$$\partial^{\kappa} \partial_{\kappa} \varphi_{\ell'}(x, x') = -\delta_{o\ell'} \delta'(x_o - x_o') \left[\frac{1}{|\bar{x} - \bar{x}'|} - \frac{a/|\bar{x}'|}{\left|\bar{x} - \frac{a^2}{|\bar{x}'|^2} \bar{x}'\right|} \right] \qquad (2\text{-}18)$$

The solution of (2-18) is

$$\varphi_{\ell'}(x, x') = -\delta_{o\ell'} \int \overset{RAC}{D}(x - x'') \delta'(x_o'' - x_o') \left[\frac{1}{|\bar{x}'' - \bar{x}'|} - \frac{a/|\bar{x}'|}{\left|\bar{x}'' - \frac{a^2}{|\bar{x}'|^2} \bar{x}'\right|} \right] d^4 x'' \qquad (2\text{-}19)$$

where $\overset{RAC}{D}(x - x'')$ is a retarded, advanced or causal Green's function.

Notice that $F_{\kappa \ell'}(x, x')$ given in the above with $\varphi_{\ell'}$ given by (2-19) does not make the Green's tensor symmetrical. Since symmetry is one of the requirements for any Green's function, we look for a way to symmetrize our tensor.

As a consequence of the equation of continuity, $\partial_{\kappa} J^{\kappa} = 0$, we can add any line to the Green's tensor which is a gradient; ie. $\partial_{\kappa} \Psi(x, x')$. Thus, we add the terms $\delta_{\kappa o} \partial_{\ell'} \varphi_{\kappa}(x, x')$ where φ_{κ} is given by (2-19), which symmetrizes the tensor.

In the Lorentz gauge, the Green's tensor is

$$G^{\text{Lorentz}}_{\kappa\ell'} = G^{\text{causal}}_{\kappa\ell'} + \partial_\kappa \varphi_{\ell'} + \delta_{\kappa o}\, \partial_{\ell'} \varphi_\kappa \qquad (\ 2\text{-}20)$$

with φ_κ (and $\varphi_{\ell'}$) given by (2-19). This is the correct Green's tensor in the Lorentz gauge. It is relativistically invariant and symmetrical.

An alternative way of writing the symmetric Green's tensor in the Lorentz gauge without having to refer to the conservation of the current is to multiply the tensor in the Coulomb gauge on the left and on the right with a projection operator which, contracted with a vector of vanishing four-divergence, gives no contribution.

For the purpose of constructing the Green's tensor for the sphere using the vector potential of the electromagnetic field, we consider the Helmholtz equation

$$\nabla^2 A_\mu (x,y,z;\omega) + \omega^2 A_\mu (x,y,z;\omega) = 0 \qquad (\ 2\text{-}21)$$

The unnormalized multipole solutions are, in the Coulomb gauge[11]

$$\vec{A}^M_L (m) = - f_L \vec{T}^M_{LL} \qquad (\ 2\text{-}22)$$

$$\vec{A}^M_L (e) = -\sqrt{\frac{L}{2L+1}}\, f_{L+1}\, \vec{T}^M_{L,L+1} + \sqrt{\frac{L+1}{2L+1}}\, f_{L-1}\, \vec{T}^M_{L,L-1} \qquad (\ 2\text{-}23)$$

$$\vec{A}^M_L (l) = \sqrt{\frac{L+1}{2L+1}}\, f_{L+1}\, \vec{T}^M_{L,L+1} + \sqrt{\frac{L}{2L+1}}\, f_{L-1}\, \vec{T}^M_{L,L-1} \qquad (\ 2\text{-}24)$$

---

11 M.E. Rose, Multipole Fields, John Wiley and Sons, Inc., New York, 1955.

where the f_L are suitable linear combinations of the spherical Bessel and Neumann functions, and the $\vec{T}^{M}_{J,L}$ are the vector spherical harmonics defined as

$$\vec{T}^{M}_{J,L}(\hat{r}) = \sum_{m,\mu} C(L\,1\,J,\,m\,\mu\,M)\,Y^{m}_{L}(\hat{r})\,\hat{\chi}_{\mu}$$

The $A_{LM}(e)$, $A_{LM}(m)$, and $A_{LM}(l)$ form a complete set of three vector fields which fulfill the electromagnetic field equations. In the Coulomb gauge the potentials $\vec{A}^{M}_{L}(m)$ and $\vec{A}^{M}_{L}(e)$ form a complete set.

The boundary conditions for the problem are that the solution be finite at the origin and the tangential components of $\vec{A}^{M}_{L}(m)$ and $\vec{A}^{M}_{L}(e)$ vanish at the boundary. The latter condition is a consequence of the boundary conditions that the tangential field vanish on the conducting sphere. We saw that the Green's tensor in the Coulomb gauge is transverse and so we make no further use of the longitudinal solution. Decomposing the electric and magnetic multipole solutions into tangential and radial parts, the boundary conditions lead to the following relations.

magnetic field: $j_L(\omega_{L,n}) = 0$ (2-25)

electric field: $d\left[(\omega_{L,m}\tfrac{r}{a})\,j_L(\omega_{L,m}\tfrac{r}{a})\right]_{r=a} = 0$ (2-26)

The normalized tangential solutions that satisfy the boundary conditions and the finiteness condition at the origin are given as

$$\vec{A}^{M}_{L}(m) = -a_{Ln}\,j_L(\omega_{L,n}\tfrac{r}{a})\,\vec{T}^{M}_{L,L}(\hat{r}) \qquad (2\text{-}27)$$

$$\vec{A}_L^{\,m}(e) = b_{L,m}\left[-\sqrt{\frac{L}{2L+1}}\; j_{L+1}\left(\omega_{L,m}\frac{r}{a}\right)\vec{T}^{\,m}_{L,L+1}(\hat{r}) + \sqrt{\frac{L+1}{2L+1}}\; j_{L-1}\left(\omega_{L,m}\frac{r}{a}\right)\vec{T}^{\,m}_{L,L-1}(\hat{r})\right] \qquad (2\text{-}28)$$

where $a_{L,n}$ and $b_{L,m}$ are normalization factors derived in the appendix.

Using these solutions, the normalized Green's tensor can be written as follows.

$$G_{\alpha\beta'}(x,x') = 2\sum_{L,M,n} a^2_{L,n}\int_c \frac{d\omega\, e^{-i\omega(t-t')}}{\frac{\omega^2_{L,n}}{a^2}-\omega^2}\, j_L\left(\omega_{L,n}\frac{r}{a}\right) j_L\left(\omega_{L,n}\frac{r'}{a}\right)\vec{T}^{\,m}_{L,L,\alpha}(\hat{r})\,\vec{T}^{\,m*}_{L,L,\beta'}(\hat{r}') +$$

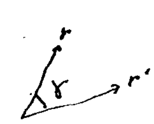
$$+\,2\sum_{L,M,m} b^2_{L,m}\int_c \frac{d\omega\, e^{-i\omega(t-t')}}{\frac{\omega^2_{L,m}}{a^2}-\omega^2}\left[\frac{L}{2L+1}\, j_{L+1}\left(\omega_{L,m}\frac{r}{a}\right) j_{L+1}\left(\omega_{L,m}\frac{r'}{a}\right)\vec{T}^{\,m}_{L,L+1,\alpha}(\hat{r})\,\vec{T}^{\,m*}_{L,L+1,\beta'}(\hat{r}') +\right.$$

$$+\,\frac{L+1}{2L+1}\, j_{L-1}\left(\omega_{L,m}\frac{r}{a}\right) j_{L-1}\left(\omega_{L,m}\frac{r'}{a}\right)\vec{T}^{\,m}_{L,L-1,\alpha}(\hat{r})\,\vec{T}^{\,m*}_{L,L-1,\beta'}(\hat{r}') \;-$$

$$-\,\frac{\sqrt{L(L+1)}}{2L+1}\, j_{L+1}\left(\omega_{L,m}\frac{r}{a}\right) j_{L-1}\left(\omega_{L,m}\frac{r'}{a}\right)\vec{T}^{\,m}_{L,L+1,\alpha}(\hat{r})\,\vec{T}^{\,m*}_{L,L-1,\beta'}(\hat{r}') \;-$$

$$\left.-\,\frac{\sqrt{L(L+1)}}{2L+1}\, j_{L-1}\left(\omega_{L,m}\frac{r}{a}\right) j_{L+1}\left(\omega_{L,m}\frac{r'}{a}\right)\vec{T}^{\,m}_{L,L-1,\alpha}(\hat{r})\,\vec{T}^{\,m*}_{L,L+1,\beta'}(\hat{r}')\right] \qquad (2\text{-}29)$$

$$G_{oo}(x,x') = \frac{\delta(t-t')}{(r^2+r'^2-2rr'\cos\gamma)^{1/2}} - \frac{\delta(t-t')}{\left(\frac{r^2r'^2}{a^2}+a^2-2rr'\cos\gamma\right)^{1/2}} \qquad (2\text{-}30)$$



The components $G_{\alpha\beta'}(x,x')$ have been given by Morse and Feshback[12]. Their expression differs only in the normalization

---

12 P.M. Morse and H. Feshback, Methods of Theoretical Physics, McGraw-Hill Book Co., New York, 1953, pp.1870-72.

of the vector spherical harmonics and the inclusion of a factor to account for finite conductivities. Their expression does not include any time dependence.

To obtain the causal Green's tensor, the integration will be performed over the contour of Fig. 2. This is a generalization of the method known in the case of unbounded space. For $t > t'$, the contour is closed in the lower half plane, and the integration performed in the clockwise direction. For $t < t'$, we close the contour in the upper half plane and integrate counter-clockwise. The integrand has poles of order one at $\omega = \pm \omega_{Ln}/a$ (magnetic), or $\omega = \pm \omega_{L,n}/a$ (electric). The residue at the negative pole is

$$k = -e^{i\frac{\omega_{Ln}}{a}(t-t')} \Big/ (2\omega_{L,n}/a) \tag{2-31}$$

and the residue at the positive pole is

$$k = -e^{-i\frac{\omega_{Ln}}{a}(t-t')} \Big/ (2\omega_{Ln}/a) \tag{2-32}$$

Then, writing $z = re^{i\theta}$, with $z = \omega$, $z_0 = \omega_{Ln}/a$, and integrating, we get, for $t > t'$,

$$\int \frac{dz\, e^{-iz(t-t')}}{(z_0 + i\varepsilon + z)(z_0 - i\varepsilon - z)} = \int_{-R}^{R} \frac{e^{-ir(t-t')}\, dr}{(z_0 + i\varepsilon + r)(z_0 - i\varepsilon - r)} + iR \int_0^{+\pi} \frac{e^{-iRe^{i\theta}(t-t')}\, e^{i\theta}\, d\theta}{(z_0 + i\varepsilon + Re^{i\theta})(z_0 - i\varepsilon - Re^{i\theta})}$$

$$= \int_{-R}^{R} \frac{e^{-ir(t-t')}\, dr}{(z_0 + i\varepsilon + r)(z_0 - i\varepsilon - r)} + \frac{i}{R} \int_0^{+\pi} \frac{e^{-i(R(t-t')\cos\theta - \theta)}}{(z_0 + i\varepsilon + Re^{i\theta})(z_0 - i\varepsilon - Re^{i\theta})}$$

The reason for choosing to close the contour in the bottom half plane is now obvious. In the bottom half plane $\sin\theta$ is always negative, so that when we take the limit $R\to\infty$ the term $e^{R(t-t')\sin\theta}$ and the integral over θ vanishes, which is required.

Thus, after taking the limits $R\to\infty$ and $\varepsilon\to 0$, we obtain

$$\int_{-\infty}^{\infty}\frac{d\omega\, e^{-i\omega(t-t')}}{\frac{\omega_{Ln}^2}{a^2}-\omega^2} = -\theta(t-t')\frac{\pi i a}{\omega_{Ln}} e^{-\frac{i\omega_{Ln}}{a}(t-t')} - \theta(t'-t)\frac{\pi i a}{\omega_{Ln}} e^{\frac{i\omega_{Ln}}{a}(t-t')}$$

In order to obtain the retarded Green's tensor, we employ the contour of Fig. 3. After performing the integration we obtain

$$\int_{-\infty}^{\infty}\frac{d\omega\, e^{-i\omega(t-t')}}{\frac{\omega_{Ln}^2}{a^2}-\omega^2} = -\theta(t-t')\frac{2\pi i a}{\omega_{Ln}}\cos\frac{\omega_{Ln}(t-t')}{a} + \theta(t'-t)(0)$$

This illustrates the basic difference between the causal and retarded Green's tensors.

The causal Green's tensor can now be written as

$$G_{\alpha\beta'}(x,x') = -2\sum_{Lmn}\frac{i\pi a}{\omega_{Ln}}\, a_{Ln}^2\left[\theta(t-t')e^{-\frac{i\omega_{Ln}}{a}(t-t')} + \theta(t'-t)e^{\frac{i\omega_{Ln}}{a}(t-t')}\right]\times$$

$$\times\, j_L\left(\omega_{Ln}\frac{r}{a}\right) j_L\left(\omega_{Ln}\frac{r'}{a}\right)\vec{T}^{m}_{L,L,\alpha}(\hat{r})\,\vec{T}^{m*}_{L,L,\beta'}(\hat{r}') +$$

$$+\,(-)2\sum_{L,M,m}\frac{i\pi a}{\omega_{L,m}}\, b_{L,m}^2\left[\theta(t-t')e^{-\frac{i\omega_{Lm}}{a}(t-t')} + \theta(t'-t)e^{\frac{i\omega_{Lm}}{a}(t-t')}\right]\times$$

$$\times\left[\frac{L}{2L+1}\, j_{L+1}\left(\omega_{Lm}\frac{r}{a}\right) j_{L+1}\left(\omega_{Lm}\frac{r'}{a}\right)\vec{T}^{m}_{L,L+1,\alpha}(\hat{r})\,\vec{T}^{m*}_{L,L+1,\beta'}(\hat{r}') +\right.$$

$$+\,\frac{L+1}{2L+1}\, j_{L-1}\left(\omega_{Lm}\frac{r}{a}\right) j_{L-1}\left(\omega_{Lm}\frac{r'}{a}\right)\vec{T}^{m}_{L,L-1,\alpha}(\hat{r})\,\vec{T}^{m*}_{L,L-1,\beta'}(\hat{r}') -$$

$$-\frac{\sqrt{L(L+1)}}{2L+1}\, j_{L+1}(\omega_{Lm}\tfrac{r}{a})\, j_{L-1}(\omega_{Lm}\tfrac{r'}{a})\, \vec{T}^{\,m}_{L,L+1,\alpha}(\hat{r})\, \vec{T}^{\,m*}_{L,L-1,\beta'}(\hat{r}') -$$

$$-\frac{\sqrt{L(L+1)}}{2L+1}\, j_{L-1}(\omega_{Lm}\tfrac{r}{a})\, j_{L+1}(\omega_{Lm}\tfrac{r'}{a})\, \vec{T}^{\,m}_{L,L-1,\alpha}(\hat{r})\, \vec{T}^{\,m*}_{L,L+1,\beta'}(\hat{r}')\Big]$$

(2-33)

The other components are given by (2-30).

Now suppose that we have two concentric spherical shells, radii r=a and r=b, with b>a. The solutions for the vector potential (in Coulomb gauge) are given by (2-22) and (2-23) for the magnetic and electric modes respectively. The radial functions f_L are suitable linear combinations of spherical Bessel and Neumann functions, chosen to satisfy the boundary conditions and normalization.

For the magnetic mode, the boundary conditions lead to

$$j_L(\omega\tfrac{a}{b}) + A\, n_L(\omega\tfrac{a}{b}) = 0 \quad \text{at } r=a \tag{2-34}$$

or $\quad A = -j_L(\omega\tfrac{a}{b})/n_L(\omega\tfrac{a}{b})$

At r=b, we obtain

$$j_L(\omega_{nL}) - \left(j_L(\omega_{Ln}\tfrac{a}{b})/n_L(\omega_{Ln}\tfrac{a}{b})\right) n_L(\omega_{Ln}) = 0 \tag{2-35}$$

where ω_{Ln} are the roots of this expression.

For the electric mode, the boundary condition at r=a leads to

$$A = -\left[j_L(\omega\tfrac{a}{b}) + \omega\tfrac{a}{b}\, j_L'(\omega\tfrac{a}{b})\right] \Big/ \left[n_L(\omega\tfrac{a}{b}) + \omega\tfrac{a}{b}\, n_L'(\omega\tfrac{a}{b})\right] \tag{2-36}$$

At r=b we obtain

$$j_L(\omega_{Lm})+\omega_{Lm}j_L'(\omega_{Lm})-\frac{j_L(\omega_{Lm}\frac{a}{b})+\omega_{Lm}\frac{a}{b}j_L'(\omega_{Lm}\frac{a}{b})}{n_L(\omega_{Lm}\frac{a}{b})+\omega_{Lm}\frac{a}{b}n_L'(\omega_{Lm}\frac{a}{b})}\left[n_L(\omega_{Lm})+\omega_{Lm}n_L'(\omega_{Lm})\right]=0 \qquad (2\text{-}37)$$

where ω_{Lm} are the roots of this expression.

The solutions for the vector potential become in this case

$$\vec{A}_{Lm_n}(m)=-\alpha_{Ln}\left[j_L(\omega_{Ln}\frac{r}{b})-\frac{j_L(\omega_{Ln}\frac{a}{b})}{n_L(\omega_{Ln}\frac{a}{b})}\,n_L(\omega_{Ln}\frac{r}{b})\right]\vec{T}^{m}_{L,L}(\hat{r}) \qquad (2\text{-}38)$$

$$\vec{A}_{Lm_m}(e)=\beta_{Lm}\left\{\sqrt{\frac{L}{2L+1}}\left[j_{L+1}(\omega_{Lm}\frac{r}{b})+\omega_{Lm}\frac{r}{b}j_{L+1}'(\omega_{Lm}\frac{r}{b})\right.\right. -$$

$$\left.\left. -\frac{j_{L+1}(\omega_{Lm}\frac{a}{b})+\omega_{Lm}\frac{a}{b}j_{L+1}'(\omega_{Lm}\frac{a}{b})}{n_{L+1}(\omega_{Lm}\frac{a}{b})+\omega_{Lm}\frac{a}{b}n_{L+1}'(\omega_{Lm}\frac{a}{b})}\left[n_{L+1}(\omega_{Lm}\frac{r}{b})+\omega_{Lm}\frac{r}{b}n_{L+1}'(\omega_{Lm}\frac{r}{b})\right]\right] \times \times\vec{T}^{m}_{L,L+1}(\hat{r}) + L+1\rightarrow L-1.\right\} \qquad (2\text{-}39)$$

with the normalization factors given by

$$\alpha_{Ln}^{-2}=\int_a^b r^2\xi_L^2(\omega_{Ln}\frac{r}{b})\,dr; \quad \beta_{Lm}^{-2}=\int_a^b r^2\left[\frac{L}{2L+1}\,\xi_{L+1}^2(\omega_{Lm}\frac{r}{b})+\frac{L+1}{2L+1}\,\xi_{L-1}^2(\omega_{Lm}\frac{r}{b})\right]dr$$

The Green's tensor is now constructed using the same prescription as before.

$$G_{\alpha\beta'}(x,x')=2\sum_{Lmn}\alpha_{Ln}^2\int\frac{d\omega\, e^{-i\omega(t-t')}}{\frac{\omega_{Ln}^2}{b^2}-\omega^2}\,\xi_L(\omega_{Ln}\frac{r}{b})\,\xi_L(\omega_{Ln}\frac{r'}{b})\,\vec{T}^{m}_{L,L,\alpha}(\hat{r})\,\vec{T}^{m*}_{L,L,\beta'}(\hat{r}')+$$

$$+2\sum_{Lmm}\beta_{Lm}^2\int\frac{d\omega\, e^{-i\omega(t-t')}}{\frac{\omega_{Lm}^2}{b^2}-\omega^2}\left[\frac{L}{2L+1}\,\xi_{L+1}(\omega_{Lm}\frac{r}{b})\,\xi_{L+1}(\omega_{Lm}\frac{r'}{b})\,\vec{T}^{m}_{L,L+1,\alpha}(\hat{r})\,\vec{T}^{m*}_{L,L+1,\beta'}(\hat{r}')+\right.$$

$$+\frac{L+1}{2L+1}\,\xi_{L-1}(\omega_{Lm}\frac{r}{b})\,\xi_{L-1}(\omega_{Lm}\frac{r'}{b})\,\vec{T}^{m}_{L,L-1,\alpha}(\hat{r})\,\vec{T}^{m*}_{L,L-1,\beta'}(\hat{r}')-$$

$$-\frac{\sqrt{L(L+1)}}{2L+1}\,\xi_{L+1}(\omega_{Lm}\frac{r}{b})\,\xi_{L-1}(\omega_{Lm}\frac{r'}{b})\,\vec{T}^{m}_{L,L+1,\alpha}(\hat{r})\,\vec{T}^{m*}_{L,L-1,\beta'}(\hat{r}')-$$

$$\left.-\frac{\sqrt{L(L+1)}}{2L+1}\,\xi_{L-1}(\omega_{Lm}\frac{r}{b})\,\xi_{L+1}(\omega_{Lm}\frac{r'}{b})\,\vec{T}^{m}_{L,L-1,\alpha}(\hat{r})\,\vec{T}^{m*}_{L,L+1,\beta'}(\hat{r}')\right] \qquad (2\text{-}40)$$

$$G_{00}(x,x') = \sum_{LM} \frac{Y^*_{Lm}(\bar{x}')\,Y_{Lm}(\bar{x})}{(2L+1)\left[1-\left(\frac{a}{b}\right)^{2L+1}\right]} \left[r_<^{L} - \frac{a^{2L+1}}{r_<^{L+1}}\right]\left[\frac{1}{r_>^{L+1}} - \frac{r_>^{L}}{b^{2L+1}}\right] \delta(x_0 - x_0') \tag{2-41}$$

and $\xi_L(\omega_{Ln} r/b)$ and $\xi_L(\omega_{Lm} r/b)$ are given by the radial parts of (2-38) and (2-39) respectively.

It is worth noting that the Green's tensor for the annulus, namely (2-40), approaches that for the sphere, namely (2-33), for large values of the radius b. This is most easily seen from the radial expressions which indicate the presence of the two boundaries for the annular region. If in (2-35) and (2-37), we take the limit $b \to \infty$, with a fixed, we recover the expressions (2-25) and (2-26) with $a \to b$. Thus for large values of b, the Green's tensor for the annulus behaves almost like that for a large sphere of radius b, and satisfies the boundary and finiteness conditions, as required.

It is known from the mathematical theory of partial differential equations that a solution of the wave equation with suitable boundary conditions is unique. Since the Green's tensors given here are just the solution operators for the wave equation, they must also be unique (up to gauge transformations). However, there are many ways of formally constructing such expressions, and some resulting tensors may be more amenable to analysis than others.

It is now shown that the following Green's tensor, which was suggested by Dr. Halpern, satisfies the requirements of the problem, although it does not introduce in an obvious way any significant simplification in the analysis.

For a particular value of L we get

$$G_{L\,\alpha,\beta'}(x,x') = \sum_{M=-L}^{L} A_L(k_o)\, j_L(k_o r_<)\left[j_L(k_o r_>)\frac{n_L(k_o a)}{j_L(k_o a)} - n_L(k_o r_>)\right] \times$$

$$\times\, \overrightarrow{\overrightarrow{T}}{}^{m}_{L,L,\alpha}(\hat{r})\, \overrightarrow{\overrightarrow{T}}{}^{m^*}_{L,L,\beta'}(\hat{r}')\, e^{-ik_o(x_o - x_o')} \qquad (2\text{-}42)$$

where $A_L(k_o)$ is a normalization factor and k_o is any number which is not an eigenvalue of our problem. This expression contains only the contribution from the magnetic mode.

The differential equation satisfied by the spherical Bessel functions is

$$\frac{d}{d\rho}\left(\rho^2 \frac{d}{d\rho} f_n\right) + (\rho^2 - n(n+1)) f_n = 0 \qquad (2\text{-}43)$$

This equation has the form of the self-adjoint equation

$$\frac{d}{d\rho}\left(p(\rho)\frac{d}{d\rho} u\right) + q(\rho) u = 0 \qquad (2\text{-}44)$$

The mathematical theory of (2-44) predicts that p(x)W=B, and B is a constant. W is the Wronskian determinant of j_L and $j_L(k_o r)\, n_L(k_o a)/j_L(k_o a) - n_L(k_o r)$. Specifically, we have

$$W\left[j_L(x),\ j_L(x)\frac{n_L}{j_L} - n_L(x)\right] = B/p(x)$$

and the radial Green's function is

$$G(x,x') = \begin{cases} -(1/B)\, j_L(x)\left[j_L(x')\frac{n_L}{j_L} - n_L(x')\right] & x < x' \\ -(1/B)\, j_L(x')\left[j_L(x)\frac{n_L}{j_L} - n_L(x)\right] & x > x' \end{cases}$$

Since the Wronskian is proportional to 1/p(x) for all values of x, it does not matter where we evaluate it. We will use the limiting forms of j_L and n_L for this.

$$\lim_{x\to 0} j_L(x) = \frac{x^L}{1\cdot 3\cdot 5\cdots(2L+1)}$$

$$\lim_{x\to 0} j_L'(x) = \frac{Lx^L}{1\cdot 3\cdot 5\cdots(2L+1)}$$

$$\lim_{x\to 0} n_L(x) = -\frac{1\cdot 3\cdot 5\cdots(2L-1)}{x^{L+1}}$$

$$\lim_{x\to 0} n_L'(x) = \frac{1\cdot 3\cdot 5\cdots(2L-1)(L+1)}{x^{L+2}}$$

$$\lim_{x\to 0} W = k_0\left[j_L(k_0r)n_L'(k_0r) - j_L'(k_0r)n_L(k_0r)\right]$$

$$= 1/k_0r^2$$

Since p(x) in our case is $k_0^2r^2$, we get

$$\beta = k_0$$

and $A_L(k_0) = -1/\beta = -1/k_0$ (2-45)

With $A_L(k_0)$ given by (2-45) we have a Green's tensor for the magnetic mode which is normalized and satisfies the given boundary conditions.

For the annular region between two concentric spheres of radii a and b respectively, we get, for the magnetic mode

$$G_{\alpha\beta'}(x,x') = \sum_{M=-L}^{L} B_L(k_0)\left[j_L(k_0r_<)\frac{n_L(k_0a)}{j_L(k_0a)} - n_L(k_0r_<)\right] x$$

$$x\left[j_L(k_0r_>)\frac{n_L(k_0b)}{j_L(k_0b)} - n_L(k_0r_>)\right]\vec{T}^{m}_{L,L,\alpha}(\hat{r})\vec{T}^{m\,a}_{L,L,\beta'}(\hat{r}')e^{-ik_0(x_0-x_0')} \quad (2\text{-}46)$$

$$\lim_{x\to 0} W = k_0\left[\frac{n_L(k_0a)}{j_L(k_0a)} - \frac{n_L(k_0b)}{j_L(k_0b)}\right]\left[j_L(k_0r)n_L'(k_0r) - j_L'(k_0r)n_L(k_0r)\right]$$

$$= \frac{k_0}{k_0^2r^2}\left[\frac{n_L(k_0a)}{j_L(k_0a)} - \frac{n_L(k_0b)}{j_L(k_0b)}\right]$$

$$= \frac{1}{k_o r^2}\left[\frac{j_L(k_ob)\,n_L(k_oa) - n_L(k_ob)\,j_L(k_oa)}{j_L(k_oa)\,j_L(k_ob)}\right]$$

and $$B_L(k_o) = -\,k_o^{-1}\left[\frac{j_L(k_oa)\,j_L(k_ob)}{j_L(k_ob)\,n_L(k_oa) - n_L(k_ob)\,j_L(k_oa)}\right] \qquad (2\text{-}47)$$

It is hoped that further investigation of this form of the Green's tensor, when the electric modes are included, will prove more useful than the eigenfunction expansion which was used to obtain the Green's tensors given earlier in this chapter. It is also now thought that this form can be made simpler by reducing the Bessel function expressions, with the sum over L, to a finite number of trigonometric expressions. Work is continuing in this direction.

CHAPTER 3

QUANTUM ELECTRODYNAMICAL RELATIONS INVOLVING THE CAUSAL GREEN'S TENSOR

In this chapter several relations from quantum theory will be discussed in an attempt to connect the basically classical Green's tensor given in the previous chapter to the more advanced theory of quantum electrodynamics. The discussion is, for the most part, concerned with the chronological product of field operators, and this concept will be considered first. It is then shown that a portion of the Green's tensor given previously satisfies a relation involving the chronological product. An expression for the zero-point energy-momentum density is then derived, again using the chronological product, and the total zero-point energy of a sphere is evaluated with this relation. The expression which results will be seen to be the same as one would expect using the simple quantum mechanical arguments of Casimir, Fierz and Boyer.

The chronological product (or the T-product) of two Boson field operators is,[13]

$$T(A_n(x)A_m(y)) = \begin{cases} A_n(x)A_m(y) & x_0 > y_0 \\ A_m(y)A_n(x) & x_0 < y_0 \end{cases} \tag{3-1}$$

[13] N.N. Bogoliubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields, trans. by G.M. Volkoff, Interscience Publishers, Inc., New York, 1959, p.142.

The T-product is thus just the ordinary product of these operators taken in a definite order which corresponds to the decrease of the time components of the arguments of the factors from left to right. It can be shown that the definition (3-1) of the T-product is covariant.[14]

The causal Green's tensor for the free electromagnetic field in unbounded space has the following connection to the T-product for the electromagnetic field operators.[15]

$$-i\,\langle T(A_m(x)A_n(y))\rangle_0 = D^c_{mn'}(x,y) \qquad (\ 3\text{-}2\ )$$

where $\langle\ \ \rangle_0$ means the vacuum expectation value of the T-product.

The relation (3-2) can be derived from the commutation relations for the electromagnetic field operators and the following relation between the positive and negative frequency commutation functions and the causal Green's function.

$$D^c(x) = \theta(x^0)\,D^{(-)}(x) - \theta(-x^0)\,D^{(+)}(x) \quad {}^{16} \qquad (\ 3\text{-}3\ )$$

In order to connect the theory of the previous chapter to quantum theory, it is now necessary to show that the Green's tensor (2-33) satisfies the relation (3-2) with the appropriate electromagnetic field operators.

---

[14] Ibid., p.143.

[15] Ibid., pp.216-217.

[16] Ibid., p.142

Consider the electromagnetic field operator

$$\vec{A}(x) = \sum_{\substack{\lambda, L, M \\ \omega_{Ln}}} \sqrt{\frac{2\pi}{\omega_{Ln}}} \left[a_{\lambda \omega_{Ln} LM} \vec{\theta}_{\lambda \omega_{Ln} LM} e^{-i\omega_{Ln} t} + a^{+}_{\lambda \omega_{Ln} LM} \vec{\theta}_{\lambda \omega_{Ln} LM} e^{i\omega_{Ln} t} \right]^{17} \qquad (3\text{-}4)$$

with $\vec{\theta}_{\lambda \omega_{Ln} LM} = a_{L\omega_{Ln}} f_{\lambda}(\omega_{Ln} r) \vec{\vec{T}}^{M}_{L,\lambda}(\theta, \varphi)$

and $a^{-\frac{1}{2}}_{L\omega_{Ln}} = \int_0^a r^2 f^2_{\lambda}(\omega_{Ln} r)\, dr$

Substituting (3-4) into the left-hand side of (3-2), we get

$$\begin{aligned}
\langle T(A_i(x) A_j(x')) \rangle_0 &= \langle (\theta(t-t') A_i(x) A_j(x') + \theta(t'-t) A_j(x) A_i(x')) \rangle_0 \\
&= \sum_{\substack{\lambda \omega_{Ln} LM \\ \lambda' \omega_{L'n'} L'M'}} \left(\frac{2\pi}{\omega_{Ln} \omega_{L'n'}} \right)^{1/2} \Big[\theta(t-t') \langle a_{\lambda \omega_{Ln} LM} \theta_{\lambda \omega_{Ln} LM;i}(\vec{x}) e^{-i\omega_{Ln} t} \times \\
&\quad \times a^{+}_{\lambda' \omega_{L'n'} L'M'} \theta^{*}_{\lambda' \omega_{L'n'} L'M';j}(\vec{x}') e^{i\omega_{L'n'} t'} \rangle_0 + \begin{matrix} \vec{x} \leftrightarrow \vec{x}' \\ i \leftrightarrow j \end{matrix} \Big] \\
&= \sum_{\lambda \omega_{Ln} LM} \left(\frac{2\pi}{\omega_{Ln}} \right) \Big[\theta(t-t') e^{-i\omega_{Ln}(t-t')} \vec{\theta}_{\lambda \omega_{Ln} LM,i}(\vec{x}) \times \\
&\quad \times \vec{\theta}^{*}_{\lambda \omega_{Ln} LM,j}(\vec{x}') + \theta(t'-t) e^{i\omega_{Ln}(t-t')} \vec{\theta}_{\lambda \omega_{Ln} LM,j}(\vec{x}) \vec{\theta}^{*}_{\lambda \omega_{Ln} LM,i}(\vec{x}) \Big]
\end{aligned} \qquad (3\text{-}5)$$

Now, consider

$$\begin{aligned}
i G^{c}_{ij}(x, x') &= -i \sum_{\lambda LM \omega_{Ln}} \frac{2i\pi}{\omega_{Ln}} \left[\theta(t-t') e^{-i\omega_{Ln}(t-t')} + \theta(t'-t) e^{i\omega_{Ln}(t-t')} \right] \times \\
&\quad \times \vec{\theta}_{\lambda \omega_{Ln} LM,i}(\vec{x}) \vec{\theta}^{*}_{\lambda \omega_{Ln} LM,j}(\vec{x}') \\
&= \sum_{\lambda \omega_{Ln} LM} \left(\frac{2\pi}{\omega_{Ln}} \right) \left[\theta(t-t') e^{-i\omega_{Ln}(t-t')} + \theta(t'-t) e^{i\omega_{Ln}(t-t')} \right] \times \\
&\quad \times \vec{\theta}_{\lambda \omega_{Ln} LM,i}(\vec{x}) \vec{\theta}^{*}_{\lambda \omega_{Ln} LM,j}(\vec{x}')
\end{aligned} \qquad (3\text{-}6)$$

---

[17] A. Messiah, Quantum Mechanics, trans. by J. Potter, North Holland Publishing Co., Amsterdam, 1966, II, p.1037.

and we have used ω_{Ln} for ω_{Ln}/a of (2-33). The index λ is summed over λ=m, λ=e for magnetic and electric modes respectively.

By performing the sum over M, it is easily seen that

$$\bar{\Theta}_{\lambda\omega_{Ln}Lm,i}(\bar{x})\,\bar{\Theta}^{*}_{\lambda\omega_{Ln}Lm,j}(\bar{x}') = \bar{\Theta}_{\lambda\omega_{Ln}Lm,j}(\bar{x}')\,\bar{\Theta}^{*}_{\lambda\omega_{Ln}Lm,i}(\bar{x})$$

since the only complex part of $\Theta_{\lambda\omega_{Ln}Lm}$ is in the vector spherical harmonic term. Thus, (3-5) is equal to (3-6) and the quantum mechanical relation (3-2) holds for the Green's tensor (2-33).

The connection of the energy-momentum tensor for the electromagnetic field to the chronological product is now given and an expression for the zero-point energy -momentum density obtained.

In analogy to the introduction of the vacuum expectation value of the current vector in quantum electrodynamics we introduce the vacuum expectation value of the energy-momentum tensor.[18] By a variational procedure, one obtains, in the bound interaction representation, the following relation.

$$\frac{1}{2}\lim_{\substack{x'\to x^+ \\ +x'\to x^-}} \frac{\langle 0_{in}|T(T^{ik}(x,x'))|0_{out}\rangle}{\langle 0_{in}|0_{out}\rangle} = \langle T^{ik}\rangle_0 \qquad (3\text{-}7)$$

The $T^{ik}(x)$ is given by

$$T_{ik}(x) = \frac{1}{4\pi}\left[F_{i\lambda}F_{\lambda k} + \frac{1}{4}\delta_{ik}F_{\lambda\sigma}F_{\lambda\sigma}\right] \qquad (3\text{-}8)$$

---

[18] B.S. DeWitt, Physical Review, 85, 653-61, 1952.

with $F^{ik} = \partial A_k / \partial x_i - \partial A_i / \partial x_k$ (3-9)

In order to obtain the zero-point energy density, we evaluate T^{oo} and obtain

$$\langle T^{oo} \rangle_o = \mathcal{E} = \lim_{x' \to x} \frac{1}{4\pi} \frac{\partial}{\partial t} \frac{\partial}{\partial t'} \langle 0 | T(\bar{A}(x) \cdot \bar{A}(x')) | 0 \rangle \qquad (3\text{-}10)$$

using (3-9) and (3-8) in (3-7).

The above procedure is much more general than the simple counting of states which was employed by Boyer in his evaluation of the zero-point energy, as well as by Casimir. One obtains not only the energy density, but also the momentum density, whereas the above two authors both evaluated only the total energy of the configurations with which they respectively dealt.

It is now possible to evaluate the zero-point energy for a sphere of radius a. We apply the relation (3-10) to the causal Green's tensor, and note that the scalar product indicated in (3-10) is obtained by taking the spur of the 3x3 Green's tensor (2-33).

The energy density is

$$\mathcal{E} = \lim_{x' \to x} \left\{ \frac{1}{2} \sum_{Lmn} a_{Ln}^2 \frac{\omega_{Ln}}{a} \left[\frac{\partial^2 \theta(t-t')}{\partial (t-t')^2} e^{-\frac{i\omega_{Ln}}{a}(t-t')} + \frac{\partial^2 \theta(t'-t)}{\partial (t-t')^2} e^{\frac{i\omega_{Ln}}{a}(t-t')} \right] \times \right.$$

$$\times j_L\left(\omega_{Ln} \frac{r}{a}\right) j_L\left(\omega_{Ln} \frac{r'}{a}\right) \bar{T}^m_{L,L}(\bar{x}) \cdot \bar{T}^m_{L,L}(\bar{x}') +$$

$$+ \frac{1}{2} \sum_{Lmm} b_{Lm}^2 \frac{\omega_{Lm}}{a} \left[\frac{\partial^2 \theta(t-t')}{\partial (t-t')^2} e^{-\frac{i\omega_{Ln}}{a}(t-t')} + \frac{\partial^2 \theta(t'-t)}{\partial (t-t')^2} e^{\frac{i\omega_{Ln}}{\alpha}(t-t')} \right] \times$$

$$\times \Big[\frac{L}{2L+1} j_{L+1}(\omega_{Lm}\tfrac{r}{a}) j_{L+1}(\omega_{Lm}\tfrac{r'}{a}) \vec{T}^{\,m}_{L,L+1}(\hat{x}) \cdot \vec{T}^{\,m*}_{L,L+1}(\hat{x}') +$$

$$+ \frac{L+1}{2L+1} j_{L-1}(\omega_{Lm}\tfrac{r}{a}) j_{L-1}(\omega_{Lm}\tfrac{r'}{a}) \vec{T}^{\,m}_{L,L-1}(\hat{x}) \cdot \vec{T}^{\,m}_{L,L-1}(\hat{x}') -$$

$$- \frac{\sqrt{L(L+1)}}{2L+1} j_{L+1}(\omega_{Lm}\tfrac{r}{a}) j_{L-1}(\omega_{Lm}\tfrac{r'}{a}) \vec{T}^{\,m}_{L,L+1}(\hat{x}) \cdot \vec{T}^{\,m*}_{L,L-1}(\hat{x}') -$$

$$- \frac{\sqrt{L(L+1)}}{2L+1} j_{L-1}(\omega_{Lm}\tfrac{r}{a}) j_{L+1}(\omega_{Lm}\tfrac{r'}{a}) \vec{T}^{\,m}_{L,L-1}(\hat{x}) \cdot \vec{T}^{\,m*}_{L,L+1}(\hat{x}') \Big] \Big\}$$

and

$$E(a) = \int_0^a r^2 dr \int d\omega \sin\theta \, \mathcal{E}(a;r,\omega)$$

$$= \frac{1}{2a} \sum_L (2L+1) \Big[\sum_n \omega_{Ln} + \sum_m \omega_{Lm} \Big] \qquad (3\text{-}11)$$

Use has been made of the fact that the vector spherical harmonics are orthonormal and the spherical Bessel functions are normalized.

This expression for the total zero-point energy is obviously infinite as can easily be seen by noting that the ω_{Ln} and ω_{Lm} are the real positive roots of the spherical Bessel functions, which are infinite in number and form a monotonic increasing sequence with all of the terms greater in absolute magnitude than one.

The same calculation, performed for the annulus using (2-40) gives, for the total energy,

$$E(a,b) = \frac{1}{2b} \sum_{L} (2L+1) \left[\sum_{n} \omega_{Ln} + \sum_{m} \omega_{Lm} \right]$$

where ω_{Lm} and ω_{Ln} are the roots of (2-35) and (2-37) respectively. This expression is also divergent.

By adding together the Green's tensors for the inner and outer regions, differentiating the resulting tensor with respect to the radius a (which is equivalent to subtracting the Green's tensor for a second configuration with the radius a increased to a+da) and evaluating the result according to (3-10), a finite value of the energy density was expected. However, as remarked before, this procedure cannot be carried out without the introduction of a suitable convergence factor.

CHAPTER 4

THE EQUIVALENCE OF THE ZERO-POINT ENERGY TO THAT OF A NEUTRINO GAS OF UNLIMITED PARTICLE NUMBER

The partition function for a gas of fermions of vanishing rest mass and unrestricted particle number is[19]

$$Z = \prod_{n}^{\infty} (1 + e^{-\beta \varepsilon_n}) \qquad (4\text{-}1)$$

where ε_n is the energy of the n^{th} state and β is commonly identified as 1/kT. In our case the same formal expression gives us the zero-point energy of the sphere with β a cutoff parameter determining the penetrability of the sphere for different particle energies. We shall see that in the limit $\beta \to 0$ we obtain the full zero-point energy of an impenetrable sphere, namely, $\frac{1}{2}\hbar\omega$ per degree of freedom.

It must be stressed that there is no implication here of a physical connection between photons and the Fermi-Dirac statistics. This particular model has been introduced as a preliminary to a further study which will be a comparison of the results of quantum field theory and statistical mechanics when applied to similar problems in physics.

The mean energy is given by

---

19 F. Reif, Fundamentals of Statistical and Thermal Physics, McGraw-Hill Book Co., Inc., New York, 1965, p.350.

$$\bar{E} = -\frac{\partial \ln Z}{\partial \beta} = \sum_n^{\infty} \varepsilon_n / (1+e^{\beta \varepsilon_n}) \qquad (4\text{-}2)$$

In the case of the sphere considered before, the energy of the nth state is given by

$$\varepsilon_n = \hbar \omega_{Ln} / a \qquad (4\text{-}3)$$

where ω_{Ln} is a zero of a Bessel function. In what follows we choose units such that h=c=1.

Now consider the change in the mean energy with a small change in the radius of the sphere.

$$\partial \bar{E}/\partial a = \sum_n \frac{d\varepsilon_n}{da}\left[\frac{1}{1+e^{\beta\varepsilon_n}} - \frac{\beta \varepsilon_n e^{\beta \varepsilon_n}}{(1+e^{\beta\varepsilon_n})^2}\right]$$

From (4-3),

$$d\varepsilon_n/da = -\varepsilon_n/a \qquad (4\text{-}4)$$

From (4-2) and (4-4) we see that, in the case of the sphere, $\beta\bar{E}$ and $\beta^2 \frac{\partial \bar{E}}{\partial a}$ are functions of (β/a) only.

We now evaluate the mean value of the total energy of the gas in a volume V. In order to do this we will make use of the formula

$$dN = 4\pi \nu^2 V d\nu = \omega^2 V d\omega / 2\pi^2 \qquad (4\text{-}5)$$

which gives the number of states in the volume V between the frequencies ω and $\omega + d\omega$. (for large volumes only)

$$\int_0^{\infty} \bar{E}\, dN = 2\int_0^{\infty} \frac{\varepsilon_n \omega^2 V d\omega}{(1+e^{\beta\varepsilon_n})2\pi^2} \qquad (4\text{-}6)$$

where we must multiply by 2 to take account of the two possible polarizations. Then (4-6) becomes

$$\frac{V}{\pi^2 a^4}\int_0^{\infty}\frac{\omega^3 d\omega}{(1+e^{\frac{\beta}{a}\omega})} = \frac{V}{\pi^2}\int_0^{\infty}\frac{x^3 dx}{(1+e^{\beta x})} \qquad \text{if } \frac{\omega}{a} = x$$

Let $y=\beta x$:

$$\frac{V}{\pi^2}\int_0^\infty \frac{\omega^3 d\omega}{1+e^{\frac{\beta\omega}{a}}} = \frac{V}{\pi^2\beta^4}\int_0^\infty \frac{y^3 dy}{1+e^y} = \frac{V}{\pi^2\beta^4}\int_0^\infty \frac{y^3 e^{-y} dy}{1+e^{-y}}$$

$$= \frac{V}{\pi^2\beta^4}\int_0^\infty y^3 e^{-y}(1-e^{-y}+e^{-2y}-e^{-3y}+\dots)dy$$

$$= \frac{V}{\pi^2\beta^4}\sum_{n=1}^\infty (-)^{n+1}\int_0^\infty y^3 e^{-ny} dy$$

$$= \frac{V}{\pi^2\beta^4}\sum_{n=1}^\infty \frac{(-)^{n+1}}{n^4}\int_0^\infty z^3 e^{-z} dz \qquad \text{if } z=ny$$

$$= \frac{V}{\pi^2\beta^4}\sum \frac{(-)^{n+1}3!}{n^4} = \frac{6V}{\pi^2\beta^4}\sum_{n=1}^\infty \frac{(-)^{n+1}}{n^4}$$

The sum over n is approximately 0.9476.

$$\bar{E}_{total} \doteq \frac{6V(0.9476)}{\pi^2\beta^4} = (0.9476)\frac{8a^3}{\pi\beta^4} \qquad (4\text{-}7)$$

for a sphere of volume $V=\frac{4}{3}\pi a^3$.

The change in mean total energy when the volume is changed infinitesimaly is now considered. Denoting the energy obtained by using Weyl's formula ($dN=4\pi\nu^2 V d\nu$) for the number of states by $\bar{u}_{weyl}$, we obtain

$$dU/da = (-4\pi a^2\bar{u}_{weyl} + 4\pi a^2\bar{u}_{weyl} - dX(a)/da) \qquad (4\text{-}8)$$

The function X is a correction factor included to take account of any deviations from Weyl's formula in the actual number of states. It can be seen from (4-8) that if the sign of dX(a)/da were known, it would be possible to determine the direction of any stresses acting on the sphere.

It has not been possible as yet to determine the functional dependence of X(a) on the radius, and thus the direction and magnitude of the resulting stresses could not be deduced with any degree of precision.

The fact that Weyl's formula does indeed deviate from the true number of states can be seen from the Table. Only the magnetic mode is considered here as zeros of the expressions involved in the electric mode were not known precisely enough to enable us to draw a conclusion. For simplicity, a sphere of unit radius is considered. The same sort of table can be constructed for arbitrary radii if enough zeros are known for j_l .

It is hoped that a computer program can be developed to determine the zeros of the spherical Bessel functions for large order and large argument. Once a large number of such zeros is known, the Table can be extended and an analytic expression obtained for X(a) by curve fitting. This problem is being pursued.

We have obtained an expression for the causal and retarded Green's functions of a perfectly conducting sphere and are therefore able to derive the expression for the energy density as prescribed by quantum field theory. This energy, as is well known, is divergent, and it is a problem that goes beyond this thesis to remove these divergences. The method outlined in this chapter, will then allow us to compare the results obtained by quantum field theory with those obtained by statistical mechanics.

Table

	No. of states from Weyl formula		True no. of states (T. M. mode only)	
ω	$d\omega = \pi$	$d\omega = 0.5$	$d\omega = \pi$	$d\omega = 0.5$
10	67	4	80	7
15	149	24	168	45
20	265	42	294	50

APPENDIX

NORMALIZATION OF SPHERICAL BESSEL FUNCTIONS

The problem of normalizing the vector potential solutions on the interval zero to a, reduces to the problem of normalizing the spherical Bessel functions, since the vector spherical harmonics which are employed are normalized to unity.

Consider only Bessel functions of the first kind. Then $J_n(\lambda x)$ satisfies

$$[\lambda x J_n'(\lambda x)]' + (\lambda^2 x - \frac{n^2}{x}) J_n(\lambda x) = 0 \qquad (\text{A-1})$$

Multiply by $2xJ_n'(\lambda x)$.

$$2x J_n'(\lambda x J_n')' + (\lambda^2 x - n^2/x) 2x J_n' J_n = 0$$

$$\text{or } (d/dx)(\lambda x J_n')^2 + (\lambda^2 x^2 - n^2)(d/dx) J_n^2 = 0 \qquad (\text{A-2})$$

Integrating (A-2) we get,

$$\int_0^a d(\lambda x J_n')^2 + \int_0^a (\lambda^2 x^2 - n^2) d(J_n^2) = 0$$

$$\text{or } (\lambda x J_n')^2 \Big|_0^a + (\lambda^2 x^2 - n^2) J_n^2 \Big|_0^a = 2\lambda^2 \int_0^a x J_n^2 \, dx \qquad (\text{A-3})$$

Using $J_0(0)=1$; $J_n(0)=0$, $n=1,2,\ldots$, we get,

$$2\lambda^2 \int_0^a x J_n^2(\lambda x)\, dx = \lambda^2 a^2 [J_n'(\lambda a)]^2 + (\lambda^2 a^2 - n^2) J_n^2(\lambda a) \qquad (\text{A-4})$$

Suppose λ_{n_j} are the positive roots of $J_n(\lambda a)=0$. Then

$$2\lambda_{nj}^2 \int_0^a x J_n^2(\lambda_{nj} x)\,dx = \lambda_{nj}^2 a^2 [J_n'(\lambda_{nj} a)]^2 + (\lambda_{nj}^2 a^2 - n^2) J_n^2(\lambda_{nj} a)$$

or $$\int_0^a x [J_n(\lambda_{nj} x)]^2 dx = \frac{a^2}{2} [J_n'(\lambda_{nj} a)]^2 \qquad (\text{A-5})$$

Replacing x by ρ, and λ_{nj} by λ_{nj}/a, we get

$J_n(\lambda_{nj})=0,$

$$\int_0^a \rho [J_n(\lambda_{nj} \tfrac{\rho}{a})]^2 d\rho = \frac{a^2}{2} [J_n'(\lambda_{nj})]^2 \qquad (\text{A-6})$$

Using the recurrence relation

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

we get

$$J_n'(\lambda_{nj}) = -J_{n+1}(\lambda_{nj})$$

and thus

$$\int_0^a \rho [J_n(\lambda_{nj} \tfrac{\rho}{a})]^2 d\rho = \frac{a^2}{2} [J_{n+1}(\lambda_{nj})]^2 \qquad (\text{A-7})$$

Now suppose that ω_{nj} are the positive roots of

$$h J_n(\omega a) + \omega a J_n'(\omega a) = 0$$

and use this relation in (A-4) to get

$$2\omega_{nj}^2 \int_0^a x [J_n(\omega_{nj} x)]^2 dx = \omega_{nj}^2 a^2 [J_n'(\omega_{nj} a)]^2 + (\omega_{nj}^2 a^2 - n^2) [J_n(\omega_{nj} a)]^2$$

or $$\int_0^a x [J_n(\omega_{nj} x)]^2 dx = \frac{a^2}{2} [J_n'^2(\omega_{nj} a) + J_n^2(\omega_{nj} a)] - \frac{n^2}{2\omega_{nj}^2} [J_n(\omega_{nj} a)]^2$$

But $$[J_n'(\omega_{nj} a)]^2 = \frac{h^2}{\omega_{nj}^2 a^2} [J_n(\omega_{nj} a)]^2$$

Thus

$$\int_0^a x [J_n(\omega_{nj} x)]^2 dx = \frac{a^2}{2} \left[\frac{h^2}{\omega_{nj}^2 a^2} J_n^2(\omega_{nj} a) + J_n^2(\omega_{nj} a)\right] - \frac{n^2}{2\omega_{nj}^2} [J_n(\omega_{nj} a)]^2$$

$$= \frac{a^2 \omega_{nj}^2 - n^2 + h^2}{2\omega_{nj}^2} [J_n(\omega_{nj} a)]^2 \qquad (\text{A-8})$$

or $$\int_0^a \rho\,[J_n(\omega_{nj}\tfrac{\rho}{a})]^2 d\rho = \frac{(\omega_{nj}^2-n^2-h^2)a^2}{2\omega_{nj}^2}[J_n(\omega_{nj})]^2 \qquad (\text{A-9})$$

Now, the roots of $J_n(\lambda_{nj})=0$ are the roots also of $j_n(\lambda_{nj})=0$, and the roots of $\frac{1}{2}J_n(x)+xJ_n'(x)=0$ are the roots of $j_n(x)+xj_n'(x)=0$. The normalization conditions are then

$$\int_0^a \rho^2[j_n(\lambda_{nj}\tfrac{\rho}{a})]^2 d\rho = \frac{\lambda_{nj}a^2}{2}[j_{n+1}(\lambda_{nj})]^2 \quad \text{for } j_n(\lambda_{nj})=0 \qquad (\text{A-10})$$

$$\int_0^a \rho^2[j_n(\omega_{nj}\tfrac{\rho}{a})]^2 d\rho = \frac{(\omega_{nj}^2-n^2+\frac{1}{4})a^2}{2\omega_{nj}}[j_n(\omega_{nj})]^2 \quad \text{for } j_n(\omega_{nj})+\omega_{nj}j_n'(\omega_{nj})=0 \qquad (\text{A-11})$$

From (A-11) we can easily deduce the two other conditions required. They are

$$\int_0^a r^2[j_{L+1}(\omega_{Ln}\tfrac{r}{a})]^2 dr = \frac{(\omega_{Ln}^2-2(L-\frac{1}{2})+(L-\frac{1}{2})^2)a^2}{2\omega_{Ln}}[j_L(\omega_{Ln})]^2 \qquad (\text{A-12})$$

$$\int_0^a r^2[j_{L-1}(\omega_{Ln}\tfrac{r}{a})]^2 dr = \frac{(\omega_{Ln}^2-2L(L-1)+L^2-\frac{1}{4})a^2}{2\omega_{Ln}}[j_L(\omega_{Ln})]^2 \qquad (\text{A-13})$$

VITA AUCTORIS

The author was born in Barrie, Ontario, on June 28, 1942 and received his early education there. In 1964, he enrolled in the Honours Physics course at the University of Waterloo, and graduated there with a B.Sc. degree in Physics in 1969.