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Positive Semidefinite Intervals for Matrix Pencils

by

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Abstract

In this thesis we are concerned with the determination of the values of t for which the resulting matrix $A+tE$ is positive semidefinite. That is, we want to find the positive semidefinite interval for the matrix pencil $A+tE$. We first present a new point of view for the case that A is positive definite to obtain the same results as in [Car88] and then use this point of view to determine the positive semidefinite interval for the case that A is negative definite. In both of these cases, the positive semidefinite interval is determined from the eigenvalues of the matrix $A^{-1}E$. We then show how to combine our results to obtain the positive semidefinite interval for the case that A is nonsingular but indefinite, and for the case when A is singular and $R(E) \subseteq R(A)$. Examples and remarks on implementation are also provided.

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CONTENTS

1. Introduction

Let A and E be real symmetric $n \times n$ matrices. The (linear) matrix pencil $A + tE$ is the function

$$t \mapsto A + tE \quad (t \in \mathbb{R}).$$

We consider the problem of determining the set T such that for $t \in T$, $A + tE$ is positive semidefinite. [Recall that a symmetric matrix is positive (negative) semidefnite if all of its eigenvalues are non-negative (non-positive), is positive (negative) definite if all of its eigenvalues are positive (negative), and is indefinite if it has both positive and negative eigenvalues.] Since the set of positive semidefinite matrices is a closed convex cone in the space of $n \times n$ matrices, T is a closed (possibly unbounded) interval. We call T the positive semidefinite interval.

One source of interest in this problem is its connection to mathematical programming. Consider first the parametric quadratic program problem ([BC86], [Rit67], [Väl85])

$$\min\{(c(t))^\top x + \frac{1}{2} x^\top C(t) x | (a_i(t))^\top x \leq b_i(t), i = 1, \cdots, m\}.$$

If $C(t) = A + tE$, then the quadratic programming is convex if and only if $A + tE$ is positive semidefinite. The positive semidefinite interval gives the values of t for which the critical points of the quadratic programming problem are guaranteed to be global minimizers.

Consider also the area of semidefinite programming ([WSV00], [VB96]) where the contraints can be written as the system of linear matrix inequalities

$$F^j(x) := F_0^{(j)} + \sum_{i=1}^{n} x_i F_i^{(j)} \succeq 0, \quad j = 1, \cdots, q$$

where $F_0^{(j)}$ and $F_i^{(j)}$ are symmetric matrices, and where $F^j(x) \succeq 0$ means that $F^j(x)$ is positive semidefinite. An application of the Coordinate Directions (CD) hit-and-run algorithm (introduced by Telgen [Tel79] and published in [Bon83]) for finding necessary constraints in the system of LMI requires the determination of T.

In this thesis we are concerned with the computation of the endpoints of T. Chapter 2 gives some background material, including Caron and Gould's results for A positive semidefinite and E of rank one or two ([CG86]), Valiaho's results for determining the inertia (number of positive, negative,and zero eigenvalues) of the matrix pencil as a function of t ([Väl88]) and Caron's results for A positive semidefinite ([Car88]). Chapter 3 discusses the case when A is nonsingular. Explicit expressions of the endpoints of the positive semidefinite interval are given when A is positive definite and when A is negative definite. In these cases, the endpoints are determined from the eigenvalues of $A^{-1}E$. Then we can combine these results to obtain the positive semidefinite interval when A is nonsingular but indefinite. Chapter 4 discusses the case when A is singular. Chapter 5 presents remarks on implementation and includes examples. Concluding remarks are given in Chapter 6.

2. Background

2.1. Preliminaries.

Throughout this paper, we shall denote the range space of a matrix M by $R(M)$, its null space by $N(M)$, and its rank by $r(M)$. For a real symmetric $n \times n$ matrix M, we say M is positive definite (and denote it by $M \succ 0$) if for all $x \in \mathbb{R}^n, x \neq 0$,

$$x^\top Mx > 0.$$

We say M is positive semidefinite (and denote it by $M \succeq 0$) if for all $x \in \mathbb{R}^n$

$$x^\top Mx \geq 0.$$

It is well known that M is positive definite if and only if all eigenvalues of M are positive and M is positive semidefinite if and only if all eigenvalues of M are nonnegative.

Given a real symmetric $n \times n$ matrix M, the Schur decomposition ([GVL83], [HJ90]) of M provides an orthogonal $n \times n$ matrix Q such that

$$Q^\top MQ = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{1}$$

where D_1 is diagonal and nonsingular, and hence $r(D_1) = r(M)$.

The inertia of a real symmetric matrix M is an ordered triple consisting of the numbers of positive, negative and zero eigenvalues of the matrix. i.e. the inertia of M is the triple

$$\mathrm{In}(M) = (\pi(M), v(M), \delta(M)),$$

where $\pi(M), v(M), \delta(M)$ denote the numbers of positive, negative and zero eigenvalues, respectively, of M. Note that ([Hoh73]) if L is nonsingular, then

$$\mathrm{In}(LML^{\top}) = \mathrm{In}(M).$$

The Schur complement ([Cot74]) $\hat{M}$ of the nonzero element m_{hk} in a matrix M with row index set I and column index set J, is

$$\hat{M} = \mathcal{S}_{hk}M = [m_{ij} - \frac{m_{ik}m_{hj}}{m_{hk}}],$$

where $i \neq h$, $j \neq k$. Note that $\hat{M}$ is an $(n-1)\times(n-1)$ matrix with rows and columns indexed by $I \setminus \{h\}$ and $J \setminus \{k\}$, respectively. Here the operator $\mathcal{S}_{hk}$ is called the pivotal condensation with the pivot m_{hk}. Note that

$$\mathcal{S}_{kh}(\mathcal{S}_{hk}M) = \mathcal{S}_{hk}(\mathcal{S}_{kh}M)$$

provided that both sides are defined. By means of pivotal condensations it is possible to determine the inertia of any real symmetric matrix. If M is of rank r, then, independent of the order of the pivots, it is possible to perform on M exactly r successive pivotal condensations ([ZA66]). Thus, we can compute $\mathrm{In}(M)$ by the following algorithm.

Algorithm 2.1. ([Cot74]) Computing the inertia of a real symmetric $n \times n$ matrix M.

Step 0. Set $C = M$, $p = q = 0$.
Step 1. If $C = 0$ (possibly vacuous), go to Step 2. Otherwise,
- (i) Select, if possible, a $c_{hh} \neq 0$. Set $C \leftarrow \mathcal{S}_{hh}C$, and set $p \leftarrow p+1$, if $c_{hh} > 0$; $q \leftarrow q+1$, if $c_{hh} < 0$.
- (ii) If every diagonal entry is zero, select a $c_{hk} \neq 0$. Set $C \leftarrow \mathcal{S}_{kh}\mathcal{S}_{hk}C$, and set $p \leftarrow p+1$, $q \leftarrow q+1$.
- (iii) Go to Step 1.

Step 2. $\mathrm{In}(M) = (p, q, n-p-q)$.

Throughout the above algorithm, we have

$$\mathrm{In}(M) = (p, q, 0) + \mathrm{In}(C).$$

Given two real symmetric $n \times n$ matrices A and E, because of continuity of the eigenvalues (Lemma 2.3), if there exists a t for which $A + tE$ is nonsingular, the solutions of the equation

$$\det(A + tE) = 0$$

divide the real line into open intervals in which the matrix pencil has constant inertia. We refer to these as intervals of constant inertia. The inertia for each interval is the inertia of $A + \hat{t}E$ where $\hat{t}$ is in the interval. When A is nonsingular, if $\det(A + tE) = 0$ then

$$\det(-\frac{1}{t}I - A^{-1}E) = 0;$$

so $-\frac{1}{t}$ is an eigenvalue of $A^{-1}E$. On the other hand, when E is nonsingular, the solutions of the equation $\det(A + tE) = 0$ are eigenvalues of $-AE^{-1}$. This gives us two points of view for the determination of T.

The continuity of eigenvalues of $A + tE$ is crucial in the following chapters. We shall state the more general Wielandt-Hoffman Theorem ([Wil65], page 104), from which one can deduce the continuity of eigenvalues of $A + tE$.

Theorem 2.2. *If $C = A + B$, where A, B, C are symmetric matrices having eigenvalues $\alpha_i, \beta_i, \gamma_i$, respectively, arranged in non-increasing order, then*

$$\sum_{i=1}^{n} (\gamma_i - \alpha_i)^2 \leq \sum_{i=1}^{n} \beta_i^2. \qquad \square$$

For each t, let $\lambda_i(t)$ $(1 \le i \le n)$ be the eigenvalues of $A + tE$ arranged in non-increasing order.

Lemma 2.3. *For each $i = 1, \cdots, n$, the function λ_i is continuous.*

Proof. Let λ_i^E $(1 \le i \le n)$ be the eigenvalues of E arranged in non-increasing order. By Theorem 2.2, we have

$$\sum_{i=1}^{n}(\lambda_i(t) - \lambda_i(t_0))^2 \le (t - t_0)^2 \sum_{i=1}^{n} (\lambda_i^E)^2$$

Hence for every i, λ_i is continuous. □

The following Theorem and Corollary give a necessary and sufficient condition to guarantee that a block symmetric matrix is positive semidefinite.

Theorem 2.4. *([Hay68]) Suppose that A is nonsingular and*

$$M = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}.$$

Then

$$\mathrm{In}(M) = \mathrm{In}(A) + \mathrm{In}(C - B^\top A^{-1} B).$$

Corollary 2.5. *Suppose that $A \succ 0$. Then*

$$\begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} \succeq 0$$

if and only if

$$C - B^\top A^{-1} B \succeq 0.$$

We are now ready to present some of the known results.

2.2. Caron and Gould's results for A positive semidefinite and E of rank one or two.

If A is positive semidefinite and E is of rank one or two, then explicit expressions for the endpoints of T, and a numerically stable method for computing them, are given in [CG86]. We will summarize these results. A symmetric matrix of rank one or two can be written as

$$\pm(uu^\top + \lambda vv^\top)$$

where u and v are linearly independent n-vectors and $\lambda = 1, 0$ or -1. However, if $A + tE$ is positive semidefinite for $t \in [a, b]$, then $A + t(-E)$ is positive semidefinite for $t \in [-b, -a]$. Therefore, we can assume without loss of generality that

$$E = uu^\top + \lambda vv^\top.$$

Let the positive semidefinite interval for $A+tE$ be $T = [\underline{t}, \bar{t}]$[1]. The expressions for $\underline{t}$ and $\bar{t}$ are derived for different cases according to the choice of E ($\lambda = 1, 0$ or -1) and the relationship between A, u, and v.

(1) $\lambda = 0$ or 1. In this case,

(a) if $u \in R(A)$ and $v \in R(A)$, then

$$\underline{t} = \frac{2}{-u^\top x - v^\top y - \sqrt{(u^\top x - v^\top y)^2 + 4(u^\top y)^2}}$$

$$\bar{t} = +\infty,$$

where $x, y \in \mathbb{R}^n$ are such that $Ax = u$ and $Ay = v$.

(b) if $u \notin R(A)$ or $v \notin R(A)$, then

$$\underline{t} = 0, \quad \bar{t} = +\infty.$$

[1] Here, and in the sequel, we will use $[a, b]$ for $\{t \in \mathbb{R} : a \le t \le b\}$. For example, if $a = -\infty$ and $b \in \mathbb{R}$, then $[a, b] = (-\infty, b]$.

(2) $\lambda = -1$. In this case,

(a) if $u \in R(A)$ and $v \in R(A)$, then

$$\underline{t} = \frac{2}{-u^\top x + v^\top y - \sqrt{(u^\top x + v^\top y)^2 - 4(v^\top x)^2}}$$

$$\bar{t} = \frac{2}{-u^\top x + v^\top y + \sqrt{(u^\top x + v^\top y)^2 - 4(v^\top x)^2}},$$

where $x, y \in \mathbb{R}^n$ are such that $Ax = u$ and $Ay = v$.

(b) if $u \in R(A)$ and $v \notin R(A)$, then

$$\underline{t} = -\frac{1}{u^\top x}, \quad \bar{t} = 0,$$

where $x \in \mathbb{R}^n$ is such that $Ax = u$.

(c) if $u \notin R(A)$ and $v \in R(A)$, then

$$\underline{t} = 0, \quad \bar{t} = \frac{1}{v^\top y},$$

where $y \in \mathbb{R}^n$ is such that $Ay = v$.

(d) if $u \notin R(A)$ and $v \notin R(A)$, but $v \in R\begin{pmatrix} A & u \end{pmatrix}$, where $\begin{pmatrix} A & u \end{pmatrix}$ denotes the matrix A augmented by u, let $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ satisfy $Ax + \alpha u = v$. Then,

$$1 - \alpha^2 = 0 \implies \underline{t} = \bar{t} = 0;$$

$$1 - \alpha^2 > 0 \implies \begin{aligned} \underline{t} &= 0 \quad \text{and} \\ \bar{t} &= \frac{1 - \alpha^2}{(v^\top - \alpha u^\top)x}; \end{aligned}$$

$$1 - \alpha^2 < 0 \implies \begin{aligned} \underline{t} &= \frac{1 - \alpha^2}{(v^\top - \alpha u^\top)x} \quad \text{and} \\ \bar{t} &= 0. \end{aligned}$$

(e) if $u \notin R(A)$, $v \notin R(A)$ and $v \notin R\begin{pmatrix} A & u \end{pmatrix}$, then

$$\underline{t} = \overline{t} = 0.$$

2.3. Valiaho's results for determining the inertia of the matrix pencil as a function of the parameter.

Valiaho ([Väl88]) extended the results in [CG86] by presenting a method to determine the inertia of $A + tE$ as a function of the parameter t. He first observed that if E is nonsingular, the inertia change points are the values of t equal to the eigenvalues of $-AE^{-1}$. Then, using Algorithm 2.1, he evaluated $\mathrm{In}(A + \hat{t}E)$, for one value $\hat{t}$ interior to each interval of constant inertia. The positive semidefinite interval is the interval with inertia $(\pi(M), 0, \delta(M))$.

If E is singular, the problem is reduced to a nonsingular case of lower dimension. As a preliminary step, by using the Lagrange's reduction ([Hoh73]), we put E into the form

$$E = L \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} L^{\top}$$

where L is nonsingular and D is nonsingular and diagonal. D is of order $r(E)$ and

$$\mathrm{In}(D) = (\pi(E), \upsilon(E), 0). \tag{2}$$

Rewrite L as $L = \begin{pmatrix} P & Q \end{pmatrix}$ so that

$$E = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P^{\top} \\ Q^{\top} \end{pmatrix} = PDP^{\top}.$$

Note that P is an $n \times r(E)$ matrix of full rank. Also, we have ([MS75])

$$R\begin{pmatrix} A & E \end{pmatrix} = R(A) + R(E) = R(A) + R(P) = R\begin{pmatrix} A & P \end{pmatrix}.$$

So

$$\operatorname{rank}\begin{pmatrix} A & E \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A & P \end{pmatrix},$$

which will be denoted by r.

The basic idea is to compute the inertia of the matrix (of order $n + r(E)$)

$$C(t) = \begin{pmatrix} A & P \\ P^\top & -t^{-1}D^{-1} \end{pmatrix}.$$

Theorem 2.4 gives us that

$$\begin{aligned} \operatorname{In}(C(t)) &= \operatorname{In}(-t^{-1}D^{-1}) + \operatorname{In}(A + P(tD)P^\top) \\ &= \operatorname{In}(-tD) + \operatorname{In}(A + tE). \end{aligned}$$

From (2), we have

$$\operatorname{In}(C(t)) = \begin{cases} (v(E), \pi(E), 0) + \operatorname{In}(A + tE), & \text{if } \ t > 0; \\ (\pi(E), v(E), 0) + \operatorname{In}(A + tE), & \text{if } \ t < 0. \end{cases}$$

Thus,

$$\operatorname{In}(A + tE) = \begin{cases} \operatorname{In}(C(t)) - (v(E), \pi(E), 0), & \text{if } \ t > 0; \\ \operatorname{In}(C(t)) - (\pi(E), v(E), 0), & \text{if } \ t < 0. \end{cases}$$

So, in order to compute $\operatorname{In}(A + tE)$, it is sufficient to compute $\operatorname{In}(C(t))$. We perform Algorithm 2.1 (Schur complement) on $C(t)$ in two phases to obtain $\operatorname{In}(C(t))$.

Phase 1: We perform Algorithm 2.1 on $C(t)$, using elements (h, k), $h, k \leq n$, as pivots as long as possible. The number of possible operations is $r(A)$. At the end of this phase, we have

$$p = \pi(A), \quad q = v(A),$$

and C of the form

$$C_1(t) = \begin{pmatrix} 0 & P_1 \\ P_1^\top & P_2 - t^{-1}D^{-1} \end{pmatrix},$$

where the zero block is of order $n - r(A)$.

Phase 2: We continue performing Algorithm 2.1 on $C_1(t)$, applying double operations $\mathcal{S}_{kh}\mathcal{S}_{hk}$ with $h \leq n,\ k > n$, as long as possible. The number of possible double operations is $r(P_1)$. So at the end of this phase, we have

$$p = \pi(A) + r(P_1), \quad q = \upsilon(A) + r(P_1),$$

and C of the form

$$C_2(t) = \begin{pmatrix} 0 & 0 \\ 0 & E_1 + t^{-1}A_1 \end{pmatrix} = t^{-1}\begin{pmatrix} 0 & 0 \\ 0 & tE_1 + A_1 \end{pmatrix},$$

where A_1 and E_1 are of order $r(E) - r(P_1)$ and the zero block on the main diagonal is of order $n - r(A) - r(P_1)$.

Since the total number of operations in Phase 1 and Phase 2 is $r(A) + r(P_1)$, we have

$$r(A) + r(P_1) = \operatorname{rank}\begin{pmatrix} A & P \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A & E \end{pmatrix} = r.$$

So,

$$r(P_1) = r - r(A).$$

Therefore,

$$p = \pi(A) + r(P_1) = r - \upsilon(A),$$

$$q = \upsilon(A) + r(P_1) = r - \pi(A).$$

Moreover, A_1 and E_1 are of order $r(A) + r(E) - r$ and the zero block on the main diagonal of $C_2(t)$ is of order $n - r$.

Now we have

$$\begin{aligned}\mathrm{In}(C(t)) &= \mathrm{In}(C_2(t)) + (p,q,0) \\ &= \begin{cases}\mathrm{In}(A_1 + tE_1) + (p,q,n-r), & \text{if } \ t>0; \\ \mathrm{In}(-(A_1+tE_1)) + (p,q,n-r), & \text{if } \ t<0.\end{cases}\end{aligned}$$

Therefore, when $t > 0$,

$$\begin{aligned}\mathrm{In}(A+tE) &= \mathrm{In}(A_1+tE_1) + (p - \upsilon(E), q - \pi(E), n-r) \\ &= \mathrm{In}(A_1+tE_1) \\ &\quad + (r - \upsilon(A) - \upsilon(E), r - \pi(A) - \pi(E), n-r);\end{aligned}$$

when $t < 0$,

$$\begin{aligned}\mathrm{In}(A+tE) &= \mathrm{In}(-(A_1+tE_1)) + (p - \pi(E), q - \upsilon(E), n-r) \\ &= \mathrm{In}(-(A_1+tE_1)) \\ &\quad + (r - \upsilon(A) - \pi(E), r - \pi(A) - \upsilon(E), n-r).\end{aligned}$$

Then we only need to compute $\mathrm{In}(A_1 + tE_1)$. Note that the matrix pencil $A_1 + tE_1$ is of order $r(A) + r(E) - r$. When E is singular,

$$r(A) + r(E) - r < n.$$

Thus, we have reduced the dimension of the problem. If E_1 is singular, we repeat the process.

While Valiaho's method can indeed determine the positive semidefinite interval, in all cases, it is a consequence of having determined all intervals of constant inertia and the inertia of the matrix pencil for some $\hat{t}$ within each interval.

However, if we are only concerned with finding when the matrix pencil is positive semidefinite, Valiaho's algorithm is more than what is necessary, because we are not interested in the matrix inertia. Note that this is the case in the analysis of linear matrix inequality constraints sets in semidefinite programming problems.

In the next section we show how the results in [CG86] can be extended more directly, focussing on the positive semidefintie interval alone.

2.4. Caron's results for A positive semidefinite.

In the technical report [Car88], Caron extended the results in [CG86] to general E, with A positive semidefinite. In case $R(E) \subseteq R(A)$, the positive semidefinite interval is obtained from the eigenvalues of the $n \times n$ matrix X satisfying $AX = E$ and $R(X) \subseteq R(A)$. In particular, if A is positive definite, then it follows that $R(E) \subseteq R(A)$ and $X = A^{-1}E$. We summarize this result in the following theorem.

Theorem 2.6. *Suppose that $R(E) \subseteq R(A)$. Let X be an $n \times n$ matrix satisfying $AX = E$ and $R(X) \subseteq R(A)$. Let δ_n be the largest eigenvalue of X and δ_1, the smallest. Then $A + tE$ is positive semidefinite if and only if $t \in [\underline{t}, \bar{t}]$ where*

$$\underline{t} = \begin{cases} -\frac{1}{\delta_n}, & \textit{if } \delta_n > 0 \\ -\infty, & \textit{otherwise} \end{cases}$$

$$\bar{t} = \begin{cases} -\frac{1}{\delta_1}, & \textit{if } \delta_1 < 0 \\ +\infty, & \textit{otherwise.} \end{cases}$$

[Car88] also discussed the case $R(E) \not\subseteq R(A)$. The results, however, are incorrect and we will present a counterexample. To state the claim, we need some notation.

The Schur decomposition of E provides an orthogonal matrix $\hat{Q}$ such that

$$\hat{Q}^\top E \hat{Q} = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -D_3 \end{pmatrix}$$

where D_1 and D_3 are diagonal positive definite matrices.

Define the $n \times n$ matrices U and V by

$$U = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_3 \end{pmatrix}.$$

Let $B = \hat{Q}^\top A \hat{Q}$. Then $A + tE$ is positive semidefinite if and only if $B + t(U - V)$ is positive semidefinite. Since $R(E) \not\subseteq R(A)$, then $R(U - V) \not\subseteq R(B)$ and therefore, either $R(U) \not\subseteq R(B)$ or $R(V) \not\subseteq R(B)$.

Now we can state the claim as in Theorem 4.1 of [Car88]: If $R(U) \not\subseteq R(B)$ and $R(V) \subseteq R(B)$, then $T = [0, t_U]$ where t_U is as given in Theorem 2.6, with A replaced by B and E replaced by $-V$.

Example 2.7. Let

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3/4 \end{pmatrix}.$$

Then B is positive semidefinite, $R(U) \not\subseteq R(B)$, and $R(V) \subseteq R(B)$. We have

$$B + (U - V) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 1/4 \end{pmatrix}$$

and

$$B - V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1/4 \end{pmatrix}.$$

So $B + (U - V)$ is positive semidefinite, but $B - V$ is not.

Let the positive semidefinite intervals for $B+(U-V)$ and $B-V$ be T and $\hat{T}$, respectively. According to Theorem 4.1 in [Car88], $T = \hat{T}$. However, in this example, we have $1 \in T$ but $1 \notin \hat{T}$.

For the case $R(U) \subseteq R(B)$ and $R(V) \not\subseteq R(B)$, we can get a similar counterexample.

In the next chapter we present new results.

3. The Positive Semidefinite Interval when A is Nonsingular

In this chapter we assume that A is nonsingular. In this case, $\det(A+tE)$ is a polynomial in t and is not identically zero. Therefore, the equation

$$\det(A + tE) = 0 \tag{3}$$

has finitely many solutions. We present the results in three sections, according to whether A is positive definite, negative definite, and indefinite.

3.1. A is positive definite.

This case has already been solved in [Car88] and summarized in our Theorem 2.6. We shall present a new point of view that will help us present new results.

When A is positive definite, the right endpoint $\bar{t}$ of T is strictly positive. Thus, if $\bar{t} \neq +\infty$, then $A + \bar{t}E \succeq 0$ and it follows that all eigenvalues of $A + \bar{t}E$ are nonnegative. Moreover, one of them must be zero. Otherwise, the continuity of the eigenvalues would imply the existance of a $\delta > 0$ with $A + (\bar{t} + \delta)E \succ 0$, contradicting the fact that $\bar{t}$ is the right endpoint. It then follows that $\det(A + \bar{t}E) = 0$. So, $\bar{t}$ must be a positive solution for (3). In other words, if (3) has no positive solutions, then $\bar{t} = +\infty$. Similarly, if (3) has no negative solutions, then $\underline{t} = -\infty$.

Lemma 3.1. *If A is positive definite and equation (3) has positive solutions, then the smallest one is the right endpoint $\bar{t}$ of the positive semidefinite interval T for $A + tE$. Otherwise, $\bar{t} = +\infty$.*

Proof. We have already shown that if (3) has no positive solutions then $\bar{t} = +\infty$. Let t_{min} be the smallest positive solution to (3). We will will first show that $A + t_{min}E \succeq 0$.

Indeed, if $A + t_{min}E$ is not positive semidefinite, then it must have a negative eigenvalue. Let $\lambda_i(t)$ $(1 \leq i \leq n)$ be eigenvalues of $A + tE$ arranged in non-increasing order. Without loss of generality, assume that $\lambda_1(t_{min}) < 0$. Since $A \succ 0$, $\lambda_1(0) > 0$. By continuity of eigenvalues, there must be some $t_0 \in (0, t_{min})$ such that $\lambda_1(t_0) = 0$. So

$$\det(A + t_0E) = 0.$$

This is a contradiction, because t_{min} is the smallest positive solution for (3). Therefore, $A + t_{min}E \succeq 0$.

We now show that $A + tE \succ 0$, for $t \in (0, t_{min})$. Indeed, for $t \in (0, t_{min})$,

$$\begin{aligned} A + tE &= t(\frac{1}{t}A + E) \\ &= t[(\frac{1}{t_{min}}A + E) + (\frac{1}{t} - \frac{1}{t_{min}})A] \\ &= t[\frac{1}{t_{min}}(A + t_{min}E) + (\frac{1}{t} - \frac{1}{t_{min}})A] \succ 0. \end{aligned}$$

It now remains to show that $A + tE$ is not positive semidefinite, for $t > t_{min}$. Since $\det(A + t_{min}E) = 0$ there exists $y \in \mathbb{R}^n$, $y \neq 0$, such that $y^\top(A + t_{min}E)y = 0$. It then follows that $y^\top Ey < 0$. So for $t > t_{min}$, $y^\top(A + tE)y = y^\top(A + t_{min}E)y + (t - t_{min})y^\top Ey < 0$. Therefore, $A + tE$ is not positive semidefinite, for $t > t_{min}$.

□

The next lemma presents the analogous result when equation (3) has negative solutions.

Lemma 3.2. *If A is positive definite and equation (3) has negative solutions, then the largest one is the left endpoint $\underline{t}$ of the positive semidefinite interval T for $A + tE$. Otherwise, $\underline{t} = -\infty$.*

Recall that if $\det(A + tE) = 0$, then $t \neq 0$ and $-1/t$ is an eigenvalue for $A^{-1}E$. The following theorem follows from Lemma 3.1 and Lemma 3.2. It is the special case of Theorem 2.6 when A is positive definite, since $X = A^{-1}E$.

Theorem 3.3. *Suppose that A is positive definite. Let δ_n (δ_1) be the largest (smallest) eigenvalue of $A^{-1}E$. The positive semidefinite interval for $A + tE$ is $T = [\underline{t}, \bar{t}]$ where*

$$\underline{t} = \begin{cases} -\frac{1}{\delta_n}, & \text{if } \delta_n > 0 \\ -\infty, & \text{otherwise} \end{cases}$$

$$\bar{t} = \begin{cases} -\frac{1}{\delta_1}, & \text{if } \delta_1 < 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

3.2. A **is negative definite.**

We first show that if E is singular or indefinite the interval is empty.

Lemma 3.4. *Suppose that A is negative definite. If* $\det(E) = 0$ *or E is indefinite, then $T = \emptyset$.*

Proof. If $\det(E) = 0$, then there exists $y \in \mathbb{R}^n$, $y \neq 0$, such that

$$y^\top E y = 0.$$

It then follows that for any t,

$$y^\top(A+tE)y<0.$$

Therefore, when $\det(E)=0$, $A+tE$ is not positive semidefinite for any t.

If E is indefinite, then there exist $u,v\in\mathbb{R}^n$, such that

$$u^\top Eu>0,$$

$$v^\top Ev<0.$$

It then follows that for $t<0$,

$$u^\top(A+tE)u<0;$$

and for $t>0$,

$$v^\top(A+tE)v<0.$$

Since $A+0E=A$ is negative definite, $T=\emptyset$. $\square$

The next result shows that when A is negative definite and E is positive definite, then $A+tE$ is positive definite for sufficiently large t.

Lemma 3.5. *Suppose that A is negative definite and E is positive definite. If*

$$t>\frac{\max\limits_{\|x\|=1} x^\top(-A)x}{\min\limits_{\|x\|=1} x^\top Ex},$$

then $A+tE\succ 0$.

Here, the numerator is the largest eigenvalue of $-A$ and the denominator is the smallest eigenvalue of E. It is easy to see this by using the diagonalization of a symmetric matrix.

Proof. For any $x \in \mathbb{R}^n$, $\|x\| = 1$, $E \succ 0$ implies $x^\top E x > 0$. Then for

$$t > \frac{\max_{\|x\|=1} x^\top(-A)x}{\min_{\|x\|=1} x^\top E x},$$

we have

$$\begin{aligned} x^\top(A+tE)x &= -x^\top(-A)x + t x^\top E x \\ &> -x^\top(-A)x + \frac{\max_{\|x\|=1} x^\top(-A)x}{\min_{\|x\|=1} x^\top E x} x^\top E x \\ &\geq -x^\top(-A)x + \max_{\|x\|=1} x^\top(-A)x \\ &\geq 0. \end{aligned}$$

Therefore, $A + tE \succ 0$. □

We can now determine the interval T.

Lemma 3.6. *Suppose that A is negative definite and E is positive definite. Let t_{max} be the largest positive solution for equation (3). Then the positive semidefinite interval for $A + tE$ is*

$$T = [t_{max}, +\infty).$$

Proof. We first show that $A + t_{max}E \succeq 0$. Indeed, if $A + t_{max}E$ is not positive semidefinite, then it must have a negative eigenvalue. By Lemma 3.5, $A + tE$ is positive definite for sufficiently large t. So, there exists some $t_1 \in (t_{max}, +\infty)$ such that $\det(A + t_1E) = 0$. This is a contradiction, because t_{max} is the largest positive solution for equation (3). Therefore,

$$A + t_{max}E \succeq 0.$$

We now show that $A + tE \succ 0$, for $t \in (t_{max}, +\infty)$. For $t \in (t_{max}, +\infty)$, we have

$$\begin{aligned} A + tE &= t(\frac{1}{t}A + E) \\ &= t[(\frac{1}{t_{max}}A + E) + (\frac{1}{t} - \frac{1}{t_{max}})A] \\ &= t[\frac{1}{t_{max}}(A + t_{max}E) + (\frac{1}{t} - \frac{1}{t_{max}})A] \succ 0. \end{aligned}$$

Finally, we show that $A + tE$ is not positive semidefinite, for $t < t_{max}$. Since $\det(A + t_{max}E) = 0$, there exists $y \in \mathbb{R}^n$, $y \neq 0$, such that

$$y^\top (A + t_{max}E)y = 0.$$

Note that $E \succ 0$ implies

$$y^\top Ey > 0.$$

So, for $t < t_{max}$, we have

$$y^\top (A + tE)y = y^\top (A + t_{max}E)y + (t - t_{max})y^\top Ey < 0,$$

and so $A + tE$ is not positive semidefinite for $t < t_{max}$.

□

The next lemma gives the corresponding result for the case that E is negative definite.

Lemma 3.7. *Suppose that A is negative definite and E is negative definite. Let t_{min} be the smallest negative solution for equation (3). Then the positive semidefinite interval for $A + tE$ is*

$$T = (-\infty, t_{min}].$$

The following theorem gives T for the case of A negative definite.

Theorem 3.8. *Suppose that A is negative definite. Let δ_n (δ_1) be the largest (smallest) eigenvalue of $A^{-1}E$. Then the positive semidefinite interval T for $A + tE$ is given as follows.*

(1) $T = \emptyset$ *if* $A^{-1}E$ *has a zero eigenvalue.*
(2) $T = \emptyset$ *if* $\delta_1 < 0 < \delta_n$.
(3) $T = [-\frac{1}{\delta_n}, +\infty)$ *if* $\delta_n < 0$.
(4) $T = (-\infty, -\frac{1}{\delta_1}]$ *if* $\delta_1 > 0$.

Proof.

(1) Since $A^{-1}E$ has a zero eigenvalue, we have $\det(A^{-1}E) = 0$. Hence $\det(E) = 0$. By Lemma 3.4, $T = \emptyset$.
(2) Let $u, v \in \mathbb{R}^n$ be eigenvectors corresponding to δ_1, δ_n. So

$$A^{-1}Eu = \delta_1 u,$$

$$A^{-1}Ev = \delta_n v.$$

Then,

$$u^\top Eu = \delta_1 u^\top Au > 0,$$

$$v^\top Ev = \delta_n v^\top Av < 0.$$

Therefore, E is indefinite. By Lemma 3.4, $T = \emptyset$.
(3) Since $\delta_n < 0$, all eigenvalues of $A^{-1}E$ are negative. Hence the equation

$$\det(A + tE) = 0$$

has no negative solutions. It then follows from A is negative definite, that for all $t < 0$, $A + tE \prec 0$. That is, for any

$x \in \mathbb{R}^n$, $x \neq 0$, and any $t < 0$, $x^\top(A + tE)x < 0$. Thus, for any $t < 0$,

$$x^\top E x > -\frac{x^\top A x}{t}.$$

Letting $t \to -\infty$, we have for any $x \in \mathbb{R}^n$, $x \neq 0$,

$$x^\top E x \geq 0.$$

That is, $E \succeq 0$.

On the other hand, $\delta_n < 0$ implies $A^{-1}E$ has no zero eigenvalue and hence $\det(A^{-1}E) \neq 0$. Thus $\det(E) \neq 0$. Therefore, $E \succ 0$. By Lemma 3.6,

$$T = [-\frac{1}{\delta_n}, +\infty).$$

(4) The proof is similar to (3) and follows from Lemma 3.7.

□

3.3. A is nonsingular and indefinite.

We now combine the results in Section 3.1 and Section 3.2 to obtain the positive semidefinite interval when A is nonsingular but indefinite. The Schur decomposition of A provides an orthogonal $n \times n$ matrix Q such that

$$Q^\top A Q = \begin{pmatrix} D_1 & 0 \\ 0 & -D_2 \end{pmatrix}, \tag{4}$$

where D_1, D_2 are diagonal and positive definite.

Let

$$Q^\top E Q = \begin{pmatrix} E_1 & E_3 \\ E_3^\top & E_2 \end{pmatrix}.$$

Then

$$Q^\top(A+tE)Q = \begin{pmatrix} D_1 + tE_1 & tE_3 \\ tE_3^\top & -D_2 + tE_2 \end{pmatrix}.$$

By Theorem 3.3 and Theorem 3.8, we can determine the positive semidefinite intervals for D_1+tE_1 and $-D_2+tE_2$. We denote them by T_1 and T_2, respectively. Let $\delta_1, \cdots, \delta_k$ be the nonzero eigenvalues of $A^{-1}E$ and $t_i = -\frac{1}{\delta_i}, \quad i = 1, \cdots, k$. Define

$$T_0 = \{t_i | 1 \leq i \leq k, \; A + t_i E \succeq 0\}.$$

Clearly, $T_0 \subseteq T \subseteq T_1 \cap T_2$.

Theorem 3.9. *When A is nonsingular and indefinite, the positive semidefinite interval T for $A + tE$ is given as follows.*

(1) *Suppose that $T_0 = \emptyset$. Then $T = \emptyset$.*
(2) *Suppose that T_0 is a singleton $\{t_0\}$.*
  (a) *If $T_1 \cap T_2 = \{t_0\}$, then $T = \{t_0\}$.*
  (b) *If $T_1 \cap T_2 = [a, b]$, then $T = \{t_0\}$.*
  (c) *If $T_1 \cap T_2 = [a, +\infty)$ and for some $t^* > t_0$, $A + t^*E \succeq 0$, then $T = [t_0, +\infty)$.*
  (d) *If $T_1 \cap T_2 = [a, +\infty)$ and for some $t^* > t_0$, $A + t^*E$ is not positive semidefinite, then $T = \{t_0\}$.*
  (e) *If $T_1 \cap T_2 = (-\infty, b]$ and for some $t^* < t_0$, $A + t^*E \succeq 0$, then $T = (-\infty, t_0]$.*
  (f) *If $T_1 \cap T_2 = (-\infty, b]$ and for some $t^* < t_0$, $A + t^*E$ is not positive semidefinite, then $T = \{t_0\}$.*
(3) *Suppose that T_0 has more than one point, say $t_{i_1}, \cdots, t_{i_r}$, where $r > 1$ and $t_{i_1} < \cdots < t_{i_r}$.*
  (a) *If $T_1 \cap T_2 = [a, b]$, then $T = [t_{i_1}, t_{i_r}]$.*
  (b) *If $T_1 \cap T_2 = [a, +\infty)$ and for some $t^* > t_{i_r}$, $A + t^*E \succeq 0$, then $T = [t_{i_1}, +\infty)$.*

(c) *If* $T_1 \cap T_2 = [a, +\infty)$ *and for some* $t^* > t_{i_r}$, $A + t^*E$ *is not positive semidefinite, then* $T = [t_{i_1}, t_{i_r}]$.

(d) *If* $T_1 \cap T_2 = (-\infty, b]$ *and for some* $t^* < t_{i_1}$, $A + t^*E \succeq 0$, *then* $T = (-\infty, t_{i_r}]$.

(e) *If* $T_1 \cap T_2 = (-\infty, b]$ *and for some* $t^* < t_{i_1}$, $A + t^*E$ *is not positive semidefinite, then* $T = [t_{i_1}, t_{i_r}]$.

Proof. The above results are straightforward. For example, we will prove 3(b).

Let $T = [\underline{t}, \bar{t}]$. In this case, if $\bar{t} < +\infty$, then $\det(A + \bar{t}E) = 0$ and $A + \bar{t}E \succeq 0$. Thus $\bar{t} \in T_0$. Since $A + t^*E \succeq 0$ and $\bar{t}$ is the right endpoint of the positive semidefinite interval, we have $\bar{t} \geq t^*$. It then follows from $t^* > t_{i_r}$ that $\bar{t} > t_{i_r}$. This is a contradiction, because t_{i_r} is the largest element in T_0. Therefore, $\bar{t} = +\infty$.

On the other hand, since $T \subseteq T_1 \cap T_2$, we have $\underline{t} > -\infty$. So $\det(A + \underline{t}E) = 0$ and $A + \underline{t}E \succeq 0$. Thus $\underline{t} \in T_0$ and hence $\underline{t} \geq t_{i_1}$. Since $A + t_{i_1}E \succeq 0$ and $\underline{t}$ is the left endpoint of the positive semidefinite interval, we have that $\underline{t} \leq t_{i_1}$. So $\underline{t} = t_{i_1}$.

Therefore, $T = [t_{i_1}, +\infty)$.

$\square$

4. The Positive Semidefinite Interval when A is Singular

In this chapter, we are concerned with the positive semidefinite interval for the matrix pencil $A + tE$ when A is singular.

The Schur decomposition of A provides an orthogonal $n \times n$ matrix Q such that

$$Q^\top A Q = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

where D_1 is diagonal and nonsingular.

Let

$$Q^\top E Q = \begin{pmatrix} E_1 & E_3 \\ E_3^\top & E_2 \end{pmatrix}.$$

Then

$$Q^\top (A + tE) Q = \begin{pmatrix} D_1 + tE_1 & tE_3 \\ tE_3^\top & tE_2 \end{pmatrix},$$

where E_1 has the dimensions of D_1.

Since D_1 is nonsingular, the positive semidefinite interval for the matrix pencil $D_1 + tE_1$, which we denote by T_1, can be found using Theorem 3.3, Theorem 3.8, or Theorem 3.9.

Let T_2 denote the positive semidefinite interval for the matrix pencil tE_2. Then,

(1) $E_2 = 0 \Rightarrow T_2 = \mathbb{R}$.
(2) $E_2 \succeq 0, E_2 \neq 0 \Rightarrow T_2 = [0, +\infty)$.
(3) $E_2 \preceq 0, E_2 \neq 0 \Rightarrow T_2 = (-\infty, 0]$.
(4) E_2 is indefinite $\Rightarrow T_2 = \{0\}$.

Note that if $A+tE$ is positive semidefinite, then both D_1+tE_1 and tE_2 are positive semidefinite. However, in general, $T_1 \cap T_2$ is not the positive semidefinite interval for $A+tE$.

Example 4.1. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the positive semidefinite interval for $A+tE$ is $T = \{0\}$. But $T_1 = T_2 = (-\infty, +\infty)$.

In the following, let's consider the case $R(E) \subseteq R(A)$, which is equivalent to $N(A) \subseteq N(E)$.

Given the Schur decomposition of A in (5), for any $n \times n$ matrix W, let

$$Q^\top W Q = \begin{pmatrix} W_1 & W_4 \\ W_3 & W_2 \end{pmatrix}.$$

Lemma 4.2. *$R(W) \subseteq R(A)$ if and only if $W_2 = 0$ and $W_3 = 0$.*

Proof. Note that $R(W) \subseteq R(A)$ if and only if for some X,

$$AX = W,$$

i.e.

$$Q^\top A Q Q^\top X Q = Q^\top W Q.$$

Let

$$Q^\top X Q = \begin{pmatrix} X_1 & X_4 \\ X_3 & X_2 \end{pmatrix}.$$

Then $R(W) \subseteq R(A)$ if and only if

$$\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 & X_4 \\ X_3 & X_2 \end{pmatrix} = \begin{pmatrix} W_1 & W_4 \\ W_3 & W_2 \end{pmatrix},$$

i.e.

$$\begin{pmatrix} D_1 X_1 & D_1 X_4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} W_1 & W_4 \\ W_3 & W_2 \end{pmatrix}.$$

Therefore, $R(W) \subseteq R(A)$ if and only if $W_2 = 0$ and $W_3 = 0$. $\square$

In Lemma 4.2, if W is symmetric, then $W_4 = W_3^\top = 0$.

By Lemma 4.2, we have for the case $R(E) \subseteq R(A)$, that

$$Q^\top E Q = \begin{pmatrix} E_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence,

$$Q^\top (A + tE) Q = \begin{pmatrix} D_1 + tE_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $A + tE \succeq 0$ if and only if $D_1 + tE_1 \succeq 0$, we have

Theorem 4.3. *Let the positive semidefinite interval for $D_1 + tE_1$ be T_1. Then the positive semidefinite interval for $A+tE$ is $T = T_1$.*

Since D_1 is nonsingular, we can use Theorem 3.3, Theorem 3.8, or Theorem 3.9 to determine T_1.

5. Implementation and Examples

When A is positive semidefinite, it is necessary to determine whether or not $R(E) \subseteq R(A)$. This can be done by checking the consistency of $AX = E$. One approach is to first determine a decomposition of A using Choleski factorization with symmetric pivoting ([DMBS79]). If $r(A) = r$, there exists a nonunique permutation matrix P and a triangular matrix R (unique for a given P) such that $P^\top AP = R^\top R$, where

$$R = \begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix},$$

and where R_{11} is a nonsingular upper triangular $r \times r$ matrix and R_{12} is an $r \times (n-r)$ matrix. Let $X_P = P^\top XP$ and $E_P = P^\top EP$.

Now, $AX = E$ is equivalent to $R^\top RX_P = E_P$. Set $Y = RX_P$. We first solve $R^\top Y = E_P$ for the matrix Y. This is done as follows. Set

$$Y = \begin{pmatrix} Y_1 & Y_4 \\ Y_3 & Y_2 \end{pmatrix}$$

and

$$E_P = \begin{pmatrix} E_{P1} & E_{P4} \\ E_{P3} & E_{P2} \end{pmatrix}.$$

Then we can rewrite $R^\top Y = E_P$ as

$$\begin{pmatrix} R_{11}^\top & 0 \\ R_{12}^\top & 0 \end{pmatrix} \begin{pmatrix} Y_1 & Y_4 \\ Y_3 & Y_2 \end{pmatrix} = \begin{pmatrix} E_{P1} & E_{P4} \\ E_{P3} & E_{P2} \end{pmatrix}.$$

Since R_{11} is nonsingular, Y_1 and Y_4 are uniquely determined by

$$R_{11}^\top Y_1 = E_{P1} \quad \text{and}$$
$$R_{11}^\top Y_4 = E_{P4}.$$

But Y_3 and Y_2 are undetermined. Furthermore, the equation

$$AX = E$$

has a solution only if Y_1 and Y_4 satisfy

$$R_{12}^{\top} Y_1 = E_{P3} \quad \text{and} \quad R_{12}^{\top} Y_4 = E_{P2}.$$

Next we solve $Y = RX_P$. Let

$$X_P = \begin{pmatrix} X_{P1} & X_{P4} \\ X_{P3} & X_{P2} \end{pmatrix}.$$

Then $Y = RX_P$ can be rewritten as

$$\begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{P1} & X_{P4} \\ X_{P3} & X_{P2} \end{pmatrix} = \begin{pmatrix} Y_1 & Y_4 \\ Y_3 & Y_2 \end{pmatrix}.$$

This implies that $Y_3 = 0$ and $Y_2 = 0$. Also, X_{P3} and X_{P2} are arbitrary; and X_{P1} and X_{P4} are the unique solutions to

$$R_{11} X_{P1} = Y_1 - R_{12} X_{P3}$$
$$R_{11} X_{P4} = Y_4 - R_{12} X_{P2},$$

respectively. We choose $X_{P3} = 0$ and $X_{P2} = 0$. The resulting matrix is

$$X = P \begin{pmatrix} X_{P1} & X_{P4} \\ X_{P3} & X_{P2} \end{pmatrix} P^{\top}.$$

We can use Householder reduction and the QR method ([GVL83]) to compute eigenvalues of the matrix X. The Householder matrix is of the form

$$H = I - \frac{2}{\|u\|^2} u u^{\top},$$

where I is the identity matrix of order n and $u \in \mathbb{R}^n$. Note that H is symmetric and orthogonal.

Suppose that $r(X) = r$. Then there exist Householder matrices $H_1, \cdots, H_r$ such that

$$H_r \cdots H_1 X = R,$$

where R is upper triangular. So

$$X = QR,$$

where Q is orthogonal.

Let $X_1 = RQ$. By Householder reduction,

$$X_1 = Q_1 R_1,$$

where Q_1 is orthogonal and R_1 is upper triangular.

Let $X_2 = R_1 Q_1$. By Householder reduction,

$$X_2 = Q_2 R_2,$$

where Q_2 is orthogonal and R_2 is upper triangular.

In general, suppose that we have $X_k = Q_k R_k$, where Q_k is orthogonal and R_k is upper triangular. Let $X_{k+1} = R_k Q_k$. By Householder reduction,

$$X_{k+1} = Q_{k+1} R_{k+1},$$

where Q_{k+1} is orthogonal and R_{k+1} is upper triangular.

Since

$$X_{k+1} = R_k Q_k = Q_k^\top X_k Q_k,$$

all X_k have the same eigenvalues as X.

Also, as $k \to +\infty$, X_k tends to an upper triangular matrix. So, for sufficiently large k, the diagonal elements of X_k are approximate values of the eigenvalues of X.

Example 5.1. Let

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Choleski decomposition of A is

$$P^\top AP = R^\top R$$

with

$$R = \begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that

$$R_{11} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

and

$$R_{12} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then

$$E_P = P^\top EP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

First we solve the equation

$$R^\top Y = E_P$$

which is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Y_1 & Y_4 \\ Y_3 & Y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

to get

$$Y_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/3 \end{pmatrix}$$

and

$$Y_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since both

$$R_{12}^{\top} Y_1 = 0$$

and

$$R_{12}^{\top} Y_4 = 0,$$

we have $R(E) \subseteq R(A)$. Set $Y_3 = 0$ and $Y_2 = 0$.

Next we solve the equation

$$Y = RX_P$$

i.e.,

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_{P1} & X_{P4} \\ X_{P3} & X_{P2} \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Setting X_{P3} and X_{P2} to zero, we get

$$X_{P1} = \begin{pmatrix} 1/4 & 0 \\ 0 & -1/9 \end{pmatrix}$$

and

$$X_{P4} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The resulting matrix X is

$$X = PX_PP^\top = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & -1/9 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, the eigenvalues of X are $1/4, -1/9$, and 0. Therefore, the positive semidefinite interval is $[-4, 9]$.

In case A is negative semidefinite, $-A$ is positive semidefinite and the above procedure applies.

The next example shows how Theorem 3.9 can be used in the case when A is indefinite.

Example 5.2. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

The matrix pencils

$$D_1 + tE_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$-D_2 + tE_2 = (-1) + t(2)$$

have positive semidefinite intervals $T_1 = [-1, 1]$ and $T_2 = [1/2, +\infty)$, yielding

$$T_1 \cap T_2 = [1/2, 1].$$

The eigenvalues of $A^{-1}E$ are:

$$\delta_1 = -1$$
$$\delta_2 = -\frac{1}{2} + \frac{\sqrt{5}}{2}$$
$$\delta_3 = -\frac{1}{2} - \frac{\sqrt{5}}{2}.$$

So

$$t_1 = -\frac{1}{\delta_1} = 1$$
$$t_2 = -\frac{1}{\delta_2} = -\frac{1}{2} - \frac{\sqrt{5}}{2}$$
$$t_3 = -\frac{1}{\delta_3} = -\frac{1}{2} + \frac{\sqrt{5}}{2}.$$

Then

$$T_0 = \{t_i | i = 1, 2, 3 \text{ and } A + t_i E \succeq 0\} = \{1, -\frac{1}{2} + \frac{\sqrt{5}}{2}\}.$$

Therefore, the positive semidefinite interval for $A + tE$ is

$$T = [-\frac{1}{2} + \frac{\sqrt{5}}{2}, 1].$$

6. Conclusion and Future Work

In this thesis, we present a summary of overview of the results on the positive semidefinite interval of the matrix pencil $A + tE$. Also, for the case that A is positive definite, we present a new point of view (Lemma 3.1 and Lemma 3.2) to get the same result as in [Car88].

For the case that A is negative definite, Theorem 3.8 shows how to determine the positive semidefinite interval from the eigenvalues of the matrix $A^{-1}E$. For the case that A is nonsingular but indefinite, we can use Theorem 3.9 to determine the positive semidefinite interval.

For the case that A is singular and $R(E) \subseteq R(A)$, the Schur decomposition of A provides an orthogonal matrix Q such that $Q^\top (A+tE)Q = \begin{pmatrix} D_1 + tE_1 & 0 \\ 0 & 0 \end{pmatrix}$, where D_1 is diagonal and nonsingular. Then the positive semidefinite interval for $A + tE$ is exactly the positive semidefinite interval for $D_1 + tE_1$.

We summarize the results with the following table.

A	E	PSD interval for $A + tE$
PSD	rank 1 or 2	[CG86]
PSD	$R(E) \subseteq R(A)$	[Car88]
PD	arbitrary	Th 3.3
ND	arbitrary	Th 3.8
indef. nonsing.	arbitrary	Th 3.9
singular	$R(E) \subseteq R(A)$	Th 4.3

Answers to the following questions require further research.

(1) How do we determine the positive semidefinite interval when A is singular and $R(E) \not\subseteq R(A)$?

(2) How do we solve the equation $\det(A + tE) = 0$ when both A and E are singular? In this case $\det(A + tE)$ may be identically zero.

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