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FINITE ELEMENT ANALYSIS
OF
SKEW SANDWICH PLATES

by
MOFREH G. MICHAIL

A Thesis

Submitted to the Faculty of Graduate Studies Through the
Department of Civil Engineering in Partial Fulfilment
of the Requirements for the Degree of
Master of Applied Science at
The University of Windsor

Thesis Advisor: Dr. Gerard R. Monforton

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FINITE ELEMENT ANALYSIS
OF
SKEW SANDWICH PLATES

ABSTRACT

by
MOFREH G. MICHAIL

The finite element method is applied to the analysis of skew sandwich plates with homogeneous faces and orthotropic cores. Using the principle of minimum total potential energy, the geometric admissibility conditions are satisfied between the elements by representing the displacement function in two ways: first as the sum of products of zero order (Bilinear) Lagrange interpolation functions to represent the membrane displacements and the sum of products of one dimensional first order Hermite interpolation polynomials to represent the transverse displacement; in the second approach both the membrane and transverse displacements are represented by one dimensional first order (Bicubic) Hermite interpolation polynomials.

The strain energy formulations for membrane, bending and coupling are presented in terms of the element geometry, the thicknesses and stiffnesses of the faces and transverse shear moduli of the orthotropic core. The stiffness matrices (Membrane, bending and coupling) and hence the total stiffness matrix for a skew sandwich plate element with orthotropic core are derived by direct application of the theorem of minimum total potential energy.

This work is done such that for a given geometry, material properties of the faces and core and loading conditions, a variety of problems of rectangular, skew, thin or sandwich plates can be solved. Comparison of results is made with existing theoretical and experimental work on skew sandwich plates.

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LIST OF SYMBOLS

a, b	plate element dimensions
A_f, A_c	surface area of the faces and core respectively
B_f, D_f	membrane and bending stiffnesses of faces respectively
C	subscript, denoting core
E_f	elastic modulus of facing material
f	subscript, denoting face ($f = 1, 2$)
G_{cxz}, G_{cyz}	shear moduli of the core
h	distance between middle surfaces of faces
$H_{ki}^{(0)}(\xi)$	zero order Hermite (Lagrange) polynomial
$H_{ki}^{(1)}(\xi)$	one-dimensional, first order Hermite polynomial
$\{H^{(0)}\}$	column array of products of zero order Hermite polynomials
$\{H^{(1)}\}$	column array of products of first order Hermite polynomials
k_{ij}	elements of stiffness matrices
$[K]$	element stiffness matrices
$\bar{[K]}$	master stiffness matrix for supported structure
$M_{\xi f}, M_{\eta f}, M_{\xi \eta f}$	face moment resultants
$N_{\xi f}, N_{\eta f}, N_{\xi \eta f}$	face stress resultants

$\bar{N}_{\xi f}, \bar{N}_{\eta f}, \bar{N}_{\xi\eta f}$	applied forces on faces
$\bar{p}_{\xi f}, \bar{p}_{\eta f}$	applied surface loads
$\{P\}$	work equivalent load vector
$\{\bar{P}\}$	master load vector for supported structure
$Q_{c\xi z}, Q_{c\eta z}$	shear stress resultants for core
q_f	transverse load over faces
t_f, t_c	thicknesses of faces ($f = 1, 2$) and core respectively
u_f, v_f	displacement components of middle surface of faces in x and y directions
$\bar{u}_f, \bar{v}_f$	displacement components of middle surface of faces in ξ and η directions.
$\tilde{u}_f, \tilde{v}_f$	displacement components of point a distance z_f from middle surface
$\tilde{u}_c, \tilde{v}_c$	displacement components of a point a distance z_c from middle surface
w	transverse displacement of the sandwich system
U_f, U_c	strain energy of faces ($f = 1, 2$), core
U_{fm}, U_{fb}	strain energy of faces due to membrane and bending respectively
W	external work
x, y	distances measured in the orthogonal coordinate system
$\{X\}$	vector of element nodal coefficients
$\{\bar{X}\}$	vector of independent degrees of freedom

z_f, z_c	distances measured normal to the middle surfaces of the faces and core
β	skew angle measured from the y axis
$\gamma_{cxz}, \gamma_{cyz}$	core transverse shear strains
ζ_{ij}	kronecker delta
Δ	general displacement component ($\Delta = u_1, v_1, u_2, v_2, w$)
$\Delta_{ij}, \Delta_{\xi ij}, \Delta_{\eta ij}, \Delta_{\xi \eta ij}$	general element nodal coefficients
$\{\Delta\}$	column array of general nodal degrees of freedom
$\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	general normal and shear strains
ν	poisson's ratio
ξ, η	distances measured in the oblique coordinate system
π_p	element potential energy
Π_p	total potential energy
$\sigma_x, \sigma_y, \tau_{xy}$	normal and shear stresses
$\sigma_\xi, \sigma_\eta, \tau_{\xi\sigma}$	normal and shear stresses in the oblique coordinate system
τ_{cxz}, τ_{cyz}	core transverse shear strains
Φ_c, Ψ_c	rotations of normal to the surface of the core

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CHAPTER I

INTRODUCTION

Sandwich construction was introduced to the aerospace industry during the second world war and since this date its use in aircraft and missile applications has become wider and wider every day. Sandwich construction has also some civilian applications which are being given a great deal of attention recently such as: building walls, flooring for houses, roofs for some important structures in the form of folded plates or shell roofs for factories. Moreover in Germany a self-supporting automobile was made from sandwich material; its construction consists of glassfiber - reinforced plastic facings bonded to a polyurethane foam core. Skew sandwich plate construction has also some applications as a structural member in skew bridges. This wide variety of applications is mainly due to its special construction which consists of two thin strong facings enclosing and bonded to a relatively thick low - strength and lightweight core. In a sandwich member, the core is used to get the high strength facings as far as possible from the neutral axis so as to assure high bending stiffness; also it transmits shear between them allowing the two faces to act as a unit.

The main advantages of sandwich construction are: its high strength and stiffness relative to its weight which makes it suitable for aircraft applications and building industry; it has good insulating properties, easy erection, less construction time and longlife with low maintenance. Making use of these advantages the Aluminum Company of Canada started to construct complete houses from sandwich panels and it

is found that the cost is about one-half the price of the usual masonry buildings. Because of the relatively small weight, the houses can easily be transported to the building site. In order to obtain lightweight sandwich construction, a low - density material must be used for the core. Most of the core materials are orthotropic in the form of paper or aluminum honeycomb, styrafoam, polyvinyle chloride foam, balsa wood or plastics. The facing material must have a relatively high - strength in order to achieve better efficiency. Aluminum alloy, heat resisted steel, reinforced plastics, or any other suitable material can be used for facings according to the type of application.

In the field of lightweight structures such as sandwich plate systems, the finite element technique as a tool for predicting the behavior of such systems has been given a large amount of attention during the past few years. The finite element method has the distinct advantage of solving practical structures with irregular configurations and complicated boundary conditions. This technique involves representing the real structure as an assemblage of a number of interconnected finite elements for which the deformation state of each element is relatively simple as compared to the deformation state of the structure as a whole. The problem is then reduced to the determination of deformations of the individual elements. The accuracy of the finite element method is directly dependent on the ability of the assumed displacement patterns of the elements to represent the real behavior of the structural system. It should be emphasized that in selecting the assumed displacement functions, geometric admissibility conditions, rigid body displacement states

and 'completeness' must be taken into consideration. The finite element analysis may involve large amounts of numerical calculations; however, these methods can be easily formulated in a matrix form and hence the problem can be conveniently solved using digital computers.

Argyris (Ref. 1) has given a useful historical survey of the literature published using this method of analysis. The only available solution for parallelogramic sandwich plates was done in 1968 by solving the governing equations (Ref. 2). However, to the author's knowledge, no solutions are yet available for skew sandwich plates using finite element techniques. The work presented gives a complete discrete element analysis for skew sandwich plates.

1.1 Historical Discussion

Since 1941 a large amount of work has been devoted to the field of sandwich construction. In this section a brief review of the most important literature that lead to this work is presented.

The discrete element methods in structural mechanics started a long time ago; in 1941, Hrennikoff (Ref. 3) introduced solutions for some problems by a 'frame work method'. Levy (Ref. 4), in 1953 introduced an initial effort to model aircraft structures by one - dimensional beams and torque boxes. In 1956 Turner, Clough, Martin and Topp (Ref. 5) refined the method to include rectangular and triangular elements in order to represent two - dimensional inplane behavior. Pappenfuss (Ref. 6) and Melosh (Ref. 7) developed discrete element modelings for rectangular plates in bending.

In 1963 Melosh (Ref. 8), stated a number of requirements such as imposing the essential geometric admissibility conditions between the elements, which should be satisfied by the assumed displacement states; one of these conditions is maintaining displacement continuity along the common edge of adjacent elements. These requirements must be satisfied in order to assure monotonic converging solutions. Pian (Ref. 9) added that slope continuity for an assembled set of plate elements in bending must also be maintained along common edges. Numerical examples based on geometrically admissible displacement states shows that, if the finite element modeling is sequentially refined, convergence of the potential energy is monotonic, but it does not necessarily converge to the true minimum potential energy. However, if the element is mathematically 'complete' then the potential energy converges to the exact minimum of the potential energy. The problem of 'completeness' has been discussed in Refs. 10 and 11 and is still a current research area. Another requirement stated in Ref. 12 is that the element displacement states should be able to represent the rigid body modes even for relatively coarse modelings so that the rigid body displacements produce very little strain energy. In selecting assumed displacement functions, Bogner (Ref. 13) presented a good discussion of the requirements that should be satisfied (i.e. geometric admissibility conditions, rigid body displacement states and completeness).

In 1966 Bogner, Fox and Schmit (Ref. 14) developed a rectangular thin plate element satisfying all the above criteria using a displacement function formed by the sum of products of one - dimensional

Hermite interpolation polynomials. This method was also applied to a skew thin plate element by Monforton and Schmit (Ref. 15).

In the area of sandwich structures Refs. 16, 17, 18 and 19 present an instructive discussion of the present and future possibilities in this field. In 1948, Reissner (Ref. 20) made important contributions for the analysis of finite deflections of rectangular sandwich plates consisting of two identical isotropic faces bonding an isotropic core. He concluded that the range that the small deflection theory is valid decreases as the core shear stiffnesses is made smaller with respect to that of the faces. In the same year Taylor (Ref. 21) used the classical theory of plates to develop a set of modified plate equations applicable to the solution of rectangular sandwich plates. He also showed that the total deflection can be represented as the sum of two parts, one due to bending, the other part due to the transverse shear. Soon after in 1949, Libove and Batdorf (Ref. 22), developed a general small deflection theory for the elastic behavior of orthotropic rectangular sandwich plates; energy expressions and differential equations for sandwich plates were given.

In 1950, Hoff (Ref. 23) employed the same assumptions as Reissner to derive a set of differential equations in order to solve the problems of bending and buckling of sandwich plates using the strain energy and a linear eigenvalue approach. In the same paper he discussed the problems that arise in selecting realistic boundary conditions for sandwich plates. Ikeda (Ref. 24), in 1955 presented a solution for the basic differential equations giving the deflection and

the stress function. His theory was developed for bending of isotropic rectangular sandwich plates taking into account the contributions of the stiffnesses of the faces and the core. Many important publications by the Forest products Laboratory greatly affected this field of study; for example Refs. 25 and 26 reported theoretical and experimental results for the deflection and stresses of simply supported, rectangular sandwich plates. In 1957, Thurstone (Ref. 27), used Hoff's energy expressions with a Lagrangian multiplier method, and derived equations for evaluating the deflections and buckling loads of clamped rectangular sandwich plates. Cheng (Ref. 28), in 1962, using a similar approach to Thurstone, developed a system of differential equations for small deflections of a rectangular sandwich plate. Eringer (Ref. 29), by the use of the theorem of minimum complementary energy carried out a useful analysis on the bending and buckling of rectangular sandwich plates having a homogeneous core. The bending rigidities of the individual faces as well as the bending rigidity of the core were taken into account.

Martin (Ref. 30), described a triangular sandwich plate element including the effects of transverse shear deformations in the core. The core was approximated as shear panels on the element edges, while the bending action was resisted solely by membrane action of the faces. In this case the slope continuity conditions does not apply; this is a result of the assumption that the faces are incapable of carrying any bending moment by themselves. The sandwich element of Martin was applied to non - sandwich plates and achieved interelement displacement continuity and at the same time included the effects of the transverse shear

deformation. In 1967, Kan and Huang (Ref. 31) made use of the method of successive approximations to solve for the small and finite deflection analysis of a rectangular clamped sandwich plate. In the same year, Alwan (Ref. 32), described the small and finite deflection behavior of rectangular simply supported sandwich plates making use of Reissner's equations. In these cases the sandwich plates were composed of an isotropic core bounded by equal thickness isotropic faces. In 1968, Guha - Thakurta (Ref. 33) and Kennedy (Ref. 2), presented a solution to the governing differential equations of a clamped parallelogramic sandwich plate. To the author's knowledge they gave the only available theoretical and experimental numerical results on clamped parallelogramic sandwich plates. In the analysis the deflection is divided into two parts, one due to bending and the other due to transverse shear. An important report on the properties of aluminum sandwich panels, with theoretical and experimental results on folded plate structures has been given by Fazio and Kennedy (Ref. 34).

Recently, Monforton (Ref. 35) developed a finite element capability for the small and large deflection analysis of sandwich plates and cylindrical shells with anisotropic faces and orthotropic cores. The elements developed satisfy the geometric admissibility conditions, completeness and rigid body mode requirements. However the elements were restricted to a rectangular geometry. To the author's knowledge, no results are yet available for skew sandwich plates using the finite element technique. The work carried out herein is an extension

of the rectangular sandwich plate element reported by Monforton and Schmit (Ref. 36); the extension involves the formulation of a finite element analysis for skew sandwich plates.

1.2 Purpose and Scope

A finite element method, based on the theorem of minimum total potential energy, for skew sandwich plates is presented in the following chapters. Potential energy expressions are given for skew sandwich plates in an oblique coordinate system. The sandwich element is composed of two elastic isotropic facings bonded to a core which is considered orthotropic and typical of that used in honeycomb sandwich construction. A linear analysis is presented for skew sandwich plates with any skew angle β ; by simply setting $\beta = 0$, the analysis is capable of solving rectangular sandwich plate problems. Also by considering a single face of the sandwich system the method is conveniently applied to describe the behavior of thin isotropic skew and rectangular plates and membranes.

Chapter II contains general expressions for the strain energy of the faces and core in terms of the skew plate geometry, the elastic stiffnesses of the faces, the transverse shear moduli of the core and the displacement components. The displacement behavior is described by five variables: the four membrane displacements $u_f, \bar{v}_f$ ($f = 1, 2$) of the two faces and the transverse displacement w . The chosen oblique coordinate system and its relations to the rectangular coordinate system

are presented. Displacement continuity conditions between the core and faces are also included. General expressions for the work done by external loads (membrane and bending) are also given in terms of the applied load and the displacement components.

In chapter III, the displacement behavior of the skew sandwich plate is described using functions which readily permit satisfaction of the geometric admissibility conditions of the principle of minimum total potential energy. The first function is a zero order Hermite (Lagrange) interpolation polynomial, the second is a first order Hermite interpolation polynomial (Osculatory). Mainly two approaches are considered to solve the problem of skew sandwich plates, one of which is based on the assumption that the membrane displacements are represented by Lagrange (Bilinear) functions and the transverse displacement function is represented as first - order Hermite interpolation polynomials. In the second approach all the displacement functions are represented as first order Hermite interpolation polynomials. For each of the two mentioned approaches, the assumed displacement functions are substituted into the general expressions for the strain energy given in chapter II, to get a discretized strain energy formulation in terms of the plate geometry, the material properties and the undetermined nodal coefficients of the element. Following a systematic derivation, the elements of the element stiffness matrices are given.

In chapter IV, a variable correlation table is used to impose geometric admissibility between elements. The selection of the indep-

endent degrees of freedom, considering the additional conditions that can be imposed due to the engineer's experience and the expected behavior of the structure is thoroughly discussed.

Several numerical examples are solved to illustrate the potential of the finite element technique in the field of sandwich plates. Linear solutions (using a Gaussin elimination method) for different problems of rectangular and skew sandwich plates are presented. Results are compared, where possible, with existing theoretical and experimental results.

Chapter V is devoted to the discussion of the results, especially the main differences between solutions using the two previously mentioned approaches (i.e. Lagrange - Hermite approach and Hermite - Hermite approach). Also the effect of grid refinement of the sandwich plate system is discussed. Some conclusions and recommendations are drawn in chapter V.

CHAPTER II

GENERAL FORMULATIONS

This chapter develops the strain energy expressions and the potential of the applied loads for the general case of skew sandwich plates. The sandwich system consists of two isotropic thin sheets bonded to a relatively thick orthotropic core. Using the chosen oblique coordinate system, the potential energy is obtained in terms of the skew plate geometry, the stiffnesses of the faces, the transverse shear moduli of the core material and the displacement components of the skew plate.

2.1 Skew Plate Oblique Coordinates and Displacement Components:

An oblique coordinate system (ξ, η, z) is chosen (Fig.1) as follows:

the ξ axis coincides with the x axis which is the horizontal edge of the plate. The η axis makes an angle β with the positive direction of y and coincides with the inclined edge of the plate. β is the angle from the y axis to the η axis measured counterclockwise.

The relations between the orthogonal coordinates (x, y, z) and the oblique coordinates (ξ, η, z) can be written as:

$$x = \xi - \eta \sin \beta; \quad y = \eta \cos \beta \quad (2.1a)$$

$$\xi = x + y \tan \beta; \eta = y \sec \beta \quad (2.1b)$$

Also the relations between the derivatives of any displacement Δ in the two coordinate systems can be expressed as:

$$\frac{\partial \Delta}{\partial x} = \frac{\partial \Delta}{\partial \xi} \quad (2.2a)$$

$$\frac{\partial \Delta}{\partial y} = \frac{\partial \Delta}{\partial \xi} \tan \beta + \frac{\partial \Delta}{\partial \eta} \sec \beta \quad (2.2b)$$

$$\frac{\partial^2 \Delta}{\partial x^2} = \frac{\partial^2 \Delta}{\partial \xi^2} \quad (2.2c)$$

$$\frac{\partial^2 \Delta}{\partial y^2} = \frac{\partial^2 \Delta}{\partial \xi^2} \tan^2 \beta + 2 \frac{\partial^2 \Delta}{\partial \xi \partial \eta} \tan \beta \sec \beta + \frac{\partial^2 \Delta}{\partial \eta^2} \sec^2 \beta \quad (2.2d)$$

$$\frac{\partial^2 \Delta}{\partial x \partial y} = \frac{\partial^2 \Delta}{\partial \xi^2} \tan \beta + \frac{\partial^2 \Delta}{\partial \xi \partial \eta} \sec \beta \quad (2.2e)$$

Also if $\bar{u}$, $\bar{v}$ and u , v are the displacement components in the two coordinate system as shown in Fig. 1, then their relations are:

$$u = \bar{u} - \bar{v} \sin \beta \quad \bar{u} = u + v \tan \beta \quad (2.3a)$$

$$v = \bar{v} \cos \beta \quad \bar{v} = v / \cos \beta = v \sec \beta \quad (2.3b)$$

u , v are in the x , y directions and $\bar{u}$, $\bar{v}$ are in ξ , η directions.

2.2 Displacement Continuity Conditions between the Core and Faces:

Assuming that the faces of the sandwich plate are thin, the displacements for the faces $\tilde{u}_f, \tilde{v}_f$ away from their middle surface in terms of u_f, v_f, w , the displacements of the middle surface can be written as (Ref. 35):

$$\tilde{u}_f(x, y, z_f) = u_f(x, y) - z_f \frac{\partial w}{\partial x} \quad (2.4a)$$

$$\tilde{v}_f(x, y, z_f) = v_f(x, y) - z_f \frac{\partial w}{\partial y} \quad (2.4b)$$

where z_f is the distance from the middle surface of the face.

The following restrictions are applied as in Ref. 37 in order to derive the relationship between the components of strain and the components of displacements in the core:

- (1) The displacements $\tilde{u}_c, \tilde{v}_c$ away from the middle surface vary linearly across the core thickness, and the transverse deflection is constant throughout the thickness of the plate, that is:

$$\tilde{u}_c(x, y, z_c) = u_c(x, y) + z_c \Phi_c \quad (2.5a)$$

$$\tilde{v}_c(x, y, z_c) = v_c(x, y) + z_c \Psi_c \quad (2.5b)$$

$$\tilde{w}_c(x, y, z_c) = w(x, y) \quad (2.5c)$$

where Φ_c, ψ_c represents the rotation of the normals to the middle surface of the core; u_c and v_c are the inplane displacements of the middle surface of the core.

(2) The transverse displacement w is small.

(3) The plate rotations are small.

The displacements $\tilde{u}, \tilde{v}$ away from the middle surface should be the same for the faces and the core at the contact plane between both the faces and the core.

From Fig. 2 these displacement continuity conditions can be expressed as:

$$\tilde{u}_c(x, y, z_c = -t_c/2) = \tilde{u}_1(x, y, z_1 = +t_1/2) \quad (2.6a)$$

$$\tilde{u}_c(x, y, z_c = +t_c/2) = u_2(x, y, z_2 = -t_2/2) \quad (2.6b)$$

$$\tilde{v}_c(x, y, z_c = -t_c/2) = \tilde{v}_1(x, y, z_1 = +t_1/2) \quad (2.6c)$$

$$\tilde{v}_c(x, y, z_c = +t_c/2) = \tilde{v}_2(x, y, z_2 = -t_2/2) \quad (2.6d)$$

The transverse deflection w is assumed to remain constant through the thickness of the sandwich plate, that is:

$$\tilde{w}_c(x, y, z_c) = \tilde{w}_1(x, y, z_1) = \tilde{w}_2(x, y, z_2) = w(x, y) \quad (2.6e)$$

It is clear that:

$$u_c = [\tilde{u}_c(x, y, z_c = -t_c/2) + \tilde{u}_c(x, y, z_c = t_c/2)] / 2 \quad (2.7a)$$

$$v_c = [\tilde{v}_c(x, y, z_c = -t_c/2) + \tilde{v}_c(x, y, z_c = t_c/2)] / 2 \quad (2.7b)$$

Using Eq. 2.4 and the continuity conditions (Eqs. 2.6) and substituting into Eqs. 2.7 gives:

$$u_c = \frac{1}{2} (u_1 + u_2) + \frac{1}{4} (t_2 - t_1) \frac{\partial w}{\partial x} \quad (2.8a)$$

$$v_c = \frac{1}{2} (v_1 + v_2) + \frac{1}{4} (t_2 - t_1) \frac{\partial w}{\partial y} \quad (2.8b)$$

Applying conditions (2.5) and using Eqs. 2.8:

$$\Phi_c = \frac{1}{t_c} [(u_2 - u_1) + \frac{1}{2} (t_2 + t_1) \frac{\partial w}{\partial x}] \quad (2.9a)$$

$$\Psi_c = \frac{1}{t_c} [(v_2 - v_1) + \frac{1}{2} (t_2 + t_1) \frac{\partial w}{\partial y}] \quad (2.9b)$$

The membrane displacements u_1, v_1, u_2, v_2 of the faces and the transverse displacement w are taken to represent the displacements

of the sandwich system. The choice of the displacement variables in this way admits transverse shear deformations in the core, also allows flexibility in choosing realistic boundary conditions; for example, it is possible to have different membrane boundary conditions for the two faces (e.g. membrane displacements prevented along lower face and membrane displacements allowed on the upper face). Also the displacement variables selected makes it possible to easily solve problems of small deflection analysis of thin isotropic plates as a special case by simply considering one face of the sandwich system.

2.3 Strain Energy of Faces:

The strain energy of the faces are derived as thin isotropic plates making use of the following assumptions:

- (1) The faces are considered thin and isotropic.
- (2) A normal to the middle surface of the facing remains normal to it and does not change its length during deformations.
- (3) The linear theory of elasticity is assumed to be valid.
- (4) The facings are considered as ordinary thin plates taking their membrane and bending rigidities into account.
- (5) The transverse normal stress is negligible.

The strain energy density for a plate in the orthogonal coordinates x, y is well known as: (Ref. 38)

$$s_f = \frac{E}{2(1-\nu^2)} \left[\epsilon_x^2 + \epsilon_y^2 + 2\nu\epsilon_x\epsilon_y + \frac{1}{2}(1-\nu)\gamma_{xy}^2 \right] \quad (2.10)$$

where the strain displacement equations are:

$$\epsilon_x = \frac{\partial u_f}{\partial x} - z_f \frac{\partial^2 w}{\partial x^2} \quad (2.11a)$$

$$\epsilon_y = \frac{\partial v_f}{\partial y} - z_f \frac{\partial^2 w}{\partial y^2} \quad (2.11b)$$

$$\gamma_{xy} = \frac{\partial v_f}{\partial x} + \frac{\partial u_f}{\partial y} - 2z_f \frac{\partial^2 w}{\partial x \partial y} \quad (2.11c)$$

Substituting the above strain - displacement equations into Eq. 2.10 and then integrating over the volume of the face, the strain energy can be separated into two parts, one concerned with the membrane displacements (U_{fm}) and the other concerned with the bending displacements (U_{fb}). The strain energy of a face due to membrane displacements becomes, in rectangular coordinates:

$$U_{fm} = \frac{E_f t_f}{2(1-\nu_f^2)} \iint_{A_f} \left[\left(\frac{\partial u_f}{\partial x} \right)^2 + \left(\frac{\partial v_f}{\partial y} \right)^2 + 2\nu_f \left(\frac{\partial u_f}{\partial x} \right) \left(\frac{\partial v_f}{\partial y} \right) + \frac{1}{2} (1 - \nu_f) \left(\frac{\partial u_f}{\partial y} + \frac{\partial v_f}{\partial x} \right)^2 \right] dx dy \quad (2.12)$$

where A_f represents the area of the face. Using the proper relations between the rectangular and oblique coordinate systems as given by Eqs. 2.1 to 2.3 and substituting the differential element $dx dy$ by $\cos \beta d\xi d\eta$ results in the following expression for membrane strain

energy in the oblique coordinate system:

$$\begin{aligned}
 U_{fm} &= \frac{E_f t_f}{2(1-\nu_f^2)} \cos \beta \int_{A_f} \int \left[\left(\frac{\partial u_f}{\partial \xi} \right)^2 + \left(\sin \beta \frac{\partial \bar{v}_f}{\partial \xi} + \frac{\partial \bar{v}_f}{\partial \eta} \right)^2 \right. \\
 &+ 2\nu_f \left(\frac{\partial u_f}{\partial \xi} \right) \left(\sin \beta \frac{\partial \bar{v}_f}{\partial \xi} + \frac{\partial \bar{v}_f}{\partial \eta} \right) + \frac{1}{2} (1-\nu_f) \left(\frac{\partial u_f}{\partial \xi} \tan \beta \right. \\
 &\left. \left. + \frac{\partial u_f}{\partial \eta} \sec \beta + \frac{\partial \bar{v}_f}{\partial \xi} \cos \beta \right)^2 \right] d\xi d\eta \quad (2.13)
 \end{aligned}$$

Let the membrane stiffness B_f be taken as

$$B_f = \frac{E_f t_f}{(1-\nu_f^2)}$$

$$\begin{aligned}
 \text{therefore: } U_{fm} &= \frac{B_f}{2} \cos \beta \int_{A_f} \int \left[\left(\frac{\partial u_f}{\partial \xi} \right)^2 + \sin^2 \beta \left(\frac{\partial \bar{v}_f}{\partial \xi} \right)^2 + \left(\frac{\partial \bar{v}_f}{\partial \eta} \right)^2 \right. \\
 &+ 2 \sin \beta \left(\frac{\partial \bar{v}_f}{\partial \xi} \right) \left(\frac{\partial \bar{v}_f}{\partial \eta} \right) + 2\nu_f \sin \beta \left(\frac{\partial u_f}{\partial \xi} \right) \left(\frac{\partial \bar{v}_f}{\partial \xi} \right) \\
 &+ 2\nu_f \frac{\partial u_f}{\partial \xi} \frac{\partial \bar{v}_f}{\partial \eta} + \frac{1}{2} (1-\nu_f) \tan^2 \beta \left(\frac{\partial u_f}{\partial \xi} \right)^2 \\
 &+ \frac{1}{2} (1-\nu_f) \sec^2 \beta \left(\frac{\partial u_f}{\partial \eta} \right)^2 + \frac{1}{2} (1-\nu_f) \cos^2 \beta \left(\frac{\partial \bar{v}_f}{\partial \xi} \right)^2 \\
 &+ (1-\nu_f) \sec \beta \tan \beta \left(\frac{\partial u_f}{\partial \xi} \right) \left(\frac{\partial u_f}{\partial \eta} \right) + (1-\nu_f) \left(\frac{\partial u_f}{\partial \eta} \right) \left(\frac{\partial \bar{v}_f}{\partial \xi} \right) \\
 &\left. + (1-\nu_f) \sin \beta \left(\frac{\partial u_f}{\partial \xi} \right) \left(\frac{\partial \bar{v}_f}{\partial \xi} \right) \right] d\xi d\eta \quad (2.14)
 \end{aligned}$$

Similarly, the strain energy in rectangular coordinates for the bending portion of the strain energy becomes:

$$U_{fb} = \frac{D_f}{2} \int \int_{A_f} \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 2\nu_f \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2(1 - \nu_f) \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dx dy \quad (2.15)$$

where $D_f = \frac{E_f t_f^3}{12(1-\nu_f^2)}$ is the flexural rigidity of the face. Again, using the proper transformation from the x, y plane to the ξ, η plane, and using Eqs. 2.1, 2.2, 2.3, the bending strain energy in oblique coordinates can be written as:

$$U_{fb} = \frac{D_f}{2 \cos^3 \beta} \int \int_{A_f} \left\{ \left(\frac{\partial^2 w}{\partial \xi^2} \right)^2 + \left(\frac{\partial^2 w}{\partial \eta^2} \right)^2 + 2(\sin^2 \beta + \nu_f \cos^2 \beta) \left(\frac{\partial^2 w}{\partial \xi^2} \right) \left(\frac{\partial^2 w}{\partial \eta^2} \right) + 4 \sin \beta \left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} \right) \left(\frac{\partial^2 w}{\partial \xi \partial \eta} \right) + [4 \sin^2 \beta + 2(1-\nu_f) \cos^2 \beta] \left(\frac{\partial^2 w}{\partial \xi \partial \eta} \right)^2 \right\} d\xi d\eta \quad (2.16)$$

2.4 Strain Energy of the Core:

In the sandwich construction the two facings are bonded to a relatively thick core. The middle surface of the core is taken as the reference surface. The elastic shearing moduli can be different in the plane directions, that is x and y , but constant through the

core thickness. The following assumptions are considered to be reasonable for most core materials:

- (1) The core is incompressible in the transverse direction $\epsilon_{cz} = 0$.
- (2) The face - parallel stiffnesses are negligible relative to the transverse shear stiffness: $\sigma_{cx} = \sigma_{cy} = \tau_{xy} = 0$
- (3) Linear variation of displacements $\tilde{u}$, $\tilde{v}_c$ through the core thickness:

$$\tilde{u}_c(x, y, z_c) = u_c(x, y) + z_c \phi_c \quad (2.17a)$$

$$v_c(x, y, z_c) = v_c(x, y) + z_c \psi_c \quad (2.17b)$$

2.4.1 Strain Displacement Equations

Considering the above assumptions, the transverse shear strain displacement equations are:

$$\gamma_{cxz} = \frac{\partial w}{\partial x} + \phi_c \quad (2.18a)$$

$$\gamma_{cyz} = \frac{\partial w}{\partial y} + \psi_c \quad (2.18b)$$

By substituting for ϕ_c and ψ_c from Eqs. 2.9, the strain displacement equations of the core can be expressed in terms of the chosen displacement components u_1 , v_1 , u_2 , v_2 and w as follows:

$$\gamma_{cxz} = \frac{1}{t_c} \left[h \frac{\partial w}{\partial x} + (u_2 - u_1) \right] \quad (2.19a)$$

$$\gamma_{cyz} = \frac{1}{t_c} \left[h \frac{\partial w}{\partial y} + (v_2 - v_1) \right] \quad (2.19b)$$

where $h = \frac{t_1 + t_2}{2} + t_c$ is the distance between the middle surfaces of the faces.

2.4.2 Strain Energy Expressions

Following the assumptions stated in sec. 2.4, the strain energy density will be only due to the transverse shear strains γ_{cxz} and γ_{cyz} . Thus the strain energy density for the core is:

$$s_c = \frac{G_{xz}}{2} \gamma_{cxz}^2 + \frac{G_{yz}}{2} \gamma_{cyz}^2 \quad (2.20)$$

where G_{xz} and G_{yz} are the transverse shear moduli of the core in the x and y directions.

The strain energy of the core is written as:

$$U_c = \iiint_{V_c} s_c \, dV \quad (2.21)$$

where V_c is the volume of the core. Using Eqs. 2.19, substituting for u_c from Eq. 2.20 into Eq. 2.21, then integrating over the thickness of the core results in the following expression for the strain energy of the core in the rectangular coordinate system:

$$U_c = \frac{1}{2t_c} \int_{A_c} \int \{ G_{xz} [h \frac{\partial w}{\partial x} + (u_2 - u_1)]^2 + G_{yz} [h \frac{\partial w}{\partial y} + (v_2 - v_1)]^2 \} d x d y \quad (2.22)$$

Where A_c represents the surface area of the core. Using the relations between rectangular and oblique coordinate systems given by Eqs. 2.1, 2.2 and 2.3 and substituting for $dx dy$ by $\cos \beta d\xi d\eta$ results in the following expression for the strain energy of the core in the oblique coordinate system:

$$\begin{aligned} U_c = & \frac{1}{2t_c} \cos \beta \int \int_{A_c} \{ u_1^2 G_{xz} + u_2^2 G_{xz} - 2u_1 u_2 G_{xz} \\ & + \bar{v}_1^2 \cos^2 \beta G_{yz} + \bar{v}_2^2 \cos^2 \beta G_{yz} - 2\bar{v}_1 \bar{v}_2 \cos^2 \beta G_{yz} \\ & + (\frac{\partial w}{\partial \xi})^2 h^2 (G_{xz} + G_{yz} \tan^2 \beta) + (\frac{\partial w}{\partial \eta})^2 h^2 G_{yz} \sec^2 \beta \\ & + 2 \frac{\partial w}{\partial \xi} (\frac{\partial w}{\partial \eta}) h^2 (G_{yz} \sec^3 \beta \tan \beta) - 2 \frac{\partial w}{\partial \xi} u_1 \cdot h \cdot G_{xz} \\ & + 2 \frac{\partial w}{\partial \xi} u_2 h G_{xz} - 2 \frac{\partial w}{\partial \xi} \bar{v}_1 h \sin \beta G_{yz} + 2 \frac{\partial w}{\partial \xi} \bar{v}_2 h \sin \beta G_{yz} \\ & - 2 \frac{\partial w}{\partial \eta} \bar{v}_1 h G_{yz} + 2 \frac{\partial w}{\partial \eta} \bar{v}_2 h G_{yz} \} d \xi d \eta \quad (2.23) \end{aligned}$$

2.5 Potential of Applied Loads:

The work done by the externally applied loads can be simply calculated by multiplying the intensity of the load by the displacement in its direction and integrating over the line or the area on which the load acts.

2.5.1 Work Done by Membrane Loads

consider the general case of membrane loads as shown in Fig. 3 acting on the facings of a sandwich plate of dimensions A and B. The work done by the external loads can be written as:

$$\begin{aligned}
 W_{fm} = & \int_0^B \int_0^A \{ p_{\xi f}(\xi, \eta) \cdot u_f(\xi, \eta) + p_{\eta f}(\xi, \eta) \cdot \bar{v}_f(\xi, \eta) \} \cos \beta \, d\xi d\eta \\
 & - \int_0^A \bar{N}_{\eta f}(\xi, 0) \cdot \bar{v}_f(\xi, 0) \, d\xi + \int_0^A \bar{N}_{\eta f}(\xi, B) \cdot \bar{v}_f(\xi, B) \, d\xi \\
 & - \int_0^B \bar{N}_{\xi f}(0, \eta) \cdot u_f(0, \eta) \, d\eta + \int_0^B \bar{N}_{\xi f}(A, \eta) \cdot u_f(A, \eta) \, d\eta \\
 & + \int_0^A \bar{N}_{\eta \xi f}(\xi, B) u_f(\xi, B) \, d\xi - \int_0^A \bar{N}_{\eta \xi f}(\xi, 0) u_f(\xi, 0) \, d\xi \\
 & + \int_0^B \bar{N}_{\xi \eta f}(A, \eta) \bar{v}_f(A, \eta) \, d\eta - \int_0^B \bar{N}_{\xi \eta f}(0, \eta) \bar{v}_f(0, \eta) \, d\eta
 \end{aligned}
 \tag{2.24}$$

2.5.2 Work Done by Lateral Loads:

The external work done by applied laterally distributed loads $q_f (\xi, \eta)$ is given by:

$$W_{fb} = \iint_{A_f} q_f (\xi, \eta) w(\xi, \eta) \cos \beta \, d\xi d\eta \quad (2.25)$$

where A_f represents the area of the face upon which the loads acts.

2.6 Total Potential Energy:

The total energy is defined as:

$$\Pi_p = U - W_e \quad (2.26)$$

where U is the total strain energy of the sandwich plate given by:

$$U = \sum_{f=1}^2 (U_{fm} + U_{fb}) + U_c \quad (2.27)$$

By summing Eqs. 2.14 and 2.16 for the two faces and Eq. 2.23 for the core a general expression for the total strain energy is available.

Also

$$W_e = \sum_{f=1}^2 (W_{fm} + W_{fb}) \quad (2.28)$$

Substituting U and W_e in Eq. 2.26, an expression for the total potential energy of a sandwich skew plate in terms of the displacement variables selected can be written.

CHAPTER III

DISCRETIZED FORMULATION

The finite element analysis of a structure can be achieved by representing the structural system as an assembly of a finite number of individual elements. The assumed modes are chosen independent for each element and the undetermined functional form is represented by the physical degrees of freedom at the corners of the elements. The elements are then collected together to form the physical structure by satisfying the geometric admissibility conditions between the elements; these conditions insure that the displacements of the discretized structure are physically compatible with the original structure. Therefore by applying the proper linking conditions, the finite element method can be used to analyze structures with cutouts, stiffening members and irregularities that are often met in practical structures. Hence this technique of finite element structural analysis has certain advantages over most other energy methods such as the Ritz method. The difficiency of the traditional methods is that the assumed displacement functions must be chosen to represent the whole structure; this is difficult for structures with complicated boundary conditions and irregular configurations. Also the finite element method has certain advantages over some other techniques such as finite difference method.

The finite element method presented herein is based on the principle of minimum total potential energy which can be stated as: of all geometrically admissible displacement states that which minimizes the total potential energy (i.e. $\delta \pi_p = 0$) satisfies the equilibrium conditions, the natural boundary conditions and is therefore the true displacement state. The accuracy of the finite element method is directly measured by the ability of the assumed modes to represent the actual behavior of the structural system. In selecting the assumed displacement functions three main factors must be taken into consideration:

(1) The rigid body displacement states: the strain energy should vanish for a set of rigid body displacements, since rigid body displacement involves no strains.

(2) The geometric admissibility conditions: the assumed displacement states should satisfy the geometric admissibility requirements. Satisfaction of the displacement boundary conditions is required because the exact nature of the energy is undefined until the boundary conditions are specified. However the stress boundary conditions do not have to be imposed since they are natural boundary conditions which are automatically satisfied at the true minimum of the potential energy. The geometric admissibility conditions include also the continuity requirements on the assumed displacement modes. From the expressions of the total potential energy π_p for skew sandwich plates presented in the previous chapter, the assumed displacement modes should satisfy

the following admissibility conditions:

- (a) The physical displacement boundary conditions must be satisfied.
- (b) The membrane displacements $u_1, \bar{v}_1, u_2, \bar{v}_2$ and the transverse deflection w must be continuous within the element and along the common edges of adjacent elements.
- (c) The first derivatives of w must be continuous within the element; also the first derivative of w (slope) normal to the edge must be continuous along the common edges of adjacent elements.

Essentially the geometric admissibility conditions insure monotonic convergence of the potential energy as long as the structural idealization is not redefined as the structure is subdivided. This criterion however does not insure convergence to the exact solution. A desirable property of any finite element solution is that each successive refinement in grid size guarantees better convergence to the true minimum of the total potential energy

(3) 'Completeness': What is meant by completeness here involves guaranteeing that the solution converges to the true minimum of the potential energy in the limit as the number of elements increases to infinity. This factor is not dealt with herein since it is a mathematical definition which is at present an area of research itself.

The following sections discuss a description of a finite element for skew sandwich plates. The assumed displacement functions are presented in two forms. In the first case the displacement modes for the membrane displacements are formed as the sum of products of Lagrange interpolation polynomials while the bending displacements are

formed as the sum of products of one-dimensional first order Hermite interpolation polynomials. In the second case all the displacement modes (i.e. membrane and bending) are formed as the sum of products of one-dimensional first-order Hermite interpolation polynomial. In both cases the assumed functions are chosen such that they permit satisfaction of the geometric admissibility conditions of the principle of minimum total potential energy. The discretized strain energy and the potential of the applied loads are expressed in terms of the independent degrees of freedom of the structural system and are presented for the two approaches. Expressions for element stiffness matrices for both approaches are also presented in terms of the geometry, the stiffnesses and the undetermined coefficients of the skew sandwich plate. By simply taking the skew angle equal to zero the rectangular sandwich plate element is readily obtained. Also by considering one face of the system the problem of a thin plate or membrane (either rectangular or skew) can be solved.

3.1 Skew Sandwich Plate Finite Element:

The geometry of the finite element is defined in terms of the chosen oblique coordinate system described in Chapter II. The skew sandwich plate element is shown in Fig. 4. The coordinates of the nodes of each element are defined by (ξ_i, η_j) where $i, j = 1, 2$.

The displacement functions selected to describe the behavior of the skew sandwich plate are represented by two dimensional interpolation expressions in terms of the two previous suggested functions (Lagrange and Hermite) in the following forms:

- (1) Zero order Hermite or Lagrange interpolation functions:

$$\Delta(\xi, \eta) = \sum_{i=1}^2 \sum_{j=1}^2 H_{oi}^{(o)}(\xi) H_{oj}^{(o)}(\eta) \Delta_{ij} \quad (3.1)$$

where:

$$H_{o1}^{(o)}(\xi) = -(\xi - a)/a \quad H_{o1}^{(o)}(\eta) = -(\eta - b)/b \quad (3.3a)$$

$$H_{o2}^{(o)}(\xi) = \xi/a \quad H_{o2}^{(o)}(\eta) = \eta/b \quad (3.3b)$$

(2) First order Hermite or oscillatory interpolation functions:

$$\begin{aligned} \Delta(\xi, \eta) = & \sum_{i=1}^2 \sum_{j=1}^2 [H_{oi}^{(1)}(\xi) H_{oj}^{(1)}(\eta) \Delta_{ij} + H_{1i}^{(1)}(\xi) H_{oj}^{(1)}(\eta) \Delta_{\xi ij} \\ & + H_{oi}^{(1)}(\xi) H_{1j}^{(1)}(\eta) \Delta_{\eta ij} + H_{1i}^{(1)}(\xi) H_{1j}^{(1)}(\eta) \Delta_{\xi \eta ij}] \end{aligned} \quad (3.2)$$

Where:

$$H_{o1}^{(1)}(\xi) = (2\xi^3 - 3a\xi^2 + a^3)/a^3 \quad (3.4a)$$

$$H_{o2}^{(1)}(\xi) = -(2\xi^3 - 3a\xi^2)/a^3 \quad (3.4b)$$

$$H_{11}^{(1)}(\xi) = (\xi^3 - 2a\xi^2 + a^2\xi)/a^2 \quad (3.4c)$$

$$H_{12}^{(1)}(\xi) = (\xi^3 - a\xi^2)/a^2 \quad (3.4d)$$

Similar expressions for $H_{ki}^{(1)}(\eta)$ are obtained by replacing ξ by η and a by b .

and Δ_{ij} , $\Delta_{\xi ij}$, $\Delta_{\eta ij}$ and $\Delta_{\xi\eta ij}$ are related to the displacements and its derivatives at the node points. For example if $\Delta = w$ then:

$$\Delta_{ij} = w_{ij} = w(\xi = \xi_i, \eta = \eta_j) \quad (3.5a)$$

$$\Delta_{\xi ij} = w_{\xi ij} = \frac{\partial w}{\partial \xi}(\xi = \xi_i, \eta = \eta_j) \quad (3.5b)$$

$$\Delta_{\eta ij} = w_{\eta ij} = \frac{\partial w}{\partial \eta}(\xi = \xi_i, \eta = \eta_j) \quad (3.5c)$$

$$\Delta_{\xi\eta ij} = w_{\xi\eta ij} = \frac{\partial^2 w}{\partial \xi \partial \eta}(\xi = \xi_i, \eta = \eta_j) \quad (3.5d)$$

The following are some of the properties of the prementioned polynomials that are explained in Ref. 14: At the node $\xi = \xi_k$

$$H_{oi}^{(0)}(\xi_k) = \zeta_{ik} \text{ where } \zeta_{ik} = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k \end{cases} \quad (3.6)$$

For the first order Hermite interpolation polynomials:

$$H_{oi}^{(1)}(\xi_k) = \zeta_{ik} \quad \frac{dH_{oi}^{(1)}(\xi_k)}{d\xi} = 0 \quad (3.7a)$$

$$H_{li}^{(1)}(\xi_k) = 0 \quad \frac{dH_{li}^{(1)}(\xi_k)}{d\xi} = \zeta_{ik} \quad (3.7b)$$

It is these properties for the first order Hermite interpolation polynomials $H_{ki}^{(1)}$ that permit satisfaction of the admissibility conditions for the transverse deflection and their normal slopes. In case of the membrane displacements of faces u_f, v_f it appears that the zero order (Lagrange) interpolation polynomials are sufficient to satisfy the admissibility conditions for the interelement continuity of the displacements (u_f, v_f only) at common edges of adjacent elements. On the other hand the Lagrange interpolation polynomials can not maintain interelement continuity of membrane face strains. For this reason and other reasons as stated in Ref. 35, it was decided to present the problem of skew sandwich plates considering the two approaches stated in section 3.1 (Lagrange - Hermite and Hermite - Hermite). In the following sections the L - H approach will be taken to represent the Lagrange - Hermite approach and the H - H approach to represent the Hermite - Hermite approach.

3.2 Element Displacement Vector:

Using the displacement functions described by Eqs. 3.1 and 3.2, the assumed element displacement vector for each of the two approaches can be described.

3.2.1 L - H Approach:

Using the displacement functions (Bilinear) described by Eq. 3.1

to represent the membrane displacements ($u_1, \bar{v}_1, u_2, \bar{v}_2$) and the (Bicubic) functions described by Eq. 3.2 to represent the transverse deflection w , results in a finite element of 32 degrees of freedom (4 for each of $u_1, \bar{v}_1, u_2, \bar{v}_2$ and 16 for w). The displacement function given by Eq. 3.1 can be written as:

$$\Delta_M = \underset{1 \times 4}{[H^{(0)}]^T} \underset{4 \times 1}{\{\Delta_M\}} \quad (3.8)$$

where $\{\Delta_M\}$ is a vector containing the four undetermined nodal membrane displacements:

$$[\Delta_M] = [\Delta_{11}, \Delta_{12}, \Delta_{22}, \Delta_{21}] \quad (3.9)$$

and $\{H^{(0)}\}$ is a vector containing the four products of the zero order (Lagrange) interpolation polynomials:

$$[H^{(0)}] = [H_{o1}^{(0)}(\xi) H_{o1}^{(0)}(\eta), H_{o1}^{(0)}(\xi) H_{o2}^{(0)}(\eta); \\ H_{o2}^{(0)}(\xi) H_{o2}^{(0)}(\eta), H_{o2}^{(0)}(\xi) H_{o1}^{(0)}(\eta)] \quad (3.10)$$

In the same manner the displacement function given by Eq. 3.2 can be written as:

$$\Delta_B = \underset{1 \times 16}{[H^{(1)}]^T} \underset{16 \times 1}{\{\Delta_B\}} \quad (3.11)$$

where $\{\Delta_B\}$ is a vector containing the sixteen undetermined nodal coefficients associated with the displacement Δ :

$$\begin{aligned} [\Delta_B] = & [\Delta_{11}, \Delta_{\xi 11}, \Delta_{\eta 11}, \Delta_{\xi \eta 11}, \Delta_{12}, \Delta_{\xi 12}, \Delta_{\eta 12}, \Delta_{\xi \eta 12}, \Delta_{22}, \\ & \Delta_{\xi 22}, \Delta_{\eta 22}, \Delta_{\xi \eta 22}, \Delta_{21}, \Delta_{\xi 21}, \Delta_{\eta 21}, \Delta_{\xi \eta 21}] \end{aligned} \quad (3.12)$$

and $\{H^{(1)}\}$ is a vector containing the sixteen products of the one - dimensional first order Hermite interpolation polynomials:

$$\begin{aligned} [H^{(1)}] = & [H_{01}^{(1)}(\xi) H_{01}^{(1)}(\eta), H_{11}^{(1)}(\xi) H_{01}^{(1)}(\eta), H_{01}^{(1)}(\xi) H_{11}^{(1)}(\eta), \\ & H_{11}^{(1)}(\xi) H_{11}^{(1)}(\eta), H_{01}^{(1)}(\xi) H_{02}^{(1)}(\eta), H_{11}^{(1)}(\xi) H_{02}^{(1)}(\eta), H_{01}^{(1)}(\xi) \\ & H_{12}^{(1)}(\eta), H_{11}^{(1)}(\xi) H_{12}^{(1)}(\eta), H_{02}^{(1)}(\xi) H_{02}^{(1)}(\eta), H_{12}^{(1)}(\xi) H_{02}^{(1)}(\eta) \\ & H_{02}^{(1)}(\xi) H_{12}^{(1)}(\eta), H_{12}^{(1)}(\xi) H_{12}^{(1)}(\eta), H_{02}^{(1)}(\xi) H_{01}^{(1)}(\eta), H_{12}^{(1)}(\xi) H_{01}^{(1)}(\eta) \\ & H_{02}^{(1)}(\xi) H_{11}^{(1)}(\eta), H_{12}^{(1)}(\xi) H_{11}^{(1)}(\eta)] \end{aligned} \quad (3.13)$$

Giving Δ_M the values of $u_1, \bar{v}_1, u_2, \bar{v}_2$ and Δ_B the value of w , the displacement functions for the set of variables can be represented as:

$$u_1 = [H^{(o)}] \{u_1\} \quad (3.14a)$$

$$\bar{v}_1 = [H^{(o)}] \{\bar{v}_1\} \quad (3.14b)$$

$$u_2 = [H^{(o)}] \{u_2\} \quad (3.14c)$$

$$\bar{v}_2 = [H^{(o)}] \{\bar{v}_2\} \quad (3.14d)$$

$$w = [H^{(1)}] \{w\} \quad (3.14e)$$

3.2.2 H - H Approach:

Using the displacement functions given by Eq. 3.2 to represent both the membrane displacements ($u_1, \bar{v}_1, u_2, \bar{v}_2$) and the transverse deflection w results in a finite element of 80 degrees of freedom (16 for each of $u_1, \bar{v}_1, u_2, \bar{v}_2$ and w). Let Δ_B (described by Eqs. 3.11, 3.12 and 3.13) be allowed to take the values $u_1, \bar{v}_1, u_2, \bar{v}_2$ and w . The displacement functions for the set of variables can be represented as:

$$u_1 = [H^{(1)}] \{u_1\} \quad (3.15a)$$

$$\bar{v}_1 = [H^{(1)}] \{\bar{v}_1\} \quad (3.15b)$$

$$u_2 = [H^{(1)}] \{u_2\} \quad (3.15c)$$

$$\bar{v}_2 = [H^{(1)}] \{\bar{v}_2\} \quad (3.15d)$$

$$w = [H^{(1)}] \{w\} \quad (3.15e)$$

3.3 Discretized Strain Energy:

The discretized strain energy formulations can be expressed in terms of the plate geometry, the membrane and bending stiffnesses of the faces, the transverse shear moduli of the core and the nodal coefficients of the element (32 for the first approach and 80 for the second). This is obtained by substituting the displacement functions given by Eqs. 3.14 and 3.15 into the strain energy expressions for the faces (Eqs. 2.14 and 2.16) and the core given by Eqs. 2.23. Integrating these expressions over the corresponding areas and summing the contributions of the two faces and the core gives:

$$U = U_1 + U_2 + U_c \quad (3.16a)$$

$$\text{and } U_1 = U_{1m} + U_{1b} \quad (3.16b)$$

$$U_2 = U_{2m} + U_{2b} \quad (3.16c)$$

where U_{1m} is the discretized strain energy for membrane displacements of face 1 and U_{1b} is the discretized strain energy for bending displacements of face 1, etc. U_c is the discretized strain energy for

the core. The contributions to the strain energy U from the faces and the core can be expressed in the form:

$$U = \frac{1}{2} \underset{1 \times n}{\{X\}}^T \underset{n \times n}{[K]} \underset{n \times 1}{\{X\}} \quad (3.17)$$

where $n = 32$ for the first approach and $n = 80$ for the second. The vector $\{X\}$ contains the nodal degrees of freedom (32 for the first approach and 80 for the second) of the element.

$[K]$ is the element stiffness matrix resulting from the contributions of the subelement stiffness matrices of the two faces and the core. The above procedure will be carried out in the following sections for both of the two previous approaches (L - H and H - H) of a skew sandwich plate element.

3.3.1 L - H Formulations:

Substituting the displacement functions given by Eqs. 3.14 into the expressions for the strain energy of the faces and the core (Eqs. 2.14, 2.16 and 2.23) presented in the previous chapter, the discretized strain energy expressions can be written in the form of the integral of products of the Lagrange and Hermite interpolation functions and the undetermined nodal coefficients of the element (u_f , $\bar{v}_f$ and w), where:

$$\underset{4 \times 1}{[u_f]} = \underset{4 \times 1}{[u_{f11}, u_{f12}, u_{f22}, u_{f21}]} \quad (3.18a)$$

$$\begin{bmatrix} \bar{v}_f \end{bmatrix}_{4 \times 1} = [\bar{v}_{f11}, \bar{v}_{f12}, \bar{v}_{f22}, \bar{v}_{f21}] \quad (3.18b)$$

$$\begin{bmatrix} w \end{bmatrix}_{16 \times 1} = [w_{11}, w_{\xi 11}, w_{\eta 11}, w_{\xi \eta 11}, w_{12}, w_{\xi 12}, w_{\eta 12}, w_{\xi \eta 12}, \\ w_{22}, w_{\xi 22}, w_{\eta 22}, w_{\xi \eta 22}, w_{21}, w_{\xi 21}, w_{\eta 21}, w_{\xi \eta 21}] \quad (3.18c)$$

By summing these expressions for the two faces and the core as given by Eqs. 3.16 substituting into the form of Eq. 3.17, the total discretized strain energy can be represented as:

$$U = U_1 + U_2 + U_c = \frac{1}{2} \begin{bmatrix} X \end{bmatrix}_{1 \times 32}^T \begin{bmatrix} K \end{bmatrix}_{32 \times 32} \begin{bmatrix} X \end{bmatrix}_{32 \times 1} \quad (3.19)$$

As stated before the vector $\{X\}$ contains the thirty two undetermined nodal coefficients of the element and can be written in the following form:

$$\begin{bmatrix} X \end{bmatrix}_{32 \times 1} = \left\{ \begin{array}{c} \begin{bmatrix} \{u_1\} \\ (4 \times 1) \end{bmatrix} \\ \begin{bmatrix} \{\bar{v}_1\} \\ (4 \times 1) \end{bmatrix} \\ \begin{bmatrix} \{u_2\} \\ (4 \times 1) \end{bmatrix} \\ \begin{bmatrix} \{\bar{v}_2\} \\ (4 \times 1) \end{bmatrix} \\ \begin{bmatrix} \{w\} \\ (16 \times 1) \end{bmatrix} \end{array} \right\} \quad (3.20)$$

The vectors $\{\Delta\}$, ($\Delta = u_1, \bar{v}_1, u_2, \bar{v}_2$) contains the four nodal displacements Δ and $\{w\}$ contains the 16 nodal coefficients associated with w . For example in this approach $\{u_1\}$ is expressed as:

$$[u_1] = [u_{111}, u_{112}, u_{122}, u_{121}] \quad (3.21)$$

and

$$[w] = [w_{11}, w_{\xi 11}, w_{\eta 11}, w_{\xi \eta 11}, w_{12}, w_{\xi 12}, w_{\eta 12}, w_{\xi \eta 12}, w_{22}, w_{\xi 22}, w_{\eta 22}, w_{\xi \eta 22}, w_{21}, w_{\xi 21}, w_{\eta 21}, w_{\xi \eta 21}] \quad (3.22)$$

The element stiffness matrix $[K]$ is a symmetric matrix and can be partitioned in the following form:

$$\begin{array}{c}
 [K] = \\
 32 \times 32
 \end{array}
 \left[\begin{array}{ccccc}
 \begin{array}{c} (u_1 u_1) \\ [K]_1^1 \\ (4 \times 4) \end{array} & \begin{array}{c} (u_1 \bar{v}_1) \\ [K]_1^1 \\ (4 \times 4) \end{array} & \begin{array}{c} (u_1 u_2) \\ [K]_1^2 \\ (4 \times 4) \end{array} & \begin{array}{c} [O] \\ (4 \times 4) \end{array} & \begin{array}{c} (u_1 w) \\ [K]_1^w \\ (4 \times 16) \end{array} \\
 \hline
 & \begin{array}{c} (\bar{v}_1 \bar{v}_1) \\ [K]_1^1 \\ (4 \times 4) \end{array} & [O] & \begin{array}{c} (\bar{v}_1 \bar{v}_2) \\ [K]_1^2 \\ (4 \times 4) \end{array} & \begin{array}{c} (\bar{v}_1 w) \\ [K]_1^w \\ (4 \times 16) \end{array} \\
 \hline
 & & \begin{array}{c} (u_2 u_2) \\ [K]_2^2 \\ (4 \times 4) \end{array} & \begin{array}{c} (u_2 \bar{v}_2) \\ [K]_2^2 \\ (4 \times 4) \end{array} & \begin{array}{c} (u_2 w) \\ [K]_2^w \\ (4 \times 16) \end{array} \\
 \hline
 & & & \begin{array}{c} (\bar{v}_2 \bar{v}_2) \\ [K]_2^2 \\ (4 \times 4) \end{array} & \begin{array}{c} (\bar{v}_2 w) \\ [K]_2^w \\ (4 \times 16) \end{array} \\
 \hline
 & & & & \\
 \hline
 \begin{array}{c} \text{(SYMMETRIC)} \end{array} & & & & \begin{array}{c} (ww) \\ [K] \\ (16 \times 16) \end{array}
 \end{array} \right]$$

(3.23)

Where all the submatrices have the dimensions as indicated in the above element stiffness matrix. The elements of the submatrices can be written in an integral form; for example:

$$\begin{aligned}
[K^{(ww)}] &= \sum_{f=1}^2 \frac{D_f}{\cos^3 \beta} \int_0^a \int_0^b \{ H_{\xi\xi}^{(1)} H_{\xi\xi}^{(1)T} + (\sin^2 \beta + \nu_f \cos^2 \beta) \\
&\quad (H_{\xi\xi}^{(1)} H_{\eta\eta}^{(1)T} + H_{\eta\eta}^{(1)} H_{\xi\xi}^{(1)T}) + H_{\eta\eta}^{(1)} H_{\eta\eta}^{(1)T} + 2 \sin \beta (H_{\xi\xi}^{(1)} \\
&\quad H_{\xi\eta}^{(1)T} + H_{\xi\eta}^{(1)} H_{\xi\xi}^{(1)T} + H_{\eta\eta}^{(1)} H_{\xi\eta}^{(1)T} + H_{\xi\eta}^{(1)} H_{\eta\eta}^{(1)T}) + [4 \sin^2 \beta \\
&\quad + 2 (1 - \nu_f) \cos^2 \beta] H_{\xi\eta}^{(1)} H_{\xi\eta}^{(1)T} \} d\xi d\eta + \frac{h^2}{t_c} \cos \beta \\
&\quad \int_0^b \int_0^a \{ H_{\xi}^{(1)} H_{\xi}^{(1)T} (G_{xz} + G_{yz} \tan^2 \beta) + H_{\eta}^{(1)} H_{\eta}^{(1)T} (G_{yz} \sec^2 \beta) \\
&\quad + (H_{\xi}^{(1)} H_{\eta}^{(1)T} + H_{\eta}^{(1)} H_{\xi}^{(1)T}) (G_{yz} \sec \beta \tan \beta) \} d\xi d\eta
\end{aligned}
\tag{3.24}$$

Note that the subscripts of $H^{(1)}$ denote partial derivatives of the elements of $H^{(1)}$ as mentioned before.

$$B_f = \frac{E_f t_f^3}{(1-\nu_f^2)} \text{ represents the membrane stiffness of the faces where } f = 1, 2.$$

$$D_f = \frac{E_f t_f^3}{12(1-\nu_f^2)} \text{ represents the flexural stiffness of the faces where } f = 1, 2.$$

G_{xz} , G_{yz} are the transverse shear moduli in the ξ and η directions.

$$h = (t_1 + t_2)/2 + t_c$$

From these expressions it can be seen that the submatrices on the diagonal are symmetric and contain contributions that do not involve coupling between the various displacement components. The off-diagonal submatrices generally are not symmetric and contain the contributions for the coupling terms in the strain energy. Formulas for all the elements of the stiffness matrix in integer form is given in Appendix B.

3.3.2 H - H Formulations:

In a similar manner substituting the displacement functions given by Eqs. 3.15 into the strain energy expressions of the faces and core (Eqs. 2.14, 2.16 and 2.23), the discretized strain energy expressions can be written in an integral form in terms of the first order Hermite interpolation functions and the undetermined nodal coefficients of the element. Summing these expressions for the two faces and the core, putting it in the form of Eq. 3.17 as:

$$U = U_1 + U_2 + U_c = \frac{1}{2} \underset{1 \times 80}{[X]^T} \underset{80 \times 80}{[K]} \underset{80 \times 1}{[X]} \quad (3.25)$$

The vector $\{X\}$ contains the eighty undetermined nodal coefficients of the element and can be partitioned as:

$$\{X\} = \begin{bmatrix} \{Z\} \\ 16 \times 1 \\ \hline \{\bar{v}_2\} \\ 16 \times 1 \\ \hline \{w\} \\ 16 \times 1 \end{bmatrix} \quad (3.26)$$

The vector $\{\Delta\}$, ($\Delta = u_1, \bar{v}_1, u_2, \bar{v}_2$ and w) contains the sixteen nodal displacements associated with each of the Δ 's, for example $\{u_1\}$ is expressed as:

$$\begin{aligned} [u_1] = & [u_{111}, u_{1\xi 11}, u_{1\eta 11}, u_{1\xi\eta 11}, u_{112}, u_{1\xi 12}, u_{1\eta 12}, \\ & u_{1\xi\eta 12}, u_{122}, u_{1\xi 22}, u_{1\eta 22}, u_{1\xi\eta 22}, u_{121}, u_{1\xi 21}, \\ & u_{1\eta 21}, u_{1\xi\eta 21}] \end{aligned} \quad (3.27)$$

Similar expressions can be written for $\{\bar{v}_1\}$, $\{u_2\}$ and $\{\bar{v}_2\}$, and $\{w\}$ is exactly as given by Eq. 3.22. The element stiffness matrix $[K]$ in this case is an (80×80) symmetric matrix and can be partitioned in the following form:

$$\begin{array}{c}
 [K] = \\
 80 \times 80
 \end{array}
 \begin{bmatrix}
 (u_1 u_1) & (u_1 \bar{v}_1) & (u_1 u_2) & [0] & (u_1 w) \\
 [K]_{11} & [K]_{1\bar{v}_1} & [K]_{1u_2} & [0] & [K]_{1w} \\
 \hline
 & (\bar{v}_1 \bar{v}_1) & [0] & (\bar{v}_1 \bar{v}_2) & (\bar{v}_1 w) \\
 & [K]_{\bar{v}_1\bar{v}_1} & [0] & [K]_{\bar{v}_1\bar{v}_2} & [K]_{\bar{v}_1w} \\
 \hline
 & & (u_2 u_2) & (u_2 \bar{v}_2) & (u_2 w) \\
 & & [K]_{2u_2} & [K]_{2\bar{v}_2} & [K]_{2w} \\
 \hline
 & & & (\bar{v}_2 \bar{v}_2) & (\bar{v}_2 w) \\
 & & & [K]_{2\bar{v}_2} & [K]_{2w} \\
 \hline
 & & & & (ww) \\
 & & & & [K]_{ww}
 \end{bmatrix}
 \quad (3.28)$$

(SYMMETRIC)

where all the submatrices are 16 x 16 dimensional arrays. The elements of the submatrices can be written in an integral form by taking the contributions of the different strain energy expressions; for example:

$$\begin{aligned}
[K]_{f f}^{(u_f u_f)} &= B_f \cos \beta \int_0^b \int_0^a \{ H_{\xi}^{(1)} H_{\xi}^{(1)T} [1 + \frac{1}{2} \tan^2 \beta (1 - \nu_f)] \\
&+ H_{\eta}^{(1)} H_{\eta}^{(1)T} [\frac{1}{2} \sec^2 \beta (1 - \nu_f)] + [\frac{1}{2} H_{\xi}^{(1)} H_{\eta}^{(1)T} (1 - \nu_f) \\
&\sec \beta \tan \beta + \frac{1}{2} H_{\eta}^{(1)} H_{\xi}^{(1)T} (1 - \nu_f) \sec \beta \tan \beta] \} d \xi d \eta \\
&+ \frac{1}{t_c} \cos \beta \int_0^b \int_0^a H^{(1)} H^{(1)T} G_{xz} d \xi d \eta \quad (3.29)
\end{aligned}$$

and $K_{ij}^{(ww)}$ is exactly the same expression as Eq. 3.24. B_f , D_f , G_{xz} , G_{yz} and h are the same as defined in the L - H approach and $f = 1, 2$.

The elements of the submatrices presented in the two approaches can be written again after integration in terms of integers representing the different combinations of Lagrange and Hermite interpolation polynomials as given in Appendix B.

3.4 Discretized Work Done:

The applied loads of the discretized structure are calculated as work equivalent loads. Substituting the assumed displacement patterns in the two approaches mentioned before into the external work expressions for the faces (Eqs. 2.24 and 2.25), the work done by loads applied to the faces can be described. For example taking a particular system of loads as shown in Fig. 5, expressions 2.24 and 2.25 leads to:

$$\begin{aligned}
W_{1m} = & - \int_0^a q_1 \cdot v_1(\xi, 0) d\xi + \int_0^a q_3 v_1(\xi, b) d\xi \\
& - \int_0^b q_2 \cdot u_1(0, \eta) d\eta + \int_0^b q_4 \cdot u_1(a, \eta) d\eta
\end{aligned} \quad (3.30a)$$

$$\begin{aligned}
W_{2m} = & - \int_0^a q_5 v_2(\xi, 0) d\xi + \int_0^a q_7 \cdot v_2(\xi, b) d\xi \\
& - \int_0^b q_6 u_2(0, \eta) d\eta + \int_0^b q_8 u_2(a, \eta) d\eta
\end{aligned} \quad (3.30b)$$

$$W_b = \int_0^b \int_0^a q \cdot w(\xi, \eta) \cdot \cos \beta d\xi d\eta \quad (3.30c)$$

where: q_1, q_2, q_3, q_4 are uniform membrane loads acting along the boundaries of a finite element of face 1. q_5, q_6, q_7, q_8 are uniform membrane loads acting along the boundaries of a finite element of face 2. q is the sum of the lateral uniformly distributed loads over faces one and two. The work done by all the loads is:

$$W = \sum_{f=1}^2 W_f = W_{1m} + W_{2m} + W_b \quad (3.31)$$

This last expression (Eq. 3.31) can be conveniently expressed as:

$$W = \begin{bmatrix} P \end{bmatrix}_{1 \times n}^T \begin{Bmatrix} X \end{Bmatrix}_{n \times 1} \quad (3.32)$$

($n = 32$ for L - H approach and 80 for H - H approach)

where $\{P\}$ is a vector containing the work equivalent loads (32 for L - H approach and 80 for H - H approach) associated with each of the undetermined nodal coefficients in the vector $\{X\}$.

3.4.1 Work Equivalent Loads in Case of L - H Approach:

Substituting the membrane displacement functions $u_f, \bar{v}_f$ (Eqs. 3.14a, b) for the two faces into Eqs. 3.30 a, b and the transverse displacement w (Eq. 3.14c) into Eq. 3.30c, and performing the indicated integrations, the work equivalent load vector can be partitioned as:

$$\begin{Bmatrix} P \\ 1 \times 32 \end{Bmatrix} = \begin{bmatrix} P_{u1} & P_{\bar{v}1} & P_{u2} & P_{\bar{v}2} & P_w \\ 1 \times 4 & 1 \times 4 & 1 \times 4 & 1 \times 4 & 1 \times 16 \end{bmatrix} \quad (3.33)$$

The vector $\{P_\Delta\}$, ($\Delta = u_1, \bar{v}_1, u_2, \bar{v}_2, w$) contains the thirty two equivalent loads associated with each of the thirty two undetermined nodal coefficients in the vector $\{X\}$ given by Eq. 3.20. For example $\{P_{u1}\}$ is expressed as:

$$\begin{Bmatrix} P_{u1} \\ 1 \times 4 \end{Bmatrix} = [-q_2 \cdot b/2, -q_2 \cdot b/2, q_4 \cdot b/2, q_4 \cdot b/2] \quad (3.34a)$$

and

$$\begin{aligned} \begin{Bmatrix} P_w \\ 1 \times 16 \end{Bmatrix} &= [qab/4, qa^2b/24, qab^2/24, qa^2b^2/144, qab/4, qa^2b/24, \\ &-qab^2/24 - qa^2b^2/144, qab/4, -qa^2b/24, -qab^2/24, qa^2b^2/144, \end{aligned}$$

cont.

$$q_{ab}/4, -q_a^2 b/24, q_{ab}^2/24, -q_a^2 b^2/144] \cdot \cos \beta \quad (3.34b)$$

3.4.2 Work Equivalent Loads in Case of H - H Approach:

In the same manner substituting the displacement functions u_f , $\bar{v}_f$ (Eqs. 3.15a, b) for the two faces into Eqs. 3.30a, b and the transverse displacement w (Eq. 3.15c) into Eq. 3.30c and integrating over the indicated limits, the work equivalent load vector can be partitioned as:

$$\begin{Bmatrix} P \end{Bmatrix}_{1 \times 80} = \begin{bmatrix} P_{u1} & P_{\bar{v}1} & P_{u2} & P_{\bar{v}2} & P_w \end{bmatrix}_{\substack{1 \times 16 & 1 \times 16 & 1 \times 16 & 1 \times 16 & 1 \times 16}} \quad (3.35)$$

The vector $\{P_\Delta\}$, ($\Delta = u_1, \bar{v}_1, u_2, \bar{v}_2, w$) contains the eighty equivalent loads (16 for each) associated with each of the eighty undetermined nodal coefficients in the vector $\{X\}$ given by Eq. 3.26. For example $\{P_{u1}\}$ is expressed as:

$$\begin{aligned} \begin{Bmatrix} P_{u1} \end{Bmatrix}_{1 \times 16} &= [-q_2 b/2, 0, -q_2 b^2/12, 0, -q_2 b/2, 0, q_2 b^2/12, 0, \\ &\quad q_4 b/2, 0, -q_4 b^2/12, 0, q_4 b/2, 0, q_4 b^2/12, 0] \end{aligned} \quad (3.36)$$

The vector $\{P_w\}$ is exactly the same as Eq. 3.34b.

The formulation presented in this work depends mainly on the principle of minimum total potential energy. The discretized potential energy of the K^{th} sandwich plate element is represented by:

$$\pi_p^{(k)} = U^{(k)} - W^{(k)} \quad (3.37)$$

where $U^{(k)}$ and $W^{(k)}$ are the discretized strain energy and work done given by Eqs. 3.19, 3.25, 3.31 for each of the two mentioned approaches. The total potential energy for the whole system is approximated by:

$$\Pi_p = \sum_{k=1}^N \pi_p^{(k)} = \frac{1}{2} \{\bar{X}\}^T [\bar{K}] \{\bar{X}\} - \{\bar{X}\}^T \{\bar{P}\} \quad (3.38)$$

where: N is the number of elements used to model the original plate.

$\{\bar{X}\}$ represents a vector contains the n independent nodal degrees of freedom for the assembeled finite element plate.

$$\{\bar{X}\}^T = \{X_1, X_2, X_3, \dots, X_i, \dots, X_n\}$$

$[\bar{K}]$ is the master stiffness matrix corresponds to these n undetermined displacements.

$\{\bar{P}\}$ is a vector of work equivalent loads for the sandwich plate element.

By the use of the principle of minimum total potential energy, the displacement state $\{\bar{X}\}$ should be chosen so as to achieve minimum Π_p that is when:

$$\frac{\partial \Pi_p}{\partial X_i} = 0, \quad i = 1, 2, \dots, n \quad (3.39)$$

This equation yields the governing matrix equation for static formulation as:

$$\{\bar{P}\} = [\bar{K}] \cdot \{\bar{X}\} \quad (3.40)$$

The choice of X_i is governed by the geometric admissibility conditions; the technique of selecting the $\{\bar{X}\}$ vector will be discussed in the next chapter.

CHAPTER IV
IMPLEMENTATION AND NUMERICAL
EVALUATION

The present chapter is concerned with two main parts. The first section is devoted to the various aspects of the implementation of the finite element analysis capability to skew sandwich plate problems. In the two previous chapters general formulations and discretized potential energy expressions were developed for a general skew sandwich plate. The displacement components of the finite element were expressed as the sum of products of Lagrange and Hermite interpolation polynomials and undetermined nodal coefficients of the element. In an assembled structure modeled by a number of finite elements, the selection of a set of independent nodal degrees of freedom satisfying the admissibility conditions of the idealized structure is of great importance. Examples are presented in this chapter to illustrate the technique used to select the independent degrees of freedom for skew sandwich plate problems so as to satisfy the admissibility conditions. The problem is formulated in terms of the membrane displacements u_f , $\bar{v}_f$ and the transverse displacement w . After deciding what the independent degrees - of - freedom are for a particular structure idealized as an assemblage of discrete elements, the method of forming a variable correlation table in terms of the chosen degrees - of - freedom for such a problem are discussed. The second part of this chapter includes several numerical examples on

plates specifically skew sandwich plates. Comparison of results are made whenever it is possible with existing solutions.

4.1 IMPLEMENTATION:

Independent degrees of freedom:

For a real structure represented by an assemblage of finite elements, numerical solutions are generated by the use of the theorem of minimum total discretized potential energy. As mentioned before this requires that the displacement patterns of a discretized structure must be represented by a set of independent nodal degrees of freedom. This set of independent degrees of freedom is obtained by linking the degrees of freedom of all the elements noting that the total number of degrees - of - freedom can be reduced by imposing various additional conditions depending upon the expected behavior of the structure. For example the displacement patterns as expressed for the L - H approach by Eqs. 3.14, it can be seen that at each node there is one degree - of - freedom associated with each of the displacements u_1 , $\bar{v}_1$, u_2 , $\bar{v}_2$ and four for w ; this makes a total of 8 degrees - of - freedom for each corner. Therefore at each node of an assemblage of finite elements, there are $8M$ possible degrees of freedom where M is the number of elements joining the node. In case of the H - H approach, all the displacement variables (u_1 , $\bar{v}_1$, u_2 , $\bar{v}_2$ and w) are expressed in terms of the first order Hermite interpolation polynomials as given by Eqs. 3.2. It can be seen from Eqs. 3.2 that at each node there are 4 degrees - of - freedom associated with each of the displacements u_1 , $\bar{v}_1$, u_2 , and $\bar{v}_2$

and four for w ; this makes a total of 20 degrees - of - freedom for each corner. Therefore at each node of an assemblage of finite elements, there are $20M$ possible degrees of freedom where M is the number of elements joining the node. Of course not all of these degrees - of - freedom are independent; this number is reduced considerably by invoking the geometric admissibility conditions, boundary conditions, symmetry conditions and additional conditions.

4.11 Geometric Admissibility:

This section discusses the selection of a set of independent degrees - of - freedom so as to satisfy geometric admissibility at the common edges of adjacent elements for the case of skew sandwich plates. As an example consider the group of four adjacent skew sandwich elements shown in Fig. 6. The geometric admissibility conditions on the common edge between elements I and II are represented by:

$$u_f^I (\xi^I = \xi, \eta^I = b) = u_f^{II} (\xi^{II} = \xi, \eta^{II} = 0) \quad (4.1a)$$

$$\bar{v}_f^I (\xi^I = \xi, \eta^I = b) = \bar{v}_f^{II} (\xi^{II} = \xi, \eta^{II} = 0) \quad (4.1b)$$

$$w^I (\xi^I = \xi, \eta^I = b) = w^{II} (\xi^{II} = \xi, \eta^{II} = 0) \quad (4.2)$$

and

$$\frac{\partial w^I}{\partial \eta} (\xi^I = \xi, \eta^I = b) = \frac{\partial w^{II}}{\partial \eta} (\xi^{II} = \xi, \eta^{II} = 0) \quad (4.3)$$

where the subscripts I and II indicates the displacement variables associated with the corresponding elements and $f = 1, 2$ corresponds to faces one and two and the coordinates ξ^I, η^I and ξ^{II}, η^{II} are the local coordinates of the corresponding element I and II.

A. Case of L - H Approach:

From Eqs. 3.1 and 3.2 it is possible to express $u_f, \bar{v}_f$ and w along the common edge of the two elements I and II as:

$$u_f^I(\xi, b) = \sum_{i=1}^2 H_{oi}^{(o)}(\xi) u_{fi2}^I \quad (4.4a)$$

$$u_f^{II}(\xi, o) = \sum_{i=1}^2 H_{oi}^{(o)}(\xi) u_{fi1}^{II} \quad (4.4b)$$

a similar expression for $\bar{v}_f$ can be written also,

$$w^I(\xi, b) = \sum_{i=1}^2 [H_{oi}^{(1)}(\xi) w_{i2}^{(I)} + H_{li}^{(1)}(\xi) w_{\xi i 2}^I] \quad (4.5a)$$

$$w^{II}(\xi, o) = \sum_{i=1}^2 [H_{oi}^{(1)}(\xi) w_{i1}^{II} + H_{li}^{(1)}(\xi) w_{\xi i 1}^{II}] \quad (4.5b)$$

Also differentiating the assumed displacement pattern for w (Eq. 4.1b) with respect to η gives:

$$\frac{\partial w^I}{\partial \eta}(\xi, b) = \sum_{i=1}^2 [H_{oi}^{(1)}(\xi) w_{\eta i 2}^I + H_{li}^{(1)}(\xi) w_{\xi \eta i 2}^I] \quad (4.6a)$$

and

$$\frac{\partial w^{II}}{\partial \eta}(\xi, o) = \sum_{i=1}^2 [H_{oi}^{(1)}(\xi) w_{\eta i 1}^{II} + H_{li}^{(1)}(\xi) w_{\xi \eta i 1}^{II}] \quad (4.6b)$$

considering bending stiffnesses of the faces, substituting Eqs. 4.4, 4.5 and 4.6 into conditions 4.1 to 4.3 and noting that $H_{oi}^{(o)}(\xi)$, $H_{oi}^{(1)}(\xi)$ and $H_{li}^{(1)}(\xi)$ are linearly independent, it follows that the geometric admissibility conditions along the common edge between elements I and II are satisfied provided that:

$$u_{fi2}^I = u_{fi1}^{II} \quad (4.7a)$$

$$\tilde{v}_{fi2}^I = \tilde{v}_{fi1}^{II} \quad (4.7b)$$

$$w_{i2}^I = w_{i1}^{II} \quad (4.7c)$$

$$w_{\xi i 2}^I = w_{\xi i 1}^{II} \quad (4.7d)$$

$$w_{\eta i 2}^I = w_{\eta i 1}^{II} \quad (4.7e)$$

$$w_{\xi \eta i 2}^I = w_{\xi \eta i 1}^{II} \quad (4.7f)$$

where $f = 1, 2$ and $i = 1, 2$

As explained before it should be noticed that prior to satisfying the geometric admissibility conditions at the common edges, there are thirty two degrees-of-freedom at the interior node common to any four elements (8 for each element); after applying the admissibility conditions, the number of independent degrees-of-freedom at the interior node is reduced to eight.

B. Case of H - H Approach:

The same procedure can be followed as in the previous part; in this case u_f , $\bar{v}_f$ and w are expressed using Eqs. 3.2 to represent the displacement patterns along the common edge of the two elements I and II that is:

$$\Delta^I(\xi, b) = \sum_{i=1}^2 [H_{oi}^{(1)}(\xi) \Delta_{i2}^I + H_{li}^{(1)}(\xi) \Delta_{\xi i 2}^I] \quad (4.8a)$$

$$\Delta^{II}(\xi, o) = \sum_{i=1}^2 [H_{oi}^{(1)}(\xi) \Delta_{i1}^{II} + H_{li}^{(1)}(\xi) \Delta_{\xi i 1}^{II}] \quad (4.8b)$$

where $\Delta = (u_f, \bar{v}_f, w)$, $f = 1, 2$

Substituting Eqs. 4.8 into Eqs. 4.2 and noting also that $H_{oi}^{(1)}(\xi)$, $H_{li}^{(1)}(\xi)$ are linearly independent, it follows that the displacement continuity conditions along the common edge between elements I and II are satisfied provided:

$$\Delta_{i2}^I = \Delta_{i1}^{II} \quad (4.9a)$$

$$\Delta_{\xi i 2}^I = \Delta_{\xi i 1}^{II} \quad (4.9b)$$

where $\Delta = u_f, \bar{v}_f$ and w , $i = 1, 2, f = 1, 2$

The continuity conditions of the slope $(\frac{\partial w}{\partial \eta})$ along the same edge are exactly the same as mentioned in the previous part A, which gives:

$$w_{\eta i 2}^I = w_{\eta i 1}^{II} \quad (4.7e)$$

$$i = 1, 2$$

$$w_{\xi \eta i 2}^I = w_{\xi \eta i 1}^{II} \quad (4.7f)$$

The same procedure can be followed for the linking conditions required to satisfy the geometric admissibility conditions at the other interfaces between the different elements shown in Fig. 6. In this approach prior to satisfying the geometric admissibility conditions at the common edges, there are eighty degrees of freedom at the interior node common to the four elements (20 for each element); after applying the admissibility conditions, the number of independent degrees - of - freedom at the interior node is reduced to forty.

4.1.2 Optional Linking Conditions:

In addition to satisfying the linking conditions resulting from geometric admissibility mentioned in the previous section, it should be noted that further reduction in the number of independent degrees - of - freedom can be achieved by imposing additional linking conditions. This depends upon the experience of the engineer and the

expected behavior of the structure. For example, in most skew sandwich plate problems the normal and shearing strains in the faces are continuous between elements. Strain continuity can be achieved in the second approach using the H - H interpolation polynomials, as indicated below. Consider the common edge between elements I and II of Fig. 6, the strain continuity conditions involves continuity requirements on $u_{f\xi}$, $u_{f\eta}$, $\bar{v}_{f\xi}$, $\bar{v}_{f\eta}$ and w_ξ in addition to those requirements given by the geometric admissibility conditions. The remaining conditions to be considered are:

$$\frac{\partial u_f^I}{\partial \eta} (\xi^I = \xi, \eta^I = b) = \frac{\partial u_f^{II}}{\partial \eta} (\xi^{II} = \xi, \eta^{II} = 0) \quad (4.10a)$$

$$\frac{\partial \bar{v}_f^I}{\partial \eta} (\xi^I = \xi, \eta^I = b) = \frac{\partial \bar{v}_f^{II}}{\partial \eta} (\xi^{II} = \xi, \eta^{II} = 0) \quad (4.10b)$$

Following the same procedure as before, the set of new linking conditions becomes:

$$u_{f\eta i 2}^I = u_{f\eta i 1}^{II}, \quad u_{f\xi\eta i 2}^I = u_{f\xi\eta i 1}^{II} \quad (4.11a)$$

$$\bar{v}_{f\eta i 2}^I = \bar{v}_{f\eta i 1}^{II}, \quad \bar{v}_{f\xi\eta i 2}^I = \bar{v}_{f\xi\eta i 1}^{II} \quad (4.11b)$$

After satisfying both the geometric admissibility and these strain

continuity conditions on all the adjacent edges of the elements shown in Fig. 6, the number of independent degrees - of - freedom at the interior node for the H - H approach is reduced from forty to twenty.

In sandwich plate constructions for certain loading conditions it is known that the membrane displacements and strains of the upper and lower faces are equal in magnitude and opposite in sign i.e. ($u_1 = -u_2$, $\bar{v}_1 = -\bar{v}_2$, $\epsilon_{1x} = -\epsilon_{2x}$, $\epsilon_{1y} = -\epsilon_{2y}$ and $\gamma_{1xy} = -\gamma_{2xy}$). Adding these conditions, the number of independent degrees of freedom at the interior node are again reduced to twelve. It can be seen from Ref. 13 that this number is the same as that involved in a thin plate problem. All the numerical results reported in this work were obtained by employing the geometric admissibility conditions. In the H - H approach the additional strain continuity conditions were involved. Moreover for certain problems symmetry can be considered and this further reduces the total number of independent degrees - of - freedom for the discretized structure as will be indicated in the numerical examples.

4.1.3 Displacement Boundary Conditions:

For a particular problem the prementioned displacement boundary conditions can be set to characterize the displacement behavior of the structure. This leads to another reduction in the number of independent degrees of freedom. The satisfaction of the displacement boundary conditions will be illustrated by means of examples.

A. L - H Approach:

Consider a clamped skew sandwich plate as shown in Fig. 7, where the bending stiffnesses of the faces are considered. Along the

boundaries of the plate all the displacements and the slope in the ξ direction along the boundaries parallel to η axis are zero; also the slope in the η direction along boundaries parallel to ξ axis are zero. This will lead to the following conditions; For the edge $\xi^I = 0$ of element I, using Eqs. 3.1 and 3.2 gives:

$$u_f^I(\xi^I = 0, \eta^I = \eta) = \sum_{i=1}^2 H_{oi}^{(0)}(\eta) u_{fli} = 0 \quad (4.12a)$$

$$\bar{v}_f^I(\xi^I = 0, \eta^I = \eta) = \sum_{i=1}^2 H_{oi}^{(0)}(\eta) \bar{v}_{fli} = 0 \quad (4.12b)$$

$$w^I(\xi^I = 0, \eta^I = \eta) = \sum_{i=1}^2 [H_{oi}^{(1)}(\eta) w_{li} + H_{li}^{(1)}(\eta) w_{\eta li}] = 0 \quad (4.12c)$$

$$\frac{\partial w^I}{\partial \xi}(\xi^I = 0, \eta^I = \eta) = \sum_{i=1}^2 [H_{oi}^{(1)}(\eta) w_{\xi li} + H_{li}^{(1)}(\eta) w_{\xi \eta li}] = 0 \quad (4.12d)$$

These conditions are satisfied if:

$$u_{fli} = \bar{v}_{fli} = 0 \quad (f = 1, 2, i = 1, 2) \quad (4.13a)$$

$$w_{li} = w_{\xi li} = w_{\eta li} = w_{\xi \eta li} = 0 \quad (i = 1, 2) \quad (4.13b)$$

Similarly for edge $\xi^{IV} = a$:

$$u_{f2i} = \bar{v}_{f2i} = 0 \quad (f = 1, 2, \quad i = 1, 2) \quad (4.14a)$$

$$w_{2i} = w_{\xi 2i} = w_{\eta 2i} = w_{\xi \eta 2i} = 0 \quad (i=1, 2) \quad (4.14b)$$

Also for the edge $\eta^I = 0$ of element I using the same procedure:

$$u_{fi1} = \bar{v}_{fi1} = 0 \quad (f = 1, 2, \quad i = 1, 2) \quad (4.15a)$$

$$w_{i1} = w_{\xi i1} = w_{\eta i1} = w_{\xi \eta i1} = 0 \quad (i = 1, 2) \quad (4.15b)$$

Similarly for a parallel edge $\eta^{III} = b$:

$$u_{fi2} = \bar{v}_{fi2} = 0 \quad (f = 1, 2, \quad i = 1, 2) \quad (4.16a)$$

$$w_{i2} = w_{\xi i2} = w_{\eta i2} = w_{\xi \eta i2} = 0 \quad (i = 1, 2) \quad (4.16b)$$

From Eqs. 4.13 to 4.16, it can be seen that if the four boundaries of a sandwich plate are clamped in this manner, there will be no degrees of freedom along the boundaries using the L - H approach.

B. Clamped Plate Using the H - H Approach:

Consider another type of a clamped skew sandwich plate problem where the flexural stiffnesses of the faces are neglected. In this case the displacements only are taken to be zero along the

boundaries of the plate shown in Fig. 7. For the edge $\xi^I = 0$ of element I using Eq. 3.2 gives:

$$u_f^I(\xi^I = 0, \eta^I = \eta) = \sum_{i=1}^2 [H_{oi}^{(1)}(\eta) u_{fli} + H_{li}^{(1)}(\eta) u_{f\eta li}] = 0 \quad (4.17a)$$

$$\bar{v}_f^I(\xi^I = 0, \eta^I = \eta) = \sum_{i=1}^2 [H_{oi}^{(1)}(\eta) \bar{v}_{fli} + H_{li}^{(1)}(\eta) \bar{v}_{f\eta li}] = 0 \quad (4.17b)$$

$$w^I(\xi^I = 0, \eta^I = \eta) = \sum_{i=1}^2 [H_{oi}^{(1)}(\eta) w_{li} + H_{li}^{(1)}(\eta) w_{\eta li}] = 0 \quad (4.17c)$$

These conditions are satisfied if:

$$u_{fli} = \bar{v}_{fli} = u_{f\eta li} = \bar{v}_{f\eta li} = 0 \quad (4.18a)$$

$$w_{li} = w_{\eta li} = 0 \quad (4.18b)$$

Similarly for the edge $\xi^{IV} = a$:

$$u_{f2i} = \bar{v}_{f2i} = u_{f\eta 2i} = \bar{v}_{f\eta 2i} = 0 \quad (4.19a)$$

$$w_{2i} = w_{\eta 2i} = 0 \quad (4.19b)$$

Using the same procedure for edge $\eta^I = 0$ gives:

$$u_{fi1} = \bar{v}_{fi1} = u_{f\xi i1} = \bar{v}_{f\xi i1} = 0 \quad (4.20a)$$

$$w_{i1} = w_{\xi i1} = 0 \quad (4.20b)$$

Similarly for a Parallel edge $\eta^{III} = b$:

$$u_{fi2} = \bar{v}_{fi2} = u_{f\xi i2} = \bar{v}_{f\xi i2} = 0 \quad (4.21a)$$

$$w_{i2} = w_{\xi i2} = 0 \quad (4.21b)$$

From Eqs. 4.18 to 4.21, the degrees - of - freedom along the boundaries can be determined.

It should be noted that the force boundary conditions need not be satisfied by the assumed displacement patterns but the displacement boundary conditions must be satisfied. This is because the principle of total minimum potential energy is used as the basis for this approximate solutions and hence force boundary conditions result from the application of the variational principle as natural boundary conditions. However the force boundary conditions can be treated as applied loads and are dealt with using the work equivalent loads as discussed in the previous chapter.

4.2 Numerical Evaluation

After deciding what the independent degrees - of - freedom are for a particular structure idealized as an assemblage of finite elements, a master stiffness matrix can be formed using the element stiffness matrix presented in Chapter III and the chosen independent degrees - of - freedom. In this work it is done by forming a table having these independent degrees - of - freedom as its coordinates; this table is called the 'variable correlation table' which is thoroughly dealt with by Bogner (Ref. 13).

In this section several numerical examples are presented for rectangular and skew sandwich plates. A number of the problems selected are from the published literature and form a basis for comparison. All the problems are done using both of the two approaches mentioned before (L- H and H - H) in order to compare the two approaches and their convergence. As mentioned before all the numerical examples presented satisfy the geometric admissibility and additional conditions stated in the previous sections. A sample computer output, showing a tabulation of the computed degrees of freedom, the displacements at the nodes for each element of a problem of a clamped skew sandwich plate is presented in Appendix D. A large variety of problems of thin rectangular and skew plates under lateral loads and membranes were done in order to check the formulations and the results showed agreement with similar solutions.

4.2.1 Rectangular Sandwich Plate Examples:

A. Fully Clamped Sandwich Plate Under Uniform Pressure (Fig.8):

A 50 in x 50 in. square sandwich plate consisting of two thin aluminum facings and a relatively thick aluminum honeycomb core is considered. The following are the properties and dimensions of the sandwich plate:

$$E_f = 10.5 \times 10^6 \quad \text{psi}$$

$$\nu_f = 0.3$$

$$t_f = 0.015 \quad \text{in}$$

$$G_{cxz} = G_{cyz} = 50,000 \quad \text{psi}$$

$$t_c = 1.0 \quad \text{in}$$

The four edges of the sandwich plate are fully clamped so as to impose the following displacement boundary conditions:

$$u_1 = v_1 = u_2 = v_2 = w = w_x = 0, \quad x = \pm A$$

$$u_1 = v_1 = u_2 = v_2 = w = w_y = 0, \quad y = \pm B$$

In this case of boundary conditions and loading the behavior of the sandwich plate is such that $u_1 = -u_2$, $u_{x1} = -u_{x2}$, ,

$v_1 = -v_2$, $v_{x1} = -v_{x2}$, Also due to symmetry about the lines

$x = 0$ and $y = 0$ only one quadrant of the system is modeled (i.e. along $x = 0 : u_1 = u_2 = 0, w_x = 0$; and along $y = 0 : v_1 = v_2 = 0, w_y = 0$).

In case of the L - H approach the sandwich plate was modeled using 4, 9 and 16 elements in the quadrant (13, 37, 40 degrees - of - freedom respectively), in the case of 16 element modeling symmetry is also considered about the diagonal. In the case of H - H approach the sandwich plate was modeled using 4 and 16 elements in the quadrant which results in 41 and 177 degrees - of - freedom respectively. If symmetry about the diagonal is considered in the 16 element modeling the number of degrees - of - freedom is reduced from 177 to 109. The centre deflection results are tabulated for all these cases for a uniformly distributed load of 1.0psi and comparison of results with Refs. 31, 35 is made as shown in table I. As noted in Ref. 35, the solution presented in Ref. 31 has a limited accuracy. Also by inspection of results in the H - H approach, the values of w_0 are in a very close agreement for 4 and 16 element modeling; this indicates that the transverse displacement can be obtained with relatively coarse modeling using the H - H approach.

B. Simply Supported Sandwich Plate Under Uniform Pressure (Fig. 9):

A 20 in x 20 in square sandwich plate constructed from two thin aluminum faces and a relatively thick orthotropic aluminum honeycomb core is considered, the material properties and dimensions of the

sandwich plate are:

$$E_f = 10 \times 10^6 \quad \text{psi}$$

$$\nu_f = 0.3$$

$$t_f = 0.02 \quad \text{in}$$

$$G_{cxz} = 75,200 \quad \text{psi}$$

$$G_{cyz} = 32,900 \quad \text{psi}$$

$$t_c = 1.0 \quad \text{in}$$

The boundaries of the sandwich plate are considered simply supported imposing the following displacement boundary conditions, where the transverse shear displacements are zero along edges:

$$v_1 = v_2 = 0, \quad w = 0 \quad x = \pm A$$

$$u_1 = u_2 = 0, \quad w = 0 \quad y = \pm B$$

A uniform transverse pressure of 1 psi is applied to the system. Also from the loading conditions $u_1 = -u_2$, $u_{x1} = -u_{x2}$, . . . , $v_1 = -v_2$, $v_{x1} = -v_{x2}$, . . . , and considering symmetry, one quadrant of the system is modeled. Additional conditions can be considered along $x=0$,

($u_1 = u_2 = 0, w_x = 0$) and along $y = 0, (v_1 = v_2 = 0, w_y = 0)$.

The first solution using the L - H approach in case of 4 and 16 elements modeling in one quadrant result in 28 and 104 degrees - of - freedom respectively. Using the H - H approach, one quadrant was modeled by 4 and 16 elements (56, 208 degrees of freedom respectively), if symmetry about the diagonal is considered the degrees - of - freedom in the 16 element modeling is reduced from 208 to 140. Results for the center deflection and its comparison with theoretical and experimental results with panel number 6 in the Forest products Labs. report (Ref. 26), shows a close agreement. Results of the two previous problems on rectangular sandwich plates are identical with the results given in Ref. 35. (Table 1)

4.2.2 SKEW SANDWICH PLATE EXAMPLES:

In this section several examples of clamped skew sandwich plates with different core properties are presented. All the problems are treated using the two previous mentioned approaches in order to compare between the results of the two solutions. Two types of 'clamped boundary conditions' are considered. Comparison of results with an existing published paper (Ref. 2) is made.

Case A: Clamped Skew Sandwich Plate Considering the Bending Rigidity of the Faces (Figs. 10, 11):

Considering a 40 in x 40 in skew sandwich plate with skew angle β , constructed of 0.025 in thick aluminum alloy faces and 1 in thick core. The material properties are taken as:

$$E_f = 10 \times 10^6 \quad \text{psi}$$

$$\nu_f = 0.32$$

$$G_{cxz} = G_{cyz} = G_c$$

Where G_c takes on different values as shown in tables. A uniformly transverse distributed load of 1 psi is applied to the system. The boundaries of the skew sandwich plate are fully clamped and bending rigidity of faces are considered so that the imposed displacement boundary conditions are taken as:

$$u_1 = \bar{v}_1 = u_2 = \bar{v}_2 = w = w_\xi = 0, \quad \xi = \pm A$$

$$u_1 = \bar{v}_1 = u_2 = \bar{v}_2 = w = w_\eta = 0, \quad \eta = \pm B$$

Due to loading and boundary conditions the behavior of the sandwich system is such that $u_1 = -u_2$, $u_{\xi 1} = -u_{\xi 2}$, . . . , $\bar{v}_1 = -\bar{v}_2$, $\bar{v}_{\xi 1} = -\bar{v}_{\xi 2}$, In case of L - H approach the sandwich plate was modeled using 4, 16, 36 elements (Figs. 10, 11); (8, 54, 78 degrees of freedom respectively), noting that in the 36 element modeling symmetry is considered about the two diagonals as shown in Fig. 10. For example the symmetry between the nodes number 7 and 19 are such that:

$$u_f^{(7)} = -u_f^{(19)} \quad (4.22a)$$

$$u_{\xi f}^{(7)} = u_{\xi f}^{(19)} \quad (4.22b)$$

$$u_{\eta f}^{(7)} = u_{\eta f}^{(19)} \quad (4.22c)$$

$$u_{\xi \eta f}^{(7)} = -u_{\xi \eta f}^{(19)} \quad (4.22d)$$

Similar relations for $\bar{v}_f$ can be written and:

$$w^{(7)} = w^{(19)} \quad (4.23a)$$

$$w_{\xi}^{(7)} = -w_{\xi}^{(19)} \quad (4.23b)$$

$$w_{\eta}^{(7)} = -w_{\eta}^{(19)} \quad (4.23c)$$

$$w_{\xi \eta}^{(7)} = w_{\xi \eta}^{(19)} \quad (4.23d)$$

These relations can be obtained from the equations of the assumed displacement patterns; also they are verified by some other problems where symmetry about diagonals are not considered. Results for centre deflection for all the cases are recorded in table 2.

Case B: Clamped Skew Sandwich Plate Neglecting the Bending Rigidity of the Faces:

The same sandwich plate dimensions with the same material properties as in case A is used, and also a transverse pressure of 1 psi is applied to the system. In this case the boundaries of the skew

sandwich plate are clamped and bending rigidity of faces are neglected so that the imposed displacement boundary conditions are taken as:

$$u_1 = \bar{v}_1 = u_2 = \bar{v}_2 = w = 0, \quad \xi = \pm A$$

$$u_1 = \bar{v}_1 = u_2 = \bar{v}_2 = w = 0, \quad \eta = \pm B$$

As explained in section (4.1.3B), these conditions are satisfied by Eqs. 4.18 to 4.21. The opposite sign relations between the displacements of the upper and lower faces is considered. The sandwich plate was discretized with the same modeling as in case A. For the L - H solution 4, 16, 36 elements modeling was used, giving (20, 44, 100 degrees - of - freedom), in the case of H - H solution the sandwich plate was discretized using 4 and 16 elements (30, 102 degrees - of - freedom), where symmetry was considered about the diagonals in both modelings. Results for centre deflection for all cases are recorded in table 3, comparison with the existing results (Ref. 2) is presented in table 4 for different core rigidities and skew angles. Inspection of results indicates a closer agreement with Ref. 2 in Case B than with Case A. Also results for centre deflection for a clamped skew sandwich plate with aspect ratio 1.5 are shown in table 5. Results for the face moments at the centre of the edges and centre of the plate and shear in the core at the centre of the edges are shown in table 6. In table 7 the force and moment resultants and the maximum moment at the centre are given; the results show a very good agreement with those of Ref. 2. The maximum force and moment resultants at the edges and their comparison with Ref. 2 are given in

table 8; again good agreements is obtained. In Figs. 17 and 18 the maximum edge moments are plotted versus the skew coordinates for skew angles 0, 15, 30, 45, 60 degrees and for $G = 10^5$ psi and 500 psi respectively; from those curves the points of maximum moments can be determined along the edges. The results indicate that the point of maximum moment starts from the centre of the edge in the rectangular case and moves towards the obtuse corner as the skew angle β increases. Also Fig. 15 shows comparison of centre deflection for $\beta = 45^\circ$ with different load intensities. Fig. 16 shows comparison of centre deflection for different skew angles and core rigidities. All the results are in very good agreement with Ref. 2.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

The analysis method presented represents an extension of the finite element displacement method to the prediction of the displacement behavior of skew sandwich plates. The finite element displacement patterns are approximated in terms of zero and first order Hermite interpolation polynomials (Bilinear Lagrange functions and Bicubical Hermite function) and the undetermined nodal displacement parameters. This results in the two previous mentioned approaches (L - H approach and the H - H approach). The end point properties of the Hermite polynomials gives the facility of satisfaction of the geometric admissibility conditions between adjacent finite elements of the displacement state of an assemblage of elements. Also, the H - H approach has the advantage of the possibility of maintaining strain continuity between adjacent elements, so that accurate stress predictions can be achieved.

The displacement behavior was described in terms of the membrane displacements of the individual faces and the transverse deflection of the whole system. Transverse shear deformations of the core were taken into account and the formulation is able to treat a wide range of boundary conditions. By simply replacing the skew angle presented in the formulations by zero, the problem of rectangular sandwich plates can be solved. Also, the analysis of thin rectangular and skew plates and membranes can be dealt with by considering only

one face of the sandwich system. The potential energy formulations were presented in terms of the plate geometry, the stiffnesses of the faces and the transverse shear moduli of the core. However, the analysis presented is limited to linear elastic material behavior and the faces are governed by the Kirchhoff - Love assumptions.

All the solutions reported were based on the theorem of total minimum potential energy. Inspection of results given in the tables and figures gives rise to many discussions and conclusions. It is seen that the centre deflection decreases with the increase in skew angle β , also the tables reveal that, as expected, deflections increase rapidly as the core rigidity decreases. In the two mentioned approaches (L - H approach and H - H approach), the results showed the convergence of the solution as the grid is refined and theoretically the potential energy converges to the true minimum potential energy as the number of elements tends to infinity. From the tables, in the case of the H - H approach the displacement results converges more rapidly than in the case of the L - H approach and even for coarse modeling in the H - H approach (4 elements), it gives good results for the centre deflection. Results showed that the L - H solution can be used in case of sandwich construction with low core rigidity to give accurate deflection results; this is due to the fact that the displacements are due mainly to transverse shear in the core. For example in case of $G_c = 500$ psi, the 4 element modeling in the L - H approach gives accurate approximation. One of the most important conclusions is that, for relatively rigid cores the H - H approach is superior to the L - H approach for the same

number of degrees of freedom. Therefore in sandwich construction using the H - H approach with a relatively coarse modeling gives better results than the L - H approach with a large number of elements even if the two approaches have the same number of total degrees - of - freedom. Results of the H - H approach for skew sandwich plates with different core rigidities are compared with Ref. 2, the results indicate good agreement.

The numerical examples presented do not provide a complete evaluation of the potential of the finite element analysis, however, they indicate that relatively few elements are required in order to accurately predict displacements and stresses of skew sandwich plates. To the author's knowledge, no available finite element analysis for skew sandwich plates exist; this attests to the importance of the proposed method.

The success of the presented formulation suggests several extensions in the same field. The method was only applied to sandwich elements connected in the same plane, an extension to solve sandwich structures using elements connected at arbitrary angles would be a significant advance. Extension of the method to more general geometries such as triangular and quadrilateral elements would have a significant importance but may involve substantial effort. Also the application of a discrete element finite displacement analysis on skew sandwich plates considering non - linearity and direct minimization of the potential energy will add a significant contribution to the problem.

The method can also be extended to the solution of skew sandwich plates with anisotropic facings such as the filamentary composite materials.

The assumption that the transverse displacement is a constant through the thickness of the sandwich system limits the possibility of imposing distinct bending boundary conditions. As stated by Ref. 35, it could be done by assuming a linear strain variation through the core and expressing the transverse deflection of the core in terms of the transverse deflection of the individual faces. Also by taking the inplane stiffnesses of the core into account would give a possibility to handle a more general class of core materials without increase in the number of degrees - of - freedom. The study of temperature and temperature gradient effects would also represent a useful extension of the proposed method. The study of dynamic response of skew sandwich plates would be of great importance in some practical applications. Also the buckling problem of this type of plates would give a required extension.

Finally, the proposed finite element analysis would give the capability of predicting some important sandwich constructions that can be modeled by rectangular and skew sandwich elements which is useful in the field of structural design.

OBLIQUE COORDINATE SYSTEM

FIGURE 1

SANDWICH PLATE SEGMENT

FIGURE 2

SYSTEM OF MEMBRANE LOADS

FIGURE 3

SKEW SANDWICH PLATE ELEMENT

FIGURE 4

SYSTEM OF APPLIED LOADS

FIGURE 5

FOUR ADJACENT SKEW SANDWICH
PLATE ELEMENT

FIGURE 6

CLAMPED SKEW SANDWICH PLATE
4 ELEMENT MODELING

FIGURE 7

CLAMPED RECTANGULAR SANDWICH
PLATE MODELING

FIGURE 8

4 ELEMENTS

PLATE

9 ELEMENTS

16 ELEMENTS

SIMPLY SUPPORTED RECTANGULAR
SANDWICH PLATE MODELING

FIGURE 9

4 ELEMENTS

16 ELEMENTS

CLAMPED SKEW SANDWICH PLATE
MODELING (4EL. AND 16 EL.)

FIGURE 10

CLAMPED SKEW SANDWICH PLATE
36 ELEMENTS MODELING

FIGURE 11

STRESS RESULTANTS RELATIONS

FIGURE 12

STRESS AND MOMENT RESULTANTS

FIGURE 13

MOMENT RESULTANTS IN THE X,Y DIRECTIONS

FIGURE 14

FIGURE 15

FIGURE 16

TABLE 1
Fully Clamped and Simply Supported Rectangular
Sandwich Plate Under Uniform Pressure
Centre Deflection in Inches
For $q = 1$ psi

Condition of Support	t_f in	$G_c \text{ lb/in}^2$		Lagrange - Hermite			Hermite - Hermite		Ref. (31)	Ref. (26)	Ref. (35)
		G_{xz}	G_{yz}	4EL	9EL	16EL	4EL	16EL			
Fully* Clamped	0.015	50,000	50,000	0.05858	0.08295	0.08848	0.0932	0.0930	.0943	—	0.0933
Simply** Supported	0.02	75,200	32,900	6.105×10^{-3}	6.190×10^{-3}	6.2607×10^{-3}	6.3×10^{-3}	6.29×10^{-3}	—	6.3×10^{-3}	6.3×10^{-3}

* a Fully Clamped Plate 50 x 50 inches

** a Simply Supported Plate 20 x 20 inches

Table 2

Fully Clamped Skew Sandwich Plate
Centre Deflection (in) For Case A
of Boundary Conditions

skew angle β	t_f (in)	G_c psi	Lagrange- Hermite approach			Hermite-Hermite	
			4EL	16EL	36EL	4EL	16EL
0.0 (Rect)	0.025	10^5	0.0010677	0.015672	0.020636	0.024331	0.023207
		10^4	0.010676	0.028027	0.031166	0.034412	0.032874
		500	0.21316	0.21877	0.22766	0.23756	0.22377
15°	0.025	10^5	0.0010081	0.014057	0.018373	0.021073	0.020647
		10^4	0.010080	0.025364	0.028228	0.03095	0.029754
		500	0.20123	0.20565	0.21401	0.22283	0.21017
30°	0.025	10^5	0.0008378	0.0099062	0.012687	0.013533	0.014264
		10^4	0.008377	0.018608	0.020742	0.022180	0.021821
		500	0.16713	0.16929	0.17616	0.18169	0.17260
45°	0.025	10^5	0.00058559	0.0050616	0.0063387	0.0061332	0.0071385
		10^4	0.0058548	0.010707	0.011911	0.012143	0.012488
		500	0.11661	0.11793	0.12260	0.12338	0.11977
60°	0.025	10^5	0.0003077	0.0015718	0.0019207	0.0017039	0.0021725
		10^4	0.0030755	0.0044831	0.0049137	0.004674	0.0050935
		500	0.06092	0.063120	0.065470	0.062624	0.063778
75°	0.025	10^5	0.85625×10^{-4}	0.00019381	0.00022472	0.00019092	0.00024659
		10^4	0.00085458	0.0010444	0.0011144	0.00096864	0.0011054
		500	0.016425	0.018054	0.018813	0.016534	0.018115

Table 3
Clamped Skew Sandwich Plate
Centre Deflection (in) Case B
of Boundary Conditions
Aspect Ratio = 1

skew angle β	t_f (in)	G_c psi	Lagrange - Hermite			Hermite-Hermite	
			4EL	16EL	36EL	4EL	16EL
0.0 (Rect.)	0.025	10^5	0.0011156	0.016237	0.020837	0.024568	0.023480
		10^4	0.011155	0.029496	0.032231	0.035508	0.034424
		500	0.22307	0.24306	0.24610	0.24830	0.24833
15°	0.025	10^5	0.001057	0.014535	0.018541	0.021568	0.020896
		10^4	0.01057	0.026734	0.029219	0.032140	0.031204
		500	0.21137	0.22879	0.23150	0.23387	0.23355
30°	0.025	10^5	0.00088966	0.01097	0.012817	0.014347	0.014462
		10^4	0.0088965	0.019717	0.021540	0.023544	0.022998
		500	0.17789	0.18903	0.19095	0.19349	0.19250
45°	0.025	10^5	0.00063897	0.0051934	0.0064115	0.0068074	0.0072762
		10^4	0.0063896	0.011471	0.012459	0.013396	0.01330
		500	0.12774	0.13241	0.13341	0.13543	0.13434
60°	0.025	10^5	0.00035145	0.0016187	0.0019527	0.0019804	0.0022427
		10^4	0.0035142	0.0048917	0.0052082	0.0054076	0.005520
		500	0.070176	0.071482	0.071804	0.072291	0.072179
75°	0.025	10^5	0.00010322	0.00020627	0.0002267	0.00022965	0.00026264
		10^4	0.0010317	0.0011693	0.0012042	0.001720	0.0012346
		500	0.020415	0.021097	0.021260	0.020559	0.021168

Table 4
Comparison of Results
For Centre Deflection of Skew
Clamped Sandwich Plate (Aspect ratio = 1)
Case B

skew angle β	t_f (in)	G_c psi	Present Solution		Ref. (2)
			4EL	16EL	
0.0 (Rect.)	0.025	10^5	0.024568	0.023480	—
		10^4	0.035508	0.034424	—
		500	0.24830	0.24833	—
15°	0.025	10^5	0.021568	0.020896	0.0210
		10^4	0.032140	0.031204	0.0322
		500	0.23387	0.23355	0.2430
30°	0.025	10^5	0.014347	0.014462	0.0146
		10^4	0.023544	0.022998	0.0237
		500	0.19349	0.19250	0.2020
45°	0.025	10^5	0.0068074	0.0072762	0.0073
		10^4	0.013396	0.01330	0.0139
		500	0.13543	0.13434	0.1420
60°	0.025	10^5	0.0019804	0.0022427	0.00229
		10^4	0.0054076	0.00520	0.00583
		500	0.072291	0.072179	0.0765
75°	0.025	10^5	0.00022965	0.00026264	—
		10^4	0.0011720	0.0012346	—
		500	0.020559	0.021168	—

Table 5
Results of a Clamped Skew
Sandwich Plate (Aspect Ratio = 1.5)

skew angle β	t_f (in.)	G_c psi	W_{max} (in.)
0.0 (Rect)	0.025	10^5	0.040190
		10^4	0.055538
		500	0.35062
15°	0.025	10^5	0.035520
		10^4	0.049921
		500	0.32818
30°	0.025	10^5	0.024038
		10^4	0.035807
		500	0.26649
45°	0.025	10^5	0.011579
		10^4	0.019644
		500	0.18099
60°	0.025	10^5	0.0033135
		10^4	0.0074927
		500	0.092987
75°	0.025	10^5	0.00034393
		10^4	0.0015016
		500	0.025660

TABLE 6
Face Moments and Shear of the Core Resultants
For
Clamped Skew Sandwich Plate
Aspect Ratio = 1

β°	t_f in	G_c psi	W Centre in	(edge) $M_{\xi f} \times 10^{-2}$ lb/in/in	(edge) $M_{\eta f} \times 10^{-2}$ lb/in/in	(Centre) $M_{\xi} \times 10^{-2}$ lb/in/in	(edge) Q_c^* lb/in
0°	0.025	10^5	0.02348	0.3753	-0.00319	-0.1856	22.27
		10^4	0.03442	0.2698	0.0934	-0.2245	16.00
		500	0.24833	-1.237	3.494	-0.9636	13.35
15°	0.025	10^5	0.02089	0.3363	0.0029	-0.1758	19.70
		10^4	0.03120	0.2483	0.1687	0.2141	14.86
		500	0.23355	-1.089	5.005	0.9599	12.72
30°	0.025	10^5	0.01446	0.2368	0.01858	-0.1487	13.66
		10^4	0.02299	0.1897	0.2691	-0.1857	11.94
		500	0.19250	-0.7048	6.337	-0.9509	10.97
45°	0.025	10^5	0.00728	0.1206	0.03429	-0.1084	7.869
		10^4	0.01330	0.1117	0.3438	-0.1459	8.377
		500	0.13434	-0.2469	6.944	-0.9444	8.364
60°	0.025	10^5	0.00224	0.0357	0.03691	-0.05918	4.685
		10^4	0.00552	0.04096	0.3346	-0.1010	5.206
		500	0.07218	0.06033	6.197	-0.9357	5.366
75°	0.025	10^5	0.00026	0.00308	0.02257	-0.0155	2.555
		10^4	0.00123	0.00367	0.2110	-0.0527	2.563
		500	0.02117	0.01022	3.522	-0.7535	2.446

* Q_c is given at the centre of the edge

TABLE 7
Maximum Stress and Moment Resultants at Centre
Aspect Ratio = 1

β°	G_c psi	N_ξ lb/in	N_η lb/in	$N_{\xi\eta}$ lb/in	$\bar{M}_x$ lb in/in	$\bar{M}_y$ lb in/in	$\bar{M}_{xy}$ lb in/in	(Present) $\bar{M}_{max}$ lb in/in	(Ref. 2) $\bar{M}_{max}$ lb in/in
0.0	10^5	38.0	38.0	0.0	38.9	38.9	0.0	38.9	—
	10^4	37.5	37.5	0.0	38.4	38.4	0.0	38.4	—
	500	39.3	39.3	0.0	40.2	40.2	0.0	40.2	—
15	10^5	37.4	37.4	10.8	36.5	37.0	1.12	37.9	37.2
	10^4	37.0	37.0	11.6	35.5	36.6	2.07	38.2	38.3
	500	38.6	38.6	12.2	37.0	38.2	2.30	39.9	39.4
30	10^5	35.5	35.5	19.4	29.6	31.5	1.69	32.5	32.0
	10^4	35.3	35.3	20.6	27.9	31.3	3.01	33.1	33.2
	500	36.4	36.4	21.4	28.6	32.3	3.24	34.1	34.0

TABLE 7 (Continued)

β°	G_c psi	N_ξ lb/in	N_η lb/in	$N_{\xi\eta}$ lb/in	$\bar{M}_x$ lb in/in	$\bar{M}_y$ lb in/in	$\bar{M}_{xy}$ lb in/in	(Present) $\bar{M}_{max}$ lb in/in	(Ref. 2) $\bar{M}_{max}$ lb in/in
45	10^5	31.9	31.9	24.0	20.0	23.1	1.55	23.7	23.3
	10^4	31.8	31.8	25.0	18.0	23.0	2.50	24.1	24.2
	500	32.2	32.2	25.4	17.9	24.4	2.68	25.3	25.4
60	10^5	25.6	25.6	23.0	10.1	13.1	0.883	13.3	13.0
	10^4	25.8	25.8	23.6	8.64	13.2	1.32	13.5	13.5
	500	25.5	25.5	23.4	8.15	14.1	1.42	14.4	14.9
75	10^5	15.2	15.2	14.9	1.79	4.02	0.300	4.06	—
	10^4	15.5	15.5	15.2	2.14	4.10	0.263	4.14	—
	500	15.3	15.3	15.0	2.40	4.06	0.221	4.10	—

TABLE 8

Maximum Stress And Total Moment Resultants At The Edges

Aspect Ratio = 1

β°	G_c psi	N_ξ lb/in	N_η lb/in	$N_{\xi\eta}$ lb/in	$\bar{M}_x$ lb/in	$\bar{M}_y$ lb/in	$\bar{M}_{xy}$ lb/in	present M_{max} lb/in	(Ref. 2) M_{max} lb/in
0	10^5	73.7	23.6	0.0	75.5	24.2	0.0	75.5	—
	10^4	72.1	23.1	0.0	73.9	23.6	0.0	73.9	—
	500	64.1	20.5	0.0	65.7	21.0	0.0	65.7	—
15	10^5	72.0	26.3	18.6	68.1	26.0	12.1	71.3	76.0
	10^4	70.8	26.0	18.6	66.7	25.8	12.2	70.1	74.0
	500	63.5	24.4	18.7	58.9	24.2	12.7	63.0	67.8
30	10^5	67.5	32.6	33.3	50.1	28.9	17.4	59.9	64.0
	10^4	66.3	31.6	32.3	49.6	28.1	16.9	58.9	60.0
	500	59.9	26.9	27.5	46.2	23.9	14.4	53.3	56.8

TABLE 8 (Continued)

β°	G psi	N_ξ lb/in	N_η lb/in	$N_{\xi\eta}$ lb/in	$\bar{M}_x$ lb/in	$\bar{M}_y$ lb/in	$\bar{M}_{xy}$ lb/in	(present) $\bar{M}_{max}$ lb/in	(Ref. 2) $\bar{M}_{max}$ lb/in
45	10^5	59.1	38.0	41.1	29.0	27.5	14.6	42.9	46.1
	10^4	57.8	41.5	43.2	25.2	30.1	14.2	42.1	50.0
	500	55.6	39.6	41.4	24.5	28.7	13.7	41.5	55.4
60	10^5	47.6	39.8	41.4	11.8	20.4	7.08	24.4	24.5
	10^4	48.6	40.5	42.2	12.1	20.7	7.28	24.9	27.0
	500	46.0	37.4	39.4	11.9	19.2	7.15	23.6	33.3
75	10^5	29.1	27.9	28.2	2.78	7.41	1.24	7.72	—
	10^4	29.3	27.8	28.2	2.87	7.38	1.37	7.77	—
	500	28.2	26.6	27.1	2.81	7.06	1.40	7.48	—

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APPENDIX 'A'

LAGRANGE AND HERMITE INTERPOLATION POLYNOMIALS

This section discusses briefly the selection of the displacement functions in order to satisfy the geometric admissibility requirements as stated in the previous chapters. The determination of the one dimensional Hermite interpolation polynomials, used to represent the displacement variables is presented following the same approach and notations as given in Ref. 14; these polynomials $H_{ki}^{(N)}(s)$ have the properties:

$$(1): \quad K = 0, \quad H_{oi}^{(N)}(s_j) = \zeta_{ij}$$

$$dH_{oi}^{(N)}(s_j)/ds = 0$$

$$d^2 H_{oi}^{(N)}(s_j)/ds^2 = 0$$

$$\vdots$$

$$d^N H_{oi}^{(N)}(s_j)/ds^N = 0$$

$$(2): \quad K = 1, \quad H_{ij}^{(N)}(s_j) = 0$$

$$dH_{li}^{(N)}(s_j)/ds = \zeta_{ij}$$

$$d^2 H_{li}^{(N)}(s_j)/ds^2 = 0$$

$$\vdots$$

$$d^N H_{lj}^{(N)}(s_j)/ds^N = 0$$

⋮

$$(N + 1): \quad K = N, \quad H_{Ni}^{(N)}(s_j) = 0$$

$$\vdots \quad dH_{Ni}^{(N)}(s_j)/ds = 0$$

$$\vdots \quad d^N H_{Ni}^{(N)}(s_j)/ds^N = \zeta_{ij}$$

where: N is the number of derivatives that the set can interpolate.

$(i, j = 1, 2)$ are indices ranging over the number of stations. K index ranging over the number of derivatives to be interpolated ($K = 0 \rightarrow N$).

s_j are specific values of the argument s , of the polynomial at the stations.

(i) Consider the case where ($N = 0$), zero order interpolation polynomials, which is defined as Lagrange interpolation function.

If $s_1 = 0, s_2 = a$ these requirements become:

$$H_{o1}^{(o)}(o) = 1$$

$$H_{o1}^{(o)}(a) = 0$$

$$H_{o2}^{(o)}(o) = 0$$

$$H_{o2}^{(o)}(a) = 1$$

Two polynomials, $H_{o1}^{(o)}(s)$ and $H_{o2}^{(o)}(s)$ having these properties is:

$$H_{o1}^{(o)}(s) = \frac{s-s_2}{s_1-s_2} = -\frac{(s-a)}{a}$$

$$H_{o2}^{(o)}(s) = \frac{s-s_1}{s_2-s_1} = \frac{s}{a}$$

These polynomials can only interpolate the function itself but not the derivatives.

In case of two dimensional problems for example in the s, r plane, where $r_1 = 0$, $r_2 = b$, the displacement function will take the form:

$$\Delta(s, r) = \sum_{i=1}^2 \sum_{j=1}^2 H_{oi}^{(o)}(s) H_{oj}^{(o)}(r) \Delta_{ij} \quad (A.1)$$

This function was taken to approximate the membrane displacement patterns of the sandwich element in the first approach as in Eq. 3.1.

(ii) If $N = 1$ and $i, j = 1, 2$ and $s_1 = 0$, $s_2 = a$ the requirements are:

$$H_{o1}^{(1)}(0) = 1 \quad d H_{o1}^{(1)}(0)/ds = 0$$

$$H_{o1}^{(1)}(a) = 0 \quad d H_{o1}^{(1)}(a)/ds = 0$$

$$H_{o2}^{(1)}(0) = 0 \quad d H_{o2}^{(1)}(0)/ds = 0$$

$$H_{o2}^{(1)}(a) = 1 \quad d H_{o2}^{(1)}(a)/ds = 0$$

$$H_{11}^{(1)}(o) = 0 \quad d H_{11}^{(1)}(o)/ds = 1$$

$$H_{11}^{(1)}(a) = 0 \quad d H_{11}^{(1)}(a)/ds = 0$$

$$H_{12}^{(1)}(o) = 0 \quad d H_{12}^{(1)}(o)/ds = 0$$

$$H_{12}^{(1)}(a) = 0 \quad d H_{12}^{(1)}(a)/ds = 1$$

A set of four polynomials satisfying these requirements are:

$$H_{o1}^{(1)}(s) = \frac{1}{a^3} (2s^3 - 3as^2 + a^3)$$

$$H_{o2}^{(1)}(s) = -\frac{1}{a^3} (2s^3 - 3as^2)$$

$$H_{11}^{(1)}(s) = \frac{1}{a^2} (s^3 - 2as^2 + a^2s)$$

$$H_{12}^{(1)}(s) = \frac{1}{a^2} (s^3 - as^2)$$

These are known as the osculatory first - order Hermite interpolation polynomials. They can interpolate a function, $f(s)$, as well as its derivative $\dot{f}(s)$, at the two stations o, a . In case of a two dimensional

problem in the s, r plane where $r_1 = 0, r_2 = b$ the assumed displacement function $\Delta(s, r)$ can be written as:

$$\begin{aligned} \Delta(s, r) = & \sum_{i=1}^2 \sum_{j=1}^2 [H_{oi}^{(1)}(s) H_{oj}^{(1)}(r) \Delta_{ij} + H_{ij}^{(1)}(s) H_{oj}^{(1)}(r) \Delta_{sij} \\ & + H_{oi}^{(1)}(s) H_{ij}^{(1)}(r) \Delta_{rij} + H_{li}^{(1)}(s) H_{lj}^{(1)}(r) \Delta_{srij}] \end{aligned}$$

This displacement function is used in Eq. 3.2 to represent the displacement patterns of the skew sandwich plate element in both cases.

APPENDIX 'B'

STIFFNESS MATRICES FORMULAS

The elements of the submatrices which makes up the element stiffness matrices in the two previous mentioned approaches (Eqs. 3.23, 3.28) are given after integration in terms of the integer portion of the different combinations of the Lagrange and first order Hermite interpolations and the dimensions and plate properties as follows.

a) L - H Approach:

$$\begin{aligned} k_{ij}^{(u_f u_f)} &= \frac{B_{f1}}{M} \left\{ \left(\frac{b}{a}\right) B_{11} q_{ij}^{(1)} + \left(\frac{a}{b}\right) B_{12} q_{ij}^{(2)} + B_{13} q_{ij}^{(3)} \right\} \\ &+ \frac{C_1}{C} abq_{ij}^{(6)} \end{aligned} \quad (B.1a)$$

$$k_{ij}^{(u_f v_f)} = \frac{B_{f1}}{M} \{ B_{21} q_{ij}^{(4)} + \frac{b}{a} B_{22} q_{ij}^{(1)} + B_{23} q_{ij}^{(5)} \} \quad (B.1b)$$

$$\begin{aligned} k_{ij}^{(v_f v_f)} &= \frac{B_{f1}}{M} \left\{ \frac{b}{a} B_{31} q_{ij}^{(1)} + \frac{a}{b} q_{ij}^{(2)} + B_{32} q_{ij}^{(3)} \right\} \\ &+ \frac{C_2}{C} abq_{ij}^{(6)} \end{aligned} \quad (B.1c)$$

$$k_{ij}^{(u_1 u_2)} = - \frac{C_1}{C} abq_{ij}^{(6)} \quad (B.1d)$$

$$k_{ij}^{(v_1 v_2)} = - \frac{C_2}{C} abq_{ij}^{(6)} \quad (B.1e)$$

$$k_{ij}^{(u_1^w)} = -\frac{C_3}{C} b q_{ij}^{(7)} a^{\infty ij} b^{\gamma ij} \quad (B.1f)$$

$$k_{ij}^{(v_1^w)} = -\frac{1}{C} (C_4 b q_{ij}^{(7)} + C_5 a q_{ij}^{(8)}) a^{\infty ij} b^{\gamma ij} \quad (B.1g)$$

$$k_{ij}^{(u_2^w)} = \frac{C_3}{C} b q_{ij}^{(7)} a^{\infty ij} b^{\gamma ij} \quad (8.1h)$$

$$k_{ij}^{(v_2^w)} = \frac{1}{C} (C_4 b q_{ij}^{(7)} + C_5 a q_{ij}^{(8)}) a^{\infty ij} b^{\gamma ij} \quad (B.1i)$$

$$\begin{aligned} k_{ij}^{(ww)} = & \sum_{f=1}^2 \left\{ \frac{D_f}{ab} \left[\left(\frac{a}{b} \right)^2 q_{ij}^{(9)} + \left(\frac{b}{a} \right)^2 q_{ij}^{(10)} \right. \right. \\ & + B_{41} \left[\left(\frac{a}{b} \right) q_{ij}^{(11)} + \left(\frac{b}{a} \right) q_{ij}^{(12)} \right] + B_{42} q_{ij}^{(13)} + B_{43} q_{ij}^{(14)} \left. \right] \\ & \frac{a^{\lambda ij} b^{\mu ij}}{B \cos^3 \beta} \left. \right\} + C_6 \left\{ C_7 \left(\frac{b}{a} \right) q_{ij}^{(16)} + C_8 \left(\frac{a}{b} \right) q_{ij}^{(17)} \right. \\ & \left. + C_9 q_{ij}^{(18)} \right\} a^{\lambda ij} b^{\mu ij} / B \end{aligned} \quad (B.1j)$$

where:

$f = 1, 2$ corresponds to face one or two.

$$M = 6 \times 6 \quad ; \quad C = 60 \times 60 \quad ; \quad B = 420 \times 420 \quad (B.2a)$$

$$B_{f1} = B_f \cos \beta \quad (B.2b)$$

$$B_{11} = 1 + \frac{1}{2} \tan^2 \beta (1 - v_f) \quad (B. 2c)$$

$$B_{12} = \frac{1}{2} \sec^2 \beta (1 - v_f) \quad (B. 2d)$$

$$B_{13} = \frac{1}{2} (1 - v_f) \sec \beta \tan \beta \quad (B. 2e)$$

$$B_{21} = \frac{1}{2} (1 - v_f) \quad (B. 2f)$$

$$B_{22} = \frac{1}{2} (1 - v_f) \sin \beta + v_f \sin \beta \quad (B. 2g)$$

$$B_{23} = v_f \quad (B. 2h)$$

$$B_{31} = \sin^2 \beta + \frac{1}{2} \cos^2 \beta (1 - v_f) \quad (B. 2i)$$

$$B_{32} = \sin \beta \quad (B. 2j)$$

$$B_{42} = \sin^2 \beta + v_f \cos^2 \beta \quad (B. 2k)$$

$$B_{41} = 2 \sin \beta \quad (B. 2l)$$

$$B_{43} = 4 \sin^2 \beta + 2(1 - v_f) \cos^2 \beta \quad (B. 2m)$$

$$C_1 = G_{xz} \cos \beta / t_c \quad (B. 2n)$$

$$C_2 = G_{yz} \cos^3 \beta / t_c \quad (\text{B. 2o})$$

$$C_3 = h \cos \beta G_{xz} / t_c \quad (\text{B. 2p})$$

$$C_4 = h \cos \beta \sin \beta G_{yz} / t_c \quad (\text{B. 2q})$$

$$C_5 = h \cos \beta G_{yz} / t_c \quad (\text{B. 2r})$$

$$C_6 = h^2 \cos \beta / t_c \quad (\text{B. 2s})$$

$$C_7 = G_{xz} + G_{yz} \tan^2 \beta \quad (\text{B. 2t})$$

$$C_8 = G_{yz} \sec^2 \beta \quad (\text{B. 2u})$$

$$C_9 = G_{yz} \sec \beta \tan \beta \quad (\text{B. 2v})$$

The constants $q_{ij}^{(k)}$ where $k = 1, 2, \dots, 18$ are elements of the matrices composed only from the integer part of the following integrals:

$$[Q^{(1)}] = \int_0^b \int_0^a H_{\xi}^{(o)} H_{\xi}^{(o)T} d\xi d\eta \quad (\text{B. 3a})$$

$$[Q^{(2)}] = \int_0^b \int_0^a H_{\eta}^{(o)} H_{\eta}^{(o)T} d\xi d\eta \quad (\text{B. 3b})$$

$$[Q^{(3)}] = \int_0^b \int_0^a (H_{\xi}^{(0)} H_{\eta}^{(0)T} + H_{\eta}^{(0)} H_{\xi}^{(0)T}) d\xi d\eta \quad (B.3c)$$

$$[Q^{(4)}] = \int_0^b \int_0^a H_{\eta}^{(0)} H_{\xi}^{(0)T} d\xi d\eta \quad (B.3d)$$

$$[Q^{(5)}] = \int_0^b \int_0^a H_{\xi}^{(0)} H_{\eta}^{(0)T} d\xi d\eta \quad (B.3e)$$

$$[Q^{(6)}] = \int_0^b \int_0^a H^{(0)} H^{(0)T} d\xi d\eta \quad (B.3f)$$

$$[Q^{(7)}] = \int_0^b \int_0^a H^{(0)} H_{\xi}^{(1)T} d\xi d\eta \quad (B.3g)$$

$$[Q^{(8)}] = \int_0^b \int_0^a H^{(0)} H_{\eta}^{(1)T} d\xi d\eta \quad (B.3h)$$

$$[Q^{(9)}] = \int_0^b \int_0^a H_{\xi\xi}^{(1)} H_{\xi\xi}^{(1)T} d\xi d\eta \quad (B.3i)$$

$$[Q^{(10)}] = \int_0^b \int_0^a H_{\eta\eta}^{(1)} H_{\eta\eta}^{(1)T} d\xi d\eta \quad (B.3j)$$

$$[Q^{(11)}] = \int_0^b \int_0^a H_{\xi\xi}^{(1)} H_{\xi\eta}^{(1)T} + H_{\xi\eta}^{(1)} H_{\xi\xi}^{(1)T} d\xi d\eta \quad (B.3k)$$

$$[Q^{(12)}] = \int_0^b \int_0^a H_{\eta\eta}^{(1)} H_{\xi\eta}^{(1)T} + H_{\xi\eta}^{(1)} H_{\eta\eta}^{(1)T} d\xi d\eta \quad (B.3l)$$

$$[Q^{(13)}] = \int_0^b \int_0^a H_{\xi\xi}^{(1)} H_{\eta\eta}^{(1)T} + H_{\eta\eta}^{(1)} H_{\xi\xi}^{(1)T} d\xi d\eta \quad (B.3m)$$

$$[Q^{(14)}] = \int_0^b \int_0^a H_{\xi\eta}^{(1)} H_{\xi\eta}^{(1)T} d\xi d\eta \quad (B.3n)$$

$$[Q^{(15)}] = \int_0^b \int_0^a H^{(1)}_{\xi} H^{(1)T}_{\xi} d\xi d\eta \quad (B.3o)$$

$$[Q^{(16)}] = \int_0^b \int_0^a H^{(1)}_{\xi} H^{(1)T}_{\xi} d\xi d\eta \quad (B.3p)$$

$$[Q^{(17)}] = \int_0^b \int_0^a H^{(1)}_{\eta} H^{(1)T}_{\eta} d\xi d\eta \quad (B.3q)$$

$$[Q^{(18)}] = \int_0^b \int_0^a H^{(1)}_{\xi} H^{(1)T}_{\eta} + H^{(1)}_{\eta} H^{(1)T}_{\xi} d\xi d\eta \quad (B.3r)$$

The values of $q_{ij}^{(k)}$, $k = 1, 2, \dots, 18$; α_{ij} , γ_{ij} , λ_{ij} and μ_{ij} are given in tables B.1 to B.4.

b) H - H Approach:

In this case all the combinations are in terms of the first order Hermite interpolation polynomials.

$$\begin{aligned} (u_f u_f)_{k_{ij}} &= [B_{f1} \{ (\frac{b}{a}) B_{11} q_{ij}^{(16)} + (\frac{a}{b}) B_{12} q_{ij}^{(17)} + B_{13} q_{ij}^{(18)} \} \\ &\quad + C_1 ab \bar{q}_{ij}^{(4)}] a^{\lambda_{ij}} b^{\mu_{ij}} / B \end{aligned} \quad (B.4a)$$

$$(u_f v_f)_{k_{ij}} = B_{f1} \{ B_{21} \bar{q}_{ij}^{(5)} + (\frac{b}{a}) B_{22} q_{ij}^{(16)} + B_{23} \bar{q}_{ij}^{(6)} \} a^{\lambda_{ij}} b^{\mu_{ij}} / B \quad (B.4b)$$

$$\begin{aligned} (v_f v_f)_{k_{ij}} &= [B_{f1} \{ (\frac{b}{a}) B_{31} q_{ij}^{(16)} + (\frac{a}{b}) q_{ij}^{(17)} + B_{32} q_{ij}^{(18)} \} \\ &\quad + C_2 ab \bar{q}_{ij}^{(4)}] a^{\lambda_{ij}} b^{\mu_{ij}} / B \end{aligned} \quad (B.4c)$$

$$k_{ij}^{(u_1 u_2)} = - \{C_1 \text{ ab } \bar{q}_{ij}^{(4)}\} a^{\lambda ij} b^{\mu ij} / B \quad (\text{B.4d})$$

$$k_{ij}^{(v_1 v_2)} = - \{C_2 \text{ ab } \bar{q}_{ij}^{(4)}\} a^{\lambda ij} b^{\mu ij} / B \quad (\text{B.4e})$$

$$k_{ij}^{(u_1 w)} = - C_3 \{b \bar{q}_{ij}^{(7)}\} a^{\lambda ij} b^{\mu ij} / B \quad (\text{B.4f})$$

$$k_{ij}^{(v_1 w)} = - \{C_4 b \bar{q}_{ij}^{(7)} + C_5 a \bar{q}_{ij}^{(8)}\} a^{\lambda ij} b^{\mu ij} / B \quad (\text{B.4g})$$

$$k_{ij}^{(u_2 w)} = \{C_3 b \bar{q}_{ij}^{(7)}\} a^{\lambda ij} b^{\mu ij} / B \quad (\text{B.4h})$$

$$k_{ij}^{(v_2 w)} = \{C_4 b \bar{q}_{ij}^{(7)} + C_5 a \bar{q}_{ij}^{(8)}\} a^{\lambda ij} b^{\mu ij} / B \quad (\text{B.4i})$$

$$\begin{aligned} k_{ij}^{(ww)} = & \sum_{f=1}^2 \left\{ \frac{D_f}{ab} \left[\left(\frac{a}{b} \right)^2 q_{ij}^{(9)} + \left(\frac{b}{a} \right)^2 q_{ij}^{(10)} \right. \right. \\ & + B_{41} \left[\left(\frac{a}{b} \right) q_{ij}^{(11)} + \left(\frac{b}{a} \right) q_{ij}^{(12)} \right] + B_{42} q_{ij}^{(13)} + B_{43} q_{ij}^{(14)} \left. \right] \\ & \left. \frac{a^{\lambda ij} b^{\mu ij}}{B \cos^3 \beta} \right\} + C_6 \left\{ C_7 \left(\frac{b}{a} \right) q_{ij}^{(16)} + C_8 \left(\frac{a}{b} \right) q_{ij}^{(17)} + C_9 q_{ij}^{(18)} \right\} \\ & \frac{a^{\lambda ij} b^{\mu ij}}{B} \end{aligned} \quad (\text{B.4j})$$

where all the $q_{ij}^{(k)}$ are the same as those given in Eqs. B3 and in addition the $\bar{q}_{ij}^{(k)}$, $k = 4, 5, \dots, 8$ are the elements of the matrices $[\bar{Q}^k]$ given by:

$$[\bar{Q}^{(4)}] = \int_0^b \int_0^a H^{(1)} H^{(1)T} d\xi d\eta \quad (B.5a)$$

$$[\bar{Q}^{(5)}] = \int_0^b \int_0^a H_{\eta}^{(1)} H_{\xi}^{(1)T} d\xi d\eta \quad (B.5b)$$

$$[\bar{Q}^{(6)}] = \int_0^b \int_0^a H_{\xi}^{(1)} H_{\eta}^{(1)T} d\xi d\eta \quad (B.5c)$$

$$[\bar{Q}^{(7)}] = \int_0^b \int_0^a H^{(1)} H_{\xi}^{(1)T} d\xi d\eta \quad (B.5d)$$

$$[\bar{Q}^{(8)}] = \int_0^b \int_0^a H^{(1)} H_{\eta}^{(1)T} d\xi d\eta \quad (B.5e)$$

The values of $\bar{q}_{ij}^{(k)}$, $k = 4, 5, \dots, 8$, λ_{ij} and μ_{ij} are given in table B.5.

TABLES B1, B2
CONSTANTS AND EXPONENTS FOR SUBMATRICES

i	j	(1) q_{ij}	(2) q_{ij}	(3) q_{ij}	(4) q_{ij}	(5) q_{ij}	(6) q_{ij}
1	1	12	12	12	9	9	4
1	2	6	-12	0	9	-9	2
1	3	-6	-6	-12	-9	-9	1
1	4	-12	6	0	-9	9	2
2	1	6	-12	0	-9	9	2
2	2	12	12	-12	-9	-9	4
2	3	-12	6	0	9	-9	2
2	4	-6	-6	12	9	9	1
3	1	-6	-6	-12	-9	-9	1
3	2	-12	6	0	-9	9	2
3	3	12	12	12	9	9	4
3	4	6	-12	0	9	-9	2
4	1	-12	6	0	9	-9	2
4	2	-6	-6	12	9	9	1
4	3	6	-12	0	-9	9	2
4	4	12	12	-12	-9	-9	4

i	j	(7) q_{ij}	(8) q_{ij}	α_{ij}	γ_{ij}	i	j	(7) q_{ij}	(8) q_{ij}	α_{ij}	γ_{ij}
1	1	-630	-630	0	0	9	1	270	270	0	0
1	2	-270	-630	0	0	9	2	630	270	0	0
1	3	-270	-270	0	0	9	3	630	630	0	0
1	4	-630	-270	0	0	9	4	270	630	0	0
2	1	105	-90	1	0	10	1	-45	-60	1	0
2	2	45	-90	1	0	10	2	-105	-60	1	0
2	3	-45	-60	1	0	10	3	105	-90	1	0
2	4	-105	-60	1	0	10	4	45	-90	1	0
3	1	-90	105	0	1	11	1	-60	-45	0	1
3	2	-60	-105	0	1	11	2	-90	45	0	1
3	3	-60	-45	0	1	11	3	-90	105	0	1
3	4	-90	45	0	1	11	4	-60	-105	0	1
4	1	15	15	1	1	12	1	10	10	1	1
4	2	10	-15	1	1	12	2	15	-10	1	1
4	3	-10	-10	1	1	12	3	-15	-15	1	1
4	4	-15	10	1	1	12	4	-10	15	1	1
5	1	-270	630	0	0	13	1	630	-270	0	0
5	2	-630	630	0	0	13	2	270	-270	0	0
5	3	-630	270	0	0	13	3	270	-630	0	0
5	4	-270	270	0	0	13	4	630	-630	0	0
6	1	45	90	1	0	14	1	-105	60	1	0
6	2	105	90	1	0	14	2	-45	60	1	0
6	3	-105	60	1	0	14	3	45	90	1	0
6	4	-45	60	1	0	14	4	105	90	1	0
7	1	60	-105	0	1	15	1	90	45	0	1
7	2	90	105	0	1	15	2	60	-45	0	1
7	3	90	45	0	1	15	3	60	-105	0	1
7	4	60	-45	0	1	15	4	90	105	0	1
8	1	-10	-15	1	1	16	1	-15	-10	1	1
8	2	-15	10	1	1	16	2	-10	15	1	1
8	3	15	10	1	1	16	3	10	15	1	1
8	4	10	-15	1	1	16	4	15	-10	1	1

TABLE B3

CONSTANTS AND EXPONENTS FOR SUBMATRICES

i	j	$q_{ij}^{(9)}$	$q_{ij}^{(10)}$	$q_{ij}^{(11)}$	$q_{ij}^{(12)}$	$q_{ij}^{(13)}$	$q_{ij}^{(14)}$	$q_{ij}^{(15)}$	λ_{ij}	μ_{ij}
1	1	786240	786240	0	0	508032	254016	24336	0	0
1	2	110880	293120	0	0	254016	21168	3432	1	0
1	3	393120	110880	0	0	254016	21168	3432	0	1
1	4	55440	55440	-35280	-35280	38808	1764	484	1	1
1	5	-786240	272160	0	0	-508032	-254016	8424	0	0
1	6	-110880	136080	0	-176400	-254016	-21168	1188	1	0
1	7	393120	-65520	0	0	42336	21168	-2028	0	1
1	8	55440	-32760	35280	35280	21168	1764	-286	1	1
1	9	-272160	-272160	0	0	508032	254016	2916	0	0
1	10	65520	136080	0	176400	-42336	-21168	-702	1	0
1	11	136080	65520	176400	0	-42336	-21168	-702	0	1
1	12	-32760	-32760	-35280	-35280	3528	1764	160	1	1
1	13	272160	-786240	0	0	-508032	-254016	8424	0	0
1	14	-65520	393120	0	0	42336	21168	-2028	1	0
1	15	136080	-110880	-176400	0	-254016	-21168	1188	0	1
1	16	-32760	55440	35280	35280	21168	1764	-286	1	1
2	2	20160	262080	0	88200	56448	28224	624	2	0
2	3	55440	55440	35280	35280	215208	1764	484	1	1
2	4	10080	26960	0	0	28224	2352	88	2	1
2	5	-110880	136080	0	176400	-254016	-21168	1188	1	0
2	6	-20160	90720	0	0	-56448	-28224	216	2	0
2	7	55440	-32760	-35280	-35280	21168	1764	-286	1	1
2	8	10080	-21840	0	0	4704	2352	-52	2	1
2	9	-65520	-136080	0	-176400	42336	21168	702	1	0
2	10	15120	45360	0	88200	14112	7056	-162	2	0
2	11	32760	32760	35280	35280	-3528	-1764	-169	1	1
2	12	-7560	-10920	-5880	-17640	-1176	-588	30	2	1
2	13	65520	-393120	0	0	-42336	-21168	2028	1	0
2	14	-15120	131040	0	0	-14112	-7056	-468	2	0
2	15	32760	-55440	-35280	-35280	-21168	-1764	286	1	1
2	16	-7560	18480	5880	17640	-7056	-588	-66	2	1
3	3	262080	20160	38200	0	56448	28224	624	0	2
3	4	26960	10080	0	0	28224	2352	88	1	2
3	5	-393120	65520	0	0	-42336	-21168	2028	0	1
3	6	-55440	32760	-35280	-35280	-21168	-1764	286	1	1
3	7	131040	-15120	0	0	-14112	-7056	-468	0	2
3	8	18480	-7560	17640	5880	-7056	-588	-66	1	2
3	9	-136080	-65520	-176400	0	42336	21168	702	0	1
3	10	32760	32760	35280	35280	-3528	-1764	-169	1	1
3	11	45360	15120	38200	0	14112	7056	-162	0	2
3	12	-10920	-7560	-17640	-5880	-1176	-588	30	1	2
3	13	136080	-110880	176400	0	-254016	-21168	1188	0	1
3	14	-32760	55440	-35280	-35280	21168	1764	-286	1	1
3	15	90720	-20160	0	0	-56448	-28224	216	0	2
3	16	-21840	10080	0	0	4704	2352	-52	1	2

TABLE B3 (Continued)

i	j	(9) q_{ij}	(10) q_{ij}	(11) q_{ij}	(12) q_{ij}	(13) q_{ij}	(14) q_{ij}	(15) q_{ij}	λ_{ij}	μ_{ij}
8	8	6720	6720	0	0	6272	3136	16	2	2
8	9	-32760	55440	35280	35280	21168	1764	-288	1	1
8	10	7560	-19480	-5880	-17640	7056	588	66	2	1
8	11	21840	-10080	0	0	-4704	-2352	52	1	2
8	12	-5040	3360	0	0	-1568	-784	-12	2	2
8	13	32760	32760	-35280	-35280	-3528	-1764	-168	1	1
8	14	-7560	-10920	5880	17640	-1176	-588	30	2	1
8	15	10920	7560	-17640	-5880	1176	588	-29	1	2
8	16	-2520	-2520	2940	2940	392	196	9	2	2
9	9	786240	786240	0	0	503032	254016	24236	0	0
9	10	-110880	-393120	0	0	-254016	-21168	-3432	1	0
9	11	-393120	-110880	0	0	-254016	-21168	-3432	0	1
9	12	55440	55440	-35280	-35280	33808	1764	484	1	1
9	13	-786240	272160	0	0	-503032	-254016	9424	0	0
9	14	110880	-136080	0	176400	254016	21168	-1188	1	0
9	15	-393120	65520	0	0	-42336	-21168	2028	0	1
9	16	55440	-32760	35280	35280	21168	1764	-286	1	1
10	10	20160	262080	0	88200	56448	28224	624	2	0
10	11	55440	55440	35280	35280	215208	1764	484	1	1
10	12	-10080	-36960	0	0	-28224	-2352	-88	2	1
10	13	110880	-136080	0	-176400	254016	21168	-1188	1	0
10	14	-20160	50720	0	0	-56448	-28224	216	2	0
10	15	55440	-32760	-35280	-35280	21168	1764	-286	1	1
10	16	-10080	21840	0	0	-4704	-2352	52	2	1
11	11	262080	20160	93200	0	56448	28224	624	0	2
11	12	-36960	-10080	0	0	-28224	-2352	-88	1	2
11	13	393120	-65520	0	0	42336	21168	-2028	0	1
11	14	-55440	32760	-35280	-35280	-21168	-1764	286	1	1
11	15	121040	-15120	0	0	-14112	-7056	-468	0	2
11	16	-18480	7560	-17640	-5880	7056	588	66	1	2
12	12	6720	6720	0	0	6272	3136	16	2	2
12	13	-55440	32760	35280	35280	-21168	-1764	286	1	1
12	14	10080	-21840	0	0	4704	2352	-52	2	1
12	15	-18480	7560	17640	5880	7056	588	66	1	2
12	16	3360	-5040	0	0	-1568	-784	-12	2	2
13	13	786240	786240	0	0	508032	254016	24236	0	0
13	14	-110880	-393120	0	0	-254016	-21168	-3432	1	0
13	15	393120	110880	0	0	254016	21168	3432	0	1
13	16	-55440	-55440	-35280	-35280	-33808	-1764	-484	1	1
14	14	20160	262080	0	-88200	56448	28224	624	2	0
14	15	-55440	-55440	35280	35280	-215208	-1764	-484	1	1
14	16	10080	36960	0	0	28224	2352	88	2	1
15	15	262080	20160	-38200	0	56448	28224	624	0	2
15	16	-36960	-10080	0	0	-28224	-2352	-88	1	2
16	16	6720	6720	0	0	6272	3136	16	2	2

TABLE B3 (Continued)

i	j	(9) q_{ij}	(10) q_{ij}	(11) q_{ij}	(12) q_{ij}	(13) q_{ij}	(14) q_{ij}	(15) q_{ij}	λ_{ij}	μ_{ij}
4	4	6720	6720	0	0	6272	3136	16	2	2
4	5	-55440	32760	35280	35280	-21168	-1764	286	1	1
4	6	-10080	21840	0	0	-4704	-2352	52	2	1
4	7	18480	-7560	-17640	-5880	-7056	-588	-66	1	2
4	8	3360	-5040	0	0	-1568	-784	-12	2	2
4	9	-32760	-32760	-35280	-35280	3528	1764	160	1	1
4	10	7560	10920	5880	17640	1176	588	-39	2	1
4	11	10920	7560	17640	5880	1176	588	-39	1	2
4	12	-2520	-2520	-2940	-2940	392	196	9	2	2
4	13	32760	-55440	35280	35280	-21168	-1764	286	1	1
4	14	-7560	18480	-5880	-17640	-7056	-588	-66	2	1
4	15	21840	-10080	0	0	-4704	-2352	52	1	2
4	16	-5040	3360	0	0	-1568	-784	-12	2	2
5	5	786240	786240	0	0	508032	254016	24336	0	0
5	6	110880	393120	0	0	254016	21168	3432	1	0
5	7	-393120	-110880	0	0	-254016	-21168	-3432	0	1
5	8	-55440	-55440	-35280	-35280	-33808	-1764	-484	1	1
5	9	272160	-786240	0	0	-508032	-254016	8424	0	0
5	10	-65520	393120	0	0	42336	21168	-2028	1	0
5	11	-136080	110880	-176400	0	254016	21168	-1180	0	1
5	12	32760	-55440	35280	35280	-21168	-1764	286	1	1
5	13	-272160	-272160	0	0	508032	254016	2916	0	0
5	14	65520	136080	0	-176400	-42336	-21168	-702	1	0
5	15	-136080	-65520	176400	0	42336	21168	702	0	1
5	16	32760	32760	-35280	-35280	-3528	-1764	-169	1	1
6	6	20160	262080	0	-88200	56448	28224	624	2	0
6	7	-55440	-55440	35280	35280	-215208	-1764	-484	1	1
6	8	-10080	-26960	0	0	-28224	-2352	-88	2	1
6	9	65520	-393120	0	0	-42336	-21168	2028	1	0
6	10	-15120	131040	0	0	-14112	-7056	-468	2	0
6	11	-32760	55440	-35280	-35280	21168	1764	-286	1	1
6	12	7560	-18480	5880	17640	7056	588	66	2	1
6	13	-65520	-136080	0	176400	42336	21168	702	1	0
6	14	15120	45360	0	-88200	14112	7056	-162	2	0
6	15	-32760	-32760	35280	35280	3528	1764	169	1	1
6	16	7560	10920	-5880	-17640	1176	588	-39	2	1
7	7	262080	20160	-88200	0	56448	28224	624	0	2
7	8	36960	10080	0	0	28224	2352	88	1	2
7	9	-136080	110880	176400	0	254016	21168	-1180	0	1
7	10	32760	-55440	-35280	-35280	-21168	-1764	286	1	1
7	11	90720	-20160	0	0	-56448	-28224	216	0	2
7	12	-21840	10080	0	0	4704	2352	-52	1	2
7	13	136080	65520	-176400	0	-42336	-21168	-702	0	1
7	14	-32760	-32760	35280	35280	3528	1764	169	1	1
7	15	45360	15120	-88200	0	14112	7056	-162	0	2
7	16	-10920	-7560	17640	5880	-1176	-588	39	1	2

TABLE B4

CONSTANTS AND EXPONENTS FOR SUBMATRICES

i	j	$q_{ij}^{(16)}$	$q_{ij}^{(17)}$	$q_{ij}^{(18)}$	λ_{ij}	μ_{ij}	i	j	$q_{ij}^{(16)}$	$q_{ij}^{(17)}$	$q_{ij}^{(18)}$	λ_{ij}	μ_{ij}
1	1	78624	78624	38200	0	0	4	4	224	224	0	2	2
1	2	6552	11088	0	1	0	4	5	546	-924	3528	1	1
1	3	11088	6552	0	0	1	4	6	728	-168	0	2	1
1	4	924	924	-3528	1	1	4	7	-126	-308	-588	1	2
1	5	27216	-78624	0	0	0	4	8	-168	-56	0	2	2
1	6	2268	-11088	-17640	1	0	4	9	-546	-546	-3528	1	1
1	7	-6552	6552	0	0	1	4	10	-182	126	588	2	1
1	8	-546	924	3528	1	1	4	11	126	-182	588	1	2
1	9	-27216	-27216	-38200	0	0	4	12	42	42	-98	2	2
1	10	2268	6552	17640	1	0	4	13	-924	546	3528	1	1
1	11	6552	2268	17640	0	1	4	14	-308	-126	-588	2	1
1	12	-546	-546	-3528	1	1	4	15	-168	728	0	1	2
1	13	-78624	27216	0	0	0	4	16	-56	-168	0	2	2
1	14	6552	-6552	0	1	0	5	5	78624	78624	-38200	0	0
1	15	-11088	2268	-17640	0	1	5	6	6552	11088	0	1	0
1	16	924	-546	3528	1	1	5	7	-11088	-6552	0	0	1
2	2	8736	2016	0	2	0	5	8	-924	-924	-3528	1	1
2	3	924	924	3528	1	1	5	9	-78624	27216	0	0	0
2	4	1232	168	0	2	1	5	10	6552	-6552	0	1	0
2	5	2268	-11088	17640	1	0	5	11	11088	-2268	-17640	0	1
2	6	3024	-2016	0	2	0	5	12	-924	546	3528	1	1
2	7	-546	924	-3528	1	1	5	13	-27216	-27216	38200	0	0
2	8	-728	168	0	2	1	5	14	2268	6552	-17640	1	0
2	9	-2268	-6552	-17640	1	0	5	15	-6552	-2268	17640	0	1
2	10	-756	1512	2940	2	0	5	16	546	546	-3528	1	1
2	11	546	546	3528	1	1	6	6	8736	2016	0	2	0
2	12	182	-126	-588	2	1	6	7	-924	-924	3528	1	1
2	13	-6552	6552	0	1	0	6	8	-1232	-168	0	2	1
2	14	-2184	-1512	0	2	0	6	9	-6552	6552	0	1	0
2	15	-924	546	-3528	1	1	6	10	-2184	-1512	0	2	0
2	16	-308	-126	588	2	1	6	11	924	-546	-3528	1	1
3	3	2016	8736	0	0	2	6	12	308	126	588	2	1
3	4	168	1232	0	1	2	6	13	-2268	-6552	17640	1	0
3	5	6552	-6552	0	0	1	6	14	-756	1512	-2940	2	0
3	6	546	-924	-3528	1	1	6	15	-546	-546	3528	1	1
3	7	-1512	-2184	0	0	2	6	16	-182	126	-588	2	1
3	8	-126	-308	588	1	2	7	7	2016	8736	0	0	2
3	9	-6552	-2268	-17640	0	1	7	8	168	1232	0	1	2
3	10	546	546	3528	1	1	7	9	11088	-2268	17640	0	1
3	11	1512	-756	2940	0	2	7	10	-924	546	-3528	1	1
3	12	-126	182	-588	1	2	7	11	-2016	3024	0	0	2
3	13	-11088	2268	17640	0	1	7	12	168	-728	0	1	2
3	14	924	-546	-3528	1	1	7	13	6552	2268	-17640	0	1
3	15	-2016	3024	0	0	2	7	14	-546	-546	3528	1	1
3	16	168	-728	0	1	2	7	15	1512	-756	-2940	0	2

TABLE B4 (Continued)

i	j	$q_{ij}^{(16)}$	$q_{ij}^{(17)}$	$q_{ij}^{(18)}$	λ_{ij}	μ_{ij}
7	16	-128	132	588	1	2
8	8	224	224	0	2	2
8	9	924	-546	3528	1	1
8	10	308	126	-588	2	1
8	11	-188	728	0	1	2
8	12	-56	-168	0	2	2
8	13	546	546	-3528	1	1
8	14	182	-126	588	2	1
8	15	126	-182	-588	1	2
8	16	42	42	98	2	2
9	9	78624	78624	88200	0	0
9	10	-6552	-11088	0	1	0
9	11	-11088	-6552	0	0	1
9	12	924	924	-3528	1	1
9	13	27216	-78624	0	0	0
9	14	-2268	11088	17640	1	0
9	15	6552	-6552	0	0	1
9	16	-546	924	3528	1	1
10	10	9736	2016	0	2	0
10	11	924	924	3528	1	1
10	12	-1232	-168	0	2	1
10	13	-2268	11088	-17640	1	0
10	14	3024	-2016	0	2	0
10	15	-546	924	-3528	1	1
10	16	728	-168	0	2	1
11	11	2016	8736	0	0	2
11	12	-168	-1232	0	1	2
11	13	-6552	6552	0	0	1
11	14	546	-924	-3528	1	1
11	15	-1512	-2184	0	0	2
11	16	126	308	-588	1	2
12	12	224	224	0	2	2
12	13	546	-924	3528	1	1
12	14	-728	168	0	2	1
12	15	126	308	588	1	2
12	16	-168	-56	0	2	2
13	13	78624	78624	-88200	0	0
13	14	-6552	-11088	0	1	0
13	15	11088	6552	0	0	1
13	16	-924	-924	-3528	1	1
14	14	8736	2016	0	2	0
14	15	-924	-924	3528	1	1
14	16	1232	168	0	2	1
15	15	2016	8736	0	0	2
15	16	-168	-1232	0	1	2
16	16	224	224	0	2	2

TABLE B5

CONSTANTS AND EXPONENTS FOR SUBMATRICES

i	j	$\bar{q}_{ij}^{(4)}$	$\bar{q}_{ij}^{(5)}$	$\bar{q}_{ij}^{(6)}$	$\bar{q}_{ij}^{(7)}$	$\bar{q}_{ij}^{(8)}$	λ_{ij}	μ_{ij}
1	1	24326	44100	44100	-32760	-32760	0	0
1	2	3432	-8820	8820	6552	-4620	1	0
1	3	3432	8820	-8820	-4620	6552	0	1
1	4	484	-1764	-1764	924	924	1	1
1	5	8424	44100	-44100	-11340	32760	0	0
1	6	1188	-8820	-8820	2268	4620	1	0
1	7	-2028	-8820	8820	2730	-6552	0	1
1	8	-286	1764	1764	-546	-924	1	1
1	9	2916	-44100	-44100	11340	11340	0	0
1	10	-702	8820	8820	-2268	-2730	1	0
1	11	-702	8820	8820	-2730	-2268	0	1
1	12	168	-1764	-1764	546	546	1	1
1	13	8424	-44100	44100	32760	-11340	0	0
1	14	-2028	8820	-8820	-6552	2730	1	0
1	15	1188	-8820	-8820	4620	2268	0	1
1	16	-286	1764	1764	-924	-546	1	1
2	1	3432	8820	-8820	-6552	-4620	1	0
2	2	624	0	0	0	-840	2	0
2	3	484	1764	1764	-924	924	1	1
2	4	88	0	0	0	168	2	1
2	5	1188	8820	8820	-2268	4620	1	0
2	6	216	0	0	0	840	2	0
2	7	-286	-1764	-1764	546	-924	1	1
2	8	-52	0	0	0	-168	2	1
2	9	702	-8820	-8820	2268	2730	1	0
2	10	-162	1470	1470	-378	-630	2	0
2	11	-162	1764	1764	-546	-546	1	1
2	12	38	-294	-294	91	126	2	1
2	13	2028	-8820	8820	6552	-2730	1	0
2	14	-468	1470	-1470	-1092	630	2	0
2	15	286	-1764	-1764	924	546	1	1
2	16	-66	294	294	-154	-126	2	1
3	1	3432	-8820	8820	-4620	-6552	0	1
3	2	484	1764	1764	924	-924	1	1
3	3	624	0	0	-840	0	0	2
3	4	88	0	0	168	0	1	2
3	5	2028	8820	-8820	-2730	6552	0	1
3	6	286	-1764	-1764	546	924	1	1
3	7	-468	-1470	1470	630	-1092	0	2
3	8	-66	294	294	-126	-154	1	2
3	9	702	-8820	-8820	2730	2268	0	1
3	10	-162	1764	1764	-546	-546	1	1
3	11	-162	1470	1470	-630	-378	0	2
3	12	38	-294	-294	126	91	1	2
3	13	1188	8820	8820	4620	-2268	0	1

TABLE B5 (Continued)

i	j	$\bar{q}_{ij}^{(4)}$	$\bar{q}_{ij}^{(5)}$	$\bar{q}_{ij}^{(6)}$	$\bar{q}_{ij}^{(7)}$	$\bar{q}_{ij}^{(8)}$	λ_{ij}	μ_{ij}
3	14	-286	-1764	-1764	-924	546	1	1
3	15	216	0	0	840	0	0	2
3	16	-52	0	0	-168	0	1	2
4	1	484	-1764	-1764	-924	-924	1	1
4	2	88	0	0	0	-168	2	1
4	3	88	0	0	-168	0	1	2
4	4	16	0	0	0	0	2	2
4	5	286	1764	1764	-546	924	1	1
4	6	52	0	0	0	168	2	1
4	7	-66	-294	-294	126	-154	1	2
4	8	-12	0	0	0	-28	2	2
4	9	168	-1764	-1764	546	546	1	1
4	10	-20	294	294	-91	-126	2	1
4	11	-32	294	294	-126	-91	1	2
4	12	6	-49	-49	21	21	2	2
4	13	286	1764	1764	924	-546	1	1
4	14	-66	-294	-294	-154	126	2	1
4	15	52	0	0	168	0	1	2
4	16	-12	0	0	-28	0	2	2
5	1	2424	-44100	44100	-11340	-32760	0	0
5	2	1188	8820	8820	2268	-4620	1	0
5	3	2028	-8820	8820	-2730	-6552	0	1
5	4	286	1764	1764	546	-924	1	1
5	5	24336	-44100	-44100	-32760	32760	0	0
5	6	3432	8820	-8820	6552	4620	1	0
5	7	-3432	8820	-8820	4620	6552	0	1
5	8	-484	-1764	-1764	-924	924	1	1
5	9	8424	44100	-44100	32760	11340	0	0
5	10	-2028	-8820	8820	-6552	-2730	1	0
5	11	-1188	-8820	-8820	-4620	2268	0	1
5	12	286	1764	1764	924	-546	1	1
5	13	2916	44100	44100	11340	-11340	0	0
5	14	-702	-8820	-8820	-2268	2730	1	0
5	15	702	8820	8820	2730	-2268	0	1
5	16	-168	-1764	-1764	-546	546	1	1
6	1	1188	-8820	-8820	-2268	-4620	1	0
6	2	216	0	0	0	-840	2	0
6	3	286	-1764	-1764	-546	-924	1	1
6	4	52	0	0	0	-168	2	1
6	5	2422	-8820	-8820	-6552	4620	1	0
6	6	624	0	0	0	840	2	0
6	7	-484	1764	1764	924	924	1	1
6	8	-88	0	0	0	168	2	1
6	9	2028	8820	-8820	6552	2730	1	0
6	10	-468	-1470	1470	-1092	-630	2	0

TABLE B5 (Continued)

i	j	$\bar{q}_{ij}^{(4)}$	$\bar{q}_{ij}^{(5)}$	$\bar{q}_{ij}^{(6)}$	$\bar{q}_{ij}^{(7)}$	$\bar{q}_{ij}^{(8)}$	λ_{ij}	μ_{ij}
6	11	-286	-1764	-1764	-924	546	1	1
6	12	66	294	294	154	-126	2	1
6	13	702	8820	8820	2268	-2730	1	0
6	14	-162	-1470	-1470	-378	630	2	0
6	15	168	1764	1764	546	-546	1	1
6	16	-39	-294	-294	-91	126	2	1
7	1	-2028	8820	-8820	2730	6552	0	1
7	2	-286	-1764	-1764	-546	924	1	1
7	3	-468	1470	-1470	630	1092	0	2
7	4	-66	-294	-294	-126	154	1	2
7	5	-3432	-8820	8820	4620	-6552	0	1
7	6	-484	1764	1764	-924	-924	1	1
7	7	624	0	0	-840	0	0	2
7	8	88	0	0	168	0	1	2
7	9	-1188	8820	8820	-4620	-2268	0	1
7	10	286	-1764	-1764	924	546	1	1
7	11	216	0	0	840	0	0	2
7	12	-52	0	0	-168	0	1	2
7	13	-702	-8820	-8820	-2730	2268	0	1
7	14	168	1764	1764	546	-546	1	1
7	15	-162	-1470	-1470	-630	378	0	2
7	16	29	294	294	126	-91	1	2
8	1	-286	1764	1764	546	924	1	1
8	2	-52	0	0	0	168	2	1
8	3	-66	294	294	126	154	1	2
8	4	-12	0	0	0	28	2	2
8	5	-484	-1764	-1764	924	-924	1	1
8	6	-88	0	0	0	-168	2	1
8	7	88	0	0	-168	0	1	2
8	8	16	0	0	0	0	2	2
8	9	-286	1764	1764	-924	-546	1	1
8	10	66	-294	-294	154	126	2	1
8	11	52	0	0	168	0	1	2
8	12	-12	0	0	-28	0	2	2
8	13	-168	-1764	-1764	-546	546	1	1
8	14	39	294	294	91	-126	2	1
8	15	-39	-294	-294	-126	91	1	2
8	16	0	49	49	21	-21	2	2
9	1	2916	-44100	-44100	-11340	-11340	0	0
9	2	702	-8820	-8820	-2268	-2730	1	0
9	3	702	-8820	-8820	-2730	-2268	0	1
9	4	168	-1764	-1764	-546	-546	1	1
9	5	3424	-44100	44100	-32760	11340	0	0
9	6	2028	-8820	8820	-6552	2730	1	0
9	7	-1188	8820	2820	4620	2268	0	1

TABLE B5 (Continued)

i	j	$\bar{q}_{ij}^{(4)}$	$\bar{q}_{ij}^{(5)}$	$\bar{q}_{ij}^{(6)}$	$\bar{q}_{ij}^{(7)}$	$\bar{q}_{ij}^{(8)}$	λ_{ij}	μ_{ij}
8	8	-286	1764	1764	924	546	1	1
9	9	2432	44100	44100	32760	32760	0	0
9	10	-3432	8820	-8820	6552	-4620	1	0
9	11	-3432	-8820	8820	-4620	6552	0	1
9	12	484	-1764	-1764	-924	-924	1	1
9	13	8424	44100	-44100	11340	-32760	0	0
9	14	-1188	8820	8820	2268	4620	1	0
9	15	2028	8820	-8820	2730	-6552	0	1
9	16	-286	1764	1764	546	924	1	1
10	1	-702	8820	8820	2268	2730	1	0
10	2	-162	1470	1470	378	630	2	0
10	3	-162	1764	1764	546	546	1	1
10	4	-39	294	294	91	126	2	1
10	5	-2028	8820	-8820	6552	-2730	1	0
10	6	-468	1470	-1470	1092	-630	2	0
10	7	286	-1764	-1764	-924	-546	1	1
10	8	66	-294	-294	-154	-126	2	1
10	9	-3432	-8820	8820	-6552	-4620	1	0
10	10	824	0	0	0	840	2	0
10	11	484	1764	1764	924	-924	1	1
10	12	-88	0	0	0	168	2	1
10	13	-1188	-8820	-8820	-2268	4620	1	0
10	14	216	0	0	0	-840	2	0
10	15	-286	-1764	-1764	-546	924	1	1
10	16	52	0	0	0	-168	2	1
11	1	-702	8820	8820	2730	2268	0	1
11	2	-162	1764	1764	546	546	1	1
11	3	-162	1470	1470	630	378	0	2
11	4	-39	294	294	126	91	1	2
11	5	-1188	-8820	-8820	4620	-2268	0	1
11	6	-286	-1764	-1764	924	-546	1	1
11	7	216	0	0	-840	0	0	2
11	8	52	0	0	-168	0	1	2
11	9	-3432	8820	-8820	-4620	-6552	0	1
11	10	484	1764	1764	-924	924	1	1
11	11	624	0	0	840	0	0	2
11	12	-88	0	0	168	0	1	2
11	13	-2028	-8820	8820	-2730	6552	0	1
11	14	286	-1764	-1764	-546	-924	1	1
11	15	-468	-1470	1470	-630	1092	0	2
11	16	66	-294	-294	-126	-154	1	2
12	1	162	-1764	-1764	-546	-546	1	1
12	2	39	-294	-294	-91	-126	2	1
12	3	39	-294	-294	-126	-91	1	2
12	4	0	-49	-49	-21	-21	2	2

TABLE B5 (Continued)

i	j	$\bar{q}_{ij}^{(4)}$	$\bar{q}_{ij}^{(5)}$	$\bar{q}_{ij}^{(6)}$	$\bar{q}_{ij}^{(7)}$	$\bar{q}_{ij}^{(8)}$	λ_{ij}	μ_{ij}
12	5	288	1764	1764	-924	546	1	1
12	6	66	294	294	-154	126	2	1
12	7	-52	0	0	168	0	1	2
12	8	-12	0	0	28	0	2	2
12	9	484	-1764	-1764	924	924	1	1
12	10	-88	0	0	0	-168	2	1
12	11	-88	0	0	-168	0	1	2
12	12	16	0	0	0	0	2	2
12	13	288	1764	1764	546	-924	1	1
12	14	-52	0	0	0	168	2	1
12	15	66	294	294	126	-154	1	2
12	16	-12	0	0	0	28	2	2
13	1	8424	44100	-44100	-32760	-11340	0	0
13	2	2028	8820	-8820	-6552	-2730	1	0
13	3	1188	8820	8820	-4620	2268	0	1
13	4	288	1764	1764	-924	546	1	1
13	5	2916	44100	44100	-11340	11340	0	0
13	6	702	8820	8820	-2268	2730	1	0
13	7	-702	-8820	-8820	2268	-2730	0	1
13	8	-169	-1764	-1764	546	-546	1	1
13	9	8424	-44100	44100	11340	32760	0	0
13	10	-1188	-8820	-8820	2268	-4620	1	0
13	11	-2028	8820	-8820	-2730	-6552	0	1
13	12	288	1764	1764	-546	924	1	1
13	13	24336	-44100	-44100	32760	-32760	0	0
13	14	-3432	-8820	8820	6552	4620	1	0
13	15	2432	-8820	8820	4620	6552	0	1
13	16	-484	-1764	-1764	924	-924	1	1
14	1	-2028	-8820	8820	6552	2730	1	0
14	2	-468	-1470	1470	1092	620	2	0
14	3	-286	-1764	-1764	924	-546	1	1
14	4	-66	-294	-294	154	-126	2	1
14	5	-702	-8820	-8820	2268	-2730	1	0
14	6	-162	-1470	-1470	378	-630	2	0
14	7	169	1764	1764	-546	546	1	1
14	8	30	294	294	-91	126	2	1
14	9	-1188	8820	8820	-2268	-4620	1	0
14	10	216	0	0	0	840	2	0
14	11	286	-1764	-1764	546	924	1	1
14	12	-52	0	0	0	-168	2	1
14	13	-3432	8820	-8820	-6552	4620	1	0
14	14	624	0	0	0	-840	2	0
14	15	-484	1764	1764	-924	-924	1	1
14	16	88	0	0	0	168	2	1
15	1	1188	-8820	-8820	-4620	-2268	0	1

TABLE B5 (Continued)

i	j	$\bar{q}_{ij}^{(4)}$	$\bar{q}_{ij}^{(5)}$	$\bar{q}_{ij}^{(6)}$	$\bar{q}_{ij}^{(7)}$	$\bar{q}_{ij}^{(8)}$	λ_{ij}	μ_{ij}
15	2	296	-1764	-1764	-924	-546	1	1
15	3	216	0	0	-840	0	0	2
15	4	52	0	0	-168	0	1	2
15	5	702	8820	8820	-2730	2268	0	1
15	6	169	1764	1764	-546	546	1	1
15	7	-162	-1470	-1470	630	-378	0	2
15	8	-39	-294	-294	126	-91	1	2
15	9	2028	-8820	8820	2730	6552	0	1
15	10	-286	-1764	-1764	546	-924	1	1
15	11	-468	1470	-1470	-630	-1092	0	2
15	12	66	294	294	-126	154	1	2
15	13	3432	8820	-8820	4620	-6552	0	1
15	14	-484	1764	1764	924	924	1	1
15	15	624	0	0	840	0	0	2
15	16	-88	0	0	168	0	1	2
16	1	-296	1764	1764	924	546	1	1
16	2	-66	294	294	154	126	2	1
16	3	-52	0	0	168	0	1	2
16	4	-12	0	0	28	0	2	2
16	5	-169	-1764	-1764	546	-546	1	1
16	6	-39	-294	-294	91	-126	2	1
16	7	39	294	294	-126	91	1	2
16	8	9	49	49	-21	21	2	2
16	9	-286	1764	1764	-546	-924	1	1
16	10	52	0	0	0	168	2	1
16	11	66	-294	-294	126	154	1	2
16	12	-12	0	0	0	-28	2	2
16	13	-484	-1764	-1764	-924	924	1	1
16	14	88	0	0	0	-168	2	1
16	15	-98	0	0	-168	0	1	2
16	16	16	0	0	0	0	2	2

APPENDIX 'C'

STRESS AND MOMENT RESULTANTS

Using the proper transformation between the oblique and rectangular coordinate systems as given by Eqs. 2.1, 2.2 and 2.3 and following the same approach of Morley (Ref. 40), the relations between the stresses in the x, y plane and that in the ξ, η plane can be written as (see Fig.12):

$$\sigma_{\xi} = \sigma_x \cos \beta + \sigma_y \sin \beta \tan \beta + 2\tau_{xy} \sin \beta \quad (c.1a)$$

$$\sigma_{\eta} = \sigma_y \sec \beta \quad (c.1b)$$

$$\tau_{\xi\eta} = \tau_{\eta\xi} = \tau_{xy} + \sigma_y \tan \beta \quad (c.1c)$$

and

$$\sigma_x = \sigma_{\xi} \sec \beta + \sigma_{\eta} \sin \beta \tan \beta - 2\tau_{\xi\eta} \tan \beta \quad (c.2a)$$

$$\sigma_y = \sigma_{\eta} \cos \beta \quad (c.2b)$$

$$\tau_{xy} = \tau_{yx} = \tau_{\xi\eta} - \sigma_{\eta} \sin \beta \quad (c.2c)$$

The stresses in the ξ, η plane can be written in terms of the displacement variables of the faces, using the stress - strain equations and the strain displacement equations as follows:

$$\begin{aligned}\sigma_{\xi f} = & \frac{E_f}{(1 - v_f^2)} \{u_x (\cos \beta + v_f \sin \beta \tan \beta) + u_y (1 - v_f) \sin \beta \\ & + v_x (1 - v_f) \sin \beta + v_y (\sin \beta \tan \beta + v_f \cos \beta)\} \\ & (c.3a)\end{aligned}$$

$$\sigma_{\eta f} = \frac{E_f}{(1 - v_f^2)} \sec \beta (v_y + v u_x) \quad (c.3b)$$

$$\begin{aligned}\sigma_{\xi \eta f} = & \frac{E_f}{(1 - v_f^2)} \left\{ \frac{(1 - v_f)}{2} (u_y + v_x) + \tan \beta (v_y + v_f u_x) \right\} \\ & (c.3c)\end{aligned}$$

Substituting for the displacements in the oblique coordinate from

Eqs. 2.2 gives:

$$\begin{aligned}\sigma_{\xi f} = & \frac{E_f}{(1 - v_f^2)} \{u_{\xi} \sec \beta + u_{\eta} \tan \beta (1 - v_f) \\ & + v_{\xi} \sec \beta \tan \beta + v_{\eta} (v_f + \tan^2 \beta)\} \\ & (c.4a)\end{aligned}$$

$$\sigma_{\eta f} = \frac{E_f}{(1 - v_f^2)} \sec \beta \{v_{\xi} \tan \beta + v_{\eta} \sec \beta + v_f u_{\xi}\} \quad (c.4b)$$

$$\begin{aligned}\sigma_{\xi \eta f} = & \frac{E_f}{(1 - v_f^2)} \left\{ u_{\xi} \frac{1}{2} (1 + v_f) \tan \beta + u_{\eta} \frac{1}{2} (1 - v_f) \sec \beta \right. \\ & \left. + v_{\xi} \left[\frac{(1 - v_f)}{2} + \tan^2 \beta \right] + v_{\eta} \sec \beta \tan \beta \right\} \\ & (c.4c)\end{aligned}$$

The stress resultants and moment resultants of the faces are:

$$N_{\xi f} = \int_{-t_f/2}^{+t_f/2} \sigma_{\xi f} dz \quad (c.5a)$$

$$N_{\eta f} = \int_{-t_f/2}^{+t_f/2} \sigma_{\eta f} dz \quad (c.5b)$$

$$N_{\xi \eta f} = \int_{-t_f/2}^{+t_f/2} \tau_{\xi \eta f} dz \quad (c.5c)$$

$$M_{\xi f} = \int_{-t_f/2}^{+t_f/2} z_f \sigma_{\xi f} dz \quad (c.5d)$$

$$M_{\eta f} = \int_{-t_f/2}^{+t_f/2} z_f \sigma_{\eta f} dz \quad (c.5e)$$

$$M_{\eta \xi f} = \int_{-t_f/2}^{+t_f/2} z_f \tau_{\eta \xi f} dz = -M_{\xi \eta} \quad (c.5f)$$

substituting for the stresses from Eqs. c.4 into Eqs. c.5 and integrating over the face thickness results in the following expressions for the stress and moment resultants in terms of the displacement components $u, \bar{v}, w$:

$$N_{\xi f} = \frac{E_f t_f}{(1-\nu_f^2)} \{u_{\xi} \sec \beta + u_{\eta} \tan \beta (1 - \nu_f) + \bar{v}_{\xi} \tan \beta + \bar{v}_{\eta} \cos \beta (\nu_f + \tan^2 \beta)\} \quad (c.6a)$$

$$N_{\eta f} = \frac{E_f t_f}{(1-\nu_f^2)} \{\bar{v}_{\xi} \tan \beta + (\bar{v}_{\eta} + \nu_f u_{\xi}) \sec \beta\} \quad (c.6b)$$

$$N_{\xi \eta f} = \frac{E_f t_f}{(1-\nu_f^2)} \left\{ u_{\xi} \frac{(1+\nu_f)}{2} \tan \beta + u_{\eta} \frac{(1-\nu_f)}{2} \sec \beta + \bar{v}_{\xi} \cos \beta \left[\frac{(1-\nu_f)}{2} + \tan^2 \beta \right] + \bar{v}_{\eta} \tan \beta \right\} \quad (c.6c)$$

and the face moment resultants will be:

$$M_{\xi f} = -D_f \sec \beta \left[\nabla^2 w - (1 - \nu_f) \frac{\partial^2 w}{\partial \eta^2} \right] \quad (c.7a)$$

$$M_{\eta f} = -D_f \sec \beta \left[\nabla^2 w - (1 - \nu_f) \frac{\partial^2 w}{\partial \xi^2} \right] \quad (c.7b)$$

$$M_{\xi \eta f} = -M_{\eta \xi f} = -D_f \sec \beta \left[\sin \beta \nabla^2 w - (1 - \nu_f) \frac{\partial^2 w}{\partial \xi \partial \eta} \right] \quad (c.7c)$$

where: $D_f = \frac{E_f t_f^3}{12(1-\nu_f^2)}$ is the flexural rigidity of faces

$$\text{and } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \sec^2 \beta \left(\frac{\partial^2}{\partial \xi^2} - 2 \sin \beta \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \right)$$

The total Moments of the sandwich $\bar{M}_\xi, \bar{M}_\eta, \bar{M}_{\xi\eta}$ are simply calculated by multiplying the stress resultants faces $N_{\xi f}, N_{\eta f}, N_{\xi\eta f}$ by the moment arm h (neglecting the moments of the faces). Also the moments $\bar{M}_x, \bar{M}_y, \bar{M}_{xy}$ can be computed using the relations given by Eqs. c.2 and hence the maximum and minimum moments can be calculated from the following equations:

$$\bar{M}_{\max} = \frac{\bar{M}_x + \bar{M}_y}{2} \pm \sqrt{\left(\frac{\bar{M}_x - \bar{M}_y}{2}\right)^2 + \bar{M}_{xy}^2}$$

$$\bar{M}_{\min}$$

In the same manner the shearing force resultants due to the core effect are given by:

$$Q_{xz} = \int_{-t_c/2}^{+t_c/2} \tau_{xz} dz \quad (\text{c.8a})$$

$$Q_{yz} = \int_{-t_c/2}^{+t_c/2} \tau_{yz} dz \quad (\text{c.8b})$$

and

$$\tau_{xz} = G_{xz} \gamma_{xz} \quad (\text{c.9a})$$

$$\tau_{yz} = G_{yz} \gamma_{yz} \quad (\text{c.9b})$$

From the strain displacement equations of the core it can be shown that:

$$\gamma_{cxz} = \frac{1}{t_c} [(u_2 - u_1) + w_{\xi} h] \quad (c.10a)$$

$$\gamma_{cyz} = \frac{1}{t_c} [(\bar{v}_2 - \bar{v}_1) \cos \beta + w_{\xi} \tan \beta h + w_{\eta} \sec \beta h] \quad (c.10b)$$

substituting Eqs. c.9 and c.10 into Eqs. c.8 and integrating gives:

$$Q_{xz} = G_{xz} [(u_2 - u_1) + w_{\xi} h] \quad (c.11a)$$

$$Q_{yz} = G_{yz} [(\bar{v}_2 - \bar{v}_1) \cos \beta + h(w_{\xi} \tan \beta + w_{\eta} \sec \beta)] \quad (c.11b)$$

From which:

$$Q_{\xi z} = G_{xz} \cos \beta [(u_2 - u_1) + w_{\xi} h] \quad (c.12a)$$

$$Q_{yz} = G_{yz} [(\bar{v}_2 - \bar{v}_1) \cos \beta + h(w_{\xi} \tan \beta + w_{\eta} \sec \beta)] \quad (c.12b)$$

Substituting in Eqs. c.6, c.7 and c.12 for the assumed displacement function as given by Eqs. 3.2 result in expressions for the stress and moment resultants in terms of the undetermined degrees of freedom.

SAMPLE COMPUTER OUT PUT

CLAMPED SKEW SANDWICH PLATE WITH 4 EL., $G = 10^4$ psi and $\text{Beta} = 30^\circ$

DEGREES OF FREEDOM FOR ELEMENT NUMBER 1																				
	U1	U1X	U1Y	U1XY	V1	V1X	V1Y	V1XY	U2	U2X	U2Y	U2XY	V2	V2X	V2Y	V2XY	W	WX	WY	WXY
NODE%1,1<	0	0	0	1	0	0	0	2	0	0	0	3	0	0	0	4	0	0	0	0
NODE%1,2<	0	5	0	6	0	7	0	8	0	9	0	10	0	11	0	12	0	0	0	0
NODE%2,2<	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
NODE%2,1<	0	0	45	46	0	0	47	48	0	0	49	50	0	0	51	52	0	0	0	0

DEGREES OF FREEDOM FOR ELEMENT NUMBER 2																				
	U1	U1X	U1Y	U1XY	V1	V1X	V1Y	V1XY	U2	U2X	U2Y	U2XY	V2	V2X	V2Y	V2XY	W	WX	WY	WXY
NODE%1,1<	0	5	0	6	0	7	0	8	0	9	0	10	0	11	0	12	0	0	0	0
NODE%1,2<	0	0	0	13	0	0	0	14	0	0	0	15	0	0	0	16	0	0	0	0
NODE%2,2<	0	0	17	18	0	0	19	20	0	0	21	22	0	0	23	24	0	0	0	0
NODE%2,1<	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44

DEGREES OF FREEDOM FOR ELEMENT NUMBER 3																				
	U1	U1X	U1Y	U1XY	V1	V1X	V1Y	V1XY	U2	U2X	U2Y	U2XY	V2	V2X	V2Y	V2XY	W	WX	WY	WXY
NODE%1,1<	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
NODE%1,2<	0	0	17	18	0	0	19	20	0	0	21	22	0	0	23	24	0	0	0	0
NODE%2,2<	0	0	0	65	0	0	0	66	0	0	0	67	0	0	0	68	0	0	0	0
NODE%2,1<	0	57	0	58	0	59	0	60	0	61	0	62	0	63	0	64	0	0	0	0

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DEGREES OF FREEDOM FOR ELEMENT NUMBER 4																				
	U1	U1X	U1Y	U1XY	V1	V1X	V1Y	V1XY	U2	U2X	U2Y	U2XY	V2	V2X	V2Y	V2XY	W	WX	WY	WXY
NODE%1,1<	0	0	45	46	0	0	47	48	0	0	49	50	0	0	51	52	0	0	0	0
NODE%1,2<	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
NODE%2,2<	0	57	0	58	0	59	0	60	0	61	0	62	0	63	0	64	0	0	0	0
NODE%2,1<	0	0	0	53	0	0	0	54	0	0	0	55	0	0	0	56	0	0	0	0

MODULUS OF ELASTICITY FACE1# 0.1000E 08
 POISSON RATIO FACE1# 0.3200
 PLATE THICKNESS FACE1# 0.2500E-01
 MODULUS OF ELASTICITY FACE2# 0.1000E 08
 POISSON RATIO FACE2# 0.3200
 PLATE THICKNESS FACE2# 0.2500E-01
 SHEAR MODULUS XZ# 0.1000E 05
 SHEAR MODULUS YZ# 0.1000E 05
 CORE THICKNESS# 0.1000E 01
 A# 0.2000E 02 B# 0.2000E 02
 SKEW ANGLE#DEGREES<# 0.3000E 02

DEGREE OF FREEDOM	LOAD	DISPLACEMENT
1	0.0	0.79582E-05
2	0.0	0.15917E-04
3	0.0	-0.79583E-05
4	0.0	-0.15917E-04
5	0.0	0.99768E-04
6	0.0	-0.82864E-05
7	0.0	0.37427E-04
8	0.0	-0.63574E-05
9	0.0	-0.99768E-04
10	0.0	0.82863E-05
11	0.0	-0.37427E-04
12	0.0	0.63575E-05
13	0.0	-0.40571E-06
14	0.0	0.27040E-06
15	0.0	0.40568E-06
16	0.0	-0.27041E-06
17	0.0	-0.21813E-04
18	0.0	0.62496E-06
19	0.0	0.11848E-03
20	0.0	0.11465E-04
21	0.0	0.21813E-04
22	0.0	-0.62499E-06
23	0.0	-0.11848E-03
24	0.0	-0.11465E-04
25	0.0	0.54297E-09
26	0.0	-0.11316E-03
27	0.0	0.34455E-04
28	0.0	0.27399E-10
29	0.0	0.18581E-08
30	0.0	-0.29498E-04
31	0.0	-0.12791E-03
32	0.0	0.33367E-10
33	0.0	0.25466E-10
34	0.0	0.11316E-03
35	0.0	-0.34455E-04
36	0.0	0.12960E-10
37	0.0	0.57025E-09
38	0.0	0.29498E-04
39	0.0	0.12791E-03
40	0.0	0.19099E-10
41	0.34641E 03	0.22180E-01
42	0.0	0.79126E-09
43	0.0	0.64847E-09
44	0.0	0.21854E-03
45	0.0	-0.21813E-04
46	0.0	-0.62502E-06
47	0.0	0.11848E-03
48	0.0	-0.11465E-04
49	0.0	0.21813E-04
50	0.0	0.62500E-06
51	0.0	-0.11848E-03
52	0.0	0.11465E-04
53	0.0	0.40562E-06
54	0.0	-0.27059E-06
55	0.0	-0.40566E-06
56	0.0	0.27027E-06
57	0.0	0.99770E-04
58	0.0	0.82864E-05
59	0.0	0.37428E-04
60	0.0	0.63574E-05
61	0.0	-0.99769E-04
62	0.0	-0.82864E-05
63	0.0	-0.37427E-04
64	0.0	-0.63575E-05
65	0.0	-0.79583E-05
66	0.0	-0.15917E-04
67	0.0	0.79583E-05
68	0.0	0.15917E-04

DISPLACEMENT RESULTS FOR ELEMENT NUMBER 1

	NODE#1,1<	NODE#1,2<	NODE#2,2<	NODE#2,1<
U1	0.0	0.0	0.54297E-09	0.0
U1X	0.0	0.99768E-04	-0.11316E-03	0.0
U1Y	0.0	0.0	0.34455E-04	-0.21813E-04
U1XY	0.79582E-05	-0.82864E-05	0.27399E-10	-0.62502E-06
V1	0.0	0.0	0.18581E-08	0.0
V1X	0.0	0.37427E-04	-0.29498E-04	0.0
V1Y	0.0	0.0	-0.12791E-03	0.11848E-03
V1XY	0.15917E-04	-0.63574E-05	0.33367E-10	-0.11465E-04
U2	0.0	0.0	0.25466E-10	0.0
U2X	0.0	-0.99768E-04	0.11316E-03	0.0
U2Y	0.0	0.0	-0.34455E-04	0.21813E-04
U2XY	-0.79583E-05	0.82863E-05	0.12960E-10	0.62500E-06
V2	0.0	0.0	0.57025E-09	0.0
V2X	0.0	-0.37427E-04	0.29498E-04	0.0
V2Y	0.0	0.0	0.12791E-03	-0.11848E-03
V2XY	-0.15917E-04	0.63575E-05	0.19099E-10	0.11465E-04
W	0.0	0.0	0.22180E-01	0.0
WX	0.0	0.0	0.79126E-09	0.0
WY	0.0	0.0	0.64847E-09	0.0
WXY	0.0	0.0	0.21854E-03	0.0

DISPLACEMENT RESULTS FOR ELEMENT NUMBER 2

	NODE#1,1<	NODE#1,2<	NODE#2,2<	NODE#2,1<
U1	0.0	0.0	0.0	0.54297E-09
U1X	0.99768E-04	0.0	0.0	-0.11316E-03
U1Y	0.0	0.0	-0.21813E-04	0.34455E-04
U1XY	-0.82864E-05	-0.40571E-06	0.62496E-06	0.27399E-10
V1	0.0	0.0	0.0	0.18581E-08
V1X	0.37427E-04	0.0	0.0	-0.29498E-04
V1Y	0.0	0.0	0.11848E-03	-0.12791E-03
V1XY	-0.63574E-05	0.27040E-06	0.11465E-04	0.33367E-10
U2	0.0	0.0	0.0	0.25466E-10
U2X	-0.99768E-04	0.0	0.0	0.11316E-03
U2Y	0.0	0.0	0.21813E-04	-0.34455E-04
U2XY	0.82863E-05	0.40568E-06	-0.62499E-06	0.12960E-10
V2	0.0	0.0	0.0	0.57025E-09
V2X	-0.37427E-04	0.0	0.0	0.29498E-04
V2Y	0.0	0.0	-0.11848E-03	0.12791E-03
V2XY	0.63575E-05	-0.27041E-06	-0.11465E-04	0.19099E-10
W	0.0	0.0	0.0	0.22180E-01
WX	0.0	0.0	0.0	0.79126E-09
WY	0.0	0.0	0.0	0.64847E-09
WXY	0.0	0.0	0.0	0.21854E-03

DISPLACEMENT RESULTS FOR ELEMENT NUMBER 3

	NODE#1,1<	NODE#1,2<	NODE#2,2<	NODE#2,1<
U1	0.54297E-09	0.0	0.0	0.0
U1X	-0.11316E-03	0.0	0.0	0.99770E-04
U1Y	0.34455E-04	-0.21813E-04	0.0	0.0
U1XY	0.27399E-10	0.62496E-06	-0.79583E-05	0.82864E-05
V1	0.18581E-08	0.0	0.0	0.0
V1X	-0.29498E-04	0.0	0.0	0.37428E-04
V1Y	-0.12791E-03	0.11848E-03	0.0	0.0
V1XY	0.33367E-10	0.11465E-04	-0.15917E-04	0.63574E-05
U2	0.25466E-10	0.0	0.0	0.0
U2X	0.11316E-03	0.0	0.0	-0.99769E-04
U2Y	-0.34455E-04	0.21813E-04	0.0	0.0
U2XY	0.12960E-10	-0.62499E-06	0.79583E-05	-0.82864E-05
V2	0.57025E-09	0.0	0.0	0.0
V2X	0.29498E-04	0.0	0.0	-0.37427E-04
V2Y	0.12791E-03	-0.11848E-03	0.0	0.0
V2XY	0.19099E-10	-0.11465E-04	0.15917E-04	-0.63575E-05
W	0.22180E-01	0.0	0.0	0.0
WX	0.79126E-09	0.0	0.0	0.0
WY	0.64847E-09	0.0	0.0	0.0
WXY	0.21854E-03	0.0	0.0	0.0

DISPLACEMENT RESULTS FOR ELEMENT NUMBER 4

	NODE#1,1<	NODE#1,2<	NODE#2,2<	NODE#2,1<
U1	0.0	0.54297E-09	0.0	0.0
U1X	0.0	-0.11316E-03	0.99770E-04	0.0
U1Y	-0.21813E-04	0.34455E-04	0.0	0.0
U1XY	-0.62502E-06	0.27399E-10	0.82864E-05	0.40562E-06
V1	0.0	0.18581E-08	0.0	0.0
V1X	0.0	-0.29498E-04	0.37428E-04	0.0
V1Y	0.11848E-03	-0.12791E-03	0.0	0.0
V1XY	-0.11465E-04	0.33367E-10	0.63574E-05	-0.27059E-06
U2	0.0	0.25466E-10	0.0	0.0
U2X	0.0	0.11316E-03	-0.99769E-04	0.0
U2Y	0.21813E-04	-0.34455E-04	0.0	0.0
U2XY	0.62500E-06	0.12960E-10	-0.82864E-05	-0.40566E-06
V2	0.0	0.57025E-09	0.0	0.0
V2X	0.0	0.29498E-04	-0.37427E-04	0.0
V2Y	-0.11848E-03	0.12791E-03	0.0	0.0
V2XY	0.11465E-04	0.19099E-10	-0.63575E-05	0.27027E-06
W	0.0	0.22180E-01	0.0	0.0
WX	0.0	0.79126E-09	0.0	0.0
WY	0.0	0.64847E-09	0.0	0.0
WXY	0.0	0.21854E-03	0.0	0.0

VITA AUCTORIS

- 1941 born in Cairo, Egypt; December 13, 1941
- 1957 in June, 1957, finished the secondary school (science and Mathematics) Fayoum Secondary School, Egypt
- 1962 In June, 1962, graduated with a Bachelor of Science in Civil Engineering (Structural Section) from the Faculty of Engineering Ain Shams University, Egypt. In July, employed as designer structural Engineer in the Ministry of Housing, Cairo, Egypt.
- 1963 In November, 1963, held a position as Demonstrator in the Structural Department, Faculty of Engineering, Ain Shams University. In the same time from 1963 - 1968, joined Cairo Engineering Office (Consulting Engineers) as part - time Engineer.
- 1968 In September, 1968, awarded a research and teaching assistantship for graduate studies at the Civil Engineering Department, University of Windsor