1-1-2006

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by

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A Thesis
Submitted to the Faculty of Graduate Studies and Research
Through Mathematics and Statistics
in Partial Fulfillment of the Requirements for
the Degree of Master of Science at the
University of Windsor

Windsor, Ontario, Canada
2006

Your file Votre référence
ISBN: 978-0-494-35968-6
Our file Notre référence
ISBN: 978-0-494-35968-6

Abstract

In this thesis, we give a realization and explicitly describe a basis and the corresponding module action for all non-integral simple torsion free A_n-modules of finite degree. This realization will mirror certain finite dimensional modules viewed in terms of a tableau formalism. In fact, the basis and module action which we defined for these realizations is described in terms of the module action on the tableau realization of finite dimensional modules.

Acknowledgements

To Dr. D.J. Britten, and Dr. F.W. Lemire I would like to express my gratitude for their patience and help while writing this thesis. In addition, I would like to thank University of Windsor Mathematics and Statistics department, Ontario Graduate Scholarship program, and National Sciences and Engineering Research Council of Canada for their financial support.

Contents

1 Introduction

Let L be a finite dimensional simple Lie algebra over the complex numbers $\mathbb{C}$, and let $\mathcal{H}$ be a Cartan subalgebra of L. An L-module V is said to be a weight module provided $V = \oplus_{\lambda \in \mathcal{H}^*} V_\lambda$, where

$$V_\lambda = \{v \in V \mid h.v = \lambda(h)v \text{ for all } h \in \mathcal{H}\}.$$

Every simple finite dimensional L-module is a weight module and is completely determined by its highest weight. However, obtaining a classification for the simple infinite dimensional L-modules is far more difficult. In fact, Lemire [9] showed that simple infinite dimensional modules need not be weight modules with respect to any Cartan subalgebra $\mathcal{H}$. As a result, at the present time, the classification of all simple L-modules seems to be beyond reach. However, a complete classification of the simple weight L-modules having finite dimensional weight spaces does exist.

A major step in this classification came when Suren Fernando [6] reduced the problem to the classification of all simple weight modules with finite dimensional weight spaces on which the root vectors act injectively. A weight L-module with this property is said to be *torsion free*. Clearly any torsion free module has the property that all of its weight spaces have the same dimension, called the degree of the module. Fernando went on to show that only the simple Lie algebras of type A and C admit simple torsion free modules of finite degree.

In [10] Mathieu classifies and provides a realization of all simple torsion free weight modules having finite degree. Mathieu's realization is very complicated, and therefore a need for an elementary realization was desirable. This was given by Britten and Lemire [3] where, using the work of Mathieu, showed that every simple torsion free module of finite degree is a submodule of the tensor product of a simple torsion free module of degree 1 and a finite dimensional module. This realization, however, does not explicitly give a basis and a module action for the simple torsion free modules.

Mathieu partitions all simple torsion free modules of finite degree into three categories, the integral regular, singular integral and non-integral regular. In this thesis, we explicitly describe a basis and a module action for all non-integral regular simple torsion free A_n-modules of finite degree. This realization is constructed by working with certain finite dimensional modules viewed in terms of a tableau formalism. Moreover, we show that describing this module action is no more difficult than determining the module action on certain finite dimensional modules.

2 Lie Algebra Background

The aim of this chapter is to review the background information on Lie algebras and their representations. This chapter will assume the reader is familiar with vector space theory and basic abstract algebra. We use for our basic reference Humphreys "Introduction to Lie Algebras and Representation Theory" [7]. Most results are stated without proof as they can be found in this basic reference.

2.1 Basic Definitions

Although the general definition of an algebra is over an arbitrary field, we restrict to algebras over the field of complex numbers $\mathbb{C}$.

Definition 2.1. Let $\mathcal{A}$ be a vector space over $\mathbb{C}$. $\mathcal{A}$ is said to be an **algebra** over $\mathbb{C}$ provided there is a bilinear binary operation $(\cdot,\cdot) : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ with (x, y) written as xy such that

1. $(ax + y)z = a(xz) + yz$, and

2. $x(by + z) = b(xy) + xz$

for all $a, b \in \mathbb{C}$ and $x, y, z \in \mathcal{A}$. This binary operation is called multiplication. Sometimes the multiplication defined on an algebra is denoted by $x * y$ or $[x, y]$.

There are two main types of algebras of interest to us, associative algebras and Lie algebras.

Definition 2.2. An **associative algebra** $\mathcal{A}$ over a field $\mathbb{C}$ is an algebra over $\mathbb{C}$ such that

1. $(xy)z = x(yz)$

for all $x, y, z \in \mathcal{A}$. If $\mathcal{A}$ contains an identity element, i.e. an element 1 such that $1x = x1 = x$ for all $x \in \mathcal{A}$, then we call $\mathcal{A}$ an **unital associative algebra**. A **subalgebra** $K \leq \mathcal{A}$ is a sub-vector space of $\mathcal{A}$ with the property that $xy \in K$ for all $x, y \in K$.

Remark 2.1. In this work all our associative algebras will be unital unless otherwise stated.

Definition 2.3. Let $G = \{g_1, \ldots, g_n\}$ be a finite group with group operation $*$. The **group algebra**, $\mathbb{C}[G]$, is the vector space over $\mathbb{C}$ having basis $\{g_1, \ldots, g_n\}$, with multiplication defined by:

$$\left(\sum_{g \in G} a_g g\right)\left(\sum_{h \in G} b_h h\right) = \sum_{g \in G, h \in G} (a_g b_h) g * h = \sum_{k \in G}\left(\sum_{g*h=k} a_g b_h\right) k$$

where $g, h \in G$ and $a_g, b_h \in \mathbb{C}$. This algebra is an associative algebra. The group algebra concept can be defined over infinite groups as well.

Example 2.1. Of particular interest to us is the group algebra $\mathbb{C}[S_{\mathcal{N}}]$, where $S_{\mathcal{N}}$ is called the **symmetric group** on the set $\mathcal{N} = \{1, \ldots, N\}$ with $N \in \mathbb{Z}_{\geq 1}$. This group is the collection of all bijective functions from $\mathcal{N}$ to $\mathcal{N}$ with group operation being composition.

Definition 2.4. Let $(\mathcal{A}, *_1)$ and $(\mathcal{B}, *_2)$ be associative algebras. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a linear map from $\mathcal{A}$ to $\mathcal{B}$ with the property that $\varphi(x *_1 y) = \varphi(x) *_2 \varphi(y)$ for all $x, y \in \mathcal{A}$. Then φ is called an **algebra homomorphism**. If φ is bijective then φ is called an **isomorphism**. In this case, $\mathcal{A}$ and $\mathcal{B}$ are said to be **isomorphic**, denoted $\mathcal{A} \cong \mathcal{B}$. When φ is bijective, and $\mathcal{A} = \mathcal{B}$ we call φ an **automorphism**.

Definition 2.5. Let $\mathcal{A}$ be an associative algebra and $I \subseteq \mathcal{A}$ be a sub-vector space of $\mathcal{A}$. Then I is a **left ideal** of $\mathcal{A}$ provided $yx \in I$ for all $x \in I$ and $y \in \mathcal{A}$. I is a **right ideal** of $\mathcal{A}$ provided $xy \in I$ for all $x \in I$ and $y \in \mathcal{A}$. I is a **two sided ideal** or simply an **ideal** provided I is both a left and right ideal.

Definition 2.6. Let $\mathcal{A}$ be an associative algebra and I be a proper ideal of $\mathcal{A}$. The **quotient algebra** is the associative algebra $\mathcal{A}/I = \{x + I \mid x \in \mathcal{A}\}$ of cosets with addition and scalar multiplication given by $a(x + I) + b(y + I) = (ax + by) + I$, and product is given by $(x + I)(y + I) = xy + I$, for all $x, y \in \mathcal{A}$ and $a, b \in \mathbb{C}$.

Definition 2.7. A **Lie algebra** over $\mathbb{C}$ is a vector space L having a multiplication $[\cdot, \cdot] : L \times L \longrightarrow L$, called a bracket operation, such that for all $x, y, z \in L$ and $a, b \in \mathbb{C}$ the following conditions are satisfied:

1. Bilinearity

   (a) $[ax + by, z] = a[x, z] + b[y, z]$

   (b) $[z, ax + by] = a[z, x] + b[z, y]$

2. $[x, x] = 0$ and

3. The Jacobi identity holds: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Remark 2.2. We notice that property 2 together with bilinearity gives us anti-commutativity in the following sense:

$$[x, y] = -[y, x]$$

for all $x, y \in L$.

Example 2.2. An associative algebra $\mathcal{A}$ with multiplication $*$ can be turned into a Lie algebra by defining a bracket operation by a commutator product: $[x, y] = x*y - y*x$, for all $x, y \in \mathcal{A}$. We denote this Lie algebra by $\mathcal{A}^-$. In particular, suppose V is a vector space over a field $\mathbb{C}$. Let $End(V)$ be the set of linear transformations from $V \longrightarrow V$ then $End(V)$ is an associative algebra under the operations of addition and composition of functions. By defining a bracket operation $[x, y] = xy - yx$ on $End(V)$ a Lie algebra is created. We denote this Lie algebra $(End(V))^-$ by $gl(V)$ and call it the **general linear algebra** on V. In the case that V is finite dimensional, after fixing a basis for V, $(dimV = n + 1)$, we may identify $gl(V)$ with the set of $(n + 1) \times (n + 1)$ matrices over $\mathbb{C}$, denoted $gl(n + 1, \mathbb{C})$.

We now introduce some basic concepts of Lie algebras.

Definition 2.8. If L and L' are two Lie algebras, then the map $\rho : L \longrightarrow L'$ is a **Lie algebra homomorphism** provided

1. $\rho(ax + by) = a\rho(x) + b\rho(y)$ and

2. $\rho([x, y]) = [\rho(x), \rho(y)]$

for all $x, y \in L$ and $a, b \in \mathbb{C}$. A Lie algebra homomorphism which is injective and surjective is said to be an **isomorphism**.

Definition 2.9. Let L be a Lie algebra. A vector subspace K of L is called a **subalgebra** of L provided it is itself a Lie algebra under the operations that it inherits from L.

Definition 2.10. Let L be a Lie algebra. A subalgebra K of L is called an **ideal** provided $[x, y] \in K$ for all $x \in L$ and $y \in K$.

Remark 2.3. Unlike associative algebras, we need not define the notion of a left ideal or a right ideal as anti-commutativity in a Lie algebra implies that any left ideal or right ideal is in fact an ideal.

Remark 2.4. For a Lie algebra L, let $[L, L]$ consist of all linear combinations of commutators $[x, y]$ for $x, y \in L$. Clearly $[L, L]$ is an ideal of L.

Definition 2.11. Let L be a Lie algebra. L is said to be **simple** provided $[L, L] \neq 0$ and the only ideals of L are (0) and L.

Example 2.3. In this work, we are interested in the special linear Lie algebra, A_n. It is the subalgebra of $gl(n+1, \mathbb{C})$ given by:

$$A_n = \{X = (x_{ij}) \in gl(n+1, \mathbb{C}) \mid Trace(X) = x_{11} + \cdots + x_{(n+1)(n+1)} = 0\}.$$

At this point, we note that A_n is closed under the commutator product because $Trace(AB) = Trace(BA)$ for all square matrices. One can show that A_n is a simple finite dimensional Lie algebra over $\mathbb{C}$.

Definition 2.12. A Lie algebra L is said to be **abelian** provided $[L, L] = 0$.

Definition 2.13. If L is a Lie algebra then the **derived series** of L is defined by

$$L^{(0)} = L\,; \qquad L^{(k+1)} = [L^{(k)}, L^{(k)}]$$

for $k \in \mathbb{Z}_{\geq 0}$. L is said to be **solvable** provided $L^{(k)} = 0$ for some $k \in \mathbb{Z}_{\geq 0}$. If I is an ideal of L then I is said to be a **solvable ideal** of L provided I is solvable as a Lie algebra.

Proposition 2.1. Let L be a Lie algebra.

1. If L is solvable then so are all subalgebras and homomorphic images of L.

2. If I and J are solvable ideals of L then so is $I + J$.

Proof. See for example Proposition 3.1 in [7] □

Proposition 2.2. For a Lie algebra L there exists a unique maximal solvable ideal which is called the radical of L and denoted **Rad L**.

Proof. Let L be a Lie algebra. Since L is finite dimensional and (0) is solvable by Zorn's lemma there exists a maximal solvable ideal of S of L. Suppose that I is another maximal solvable ideal of L. By Proposition 2.1 part 2 we have that $S + I$ is a solvable ideal of L. By maximality, $S + I = S$ or $I \subseteq S$ and uniqueness is shown. Therefore, every finite dimensional Lie algebra contains a unique maximal solvable ideal.

□

Definition 2.14. A Lie algebra L is said to be **semisimple** provided *Rad* $L = (0)$.

Remark 2.5. Every simple Lie algebra is semisimple. To see this, let L be a simple Lie algebra. Then the ideal $L^{(1)} \neq (0)$ and hence $L^{(1)} = L$. Therefore, $L^{(k)} = L \neq (0)$ for all k. Therefore L is not solvable, since the only ideals of L are (0) and L, that is, *Rad* $L = (0)$, i.e. L is semisimple.

Definition 2.15. Let L be a Lie algebra. For $x \in L$ define $ad_x : L \to L$ by $ad_x(y) = [x, y]$ for all $y \in L$. ad_x is called the **adjoint action** of x on L.

Proposition 2.3. Let L be a Lie algebra. Then $Ad_L = \{ad_x \mid x \in L\}$ is a Lie subalgebra of $gl(L)$, in particular $[ad_x, ad_y] = ad_{[x,y]}$ for all $x, y \in L$.

Proof. By the bilinearity of $[\cdot,\cdot]$ we have that ad_x is a linear map, $ad_x + ad_y = ad_{x+y}$ and $b(ad_x) = ad_{bx}$ for all $x, y \in L$ and $b \in \mathbb{C}$. Lastly,

$$\begin{aligned}
[ad_x, ad_y](z) &= ad_x ad_y(z) - ad_y ad_x(z) \\
&= ad_x([y, z]) - ad_y([x, z]) \\
&= [x, [y, z]] - [y, [x, z]] \\
&= [x, [y, z]] + [[x, z], y] \\
&= [[x, y], z] \\
&= ad_{[x,y]}(z)
\end{aligned}$$

and therefore, Ad_L is a Lie subalgebra of $gl(L)$. □

Theorem 2.1. (Lie's Theorem) Let L be a solvable subalgebra of $gl(V)$ with V finite dimensional. Then the matrices of L relative to a suitable basis of V are upper triangular.

Proof. See for example, Corollary 4.1 (A) in [7] □

2.2 Killing Form

From this point on we restrict ourselves to finite dimensional Lie algebras over $\mathbb{C}$.

Definition 2.16. Let L be a Lie algebra. The **Killing form** on L is a symmetric bilinear associative form defined by:

$$\mathcal{K} : L \times L \to \mathbb{C},$$

such that

$$\mathcal{K}(x, y) = Trace(ad_x ad_y).$$

Remark 2.6. The bilinearity of the Killing form is a result of linearity of the ad and trace operators. Symmetry is due to symmetry of the trace operator. Associativity is easy to prove: Take $x, y, z \in L$ then,

$$\begin{aligned}
\mathcal{K}([x,y],z) &= Tr(ad_{[x,y]}ad_z) \\
&= Tr([ad_x, ad_y]ad_z) && (\textit{Proposition } 2.3) \\
&= Tr(ad_x ad_y ad_z - ad_y ad_x ad_z) \\
&= Tr(ad_x ad_y ad_z - ad_x ad_z ad_y) \\
&= Tr(ad_x ad_{[y,z]}) && (\textit{Proposition } 2.3) \\
&= \mathcal{K}(x,[y,z]).
\end{aligned}$$

Definition 2.17. Let L be a Lie algebra. A bilinear form $(\cdot,\cdot) : L \times L \to \mathbb{C}$ is **non degenerate** provided $(x, y) = 0$ for all $y \in L$ implies $x = 0$.

Theorem 2.2. *L is a semisimple Lie algebra if and only if the Killing form $\mathcal{K}$ on L is non degenerate.*

Proof. See for example Theorem 5.1 in [7] □

2.3 Rootspace decomposition for semisimple Lie algebras

In this section, we briefly review the structure theory of finite dimensional semisimple Lie algebras over $\mathbb{C}$.

Definition 2.18. Let L be a semisimple Lie algebra. A subalgebra T of L is called **toral** provided for every $x \in T$, ad_x is diagonalizable.

Proposition 2.4. There exists a maximal toral subalgebra in every finite dimensional semisimple Lie algebra.

Proof. See for example Section 8.1 in [7] □

Lemma 2.1. Let L be a semisimple Lie algebra. A toral subalgebra of L is abelian.

Proof. See for example Lemma 8.1 in [7] □

Definition 2.19. Let L be a semisimple Lie algebra. A **Cartan subalgebra,** $\mathcal{H}$ of L is a maximal toral subalgebra of L.

Remark 2.7. Fix a Cartan subalgebra, $\mathcal{H}$ of L. Denote the vector space of linear functionals on $\mathcal{H}$ by $\mathcal{H}^*$. For each $\alpha \in \mathcal{H}^*$, define $L_\alpha = \{x \in L \mid [h,x] = \alpha(h)x \ \forall \ h \in \mathcal{H}\}$. Since $\mathcal{H}$ is abelian, Proposition 2.3 tells us that $ad_L(\mathcal{H}) = \{ad_L h \mid h \in \mathcal{H}\}$ is a commuting family of semisimple endomorphisms of L. A standard result in linear algebra implies that $ad_L(\mathcal{H})$ is simultaneously diagonalizable. That is, we have the following decomposition for our semisimple Lie algebra L:

$$L = \bigoplus_{\alpha \in \mathcal{H}^*} L_\alpha.$$

Proposition 2.5. Let $\mathcal{H}$ be a Cartan subgalgebra for a semisimple Lie algebra L. Then $\mathcal{H} = L_0$.

Proof. See for example Corollary 8.1 in [7] □

Definition 2.20. Let L be a semisimple Lie algebra. If $0 \neq \alpha \in \mathcal{H}^*$ and $L_\alpha \neq 0$ then α is said to be a **root** of L relative to $\mathcal{H}$. The set of roots of L relative to $\mathcal{H}$ is denoted by Φ. For each $\alpha \in \Phi$, L_α is a **root space** of L with respect to $\mathcal{H}$. The non-zero vectors in L_α are called **root vectors.**

We have arrived at the standard **root space decomposition** of L:

$$L = \mathcal{H} \bigoplus_{\alpha \in \Phi} L_\alpha.$$

Lemma 2.2. Let L be a semi-simple Lie algebra. If $\alpha, \beta \in \Phi$ with $\alpha + \beta \neq 0$ then L_α is orthogonal to L_β with respect to the Killing form $\mathcal{K}$ of L. In other words, $\mathcal{K}(x,y) = 0$ for all $x \in L_\alpha$ and $y \in L_\beta$.

Proof. Take $x \in L_\alpha$, $y \in L_\beta$. Since $\alpha \neq \beta$, select an element $h \in \mathcal{H}$ such that $\alpha(h) \neq \beta(h)$.

$$\begin{aligned}\alpha(h)\mathcal{K}(x,y) &= \mathcal{K}(\alpha(h)x,y)\\ &= \mathcal{K}([h,x],y)\\ &= -\mathcal{K}([x,h],y)\\ &= -\mathcal{K}(x,[h,y])\\ &= -\mathcal{K}(x,\beta(h)y)\\ &= -\beta(h)\mathcal{K}(x,y).\end{aligned}$$

Therefore, $(\alpha(h)+\beta(h))\mathcal{K}(x,y)=0$, i.e. $\mathcal{K}(x,y)=0$. □

Lemma 2.3. The restriction of the Killing form to $\mathcal{H}$ is non-degenerate.

Proof. Assume $L \neq (0)$. Since L is semisimple by Theorem 2.2, $\mathcal{K}$ is non-degenerate on L and so we may take $0 \neq x \in L_0$ and $y \in L$ with $\mathcal{K}(x,y) \neq 0$. Now, $y = y_0 + \sum_{\alpha\in\Phi} y_\alpha$ with $y_0 \in L_0$ and $y_\alpha \in L_\alpha$ for $\alpha \in \Phi$. Since L_0 is orthogonal to L_α for all $\alpha \in \Phi$ we have,

$$\begin{aligned}0 \neq \mathcal{K}(x,y) &= \mathcal{K}(x, y_0 + \sum_{\alpha\in\Phi} y_\alpha)\\ &= \mathcal{K}(x,y_0).\end{aligned}$$

Therefore, there exists a $y_0 \in L_0$ with $\mathcal{K}(x,y_0) \neq 0$, i.e. $\mathcal{K}$ restricted to L_0 is nondegenerate. Since $\mathcal{H} = L_0$ we are done. □

Remark 2.8. In light of Lemma 2.3 we may identify $\mathcal{H}$ with $\mathcal{H}^*$ by using the Killing form as follows: For $\phi \in \mathcal{H}^*$ assign a unique element $t_\phi \in \mathcal{H}$ satisfying $\phi(h) = \mathcal{K}(t_\phi, h)$ for all $h \in \mathcal{H}$. In particular, Φ corresponds to the subset $\{t_\alpha \mid \alpha \in \Phi\}$ of $\mathcal{H}$.

Theorem 2.3. Let L be a semisimple Lie algebra with Φ being the set of roots in L relative to a fixed Cartan subalgebra $\mathcal{H}$.

1. Φ spans $\mathcal{H}^*$.

2. If α, β and $\alpha+\beta \in \Phi$ then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.

3. If $\alpha \in \Phi$ and $c\alpha \in \Phi$ then $c = \pm 1$ and L_α is one dimensional.

4. Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$. Let r, q be the largest integers for which $\beta - r\alpha$, $\beta + q\alpha$ are roots. Then for $-r \leq i \leq q$, $\beta + i\alpha \in \Phi$ and $\beta(h_\alpha) = r - q$.

5. If $\alpha \in \Phi$ then for any $0 \neq x_\alpha \in L_\alpha$ there exists elements $y_\alpha \in L_{-\alpha}$ and $h_\alpha \in \mathcal{H}$ such that $Span_{\mathbb{C}}\{h_\alpha, x_\alpha, y_\alpha\} \backsimeq sl(2, \mathbb{C})$.

6. $h_\alpha = \frac{2t_\alpha}{\mathcal{K}(t_\alpha, t_\alpha)}$ and $h_\alpha = -h_{-\alpha}$.

7. L is generated as a Lie algebra by the root spaces L_α.

Proof. See for example Proposition 8.3 and 8.4 in [7] □

Since the Killing form is non-degenerate on $\mathcal{H}$ the correspondence between Φ and $\{t_\alpha \mid \alpha \in \Phi\} \subset \mathcal{H}$ allows us to define an inner product on $E = Span_{\mathbb{R}}(\Phi)$:

$$(\mu, \nu) = \mathcal{K}(t_\mu, t_\nu).$$

We refer to E as the **Euclidean space** spanned by Φ.

Theorem 2.4. Let L be a semisimple Lie algebra. Let $\mathcal{H}$ be a Cartan subalgebra of L, Φ the set of roots of L relative to $\mathcal{H}$ and $E = Span_{\mathbb{R}}(\Phi)$. The following properties hold:

1. Φ is finite, spans E and does not contain 0,

2. If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$,

3. If $\alpha, \beta \in \Phi$ then $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$, and

4. If $\alpha, \beta \in \Phi$ then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Proof. See for example Theorem 8.5 in [7] □

Definition 2.21. Let L be a semisimple Lie algebra with Cartan subalgebra $\mathcal{H}$. Φ denote the set of roots of L relative to $\mathcal{H}$ and $E = Span_{\mathbb{R}}(\Phi)$. If Φ satisfies properties 1 to 4 in Theorem 2.4 then Φ is said to be a **root system**. If $\Delta \subset \Phi$ such that

1. Δ is a basis for E, and

2. for every $\beta \in \Phi$, β can be expressed as an integral linear combination of elements from Δ where all coefficients are non-negative or non-positive.

then Δ is a **base** for Φ and the elements in Δ are called **simple roots**.

Theorem 2.5. Let L be a semisimple Lie algebra with root system Φ. Then Φ has a base Δ.

Proof. See for example Theorem 10.1.2 [7] ☐

Definition 2.22. Let L be a semisimple Lie algebra with root system Φ and base Δ. Let

$$\Phi^+(\Delta) = \{\beta \in \Phi \mid \beta = \sum_{i=1}^{n} k_i \alpha_i \quad \alpha_i \in \mathbb{Z}_{\geq 0}\},$$

and

$$\Phi^-(\Delta) = \{\beta \in \Phi \mid \beta = \sum_{i=1}^{n} k_i \alpha_i \quad \alpha_i \in \mathbb{Z}_{\leq 0}\},$$

where $\Phi^+(\Delta)$ is referred to as the **positive roots** of Φ and $\Phi^-(\Delta)$ is referred to as the **negative roots** of Φ.

Remark 2.9. Clearly by definition the set of positive and negative roots of Φ partition Φ.

2.4 The Weyl Group

Definition 2.23. Let L be a semisimple Lie algebra, with root system Φ. Let E be the Euclidean space spanned by Φ. For each $\alpha \in \Phi$, let $\sigma_\alpha : E \to E$ denote the reflection in the hyperplane perpendicular to α. i.e.

$$\sigma_\alpha(\gamma) = \gamma - 2\frac{(\gamma, \alpha)}{(\alpha, \alpha)}\alpha$$

for all $\gamma \in E$. Define the **Weyl group**, denoted $\mathcal{W}$, to be the group generated by $\{\sigma_\alpha \mid \alpha \in \Phi\}$.

Proposition 2.6. Let L be a semisimple Lie algebra, with root system Φ. Let Δ be a base for Φ. Then $\mathcal{W}$ is generated by the set $\{\sigma_\alpha \mid \alpha \in \Delta\}$.

Proof. See for example Theorem 10.3 in [7]. □

Definition 2.24. For a semisimple Lie algebra L with root system Φ with respect to the Cartan subalgebra $\mathcal{H}$, fix a base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ with basis $\{h_1, \ldots, h_n\}$ of $\mathcal{H}$, where $h_i = h_{\alpha_i}$. Obtain the dual basis for $\mathcal{H}^*$ by choosing, for each i, $\omega_i \in \mathcal{H}^*$ given by $\omega_i(h_j) = \delta_{ij}$ and extending linearly. We call $\{\omega_1, \ldots, \omega_n\}$ the **fundamental basis** for $\mathcal{H}^*$ relative to Δ, and the ω_i's are called the **fundamental weights**. Notice the fundamental weights are defined with respect to Δ, i.e. if you change Δ the fundamental weights change.

Example 2.4. For A_n the fundamental weights are given by:

$$\omega_i = \sum_{j=1}^{i-1} \frac{j(n-i+1)}{n+1}\alpha_j + \sum_{k=i}^{n} \frac{i(n-k+1)}{n+1}\alpha_k$$

for $i = 1, \ldots n$.

Definition 2.25. Let L be a semisimple Lie algebra, with Cartan subalgebra $\mathcal{H}$ and root system Φ. Let $\omega_1, \ldots, \omega_n$ be the fundamental weights with respect to a fixed base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. Define

$$\rho = \sum_{i=1}^{n} \omega_i.$$

We now define another useful action of the Weyl group.

Definition 2.26. Let L be a semisimple Lie algebra, with root system Φ and fixed base Δ. Let E be the Euclidean space spanned by Φ, and $\mathcal{W}$ be the Weyl group of L. Define the **affine action** of $\mathcal{W}$ on E to be $\cdot : \mathcal{W} \times E \to E$ given by

$$\sigma \cdot \gamma = \sigma(\gamma + \rho) - \rho$$

2.5 Structure of A_n

Our algebra of interest is the special linear Lie algebra given by

$$A_n = \{X = (x_{ij}) \in g\ell(n+1, \mathbb{C}) \mid Trace\ X = x_{11} + \cdots + x_{(n+1)(n+1)} = 0\}.$$

We give a description of its root space decomposition.

The set of all diagonal matrices in A_n forms a Cartan subalgebra, which we will denote by $\mathcal{H}$. Define the linear functional $\epsilon_i : \mathcal{H} \to \mathbb{C}$ by $\epsilon_i(M) = m_{ii}$ where $M = (m_{ij}) \in \mathcal{H}$, for $i = 1, \ldots, n+1$.

The roots for A_n can be expressed in terms of the ϵ_i's as follows:

$$\Phi = \{\pm(\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq n+1\}.$$

Define $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \ldots, n$. Then $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a set of simple roots for Φ. Since, $\Phi = \{\pm(\epsilon_i - \epsilon_j) = \pm(\alpha_i + \cdots + \alpha_{j-1}) \mid 1 \leq i < j \leq n+1\}$, we easily see that

$$\Phi^+(\Delta) = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n+1\}$$

and

$$\Phi^-(\Delta) = \{\epsilon_j - \epsilon_i \mid 1 \leq i < j \leq n+1\}.$$

A basis for A_n can be defined in terms of the standard matrix units as follows:

$$X_\alpha = E_{i,j} \qquad \text{for } \alpha = \epsilon_i - \epsilon_j \in \Phi^+(\Delta),$$

$$Y_\alpha = E_{j,i} \qquad \text{for } \alpha = \epsilon_j - \epsilon_i \in \Phi^-(\Delta), \textit{ and}$$

$$H_{\alpha_i} = E_{i,i} - E_{i+1,i+1} \text{ for } i = 1, \ldots, n.$$

For $\alpha \in \Phi^+(\Delta)$ we have $L_\alpha = \mathbb{C}X_\alpha$ and so X_α is a root vector. For $\alpha \in \Phi^-(\Delta)$ we have $L_\alpha = \mathbb{C}Y_\alpha$ and so Y_α is a root vector. The rootspace decomposition for A_n is given by:

$$A_n = \mathcal{H} \bigoplus_{\alpha \in \Phi^+(\Delta)} \mathbb{C}X_\alpha \bigoplus_{\alpha \in \Phi^-(\Delta)} \mathbb{C}Y_\alpha.$$

2.6 Tensor algebras

In this section our goal is to introduce the notion of a tensor algebra. This algebra will be central in the construction of the so called universal enveloping algebra and free Lie algebra.

Definition 2.27. Let V_1 and V_2 be two vector spaces over a field K with basis $\mathcal{B}_{V_1} = \{v_1, \cdots, v_m\}$ and $\mathcal{B}_{V_2} = \{u_1, \cdots, u_n\}$ for V_1 and V_2 respectively. Then the **tensor product** of V_1 with V_2, denoted $V_1 \otimes V_2$ is the vector space having basis:

$$\{v_i \otimes u_j | \; i = 1, \cdots, m, \; j = 1, \cdots, n \; \}$$

where

$$v_i \otimes (\sum_{j=1}^{n} b_j u_j) = \sum_{j=1}^{n} b_j (v_i \otimes u_j)$$
$$(\sum_{i=1}^{m} a_i v_i) \otimes u_j = \sum_{i=1}^{m} a_i (v_i \otimes u_j)$$

for all $v_i \in V_1$, $u_j \in V_2$ and $a_i, b_j \in K$. This definition may be extended to $\otimes^N V_i := V_1 \otimes \cdots \otimes V_N$ and it is called the **N-fold tensor product**. Any $\beta \in \bigotimes^N V_i$ is said to be a **simple tensor** provided $\beta = \beta_1 \otimes \cdots \otimes \beta_N$ where $\beta_i \in V_i$ for $i = 1, \ldots, N$.

Definition 2.28. Let V be a finite dimensional vector space over $\mathbb{C}$ with basis $\{v_1, \ldots, v_n\}$. For $k \in \mathbb{N}$ define $T^0 V = \mathbb{C}$ and $T^k V = \bigotimes^k V$ (the k-fold tensor product of V with itself) for $k \geq 1$. Let $T(V) = \sum_{k=0}^{\infty} \oplus T^k V$ as a vector space. A basis for $T(V)$ is $\{1, v_{i_1} \otimes \cdots \otimes v_{i_k} \mid k \in \mathbb{Z}_{i>0} \; ; \; i_j = 1, \ldots, n\}$. Define a multiplication on the basis elements by juxtaposition and extend linearly:

$$(v_{i_1} \otimes \cdots \otimes v_{i_k})(v_{j_1} \otimes \cdots \otimes v_{j_l}) = v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_{j_1} \otimes \cdots \otimes v_{j_l}$$

With this multiplication $T(V)$ is an associative algebra with 1 and is called the **tensor algebra** on V.

2.7 Universal Enveloping Algebra

The universal enveloping algebra of a Lie algebra is a central object of study in representation theory. In this section we give a brief description of it, and we follow this section with an introduction to representations.

Definition 2.29. Let $\mathcal{A}$ and $\mathcal{B}$ be two associative algebras over $\mathbb{C}$. The map $\rho : \mathcal{A} \to \mathcal{B}$ is an **algebra homomorphism** provided:

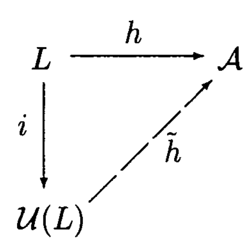
1. $\rho(ax + by) = a\rho(x) + b\rho(y)$, and
2. $\rho(xy) = \rho(x)\rho(y)$

for all $x, y \in \mathcal{A}$ and $a, b \in \mathbb{C}$.

Definition 2.30. A **universal enveloping algebra** of L is a pair $(\mathcal{U}(L), i)$ where $\mathcal{U}(L)$ is an associative algebra with 1, $i : L \to \mathcal{U}(L)$ is a linear map satisfying:

$$i([x, y]) = i(x)i(y) - i(y)i(x)$$

for all $x, y \in L$, and the following holds: for any associative algebra $\mathcal{A}$ with 1 and any Lie algebra homomorphism $h : L \to \mathcal{A}^-$ there is a unique associative algebra homomorphism $\tilde{h} : \mathcal{U}(L) \to \mathcal{A}$ such that $\tilde{h}(1) = 1$ and $h = \tilde{h} \circ i$.



Remark 2.10. We outline the existence and uniqueness of a universal enveloping algebra of L. Construct $T(L)$ using only the vector space structure of L. Let J be the ideal of $T(L)$ generated by $\{x \otimes y - y \otimes x - [x, y] \mid x, y \in L\}$. Let $\mathcal{U}(L) = T(L)/J$. Define $\pi : T(L) \to \mathcal{U}(L)$ be the canonical homomorphism, and $i : L \to \mathcal{U}(L)$ be restriction of π to L. It follows that $(\mathcal{U}(L), i)$ is a universal enveloping algebra of L and is in fact unique.

Proof. See for example Section 17.2 in [7] □

Notation. To simplify notation, when working with the universal enveloping algebra, xy is to be interpreted as $x \otimes y$.

The following Theorem has been specialized to A_n but holds for any Lie algebra.

Theorem 2.6. (Poincaré-Birkhoff-Witt) Let $\mathcal{U}(A_n)$ be the universal enveloping algebra of A_n. Let $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ be an ordered list of the positive and negative root vectors as described in Section 2.5 and $H_1, \ldots, H_n$ be an ordered list of the H_{α_i}'s also described in Section 2.5. Then

$$\{X_1^{m_1} \cdots X_k^{m_k} Y_1^{l_1} \cdots Y_k^{l_k} H_1^{k_1} \cdots H_n^{k_n} \mid m_i, l_i, k_i \in \mathbb{Z}_{\geq 0}\}$$

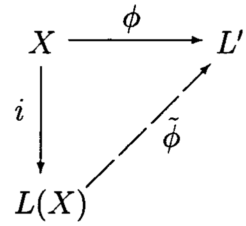
is a basis of $\mathcal{U}(A_n)$.

Proof. See for example Theorem 17.3 in [7] □

2.8 Serre Relations

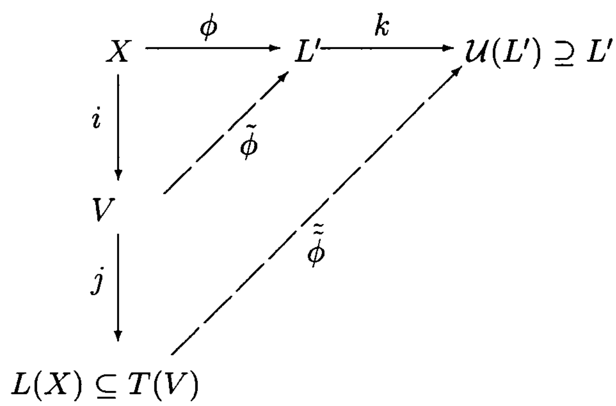
In this section we briefly review the generator/relations realization of any semisimple Lie algebra L as given by Serre. For more details see Section 18 in [7]. This realization provides a computational means to verify whether a map $\rho : L \to gl(V)$ is a Lie algebra homomorphism. In the next section we will see that such a Lie algebra homomorphism will be referred to as a representation.

Definition 2.31. If X is a set then the **free Lie algebra** generated on X consists of a pair $(i, L(X))$ where $L(X)$ is a Lie algebra and $i : X \longrightarrow L(X)$ is a map such that if $\phi : X \longrightarrow L'$ is a map into a Lie algebra L' then there exists a unique Lie algebra homomorphism $\tilde{\phi} : L(X) \longrightarrow L'$ such that $\tilde{\phi} \circ i = \phi$.



This property determines the pair $(i, L(X))$ uniquely up to isomorphism and is known as the **universal mapping property**.

Remark 2.11. We outline the existence and uniqueness of such an algebra. The reader is encouraged to refer to the diagram below while reading through this construction. Let X be a set whose elements form a basis for the vector space V over $\mathbb{C}$. Form the tensor algebra $T(V)$, which when endowed with the bracket operation has a Lie algebra structure. Viewing $T(V)$ in terms of it's Lie algebra structure, we see that $T(V)$ contains the subalgebra generated by X, which we denote by $L(X)$. Given a map $\phi : X \to L'$ where L' is a Lie algebra, define the injection map $i : X \to V$. There exists a unique linear map $\tilde{\phi} : V \to L'$ such that $\tilde{\phi} \circ i = \phi$. Define injection maps $j : V \to T(V)$ and $k : L' \to \mathcal{U}(L')$ respectively. Then there exists a unique associative algebra homomorphism $\tilde{\tilde{\phi}} : T(V) \to \mathcal{U}(L')$ whose restriction to $L(X)$ is a Lie algebra homomorphism. Uniqueness follows from the definition and is easily verified.



Definition 2.32. Let L be a semisimple Lie algbera. The **Cartan matrix** of L is given by

$$\mathcal{C}(L) = (< \alpha_i, \alpha_j >) = \left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \right)$$

where $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ are the simple roots.

Remark 2.12. Taking the root system defined in Section 2.5 we have,

$$\mathcal{C}(A_n) = \begin{pmatrix} 2 & -1 & 0 & & \dots & 0 & 0 \\ -1 & 2 & -1 & & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

Lemma 2.4. The Cartan matrix is independent of the choice of Φ.

Proof. See for example Theorem 10.3 (B) in [7] □

Lemma 2.5. Let L be a semisimple Lie algebra. L is determined up to isomorphism by it's Cartan matrix.

Proof. See for example Proposition 11.1 in [7] □

The following theorem will be crucial in later sections. Since it will be applied directly to the Lie algebra A_n, we state it in terms of this algebra.

Theorem 2.7. (Serre) Let $\mathcal{H}$ be a Cartan subalgebra for A_n, Φ the set of roots of A_n relative to $\mathcal{H}$ with base $\Delta = \{\alpha_1, \dots, \alpha_n\}$. Let $X = \{x_{\alpha_i}, y_{\alpha_i}, h_{\alpha_i} \mid \alpha_i \in \Delta\}$. Then A_n is isomorphic to the free Lie algebra $L(X)$ subject to the following relations:

1. $[h_{\alpha_i}, h_{\alpha_j}] = 0$,
2. $[x_{\alpha_i}, y_{\alpha_j}] - \delta_{ij} h_{\alpha_i} = 0$,
3. $[h_{\alpha_i}, x_{\alpha_j}] - c_{ji} x_{\alpha_j} = 0$,
4. $[h_{\alpha_i}, y_{\alpha_j}] + c_{ji} y_{\alpha_j} = 0$,
5. $(adx_{\alpha_i})^{1-c_{ji}}(x_{\alpha_j}) = 0, \ \ i \neq j$, and
6. $(ady_{\alpha_i})^{1-c_{ji}}(y_{\alpha_j}) = 0, \ \ i \neq j$

where c_{ij} is the (i, j) entry of $\mathcal{C}(A_n)$.

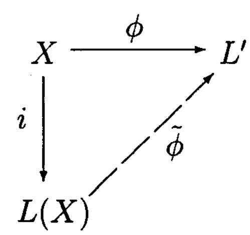
Proof. See for example Theorem 18.3 in [7] □

Definition 2.33. The relations 1 through 6 of Theorem 2.7 are called the **Serre relations** of A_n.

Before closing this section we emphasize our application of Serre's Theorem.

Let $L(X)$ be the free Lie algebra generated by $X = \{x_{\alpha_i}, y_{\alpha_i}, h_{\alpha_i} \mid i = 1, \ldots, n\}$ as in Theorem 2.7. Let I be the ideal of $L(X)$ generated by the elements of $L(X)$ obtained by taking the left hand side of relations 1 through 6. Then $A_n \cong L(X)/I$.

For an arbitrary Lie algebra L' and a map $\phi : X \to L'$ there exists a unique Lie algebra homomorphism $\tilde{\phi} : L(X) \to L'$ given by the commutative diagram:



If $I \subseteq Ker(\tilde{\phi})$, then ϕ determines a unique Lie algebra homomorphism on A_n :

$$\bar{\phi} : A_n = L(X)/I \to L'$$

such that $\bar{\phi}(h_{\alpha_i}) = \phi(h_{\alpha_i})$, $\bar{\phi}(x_{\alpha_i}) = \phi(x_{\alpha_i})$ and $\bar{\phi}(y_{\alpha_i}) = \phi(y_{\alpha_i})$, for $i = 1, \ldots n$.

2.9 Representation theory of semisimple Lie algebras

A representation of a Lie algebra is a special Lie algebra homomorphism. Throughout this section L denotes a semisimple Lie algebra.

Definition 2.34. A **representation of a Lie algebra** L is a pair (ρ, V) where V is a vector space and $\rho : L \longrightarrow gl(V)$ is a Lie algebra homomorphism. In this case, V is called the representation space of ρ.

Example 2.5. Since A_n is a Lie subalgebra of $gl(\mathbb{C}^{n+1}) = gl(n+1, \mathbb{C})$, we see that the injection map:

$$i : A_n \to gl(n+1, \mathbb{C})$$

is a representation of A_n on $\mathbb{C}^{n+1}$. For this reason $\mathbb{C}^{n+1}$ is called the **natural representation space** of A_n, and we denote it by $\mathcal{V}$.

We also have representations of associative algebras.

Definition 2.35. A **representation of an associative algebra** $\mathcal{A}$ is a pair (ρ, V) where V is a vector space and $\rho : \mathcal{A} \to End(V)$ is an associative algebra homomorphism. In this case, V is called the representation space of ρ.

One can view representations from the point of view of modules.

Definition 2.36. A vector space V with an operation $L \times V \longrightarrow V$ (denoted $(x, v) = x.v$) is called a L-**module** if the following conditions are satisfied:

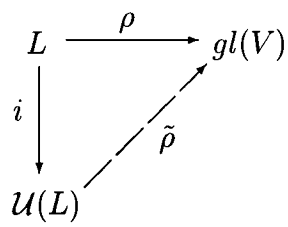
1. $(ax + by).v = a(x.v) + b(y.v)$,
2. $x.(av + bw) = a(x.v) + b(x.w)$, and
3. $[xy].v = x.y.v - y.x.v$

for all $x, y \in L;\ \ v, w \in V$ and $a, b \in \mathbb{C}$.

Remark 2.13. The notions of modules and representations are interchangeable in the following sense. Suppose (ρ, V) is a representation of L. We may view V as an L-module via the action $x.v = \rho(x)(v)$. Clearly, conditions $1, 2$ and 3 are satisfied. Conversely, given a L-module V, define $\rho : L \longrightarrow gl(V)$ by setting $\rho(x)(v) = x.v$. Due to this correspondence we will use the phrases L-module and L-representation interchangeably throughout this work.

Definition 2.37. Let V be an L-module and W be a subspace of V. W is said to be a **sub-module** of V provided $x.w \in W$ for all $x \in L$ and $w \in W$. V is said to be a **simple** L-module provided it has no non-zero proper sub-modules. Simple modules viewed in terms of their representations are said to be **irreducible**. Lastly, V is **completely reducible** provided V is the direct sum of simple L sub-modules.

Remark 2.14. There is a one to one correspondence between representation of L and representations of $\mathcal{U}(L)$: If V is a L-module and (ρ, V) is the associated representation, then we have a commutative diagram:



The existence of $\tilde{\rho}$ turns V into a module for the associative algebra $\mathcal{U}(L)$. Conversely, if V is a module for the associative algebra $\mathcal{U}(L)$ then the existence of the injection map, $i : L \to \mathcal{U}(L)$ turns V into a module for L. Lastly, this one to one correspondence preserves irreducibility as any submodule of V under ρ is a submodule of V under $\tilde{\rho}$ and vice versa.

Definition 2.38. Let V and W be two L-modules. A **homomorphism of L-modules** is a linear map $\psi : V \to W$ such that $\psi(x.v) = x.\psi(v)$ for all $x \in L$ and $v \in V$. When ψ is an isomorphism of vector spaces we call it an **isomorphism of L-modules**.

Theorem 2.8. (Weyl) Let (ρ, V) be a finite dimensional representation of a semisimple Lie algebra. Then V is completely reducible.

Proof. See for example Theorem 6.3 in [7] □

As a result of Weyl's Theorem, for a semisimple Lie algebra L, the study of finite dimensional L-modules reduces to the study of the simple L-modules.

Definition 2.39. Let V be a finite dimensional L-module, $\mathcal{H}$ a fixed Cartan subalgebra of L, $\lambda \in \mathcal{H}^*$ and $V_\lambda = \{v \in V \mid h.v = \lambda(h).v \text{ for all } h \in \mathcal{H}\}$. If $V_\lambda \neq 0$, then V_λ is called a **weight space** of V, λ is called a **weight** of V, and the elements $0 \neq v \in V_\lambda$ are called the **weight vectors**. The **support** of the module V, denoted $\operatorname{Supp} V$ is defined to be

$$\operatorname{Supp} V = \{\lambda \in \mathcal{H}^* \mid V_\lambda \neq 0\}.$$

That is, $\operatorname{Supp} V$ is the set of all linear functionals corresponding to weight spaces in V.

Definition 2.40. An L-module V is said to admit a **weight space decomposition** provided

$$V = \bigoplus_{\lambda \in Supp\ V} V_\lambda.$$

Theorem 2.9. Let V be an arbitrary L-module.

1. If V is finite dimensional then V has at least one weight.
2. If α is a root of L and λ is a weight of V then $L_\alpha V_\lambda \subseteq V_{\lambda+\alpha}$
3. If V is finite dimensional then V admits a weight space decomposition.

Proof. (1) Recall that $\mathcal{H}$ is abelian (Lemma 2.1) and hence solvable. Therefore $\rho(\mathcal{H})$ being the homomorphic image of $\mathcal{H}$ is a solvable subalgebra of $gl(V)$ (Proposition 2.1). Since V is finite dimensional we have by Lie theorem (Theorem 2.1) that there exists a $\lambda \in \mathcal{H}^*$ such that for some $0 \neq v_0 \in V$, $\rho(h)v_0 = \lambda(h)v_0$ for all $h \in \mathcal{H}$.

(2) Take $x \in L_\alpha, v \in V_\lambda$ and $h \in \mathcal{H}$ then

$$h.x.v = x.h.v + [h,x].v = (\lambda(h) + \alpha(h))x.v.$$

(3) By Weyl's Theorem we may assume that V is simple. Let

$$\tilde{V} = \bigoplus_{\lambda \in \mathcal{H}^*} V_\lambda.$$

By part (1) and (2), $\tilde{V}$ is a non-zero L-module. Simplicity of V implies that $V = \tilde{V}$ and therefore V is a weight module. □

Definition 2.41. Let L be a semisimple Lie algebra with root system Φ, with simple roots Δ and positive roots $\Phi^+(\Delta)$. The **integral root lattice**, denoted Q, is defined to be

$$Q = \{\sum_{\alpha \in \Phi} k_\alpha \alpha \mid k_\alpha \in \mathbb{Z}\}.$$

Define

$$Q^+ = \{ \sum_{\alpha \in \Phi^+(\Delta)} k_\alpha \alpha \mid k_\alpha \in \mathbb{Z}_{\geq 0}\}$$

and

$$Q^- = \{ \sum_{\alpha \in \Phi^+(\Delta)} k_\alpha \alpha \mid k_\alpha \in \mathbb{Z}_{\leq 0}\}.$$

Remark 2.15.

$$\mathcal{U}(L) = \bigoplus_{\gamma \in Q} \mathcal{U}(L)_\gamma$$

where $\mathcal{U}(L)_\gamma$ denotes the γ weight space of $\mathcal{U}(L)$ with respect to the adjoint representation of L on $\mathcal{U}(L)$. Also observe that $\mathcal{U}(L)_0 = \{u \in \mathcal{U}(L) \mid [h, u] = 0 \text{ for all } h \in \mathcal{H}\}$. This is exactly the centralizer of $\mathcal{H}$ in $\mathcal{U}(L)$ and hence $\mathcal{U}(L)_0$ is a submodule.

Definition 2.42. Let Φ be a root system of L with base Δ and positive roots $\Phi^+(\Delta)$. Let V be a L-module. A **maximal vector** of weight λ in V is a non-zero weight vector $v^+ \in V_\lambda$ such that $x.v^+ = 0$ for all $x \in L_\alpha$ and all $\alpha \in \Phi^+(\Delta)$.

Definition 2.43. Let v^+ be a maximal vector of weight λ. A L-module is said to be of **highest weight** λ provided it is generated by v^+. That is, V is a L-module of highest weight λ provided $V = \mathcal{U}(L).v^+$ and v^+ is a maximal vector.

Theorem 2.10. Let Φ be a root system of L with base $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. $\Phi^+(\Delta) = \{\beta_1, \ldots, \beta_m\}$ be the positive roots, and $\{x_{\pm\beta_1}, \ldots, x_{\pm\beta_m}\}$ be a fixed set of root vectors. Let V be a L-module with highest weight λ and maximal vector $v^+ \in V_\lambda$. Then:

1. V is spanned by the vectors $x^{i_1}_{-\beta_1} \cdots x^{i_m}_{-\beta_m}.v^+$ where $i_j \in \mathbb{Z}^+$ and x_{β_i} are fixed nonzero root vectors in L_{β_i}.

2. The weights of V are of the form $\mu = \lambda - \sum_{i=1}^{l} k_i \alpha_i$ for $k_i \in \mathbb{Z}^+$.

3. For each $\mu \in \mathcal{H}^*$, V_μ is finite dimensional and V_λ has dimension one.

4. Each submodule of V is the direct sum of its weight spaces.

5. If V is simple then v^+ is the unique maximal vector in V up to a non zero scalar multiple.

6. For every $\lambda \in \mathcal{H}^*$ there exists a unique simple highest weight L-module of weight λ.

Proof. See for example Theorem 20.2, Corollary 20.2 and Theorem 20.3 (A) and (B) in [7] □

As a result of part 6 of the previous Theorem we make the following definition.

Definition 2.44. Let $\mathcal{H}$ be a Cartan subalgebra of L. For each $\lambda \in \mathcal{H}^*$, denote the simple L-module having highest weight λ by $V(\lambda)$.

Proposition 2.7. Let $\mathcal{H}$ be a Cartan subalgebra of L. Let Φ be root system of L with base Δ. Let V be a finite dimensional simple L-module. Then $V = V(\lambda)$ for some $\lambda \in \mathcal{H}^*$.

Proof. Let $\Phi(\Delta)^+$ be the positive roots with respect $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. By Proposition 2.9 , since V is finite dimensional, V admits a weight space decomposition. Also, since V is finite dimensional, we must have that $\operatorname{Supp} V$ is a finite set. If $\lambda_0 \in \operatorname{Supp} V$, then the set

$$\{\lambda_0 + \sum_{i=1}^{n} k_i\alpha_i \in \operatorname{Supp} V \mid k_i \in \mathbb{Z}_{\geq 0} \text{ for each } i\}$$

is also finite. We can therefore choose $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}$ such that

$$\lambda = \lambda_0 + \sum_{i=1}^{n} m_i\alpha_i \in \operatorname{Supp} V$$

and for any sequence $(k_1, \ldots, k_n) \in \mathbb{Z}_{\geq 0}$ with $(k_1, \ldots, k_n) \neq (m_1, \ldots, m_n)$ and $k_i \geq m_i$ for all i, we have

$$\lambda_0 + \sum_{i=1}^{n} k_i\alpha_i \notin \operatorname{Supp} V$$

Let $v^+ \in V_\lambda$ with $v^+ \neq 0$. Let $\beta \in \Phi(\Delta)^+$. Then $\beta = \sum_{i=1}^{n} b_i\alpha_i$ for some $b_i \in \mathbb{Z}_{\geq 0}$. Therefore $x_\beta v^+$ has weight equal to $\lambda_0 + \sum_{i=1}^{n}(m_i + b_i)\alpha_i$. Since $\beta \neq 0$ we have $(m_1 + b_1, \ldots, m_n + b_n) \neq (m_1, \ldots, m_n)$. Further, for each i, $m_i + b_i \geq m_i$ and hence

$$\lambda_0 + \sum_{i=1}^{n}(m_i + b_i)\alpha_i \notin \operatorname{Supp} V$$

Therefore $x_\beta v^+ = 0$, which implies v^+ is a maximal vector. Since the highest weight module generated by v^+ is a submodule of V, and V is simple, we must have that V is itself generated by v^+. Therefore $V = V(\lambda)$. □

Definition 2.45. For a semisimple Lie algebra L with root system Φ with respect to the Cartan subalgebra $\mathcal{H}$, fix a base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ with basis $\{h_{\alpha_1}, \ldots, h_{\alpha_n}\}$ of $\mathcal{H}$. Let $\{\omega_1, \ldots, \omega_n\}$ be the fundamental basis for $\mathcal{H}^*$. A weight which is expressible as a nonnegative integral linear combination of the ω_i's is said to be a **dominant integral weight** or simply **dominant integral**.

Theorem 2.11. Let L be a semisimple Lie algebra, with Cartan subalgebra $\mathcal{H}$. For each $\lambda \in \mathcal{H}^*$, the simple highest weight L-module $V(\lambda)$ is finite dimensional if and only if λ is a dominant integral weight.

Proof. See for example Theorem 21.1 and Theorem 21.2 in [7] □

Corollary 2.1. Let L be a semisimple Lie algebra, with Cartan subalgebra $\mathcal{H}$. Every finite dimensional simple L-module is some $V(\lambda)$ where λ is a dominant integral weight.

Proof. If V is any finite dimensional simple L-module, then by Theorem 2.10 part 7, $V = V(\lambda)$ for some $\lambda \in \mathcal{H}^*$. Due to the previous theorem, λ must be a dominant integral weight. □

3 Mathieu's classification of simple torsion free A_n-modules of finite degree

We now move onto the work of Mathieu [10] who classifies the so called simple torsion free modules for the type A and C Lie algebras. In the next several sections we will be introducing the required background information so that we may begin reviewing Mathieu's classification.

3.1 Admissible Modules

Definition 3.1. Let L be a semisimple Lie algebra, and V a L-module admitting a weight space decomposition. For each $\nu \in \operatorname{Supp} V$, the **multiplicity** of ν in V, denoted $m_V(\nu)$, is the dimension of the ν weight space in V. That is,

$$m_V(\nu) = dim V_\nu.$$

Notice that Theorem 2.10 implies that if V is a λ highest weight module then $m_V(\lambda) = 1$ and for all $\nu \in \operatorname{Supp} V$ we have $m_V(\nu) < \infty$.

Definition 3.2. Let L be a semisimple Lie algebra, and V be a L-module admitting a weight space decomposition. V is **admissible** provided V is infinite dimensional, the set of roots of V are contained in the union of a finite number of Q-cosets, and there exists an $N \in \mathbb{Z}_{\geq 0}$ such that for all $\nu \in \operatorname{Supp} V$, $m_V(\nu) \leq N$.

Definition 3.3. Let L be a semisimple Lie algebra, and V be an admissible L-module. Define the **degree** of V, denoted $\deg V$ by

$$\deg V = \max\{m_V(\nu) \mid \nu \in \operatorname{Supp} V\}$$

Definition 3.4. Let $\mathcal{A}$ be an associative algebra, and V be a submodule of $\mathcal{A}$. An **ascending chain** of submodules is a finite sequence $\mathcal{C} = (W_0, \ldots, W_k)$ consisting of submodules of V such that

$$W_0 \subset W_1 \subset \cdots \subset W_k$$

where all inclusions are proper. The number k is called the length of the ascending chain $\mathcal{C}$, and is denoted by $\mathfrak{l}(\mathcal{C})$.

Definition 3.5. Let $\mathcal{A}$ be an associative algebra, and V be an $\mathcal{A}$-module. Define the **length** of V to be the (possibly infinite) value

$$\text{Length}(V) = \sup\{k \in \mathbb{Z}_{>0} \mid \mathfrak{l}(\mathcal{C}) = k \; for\ some\ ascending\ chain\ \mathcal{C}\ of\ submodules\ of\ V\}$$

Theorem 3.1. (Jordan-Hölder) Let $\mathcal{A}$ be an associative algebra, and V be a submodule of $\mathcal{A}$. If Length$(V) = k < \infty$ then there exists an ascending chain

$$W_0 \subset W_1 \subset \cdots \subset W_k$$

such that $W_0 = (0)$, $W_k = V$ and for each $1 \leq i \leq k$ the module W_i/W_{i-1} is simple. Such a sequence is called a **composition series** of V. Further, if $W_0 \subset \cdots \subset W_k$ and $U_0 \subset \cdots \subset U_k$ are two composition series of V, then the semisimple modules

$$U = \bigoplus_{i=1}^{k} U_i/U_{i-1} \quad \textit{and} \quad W = \bigoplus_{i=1}^{k} W_i/W_{i-1}$$

are equivalent.

Proof. See for example Theorem 3.5 in [8] □

Lemma 3.1. (Mathieu) Let L be a finite dimensional simple Lie algebra, and V be an admissible L-module. Then V has finite length.

Proof. See Lemma 3.3 in [10] □

3.2 Torsion Free Modules

Definition 3.6. Let L be a semisimple Lie algebra. An L-module V is said to be **torsion free** provided it has a weight space decomposition with respect to a Cartan subalgebra $\mathcal{H}$ of L, and the root vectors of L act injectively on V.

Proposition 3.1. (Fernando) Let L be a semisimple Lie algebra with Cartan subalgebra $\mathcal{H}$, and V be a simple L-module admitting a weight space decomposition. Then V is torsion free if and only if $\operatorname{Supp} V = \lambda + Q$ for some $\lambda \in \mathcal{H}^*$.

Proof. See for example Corollary 1.4 in [10] □

Naturally, torsion free modules are infinite dimensional. Using the above Proposition we show that for a simple torsion free module of finite degree every weight space has the same dimension. That is, simple torsion free modules of finite degree are admissible.

Proposition 3.2. Let L be a semisimple Lie algebra, and V be a simple torsion free L-module of finite degree. Then there exists an $N \in \mathbb{Z}_{\geq 0}$ such that $\dim V_\nu = N$ for all $\nu \in \operatorname{Supp} V$. In particular, V is admissible.

Proof. Let $\mathcal{H}$ be a Cartan subalgebra of L, and let Φ be the root system of L with respect to $\mathcal{H}$. Let $\Phi^+(\Delta) = \{\beta_1, \dots, \beta_m\}$ be the set of positive roots. Let $\phi : L \to gl(V)$ be the map defining the action of L on V. By the previous proposition, we have that $\operatorname{Supp} V = \lambda + Q$ for some $\lambda \in \mathcal{H}^*$. Let $\nu, \gamma \in \operatorname{Supp} V$. Then $\gamma - \nu \in Q$, and hence

$$\gamma = \nu + \sum_{i=1}^{m} k_i\beta_i - \sum_{j=1}^{m} l_i\beta_i$$

for some $k_1, \dots, k_m, l_1, \dots, l_m \in \mathbb{Z}_{\geq 0}$. Set

$$\sigma = \phi(x_{\beta_1})^{k_1} \dots \phi(x_{\beta_m})^{k_m} \phi(x_{-\beta_1})^{l_1} \dots \phi(x_{-\beta_m})^{l_m}$$

then $\sigma \in gl(V)$ is an injective linear map. Further, for any $v \in V_\nu$ we have that $\sigma(v) \in V_\gamma$. We can therefore find a injective linear map between any two weight spaces of V. Thus all weight spaces of V must have the same dimension. Since V is assumed to have finite degree, we have our result. □

The reader is encouraged to pay close attention to the following example, as it will be used in later sections to motivate our methods of constructing certain torsion free A_n-modules.

Example 3.1. Let $V = Span_{\mathbb{C}}\{x_1^{a_1} \cdots x_{n+1}^{a_{n+1}} \mid a_i \in \mathbb{C}\}$. V is an A_n-module which contains submodules of interest to us. Rather then viewing the module action on V in terms of the operators E_{ij}, we will view the module action in terms of the operators $x_i\partial_j$, where x_i acts on V as multiplication by x_i and ∂_j acts on V by partial differentiation with respect to x_j. This is justified by the algebra homomorphism given by

$$\phi : gl(n+1, \mathbb{C}) \longrightarrow End_{\mathbb{C}}(V) \text{ where } \phi(E_{ij}) = x_i\partial_j$$

Let $k \in \mathbb{Z}_{\geq 0}$ and $\bar{k} = (k, 0, \dots, 0) \in \mathbb{C}^{n+1}$. Then

$$M(\bar{k}) = Span_{\mathbb{C}}\{x_1^{k-l_1} x_2^{l_1-l_2} \cdots x_{n+1}^{l_n} \mid 0 \leq l_n \leq l_{n-1} \cdots \leq l_1 \leq k\} \cong V(k\omega_1).$$

V also contains simple torsion free sub-modules of degree one. Fix $\bar{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1}$ with each $a_i \in \mathbb{C} \setminus \mathbb{Z}$. Then

$$M(\bar{a}) = Span_{\mathbb{C}}\{x_1^{a_1+k_1} \cdots x_{n+1}^{a_{n+1}+k_{n+1}} \mid k_i \in \mathbb{Z} \text{ and } \sum_{i=1}^{n+1} k_i = 0\}$$

is a simple torsion free module having all one dimensional weight spaces.

Proof. See Example 1.4 in [2] □

The following theorem will be important in later sections.

Theorem 3.2. Every simple torsion free A_n-module of degree one is isomorphic to $M(\bar{a})$ for some choice of $\bar{a} = (a_1, \ldots, a_{n+1})$ with $a_i \in \mathbb{C} \setminus \mathbb{Z}$.

Proof. See main result in [4] □

3.3 The Central Character

Proposition 3.3. (Schur's Lemma) Let L be a semisimple Lie algebra, and V be a simple L-module with action given by $\phi : L \to gl(V)$. If $\pi \in gl(V)$ such that $[\pi, \phi(x)] = 0$ for all $x \in L$, then there exists a $c \in \mathbb{C}$ such that $\pi(v) = cv$ for all $v \in V$. i.e. π acts as multiplication by some scalar.

Proof. See for example Lemma 6.1 in [7] □

Definition 3.7. Let L be a Lie algebra, and $\mathcal{U}(L)$ be the universal enveloping algebra of L. The **centre** of $\mathcal{U}(L)$, denoted $Z(\mathcal{U}(L))$ is defined to be

$$Z(\mathcal{U}(L)) = \{z \in \mathcal{U}(L) \mid xz - zx = 0 \; for \; all \; x \in \mathcal{U}(L)\}$$

Definition 3.8. Let L be a semisimple Lie algebra, and $Z(\mathcal{U}(L))$ be the centre of the universal enveloping algebra of L. An algebra homomorphism $\chi : Z(\mathcal{U}(L)) \to \mathbb{C}$ is called a **central character**. If M is a $\mathcal{U}(L) - module$ with the property that there exists a central character χ_M for which $zu = \chi_M(z)u$ for all $z \in Z(\mathcal{U}(L))$ and all $u \in M$, then M is said to **admit a central character**, and χ_M is called the **central character of** M.

Proposition 3.4. Let L be a semisimple Lie algebra. Let V be a simple L-module, then V admits a central character.

Proof. Suppose the action of V on L is given by the map $\phi : L \to gl(V)$. Let $z \in Z(\mathcal{U}(L))$. Then for any $x \in L$, we have

$$[\phi(z), \phi(x)] = \phi(z)\phi(x) - \phi(x)\phi(z) = \phi(zx - xz) = \phi(0) = 0.$$

Then by Schur's lemma, we have that for each $z \in Z(\mathcal{U}(L))$ $\phi(z)(v) = c_z v$ for some $c_z \in \mathbb{C}$ and all $v \in V$. Define $\chi : Z(\mathcal{U}(L)) \to \mathbb{C}$ by $\chi(z) = c_z$. Clearly, since ϕ is an algebra homomorphism, we have that χ is an algebra homomorphism. Hence χ is the central character of V. □

Corollary 3.1. Let L be a semisimple Lie algebra, with Cartan subalgebra $\mathcal{H}$. Then for any $\lambda \in \mathcal{H}^*$ the simple highest weight module $V(\lambda)$ admits a central character, which we will denote by χ_λ.

Proof. By Proposition 3.4, since $V(\lambda)$ is simple, it admits a central character. □

Theorem 3.3. (Harish-Chandra) Let L be a semisimple Lie algebra with Cartan subalgebra $\mathcal{H}$ and Weyl group $\mathcal{W}$. Let $\lambda, \mu \in \mathcal{H}^*$. Then $\chi_\lambda = \chi_\mu$ if and only if there exists $\sigma \in \mathcal{W}$ such that $\sigma(\lambda + \rho) - \rho = \mu$.

Proof. See Theorem 23.3 in [7] □

Proposition 3.5. Let L be a semisimple Lie algebra with Cartan subalgebra $\mathcal{H}$, and $Z(\mathcal{U}(L))$ be the centre of the universal enveloping algebra of L. If $\chi : Z(\mathcal{U}(L)) \to \mathbb{C}$ is an algebra homomorphism then $\chi = \chi_\lambda$ for some $\lambda \in \mathcal{H}^*$.

Proof. See for example Proposition 7.4.8 in [5] □

3.4 Coherent Families

Fernando [6] showed that the only finite dimensional simple Lie algebras which admit torsion free modules of finite degree are the Lie algebras of type A and type C. The work of Mathieu [10] classifies the simple torsion free modules of finite degree occurring in type A and type C Lie algebras. In the next section we will restrict ourselves to Mathieu's classification of the simple torsion free modules of finite degree for type A Lie algebras. Mathieu's classification requires the notion of a semisimple irreducible coherent family, and it will be the aim of this section to introduce such a concept.

We remind that reader that as defined in section 2.7, $\mathcal{U}(L)$ denotes the universal enveloping algebra of L and $\mathcal{U}(L)_0$ denotes the zero weight space of $\mathcal{U}(L)$ with respect to the adjoint action of the Cartan subalgebra $\mathcal{H}$.

Definition 3.9. Let L be a finite dimensional simple Lie algebra with Cartan subalgebra $\mathcal{H}$. A **coherent family** $\mathcal{M}$ is an admissible L-module of degree d such that

1. $\operatorname{Supp}\mathcal{M} = \mathcal{H}^*$;
2. $\dim \mathcal{M}_\lambda = d$ for all $\lambda \in \mathcal{H}^*$; and
3. for any $u \in \mathcal{U}(L)_0$ there exists a polynomial $p(x)$ such that $p(\lambda) = Tr\, u|_{\mathcal{M}_\lambda}$ for all $\lambda \in \mathcal{H}^*$.

We say $\mathcal{M}$ is **irreducible** provided there exists a $\lambda \in \mathcal{H}^*$ such that the $\mathcal{U}(L)_0$ module $\mathcal{M}_\lambda$ is simple.

Definition 3.10. Let L be a finite dimensional simple Lie algebra with Cartan subalgebra $\mathcal{H}$ and root system Φ. Let Q be the integral root lattice with respect to Φ. Let $\mathcal{M}$ be a coherent family of L. Then for $\mu \in \mathcal{H}^*$

$$\mathcal{M}[\mu] := \textstyle\sum_{\nu \in \mu + Q} \mathcal{M}_\nu.$$

Definition 3.11. A coherent family $\mathcal{M}$ of L is said to be **semisimple** provided for each $\mu \in \mathcal{H}^*$, the module $\mathcal{M}[\mu]$ is semisimple.

Lemma 3.2. (Mathieu) Let L be a finite dimensional simple Lie algebra, and V be a simple admissible L-module with degree d. Then the following hold:

1. there exists a unique semisimple irreducible coherent family $\mathcal{M}$ of degree d such that V is a submodule of $\mathcal{M}$;

2. if V' is any infinite dimensional submodule of $\mathcal{M}$ then V' is admissible, and $\deg V' = d$; and

3. all simple submodules of $\mathcal{M}$ have the same central character.

Proof. See Proposition 4.8 in [10] □

3.5 Classification of coherent families for $sl(n+1)$

We will be restricting ourselves to Mathieu's classification of all simple torsion free A_n-modules of finite degree. Recall in section 2.3 Φ denotes a root system for A_n with base Δ and E stands for the Euclidean space spanned by Φ. In section 2.4 we define $\{\omega_1, \dots, \omega_n\}$ to be the fundamental basis for $\mathcal{H}^*$ with $\rho = \sum_{i=1}^{n} \omega_i$. $\mathcal{W}$ will be the Weyl group of A_n, and for any $\sigma \in \mathcal{W}$ and $\gamma \in E$ $\sigma \cdot \gamma$ denotes the affine action. From section 2.5 $\mathcal{H}$ denotes a Cartan subalgebra of $A_n = sl(n+1)$ with basis given by $\{h_1, \dots, h_n\}$. For $x, y \in \mathbb{C}$ $x \succ y$ means that $x - y \in \mathbb{Z}_{>0}$ and $x \not\succ y$ will indicate that $x - y \notin \mathbb{Z}_{>0}$. Let $P = \{\lambda \in \mathcal{H}^* \mid \lambda(h_i) \in \mathbb{Z} \text{ for } i = 1, \dots, n\}$ and $P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for } i = 1, \dots, n\}$.

Lemma 3.3. Let $V(\lambda)$ be an admissible λ-highest weight A_n module, and let $A = \{i \mid (\lambda + \rho)(h_i) \notin \mathbb{Z}_{>0}\}$. Then one of the following three assertions holds:

1. $A = \{1\}$ or $A = \{n\}$.

2. $A = \{i\}$ for some $1 < i < n$ and $(\lambda+\rho)(h_{i-1} + h_i) \in \mathbb{Z}_{>0}$ or $(\lambda+\rho)(h_i + h_{i+1}) \in \mathbb{Z}_{>0}$.

3. $A = \{i, i+1\}$ for some $1 \leq i < n$ and $(\lambda + \rho)(h_i + h_{i+1}) \in \mathbb{Z}_{>0}$.

Proof. See Lemma 8.1 in [10] □

Definition 3.12. A k-tuple $m = (m_1, \dots, m_k) \in \mathbb{C}^k$ is called **ordered** if $m_i \succ m_{i+1}$, i.e. $m_i - m_{i+1} \in \mathbb{Z}_{>0}$ for $i = 1, \dots, k-1$.

Remark 3.1. Notice that if $m = (m_1, \dots, m_k)$ is an ordered sequence then $m_i - m_j \in \mathbb{Z}_{>0}$ provided $i < j$, which implies $m_i - m_j \in \mathbb{Z}$ for any i, j.

Definition 3.13. An **$sl(n+1)$-sequence** is a $n+1$-tuple $m = (m_1, \dots, m_{n+1}) \in \mathbb{C}^{n+1}$ such that $\sum_{i=1}^{n+1} m_i = 0$.

Notation. Let $\mathcal{P}$ be the set of all $sl(n+1)$-sequences which are not ordered but become ordered after removing one term. Let $\mathcal{P}^+$ be the set of all sequences in $\mathcal{P}$ which become ordered by removing the first term. Let $\mathcal{P}^-$ be the set of all sequences in $\mathcal{P}$ which become ordered by removing the last term.

Proposition 3.6. A weight $\lambda \in \mathcal{H}^*$ of $sl(n+1)$ can be associated bijectively with the $sl(n+1)$ sequence $m(\lambda) = (m_1(\lambda), \dots, m_{n+1}(\lambda))$ where $(\lambda+\rho)(h_i) = m_i(\lambda) - m_{i+1}(\lambda)$ for $i = 1, \dots, n$ and $\sum_{i=1}^{n+1} m_i(\lambda) = 0$.

Proof. Let $\lambda \in \mathcal{H}^*$ be a weight of $sl(n+1)$. Then there exists a unique $sl(n+1)$ sequence determined by

$$(\lambda+\rho)(h_i) = m_i(\lambda) - m_{i+1}(\lambda) \text{ for } i = 1, \dots, n \text{ and } \sum_{i=1}^{n+1} m_i(\lambda) = 0$$

After setting each $m_i(\lambda)$ to m_i, these conditions create $n+1$ equations in $n+1$ unknowns given by

$$\begin{pmatrix} 1 & -1 & 0 & & \dots & 0 & 0 \\ 0 & 1 & -1 & & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & -1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \\ m_{n+1} \end{pmatrix} = \begin{pmatrix} (\lambda+\rho)(h_1) \\ (\lambda+\rho)(h_2) \\ \vdots \\ (\lambda+\rho)(h_n) \\ 0 \end{pmatrix} \quad (1)$$

Since

$$\begin{pmatrix} 1 & -1 & 0 & & \dots & 0 & 0 \\ 0 & 1 & -1 & & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & -1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{pmatrix} \overset{\text{row reduce}}{\rightsquigarrow} \begin{pmatrix} 1 & -1 & 0 & & \dots & 0 & 0 \\ 0 & 1 & -1 & & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 & n+1 \end{pmatrix}$$

the matrix has non-zero determinant and is therefore invertible which implies there exists a unique solution $(m_1, \dots, m_{n+1})$, which is necessarily an $sl(n+1)$-sequence.

Conversely given any $sl(n+1)$-sequence it can be associated to the weight $\lambda \in \mathcal{H}^*$ by setting $\lambda = \sum_{i=1}^{n}(m_i - m_{i+1})\omega_i - \rho$. □

Remark 3.2. Let $\lambda = \sum_{i=1}^{n} m_i\omega_i$ with corresponding $sl(n+1)$-sequence $m(\lambda) = (m_1(\lambda), \ldots, m_{n+1}(\lambda))$. Solving equation (2) in the proof of Proposition 3.6 one finds that for $i = 1, \ldots, n+1$

$$m_i(\lambda) = \sum_{j=1}^{i-1} \frac{-j}{n+1} m_j + \sum_{k=i}^{n} \frac{n-k+1}{n+1} m_k + \frac{n}{2} - (i-1).$$

Note that this sequence is not in general ordered. However, when λ is dominant integral, λ corresponds to an ordered $sl(n+1)$-sequence.

Proposition 3.7. Let λ be a weight for A_n. $m(\lambda) \in \mathcal{P}$ if and only if λ satisfies one of the three conditions in Lemma 3.3.

Proof. Let $A = \{i \mid (\lambda+\rho)(h_i) \notin \mathbb{Z}_{>0}\}$ and $m(\lambda) = (m_1, \ldots, m_{n+1}) \in \mathcal{P}$. Recall by Proposition 3.6 that $(\lambda+\rho)(h_i) = m_i - m_{i+1}$ for $i = 1, \ldots, n$.

$$\begin{aligned} A = \{1\} &\iff (\lambda+\rho)(h_1) \notin \mathbb{Z}_{>0} \text{ and } (\lambda+\rho)(h_i) \in \mathbb{Z}_{>0} \text{ for } i = 2, \ldots, n \\ &\iff m_1 - m_2 \notin \mathbb{Z}_{>0} \text{ and } m_i - m_{i+1} \in \mathbb{Z}_{>0} \text{ for } i = 2, \ldots, n \\ &\iff m(\lambda) \text{ with } m_1 \text{ removed is ordered.} \end{aligned}$$

$$\begin{aligned} A = \{n\} &\iff (\lambda+\rho)(h_n) \notin \mathbb{Z}_{>0} \text{ and } (\lambda+\rho)(h_i) \in \mathbb{Z}_{>0} \text{ for } i = 1, \ldots, n-1 \\ &\iff m_n - m_{n+1} \notin \mathbb{Z}_{>0} \text{ and } m_i - m_{i+1} \in \mathbb{Z}_{>0} \text{ for } i = 1, \ldots, n-1 \\ &\iff m(\lambda) \text{ with } m_{n+1} \text{ removed is ordered.} \end{aligned}$$

For some $1 \leq i < n$

$$
\begin{aligned}
A = \{i, i+1\} \text{ and } (\lambda+\rho)(h_i + h_{i+1}) \in \mathbb{Z}_{>0} \\
\iff & (\lambda+\rho)(h_i) \notin \mathbb{Z}_{>0}\ , (\lambda+\rho)(h_{i+1}) \notin \mathbb{Z}_{>0}\ , \\
& (\lambda+\rho)(h_j) \in \mathbb{Z}_{>0} \text{ for } j \neq i, i+1 \text{ and} \\
& (\lambda+\rho)(h_i + h_{i+1}) \in \mathbb{Z}_{>0} \\
\iff & m_i - m_{i+1}, m_{i+1} - m_{i+2} \notin \mathbb{Z}_{>0}, m_j - m_{j+1} \in \mathbb{Z}_{>0} \\
& \text{for } j \neq i, i+1 \text{ and } m_i - m_{i+2} \in \mathbb{Z}_{>0} \\
\iff & m(\lambda) \text{ with } m_{i+1} \text{ removed is ordered}
\end{aligned}
$$

For some $1 < i < n$

$$
\begin{aligned}
A = \{i\} \text{ and } (\lambda+\rho)(h_{i-1} + h_i) \in \mathbb{Z}_{>0} \textit{ or } (\lambda+\rho)(h_i + h_{i+1}) \in \mathbb{Z}_{>0} \\
\iff & m_i - m_{i+1} \notin \mathbb{Z}_{>0}, m_j - m_{j+1} \in \mathbb{Z}_{>0} \text{ for } j \neq i \text{ and} \\
& m_{i-1} - m_{i+1} \in \mathbb{Z}_{>0} \text{ or } m_i - m_{i+2} \in \mathbb{Z}_{>0} \\
\iff & (m_i - m_{i+1} \notin \mathbb{Z}_{>0}, m_j - m_{j+1} \in \mathbb{Z}_{>0} \text{ for } j \neq i \text{ and} \\
& m_{i-1} - m_{i+1} \in \mathbb{Z}_{>0}) \quad \text{or} \quad (m_i - m_{i+1} \notin \mathbb{Z}_{>0}, m_j - m_{j+1} \in \mathbb{Z}_{>0} \\
& \text{for } j \neq i \text{ and } m_i - m_{i+2} \in \mathbb{Z}_{>0}) \\
\iff & m(\lambda) \text{ with } m_i \text{ removed is ordered or } m(\lambda) \text{ with } m_{i+1} \text{ removed is} \\
& \text{ordered.}
\end{aligned}
$$

Therefore one of the three assertions in Lemma 3.3 hold if and only if $m(\lambda)$ becomes ordered after eliminating one term if and only if $m(\lambda) \in \mathcal{P}$. □

Proposition 3.8. Let λ be a weight for A_n. $V(\lambda)$ is admissible if and only if $m(\lambda) \in \mathcal{P}$.

Proof. See Proposition 8.4 in [10] □

Definition 3.14. The action of the Weyl group $\mathcal{W} \simeq S_{n+1}$ on elements in $\mathcal{P}$ is defined by

$$\sigma(m_1(\lambda), \ldots, m_{n+1}(\lambda)) = (m_{\sigma(1)}(\lambda), \ldots, m_{\sigma(n+1)}(\lambda))$$

for all $\sigma \in S_{n+1}$ and $m(\lambda) = (m_1(\lambda), \cdots, m_{n+1}(\lambda)) \in \mathcal{P}$. Take $m(\lambda) \in \mathcal{P}$. The S_{n+1} **orbit of** $m(\lambda)$ is the set defined by

$$\mathcal{W}(m(\lambda)) = \{\sigma m(\lambda) \mid \sigma \in S_{n+1}\}.$$

Definition 3.15. Let $\lambda \in \mathcal{H}^*$ be an admissible weight for $sl(n+1)$ with corresponding $sl(n+1)$-sequence $m(\lambda)$. The **central character associated with the weight** λ is denoted by $\chi(\lambda)$ and defined to be

$$\chi(\lambda) = \mathcal{W}(m(\lambda)) \bigcap \mathcal{P}.$$

Definition 3.16. Let $\lambda \in \mathcal{H}^*$ be an admissible weight for $sl(n+1)$. $\chi(\lambda)$ will denote the central character of $\mathcal{P}$ associated with the weight λ. Take $m(\lambda) = (m_1, \ldots, m_{n+1}) \in \chi(\lambda)$. $m(\lambda)$ is said to be

1. **integral** provided $m_i - m_j \in \mathbb{Z}$ for all $1 \leq i, j \leq n+1$.

2. **non-integral** provided there exists indices i, j such that $m_i - m_j \notin \mathbb{Z}$.

3. **regular** provided $m_i - m_j \neq 0$ for all $1 \leq i \neq j \leq n+1$.

4. **singular** provided there exists distinct indices i, j such that $m_i - m_j = 0$.

Remark 3.3. Notice that if $m(\lambda)$ is integral (respectively non-integral, regular or singular) then all the elements in $\mathcal{W}(m(\lambda))$ are integral (respectively non-integral, regular or singular). For this reason we often refer to the set $\chi(\lambda)$ as being integral, non-integral, regular or singular.

Proposition 3.9. Let $\lambda \in \mathcal{H}^*$ be a weight for $sl(n+1)$ with $m(\lambda) = (m_1, \ldots, m_{n+1}) \in \chi(\lambda)$.

1. If $m(\lambda)$ is singular then $m(\lambda)$ is integral and there are exactly two distinct indices i, j such that $m_i = m_j$.

2. If $m(\lambda)$ is non-integral then $m(\lambda)$ is regular and if $n \neq 1$ there exists a unique index i such that $m_j - m_k \in \mathbb{Z}$ for all $j \neq i \neq k$.

Proof. Suppose $m(\lambda)$ is singular. Therefore there exists two distinct indices i, j such that $m_i = m_j$. Since $m(\lambda) \in \mathcal{P}$ by removing m_i or m_j the resulting subsequences must be ordered. Without loss of generality assume m_i is eliminated, then $m_1 \succ \cdots \succ m_{i-1} \succ m_{i+1} \succ \cdots \succ m_j \succ \cdots \succ m_{n+1}$ and therefore $m_k - m_l \in \mathbb{Z}$ for all $1 \leq k \neq i \neq l \leq n+1$. Since $m_i = m_j$ we have $m_k - m_l \in \mathbb{Z}$ for all $1 \leq k, l \leq n+1$ and so $m(\lambda)$ is integral.

Now we need to show there is a unique set of distinct indices i, j such that $m_i = m_j$. Suppose there are three terms in $m(\lambda)$ which are equal. Without loss of generality suppose $m_i = m_j = m_k$ with $i < j < k$. Since $m(\lambda) \in \mathcal{P}$ it must be the case that eliminating one of these three terms will result in an ordered subsequence. Without loss of generality suppose we eliminate m_i. By Remark 3.1 $0 = m_j - m_k \in \mathbb{Z}_{>0}$, which is a contradiction. Therefore we cannot have three terms equal in $m(\lambda)$. Also if we had four distinct indices i, j, k, l such that $m_i = m_j \neq m_k = m_l$ then more then one term would need to be eliminated in order for an ordered subsequence to result. Therefore, there exists a unique set of distinct indices i, j such that $m_i = m_j$.

Moving onto part 2, suppose that $m(\lambda)$ is non-integral. Therefore there exists distinct indices i, j such that $m_i - m_j \notin \mathbb{Z}$. Since $m(\lambda) \in \mathcal{P}$ we must eliminate m_i or m_j. Without loss of generality assume that m_i must be eliminated. Therefore $m_1 \succ m_2 \succ \cdots \succ m_{i-1} \succ m_{i+1} \succ \cdots \succ m_{n+1}$ which implies $m_k - m_l \in \mathbb{Z}_{>0}$ for all $k \neq i \neq l$ with $k < l$. Therefore $m_k \neq m_l$ for all distinct indices k, l with $k \neq i \neq l$. By assumption we also have $m_k \neq m_i$ for all $k \neq i$. Therefore $m_k \neq m_l$ for any distinct indices k, l which implies that $m(\lambda)$ is regular.

Now we need to show there exists a unique index r such that $m_i - m_j \in \mathbb{Z}$ for all $i \neq r \neq j$. $m(\lambda)$ is non-integral and therefore there exists distinct indices r, s such that $m_r - m_s \notin \mathbb{Z}$. Since $m(\lambda) \in \mathcal{P}$ eliminating one of m_r or m_s will result in an ordered subsequence. Without loss of generality assume we eliminate m_r, and therefore, $m_i - m_j \in \mathbb{Z}$ for all $i \neq r \neq j$. Suppose there exists an index $r' \neq r$ such that $m_i - m_j \in \mathbb{Z}$ for all $i \neq r' \neq j$. Therefore we know that $m_i - m_j \in \mathbb{Z}$ for $i \neq r \neq j$ and $m_i - m_j \in \mathbb{Z}$ for $i \neq r' \neq j$. In particular, for an $i \neq r, r'$, $m_r - m_i \in \mathbb{Z}$ and $m_i - m_{r'} \in \mathbb{Z}$ which implies $m_r - m_{r'} \in \mathbb{Z}$. But then we have $m_i - m_j \in \mathbb{Z}$ for any i, j and therefore, $m(\lambda)$ is integral. This contradiction implies that there exists a unique index r such that $m_i - m_j \in \mathbb{Z}$ for any $i \neq r \neq j$.

□

Proposition 3.10. The integral regular, non-integral regular and singular integral elements in $\mathcal{P}$ partition $\mathcal{P}$.

Proof. By the above proposition we see that any element in $\mathcal{P}$ is either integral and regular, non-integral and regular or singular and integral. We now have exactly three types of central characters occurring in $\mathcal{P}$. By definition any element in $\mathcal{P}$ cannot be singular and regular. Also, if an element in $\mathcal{P}$ were non-integral it could not be singular as singular implies integral. Therefore the integral regular, non-integral regular and singular integral characters occurring in $\mathcal{P}$ must partition $\mathcal{P}$. □

Definition 3.17. Let m and m' be two distinct elements in $\mathcal{P}$. There is an **oriented edge** from m to m', denoted $m \to m'$, provided there is an index i such that $m_i - m_{i+1} \notin \mathbb{Z}_{>0}$ and $m' = s_i m$, where s_i is the transposition interchanging position i and position $i+1$. If in addition $m_i - m_{i+1} \notin \mathbb{Z}$ there will also be an oriented edge from m' to m and we write $m \leftrightarrow m'$. A **connected component** is a set of elements in $\mathcal{P}$ such that for any two elements in the set say v and v' there exists a sequence of vertices $v = v_1, v_2, \ldots, v_k = v'$ such that v_i and v_{i+1} are joined by an oriented edge.

Remark 3.4. We now focus our attention on the central characters in $\mathcal{P}$ which are non-integral. Let $\chi(\lambda)$ be the central character of $\mathcal{P}$ associated with the weight λ. Take $m(\lambda) \in \chi(\lambda)$ such that $m(\lambda) = (m_1, m_2, \ldots, m_{n+1})$ with i being the unique index such that $m_j - m_k \in \mathbb{Z}$ for $j \neq i \neq k$. Let s_i be the transposition which exchanges position i and position $i+1$. Define $c_{ik} = s_k s_{k+1} \ldots s_{i-1}$ for $k < i$ and $c_{ik} = s_{k-1} s_k \ldots s_i$ for $i < k$. Set $\chi(i) = m(\lambda)$ and $\chi(k) = c_{ik} m(\lambda)$ for $1 \leq k \neq i \leq n+1$. Notice by definition of $\chi(k)$ removing the k^{th} term in $\chi(k)$ results in an ordered $sl(n+1)$-

sequence. Later we will show that $\chi(1), \ldots, \chi(n+1)$ form a complete list of the non-integral central characters appearing in $\mathcal{P}$ which are associated to the weight λ.

Example 3.2. Let (m_1, m_2, m_3) be an ordered sequence and suppose $m' - m_i \notin \mathbb{Z}$ for all $i = 1, 2, 3$.

Now, $(1,2)(m', m_1, m_2, m_3) = (m_1, m', m_2, m_3)$, therefore,

$$(m', m_1, m_2, m_3) \leftrightarrow (m_1, m', m_2, m_3).$$

Similarly,

$$(m_1, m', m_2, m_3) \leftrightarrow (m_1, m_2, m', m_3) \leftrightarrow (m_1, m_2, m_3, m').$$

Therefore

$$(m', m_1, m_2, m_3), (m_1, m', m_2, m_3), (m_1, m_2, m', m_3), \text{ and } (m_1, m_2, m_3, m')$$

make up the connected component all corresponding to the same non-integral central character.

Lemma 3.4. Let $m(\lambda) = (m_1, \ldots, m_{n+1})$ be a $sl(n+1)$-sequence in $\mathcal{P}$ such that $m_j - m_k \in \mathbb{Z}$ for $j \neq i \neq k$. Define $\chi(i) = m(\lambda)$ and $\chi(k) = c_{ik}m(\lambda)$ for $1 \leq k \neq i \leq n+1$. Let $\chi(\lambda)$ be the non-integral central character occurring in $\mathcal{P}$ associated with the weight λ. Then $\chi(\lambda)$ consists of exactly $n+1$ elements which form the connected component defined as follows:

$$\chi(1) \leftrightarrow \chi(2) \leftrightarrow \cdots \chi(i) \leftrightarrow \chi(i+1) \leftrightarrow \cdots \chi(n+1).$$

Moreover $\chi(1) \in \mathcal{P}^+$ and $\chi(n+1) \in \mathcal{P}^-$.

Proof. See Lemma 8.3 in [10] □

Definition 3.18. Let $\mathcal{M}$ be a semisimple irreducible coherent family. $m(\mathcal{M})$ is defined to be the set of all $sl(n+1)$-sequences $m(\lambda)$ such that $\lambda \notin P^+$ and $V(\lambda)$ is a submodule of $\mathcal{M}$.

Remark 3.5. Notice that by Proposition 3.8 each $m(\lambda)$ in $m(\mathcal{M})$ must be in $\mathcal{P}$ as $m(\lambda)$ corresponds to the simple admissible λ highest weight module. Each element in $m(\mathcal{M})$ corresponds to a simple admissible highest weight submodule of $\mathcal{M}$. By Lemma 3.2 part 4 all these submodules have the same central character. As a result the elements in $m(\mathcal{M})$ can either be all integral and regular, non-integral and regular or singular and integral. The next Theorem establishes a correlation between the elements in $m(\mathcal{M})$ and the connected components.

Theorem 3.4. Let $\mathcal{M}$ be an irreducible semi-simple coherent family.

1. $m(\mathcal{M})$ contains exactly one connected component.

2. There is a bijection between the set of irreducible semisimple coherent families and the set of connected components of $\mathcal{P}$.

Proof. See Theorem 8.6 in [10] □

Remark 3.6. By Theorem 3.4 and Lemma 3.4, if $m(\mathcal{M})$ consists of non-integral central characters then there are exactly $n+1$ simple admissible highest weight modules which occur as submodules in $\mathcal{M}$. All of these submodules have the same non-integral central character. Furthermore, each of these submodules corresponds to a unique $\chi(k)$ in the connected component $\chi(1) \leftrightarrow \chi(2) \leftrightarrow \cdots \chi(i) \leftrightarrow \chi(i+1) \leftrightarrow \cdots \chi(n+1)$. By part 2 of Theorem 3.4 each connected component uniquely determines the semi-simple irreducible coherent family which the $n+1$ simple admissible highest weight submodules of $\mathcal{M}$. Hence for each non-integral central character there exists a unique irreducible semi-simple coherent family for this central character.

Theorem 3.5. Let V be a simple torsion free A_n-module of finite degree having a non-integral central character. V is determined up to equivalence by it's central character and weight lattice.

Proof. V has non-integral central character χ_λ. Let $m(\lambda)$ be the corresponding $sl(n+1)$-sequence in $\mathcal{P}$. By Theorem 3.4 part 1, $m(\lambda)$ is part of a unique connected component. By Theorem 3.4 part 2, this connected component uniquely determines the semisimple irreducible coherent family $\mathcal{M}$ which V is a submodule of. Since V is simple torsion free module by Proposition 3.1 Supp$V = \lambda + Q$ for some $\lambda \in \mathcal{H}^*$ and $V = \mathcal{M}[\lambda]$ is the submodule of $\mathcal{M}$ $\square$

Theorem 3.6. Let $a \notin \mathbb{Z}$ and $m_i \in \mathbb{Z}_{>0}$ for $i = 2, \ldots, n+1$. Let $\lambda = a\omega_1 + \sum_{i=2}^{n+1} m_i\omega_i$. For the central character χ_λ and any weight lattice corresponding to a torsion free module there exists a unique simple torsion free A_n-module and it has degree equal to the dimension of the A_{n-1} module with highest weight $\sum_{i=1}^{n} m_{i+1}\omega_i$.

Proof. See Theorem 11.4 in [10] $\square$

4 Tableau Background

$\mathcal{V}$ denotes the natural representation space of A_n defined in Example 2.5, and $\otimes^N\mathcal{V}$ denotes the $N-fold$ tensor product of $\mathcal{V}$ outlined in section 2.6. The aim of this chapter is to review the realization of finite dimensional simple A_n modules as particular submodules of $\otimes^N\mathcal{V}$. The key to this realization is the notion of Young symmetrizers, which are certain elements in the group algebra $\mathbb{C}[\mathcal{S}_N]$.

4.1 Basic Definitions

We first introduce some basic terminology and notation to familiarize the reader with the notion of a tableau.

Definition 4.1. A sequence of positive integers $\pi = \{\pi_1 \geq \pi_2 \geq \cdots \geq \pi_p\}$ is called a **partition of N** if and only if

$$\sum_{i=1}^{p} \pi_i = N$$

The π_i's are called the **parts** of π, and the set of all partitions of N is denoted by $\prod(N)$. If several parts of π are equal, suppose a_i parts are equal to i, this is denoted by $\pi = \{N^{a_N}, (N-1)^{a_{N-1}}, \cdots, 1^{a_1}\}$. For notational convenience we set $\pi_i = 0$ for $i > p$.

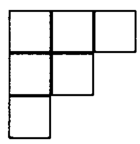
Example 4.1. $\{3 \geq 3 \geq 2 \geq 2\} \in \prod(10)$ will be written as $\{3^2, 2^2\}$.

For the remainder of this work we make the following assumptions. N will denote a positive integer and $\mathcal{N} = \{1, \ldots, N\}$. For $\pi = \{\pi_1 \geq \cdots \geq \pi_p\} \in \prod(N)$ let $|\pi| = \pi_1 + \cdots + \pi_p = N$. When $\mathcal{N} = \varnothing$, there is just one partition in $\prod(N)$, the partition with zero parts.

Example 4.2. Suppose $\pi = \{3, 2, 1^2\} \in \prod(7)$ then π has 4 parts and $|\pi| = 7$.

Definition 4.2. For every $\pi \in \prod(N)$, the associated **Young frame** or **Ferrers diagram**, denoted by $\mathcal{F}(\pi)$, is an array of boxes with π_i boxes in the ith row and each row of boxes is left justified.

Example 4.3. The Young frame having the partition $\pi = \{3, 2, 1\}$ is



Definition 4.3. If $\pi \in \prod(N)$ then a **Young tableau** having frame $\mathcal{F}(\pi)$ is obtained by inserting the elements of $\mathcal{N}$ bijectively into the boxes.

Example 4.4. Suppose $\mathcal{N}$=$\{1, 2, 3, 4, 5, 6\}$ with $\pi = \{3, 2, 1\} \in \prod(6)$. Then

1	2	4
3	5	
6		

is a Young tableau corresponding to $\mathcal{F}(\pi)$.

Definition 4.4. Any Young tableau with underlying partition $\pi \in \prod(N)$ is said to have **content** $\{1, \ldots, N\}$ and **shape** π.

Definition 4.5. A Young tableau is called **standard** provided the entries strictly increase from top to bottom and left to right.

The symmetric group on $\mathcal{N}$, denoted $S_{\mathcal{N}}$, is the collection of all one to one, onto functions from $\mathcal{N}$ to $\mathcal{N}$, with the group operation being composition of functions. Given a Young tableau we associate two subgroups of $S_{\mathcal{N}}$.

Definition 4.6. Fix a $\pi \in \prod(N)$ with associated Young tableau τ. The **row group** of τ, denoted by $\mathcal{R}_\tau$, is the set of all permutations in $S_\mathcal{N}$ which permute only the elements of $\mathcal{N}$ lying in the same row in τ. The **column group** of τ, denoted by $\mathcal{C}_\tau$, is the permutations in $S_\mathcal{N}$ which permute only the elements in $\mathcal{N}$ lying in the same column in τ.

We now define an element in the group algebra $\mathbb{C}[S_\mathcal{N}]$ known as a Young symmetrizer.

Definition 4.7. Let τ be a Young tableau with underlying partition $\pi \in \prod(N)$, row group $\mathcal{R}_\tau$, and column group $\mathcal{C}_\tau$. The **Young symmetrizer** of a Young tableau τ, denoted g_τ, is defined to be:

$$g_\tau = \left(\sum_{\gamma \in \mathcal{C}_\tau} sgn(\gamma)\gamma\right)\left(\sum_{\psi \in \mathcal{R}_\tau} \psi\right) = \sum_{\substack{\gamma \in \mathcal{C}_\tau \\ \psi \in \mathcal{R}_\tau}} sgn(\gamma)\gamma\psi$$

where $sgn(\gamma)$ takes on a value of $+1$ when γ is an even permutation and -1 when γ is an odd permutation.

4.2 Viewing $\otimes^N \mathcal{V}$ as an $S_\mathcal{N}$-module and an A_n-module

In this section we define actions on $\otimes^N \mathcal{V}$ in order to view $\otimes^N \mathcal{V}$ as an $S_\mathcal{N}$-module and an A_n-module. Inside $\otimes^N \mathcal{V}$ we will be considering a particular submodule, denoted $g_\pi(\otimes^N \mathcal{V})$, which we be crucial in our goal of realizing all simple torsion free A_n-module of finite degree having a non-integral central character.

Fix a basis $\{e_i \mid i = 1, \ldots, n+1\}$ for $\mathcal{V}$. Then a basis for $\otimes^N \mathcal{V}$ is given by $\{e_{j_1} \otimes \cdots \otimes e_{j_N} \mid j_i \in \{1, \ldots, n+1\}\}$.

Definition 4.8. For any $\sigma \in S_\mathcal{N}$ the action of σ on a basis vector of $\otimes^N \mathcal{V}$ is

$$\sigma(e_{j_1} \otimes \cdots \otimes e_{j_N}) = e_{j_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{j_{\sigma^{-1}(N)}}.$$

Extending this action linearly we have an action of $S_\mathcal{N}$ on $\otimes^N \mathcal{V}$.

We now give an example to illustrate that this definition is equivalent to permuting the positions of the factors of the simple tensors by $\sigma \in S_\mathcal{N}$.

Example 4.5. Consider $\otimes^3\mathcal{V}$. If $\sigma = (123)$ then $\sigma^{-1} = (132)$. For a basis vector $e_{t_1} \otimes e_{t_2} \otimes e_{t_3}$ of $\otimes^N\mathcal{V}$, consider the action of σ on $e_{t_1} \otimes e_{t_2} \otimes e_{t_3}$:

$$\sigma(e_{t_1} \otimes e_{t_2} \otimes e_{t_3}) = e_{t_{\sigma^{-1}(1)}} \otimes e_{t_{\sigma^{-1}(2)}} \otimes e_{t_{\sigma^{-1}(3)}} = e_{t_3} \otimes e_{t_1} \otimes e_{t_2}.$$

Now we permute the positions of the simple basis tensor by σ. Therefore, σ moves the first factor of our simple basis tensor to the second position, the second factor to the third position and third factor to the first position giving:

$$\sigma(e_{t_1} \otimes e_{t_2} \otimes e_{t_3}) = e_{t_3} \otimes e_{t_1} \otimes e_{t_2}.$$

Definition 4.9. The **canonical tableau** with underlying partition $\pi \in \prod(N)$, denoted τ_π, is the standard tableau constructed by inserting the elements of $\mathcal{N}$ in order from smallest to largest into the frame $\mathcal{F}(\pi)$, beginning with the first row, then the second row and so forth, proceeding from left to right.

Notation. For the remainder of this work to simplify notation we fix τ_π to be the canonical tableau with underlying partition π and we denote R_{τ_π}, C_{τ_π} and g_{τ_π} by $\mathcal{R}_\pi, \mathcal{C}_\pi$ and g_π respectively.

Example 4.6. If $\pi = \{3, 2, 1\}$ then

$$\tau_\pi = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \cline{1-2} 6 & & \\ \cline{1-1}\end{array}$$

is the canonical tableau with underlying partition π. Moreover,

$$\mathcal{R}_\pi = \mathcal{S}_{\{1,2,3\}} \times \mathcal{S}_{\{4,5\}} \times \mathcal{S}_{\{6\}},$$

and

$$\mathcal{C}_\pi = \mathcal{S}_{\{1,4,6\}} \times \mathcal{S}_{\{2,5\}} \times \mathcal{S}_{\{3\}}.$$

Lemma 4.1. Let τ and τ' be Young tableaux with underlying shape $\pi \in \prod(N)$. Then

$$g_\tau(\otimes^N\mathcal{V}) \simeq g_{\tau'}(\otimes^N\mathcal{V}).$$

Proof. See for example Lemma 2.37 in [1] $\square$

Remark 4.1. As a result of the above Lemma for the remainder of this work we will be working with the Young symmetrizer g_π.

Definition 4.10. Let $e_1, \ldots, e_{n+1}$ be the standard basis of $\mathcal{V}$. For every simple basis tensor $\beta = e_{t_1} \otimes \cdots \otimes e_{t_N}$ in $\otimes^N \mathcal{V}$, a substitution of the factors of β into the tableau τ_π is made. For $k = 1, \ldots, N$ put the subscript t_k, in the box of τ_π holding the entry k. We call this the **generalized tableau** for β, denoted $\mathcal{T}_\pi(\beta)$. $\mathcal{T}_\pi(\beta)$ is said to have **content** $\{t_1^{m_1}, \ldots, t_N^{m_N}\}$ provided $\mathcal{T}_\pi(\beta)$ has m_k boxes filled with t_k, for $k = 1, \ldots, N$. The **shape** of $\mathcal{T}_\pi(\beta)$ is the underlying partition π.

Example 4.7. If $\pi = \{3, 2, 1\} \in \prod(6)$ and $\beta = e_1 \otimes e_3 \otimes e_4 \otimes e_2 \otimes e_4 \otimes e_2 \in \otimes^6 \mathcal{V}$ then the generalized tableau for β is

$$\mathcal{T}_\pi(\beta) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 4 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$$

where $\mathcal{T}_\pi(\beta)$ has content $\{1, 2^2, 3, 4^2\}$ and shape $\pi = \{3, 2, 1\}$.

Remark 4.2. There is a bijective correspondence between simple basis tensors coming out of $\otimes^N \mathcal{V}$ and the collection of all generalized tableaux with shape π and content $\{t_1^{m_1}, \ldots, t_N^{m_N}\}$ where $t_i \in \{1, \ldots, n+1\}$. Therefore, for the remainder of this work we will refer to simple basis tensors coming out of $\otimes^N \mathcal{V}$ and it's corresponding generalized tableaux interchangeably.

Definition 4.11. Let $\pi = \{\pi_1 \geq \cdots \geq \pi_p\} \in \prod(N)$ and $\beta = e_1 \otimes \cdots e_{\pi_1} \otimes e_{\pi_1+1} \otimes \cdots \otimes e_N$ be a simple tensor out of $\otimes^N \mathcal{V}$. Let $\pi' = \{\pi_2 \geq \cdots \geq \pi_p\}$ and $\beta' = e_{\pi_1+1} \otimes \cdots \otimes e_N$. $\mathcal{T}_{\pi'}(\beta')$ is the **row diminished** tableau of $\mathcal{T}_\pi(\beta)$ which, to simplify notation, is denoted by $\widetilde{\mathcal{T}_\pi(\beta)}$.

Example 4.8. Suppose

$$\mathcal{T}_\pi(\beta) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 4 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$$

then

$$\widetilde{\mathcal{T}_\pi(\beta)} = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$$

For this work we will be viewing the action of $S_\mathcal{N}$ on generalized tableaux rather then simple tensors.

Definition 4.12. Let $\pi \in \prod(N)$ and β be a basis tensor out of $\otimes^N \mathcal{V}$. For any $\sigma \in S_\mathcal{N}$

$$\sigma \mathcal{T}_\pi(\beta) = \mathcal{T}_\pi(\sigma\beta).$$

Extending this action linearly we again have an action of $S_\mathcal{N}$ on $\otimes^N \mathcal{V}$.

Example 4.9. Suppose $\pi = \{3, 2, 1\}$ and $\beta = e_1 \otimes e_3 \otimes e_4 \otimes e_2 \otimes e_4 \otimes e_2 \in \otimes^6 \mathcal{V}$.

$$\mathcal{T}_\pi(\beta) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 4 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$$

Let $\sigma = (123)(45)$ then $\sigma(e_1 \otimes e_3 \otimes e_4 \otimes e_2 \otimes e_4 \otimes e_2) = e_4 \otimes e_1 \otimes e_3 \otimes e_4 \otimes e_2 \otimes e_2$ and we have:

$$\mathcal{T}_\pi(\sigma\beta) = \begin{array}{|c|c|c|}\hline 4 & 1 & 3 \\ \hline 4 & 2 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$$

Remark 4.3. Instead of taking a simple basis vector β out of $\otimes^N \mathcal{V}$ and relating it to it's corresponding generalized tableau $\mathcal{T}_\pi(\beta)$, we will often suppress the π and β and simply write $\mathcal{T}$, explicitly giving the shape and content of $\mathcal{T}$. Also for any $\sigma \in S_\mathcal{N}$

when we write $\sigma\mathcal{T}$ we mean that σ is acting on the corresponding simple basis tensor of $\mathcal{T}$.

Definition 4.13. Let $\pi = \{\pi_1 \geq \cdots \geq \pi_p\} \in \prod(N)$. Let $\mathcal{T}$ be a generalized tableau having shape π. For $i = 1, \ldots, p$ and $j = 1, \ldots, \pi_i$, $\mathcal{T}[i, j]$ will denote the index which occurs in the intersection of the i^{th} row and j^{th} column of $\mathcal{T}$.

Definition 4.14. Let $\pi \in \prod(N)$ and β be a simple basis tensor out of $\otimes^N \mathcal{V}$. $\mathcal{T}_\pi(\beta)$ is called π **semi-standard** provided for all indices $i < i'$, $\mathcal{T}_\pi(\beta)[i, j] < \mathcal{T}_\pi(\beta)[i', j]$ and for all indices $j < j'$, $\mathcal{T}_\pi(\beta)[i, j] \leq \mathcal{T}_\pi(\beta)[i, j']$. The set of all π semi-standard generalized tableaux is denoted by $\mathcal{S}_\pi(N)$. $\mathcal{T}_\pi(\beta)$ is said to be **non** π **semi-standard** provided there exists indices $i < i'$ such that $\mathcal{T}_\pi(\beta)[i, j] \geq \mathcal{T}_\pi(\beta)[i', j]$ or there exists indices $j < j'$ such that $\mathcal{T}_\pi(\beta)[i, j] > \mathcal{T}_\pi(\beta)[i, j']$.

Notation. For $i \in \{1, \ldots, n+1\}$, $l \in \mathbb{Z}$ and $K_i \in \mathbb{Z}_{>0}$ define $\boxed{K_i}$ to stand for a 1 row tableau having K_i boxes each containing the value i. When we write $\boxed{K_i + l}$ we are indicating a 1 row tableau having $K_i + l$ boxes each containing the value of i. $\boxed{K_1}\boxed{K_2}\ \cdots\ \boxed{K_{n+1}}$ stands for a one row tableau containing K_1 1's followed by K_2 2's, and so on.

Example 4.10. If $K_1 = 3$ and $K_2 = 2$ then

$$\boxed{K_1}\boxed{K_2} = \boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\,,$$

and if $l = 2$ then

$$\boxed{K_1 + l}\boxed{K_2} = \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\,.$$

We now define an action of $gl(n+1, \mathbb{C})$ on $\otimes^N \mathcal{V}$.

Definition 4.15. Let $I = \{1, \ldots, n+1\}^N$. For each $A \in gl(n+1, \mathbb{C})$, and $t = \sum_{(i_1, \ldots, i_N) \in I} (\alpha_{i_1 \ldots i_N}) e_{i_1} \otimes \cdots \otimes e_{i_N}$ where $\alpha_{i_1 \ldots i_N} \in \mathbb{C}$, an action of $gl(n+1, \mathbb{C})$ on $\otimes^N \mathcal{V}$ is as follows,

$$A(t) = \sum_{(i_1,\ldots,i_N)\in I} \alpha_{i_1 \ldots \alpha_{i_N}} \sum_{j=1}^{N} e_{i_1} \otimes \cdots \otimes (Ae_{i_j}) \otimes \cdots \otimes e_{i_N}.$$

Extending this action linearly defines a $gl(n+1,\mathbb{C})$ module structure on $\otimes^N \mathcal{V}$, and restricting this action to A_n, $\otimes^N \mathcal{V}$ becomes an A_n-module.

Remark 4.4. In later sections we will be interested in the action of $gl(n+1,\mathbb{C})$ on simple basis tensors coming out of $\otimes^N \mathcal{V}$. For the simple basis tensor $e_{t_1} \otimes \cdots \otimes e_{t_N}$, and the standard matrix unit $E_{ij} \in gl(n+1,\mathbb{C})$ we have

$$E_{ij}(e_{t_1} \otimes \cdots \otimes e_{t_N}) = \sum_{k=1}^{N} e_{t_1} \otimes \cdots \otimes (E_{ij}e_{t_k}) \otimes \cdots \otimes e_{t_N}.$$

Remark 4.5. Observe that for any $E_{ij} \in gl(n+1,\mathbb{C})$, $\sigma \in S_N$ and simple basis tensor $\beta \in \otimes^N \mathcal{V}$

$$E_{ij}(\sigma(\beta)) = \sigma(E_{ij}(\beta)).$$

Therefore

$$g_\pi(\otimes^N \mathcal{V}) = Span_{\mathbb{C}}\{g_\pi(e_{j_1} \otimes \cdots \otimes e_{j_N}) \mid j_i \in \{1,\ldots,n+1\}\}$$

is an A_n submodule of $\otimes^N \mathcal{V}$.

We can also view the action of $gl(n+1,\mathbb{C})$ on generalized tableaux.

Definition 4.16. Consider $\beta = e_{t_1} \otimes \cdots \otimes e_{t_N} \in \otimes^N \mathcal{V}$. The action of $gl(n+1,\mathbb{C})$ on $\mathcal{T}_\pi(\beta)$ is as follows

$$E_{ij}\mathcal{T}_\pi(\beta) = \sum_{k=1}^{N} \mathcal{T}_\pi(\beta_k)$$

where $\beta_k = e_{t_1} \otimes \cdots \otimes E_{ij}e_{t_k} \otimes \cdots \otimes e_{t_N}$.

Example 4.11. For $\beta = e_3 \otimes e_1 \otimes e_4 \otimes e_2 \otimes e_2$

$$\mathcal{T}_\pi(\beta) = \begin{array}{|c|c|c|}\hline 3 & 1 & 4 \\ \hline 4 & 2 & \\ \cline{1-2} 2 & & \\ \cline{1-1}\end{array}$$

Consider the action of E_{12} on $\mathcal{T}_\pi(\beta)$. In this case $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$, since $E_{12}e_j = 0$ for $j \neq 2$. Therefore, we have

$$E_{12}\mathcal{T}_\pi(\beta) = \begin{array}{|c|c|c|}\hline 3 & 1 & 4 \\ \hline 4 & 1 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array} \quad + \quad \begin{array}{|c|c|c|}\hline 3 & 1 & 4 \\ \hline 4 & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

4.3 Classical Results

In this section we review two main results about the A_n-module $g_\pi(\otimes^N \mathcal{V})$. The first result describes a basis for $g_\pi(\otimes^N \mathcal{V})$ and the second result realizes every finite dimensional simple A_n-module as the module $g_\pi(\otimes^N \mathcal{V})$ for an appropriate choice of π.

Lemma 4.2. If $\mathcal{T}$ and $\mathcal{T}'$ are two generalized tableaux with underlying partition π such that $\mathcal{T} = p\mathcal{T}'$ for some $p \in \mathcal{R}_\pi$ then $g_\pi(\mathcal{T}) = g_\pi(\mathcal{T}')$.

Proof. For any $p \in \mathcal{R}_\pi$, we have that

$$\sum_{\psi \in \mathcal{R}_\pi} \psi \circ p = \sum_{\psi \in \mathcal{R}_\pi} \psi$$

and therefore,

$$g_\pi(\mathcal{T}) = g_\pi(p\mathcal{T}') = \left(\sum_{\gamma \in \mathcal{C}_\pi} sgn(\gamma)\gamma\right)\left(\sum_{\psi \in \mathcal{R}_\pi} \psi \circ p\right)\mathcal{T}' = g_\pi(\mathcal{T}').$$

□

Lemma 4.3. If a generalized tableau $\mathcal{T}$ of shape π is such that $\mathcal{T}$ has a column containing two equal elements then

$$(\sum_{\sigma \in C_\pi} sgn(\sigma)\sigma)\mathcal{T} = 0.$$

Proof. Suppose that $(q, r) \in \mathcal{C}_\pi$ such that (q, r) interchanges two equal elements in the same column in $\mathcal{T}$. Since $\{\sigma(q, r) \mid \sigma \in \mathcal{C}_\pi\} = \mathcal{C}_\pi$ we have:

$$\sum_{\sigma \in \mathcal{C}_\pi} sgn(\sigma)\sigma\mathcal{T} = \sum_{\sigma \in \mathcal{C}_\pi} sgn(\sigma(q,r))\sigma(q,r)\mathcal{T} = -\sum_{\sigma \in \mathcal{C}_\pi}(sgn(\sigma))\sigma\mathcal{T}$$

and therefore,

$$\sum_{\sigma \in \mathcal{C}_\pi} sgn(\sigma)\sigma\mathcal{T} = 0.$$

□

Theorem 4.1. $\{g_\pi(\mathcal{T}) \mid \mathcal{T} \in \mathcal{S}_\pi(N)\}$ is a basis for $g_\pi(\otimes^N \mathcal{V})$.

Proof. See for example Theorem 8.11 in [1] □

For any non π semi-standard generalized tableau T, $g_\pi(T) \in g_\pi(\otimes^N \mathcal{V})$. By Theorem 4.1, $g_\pi(T)$ can be expressed as a linear combination of the elements in $\{g_\pi(\mathcal{T}) \mid \mathcal{T} \in \mathcal{S}_\pi(N)\}$. In this event $g_\pi(T)$ is said to be **straightened**.

Recall, by combining Proposition 2.7 and Theorem 2.11 we know that every finite dimensional simple A_n-module is some $V(\lambda)$ where λ is dominant integral. The following Theorem shows how we can realize every finite dimensional simple A_n-module as a particular submodule of $\otimes^N \mathcal{V}$.

Theorem 4.2. Suppose $\lambda = \sum_{i=1}^n h_i\omega_i$ is a dominant integral weight for A_n. Set $\pi_k = \sum_{j=k}^n h_j$ and let $N = \sum_{i=1}^n \pi_i$ then $\pi = \{\pi_1 \geq \cdots \geq \pi_n\} \in \prod(N)$. The finite dimensional simple A_n module $V(\lambda)$ with highest weight λ is isomorphic to $g_\pi(\otimes^N \mathcal{V})$. In particular, the highest weight vector in $g_\pi(\otimes^N \mathcal{V})$ is $g_\pi(\mathcal{T}^+)$ where $\mathcal{T}^+$ is the π semi-standard generalized tableau having i^{th} row filled with the values i.

Proof. See for example Theorem 2.33 in [1].

□

4.4 Ordering on Tableaux

In this section we define an ordering on generalized tableaux and review some resulting properties. This ordering will assist us in determining which π semi-standard generalized tableaux appear in the expansion of a Young symmetrizer, g_π, acting on an arbitrary π semi-standard generalized tableau.

Definition 4.17. Let β_1 and β_2 be basis tensors for $\otimes^N \mathcal{V}$. $\beta_1 < \beta_2$ if there exists $\gamma \in \mathcal{C}_\pi$ such that $\mathcal{T}_\pi(\beta_2)$ can be obtained from $\mathcal{T}_\pi(\gamma\beta_1)$ by successively interchanging pairs of entries in the same row of $\mathcal{T}_\pi(\gamma\beta_1)$ such that at each stage the entry with the smaller value is moved to a column which is further left while still lying in it's original row. In this case, we say $\mathcal{T}_\pi(\beta_1) < \mathcal{T}_\pi(\beta_2)$.

Example 4.12. Let $\pi = \{3,2,1\} \in \prod(6)$. Consider $\beta_1 = e_1 \otimes e_2 \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_4$ and $\beta_2 = e_1 \otimes e_2 \otimes e_4 \otimes e_2 \otimes e_3 \otimes e_4 \in \otimes^6 \mathcal{V}$. Then

$$\mathcal{T}_\pi(\beta_1) = \begin{array}{ccc} 1 & 2 & 2 \\ 3 & 4 & \\ 4 & & \end{array} \qquad \mathcal{T}_\pi(\beta_2) = \begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & \\ 4 & & \end{array}$$

$$\mathcal{T}_\pi((2,5)\beta_1) = \begin{array}{ccc} 1 & 4 & 2 \\ 3 & 2 & \\ 4 & & \end{array} \longrightarrow \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 2 & \\ 4 & & \end{array} \longrightarrow \begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & \\ 4 & & \end{array} = \mathcal{T}_\pi(\beta_2),$$

and therefore $\beta_1 < \beta_2$. However,

$$\mathcal{T}_\pi(\beta_3) = \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 4 & \\ 2 & & \end{array}$$

is not related to $\mathcal{T}_\pi(\beta_1)$ because there does not exist a $\gamma \in \mathcal{C}_\pi$ such that the row sets of $\mathcal{T}_\pi(\gamma\beta_3)$ coincide with the row sets of $\mathcal{T}_\pi(\beta_1)$.

Lemma 4.4. 1. Let β_1 and β_2 be π semi-standard basis tensors for $\otimes^N \mathcal{V}$. If $qp\beta_2 = \beta_1$ for $q \in \mathcal{C}_\pi$ and $p \in \mathcal{R}_\pi$, then either $\beta_1 = \beta_2$ or $\beta_1 < \beta_2$.

2. If β is a π semi-standard basis tensor for $\otimes^N \mathcal{V}$ then the coefficient of β, when $g_\pi(\beta)$ is written as a linear combination of basis tensors, is nonzero. In particular, $g_\pi(\beta) \neq 0$.

Proof. See Lemma 8.8 in [1] ☐

Remark 4.6. By part 2 of the above Lemma for any $\mathcal{T} \in \mathcal{S}_\pi(N)$, $\mathcal{T}$ appears in $g_\pi(\mathcal{T})$ with non-zero coefficient.

With the groundwork set, in the next section we will realize all torsion free A_n-modules of degree one by working with finite dimensional simple A_n-modules viewed in terms of a tableau formalism.

5 Motivating Example

In this section we show that all simple torsion free A_n-modules of degree one can be obtained by a "complex continuation" of the simple A_n modules $V(K\omega_1)$ for $K \in \mathbb{Z}_{\geq 0}$. For each $K \in \mathbb{Z}_{>0}$, in this realization a basis for $V(K\omega_1)$ consists of the vectors $g_\pi(\boxed{K_1}\boxed{K_2}\ldots\boxed{K_{n+1}})$ where $\sum_{i=1}^{n+1} K_i = K$, $\pi = \{K\} \in \prod(K)$ and $K_i \in \mathbb{Z}_{\geq 0}$. For all $K \in \mathbb{Z}_{>0}$ these are representations, and therefore the operators satisfy the Serre relations for all bases elements. The coefficients of the bases elements in the Serre relations can be viewed as polynomials in the integer variables $K_1, \ldots, K_{n+1}$ which are identically zero. For example consider the Serre relation $[E_{11} - E_{22}, E_{23}] + E_{23}$.

$$
\begin{aligned}
&([E_{11} - E_{22}, E_{23}] + E_{23})g_\pi(\boxed{K_1}\boxed{K_2}\ldots\boxed{K_{n+1}}) \\
&\qquad = ((E_{11} - E_{22})E_{23} - E_{23}(E_{11} - E_{22}) + E_{23})g_\pi(\boxed{K_1}\boxed{K_2}\ldots\boxed{K_{n+1}}) \\
&\qquad = K_3(K_1 - (K_2 + 1))g_\pi(\cdots\boxed{K_2+1}\boxed{K_3-1}\cdots) \\
&\qquad - K_3(K_1 - K_2)g_\pi(\cdots\boxed{K_2+1}\boxed{K_3-1}\cdots) \\
&\qquad + K_3 g_\pi(\cdots\boxed{K_2+1}\boxed{K_3-1}\cdots) \\
&\qquad = (K_3(K_1 - (K_2 + 1)) - K_3(K_1 - K_2) + K_3)g_\pi(\cdots\boxed{K_2+1}\boxed{K_3-1}\cdots) \\
&\qquad = 0 \cdot g_\pi(\cdots\boxed{K_2+1}\boxed{K_3-1}\cdots).
\end{aligned}
$$

Therefore, $(K_3(K_1 - (K_2 + 1)) - K_3(K_1 - K_2) + K_3)$ is a polynomial in K_1, K_2 and K_3 which is identically zero.

The idea is to construct new representations by "complexifying" the parameters K_i. With appropriate conditions on the new parameters, these A_n-modules are simple torsion free degree one, and in fact all such modules can be realized in this manner.

We first establish the connection between two realizations of the simple finite dimensional A_n-module $V(K\omega_1)$.

For each $K \in \mathbb{Z}_{\geq 0}$ we fix $\pi = \{K\} \in \prod(K)$ then

$$R_\pi = \sum_{\sigma \in S_K} \sigma \ , \ C_\pi = \epsilon \ , \text{and } g_\pi = \sum_{\sigma \in S_K} \sigma.$$

Consider the following A_n module:

$$g_\pi(\otimes^K \mathcal{V}) = Span_{\mathbb{C}}\{g_\pi(\boxed{K_1}\boxed{K_2}\ldots\boxed{K_{n+1}}) \mid \sum_{i=1}^{n+1} K_i = K \ ; \ K_i \in \mathbb{Z}_{\geq 0}\}.$$

By Theorem 4.2 this module is simple with highest weight $K\omega_1$ and maximal vector $v^+ = g_\pi(\ \boxed{K_1 = K}\ )$, and hence isomorphic to $V(K\omega_1)$. All the weight spaces are one dimensional with weight vector $g_\pi(\boxed{K_1}\boxed{K_2}\ldots\boxed{K_{n+1}})$ having weight $\sum_{i=1}^{n}(K_i - K_{i+1})\omega_i$.

On the other hand, recall the following simple finite dimensional A_n-module from Example 3.1:

$$M(\bar{k}) = Span_{\mathbb{C}}\{x_1^{k-l_1} x_2^{l_1-l_2} \cdots x_{n+1}^{l_n} \mid 0 \leq l_n \leq l_{n-1} \cdots \leq l_1 \leq K\}$$

where $\bar{k} = (K, 0, \ldots, 0)$.

Lemma 5.1. Assuming the notation above, $M(\bar{k})$ and $g_\pi(\otimes^K \mathcal{V})$ are isomorphic as A_n-modules when $\bar{k} = (K, 0, \ldots, 0)$.

Proof. Let

$$\psi : M(\bar{k}) \to g_\pi(\otimes^N \mathcal{V})$$

given by

$$\psi(x_1^{K-l_1} x_2^{l_1-l_2} \ldots x_{n+1}^{l_n}) = g_\pi(\boxed{K_1}\boxed{K_2}\ldots\boxed{K_{n+1}})$$

where $K_1 = K - l_1,\ K_i = l_{i-1} - l_i,$ for $i = 2, \ldots, n$ and $K_{n+1} = l_n$.

We will show that ψ is in fact an isomorphism of $gl(n+1,\mathbb{C})$-modules. Clearly ψ is an isomorphism of vector spaces as we are mapping basis vector to basis vector, and so we need only show $\psi(E_{ij}.v) = E_{ij}.\psi(v)$ for all $E_{ij} \in gl(n+1,\mathbb{C})$ and $v \in M(\bar{k})$. Take $E_{ij} \in gl(n+1,\mathbb{C})$. Set $l_0 = K$ and $l_{n+1} = 0$.

$$\begin{aligned}
E_{ij}\psi(x_1^{K-l_1}x_2^{l_1-l_2}\dots x_{n+1}^{l_n}) &= E_{ij}g_\pi(\boxed{K_1}\boxed{K_2}\dots\boxed{K_{n+1}})\\
&= g_\pi(E_{ij}\boxed{K_1}\boxed{K_2}\dots\boxed{K_{n+1}})\\
&= K_j g_\pi(\dots\boxed{K_i+1}\dots\boxed{K_j-1}\dots)\\
&= (l_{j-1}-l_j)\psi(\cdots x_i^{l_{i-1}-l_i+1}\cdots x_j^{l_{j-1}-l_j-1}\cdots)\\
&= \psi((l_{j-1}-l_j)\cdots x_i^{l_{i-1}-l_i+1}\cdots x_j^{l_{j-1}-l_j-1}\cdots)\\
&= \psi(E_{ij}x_1^{K-l_1}x_2^{l_1-l_2}\dots x_{n+1}^{l_n}).
\end{aligned}$$

Restricting ψ to the elements in A_n, it follows that $M(\bar{k})$ and $g_\pi(\otimes^N\mathcal{V})$ are isomorphic as A_n-modules.

□

Recall,

$$E_{ij}g_\pi(\boxed{K_1}\boxed{K_2}\dots\boxed{K_{n+1}}) = K_j g_\pi(\dots\boxed{K_i+1}\dots\boxed{K_j-1}\dots).$$

Since $g_\pi(\otimes^K V)$ is a module the Serre relations must be satisfied (comment proceeding Theorem 2.7). It is obvious from the action of E_{ij} that the coefficients of the basis vectors in the Serre relations will result in polynomials in $K_1,\dots,K_{n+1}$. This result along with the next lemma will assist us in our goal of realizing all simple torsion free A_n-modules of degree one.

Lemma 5.2. For $i = 1,\dots,n+1$ fix $N_i \in \mathbb{Z}_{>0}$. Let $f \in \mathbb{C}[x_1,\dots,x_{n+1}]$ such that $f(\bar{x}) = 0$ for all $\bar{x} \in \{(k_1,\dots,k_{n+1}) \in \mathbb{Z}^{n+1} \mid k_i \geq N_i,\ i = 1,\dots,n+1\}$ then $f(\bar{x}) = 0$ in $\mathbb{C}[x_1,\dots,x_{n+1}]$.

Proof. We induct on n. For $n = 1$, f is a polynomial in one variable. Let $f(\bar{x}) \in \mathbb{C}[x_1]$. By assumption for some $N_1 \in \mathbb{Z}_{>0}$, $f(k_1) = 0$ for all $k_1 \geq N_1$. Therefore, f has infinitely many roots and therefore $f(x_1) = 0$.

Now assume that the Lemma is true for $n \geq 1$. Consider $f(x_1, \ldots, x_{n+1}) \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ such that we write $f(x_1, \ldots, x_{n+1})$ as

$$f(x_1, \ldots, x_{n+1}) = \sum_{i=0}^{q} P_i(x_1, \ldots, x_n) x_{n+1}^i$$

where each $P_i(x_1, \ldots, x_n)$ is a polynomial in $\mathbb{C}[x_1, \ldots, x_n]$.

For each x_i substitute $k_i \in \mathbb{Z}$ such that $k_i \geq N_i$. Then

$$f(k_1, \ldots, k_n, x_{n+1}) = \sum_{i=0}^{q} P_i(k_1, \ldots, k_n) x_{n+1}^i$$

is a polynomial in one variable with infinitely many roots, namely, $x_{n+1} \geq N_{n+1}$. Therefore,

$$P_i(k_1, \ldots, k_n) = 0$$

for all $k_i \geq N_i$. By the inductive hypothesis we have

$$P_i(x_1, \ldots, x_n) = 0.$$

Therefore,

$$f(x_1, \ldots, x_{n+1}) = \sum_{i=0}^{q} P_i(x_1, \ldots, x_n) x_{n+1}^i = \sum_{i=0}^{q} 0 \cdot x_{n+1}^i = 0.$$

$\square$

We now introduce our modified tableau construction of a simple torsion free A_n-module of degree one. Fix $\bar{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{C}$ such that each $a_i \in \mathbb{C} \setminus \mathbb{Z}$. Define a vector space $\hat{M}(\bar{a})$ over $\mathbb{C}$ to have a formal basis

$$\mathcal{B} = \{v(\bar{a} + \bar{M}) \mid \bar{M} = (M_1, \ldots, M_{n+1}) \in \mathbb{Z}^{n+1}, \sum_{i=1}^{n+1} M_i = 0\}.$$

Next we define a module structure on $\hat{M}(\bar{a})$ by defining the action on $\hat{M}(\bar{a})$ analogous to our finite module $g_\pi(\otimes^N \mathcal{V})$. Let e_i stand for an $n+1$-tuple which has a zero in every co-ordinate except in the i^{th} co-ordinate, which has a value of 1.

$$E_{ij}v(\bar{a}+\bar{M}) = (a_j + M_j)v(\bar{a}+\bar{M}+e_i-e_j)$$

The following Theorem shows that we have now constructed a simple torsion free A_n-module of degree one.

Theorem 5.1. Fix $\bar{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{C}$ such that each $a_i \in \mathbb{C} \setminus \mathbb{Z}$. $\hat{M}(\bar{a})$ is a simple torsion free A_n-module of degree one isomorphic to $M(\bar{a})$, where $M(\bar{a})$ is described in Example 3.1.

Proof. For $\bar{a} = (a_1, \ldots, a_{n+1})$, we have the simple torsion free degree one A_n-module,

$$M(\bar{a}) = Span_{\mathbb{C}}\{x_1^{a_1-k_1}x_2^{a_2+k_1-k_2}\ldots x_n^{a_n+k_{n-1}-k_n}x_{n+1}^{a_{n+1}+k_n} \mid k_i \in \mathbb{Z}\}.$$

Let $\psi : M(\bar{a}) \to \hat{M}(\bar{a})$ given by:

$$\psi(x_1^{a_1-k_1}x_2^{a_2+k_1-k_2}\ldots x_n^{a_n+k_{n-1}-k_n}x_{n+1}^{a_{n+1}+k_n}) = v(\bar{a}+\bar{M})$$

where $M_1 = -k_1$ and $M_i = k_{i-1} - k_i$ for $i = 2, \ldots, n+1$ with $k_{n+1} = 0$.

Now $M(\bar{a})$ and $\hat{M}(\bar{a})$ are isomorphic as vectors spaces. Since $M(\bar{a})$ is a module and $\hat{M}(\bar{a})$ has an action defined on it, it suffices to show that for the generators E_{ij}, ψ satisfies the module homomorphism condition:

$$\psi(E_{ij}v) = E_{ij}\psi(v),$$

where $v = x_1^{a_1-k_1}x_2^{a_2+k_1-k_2}\ldots x_n^{a_n+k_{n-1}-k_n}x_{n+1}^{a_{n+1}+k_n}$.

$$\begin{aligned}
&\psi(E_{ij}x_1^{a_1-k_1}x_2^{a_2+k_1-k_2}\ldots x_n^{a_n+k_{n-1}-k_n}x_{n+1}^{a_{n+1}+k_n}) \\
&\quad = \psi((a_j + k_{j-1} - k_j)\ldots x_i^{a_i+k_{i-1}-k_i+1}\ldots x_j^{a_j+k_{j-1}-k_j-1}\ldots) \\
&\quad = (a_j + k_{j-1} - k_j)\psi(\ldots x_i^{a_i+k_{i-1}-k_i+1}\ldots x_j^{a_j+k_{j-1}-k_j-1}\ldots) \\
&\quad = (a_j + M_j)v(\bar{a}+\bar{M}+e_i-e_j) \\
&\quad = E_{ij}v(\bar{a}+\bar{M}) \\
&\quad = E_{ij}\psi(x_1^{a_1-k_1}x_2^{a_2+k_1-k_2}\ldots x_n^{a_n+k_{n-1}-k_n}x_{n+1}^{a_{n+1}+k_n}).
\end{aligned}$$

□

We have now achieved our goal for this section. By Theorem 3.2 every simple torsion free A_n-module of degree one is isomorphic to $M(\bar{a})$ for an appropriate choice of $\bar{a} = (a_1, \ldots, a_{n+1})$. For any such $\bar{a}$, by Theorem 5.1, we can construct a simple torsion free degree one A_n-module, $\hat{M}(\bar{a})$, which is isomorphic to $M(\bar{a})$. Therefore, we have just realized every simple torsion free A_n-module of degree one using a tableau formalism. Clearly the construction by Britten and Lemire [3] provides a better realization then the one we have constructed. For torsion free modules having degree greater then 1, Britten and Lemire [3] showed that these torsion free modules occur as submodules in $M(\bar{a}) \otimes V(\lambda)$ for appropriate choices of $\bar{a}$ and λ. The problem with this realization is that a basis and a module action is not described. Generalizing the results from this section we will give a basis and a module action for realizing all non-integral simple torsion free A_n-module having finite degree. Moreover, this module action will be defined by working with certain finite dimensional modules, and therefore will be no more difficult then determining the modules action for finite dimensional modules.

6 Action of operators E_{ij} on finite dimensional modules in tableau form

In section 5, we showed that starting with simple finite dimensional A_n-modules, viewed in terms of tableau formalism, we can construct simple torsion free A_n-modules of degree one by applying a "complex continuation". Motivated by this success, the goal of this work is to generalize this construction to obtain all simple torsion free A_n-modules of finite degree having a non-integral central character. An important step in this generalization is examining the coefficients which appear in the action of the operators E_{ij} on certain basis vectors for the modules $g_\pi(\otimes^N \mathcal{V})$. This analysis will be the focus of this chapter.

6.1 Setup

We begin this section by defining these special basis vectors and discussing a general method for examining the coefficients which appear in the action of the operators E_{ij} on these special basis vectors.

Recall from section 4.2 we defined $\boxed{K_1}\boxed{K_2} \quad \cdots \quad \boxed{K_{n+1}}$ to stand for a one row tableau containing K_1 1's followed by K_2 2's, and so on. $\mathcal{V}$ stands for the natural representation space of A_n which was defined in Example 2.5.

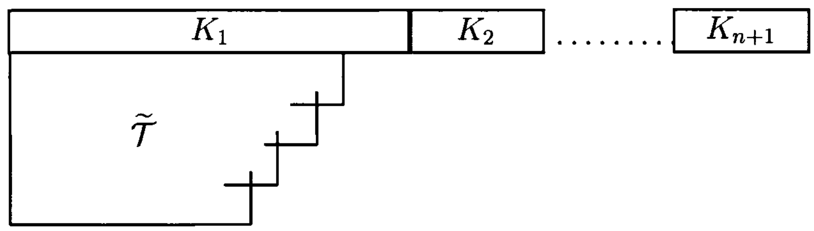
For all partitions $\pi = \{\pi_1 \geq \cdots \geq \pi_p\}$ with $\tilde{\pi} = \{\pi_2 \geq \cdots \geq \pi_p\}$ fixed and $\pi_1 \gg \pi_2$ variable, we consider certain basis vectors coming out of the modules $g_\pi(\otimes^N \mathcal{V}) = Span_{\mathbb{C}}\{g_\pi(T) \mid T \in \mathcal{S}_\pi(N)\}$.

Definition 6.1. Fix $\tilde{\pi} = \{\pi_2 \geq \cdots \geq \pi_p\} \in \prod(M)$ and $\pi_1 \in \mathbb{Z}_{>0}$ such that $\pi_1 \gg \pi_2$ and π_1 is variable. Let $\pi = \{\pi_1 \geq \cdots \geq \pi_p\}$. Let $\mathcal{T} \in \mathcal{S}_\pi(N)$ with top row $\boxed{K_1}\cdots\boxed{K_{n+1}}$ with each K_i chosen large enough such that the action of the Serre relations on $g_\pi(\mathcal{T})$ is non-zero. Then $\mathcal{T}$ is said to be **core** and $g_\pi(\mathcal{T})$ is said to be a **core basis vector**. $\mathcal{K} \in \mathbb{Z}_{>0}$ will denote a lower bound on $K_1, \ldots, K_{n+1}$ such that $\mathcal{T}$ is core.

Remark 6.1. The core basis vectors come from an infinite number of finite dimensional representations. We fix $\tilde{\pi} = \{\pi_2 \geq \cdots \geq \pi_p\} \in \prod(M)$ and consider all partitions $\pi = \{\pi_1 \geq \cdots \geq \pi_p\}$ with variable $\pi_1 \in \mathbb{Z}_{>0}$ such that $\pi_1 \gg \pi_2$. As long as each $K_i \geq \mathcal{K}$ the corresponding core basis vectors satisfy all results in this chapter. As the Serre relations are generated by a finite number of operators, the existence of $\mathcal{K}$ is guaranteed.

Fix $\pi = \{\pi_1 \gg \pi_2 \geq \cdots \geq \pi_p\} \in \prod(N)$ and define τ_π to be the corresponding canonical tableau with row group $\mathcal{R}_\pi$, column group $\mathcal{C}_\pi$ and Young symmetrizer g_π. Fix a π semi-standard generalized tableau with underlying Young frame $\mathcal{F}(\pi)$, and content$\{1^{K_1}, 2^{K_2+m_2}, \ldots, (n+1)^{K_{n+1}+m_{n+1}}\}$ to be

$\mathcal{T} =$



with $\widetilde{\mathcal{T}}$ a fixed row diminished tableau of $\mathcal{T}$ with content $\{2^{m_2}, \ldots, (n+1)^{m_{n+1}}\}$, $K_i \gg \sum_{i=2}^{p} \pi_i$ for $i = 1, \ldots, n+1$, and $\pi_1 = \sum_{j=1}^{n+1} K_j$. Naturally, the condition that $K_i \gg \sum_{i=2}^{p} \pi_i$ imposes a size constraint on π_1, that is $\pi_1 \gg (p-1)\sum_{j=2}^{p} \pi_j$.

Remark 6.2. For the remainder of this chapter $\mathcal{T}$ will stand for the fixed core π semi-standard generalized tableau defined above.

We wish to examine the action of E_{ij} on $g_\pi(\mathcal{T})$.

$$E_{ij} g_\pi(\mathcal{T}) = \sum_r g_\pi(\mathcal{T}_r) \tag{2}$$

where each $\mathcal{T}_r$ is a π generalized tableaux not necessarily semi-standard. Each of the generalized tableaux appearing in the right hand side of equation (2) which are non π semi-standard must be straightened. To do this, suppose $\mathcal{T}_{n.s.}$ is an arbitrary one of these non π semi-standard generalized tableaux. Since $g_\pi(\mathcal{T}_{n.s.}) \in g_\pi(\otimes^N \mathcal{V})$, by Theorem 4.1, $g_\pi(\mathcal{T}_{n.s.})$ has a unique expansion with respect to the basis $\{g_\pi(\mathcal{T}_k) \mid \mathcal{T}_k \in \mathcal{S}_\pi(N)\}$. That is,

$$g_\pi(\mathcal{T}_{n.s.}) = \sum_k c_k g_\pi(\mathcal{T}_k) \qquad \text{where each } c_k \in \mathbb{C} \text{ and each } \mathcal{T}_k \in \mathcal{S}_\pi(N). \tag{3}$$

We now expand all the terms in (3) with respect to the basis elements of $\otimes^N \mathcal{V}$, namely, $\{e_{i_1} \otimes \cdots \otimes e_{i_N} \mid i_j \in \{1, \ldots, n+1\}\}$. To determine the values of the coefficients c_k we observe that any π semi-standard generalized tableau, say $\mathcal{T}_s$, must appear with the same coefficient on both sides of the equation. We then need to know the number of times $\mathcal{T}_s$ appears in $g_\pi(\mathcal{T}_{n.s.})$ and in each of the basis vectors $g_\pi(\mathcal{T}_k)$, when $g_\pi(\mathcal{T}_{n.s.})$ and $g_\pi(\mathcal{T}_k)$ are both expressed with respect to the basis tensors of $\otimes^N \mathcal{V}$. To solve these problems, in later sections several counting properties will be introduced. However, for certain values of i and j the dependence on $K_1, \ldots, K_{n+1}$ when straightening $E_{ij} g_\pi(\mathcal{T})$ requires no advanced counting properties. We break the problem into the following 3 cases:

(1) $E_{i1} g_\pi(\mathcal{T})$ for i arbitrary

(2) $E_{ij} g_\pi(\mathcal{T})$ for $i \neq 1 \neq j$, and

(3) $E_{1j} g_\pi(\mathcal{T})$ for j arbitrary

In the next section we will be concerned with cases (1) and (2).

6.2 Cases (1) and (2)

The goal of this section is to determine the dependence of the coefficients in equation (3) on $K_1, \ldots, K_{n+1}$ when straightening $E_{ij} g_\pi(\mathcal{T})$, for the following two situations:

(1) $E_{i1} g_\pi(\mathcal{T})$ for i arbitrary

(2) $E_{ij} g_\pi(\mathcal{T})$ for $i \neq 1 \neq j$

We remind the reader that $g_\pi(\mathcal{T})$ represents a special type of basis vector in $\{g_\pi(T) \mid T \in \mathcal{S}_\pi(N)\}$ which was defined explicitly in section 6.1.

Notation. As we are interested in examining the action of E_{ij} on $g_\pi(\mathcal{T})$, we introduce a notation to keep track of what factor E_{ij} is acting on in $\mathcal{T}$. Define $E_{ij}^{kl}\mathcal{T}$ to be the result of the operator E_{ij} acting on the element in $\mathcal{T}$ located in the k^{th} row and l^{th} column. Therefore, for $\pi = \{\pi_1 \geq \cdots \geq \pi_p\} \in \prod(N)$

$$E_{ij}\mathcal{T} = \sum_{\substack{k=1,\ldots,p \\ l=1,\ldots,\pi_k}} E_{ij}^{kl}\mathcal{T}.$$

Notation. Let $\mathcal{T}_j^i$ denote the π semi-standard generalized tableau identical to $\mathcal{T}$ except in the top row, which has an extra i and exactly one less j.

Lemma 6.1. Let $M = 0$ if $j = 1$ and $M = \sum_{r=1}^{j-1} K_r$ if $j > 1$.

$$\sum_{l=M+1}^{M+K_j} E_{ij}^{1l} g_\pi(\mathcal{T}) = K_j g_\pi(\mathcal{T}_j^i).$$

Proof. $E_{ij}^{1l}\mathcal{T} = 0$ unless $l = M+1, \ldots, M+K_j$. For any $l = M+1, \ldots, M+K_j$ there exists a $p \in \mathcal{R}_\pi$ such that $pE_{ij}^{1l}\mathcal{T} = \mathcal{T}_j^i$. By Lemma 4.2, $g_\pi(E_{ij}^{1l}\mathcal{T}) = g_\pi(\mathcal{T}_j^i)$ for $l = M+1, \ldots, M+K_j$. Therefore,

$$\sum_{l=M+1}^{M+K_j} E_{ij}^{1l} g_\pi(\mathcal{T}) = \sum_{l=M+1}^{M+K_j} g_\pi(E_{ij}^{1l}\mathcal{T}) = \sum_{l=M+1}^{M+K_j} g_\pi(\mathcal{T}_j^i) = K_j g_\pi(\mathcal{T}_j^i).$$

□

Now we consider the operator E_{i1} acting on $g_\pi(\mathcal{T})$.

Lemma 6.2.

$$E_{i1} g_\pi(\mathcal{T}) = K_1 g_\pi(\mathcal{T}_1^i).$$

Proof. Since $\mathcal{T}$ is semistandard, the index 1 only appears in the first K_1 positions of the first row, and so, $E_{i1}^{kl}\mathcal{T}$ is non-zero only when acting on elements in $\boxed{K_1}$. Therefore we have the following decomposition:

$$\begin{aligned} E_{i1} g_\pi(\mathcal{T}) &= g_\pi(\sum_{l=1}^{K_1} E_{i1}^{1l}\mathcal{T}) \\ &= \sum_{l=1}^{K_1} E_{i1}^{1l} g_\pi(\mathcal{T}) \\ &= K_1 g_\pi(\mathcal{T}_1^i). \quad (\textit{Lemma } 6.1) \end{aligned}$$

Observe that $\mathcal{T}_1^i = \mathcal{T}$ when $i = 1$. □

Before moving onto Case (2) we introduce some notation and several Lemmas.

Notation. Recall from definition 4.13 that $\mathcal{T}[i,j]$ denotes the index which occurs in the intersection of the i^{th} row and j^{th} column of $\mathcal{T}$.

Many times the action of elements from $\mathcal{R}_\pi$ on $\mathcal{T}$ will leave $\mathcal{T}$ fixed. In particular, we are interested when this property occurs on the first row of $\mathcal{T}$.

Notation. Recall $\mathcal{N} = \{1, \ldots, N\}$. Let $\hat{\mathcal{N}} = \{\pi_1 + 1, \ldots, N\}$ and so, the symmetric group $S_{\hat{\mathcal{N}}}$ can be embedded into the symmetric group $S_{\mathcal{N}}$. We remove from the canonical tableau, τ_π, the first row to obtain the row diminished canonical tableau $\widetilde{\tau}_\pi$. Let $\hat{\mathcal{R}}_\pi$ be the row group of $\widetilde{\tau}_\pi$ and $\hat{\mathcal{C}}_\pi$ be the column group of $\widetilde{\tau}_\pi$. Then $\hat{g}_\pi$ will denote the corresponding Young symmetrizer explicitly given by

$$\hat{g}_\pi = \sum_{\substack{\gamma \in \hat{\mathcal{C}}_\pi \\ \psi \in \hat{\mathcal{R}}_\pi}} sgn(\gamma)\gamma\psi.$$

Definition 6.2. Let $p \in \mathcal{R}_{\pi_1}$, $M = \sum_{i=1}^{m-1} K_i$. p is said to act **block invariant** on $\boxed{K_m}$ provided:

$$(p\mathcal{T})[1,i] = m \;\; for \;\; i = M+1, \ldots, M+K_m.$$

p is said to act **block invariant on the first row of $\mathcal{T}$** if $(p\mathcal{T})[1,i] = \mathcal{T}[1,i]$ for $i = 1, \ldots, \pi_1$. $S = \{p \in \mathcal{R}_{\pi_1} \mid (p\mathcal{T})[1,i] = \mathcal{T}[1,i] \text{ for } i = 1, \ldots, \pi_1\}$ is a subgroup of $\mathcal{R}_{\pi_1}$, is called the **stabilizer of the top row of $\mathcal{T}$**.

Remark 6.3. Notice that $|S| = K_1! \cdots K_{n+1}!$. To simplify notation, for the remainder of this work set

$$\mathbf{K!} := K_1! \cdots K_{n+1}!.$$

Lemma 6.3. Let $\pi = \{\pi_1 \geq \cdots \geq \pi_p\} \in \prod(N)$, T a π generalized tableau, having top row $\boxed{K_1} \cdots \boxed{K_{n+1}}$, with each $K_i \gg \sum_{j=2}^{p} \pi_j$, and row diminished tableau, $\widetilde{T}$, of T having content $\{2^{t_2}, \ldots, (n+1)^{t_{n+1}}\}$ where $t_i \in \mathbb{Z}_{\geq 0}$. If there exists a $q \in \mathcal{C}_\pi$ and a $p \in \mathcal{R}_\pi$ such that qpT is semi-standard, then $q \in \hat{\mathcal{C}}_\pi$ and p must act block invariant on the top row of T.

Proof. Let $q \in \mathcal{C}_\pi$ and $p \in \mathcal{R}_\pi$ such that qpT is π semi-standard. First consider the action of p on T, and suppose p is not block invariant on the top row of T. Therefore

pT is not a π semi-standard generalized tableau, and we must find a $q \in \mathcal{C}_\pi$ such that qpT is semi-standard. Suppose p does not act block invariant on $\boxed{K_1}$. Since no 1's lie below the 1^{st} row, there does not exist a $q \in \mathcal{C}_\pi$ such that qpT is a π semi-standard generalized tableau. Therefore, p must act block invariant on $\boxed{K_1}$. Suppose p does not act block invariant on $\boxed{K_i}$ for some $i = 2, \dots, n+1$. Since $K_1 \gg \sum_{j=2}^{p} \pi_j$, there does not exist a $q \in \mathcal{C}_\pi$ such that qpT is semi-standard. Therefore, p must act block invariant on the top row of T.

Now consider the action of q on pT. Suppose $q \notin \hat{\mathcal{C}}_\pi$. Therefore q must permute the top row of pT non-trivially, and since pT has top $\boxed{K_1}\cdots\boxed{K_{n+1}}$, q will permute a 1 into $\widetilde{T}$, creating a non semi-standard tableau. Therefore, $q \in \hat{\mathcal{C}}_\pi$.

□

Notation. Let T and T' be π generalized tableaux. $[T : g_\pi(T')]$ will denote the number of times T appears in $g_\pi(T')$ when $g_\pi(T')$ is expressed as a sum of generalized tableaux written in terms of the basis elements of $\otimes^N \mathcal{V}$.

Lemma 6.4. Let $\pi = \{\pi_1 \geq \cdots \geq \pi_p\} \in \prod(N)$, T be a π generalized tableau having top row $\boxed{K_1}\cdots\boxed{K_{n+1}}$, with each $K_i \gg \sum_{j=2}^{p} \pi_j$, and row diminished tableau, $\widetilde{T}$, of T having content $\{2^{t_2}, \dots, (n+1)^{t_{n+1}}\}$ where $t_i \in \mathbb{Z}_{\geq 0}$. Let T_s be a π semi-standard generalized tableau having shape and content identical to T. Then

$$[T_s : g_\pi(T)] = \mathbf{K}![T_s : \hat{g}_\pi(T)].$$

Proof. $\mathcal{R}_\pi = \mathcal{R}_{\pi_1} \times \hat{\mathcal{R}}_\pi$. Let $S \subset \mathcal{R}_{\pi_1}$ be stabilizer of the top row of T. Let $\sigma_0 = id$, $\sigma_i \in \mathcal{R}_{\pi_1}$ for $i = 1, \dots, l$ be transversals for $\mathcal{R}_{\pi_1}/S$. That is, $\mathcal{R}_{\pi_1} = \sigma_0 S \uplus \cdots \uplus \sigma_l S$. Let $\mu_0 = id$, $\mu_i \in \mathcal{C}_\pi$ for $i = 1, \dots, r$ be transversals for $\mathcal{C}_\pi/\hat{\mathcal{C}}_\pi$. That is, $\mathcal{C}_\pi = \mu_0 \hat{\mathcal{C}}_\pi \uplus \cdots \uplus \mu_r \hat{\mathcal{C}}_\pi$.

$$\begin{aligned} g_\pi(T) &= \left(\sum_{\gamma \in \mathcal{C}_\pi} sgn(\gamma)\gamma\right)\left(\sum_{p \in \mathcal{R}_\pi} p\right)(T) \\ &= \left(\sum_{\gamma \in \hat{\mathcal{C}}_\pi} sgn(\gamma)\gamma + \sum_{i=1}^{r}\sum_{\gamma \in \hat{\mathcal{C}}_\pi} sgn(\mu_i) sgn(\gamma)\mu_i\gamma\right)\left(\sum_{p \in S \times \hat{\mathcal{R}}_\pi} p + \sum_{i=1}^{l}\sum_{s \in S \times \hat{\mathcal{R}}_\pi} \sigma_i s\right)(T) \end{aligned}$$

We are interested in the π semi-standard generalized tableaux appearing in the expansion of $g_\pi(T)$. Examining the right hand side of the above equation, notice for $i \geq 1$ $\mu_i\gamma \notin \hat{\mathcal{C}}_\pi$ and $\sigma_i s \notin S$. By Lemma 6.3, the action of either of these elements on T will result in a non π semi-standard generalized tableau. Therefore,

$$\left(\sum_{\gamma\in\hat{\mathcal{C}}_\pi} sgn(\gamma)\gamma\right)\left(\sum_{s\in S}\sum_{p\in\hat{\mathcal{R}}_\pi} ps\right)$$

are the only terms in g_π whose action on T can create π semi-standard generalized tableaux. Since

$$\begin{aligned}\left(\sum_{\gamma\in\hat{\mathcal{C}}_\pi} sgn(\gamma)\gamma\right)\left(\sum_{s\in S}\sum_{p\in\hat{\mathcal{R}}_\pi} ps\right)(T) &= \mathbf{K!}\left(\sum_{\gamma\in\hat{\mathcal{C}}_\pi} sgn(\gamma)\gamma\right)\left(\sum_{p\in\hat{\mathcal{R}}_\pi} p\right)(T)\\ &= \mathbf{K!}\hat{g}_\pi(T),\end{aligned}$$

any semi-standard appearing in $g_\pi(T)$ appears exactly $\mathbf{K!}$ times more then in $\hat{g}_\pi(T)$. □

Lemma 6.5. Let $\pi = \{\pi_1 \geq \cdots \geq \pi_p\} \in \prod(N)$, T be a π generalized tableau having top row $\boxed{K_1}\cdots\boxed{K_{n+1}}$, with each $K_i \gg \sum_{j=2}^{p}\pi_j$, and row diminished tableau, $\widetilde{T}$, of T having content $\{2^{t_2}, \ldots, (n+1)^{t_{n+1}}\}$ where $t_i \in \mathbb{Z}_{\geq 0}$. Let $\{S_1, \ldots, S_k\}$ be a set of semi-standard generalized tableaux such that

$$(1) \quad \hat{g}_\pi(\widetilde{T}) = \sum_{i=1}^{k} c_i\hat{g}_\pi(S_i).$$

Then letting T_i be the π semi-standard generalized tableau having top row identical to top row of T, and having a row diminished tableau S_i, for $i = 1, \ldots, k$, it follows that

$$g_\pi(T) = \sum_{i=1}^{k} c_i g_\pi(T_i),$$

where the c_i's are as in (1). As a result, each c_i is a constant number independent of $K_1, \ldots, K_{n+1}$.

Proof. By assumption,

$$\hat{g}_\pi(\widetilde{T}) - \sum_{i=1}^{k} c_i \hat{g}_\pi(S_i),$$

has no semi-standard generalized tableaux when expressed with respect to the bases of $\otimes^N \mathcal{V}$. Therefore,

$$(1) \quad \hat{g}_\pi(T) - \sum_{i=1}^{k} c_i \hat{g}_\pi(T_i),$$

has no semi-standard generalized tableaux when expressed with respect to the bases of $\otimes^N \mathcal{V}$.

Suppose,

$$(2) \quad g_\pi(T) - \sum_{i=1}^{k} c_i g_\pi(T_i),$$

has a π semi-standard generalized tableaux, $\hat{T}$, when (2) is expressed with respect to the bases of $\otimes^N \mathcal{V}$.

By Lemma 6.4,

$$[\hat{T} : g_\pi(T)] = \mathbf{K!}[\hat{T} : \hat{g}_\pi(T)],$$

and

$$[\hat{T} : g_\pi(T_i)] = \mathbf{K!}[\hat{T} : \hat{g}_\pi(T_i)] \text{ for } i = 1, \ldots, k.$$

Therefore, the coefficient in front of $\hat{T}$ in (2) is,

$$\mathbf{K!} \times [\hat{T} : \hat{g}_\pi(T)] - \mathbf{K!} \sum_{i=1}^{k} c_i [\hat{T} : \hat{g}_\pi(T_i)].$$

By combining this with (1)

$$[\hat{T} : g_\pi(T)] - \sum_{i=1}^{k} c_i[\hat{T} : g_\pi(T_i)] = \mathbf{K!} \times [\hat{T} : \hat{g}_\pi(T)] - \mathbf{K!}\sum_{i=1}^{k} c_i[\hat{T} : \hat{g}_\pi(T_i)]$$
$$= \mathbf{K!}([\hat{T} : \hat{g}_\pi(T)] - \sum_{i=1}^{k} c_i[\hat{T} : \hat{g}_\pi(T_i)])$$
$$= \mathbf{K!} \times 0$$
$$= 0.$$

Therefore, when expressing (2) with respect to the bases of $\otimes^N \mathcal{V}$, $\hat{T}$ appears with coefficient 0, which implies (2) has no π semi-standard generalized tableaux when expressed with respect to the bases of $\otimes^N \mathcal{V}$. Therefore

$$g_\pi(T) = \sum_{i=1}^{k} c_i g_\pi(T_i),$$

where the c_i's are as they were in (1). Since the c_i's came out of

$$\hat{g}_\pi(\widetilde{T}) = \sum_{i=1}^{k} c_i \hat{g}_\pi(S_i),$$

and all tableaux involved in the above equation are independent of $K_1, \ldots, K_{n+1}$, it must follow that each c_i is a constant number independent of $K_1, \ldots, K_{n+1}$.

□

We are now in a position to examine Case (2).

Lemma 6.6. For $i \neq 1 \neq j$,

$$E_{ij} g_\pi(\mathcal{T}) = K_j g_\pi(\mathcal{T}_j^i) + \sum_r c_r g_\pi(\mathcal{T}_r),$$

where each c_r is a constant number independent of $K_1, \ldots, K_{n+1}$, and each $\mathcal{T}_r$ is a π semi-standard generalized tableau having top row identical to the top row of $\mathcal{T}$.

Proof. Let $M = \sum_{i=1}^{j-1} K_i$. Since, $E_{ij}^{1l}\mathcal{T} \neq 0$ only when $l = M+1, \ldots, M+K_j$ we have

$$E_{ij}\mathcal{T} = \sum_{l=M+1}^{M+K_j} E_{ij}^{1l}\mathcal{T} + \sum_{\substack{k=2,\ldots,p \\ l=1,\ldots,\pi_k}} E_{ij}^{kl}\mathcal{T}.$$

Therefore

$$\begin{aligned} E_{ij}g_\pi(\mathcal{T}) &= g_\pi\left(\sum_{l=M+1}^{M+K_j} E_{ij}^{1l}\mathcal{T} + \sum_{\substack{k=2,\ldots,p \\ l=1,\ldots,\pi_k}} E_{ij}^{kl}\mathcal{T}\right) \\ &= \sum_{l=M+1}^{M+K_j} g_\pi(E_{ij}^{1l}\mathcal{T}) + \sum_{\substack{k=2,\ldots,p \\ l=1,\ldots,\pi_k}} g_\pi(E_{ij}^{kl}\mathcal{T}) \\ &= K_j g_\pi(\mathcal{T}_j^i) + \sum_{\substack{k=2,\ldots,p \\ l=1,\ldots,\pi_k}} g_\pi(E_{ij}^{kl}\mathcal{T}) \quad (\textit{Lemma } 6.1), \end{aligned}$$

Now consider the second term in the right hand side of the above equation.

For $k > 1$, $E_{ij}^{kl}\mathcal{T}$ has top row identical to $\mathcal{T}$, with row diminished tableau, $\widetilde{E_{ij}^{kl}\mathcal{T}}$, of $E_{ij}^{kl}\mathcal{T}$ having content $\{2^{m_2}, \ldots, i^{m_i+1}, \ldots, j^{m_j-1}, \ldots, (n+1)^{m_{n+1}}\}$. By Theorem 4.1,

$$\hat{g}_\pi(\widetilde{E_{ij}^{kl}\mathcal{T}}) = \sum_r c_r \hat{g}_\pi(S_r),$$

where each S_r is a semi-standard generalized tableau and each $c_r \in \mathbb{C}$. Let $\mathcal{T}_r$ be the π semi-standard generalized tableau with top row identical to the top row of $\mathcal{T}$, and row diminished tableau S_r. By Lemma 6.5,

$$g_\pi(E_{ij}^{kl}\mathcal{T}) = \sum_r c_r g_\pi(\mathcal{T}_r),$$

where each c_r is a constant number independent of $K_1, \ldots, K_{n+1}$.

Since our k and l were arbitrary, this results holds for all $k > 1$ and l, and so,

$$\sum_{\substack{k=2,\ldots,p \\ l=1,\ldots,\pi_k}} g_\pi(E_{ij}^{kl}\mathcal{T}) = \sum_r c_r g_\pi(\mathcal{T}_r),$$

where each c_r is a constant number independent of $K_1, \ldots, K_{n+1}$.

Combining our results,

$$E_{ij} g_\pi(\mathcal{T}) = K_j g_\pi(\mathcal{T}_j^i) + \sum_r c_r g_\pi(\mathcal{T}_r)$$

which completes the proof.

□

We have now achieved our goal for this section and summarize below.

Case (1): For arbitrary i,

$$E_{i1} g_\pi(\mathcal{T}) = K_1 g_\pi(\mathcal{T}_1^i).$$

Case (2): For $i \neq 1 \neq j$,

$$E_{ij} g_\pi(\mathcal{T}) = K_j g_\pi(\mathcal{T}_j^i) + \sum_r c_r g_\pi(\mathcal{T}_r),$$

where each c_r is a constant number independent of $K_1, \ldots, K_{n+1}$, and each $\mathcal{T}_r \in \mathcal{S}_\pi(N)$ having top row identical to top row of $\mathcal{T}$.

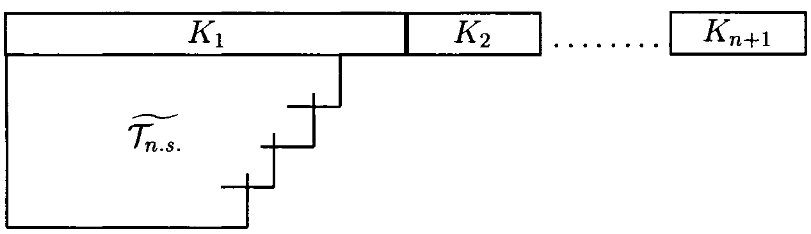
6.3 Introducing Case (3)

Analyzing $E_{1j} g_\pi(\mathcal{T})$ is more complicated. This section outlines the difficulties which arise in this situation, and the need for several counting properties.

Consider the decomposition of $E_{1j}g_\pi(\mathcal{T})$:

$$\begin{aligned} g_\pi(E_{1j}\mathcal{T}) &= g_\pi\Big(\sum_{l=1,\dots,\pi_1} E_{1j}^{1l}\mathcal{T}\Big) + g_\pi\Big(\sum_{\substack{k=2,\dots,p\\ l=1,\dots,\pi_k}} E_{1j}^{kl}\mathcal{T}\Big) \\ &= \sum_{l=1,\dots,\pi_1} g_\pi(E_{1j}^{1l}\mathcal{T}) + \sum_{\substack{k=2,\dots,p\\ l=1,\dots,\pi_k}} g_\pi(E_{1j}^{kl}\mathcal{T}) \\ &= K_j g_\pi(\mathcal{T}_j^1) \;+ \sum_{\substack{k=2,\dots,p\\ l=1,\dots,\pi_k}} g_\pi(E_{1j}^{kl}\mathcal{T}) \qquad (\textit{Lemma } 6.1), \end{aligned} \tag{4}$$

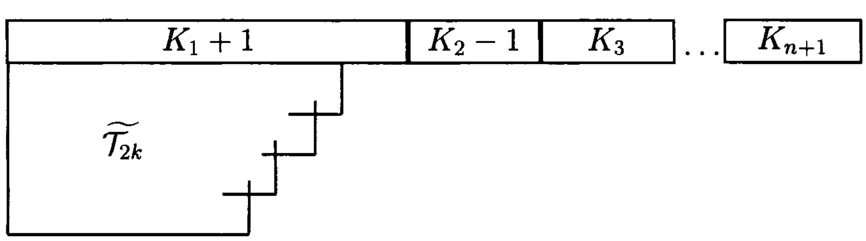
Unfortunately for $k \geq 2$, $E_{1j}^{kl}\mathcal{T}$ does not create a generalized tableau row equivalent to some π semi-standard generalized tableau. For $k \geq 2$ the action of E_{1j}^{kl} on $\mathcal{T}$ creates non π semi-standard generalized tableaux of the form:

$\mathcal{T}_{n.s.} =$


where $\widetilde{\mathcal{T}_{n.s.}}$ is the row diminished tableau of $\mathcal{T}_{n.s.}$ identical to $\widetilde{\mathcal{T}}$ except the k^{th} row, l^{th} column has a 1 in place of a j, i.e. $\widetilde{\mathcal{T}_{n.s.}} = \widetilde{E_{1j}^{kl}\mathcal{T}}$.

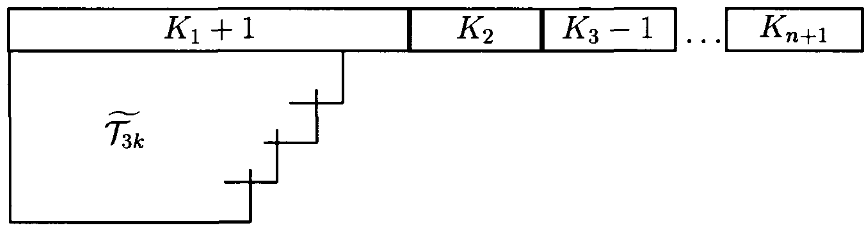
Recall, $\widetilde{\mathcal{T}}$ has content $\{2^{m_2}, \dots, (n+1)^{m_{n+1}}\}$. For $i = 2, \dots, n+1$, define f_i to be the number of π semi-standard generalized tableaux having shape equal to the shape of $\widetilde{\mathcal{T}}$ and content $\{2^{m_2}, \dots, j^{m_j-1}, \dots, i^{m_i+1}, \dots, (n+1)^{m_{n+1}}\}$.

Define the following π semi-standard generalized tableaux:

$\mathcal{T}_{2k} =$


for $k = 1, \ldots, f_2$.

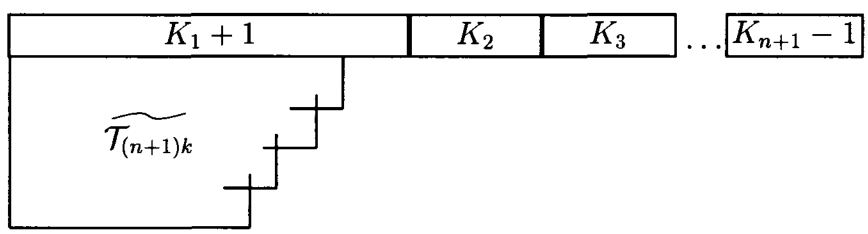
$\mathcal{T}_{3k} =$



for $k = 1, \ldots, f_3$.

$$\vdots$$

$\mathcal{T}_{(n+1)k} =$



for $k = 1, \ldots, f_{n+1}$.

where $\widetilde{\mathcal{T}_{ik}}$ is a row diminished tableau of $\mathcal{T}_{ik}$ having content $\{2^{m_2}, \ldots, j^{m_j-1}, \ldots, i^{m_i+1}, \ldots, (n+1)^{m_{n+1}}\}$ for $i = 2, \ldots, n+1$ and $k = 1, \ldots, f_i$, necessarily having each $K_i \gg \sum_{j=2}^{p} \pi_j$.

Lemma 6.7. Let $p \in \mathcal{R}_\pi$ and $q \in \mathcal{C}_\pi$. If $qp\mathcal{T}_{ik}$ is a π semi-standard generalized tableau then $q \in \hat{\mathcal{C}}_\pi$ and p must act block invariant on the first row of $\mathcal{T}_{ik}$.

Proof. This is just a special case of Lemma 6.3. □

Remark 6.4. The above lemma implies that when $i \neq r$, $\mathcal{T}_{ik}$ does not appear in $g_\pi(\mathcal{T}_{rs})$ when expressed in terms of the bases for $\otimes^N \mathcal{V}$.

Lemma 6.8.

$$g_\pi(\mathcal{T}_{n.s.}) = \sum_{i=2}^{n+1}\sum_{k=1}^{f_i} c_{ik} g_\pi(\mathcal{T}_{ik}) \text{ where } c_{ik} \in \mathbb{C}.$$

Proof. First consider the π semi-standards appearing in expansion of $g_\pi(\mathcal{T}_{n.s.})$. Let $q \in \mathcal{C}_\pi$ and $p \in \mathcal{R}_\pi$ such that $qp\mathcal{T}_{n.s.}$ is semi-standard. Suppose p has already acted on $\mathcal{T}_{n.s.}$ and consider the action of q on $p\mathcal{T}_{n.s.}$. The 1 in $\widetilde{p\mathcal{T}_{n.s.}}$ must be permuted into the top row of $p\mathcal{T}_{n.s.}$. We claim that 1 is the only element which can be permuted out of $\widetilde{p\mathcal{T}_{n.s.}}$. Suppose another element besides 1 were permuted into the top row of $p\mathcal{T}_{n.s.}$. Since $K_1 \gg \sum_{j=2}^{p} \pi_j$, the top row would not be weakly increasing from left to right, and hence a semi-standard would not result. Therefore, only the 1 from $\widetilde{p\mathcal{T}_{n.s}}$ can be permuted to the top row of $p\mathcal{T}_{n.s.}$, and must be replaced by a $2, \ldots, n+1$. Therefore, any π semi-standard generalized tableau appearing in the expansion of $g_\pi(\mathcal{T}_{n.s.})$ is of the form $\mathcal{T}_{ik}$ for some $i = 2, \ldots, n+1$ and $k = 1, \ldots, f_i$.

Now consider the types of semi-standard generalized tableaux appearing in $g_\pi(\mathcal{T}_{ik})$. By Lemma 6.7, any semi-standard tableau in $g_\pi(\mathcal{T}_{ik})$ must have top row identical to the top row of $\mathcal{T}_{ik}$, and therefore must be of the form $\mathcal{T}_{il}$ for some $l = 1, \ldots, f_i$.

Define π semi-standard generalized tableaux, T_r, having the same content as $\mathcal{T}_{n.s.}$ but not equal to any of the $\mathcal{T}_{ik}$'s. Suppose there exist coefficients c_{ik} and c_r such that

$$g_\pi(\mathcal{T}_{n.s.}) = \sum_{i=2}^{n+1}\sum_{k=1}^{f_i} c_{ij} g_\pi(\mathcal{T}_{ik}) + \sum_r c_r g_\pi(T_r).$$

Using the partial ordering on tableaux from Definition 4.17 let $\hat{T}_r$ be maximal among the T_r's. By Lemma 4.4, $\hat{T}_r$ appears with a non-zero coefficient when expressing $\sum_r c_r g_\pi(T_r)$ in terms of the bases of $\otimes^N \mathcal{V}$. By the above argument, for any i, k, $\hat{T}_r$ does not appear in $g_\pi(\mathcal{T}_{n.s.})$ and $g_\pi(\mathcal{T}_{ik})$ when expressed with respect to the bases of $\otimes^N \mathcal{V}$. This contradiction implies that

$$g_\pi(\mathcal{T}_{n.s.}) = \sum_{i=2}^{n+1}\sum_{k=1}^{f_i} c_{ik} g_\pi(\mathcal{T}_{ik})$$

□

We wish to solve for the coefficients c_{ik} to determine their dependence on $K_1, \dots, K_{n+1}$. Before doing this, we introduce some counting properties.

6.4 Counting Properties

In this section we discuss several counting properties which will aid us in answering the following two questions:

1. How many times does $\mathcal{T}_{ij}$ appear in the expansion of $g_\pi(\mathcal{T}_{kl})$?

2. How many times does $\mathcal{T}_{ij}$ appear in the expansion of $g_\pi(\mathcal{T}_{n.s.})$?

We again remind the reader that $\mathcal{T}_{n.s.}$ and $\mathcal{T}_{ij}$ are the generalized tableaux defined explicitly in section 6.3.

Lemma 6.9. The number of times $\mathcal{T}_{ij}$ appears in the expansion of $g_\pi(\mathcal{T}_{ik})$ is

$$\frac{(K_1+1)}{K_i}\mathbf{K!} \times M$$

where M is a constant number, possibly zero, independent of $K_1, \dots, K_{n+1}$.

Proof. By Lemma 6.4,

$$\begin{aligned}[][\mathcal{T}_{ij} : g_\pi(\mathcal{T}_{ik})] &= (K_1+1)!K_2! \cdots K_{i-1}!(K_i-1)!K_{i+1}! \cdots K_{n+1}![\mathcal{T}_{ij} : \hat{g}_\pi(\mathcal{T}_{ik})] \\ &= \frac{K_1+1}{K_i}\mathbf{K!}[\mathcal{T}_{ij} : \hat{g}_\pi(\mathcal{T}_{ik})].\end{aligned}$$

Since $\hat{g}_\pi$ involves permutations which act on $\widetilde{\mathcal{T}_{ik}}$, $M = [\mathcal{T}_{ij} : \hat{g}_\pi(\mathcal{T}_{ik})]$ must be the same constant number for all values of $K_1, \dots, K_{n+1}$ chosen sufficiently large.

□

Recall, for $i \in \{1, \dots, n+1\}$, $k \in \mathbb{Z}$ and $K_i \in \mathbb{Z}_{>0}$, define $\boxed{K_i}$ to be a one row tableau having K_i boxes, where each box contains the value i. $\boxed{K_i + k}$ will indicate a 1 row tableau having $K_i + k$ boxes each containing the value of i. $\boxed{K_1}\boxed{K_2}\ \cdots\ \boxed{K_{n+1}}$ stands for a one row tableau containing K_1 1's followed by K_2 2's, and so on.

Notation. Let $\mathcal{T}_{n.s.}(1,i)$ denote a π generalized tableau obtained from the π generalized tableau $\mathcal{T}_{n.s.} = E^{kl}_{1j}\mathcal{T}$, by replacing the top row by $\boxed{K_1+1}\cdots\boxed{K_i-1}\cdots\boxed{K_{n+1}}$ and the 1 in $\widetilde{\mathcal{T}_{n.s.}}$ by i.

Lemma 6.10. Let $p = p_1 p_1^*$ with $p_1 \in \mathcal{R}_{\pi_1}$ and $p_1^* \in \hat{\mathcal{R}}_\pi$. Without loss of generality suppose the 1 in $\widetilde{p_1^*\mathcal{T}_{n.s.}}$ is located in the l^{th} column. Then

1. If there exists a $q \in \mathcal{C}_\pi$ such that $qp\mathcal{T}_{n.s.} = \mathcal{T}_{ij}$ then the top row of $p\mathcal{T}_{n.s.}$ is of the form,

$$\boxed{(l-1)_1}\boxed{1_i}\boxed{K_1-(l-1)}\boxed{K_2}\cdots\boxed{K_{n+1}}$$

2. The number of $p_1 \in \mathcal{R}_{\pi_1}$ such that $p_1 p_1^*\mathcal{T}_{n.s.}$ has top row

$$\boxed{(l-1)_1}\boxed{1_i}\boxed{K_1-(l-1)}\boxed{K_2}\cdots\boxed{K_{n+1}}$$

   is **K!**.

3. There exists at most one $q \in \mathcal{C}_\pi$ such that $qp\mathcal{T}_{n.s.} = \mathcal{T}_{ij}$.

4. If there is a $q \in \mathcal{C}_\pi$ such that $qp_1p_1^*\mathcal{T}_{n.s.} = \mathcal{T}_{ij}$, then there is a unique $q' \in \hat{\mathcal{C}}_\pi$ such that $q'p_1^*\mathcal{T}_{n.s.}(1,i) = \mathcal{T}_{ij}$.

Proof. (1) Consider the action of q on $p_1p_1^*\mathcal{T}_{n.s.}$. Suppose an element other then 1 were permuted into the top row of $p_1p_1^*\mathcal{T}_{n.s.}$. Since $K_1 \gg \sum_{r=2}^{p}\pi_r$, the top row of $qp_1p_1^*\mathcal{T}_{n.s.}$ would not be weakly increasing from left to right, and therefore, not in semi-standard form. Therefore, only the 1 in $\widetilde{p_1p_1^*\mathcal{T}_{n.s.}}$ may be permuted into the top row of $p_1p_1^*\mathcal{T}_{n.s.}$. Since an i from the top row of $p_1p_1^*\mathcal{T}_{n.s.}$ must also be permuted into $\widetilde{p_1p_1^*\mathcal{T}_{n.s.}}$ it follows that the top row of $p_1p_1^*\mathcal{T}_{n.s.}$ is of the form,

$$\boxed{(l-1)_1}\boxed{1_i}\boxed{K_1-(l-1)}\boxed{K_2}\cdots\boxed{K_{n+1}}.$$

(2) Let S be the stabilizer of the top row of $p_1^*\mathcal{T}_{n.s.}$ and

$$P_l = \{\sigma \in \mathcal{R}_{\pi_1} \mid \sigma(\boxed{K_1}\cdots\boxed{K_{n+1}}) = \boxed{(l-1)_1}\boxed{1_i}\boxed{K_1-(l-1)}\boxed{K_2}\cdots\boxed{K_{n+1}}\}.$$

Fix $\sigma_0 \in P_l$. Let $f : S \to P_l$ be a map, given by $f(\sigma) = \sigma_0 \circ \sigma$. We claim that f is a bijective map. Take $\sigma_1, \sigma_2 \in S$ then $f(\sigma_1) = f(\sigma_2)$ implies $\sigma_0 \circ \sigma_1 = \sigma_0 \circ \sigma_2$. Multiplying both sides by σ_0^{-1} gives $\sigma_1 = \sigma_2$ and therefore f is one to one. To see that f is onto take an arbitrary $\sigma_l \in P_l$. Then $f(\sigma_0^{-1}\sigma_l) = \sigma_l$ and therefore, f is onto. f is a bijective map between finite sets S and P_l and therefore $|P_l| = |S| = \mathbf{K!}$.

(3) Assume $p \in \mathcal{R}_\pi$ is such that there exists a $q \in \mathcal{C}_\pi$ with $qp\mathcal{T}_{n.s.} = \mathcal{T}_{ij}$. Clearly, the columns of $p\mathcal{T}_{n.s.}$ must have all distinct entries. Since there is only one way to order each column such that it is strictly increasing, reading top to bottom, this q must be unique.

(4) We are assuming without loss of generality that the 1 in $\widetilde{p\mathcal{T}_{n.s.}}$ is located in the l^{th} column. Since there exists a $q \in \mathcal{C}_\pi$ such that $qp\mathcal{T}_{n.s.} = \mathcal{T}_{ij}$, by part 1 the top row of $p\mathcal{T}_{n.s.}$ must have the form:

$$\boxed{(l-1)_1}\boxed{1_i}\boxed{K_1-(l-1)}\boxed{K_2}\cdots\boxed{K_{n+1}}.$$

Now $q = q_1 \cdots q_r$ with $q_i \in \hat{\mathcal{C}}_\pi$ for $i \neq l$ and $q_l \in \mathcal{C}_\pi \setminus \hat{\mathcal{C}}_\pi$ such that q_i orders the elements in the i^{th} column of $p\mathcal{T}_{n.s.}$ so that they are strictly increasing from top to bottom. $p\mathcal{T}_{n.s.}$ and $p_1^*\mathcal{T}_{n.s.}(1,i)$ only differ in the 1^{st} entry in the l^{th} column and without loss of generality, suppose the k^{th} entries in the l^{th} column. That is, the l^{th} column of $p\mathcal{T}_{n.s.}$ and $p_1^*\mathcal{T}_{n.s.}(1,i)$ are

$$\boxed{\begin{matrix} \mathrm{i} \\ \vdots \\ 1 \\ \vdots \end{matrix}} \qquad \boxed{\begin{matrix} 1 \\ \vdots \\ \mathrm{i} \\ \vdots \end{matrix}}$$

respectively, where the entries not listed imply that they are identical in $p\mathcal{T}_{n.s.}$ and $p_1^*\mathcal{T}_{n.s.}(1,i)$.

We need to find a $q' \in \hat{\mathcal{C}}_\pi$ such that $q'p_1^*\mathcal{T}_{n.s.}(1,i) = \mathcal{T}_{ij}$. Let s be the unique permutation in $\mathcal{C}_\pi$ which interchanges the 1^{st} and k^{th} entry in the l^{th} column of $p_1^*\mathcal{T}_{n.s.}(1,i)$. Let $q_l' = q_l s$. $q_l' \in \hat{\mathcal{C}}_\pi$ since the 1 in the l^{th} column of $p_1^*\mathcal{T}_{n.s.}(1,i)$ is already in the correct position. Then $q' = q_1 \ldots q_{l-1}q_l'q_{l+1}\ldots q_r \in \hat{\mathcal{C}}_\pi$ is such that $q'p_1^*\mathcal{T}_{n.s.}(1,i) = \mathcal{T}_{ij}$. By part 3 q' is unique.

□

Lemma 6.11. Let $\mathcal{T}_{n.s.}$ and $\mathcal{T}_{ij}$ be the π generalized tableaux defined explicitly in section 6.3. Let $p_1 \in \mathcal{R}_{\pi_1}$ and $p_1^* \in \hat{\mathcal{R}}_\pi$. Assume the 1 in $\widetilde{p_1^*\mathcal{T}}_{n.s.}$ is located in the l^{th} column. Define the set

$$P_l = \{\sigma \in \mathcal{R}_{\pi_1} \mid \sigma(\boxed{K_1}\cdots\boxed{K_{n+1}}) = \boxed{(l-1)_1}\boxed{1_i}\boxed{K_1-(l-1)}\boxed{K_2}\cdots\boxed{K_{n+1}}\}.$$

Then,

$$[\mathcal{T}_{ij} : g_\pi(\mathcal{T}_{n.s.})] = \mathbf{K!} \times M,$$

where M is a constant number independent of $K_1, \ldots, K_{n+1}$.

Proof.

$$\begin{aligned}
[\mathcal{T}_{ij} : g_\pi(\mathcal{T}_{n.s.})] &= [\mathcal{T}_{ij} : \sum_{\gamma\in\mathcal{C}_\pi}\sum_{\rho\in\mathcal{R}_\pi} sgn(\gamma)\gamma\rho\mathcal{T}_{n.s.}] \\
&= [\mathcal{T}_{ij} : \sum_{\rho\in\mathcal{R}_\pi} sgn(\gamma_\rho)\gamma_\rho\rho\mathcal{T}_{n.s.}] && \textit{(Lemma 6.10 part 3)} \\
&= [\mathcal{T}_{ij} : \sum_{\rho_1\in P_l}\sum_{\rho_1^*\in\hat{\mathcal{R}}_\pi} sgn(\gamma_\rho)\gamma_\rho\rho_1\rho_1^*\mathcal{T}_{n.s.}] && \textit{(Lemma 6.10 part 1)} \\
&= \mathbf{K!}[\mathcal{T}_{ij} : \sum_{\rho_1^*\in\hat{\mathcal{R}}_\pi} sgn(\gamma_\rho)\gamma_\rho\rho_1\rho_1^*\mathcal{T}_{n.s.}] && \textit{(Lemma 6.10 part 2)} \\
&= \mathbf{K!}[\mathcal{T}_{ij} : \sum_{\rho_1^*\in\hat{\mathcal{R}}_\pi} sgn(\gamma_\rho')\gamma_\rho'\rho_1^*\mathcal{T}_{n.s.}(1,i)] && \textit{(Lemma 6.10 part 4)} \\
&= \mathbf{K!}[\mathcal{T}_{ij} : \hat{g}_\pi(\mathcal{T}_{n.s.}(1,i))].
\end{aligned}$$

$[\mathcal{T}_{ij} : \hat{g}_\pi(\mathcal{T}_{n.s.}(1,i))]$ is a constant number independent of $K_1, \ldots, K_{n+1}$. □

With Lemma 6.9 and Lemma 6.11 answering questions one and two, we move onto examining Case (3).

6.5 Case (3)

Before we discuss Case (3) recall, $\mathcal{T}$ was the fixed core π semi-standard generalized tableau defined in section 6.1, and $\mathcal{T}_{n.s}$ and $\mathcal{T}_{ik}$ for $i = 2, \ldots, n+1$ and $k = 1, \ldots, f_i$ are the π generalized tableaux defined in section 6.3.

In Case (3) we consider the action of E_{1j} on $g_\pi(\mathcal{T})$, which in section 6.3 yielded the following decomposition:

$$g_\pi(E_{1j}\mathcal{T}) = K_j g_\pi(\mathcal{T}_j^1) \; + \sum_{\substack{k=2,\ldots,p \\ l=1,\ldots,\pi_k}} g_\pi(E_{1j}^{kl}\mathcal{T}).$$

We still need to straighten the sum of terms in the right hand side of the above equation. As $\mathcal{T}_{n.s.}$ represents $E_{ij}^{kl}\mathcal{T}$ for arbitrary integers $k \geq 2$ and l our first goal is to straighten $g_\pi(\mathcal{T}_{n.s.})$.

Before discussing a straightening algorithm for $g_\pi(\mathcal{T}_{n.s.})$, we first make the following definition.

Definition 6.3. Let $(\mathcal{C}, \leq)$ be a partially ordered set. Let $\mathcal{C}_1$ be the set of all maximal elements in $\mathcal{C}$, and for $k > 1$, $\mathcal{C}_k$ be the set of all maximal elements in $\mathcal{C} \setminus \bigcup_{r=1}^{k-1} \mathcal{C}_r$. $\mathcal{C}$ is said to have k**-layers** provided $\mathcal{C} = \bigcup_{i=1}^{k} \mathcal{C}_i$. The i^{th}**-layer** of $\mathcal{C}$ is $\mathcal{C}_i$. $\mathcal{C}_i$ is said to be in an **upper-layer** to $\mathcal{C}_j$ provided $i < j$.

Theorem 6.1. Let $\mathcal{T}_{n.s.}$ and $\mathcal{T}_{ik}$ be the π generalized tableaux defined in section 6.3.

$$g_\pi(\mathcal{T}_{n.s.}) = \sum_{i=2}^{n+1} \sum_{k=1}^{f_i} \frac{K_i}{K_1 + 1} M_{ik} g_\pi(\mathcal{T}_{ik}),$$

where each M_{ik} is a constant number independent of $K_1, \ldots, K_{n+1}$.

Proof. Fix an $i = 2, \ldots, n+1$. Let $\mathcal{C}^i = \{\mathcal{T}_{ik} \mid k = 1, \ldots, f_i\}$. $(\mathcal{C}^i, \leq)$ is a partially ordered set with $\leq$ defined in Definition 4.17. Re-index $\mathcal{C}^i$ in terms of layers. That is suppose $\mathcal{C}^i$ has t-layers. Then $\mathcal{C}^i = \bigcup_{l=1}^{t} \mathcal{C}_l$, with $\mathcal{C}_l = \{S_{l1}, \ldots, S_{lm_l}\}$, where $m_l \in \mathbb{Z}_{>0}$

and $\sum_{l=1}^{t} m_l = f_i$.

It suffices to show that there exists coefficients M_{lq} for $l = 1, \ldots, t$ and $q = 1, \ldots, m_l$, where each M_{lq} is a constant number independent of $K_1, \ldots, K_{n+1}$, such that when expressing

$$g_\pi(\mathcal{T}_{n.s.}) - \sum_{l=1}^{t}\sum_{q=1}^{m_l} \frac{K_i}{K_1+1} M_{lq} g_\pi(S_{lq})$$

in terms of a basis of $\otimes^N \mathcal{V}$, no elements in $\mathcal{C}^i$ appear. The justification of this claim will be withheld until the end of this proof. The reason for this, is that by this time, the reader will then be familiar with our straightening algorithm, which will make the justification of our claim easier to describe.

We show by induction that we can remove all occurrences of S_{lq} for $l = 1, \ldots t$ and $q = 1, \ldots, m_l$. The inductive parameter is the layer index l. We begin by removing all occurrences of S_{1j} for $j = 1, \ldots, m_1$. At the conclusion of this step, the first layer of $\mathcal{C}^i$ will be removed.

Without loss of generality, take $S_{11} \in \mathcal{C}_1$. By Lemma 6.11, Lemma 6.9 and Remark 4.6

$$[S_{11} : g_\pi(\mathcal{T}_{n.s.})] = \mathbf{K!} \times Q_{11} \text{ and } [S_{11} : g_\pi(S_{11})] = \tfrac{K_1+1}{K_i}\mathbf{K!} \times R_{11},$$

where Q_{11} is a constant number and R_{11} is a non-zero constant number, both independent of $K_1, \ldots, K_{n+1}$.

Therefore when

$$g_\pi(\mathcal{T}_{n.s.}) - \frac{[S_{11} : g_\pi(\mathcal{T}_{n.s.})]}{[S_{11} : g_\pi(S_{11})]} g_\pi(S_{11}) = g_\pi(\mathcal{T}_{n.s.}) - \frac{K_i}{K_1+1}\frac{Q_{11}}{R_{11}} g_\pi(S_{11}) \qquad (5)$$

is expressed with respect to a basis for $\otimes^N \mathcal{V}$, no occurrences of S_{11} appear. Since all elements in $\mathcal{C}_1$ are maximal, by Lemma 4.4, $S_{1q} \in \mathcal{C}_1$ does not appear in $g_\pi(S_{1l})$ for $l \neq q$. Therefore in a similar fashion, for $q = 1, \ldots, m_1$, we may subtract terms of the form $\frac{K_i}{K_1+1}\frac{Q_{1q}}{R_{1q}}g_\pi(S_{1q})$ to (5), where each Q_{1q} is a constant number and each R_{1q} is a non-zero constant number, all of which are independent of $K_1, \ldots, K_{n+1}$. As a result, the elements in $\mathcal{C}_1$ do not appear in

$$g_\pi(\mathcal{T}_{n.s.}) - \sum_{q=1}^{m_1} \frac{K_i}{K_1+1}\frac{Q_{1q}}{R_{1q}} g_\pi(S_{1q}),$$

when expressed with respect to a basis for $\otimes^N \mathcal{V}$.

Assume the Theorem holds for $\mathcal{C}^i$ having less then t layers. That is, for $l = 1, \ldots, t-1$ and $q = 1, \ldots, m_l$, let M_{lq} be a constant number independent of $K_1, \ldots, K_{n+1}$, and suppose when

$$g_\pi(\mathcal{T}_{n.s.}) - \sum_{l=1}^{t-1}\sum_{q=1}^{m_l} \frac{K_i}{K_1+1} M_{lq} g_\pi(S_{lq})$$

is expressed with respect to a basis for $\otimes^N \mathcal{V}$, no elements in $\mathcal{C}_1 \bigcup \cdots \bigcup \mathcal{C}_{t-1}$ appear.

Take $S_{t1} \in \mathcal{C}_t$. By Lemma 6.11

$$[S_{t1} : g_\pi(\mathcal{T}_{n.s.})] = \mathbf{K}! \times Q_{t1}$$

where Q_{t1} is a constant number independent of $K_1, \ldots, K_{n+1}$.
By Lemma 6.9 and Remark 4.6

$$[S_{t1} : g_\pi(S_{t1})] = \tfrac{K_1+1}{K_i}\mathbf{K}! \times R_{t1}$$

and

$$[S_{t1} : g_\pi(S_{lq})] = \tfrac{K_1+1}{K_i}\mathbf{K}! \times R_{lq}$$

for $l = 1, \ldots, t-1$ and $q = 1, \ldots, m_l$, where R_{t1} is a non-zero constant number and each R_{lq} is a constant number, all independent of $K_1, \ldots, K_{n+1}$.

We want to find a value for $\hat{M}_t$ in the following expression,

$$g_\pi(\mathcal{T}_{n.s.}) - \sum_{l=1}^{t-1}\sum_{q=1}^{m_l} \frac{K_i}{K_1+1} M_{lq} g_\pi(S_{lq}) - \hat{M}_t g_\pi(S_{t1}),$$

such that when it is expressed with respect to a basis for $\otimes^N \mathcal{V}$, no occurrences of S_{t1} appear. Therefore we solve for $\hat{M}_t$ such that,

$$\begin{aligned}
&[S_{t1} : g_\pi(\mathcal{T}_{n.s.})] - \sum_{l=1}^{t-1}\sum_{q=1}^{m_l} \frac{K_i}{K_1+1} M_{lp}[S_{t1} : g_\pi(S_{lq})] - \hat{M}_{t1}[S_{t1} : g_\pi(S_{lq})] \\
&= \mathbf{K}!Q_{t1} - \sum_{l=1}^{t-1}\sum_{q=1}^{m_l} \frac{K_i}{K_1+1} M_{lq} \frac{K_1+1}{K_i} \mathbf{K}!R_{lq} - \hat{M}_{t1} \frac{K_1+1}{K_i} \mathbf{K}!R_{t1} \\
&= 0.
\end{aligned}$$

Which gives,

$$\hat{M}_{t1} = \frac{K_i}{K_1+1}\left(\frac{Q_{t1}}{R_{t1}} - \sum_{l=1}^{t-1}\sum_{q=1}^{m_l} \frac{M_{lq}R_{lq}}{R_{t1}}\right).$$

Let $M_{t1} = \frac{Q_{t1}}{R_{t1}} - \sum_{l=1}^{t-1}\sum_{q=1}^{m_l} \frac{M_{lq}R_{lq}}{R_{t1}}$, and therefore, $\hat{M}_{t1} = \frac{K_i}{K_1+1} M_{t1}$. Since all the terms in M_{t1} are constant numbers independent of $K_1, \ldots, K_{n+1}$, it follows that M_{t1} is a constant number independent of $K_1, \ldots, K_{n+1}$.

We now have,

$$g_\pi(\mathcal{T}_{n.s.}) - \sum_{l=1}^{t-1}\sum_{q=1}^{m_l} \frac{K_i}{K_1+1} M_{lq} g_\pi(S_{lq}) - \frac{K_i}{K_1+1} M_{t1} g_\pi(S_{t1}) \tag{6}$$

has no elements in $\mathcal{C}_1 \bigcup \cdots \bigcup \mathcal{C}_{t-1} \bigcup \{S_{t1}\}$ when it is expressed with respect to a basis for $\otimes^N \mathcal{V}$. By Lemma 4.4, S_{tl} does not appear in $g_\pi(S_{tq})$ for $l \neq q$. Therefore, we may subtract terms of the form $g_\pi(S_{tq})$ for $q = 2, \ldots, m_t$ to (6), and when (6) is expressed with respect to a basis for $\otimes^N \mathcal{V}$, it still will not contain the elements in $\mathcal{C}_1 \bigcup \cdots \bigcup \mathcal{C}_{t-1} \bigcup \{S_{t1}\}$. Choosing coefficients in front of each $g_\pi(S_{tq})$ as we did for $g_\pi(S_{t1})$ we conclude that,

$$g_\pi(\mathcal{T}_{n.s.}) - \sum_{l=1}^{t}\sum_{q=1}^{m_l} \frac{K_i}{K_1+1} M_{lq} g_\pi(S_{lq}) \tag{7}$$

has no elements in $\mathcal{C}^i = \mathcal{C}_1 \bigcup \cdots \bigcup \mathcal{C}_t$ when it is expressed with respect to the basis for $\otimes^N \mathcal{V}$, and each M_{lq} is a constant number independent of $K_1, \ldots, K_{n+1}$.

We have now eliminated all the elements in $\mathcal{C}^i$ from $g_\pi(\mathcal{T}_{n.s.})$. Now that the reader is familiar with our straightening algorithm, we justify the claim made at the beginning of this proof. Let $\mathcal{C} = \{\mathcal{T}_{ij} \mid i = 2, \ldots, n+1 \text{ and } j = 1, \ldots, f_i\}$. Suppose $r \neq i$ and we consider the set $\mathcal{C}^r = \{\mathcal{T}_{rk} \mid k = 1, \ldots, f_r\}$. By Remark 6.4, for any $\mathcal{T}_{ik} \in \mathcal{C}^i$, none of the elements in $\mathcal{C}^r$ appear in $g_\pi(\mathcal{T}_{ik})$, when $g_\pi(\mathcal{T}_{ik})$ is expressed with respect to a basis for $\otimes^N \mathcal{V}$. That is, when $\sum_{l=1}^{t}\sum_{q=1}^{m_l} \frac{K_i}{K_1+1} M_{lq} g_\pi(S_{lq})$ is expressed with respect to a basis for $\otimes^N \mathcal{V}$, none of the elements in $\mathcal{C}^r$ appear. Therefore, adding terms of the form $g_\pi(\mathcal{T}_{rk})$ for $\mathcal{T}_{rk} \in \mathcal{C}^r$ to (7), and expressing this new equation with respect to a basis for $\otimes^N \mathcal{V}$, will still have no occurrences of the elements in $\mathcal{C}^i$. By Remark 6.4, we also have that for any $\mathcal{T}_{rk} \in \mathcal{C}^r$, none of the elements in $\mathcal{C}^i$ appear in $g_\pi(\mathcal{T}_{rk})$. Therefore, the process of eliminating every element in $\mathcal{C}^r$ from $g_\pi(\mathcal{T}_{n.s.})$ is independent of eliminating the elements in $\mathcal{C}^i$ from $g_\pi(\mathcal{T}_{n.s.})$. Not only is this process independent, but it is also done in an identical manner. As the number of times any element in $\mathcal{C}$ appears in $g_\pi(\mathcal{T}_{n.s.})$ is $\mathbf{K}!$ multiplied by some constant number which is independent of $K_1, \ldots, K_{n+1}$, the process of eliminating every element in $\mathcal{C}^r$ from $g_\pi(\mathcal{T}_{n.s.})$ is identical to the straightening algorithm we have just presented. Since $\mathcal{C} = \biguplus_{i=2}^{n+1} \mathcal{C}^i$, and by Lemma 6.8, the elements in $\mathcal{C}$ form a complete list of π semi-standard generalized tableaux needed to straighten $g_\pi(\mathcal{T}_{n.s.})$, we have our result.

$\square$

We are now in a position to examine Case (3). Recall,

$$g_\pi(E_{1j}\mathcal{T}) = K_j g_\pi(\mathcal{T}_j^1) + \sum_{\substack{k=2,\ldots,p \\ l=1,\ldots,\pi_k}} g_\pi(E_{1j}^{kl}\mathcal{T}).$$

By Theorem 6.1,

$$g_\pi(E_{1j}^{kl}\mathcal{T}) = \sum_{r=2}^{n+1}\sum_{s=1}^{f_r} \frac{K_r}{K_1+1} M_{rs} g_\pi(\mathcal{T}_{rs})$$

where each M_{rs} is a constant number independent from $K_1, \ldots, K_{n+1}$.

Therefore,

$$\sum_{\substack{k=2,\ldots,p \\ l=1,\ldots,\pi_k}} g_\pi(E_{1j}^{kl}\mathcal{T}) = \sum_{r=2}^{n+1}\sum_{s=1}^{f_r} \frac{K_r}{K_1+1} C_{rs} g_\pi(\mathcal{T}_{rs})$$

where C_{rs} is a constant number independent of $K_1, \ldots, K_{n+1}$. Notice that we changed the coefficients from M_{rs} to C_{rs}, as the coefficients M_{rs} came from Theorem 6.1, which corresponded to straightening an arbitrary but fixed $g_\pi(E_{1j}^{kl}\mathcal{T})$.

Combining these results

$$\begin{aligned} g_\pi(E_{1j}\mathcal{T}) &= K_j g_\pi(\mathcal{T}_j^1) \;+ \sum_{\substack{k=2,\ldots,p \\ l=1,\ldots,\pi_k}} g_\pi(E_{1j}^{kl}\mathcal{T}) \\ &= K_j g_\pi(\mathcal{T}_j^1) \;+ \sum_{r=2}^{n+1}\sum_{s=1}^{f_r} \frac{K_r}{K_1+1} C_{rs} g_\pi(\mathcal{T}_{rs}), \end{aligned}$$

where for $r = 2, \ldots, n+1$ and $s = 1, \ldots, f_r$, C_{rs} is a constant number independent of $K_1, \ldots, K_{n+1}$.

6.6 Summary

We have now achieved our goal for this chapter and summarize our results below.

Case (1): For arbitrary $1 \le i \le n+1$,

$$E_{i1} g_\pi(\mathcal{T}) = K_1 g_\pi(\mathcal{T}_1^i),$$

Notice when $i = 1$ $\mathcal{T}_1^i = \mathcal{T}$.

Case (2): For $1 < i, j \leq n+1$,

$$E_{ij} g_\pi(\mathcal{T}) = K_j g_\pi(\mathcal{T}_j^i) + \sum_r c_r g_\pi(\mathcal{T}_r),$$

where each c_r is a constant number independent of $K_1, \ldots, K_{n+1}$, and each $\mathcal{T}_r \in \mathcal{S}_\pi(N)$ having top row identical to top row of $\mathcal{T}$.

Case (3): For $1 < j \leq n+1$,

$$E_{1j} g_\pi(\mathcal{T}) = K_j g_\pi(\mathcal{T}_j^1) \ + \sum_{r=2}^{n+1} \sum_{s=1}^{f_r} \frac{K_r}{K_1 + 1} C_{rs} g_\pi(\mathcal{T}_{rs}),$$

where each C_{rs} is a constant number independent of $K_1, \ldots, K_{n+1}$, and each $\mathcal{T}_{rs} \in \mathcal{S}_\pi(N)$ which we defined explicitly in section 6.3.

Remark 6.5. Let $g_\pi(\mathcal{T})$ be a basis vector for the A_n-module $g_\pi(\otimes^N \mathcal{V})$ where $\mathcal{T} \in \mathcal{S}_\pi(N)$ having top row $\boxed{K_1} \cdots \boxed{K_{n+1}}$ with each K_i is chosen sufficiently large. Let $S_1, \ldots, S_6$ be the Serre relations for A_n defined in definition 2.33. The Serre relations are sums and differences of monomials in the operators E_{ij}. Therefore, for $r = 1, \ldots, 6$

$$S_r g_\pi(\mathcal{T}) = \sum_l \frac{P_l(K_1, \ldots, K_{n+1})}{Q_l(K_1)} g_\pi(\mathcal{T}_l)$$

where for all values of l, P_l and Q_l are polynomials in $K_1, \ldots, K_{n+1}$ and K_1 respectively, and $\mathcal{T}_l \in \mathcal{S}_\pi(N)$. In addition, as K_1 was chosen to be sufficiently large, Q_l is a non-zero polynomial.

7 Realization of non-integral simple torsion free A_n-modules

Recall in our motivating example from section 5 through a complex continuation we realized all simple torsion free A_n-modules of degree one. Following these methods the goal of this chapter is a realization of all simple torsion free A_n-modules of finite degree having a non-integral central character. As in the case of our motivating example we look to the finite dimensional simple highest weight A_n-modules as the framework for our construction.

Recall from section 4.3 we may use tableau formalism to construct finite dimensional simple highest weight A_n-modules. For $\pi = \{\pi_1 \geq \cdots \geq \pi_p\} \in \prod(N)$. Set $\lambda = \sum_{i=1}^{n} m_i\omega_i$ with $m_i = \pi_i - \pi_{i+1}$ for $i = 1, \ldots, n$ and $m_{n+1} = \pi_{n+1}$. By Theorem 4.2, $g_\pi(\otimes^N \mathcal{V})$ is isomorphic to the finite dimensional simple A_n-module $V(\lambda)$ with highest weight λ. In addition we know that $g_\pi(\otimes^N \mathcal{V})$ has highest weight vector $g_\pi(T^+)$ where T^+ is the π semi-standard generalized tableau having i^{th} row filled entirely with the value i for $i = 1, \ldots, n{+}1$. Lastly by Theorem 4.1 $\{g_\pi(T) \mid T \in \mathcal{S}_\pi(N)\}$ is a basis for $g_\pi(\otimes^N \mathcal{V})$.

We wish to construct a torsion free A_n-module having central character $\chi_{a\omega_1+m_2\omega_2+\cdots+m_n\omega_n}$ where $a \in \mathbb{C} \setminus \mathbb{Z}$ and $m_i \in \mathbb{Z}_{\geq 0}$ for $i = 2, \ldots, n$. Fix $\tilde{\pi} = \{\pi_2 \geq \cdots \geq \pi_p\} \in \prod(M)$ and let $\pi_1 \in \mathbb{Z}_{>0}$ such that $\pi_1 \gg \pi_2$ and π_1 is variable. First fix a vector $\bar{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1}$ where each $a_i \in \mathbb{C} \setminus \mathbb{Z}$ such that $\sum_{i=1}^{n+1} a_i - \pi_2 = a$. We introduce a formal symbol $v(\bar{a}+\bar{M}, \widetilde{\mathcal{T}})$, where $\bar{M} = (M_1, \ldots, M_{n+1}) \in \mathbb{Z}^{n+1}$ with $\sum_{i=1}^{n+1} M_i = 0$ and $\widetilde{\mathcal{T}} \in \mathcal{S}_{\tilde{\pi}}(M)$ with no index having a value of 1. We are viewing $v(\bar{a}+\bar{M}, \widetilde{\mathcal{T}})$ as corresponding to a core basis vector. Formally define the vector space $V(\bar{a}, \tilde{\pi})$ to have basis

$$\mathcal{B} = \{v(\bar{a} + \bar{M}, \widetilde{\mathcal{T}}) \mid \bar{M} \in \mathbb{Z}^{n+1}\ ,\ \sum_{i=1}^{n+1} M_i = 0 \text{ and } \widetilde{\mathcal{T}} \in \mathcal{S}_{\tilde{\pi}}(M)\}.$$

Define the action of the operators E_{ij} on $V(\bar{a}, \tilde{\pi})$ analogous to it's action on the core basis vectors, which we outlined in section 6.6. For $1 \leq i \leq n+1$, e_i will denote the $n{+}1$-tuple having a zero in every co-ordinate other then the i^{th} co-ordinate, which contains the value 1.

1. For $1 \leq i \leq n+1$

$$E_{ii}v(\bar{a} + \bar{M}, \widetilde{\mathcal{T}}) = (a_i + M_i + k_i)v(\bar{a} + \bar{M}, \widetilde{\mathcal{T}})$$

   where k_i is equal to the number of i's occurring in $\widetilde{\mathcal{T}}$.

2. For $1 < i \leq n+1$

$$E_{i1}v(\bar{a}+\bar{M},\widetilde{\mathcal{T}}) = (a_1+M_1)v(\bar{a}+\bar{M}+e_i-e_1,\widetilde{\mathcal{T}}).$$

3. For $1 < i,j \leq n+1$

$$E_{ij}v(\bar{a}+\bar{M},\widetilde{\mathcal{T}}) = (a_j+M_j)v(\bar{a}+\bar{M}+e_i-e_j,\widetilde{\mathcal{T}}) + \sum_r c_r v(\bar{a}+\bar{M},\widetilde{\mathcal{T}_r})$$

where each $\widetilde{\mathcal{T}_r} \in \mathcal{S}_{\tilde{\pi}}(M)$, and the coefficients c_r correspond to the coefficients which occur in case (2) of section 6.6

4. For $1 < j \leq n+1$

$$E_{1j}v(\bar{a}+\bar{M},\widetilde{\mathcal{T}}) = (a_j+M_j)v(\bar{a}+\bar{M}+e_1-e_j,\widetilde{\mathcal{T}}) + \sum_{r=2}^{n+1}\sum_{s=1}^{f_r}\frac{a_r+M_r}{a_1+M_1+1}C_{rs}v(\bar{a}+\bar{M}+e_r-e_j,\widetilde{\mathcal{T}_{rs}})$$

where each $\widetilde{\mathcal{T}_{rs}} \in \mathcal{S}_{\tilde{\pi}}(M)$ was defined explicitly in section 6.3, and the coefficients C_{rs} are the coefficients which occur in case (3) of section 6.6.

Remark 7.1. It is important that the reader notice that this action is identical to the action we outlined in section 6.6 except each K_i is substituted with an $a_i + M_i$. Also the basis vectors for $V(\bar{a},\tilde{\pi})$ are weight vectors for the Cartan subalgebra $\mathcal{H}$ of A_n. The weight of each $v(\bar{a}+\bar{M},\widetilde{\mathcal{T}})$ is

$$\sum_{i=1}^{n}(a_i - a_{i+1} + M_i - M_{i+1} + k_i - k_{i+1})\omega_i$$

where k_i is the number of i's occurring in $\widetilde{\mathcal{T}}$. Notice since $\widetilde{\mathcal{T}}$ does not contain an index with a value of 1, we must have $k_1 = 0$.

$V(\bar{a},\tilde{\pi})$ has weight lattice contained in

$$(a_1 - a_2 - \pi_2)\omega_1 + \sum_{i=2}^{n}(a_i - a_{i+1} + \pi_i - \pi_{i+1})\omega_i + Q.$$

Lemma 7.1. Let $a \in \mathbb{C} \setminus \mathbb{Z}$ and fix $\bar{a} = (a_1, \dots, a_{n+1}) \in \mathbb{C}^{n+1}$ with each $a_i \in \mathbb{C} \setminus \mathbb{Z}$ such that $\sum_{i=1}^{n+1} a_i - \pi_2 = a$. Then $V(\bar{a}, \tilde{\pi})$ is an A_n-module.

Proof. Let $S_1, \dots, S_6$ be a complete list of Serre relations for A_n defined in definition 2.33. Since the action of the operators E_{ij} on $V(\bar{a}, \tilde{\pi})$ were defined analogously to the action of the operators E_{ij} on the core basis vectors coming out of our finite dimensional modules $g_\pi(\otimes^N \mathcal{V})$ by Remark 6.5

$$S_i v(\bar{a} + \bar{M}, \widetilde{\mathcal{T}}) = \sum_l \frac{P_l(a_1 + M_1, \dots, a_{n+1} + M_{n+1})}{Q_l(a_1 + M_1)} v(\bar{a} + \bar{M}, \widetilde{\mathcal{T}_l})$$

where for each index of l we have $\widetilde{\mathcal{T}_l} \in \mathcal{S}_{\tilde{\pi}}(M)$, P_l is a polynomial in $n+1$ variables evaluated at $a_1 + M_1, \dots, a_{n+1} + M_{n+1}$ and Q_l is a polynomial in one variable evaluated at $a_1 + M_1$. Observe that the roots of the polynomial Q_l are all integers and since $a_1 \in \mathbb{C} \setminus \mathbb{Z}$ we have that $Q_l(a_1 + M_1) \neq 0$ for all l. Furthermore, each $\frac{P_l(a_1+M_1,\dots,a_{n+1}+M_{n+1})}{Q_l(a_1+M_1)}$ corresponds to $\frac{P_l(K_1,\dots,K_{n+1})}{Q_l(K_1)}$ occurring in

$$S_i g_\pi(\mathcal{T}) = \sum_l \frac{P_l(K_1, \dots, K_{n+1})}{Q_l(K_1)} g_\pi(\mathcal{T}_l).$$

Since each $g_\pi(\mathcal{T})$ are basis vectors for some A_n-module $g_\pi(\otimes^N \mathcal{V})$

$$S_i g_\pi(\mathcal{T}) = 0 \text{ for } i = 1, \dots, 6.$$

Therefore for every l,

$$P_l(K_1, \dots, K_{n+1}) = 0 \text{ for all } K_1, \dots, K_{n+1} \text{ larger then } \mathcal{K}.$$

By Lemma 5.2

$$P_l(a_1 + M_1, \dots, a_{n+1} + M_{n+1}) = 0 \text{ for all } a_1 + M_1, \dots, a_{n+1} + M_{n+1} \text{ in } \mathbb{C} \setminus \mathbb{Z}.$$

Therefore,

$$S_i v(\bar{a} + \bar{M}, \widetilde{\mathcal{T}}) = 0$$

for all basis vectors $v(\bar{a} + \bar{M}, \widetilde{\mathcal{T}})$ of $V(\bar{a}, \tilde{\pi})$ and all Serre relations $S_1, \dots, S_6$. By the comment proceeding Theorem 2.7, $V(\bar{a}, \tilde{\pi})$ is an A_n-module. □

Lemma 7.2. Let $a \in \mathbb{C}\backslash\mathbb{Z}$ and fix $\bar{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1}$ with each $a_i \in \mathbb{C}\backslash\mathbb{Z}$ such that $\sum_{i=1}^{n+1} a_i - \pi_2 = a$. The degree of $V(\bar{a}, \tilde{\pi})$ is equal to the dimension of the finite dimensional simple A_{n-1}-module having highest weight $\lambda = \sum_{i=2}^{n}(\pi_i - \pi_{i+1})\omega_{i-1}$.

Proof. Let μ be a weight for A_n. Let $v(\bar{a} + \bar{M}, \widetilde{\mathcal{T}}) \in V_\mu$ with $\widetilde{\mathcal{T}}$ having content $\{2^{k_2}, \ldots, (n+1)^{k_{n+1}}\}$.

Therefore, $v(\bar{a} + \bar{M}, \widetilde{\mathcal{T}})$ has weight

$$\sum_{i=1}^{n}(a_i - a_{i+1} + M_i - M_{i+1} + k_i - k_{i+1})\omega_i$$

For any $\widetilde{\mathcal{T}'} \in \mathcal{S}_{\tilde{\pi}}(M)$ with content $\{2^{l_2}, \ldots, (n+1)^{l_{n+1}}\}$, let $\bar{M}' = \bar{M} + (0, k_2, \ldots, k_{n+1}) + (0, -l_2, \ldots, -l_{n+1})$. Then $v(\bar{a} + \bar{M}', \widetilde{\mathcal{T}'}) \in V_\mu$. Also for a fixed $\widetilde{\mathcal{T}} \in \mathcal{S}_{\tilde{\pi}}(M)$ there corresponds a unique choice for $\bar{M}$ such that $v(\bar{a} + \bar{M}, \widetilde{\mathcal{T}})$ is in V_μ.

Therefore, the dimension of V_μ is equal to the number of ways a Young frame with underlying partition $\tilde{\pi}$ can be filled with the values $2, \ldots, n+1$ in semi-standard fashion. However, this is equal to the number of ways a Young frame with underlying partition $\tilde{\pi}$ can be filled with the values $1, \ldots, n$ in semi-standard fashion. By Theorem 4.1 this is the dimension of $V(\sum_{i=2}^{n}(\pi_i - \pi_{i+1})\omega_{i-1})$, the finite dimensional simple A_{n-1}-module of highest weight $\sum_{i=2}^{n}(\pi_i - \pi_{i+1})\omega_{i-1}$.

As V_μ was arbitrarily chosen we have shown that the dimension of each weight space is equal to the dimension of $V(\lambda)$ and we have our result. □

Lemma 7.3. Fix $\tilde{\pi} = \{\pi_2 \geq \cdots \geq \pi_p\} \in \prod(M)$. Let $a \in \mathbb{C} \setminus \mathbb{Z}$ and fix $\bar{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1}$ with each $a_i \in \mathbb{C} \setminus \mathbb{Z}$ such that $\sum_{i=1}^{n+1} a_i - \pi_2 = a$. Then $V(\bar{a}, \tilde{\pi})$ has central character χ_λ where $\lambda = a\omega_1 + \sum_{i=2}^{n}(\pi_i - \pi_{i+1})\omega_i$.

Proof. Let $\pi = \{\pi_1 \geq \cdots \geq \pi_p\}$ with $\pi_1 \gg \pi_2$ be variable and $\tilde{\pi}$ is fixed. Let $\{h_1, \ldots, h_n\}$ be a basis for the Cartan subalgebra $\mathcal{H}$ of A_n. We want to show that the central character of $V(\bar{a}, \tilde{\pi})$ is χ_λ. First consider $g_\pi(\otimes^N \mathcal{V})$, the finite dimensional simple A_n-module of highest weight $\mu = \sum_{i=1}^{n}(\pi_i - \pi_{i+1})\omega_i$. $g_\pi(\otimes^N \mathcal{V})$ admits a central character χ_μ. By definition, for any $v \in g_\pi(\otimes^N \mathcal{V})$ and any $z \in Z(\mathcal{U}(A_n))$

$$z.v = \chi_\mu(z)v.$$

In particular this is true for the maximal vector $v^+ \in g_\pi(\otimes^N \mathcal{V})$. Take a $z = \sum_l h_{1l}^{p_{1l}} \cdots h_{nl}^{p_{nl}} \in Z(\mathcal{U}(A_n))$, where $p_{il} \in \mathbb{Z}_{\geq 0}$ and $h_{il} \in \{h_1, \ldots, h_n\}$.

$$z.v^+ = \sum_l \mu(h_{1l})^{p_{1l}} \cdots \mu(h_{nl})^{p_{nl}} v^+ = \chi_\mu(z)v^+.$$

Therefore for $z \in Z(\mathcal{U}(A_n))$

$$\begin{aligned}\chi_\mu(z) &= \sum_l \mu(h_{1l})^{p_{1l}} \cdots \mu(h_{nl})^{p_{nl}} \\ &= \sum_l (\pi_1 - \pi_2)^{p_{1l}} \cdots (\pi_n - \pi_{n+1})^{p_{nl}} \\ &= \sum_l (\sum_{i=1}^{n+1} K_i - \pi_2)^{p_{1l}} (\pi_2 - \pi_3)^{p_{2l}} \cdots (\pi_n - \pi_{n+1})^{p_{nl}}\end{aligned}$$

where $\sum_{i=1}^{n+1} K_i = \pi_1$.

As $\tilde{\pi} = \{\pi_2 \geq \cdots \geq \pi_p\}$ is fixed and we are varying $\pi_1 = \sum_{i=1}^{n+1} K_i$

$$\chi_\mu(z) = f_z(\sum_{i=1}^{n+1} K_i)$$

where f_z is a polynomial in the variable $\sum_{i=1}^{n+1} K_i$.

By Schur Lemma 3.3

$$z.v = f_z(\sum_{i=1}^{n+1} K_i)v \quad \text{for all } v \in g_\pi(\otimes^N \mathcal{V})$$

In particular for all core basis vectors v in the finite dimensional modules $g_\pi(\otimes^N \mathcal{V})$

$$z.v = f_z(\sum_{i=1}^{n+1} K_i)v.$$

The action of z on the basis vectors in $V(\bar{a}, \tilde{\pi})$ were defined identically to it's action on the core basis vectors in our finite dimensional modules $g_\pi(\otimes^N \mathcal{V})$, with the exception that each K_i is substituted with a $a_i + M_i$. Therefore

$$z.v = f_z(a + \pi_2)v \qquad \text{for all } v \in V(\bar{a}, \tilde{\pi})$$

where $\sum_{i=1}^{n+1} a_i - \pi_2 = a$.

Therefore $V(\bar{a}, \tilde{\pi})$ has central character χ_λ where $\lambda = a\omega_1 + \sum_{i=2}^{n+1}(\pi_i - \pi_{i+1})\omega_i$.

□

Lemma 7.4. Fix $\tilde{\pi} = \{\pi_2 \geq \cdots \geq \pi_p\} \in \prod(M)$. Let $a \in \mathbb{C} \setminus \mathbb{Z}$ and fix $\bar{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1}$ with each $a_i \in \mathbb{C} \setminus \mathbb{Z}$ such that $\sum_{i=1}^{n+1} a_i - \pi_2 = a$. Let $\lambda = a\omega_1 + \sum_{i=2}^{n}(\pi_i - \pi_{i+1})\omega_i$. The central character χ_λ of $V(\bar{a}, \tilde{\pi})$ is non-integral.

Proof. Suppose the sequence associated with $\lambda = a\omega_1 + \sum_{i=2}^{n}(\pi_i - \pi_{i+1})\omega_i$ is $m(\lambda) = (m_1(\lambda), \ldots, m_{n+1}(\lambda))$. Then for $i = 1, \ldots, n$, $(\lambda + \rho)(h_{\alpha_i}) = m_i(\lambda) - m_{i+1}(\lambda)$.

$m(\lambda)$ is non-integral provided there exists indices j, k such that $m_j(\lambda) - m_k(\lambda) \notin \mathbb{Z}$.

Since

$$m_1(\lambda) - m_2(\lambda) = a + 1 \notin \mathbb{Z},$$

χ_λ is a non-integral central character.

□

Lemma 7.5. Let $\tilde{\pi} = \{\pi_2 \geq \cdots \geq \pi_p\} \in \prod(M)$. Let $a \in \mathbb{C} \setminus \mathbb{Z}$ and fix $\bar{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1}$ with each $a_i \in \mathbb{C} \setminus \mathbb{Z}$ such that $\sum_{i=1}^{n+1} a_i - \pi_2 = a$. The operator E_{j1} acts injectively on $V(\bar{a}, \tilde{\pi})$.

Proof. It suffices to show that E_{j1} acts injectively on an arbitrary weight space of $V(\bar{a}, \tilde{\pi})$. Recall, in the proof of Lemma 7.2, for any weight μ, the weight space V_μ has a basis labelled by the elements $\widetilde{\mathcal{T}} \in S_{\tilde{\pi}}(M)$. That is, for each $\widetilde{\mathcal{T}} \in S_{\tilde{\pi}}(M)$ there exists a unique $\bar{M}_{\widetilde{\mathcal{T}}}$ such that $v(\bar{a} + \bar{M}_{\widetilde{\mathcal{T}}}, \widetilde{\mathcal{T}}) \in V_\mu$, and $\{v(\bar{a} + \bar{M}_{\widetilde{\mathcal{T}}}, \widetilde{\mathcal{T}}) \mid \widetilde{\mathcal{T}} \in S_{\tilde{\pi}}(M)\}$

is a basis of V_μ.

For any basis vector $v(\bar{a}+\bar{M}_{\widetilde{T}}, \widetilde{T})$ of V_μ

$$E_{j1}v(\bar{a}+\bar{M}_{\widetilde{T}}, \widetilde{T}) = (a_1 + M_1)v(\bar{a}+\bar{M}_{\widetilde{T}} + e_j - e_1, \widetilde{T})$$

Therefore, for an arbitrary non-zero linear combination of elements in V_μ, say $\sum_{\widetilde{T}\in S_{\tilde{\pi}}(M)} c_{\widetilde{T}} v(\bar{a}+\bar{M}_{\widetilde{T}}, \widetilde{T})$

$$E_{j1} \sum_{\widetilde{T}\in S_{\tilde{\pi}}(M)} c_{\widetilde{T}} v(\bar{a}+\bar{M}_{\widetilde{T}}, \widetilde{T}) = \sum_{\widetilde{T}\in S_{\tilde{\pi}}(M)} c_{\widetilde{T}}(a_1 + (\bar{M}_{\widetilde{T}})_1)v(\bar{a}+\bar{M}_{\widetilde{T}} + e_j - e_1, \widetilde{T}).$$

As $\sum_{\widetilde{T}\in S_{\tilde{\pi}}(M)} c_{\widetilde{T}} v(\bar{a}+\bar{M}_{\widetilde{T}}, \widetilde{T})$ is non-zero, there exists a $\widetilde{T} \in S_{\tilde{\pi}}(M)$ such that $c_{\widetilde{T}} \neq 0$. Since $(a_1 + (\bar{M}_{\widetilde{T}})_1) \neq 0$ for all $\widetilde{T} \in S_{\tilde{\pi}}(M)$, it follows that

$$E_{j1} \sum_{\widetilde{T}\in S_{\tilde{\pi}}(M)} c_{\widetilde{T}} v(\bar{a}+\bar{M}_{\widetilde{T}}, \widetilde{T}) = \sum_{\widetilde{T}\in S_{\tilde{\pi}}(M)} c_{\widetilde{T}}(a_1 + (\bar{M}_{\widetilde{T}})_1)v(\bar{a}+\bar{M}_{\widetilde{T}} + e_j - e_1, \widetilde{T}) \neq 0.$$

As $\{v(\bar{a}+\bar{M}_{\widetilde{T}} + e_j - e_1, \widetilde{T}) \mid \widetilde{T} \in S_{\tilde{\pi}}(M)\}$ is a basis for the weight space $V_{\mu-\alpha}$, where $\alpha = \epsilon_j - \epsilon_1$ was defined in section 2.5, E_{j1} acts injectively on V_μ.

□

Lemma 7.6. Let $\tilde{\pi} = \{\pi_2 \geq \cdots \geq \pi_p\} \in \prod(M)$. Let $a \in \mathbb{C} \setminus \mathbb{Z}$ and fix $\bar{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1}$ with each $a_i \in \mathbb{C} \setminus \mathbb{Z}$ such that $\sum_{i=1}^{n+1} a_i - \pi_2 = a$. Then $V(\bar{a}, \tilde{\pi})$ is torsion free and simple.

Proof. By Lemma 7.2, $V(\bar{a}, \tilde{\pi})$ has bounded weight spaces, by construction, $V(\bar{a}, \tilde{\pi})$ is infinite dimensional, and by Remark 7.1 the weights of $V(\bar{a}, \tilde{\pi})$ are contained in exactly one Q-coset, and therefore $V(\bar{a}, \tilde{\pi})$ is admissible. By Lemma 3.1, $V(\bar{a}, \tilde{\pi})$ has finite length. By Theorem 3.1 a composition series exists and therefore $V(\bar{a}, \tilde{\pi})$ contains a simple submodule V'. V' is a submodule of $V(\bar{a}, \tilde{\pi})$, and by Lemma 7.5, V' is infinite dimensional, and therefore is admissible. V' has central character

$$\chi_\nu \text{ where } \nu = a\omega_1 + \sum_{i=2}^{n} (\pi_i - \pi_{i+1})\omega_i$$

and weight lattice contained in

$$(a_1 - a_2 - \pi_2)\omega_1 + \sum_{i=2}^{n}(a_i - a_{i+1} + \pi_i - \pi_{i+1})\omega_i + Q$$

Fix $\bar{a}' = (a'_1, \ldots, a'_{n+1}) \in \mathbb{C}^{n+1}$ with each $a'_i \in \mathbb{C} \setminus \mathbb{Z}$. Recall from Example 3.1, $M(\bar{a}')$ denotes a simple torsion free A_n-module of degree one. Let $\pi_1 \in \mathbb{Z}$ with $\pi_1 \geq \pi_2$. By Theorem 1.15 part 3 [3]

$$M(\bar{a}') \otimes V(\sum_{i=1}^{n}(\pi_i - \pi_{i+1})\omega_i)$$

is torsion free. By Theorem 3.4 [3], $M(\bar{a}') \otimes V(\sum_{i=1}^{n}(\pi_i - \pi_{i+1})\omega_i)$ contains a simple torsion free submodule W which has central character

$$\chi_\mu \text{ where } \mu = (\sum_{i=1}^{n+1} a'_i + \pi_1 - \pi_2)\omega_1 + \sum_{i=2}^{n}(\pi_i - \pi_{i+1})\omega_i$$

and weight lattice

$$\sum_{i=1}^{n}(a'_i - a'_{i+1} + \pi_i - \pi_{i+1})\omega_i + Q.$$

Notice by letting $a'_1 = a_1 - \pi_1$ and $a'_i = a_i$ for $i > 1$, W and V' are both simple admissible modules having the same central character and weights contained $\sum_{i=1}^{n}(a'_i - a'_{i+1} + \pi_i - \pi_{i+1})\omega_i + Q$. The semi-simple irreducible coherent families constructed from W and V' both have the same central character and by Remark 3.6 these coherent families are isomorphic and hence W and V' are isomorphic. Therefore V' is a simple torsion free A_n-module having central character

$$\chi_\nu \text{ where } \nu = a\omega_1 + \sum_{i=2}^{n}(\pi_i - \pi_{i+1})\omega_i$$

and weight lattice

$$(a_1 - a_2 - \pi_2)\omega_1 + \sum_{i=2}^{n}(a_i - a_{i+1} + \pi_i - \pi_{i+1})\omega_i + Q.$$

By Theorem 3.6 the degree of V' is equal to the dimension of the finite dimensional A_{n-1}-module having highest weight $\sum_{i=2}^{n}(\pi_i - \pi_{i+1})\omega_{i-1}$. By Lemma 7.2 this is also the degree of $V(\bar{a}, \tilde{\pi})$. Therefore $V(\bar{a}, \tilde{\pi}) = V'$ which implies $V(\bar{a}, \tilde{\pi})$ is simple and torsion free. □

We have now arrived at the goal of our work. We have a realization of any simple torsion free A_n-module having finite degree and a non-integral central character. Moreover, this realization was achieved by working exclusively with finite dimensional A_n-modules. We present this realization in the next Theorem.

Theorem 7.1. (Main Theorem) Let $a \in \mathbb{C} \setminus \mathbb{Z}$ and $m_i \in \mathbb{Z}_{\geq 0}$ for $i = 2, \ldots, n$. A simple torsion free A_n-module of finite degree having a non-integral central character $\chi_{a\omega_1 + \sum_{i=2}^{n} m_i\omega_i}$ can be realized in the following manner.

Fix $\tilde{\pi} = \{\pi_2 \geq \cdots \geq \pi_p\} \in \prod(M)$ such that $m_i = \pi_i - \pi_{i+1}$ for $i = 2, \ldots, n$. Fix $\bar{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1}$ with each $a_i \in \mathbb{C} \setminus \mathbb{Z}$ such that $\sum_{i=1}^{n+1} a_i - \pi_2 = a$.

Let $V(\bar{a}, \tilde{\pi})$ be the vector space with basis

$$\mathcal{B} = \{v(\bar{a} + \bar{M}, \widetilde{T}) \mid \bar{M} \in \mathbb{Z}^{n+1}, \sum_{i=1}^{n+1} M_i = 0 \text{ and } \widetilde{T} \in \mathcal{S}_{\tilde{\pi}}(M)\}.$$

Define the action of the root vectors E_{ij} on basis vectors of $V(\bar{a}, \tilde{\pi})$ to be

1. For $1 \leq i \leq n+1$

$$E_{ii}v(\bar{a} + \bar{M}, \widetilde{T}) = (a_i + M_i + k_i)v(\bar{a} + \bar{M}, \widetilde{T})$$

   where k_i is equal to the number of i's occurring in $\widetilde{T}$.

2. For $1 < i \leq n+1$

$$E_{i1}v(\bar{a} + \bar{M}, \widetilde{T}) = (a_1 + M_1)v(\bar{a} + \bar{M} + e_i - e_1, \widetilde{T}).$$

3. For $1 < i, j \leq n+1$

$$E_{ij}v(\bar{a} + \bar{M}, \widetilde{T}) = (a_j + M_j)v(\bar{a} + \bar{M} + e_i - e_j, \widetilde{T}) + \sum_r c_r v(\bar{a} + \bar{M}, \widetilde{T}_r)$$

   where each $\widetilde{T}_r \in \mathcal{S}_{\tilde{\pi}}(M)$, and the coefficients c_r correspond to the coefficients which occur in case (2) of section 6.6

4. For $1 < j \leq n+1$

$$E_{1j}v(\bar{a}+\bar{M},\widetilde{T}) = (a_j+M_j)v(\bar{a}+\bar{M}+e_1-e_j,\widetilde{T}) + \sum_{r=2}^{n+1}\sum_{s=1}^{f_r}\frac{a_r+M_r}{a_1+M_1+1}C_{rs}v(\bar{a}+\bar{M}+e_r-e_j,\widetilde{T_{rs}})$$

where each $\widetilde{T_{rs}} \in \mathcal{S}_{\tilde{\pi}}(M)$ was defined explicitly in section 6.3, and the coefficients C_{rs} are the coefficients which occur in case (3) of section 6.6.

Then $V(\bar{a},\tilde{\pi})$ is a simple torsion free A_n-module having a non-integral central character $\chi_{a\omega_1+\sum_{i=2}^{n} m_i\omega_i}$ with degree equal to the dimension of the finite dimensional A_{n-1} module having highest weight $\sum_{i=2}^{n} m_i\omega_{i-1}$.

Proof. By Lemma 7.1 and Lemma 7.6, $V(\bar{a},\tilde{\pi})$ is a simple torsion free A_n-module. By Lemma 7.3 and Lemma 7.4, $V(\bar{a},\tilde{\pi})$ has a non-integral central character

$$\chi_{a\omega_1+\sum_{i=2}^{n}(\pi_i-\pi_{i+1})\omega_i} = \chi_{a\omega_1+\sum_{i=2}^{n} m_i\omega_i}.$$

By Lemma 7.2, the degree of $V(\bar{a},\tilde{\pi})$ is equal to the dimension of the finite dimensional A_{n-1} module having highest weight $\sum_{i=2}^{n} m_i\omega_{i-1}$. □

8 Future Research

Mathieu [10] classified all simple torsion free A_n-modules having finite degree. In particular, Mathieu partitioned all such modules into 3 types: integral regular, singular integral and non-integral regular. In Theorem 7.1, we gave a realization and explicitly described a basis and a module action for the simple torsion free A_n-modules in the non-integral regular case. We believe that for an appropriate choice of $\bar{a} \in \mathbb{C}^{n+1}$, by using the module constructed in this work, a complete realization with an explicit basis and module action described will be obtained for the singular integral type. However, for the integral regular case, a realization using a tableau formalism is somewhat more problematic. We can find an $\bar{a} \in \mathbb{C}^{n+1}$ such that the module $V(\bar{a},\tilde{\pi})$ admits an integral regular central character. However, in this situation $V(\bar{a},\tilde{\pi})$ is not simple. The problem here is to determine a decomposition of $V(\bar{a},\tilde{\pi})$.

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