Perron-Frobenius theory and KMS states on higher-rank graph C*-Algebras

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by

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Abstract

In this thesis, we study the Perron-Frobenius theory for irreducible matrices and irreducible family of commuting matrices in detail. We then apply it to study the KMS states of the C^*-algebras of k-graphs. To be more precise, we define the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ and C^*-algebra $C^*(\Lambda)$ for a k-graph Λ. For $r \in (0,\infty)^k$, there is a natural one-parameter C^*-dynamical system $(\mathcal{T}C^*(\Lambda), \alpha^r, \mathbb{R})$ induced from the gauge action of $\mathbb{T}^k$ on $\mathcal{T}C^*(\Lambda)$. We study the KMS states on the dynamical system $(\mathcal{T}C^*(\Lambda), \alpha^r, \mathbb{R})$. With suitable selections of $r \in (0,\infty)^k$ and $\beta \in (0,\infty)$, with emphasis on Λ being strongly connected, it is shown that the KMS states are closely related to the unimodular Perron-Frobenius eigenvector of Λ.

Dedication

For my Parents

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Contents

CHAPTER 1

Introduction

The main aims of this thesis are to study the Perron-Frobenius theory (Chapter 2 to Chapter 4) and the Kubo-Martin-Schwinger (KMS) states on the C^*-algebras of k-graphs (Chapter 6).

Chapter 2 and Chapter 3 are mainly referred from [**1**]. In Chapter 2, we study primitive matrices. For a primitive matrix T we show the existence of a simple and positive eigenvalue r which is the unique eigenvalue satisfying $r = \rho(T)$. This result is known as the Perron-Frobenius theorem for primitive matrices (see Theorem 2.2.1).

Let $T \in M_n(\mathbb{R})$ be a non-negative matrix. Define a directed graph E associated with T in such a way that the indices of T are the vertices of E and $T(u, v)$ is the number of edges from the vertex v to the vertex u. In fact, T is called the coordinate matrix of E. In Chapter 3, we give a classification of indices of a non-negative matrix into different self-communicating classes. The directed graphs associated to sub-matrices corresponding to self-communicating classes turn out to be strongly connected; such sub-matrices are termed as irreducible matrices. It comes into notice that primitive matrices are a particular case of irreducible matrices. We give the Perron-Frobenius theorem for irreducible matrices (see Theorem 3.4.1) which is a weaker version of the Perron-Frobenius theorem for primitive matrices. Some differences between these two theorems are also mentioned in the last theorem of Chapter 3.

In Chapter 4, we study higher-rank graphs (also known as k-graphs) which are, in fact, the higher-dimensional analogues of directed graphs. For a k-graph, there are k coordinate matrices. Our main focus is the study of strongly connected k-graphs, in which case the family of coordinate matrices turns out to be a family of irreducible matrices. We study the Perron-Frobenius theory for an irreducible family of matrices and later we give a Perron-Frobenius theorem for strongly connected k-graphs.

The Perron-Frobenius theory we study here has some applications in the study of KMS states on the C^*-algebras of graphs, which we see in Chapter 6. First we study the required background in Chapter 5 with an example of KMS states called Gibbs states.

In Chapter 6, to warm up we start with studying the KMS states on C^*-algebras of directed graphs. The last section of this chapter is of the most importance. We define the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ and graph C^*-algebra $C^*(\Lambda)$ for a k-graph Λ. We find out that $C^*(\Lambda)$ is a quotient of $\mathcal{T}C^*(\Lambda)$. Let $r \in (0,\infty)^k$. We give the subinvariance relation for the dynamical system $(\mathcal{T}C^*(\Lambda), \alpha^r, \mathbb{R})$ in Proposition 6.3.14. Let ω be a KMS_β state of $(\mathcal{T}C^*(\Lambda), \alpha^r, \mathbb{R})$. If Λ is strongly connected with coordinate matrices A_i's, then $\beta r_i \geq \ln \rho(A_i)$. Moreover, if ω is a KMS_β state of $C^*(\Lambda)$, then $\beta r_i = \ln \rho(A_i)$ and consequently $m^\omega = (\omega(t_v)) \in [0,1]^{\Lambda^0}$ turns out to be the unimodular Perron-Frobenius eigenvector of Λ. If $\beta r_i > \ln \rho(A_i)$, then we construct all possible KMS_β states of $(\mathcal{T}C^*(\Lambda), \alpha^r, \mathbb{R})$ in Theorem 6.3.22. We finish with providing a sufficient condition that $(\mathcal{T}C^*(\Lambda), \alpha^r, \mathbb{R})$ has a unique KMS_1 state in Theorem 6.3.23; followed by a an example of computing the KMS_1 states of a 1-graph.

Author's contribution. The results in this thesis are primarily from [**1**], [**3**], [**9**] and [**10**]. Additionally, the background of C^*-algebra and k-graphs are referred from various references mentioned in the relevant sections. The author's main contribution in this thesis has been to provide the full details of the main results presented in this thesis. In addition to that the author studies the behavior of canonical form of irreducible matrices in detail in Section 3.3. Among the results whose proofs have been substantially expanded, we can mention Theorems 2.2.1, 3.3.14, 3.4.5, Proposition 4.1.5, Gibbs states in Section 5.3, Example 6.2.2, Propositions 6.2.15, 6.3.13, 6.3.14, Theorems 6.3.22, 6.3.23 and Example 6.3.24.

CHAPTER 2

Perron-Frobenius Theory For Primitive Matrices

In this chapter we introduce primitive matrices and give a detailed proof of the Perron-Frobenius theorem for primitive matrices.

2.1. Preliminaries

NOTATION 2.1.1. (a) We denote the set of all $n \times m$ matrices over a field $\mathbb{F}$ by $M_{n\times m}(\mathbb{F})$. If $n = m$ then we write $M_{n\times n}(\mathbb{F})$ as $M_n(\mathbb{F})$.

(b) We denote $x \in \mathbb{F}^n$ as a column vector $x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^t$.

(c) For $T = [t_{ij}] \in M_n(\mathbb{F})$, its k^{th} power is denoted as $T^k = [t_{ij}^{(k)}]$, where $t_{ij}^{(k)}$ is the $(i,j)^{th}$ entry of T^k.

DEFINITION 2.1.1. (a) A matrix $T = [t_{ij}]$ in $M_{n\times m}(\mathbb{R})$ is said to be *non-negative* if $t_{ij} \geq 0$ for every $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$ and we write $T \succeq 0$. If $T \succeq 0$ and $T \neq 0$, then we write $T \succneqq 0$. Moreover, if $t_{ij} > 0$ for every $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, then T is called *strictly positive* and we write $T \succ 0$.

(b) For $A = [a_{ij}]$ and $B = [b_{ij}]$ in $M_{n\times m}(\mathbb{F})$, we say $A \preceq B$ if $a_{ij} \leq b_{ij}$ and $A \prec B$ if $a_{ij} < b_{ij}$ for every $i = 1, 2, \dots, n$ and $j = 1, 2, , \dots, m$.

LEMMA 2.1.1. *Let $A, B \in M_{n\times m}(\mathbb{R})$, $C \in M_{m\times s}(\mathbb{R})$ and $D \in M_{t\times n}(\mathbb{R})$ such that $C \succeq 0$ and $D \succeq 0$. Then $A \preceq B$ implies $AC \preceq BC$ and $DA \preceq DB$.*

PROOF. Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{jk}]$ such that $A \preceq B$.
Then for every $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ and $k = 1, 2, \dots, s$,

$$\begin{aligned} a_{ij} \leq b_{ij} &\implies a_{ij}c_{jk} \leq b_{ij}c_{jk}, \qquad\qquad (\text{as } c_{jk} \geq 0) \\ &\implies \sum_{j=1}^{m} a_{ij}c_{jk} \leq \sum_{j=1}^{m} b_{ij}c_{jk} \end{aligned}$$

$$\implies AC(i,k) \leq BC(i,k) \qquad \text{(by the definition of matrix multiplication)}$$

$$\implies AC \preceq BC.$$

Similarly, $DA \preceq DB$. □

LEMMA 2.1.2. *If the sum of elements of each row (respectively column) of a matrix $A \in M_n(\mathbb{C})$ is equal, then the sum is a right (respectively left) eigenvalue of A.*

PROOF. Case 1: If the sum of elements of each row of the matrix A is equal to r, then $x \in M_{n\times 1}(\mathbb{C})$ with each entry equal to 1 is the right eigenvector associated to r, i.e., $Ax = rx$. Hence r is a right eigenvalue of A.

Case 2: If the sum of elements of each column of the matrix A is equal to s, then $y \in M_{1\times n}(\mathbb{C})$ with each entry equal to 1 is the left eigenvector associated to s, i.e., $yA = sy$. Hence s is a left eigenvalue of A. □

DEFINITION 2.1.2. Let A be a non-empty subset of a metric space and a function f from A to $\mathbb{R}$ is said to be *upper semi-continuous* on A if $\limsup_{k\to\infty} f(x_k) \leq f(x_0)$ for any $x_0 \in A$ and any sequence $\{x_k\}$ in A such that $x_k \to x_0$ as $k \to \infty$.

We have the following result from Appendix C of [**1**] which is used later in the proof of Theorem 2.2.1.

LEMMA 2.1.3. *An upper semi-continuous function defined on a non-empty compact metric space A attains its supremum at some point x_0 in A.*

PROOF. Let $a = \sup_{x\in A} f(x)$ for an upper semi-continuous function f on A. Notice that a is finite. To the contrary let us suppose a is infinite. Then there exists a sequence $\{x_n\}$ in A such that

$$\lim_{n\to\infty} f(x_n) = +\infty. \tag{2.1.1}$$

Since A is compact, there exists a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ in A which converges to a point $x_0 \in A$. Since f is upper semi-continuous on A,

$$\lim_{k\to\infty} x_{n_k} = x_0 \implies \limsup_{k\to\infty} f(x_{n_k}) \le f(x_0).$$

Hence by (2.1.1), $+\infty = \limsup_{k\to\infty} f(x_{n_k}) \le f(x_0)$ implies $f(x_0) = +\infty$ which is a contradiction. Hence $a = \sup_{x\in A} f(x)$ is finite. Then by definition of supremum there exists a sequence $\{x_k\}$ in A such that $\lim_{k\to\infty} f(x_k) = a$. Since A is compact and $\{x_k\}$ is a sequence in A, there exists a subsequence $\{x_{k_i}\}_i$ of $\{x_k\}_k$ which is convergent, say to $x_0 \in A$. Clearly $\lim_{k\to\infty} f(x_k) = a$ implies $\lim_{i\to\infty} f(x_{k_i}) = a$. Then

$$a = \limsup_{i\to\infty} f(x_{k_i}) \le f(x_0) \le \sup_{x\in A} f(x) = a.$$

Hence $f(x_0) = a$. □

REMARK 2.1.4. $a = \limsup_{i\to\infty} f(x_{k_i}) \le f(x_0)$ also implies that a has to be finite.

2.2. Perron-Frobenius Theorem for Primitive Matrices

DEFINITION 2.2.1. A square non-negative matrix T is said to be a *primitive matrix* if there exists a positive integer k such that $T^k \succ 0$.

DEFINITION 2.2.2. Let $T \in M_n(\mathbb{C})$ and $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ be its eigenvalues. Then the *spectral radius* of T, denoted by $\rho(T)$ is defined as $\rho(T) = \max_{1\le i\le n}\{|\lambda_i|\}$.

THEOREM 2.2.1. ***(The Perron-Frobenius theorem for primitive matrices)*** *Let T be a primitive matrix. Then we have the following properties:*

(a) There exists a real and positive eigenvalue r.

(b) $r = \rho(T)$. Moreover, if $r = |\lambda_i|$ for some eigenvalue λ_i, then $r = \lambda_i$.

(c) The eigenvector space associated to r is one dimensional.

(d) There exist left and right eigenvectors associated to r which are strictly positive.

(e) If $0 \preceq B \preceq T$ and β is an eigenvalue of B, then $|\beta| \le r$. Moreover, $|\beta| = r$ implies $B = T$.

(f) r is a simple eigenvalue.

Proof. Let $A := \{x \in \mathbb{R}^n : x \succeq 0 \text{ and } ||x|| = 1\}$, where $||.||$ is the ℓ_1 norm given as $||x|| = \sum_{i=1}^{n} |x_i|$. Also let $T = [t_{ij}]$. Define a function $s : A \to \mathbb{R}$ as

$$s(x) = \min_{1\le i\le n} \frac{(Tx)_i}{x_i} \qquad \text{for every } x \in A.$$

The fraction $\frac{(Tx)_i}{x_i}$ can be infinite if $x_i = 0$. But since $x \in A$, one has $0 \le s(x) < \infty$.

(a) Now

$$s(x) = \min_{1\le i\le n} \frac{(Tx)_i}{x_i} \le \frac{(Tx)_i}{x_i} \qquad \text{for every } i = 1, 2, \ldots, n$$

$$\implies xs(x) \preceq Tx.$$

Let $\mathbf{1}^t \in \mathbb{R}^n$ be the row vector with all entries equal to 1. Then by Lemma 2.1.1, the above inequality yields

$$\mathbf{1}^t xs(x) \preceq \mathbf{1}^t Tx. \tag{2.2.1}$$

Let $K := \max_{1\le j\le n} \sum_{i=1}^{n} t_{ij}$. Clearly K is finite and independent of x. Then using Lemma 2.1.1 and the inequality (2.2.1), we get

$$\mathbf{1}^t T \preceq \mathbf{1}^t K \implies \mathbf{1}^t Tx \preceq \mathbf{1}^t Kx \implies \mathbf{1}^t xs(x) \preceq \mathbf{1}^t xK \implies s(x) \le K.$$

Hence $\sup_{x\in A} s(x) \le K < \infty$. Now let

$$r := \sup_{x\in A} s(x). \tag{2.2.2}$$

Let $u \in \mathbb{R}^n$ be the column vector with all entries equal to $\dfrac{1}{n}$. Then $u \in A$. Since T is a primitive matrix, it can't have a column with all zeros. So

$$s(u) = \min_{1 \le i \le n} \frac{(Tu)_i}{u_i} = \min_{1 \le i \le n} \frac{\sum_{j=1}^{n} t_{ij} \frac{1}{n}}{\frac{1}{n}} = \min_{1 \le i \le n} \sum_{j=1}^{n} t_{ij} > 0.$$

Thus, $0 < s(u) \le r < \infty$. This shows that r is a positive finite number.

Claim 1: The function $s : A \to \mathbb{R}$ is an upper semi-continuous function.
Let $x_0 \in A$ and $\{x_l\}$ be a sequence in A such that $x_l \to x_0$ as $l \to \infty$.

Since T is a finite dimensional matrix, it is a bounded and so a continuous linear operator. So we have, as $l \to \infty$

$$x_l \to x_0 \implies Tx_l \to Tx_0 \implies (Tx_l)_i \to (Tx_0)_i \implies \frac{(Tx_l)_i}{(x_l)_i} \to \frac{(Tx_0)_i}{(x_0)_i}. \tag{2.2.3}$$

$$\begin{aligned}
&\text{Since } s(x_l) \le \frac{(Tx_l)_i}{(x_l)_i} \qquad \text{for every } i = 1, 2, \ldots, n \\
&\implies \limsup_{l \to \infty} s(x_l) \le \limsup_{l \to \infty} \frac{(Tx_l)_i}{(x_l)_i} \\
&\implies \limsup_{l \to \infty} s(x_l) \le \frac{(Tx_0)_i}{(x_0)_i} \qquad \text{(by (2.2.3))} \\
&\implies \limsup_{l \to \infty} s(x_l) \le \min_{1 \le i \le n} \frac{(Tx_0)_i}{(x_0)_i} \\
&\implies \limsup_{l \to \infty} s(x_l) \le s(x_0).
\end{aligned}$$

Claim 2: The set A is compact. Since A is bounded, it suffices to show that A is closed. Let $\{x_k\}$ be a sequence in A such that $x_k \to x$ as $k \to \infty$ in the topology induced by ℓ_1 norm. Since $\big| ||x_k|| - ||x|| \big| \le ||x_k - x||$, one can conclude $||x_k|| \to ||x||$ as $k \to \infty$. Thus $||x_k|| = 1$ for every k implies $||x|| = 1$. Also $x_k \ge 0$ implies $(x_k)_i \ge 0$ for every $k \in \mathbb{N}$ and $i = 1, 2, \ldots, n$. Now

$$x_n \to x \implies (x_n)_i \to (x)_i \implies (x)_i \ge 0 \qquad \text{for every } i = 1, 2, \ldots, n.$$

Hence $x \succeq 0$ and so $x \in A$. This shows that A is compact. By Lemma 2.1.3, $s(x)$ attains its supremum in A. Thus there exists $\hat{x} \in A$, such that

$$r = \min_{1 \le i \le n} \frac{(T\hat{x})_i}{(\hat{x})_i}.$$

In particular we have

$$r \le \frac{(T\hat{x})_i}{(\hat{x})_i} \qquad (i = 1, 2, \ldots, n)$$

$$\Longrightarrow \hat{x}r \preceq T\hat{x} \qquad \text{or} \qquad T\hat{x} - \hat{x}r \succeq 0. \tag{2.2.4}$$

Let $z = T\hat{x} - \hat{x}r$. Then $z \succeq 0$. We shall show that z has all entries equal to zero. Let us suppose there exists $i \in \{1, 2, \ldots, n\}$ such that $(z)_i > 0$. Then

$$(T\hat{x} - \hat{x}r)_i > 0 \implies T\hat{x} - \hat{x}r \gneqq 0.$$

Since T is primitive there exists $k \in \mathbb{Z}^+$ such that $T^k \succ 0$ and so

$$\begin{aligned}
& T^k(T\hat{x} - \hat{x}r) \succ 0 \\
\Longrightarrow\ & T(T^k\hat{x}) \succ r(T^k\hat{x}) \\
\Longrightarrow\ & (TT^k\hat{x})_i > r(T^k\hat{x})_i \qquad (i = 1, 2, \ldots, n) \\
\Longrightarrow\ & \min_{1 \le i \le n} \frac{(TT^k\hat{x})_i}{(T^k\hat{x})_i} > r \\
\Longrightarrow\ & \min_{1 \le i \le n} \frac{\left(T\frac{T^k\hat{x}}{||T^k\hat{x}||}\right)_i}{\left(\frac{T^k\hat{x}}{||T^k\hat{x}||}\right)_i} > r.
\end{aligned}$$

Since $\hat{x} \in A$ and $T^k \succ 0$ implies $T^k\hat{x} \gneqq 0$. Thus we get $\dfrac{T^k\hat{x}}{||T^k\hat{x}||} \in A$ such that $s\left(\frac{T^k\hat{x}}{||T^k\hat{x}||}\right) = \min_{1 \le i \le n} \frac{\left(T\frac{T^k\hat{x}}{||T^k\hat{x}||}\right)_i}{\left(\frac{T^k\hat{x}}{||T^k\hat{x}||}\right)_i} > r$, which is a contradiction to the definition of r in (2.2.2). Hence $z = 0$, i.e.,

$$T\hat{x} = r\hat{x}. \tag{2.2.5}$$

This shows that r is an eigenvalue of T which is real and positive.

(b) Let λ be any eigenvalue of T. Then there exists a norm-1 vector $x \neq 0$ such that $Tx = \lambda x$. Now for every $i = 1, 2, \ldots, n$

$$\lambda x_i = (Tx)_i \implies |\lambda x_i| = |(Tx)_i| \implies |\lambda||x_i| = \Big|\sum_{j=1}^{n} t_{ij}x_j\Big|$$

$$\implies |\lambda| \leq \min_{1\leq i\leq n} \frac{\Big|\sum_{j=1}^{n} t_{ij}x_j\Big|}{|x_i|} \leq \min_{1\leq i\leq n} \frac{\sum_{j=1}^{n} t_{ij}|x_j|}{|x_i|} = \min_{1\leq i\leq n} \frac{(Tx_+)_i}{(x_+)_i}, \tag{2.2.6}$$

where $x_+ = (|x_i|) \in M_n(\mathbb{R})$ and clearly $x_+ \in A$. Hence from (2.2.2)

$$|\lambda| \leq \min_{1\leq i\leq n} \frac{(Tx_+)_i}{(x_+)_i} \leq r.$$

Since λ is an arbitrary eigenvalue of T, the above inequality shows that $r = \max\limits_{1\leq i\leq n} |\lambda_i| = \rho(T)$ (using Definition 2.2.2).

Now suppose $|\lambda| = r$. From (2.2.6), for every $i = 1, 2, \ldots, n$ we have

$$\frac{(Tx_+)_i}{(x_+)_i} \geq |\lambda| \implies (Tx_+)_i \geq |\lambda||x_i| = r|x_i| \implies Tx_+ \succeq rx_+.$$

This inequality is completely similar to (2.2.4). So by same arguments we can get

$$Tx_+ = rx_+.$$

i.e., x_+ is an eigenvector of T associated to r. Moreover, for some $k > 0$ we have $T^k \succ 0$. Thus

$$T^k x_+ = r^k x_+ \implies (T^k x_+)_i = r^k|x_i| \implies \sum_{j=1}^{n} t_{ij}^{(k)}|x_j| = r^k|x_i|$$

$$\implies \sum_{j=1}^{n} t_{ij}^{(k)}|x_j| = |\lambda^k x_i| = \Big|\sum_{j=1}^{n} t_{ij}^{(k)}x_j\Big| \qquad (\text{as } |\lambda| = r).$$

Thus x_j's have the same direction, i.e., $x_j = |x_j|e^{i\theta}$ for every $j = 1, 2, \ldots, n$. Since λ is an eigenvalue of T as assumed, i.e., $\lambda x = Tx$, for every $i = 1, 2, \ldots, n$

$$\lambda x_i = (Tx)_i \implies \lambda x_i = \sum_{j=1}^{n} t_{ij} x_j \implies \lambda |x_i| e^{i\theta} = \sum_{j=1}^{n} t_{ij} |x_j| e^{i\theta}$$

$$\implies \lambda |x_i| = \sum_{j=1}^{n} t_{ij} |x_j| \implies \lambda x_+ = T x_+.$$

Since T is primitive, there exists $k > 0$ such that $T^k \succ 0$. Also $\lambda^k x_+ = T^k x_+$. Now

$$T^k \succ 0 \text{ and } x_+ \succeq 0 \implies T^k x_+ \succ 0 \implies \lambda^k x_+ \succ 0 \implies x_+ \succ 0.$$

Hence $\lambda x_+ = T x_+$ and $x_+ \succ 0$ implies λ is real and positive. Thus $|\lambda| = r$ implies $\lambda = r$.

(c) From (2.2.5) we already have an eigenvector $\hat{x}$ of T associated to r. Let $x \neq 0$ be another eigenvector of T associated to r. Then by arguments in part (b), x_+ is also an eigenvector of T associated to r and satisfies $x_+ \succ 0$. Let $c \in \mathbb{C}$. Then $\eta = \hat{x} - cx$ is also an eigenvector of T associated to r.

Let us assume that $\eta \neq 0$. So η is an non zero eigenvector of T and thus by arguments in part (b), η_+ is also an eigenvector of T associated to r and $\eta_+ \succ 0$.

Since c is an arbitrary complex number, we can choose c in such a way that at least for one j, $\hat{x}_j - cx_j = 0$; i.e., $\eta \neq 0$ but at least one entry of η is zero. But this contradicts the fact that $\eta_+ \succ 0$. Hence $\eta = 0$. This gives us $\hat{x} = cx$, this shows that every eigenvector of T associated to r is a multiple of $\hat{x}$. Hence eigenvector space associated to r is one dimensional.

(d) Since T is primitive, there exists $k > 0$ such that $T^k \succ 0$. From (2.2.5) we have $T\hat{x} = r\hat{x}$, i.e., $\hat{x}$ is an right eigenvector of T associated to r and $\hat{x} \succneqq 0$. Then also one can get $T^k \hat{x} = r^k \hat{x}$ which implies $\hat{x} \succ 0$.

Since transpose of a primitive matrix is also primitive and eigenvalues also remain same, r is also an eigenvalue of T^t possessing same properties in parts (a)-(c), hence there exists a right eigenvector x of T^t associated to r such that $x \succ 0$. Now $T^t x =$

$rx \implies x^tT = rx^t$, implies x^t is the left eigenvector of T associated to r which is strictly positive.

(e) Let $0 \preceq B \preceq T$ and β be an eigenvalue of B. Let $y \neq 0$ be a right eigenvector of B associated to β, i.e., $By = \beta y$. Then for $B = [b_{ij}]$, $y = [y_i]$ and for every $i = 1, 2, \ldots, n$

$$\beta y_i = \sum_{j=1}^{n} b_{ij} y_j \implies |\beta y_i| = \Big|\sum_{j=1}^{n} b_{ij} y_j\Big| \leq \sum_{j=1}^{n} |b_{ij}||y_j| = \sum_{j=1}^{n} b_{ij}|y_j| \implies |\beta| y_+ \preceq B y_+.$$

Since $B \preceq T$, by Lemma 2.1.1, $By_+ \preceq Ty_+$. So

$$|\beta| y_+ \preceq B y_+ \preceq T y_+. \tag{2.2.7}$$

Let x^t be a left strictly positive eigenvector of T associated to r. Then using Lemma 2.1.1 and the Inequality (2.2.7), we get

$$|\beta| x^t y_+ \preceq x^t T y_+ = r x^t y_+ \implies |\beta| \leq r \qquad (\text{as } x^t y_+ \succ 0).$$

Now suppose $|\beta| = r$, then from (2.2.7), $ry_+ \preceq Ty_+$. This is the same inequality as in (2.2.4), so using the same arguments we can get $ry_+ = Ty_+$ and $y_+ \succ 0$. Using (2.2.7)

$$\begin{aligned} & Ty_+ \succeq By_+ \succeq |\beta| y_+ = r y_+ = T y_+ \\ & \implies By_+ = Ty_+ \end{aligned} \tag{2.2.8}$$

Since $B \preceq T$, (2.2.8) is valid if, and only if, $B = T$.

(f) Recall that, for $A \in M_n(\mathbb{F})$, we have $A(\operatorname{Adj} A) = (\det A)I$, where I is the identity matrix of $M_n(\mathbb{F})$. Thus

$$\begin{aligned} & (rI - T)\operatorname{Adj}(rI - T) = \det(rI - T)I \\ \text{and} \qquad & \operatorname{Adj}(rI - T)(rI - T) = \det(rI - T)I. \end{aligned}$$

Since r is an eigenvalue of T, $\det(rI - T) = 0$. So we get

$$(rI - T)\,\mathrm{Adj}(rI - T) = 0. \tag{2.2.9}$$

$$\mathrm{Adj}(rI - T)(rI - T) = 0. \tag{2.2.10}$$

Now (2.2.9) shows that any column of $\mathrm{Adj}(rI - T)$ is either:

(i) a right eigenvector of T associated to r or

(ii) a column of zeros.

Similarly (2.2.10) shows that any row of $\mathrm{Adj}(rI - T)$ is either:

(i) a left eigenvector of T associated to r or

(ii) a row of zeros.

Combining these factors, we can conclude that any column and any row of $\mathrm{Adj}(rI-T)$ is either an eigenvector of T associated to r or the zero vector. But part (c) and part (d) assure that there exist strictly positive left and right eigenvectors of T associated to r and every eigenvector of T associated to r is unique to constant multiple. So no non-zero eigenvector can have a zero entry in it. Thus $\mathrm{Adj}(rI - T)$ is either

(i) a strictly positive matrix (every row and column is an eigenvector) or

(ii) a zeros matrix.

In what follows, we shall show that one entry of $\mathrm{Adj}(rI - T)$ is positive which gives that the case (i) holds.

The $(n, n)^{th}$ entry of $\mathrm{Adj}(rI - T)$ is $\det(rI(n|n) - T(n|n))$, where $T(n|n)$ is the matrix obtained from T by eliminating n^{th} row and n^{th} column and $I(n|n)$ is the corresponding identity matrix. Clearly

$$0 \preceq \begin{bmatrix} T(n|n) & \mathbf{0} \\ \mathbf{0}^t & 0 \end{bmatrix} \precneqq T$$

and the equality is $\precneqq$ because being primitive T cannot have a column or row of all zeros.

Let $B = \begin{bmatrix} T(n|n) & \mathbf{0} \\ \mathbf{0}^t & 0 \end{bmatrix}$. Then using arguments in part (e) we can conclude that no eigenvalue of B (so of $T(n|n)$) can have modulus greater than r. I.e., if $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are the eigenvalues of $T(n|n)$, then $0 \leq |\lambda_i| < r$ for every $i = 1, 2, \ldots, n-1$. Also

$$\det(xI(n|n) - T(n|n)) = 0 \implies \prod_{i=1}^{n-1}(x - \lambda_i) = 0$$

- If λ_i is real, then $r - \lambda_i > 0$ implies $\det(rI(n|n) - T(n|n)) > 0$.
- If λ_i is complex, then we must have eigenvalues in conjugates; i.e., we have factors like
  $(r - \lambda_i)(r - \bar{\lambda}_i) = r^2 - |\lambda_i|^2 > 0$. This implies $\det(rI(n|n) - T(n|n)) > 0$.

Hence $\det(rI(n|n) - T(n|n)) > 0$, so $(n, n)^{th}$ entry of $\operatorname{Adj}(rI - T)$ is positive. Thus Case (i) holds, i.e., $\operatorname{Adj}(rI - T) \succ 0$.

Write $\phi(x) = \det(xI - T)$ and differentiate $(xI - T)\operatorname{Adj}(xI - T) = \phi(x)I$ with respect to x. We have

$$\begin{aligned} &\frac{d}{dx}(xI - T)\operatorname{Adj}(xI - T) + (xI - T)\frac{d}{dx}\operatorname{Adj}(xI - T) = \phi'(x)I \\ \implies &\operatorname{Adj}(xI - T) + (xI - T)\frac{d}{dx}\operatorname{Adj}(xI - T) = \phi'(x)I. \end{aligned}$$

Put $x = r$ and multiply from right by $\hat{x}$ to get

$$\begin{aligned} &\operatorname{Adj}(rI - T)\hat{x} + (rI - T)\hat{x}\frac{d}{dx}\operatorname{Adj}(xI - T)|_{x=r} = \hat{x}\phi'(r) \\ \implies &\operatorname{Adj}(rI - T)\hat{x} = \hat{x}\phi'(r). \end{aligned}$$

Since $\operatorname{Adj}(rI - T) \succ 0$ and $\hat{x} \succ 0$, we have $\phi'(r) > 0$ which implies that r is not a root of $\phi'(x)$. This shows that the algebraic multiplicity of r in $\det(xI - T)$ is one. Hence r is a simple eigenvalue. □

The eigenvalue r obtained in Theorem 2.2.1 also has some more special properties which we discuss in following two corollaries.

COROLLARY 2.2.2.

$$\min_{1\le i\le n}\sum_{j=1}^{n} t_{ij} \le r \le \max_{1\le i\le n}\sum_{j=1}^{n} t_{ij}$$

and the equality holds on either side implies the sum of all rows are same.

PROOF. From Theorem 2.2.1(a), we have $0 \le s(u) \le r \le k$, where $u \in \mathbb{R}^n$ be the column vector with all entries equal to $\frac{1}{n}$, i.e.,

$$\min_{1\le j\le n}\sum_{i=1}^{n} t_{ij} \le r \le \max_{1\le i\le n}\sum_{j=1}^{n} t_{ij} < \infty$$

$$\Longrightarrow r \le \max_{1\le i\le n}\sum_{j=1}^{n} t_{ij}. \tag{2.2.11}$$

Since T is primitive, T^t (transpose of T) is also primitive with same unique highest modulus eigenvalue r. Thus again from Theorem 2.2.1(a), we get

$$\min_{1\le j\le n}\sum_{i=1}^{n} t^t_{ij} \le r \le \max_{1\le i\le n}\sum_{j=1}^{n} t^t_{ij}$$

$$\Longrightarrow \min_{1\le j\le n}\sum_{i=1}^{n} t_{ji} \le r \le \max_{1\le i\le n}\sum_{j=1}^{n} t_{ji}$$

$$\Longrightarrow \min_{1\le j\le n}\sum_{i=1}^{n} t_{ji} \le r \text{ namely } \min_{1\le i\le n}\sum_{j=1}^{n} t_{ij} \le r. \tag{2.2.12}$$

Combining (2.2.11) and (2.2.12), we get

$$\min_{1\le i\le n}\sum_{j=1}^{n} t_{ij} \le r \le \max_{1\le i\le n}\sum_{j=1}^{n} t_{ij}.$$

Now we shall show that if one side of equality holds then the sum of all rows are equal.

<u>Case 1</u>: Suppose that $\min_{1\le i\le n}\sum_{j=1}^{n} t_{ij} = r$. To the contrary, assume that not all row sums are the same. Let $\tilde{T}$ be a matrix such that $\tilde{T} \preceq T$ and all row sums of $\tilde{T}$ are equal to r. By Lemma 2.1.2, r is an eigenvalue of $\tilde{T}$. So by Theorem 2.2.1(e), $\tilde{T} = T$,

which is a contradiction to the fact that not all row sums of T are the same. Hence all row sums of T are equal and are equal to r.

Case 2: Suppose that $r = \max_{1\le i\le n} \sum_{j=1}^{n} t_{ij}$. To the contrary, assume that not all row sums are the same. Let $\bar{T} = [\bar{t}_{ij}]$ be a matrix such that $T \preceq \bar{T}$ and all row sums of $\bar{T}$ are equal to r. So again by Lemma 2.1.2, r is an eigenvalue of $\bar{T}$. Since $\bar{T}$ has positive entries at least in the same positions as of T (it may has more positive entries), $\bar{T}$ is primitive. Hence by above theorem, there exists a unique highest modulus eigenvalue of $\bar{T}$, say r'. Then by above result of this corollary we get

$$r = \min_{1\le i\le n} \sum_{j=1}^{n} \bar{t}_{ij} \le r' \le \max_{1\le i\le n} \sum_{j=1}^{n} \bar{t}_{ij} = r.$$

This implies $r' = r$. Hence from Theorem 2.2.1(e) $\bar{T} = T$, which is a contradiction to the fact that T does not have all row sums equal to r.

Hence if either of the equality holds, then all row sums are equal (and are actually equal to r). □

Corollary 2.2.3. *Let v^t and w be positive left and right eigenvectors of T associated to r, which are normed so that $v^t w = 1$. Then*

$$\frac{\operatorname{Adj}(rI - T)}{\phi'(r)} = wv^t.$$

Proof. In Theorem 2.2.1(f), we proved that every row (column) of $\operatorname{Adj}(rI - T)$ is a multiple of an left (right) positive eigenvector of T associated to r. In particular when every row is a multiple of a left positive eigenvector, let us say that eigenvector be $x^t = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$ and when every column is a multiple of a right positive eigenvector, let us say that eigenvector be $y = \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix}^t$. Then the matrix

$\operatorname{Adj}(rI - T)$ must be of the form:

$$\operatorname{Adj}(rI - T) = \begin{bmatrix} y_1x_1 & y_1x_2 & \dots & y_1x_n \\ y_2x_1 & y_2x_2 & \dots & y_2x_n \\ \vdots & \vdots & \dots & \vdots \\ y_nx_1 & y_nx_2 & \dots & y_nx_n \end{bmatrix}.$$

Thus

$$\operatorname{Adj}(rI - T) = yx^t.$$

But by Theorem 2.2.1(c), eigenvectors associated to r are unique up to constant multiples. Thus there are c_1, $c_2 > 0$ such that $y = c_1w$ and $x^t = c_2v^t$. Hence

$$\operatorname{Adj}(rI - T) = c_1c_2wv^t. \tag{2.2.13}$$

Now in the proof of part (f), we got

$$\operatorname{Adj}(xI - T) + (xI - T)\frac{d}{dx}\operatorname{Adj}(xI - T) = \phi'(x)I.$$

putting $x = r$ and pre-multiplying v^t

$$\begin{aligned} & v^t\operatorname{Adj}(rI - T) + v^t(rI - T)\frac{d}{dx}\operatorname{Adj}(xI - T)|_{x=r} = v^t\phi'(r) \\ \Longrightarrow\ & v^t.\operatorname{Adj}(rI - T) = v^t\phi'(r) \implies v^tc_1c_2wv^t = v^t\phi'(r) \qquad \text{using}(2.2.13) \\ \Longrightarrow\ & \phi'(r) = c_1c_2, \end{aligned}$$

as desired. □

From Theorem 2.2.1(c) the eigenvector space associated to r is one dimensional. Suppose that there exists another eigenvector $y \in A$ of T associated to r such that $y \succ 0$ and $||y|| = 1$. Then being non-negative and of same norm in the same one-dimensional eigenvector space of T associated to r, $y = \hat{x}$. Thus such $\hat{x}$ with $||\hat{x}|| = 1$ is unique.

Definition 2.2.3. The eigenvalue r of T obtained in Theorem 2.2.1 is called the *Perron-Frobenius (PF) eigenvalue* and its corresponding unique positive (left or right) eigenvector $\hat{x}$ with $||\hat{x}|| = 1$, is called the *Unimodular Perron-Frobenius (UPF) eigenvector*.

The norm we used in Theorem 2.2.1 is ℓ_1-norm over $\mathbb{R}^n$. In this thesis we usually use the same ℓ_1-norm over $\mathbb{R}^n$, otherwise specified.

CHAPTER 3

Perron-Frobenius Theory For Irreducible Matrices

In this chapter we study the structure of non-negative matrices with the help of graph theory and then we introduce irreducible matrices. Later we state and prove the Perron-Frobenius theorem for irreducible matrices.

3.1. Some Basics of Graph Theory

DEFINITION 3.1.1. A *directed graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 and two functions $r, s : E^1 \to E^0$. The elements of E^0 are called vertices and elements of E^1 are called edges. For each edge e we call $s(e)$ the source of e and $r(e)$ the range of e.

An edge $e \in E^1$ can therefore be thought of as traveling from $s(e)$ to $r(e)$.
Let $v, u \in E^0$ and $A \subseteq E^*$. Then we define $vA := \{e \in A : r(e) = v\}$, $Au := \{e \in A : s(e) = v\}$, and $vAu := \{e \in A : s(e) = u, r(e) = v\}$.

DEFINITION 3.1.2. The *coordinate matrix* A of a graph $E = (E^0, E^1, r, s)$ is defined as a matrix with entries $A(v, u) = \left|vE^1u\right|$, where $\left|vE^1u\right|$ denotes the cardinality of the set vE^1u.

DEFINITION 3.1.3. A graph is called *row finite* if vE^1 is a finite set for every $v \in E^0$. That is, in the coordinate matrix of the graph, the sum of entries in every row is finite.

DEFINITION 3.1.4. Let E be a directed graph and $u, v \in E^0$.

(a) If $vE^1 = \emptyset$, then v is called a *source*.
(b) If $E^1u = \emptyset$, then u is called a *sink*.

In this thesis we assume every graph is row-finite and has no sources.

DEFINITION 3.1.5. A *path* in a directed graph E is of the form $\mu = \mu_n \mu_{n-1} \dots \mu_2 \mu_1$, where $\mu_1, \mu_2, \dots, \mu_n$ are edges satisfying $r(\mu_i) = s(\mu_{i+1})$, $i = 1, 2, \dots, n-1$.

Note that n is called the *length* of μ. If $n = 1$, then the path μ is just an edge.

We extend the definitions of r and s to paths by defining $r(\mu) = r(\mu_n)$ and $s(\mu) = s(\mu_1)$.

DEFINITION 3.1.6. We denote by E^n the set of all paths of length n. Define $E^* = \bigcup_{n \geq 1} E^n$, which is called *path space* of E.

DEFINITION 3.1.7. A directed graph E is said to be *strongly connected* if for every $u, v \in E^0$, $uE^*v \neq \emptyset$.

3.2. Structure of Non-Negative Matrices

In this section, we classify the indices of non-negative matrices and study the structure of non-negative matrices.

DEFINITION 3.2.1. A sequence $(i, i_1, i_2, \dots, i_{t-1}, j)$ for $t \geq 1$ $(i_0 = i)$ from the index set $\{1, 2, \dots, n\}$ of a non-negative matrix T is said to form a *chain of length t between the ordered pair* (i, j) if

$$t_{ii_1} t_{i_1 i_2} t_{i_2 i_3} \dots t_{i_{t-1} j} > 0.$$

If $i = j$, such a chain is called *a cycle of length t between i and itself.*

NOTE 3.2.1. In Definition 3.2.1, we may WLOG assume that for a fixed pair (i, j) $i \neq i_k$ and $j \neq i_k$ for every $k = 1, 2, \dots, t-1$ to obtain a minimal length chain or cycle.

Let T be a non-negative matrix with index set $\{1, 2, \dots, n\}$. We define a directed graph E associated to T as follows: $\{1, 2, \dots, n\}$ are the vertices of E and the $(i, j)^{th}$ entry of T denotes the number of edges from the vertex j to the vertex i.

DEFINITION 3.2.2. Let i, j be arbitrary indices from the index set $\{1, 2, \dots, n\}$ of matrix T. We say that j *leads to* i and write $i \leftarrow j$ if there exists a chain

$(i, i_1, i_2, \dots, i_{k-1}, j)$ of length k between the ordered pair (i, j). If j does not lead to i, we write $i \nleftarrow j$. We say that i *and* j *communicate* if $i \leftarrow j$ and $j \leftarrow i$ and we write $i \leftrightarrow j$.

REMARK 3.2.2. If there exists a chain $(i, i_1, i_2, \dots, i_{k-1}, j)$ between (i, j), then one has $t_{ij}^k \geq t_{ii_1} t_{i_1 i_2} t_{i_2 i_3} \dots t_{i_{t-1} j} > 0$. So j leads to i if, and only if, there exists $k \geq 1$ such that $t_{ij}^k > 0$. Let E be the directed graph associated to a non-negative matrix T with index set $\{1, 2, \dots, n\}$. From Definition 3.1.5, a chain between (i, j) of length t is actually a path of length t from the vertex j to vertex i.

Now we shall classify the indices of a non-negative matrix in two different classes as follows:

DEFINITION 3.2.3. (a) If $i \leftarrow j$ and $j \nleftarrow i$, then the index j is called *inessential.* An index which leads to no index at all is also called *inessential* (this arises when the matrix T has a column of zeros).

(b) An index j is called *essential* if there is at least one i such that $i \leftarrow j$ and for every i such that $i \leftarrow j$ one can have $j \leftarrow i$.

If E is the directed graph associated to a non-negative matrix T. A vertex v is inessential if either v is a sink or there exist a vertex u such that $uE^*v \neq \emptyset$ but $vE^*u = \emptyset$. Whereas, a vertex v is essential if v is not a sink and for every index u such that $uE^*v \neq \emptyset$ one has $vE^*u \neq \emptyset$.

Classification of Indices. (a) We can now divide all essential indices in essential classes in such a way that all the indices belonging to one class communicate with each other but can not lead to any index outside the class.

(b) Also, all inessential indices which communicate with some indices can also be divided into inessential classes in such a way that all inessential indices which communicate with each other will belong to same class (indices in an inessential class may lead to some indices outside the class).

Since classification of indices of a non-negative matrix only depends on the position of positive elements, one can conclude that any two non-negative matrices having positive entries at same corresponding places will have the same classification of their indices. In order to unify the classification we introduce the following definition:

DEFINITION 3.2.4. The *incidence matrix* of a non-negative matrix $T = [t_{ij}]$ is defined as

$$\tilde{T}(i,j) = \begin{cases} 1, & if \quad t_{ij} \neq 0. \\ 0, & if \quad t_{ij} = 0. \end{cases}$$

Hence any two non-negative matrices with the same incidence matrix will have the same index classification.

In the following example, we will see how to easily classify and group indices with the help of graphs.

EXAMPLE 3.2.3. A non-negative matrix T has incidence matrix

$$\tilde{T} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

The associated graph is given as:

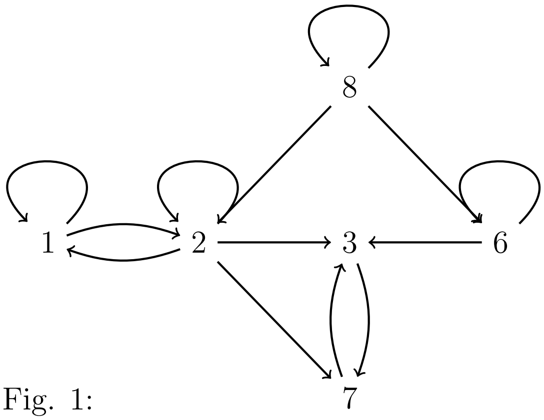


Fig. 1:

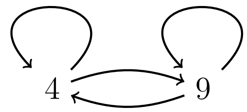


Fig. 2:

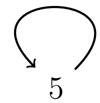


Fig. 3:

- Figure 1 shows that $\{3, 7\}$ is an *essential* class, while $\{1, 2\}$, $\{6\}$ and $\{8\}$ are *inessential* class.
- Figure 2 shows $\{4, 9\}$ is an *essential* class.
- Figure 3 shows $\{5\}$ is an *essential* class.

Now with the help of classification of indices, we shall seek a canonical form of a non-negative matrix as follows:

Canonical Form. A non-negative matrix T may be put into the canonical form by first relabeling the indices in a specific manner by keeping the following rules into consideration:

- Relabeling the indices using the same indexing set $\{1, 2, \dots, n\}$.
- Rewriting T by performing a simultaneous permutations of rows and columns of the matrix.

Simultaneous permutations of rows and columns only transform the original matrix T to another matrix T_c such that $T = P_m^{-1} \dots P_2^{-1} P_1^{-1} T_c P_1 P_2 \dots P_m$, where P_i is a square invertible matrix for $i = 1, 2, \dots, m$. This transformation of T does not alter its powers (i.e., powers of T are also similarly transformed) and the set of its eigenvalues also remains unchanged.

The canonical form is attained by first taking the indices of one essential class (if any) and renumbering them consecutively using the lowest integers and following by the indices of other essential classes, if any, until the essential classes exhausted. The numbering is then extended to inessential classes (if any) which are arranged in an order such that an inessential class occurring earlier(thus higher in arrangement) does not lead to any inessential class occurring later (i.e., any element of an inessential class higher in arrangement should not lead to any element of inessential class occurring later).

EXAMPLE 3.2.4. For the matrix T in Example 3.2.3, the essential classes are $\{5\}$, $\{4,9\}$, $\{3,7\}$ and inessential classes are $\{1,2\}$, $\{6\}$, $\{8\}$. Note that class $\{8\}$ can not be ordered before $\{1,2\}$ or $\{6\}$ as 8 leads to 2 and 6. Hence

$$\tilde{T}_c = \begin{array}{c} \\ 5 \\ 4 \\ 9 \\ 3 \\ 7 \\ 1 \\ 2 \\ 6 \\ 8 \end{array}\begin{array}{c} \begin{array}{ccccccccc} 5 & 4 & 9 & 3 & 7 & 1 & 2 & 6 & 8 \end{array} \\ \left(\begin{array}{c|cc|cc|cc|c|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array}.$$

The canonical form of a non-negative matrix T consists of square diagonal blocks corresponding to communicating classes, zero to the below of each communicating class block but at least one non-zero entry above each inessential class unless it corresponding to an index which leads to no other index. Thus we can write more general version of the canonical form of T as

$$\tilde{T}_c = \left[\begin{array}{cccc|c} T_1 & 0 & \dots & 0 & \\ 0 & T_2 & \dots & 0 & \\ \vdots & \vdots & \ddots & \vdots & R \\ 0 & 0 & \dots & T_z & \\ \hline & & \mathbf{0} & & Q \end{array}\right],$$

where the T_i, $i = 1, 2, \dots, z$, correspond to the z essential classes and Q corresponds to the inessential classes with $R \neq 0$ in general and Q itself having structure analogous to T such that there may be non-zero elements to the above of its diagonal blocks.

$$Q = \begin{bmatrix} Q_1 & & & \\ & Q_2 & & \mathbf{S} \\ & & \ddots & \\ & \mathbf{0} & & Q_w \end{bmatrix}.$$

Now, in most applications we are interested in the behavior of powers of T.

$$T^k = \left[\begin{array}{cccc|c} T_1^k & 0 & \dots & 0 & \\ 0 & T_2^k & \dots & 0 & \\ \vdots & \vdots & \ddots & \vdots & R_k \\ 0 & 0 & \dots & T_z^k & \\ \hline & & \mathbf{0} & & Q^k \end{array}\right] \text{ and } Q^k = \begin{bmatrix} Q_1^k & & & \\ & Q_2^k & & S_k \\ & & \ddots & \\ & \mathbf{0} & & Q_w^k. \end{bmatrix}.$$

It follows that, in order to study the behavior of powers of T, it will be considerable to study the powers of the diagonal block sub-matrices corresponding to self communicating classes. The evolution of R_k and S_k is complicated, in fact it will be sufficient if we only want to study the essential indices.

Definition 3.2.5. A sub-matrix corresponding to a single self-communicating class is called *irreducible*.

NOTE 3.2.5. In general, there are cases in which all indices of a non-negative matrix fall into non-self-communicating classes; for example $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. But we need to show that, normally there is at least one self-communicating (indeed essential) class of indices present for any non-negative matrix T.

LEMMA 3.2.6. *A non-negative matrix $T \in M_n(\mathbb{R})$ with at least one positive entry in each row possesses at least one essential class of indices.*

PROOF. Suppose that every row has at least one positive entry but all indices are inessential. This implies that for any index $j \in \{1, 2, \ldots, n\}$ there is at least one $i \in \{1, 2, \ldots, n\}$ such that $i \leftarrow j$ but $j \not\leftarrow i$. Let i_1 be any index. Then by assumption we can find i_2 such that $i_2 \leftarrow i_1$ but $i_1 \not\leftarrow i_2$. Similarly we can find i_3 such that $i_3 \leftarrow i_2$ but $i_2 \not\leftarrow i_3$. Repeating the same argument, we can find a sequence $i_1, i_2, \ldots, i_{n+1}$ such that

$$\begin{aligned} &i_{n+1} \leftarrow i_n \leftarrow \cdots \leftarrow i_2 \leftarrow i_1 \text{ but } i_k \not\leftarrow i_{k+1} \text{ and hence} \\ &i_{k-1} \not\leftarrow i_{k+1}, i_{k-2} \not\leftarrow i_{k+1}, \ldots, i_3 \not\leftarrow i_{k+1}, i_2 \not\leftarrow i_{k+1} \text{ and } i_1 \not\leftarrow i_{k+1}, \end{aligned} \tag{3.2.1}$$

for $k = 1, 2, \ldots, n$. i.e., an index in the sequence cannot lead to any index occurring prior to that index.

Since the sequence $i_1, i_2, \ldots, i_{n+1}$ is a set of $n + 1$ indices, each chosen from the set $\{1, 2, \ldots, n\}$. Hence by Pigeonhole principle at least one index repeats in the sequence which contradicts the fact (3.2.1). □

DEFINITION 3.2.6. If $i \leftarrow i$, then $d(i)$ is called the *period* of the index i if it is the largest common divisor of those $k \in \mathbb{Z}^+$ for which $t_{ii}^{(k)} > 0$.

DEFINITION 3.2.7. Let $\{a_1, a_2, a_3, \ldots\}$ be an infinite set of positive integers. If d_k is the greatest common divisor of $a_1, a_2, \ldots, a_k$, then *the greatest common divisor of the infinite set* $\{a_1, a_2, a_3, \ldots\}$ is defined by $d = \lim_{k \to \infty} d_k$.

The limit $d \geq 1$ clearly exists since $\{d_k\}$ is a decreasing sequence of integers greater than equal to 1 and it must be attained after a finite number of k's.

LEMMA 3.2.7. *Let $T \in M_n(\mathbb{R})$ such that $T \succeq 0$ and i, j be any two indices such that $i \leftrightarrow j$. Then $d(i) = d(j)$.*

PROOF. It is sufficient to show that $d(i) \leq d(j)$. Since $i \leftrightarrow j$, there exist $p, q \in \mathbb{Z}^+$ such that $t_{ij}^{(p)} > 0$ and $t_{ji}^{(q)} > 0$. Now, $t_{ii}^{(p+q)} \geq t_{ij}^{(p)} t_{ji}^{(q)} > 0$ implies $d(i)\big|p+q$. Let $s \in \mathbb{Z}^+$ be such that $t_{jj}^{(s)} > 0$ (such an s exists as $i \leftrightarrow j$ implies $j \leftarrow j$). Now, $t_{ii}^{(p+q+s)} > t_{ij}^{(p)} t_{jj}^{(s)} t_{ji}^{(q)} > 0$ implies $d(i)\big|p+q+s$. Thus $d(i)\big|s$. Hence $d(i)$ divides any positive integer s for which $t_{jj}^{(s)} > 0$. So by the definition of $d(j)$, $d(i) \leq d(j)$. $\square$

Lemma 3.2.7 shows that the period of all indices in a communicating class (essential or inessential) is same. So define period of a commuting class as:

DEFINITION 3.2.8. The *period of a communicating class* is defined as the period of any index in the class.

EXAMPLE 3.2.8. Consider the same matrix T from Example 3.2.3. The *Periods* of communicating classes are given as below:

Essential classes	Inessential classes
$\{5\}$ has period 1 as $t_{55} > 0$.	$\{1, 2\}$ has period 1 as $t_{22} > 0$.
$\{4, 9\}$ has period 1 as $t_{44} > 0$.	$\{6\}$ has period 1 as $t_{66} > 0$.
$\{3, 7\}$ has period 2 as $t_{33} = 0$ and $t_{33}^{(k)} >$ 0, for $k = 2n$, $n \in \mathbb{N}$.	$\{8\}$ has period 1 as $t_{88} > 0$.

3.3. Irreducible Matrices

We now give a more precise definition of irreducible matrices than in Definition 3.2.5. We introduce a canonical form of irreducible matrices on the basis of classification of indices of irreducible matrices and study the behavior of subclasses given by the canonical form.

DEFINITION 3.3.1. A non-negative matrix $T = [t_{ij}] \in M_n(\mathbb{R})$ is called *irreducible* if for every pair of indices (i, j) from its index set $\{1, 2, \ldots, n\}$, there exists a positive integer $m = m(i, j)$ such that $t_{ij}^{(m)} > 0$.

In terms of graph theory, a non-negative matrix T is said to be irreducible if the directed graph associated to T is strongly connected.

According to the classification of indices of non-negative matrices it is clear that all indices of an irreducible matrix are essential. So there is only one communicating class that is the entire index set. Moreover, by Lemma 3.2.7 and Definition 3.2.8 we have the following definition.

DEFINITION 3.3.2. The *period* of an irreducible matrix is defined as the period of any of its index.

DEFINITION 3.3.3. An irreducible matrix is said to be *cyclic* (or *periodic*) with period d if $d > 1$. It is said to be *acyclic* (or *aperiodic*) if $d = 1$.

In view of Definition 3.2.6 and Definition 3.3.2 the following lemma supports the definition of the period of an irreducible matrix. This lemma will be used in classification of indices of an irreducible matrix.

LEMMA 3.3.1. *Let $T = [t_{ij}]$ be an irreducible matrix of period d. Then there is $N_0 \in \mathbb{N}$, depending on i such that $t_{ii}^{(kd)} > 0$ for every integers $k \geq N_0$.*

To prove this lemma we need the following lemma.

LEMMA 3.3.2. *A semigroup S of positive integers contains all but finitely many positive multiples of its greatest common divisor (*gcd*).*

PROOF. Let d be the gcd of all elements of S. Dividing every element of S by d, we can reduce the problem to the case $d = 1$. From Definition 3.2.7 $d = 1$ must be the gcd of some finite elements of S, say, $a_1, a_2, \ldots, a_k$. Hence we can write $1 = \sum_{i=1}^{k} a_i\gamma_i$, where $\gamma_i \in \mathbb{Z}$. Then some of $a_i\gamma_i$'s are positive and some are negative. Let m be the sum of positive $a_i\gamma_i$'s and n be the sum of negative $a_i\gamma_i$'s. That is, $m - n = 1$, clearly

$m, n > 0$ and are linear combinations of a_i's. So $m, n \in S$ as S is a semigroup. Let $q \in \mathbb{Z}$ such that $q \geq n(n-1)$. By division algorithm we have $q = an + b$, where $a, b \in \mathbb{Z}$, $0 \leq b < n$ and $a \geq (n-1)$. Since $m - n = 1$,

$$q = an + b = an + b(m-n) = (a-b)n + bm \in S$$

as S is a semigroup and $m, n \in S$. Hence all $q \geq n(n-1)$ belong to S. That is, all multiples of $d = 1$ but finitely many belongs to S. ☐

Proof of Lemma 3.3.1. Let $A := \{kd : k \in \mathbb{Z}^+ \text{ and } t_{ii}^{(kd)} > 0\}$. Since d is the period of T, there exists $l > 0$ such that $t_{ii}^{(ld)} > 0$. This implies $ld \in A$, so $A \neq \emptyset$. Let $kd, sd \in A$. Then $t_{ii}^{(kd)} > 0$ and $t_{ii}^{(sd)} > 0$. So $t_{ii}^{(kd+sd)} = t_{ii}^{[d(k+s)]} \geq t_{ii}^{(kd)} t_{ii}^{(sd)} > 0$. This shows that $kd + sd \in A$. Hence A is closed under addition and clearly d is the gcd of elements of A. So by Lemma 3.3.2, A must contain all sufficiently large multiples of d, i.e., $t_{ii}^{(kd)} > 0$ for $k \geq N_0$, where $N_0 > 0$ is some integer. ☐

The following theorem gives us a unique way of classifying indices of an irreducible matrix to construct a canonical form.

Theorem 3.3.3. *Let j be any fixed index from the index set $\{1, 2, \ldots, n\}$ of an irreducible matrix $T = [t_{ij}]$ with period d. Then for every index i, there exists a unique integer r_i in the range $0 \leq r_i < d$ such that*

(a) $t_{ij}^{(s)} > 0$ implies $s \equiv r_i (\text{mod } d)$,
(b) $t_{ij}^{(kd+r_i)} > 0$ for $k \geq N(i)$, where $N(i)$ is an integer dependent on i.

Proof. Let $j \in \{1, 2, \ldots, n\}$ be fixed and $i \in \{1, 2, \ldots, n\}$ be an arbitrary index.

(a) For the indices i and j, there exists $s, r, p \in \mathbb{Z}^+$ such that $t_{ij}^{(s)} > 0$, $t_{ij}^{(r)} > 0$ and $t_{ji}^{(p)} > 0$. Then $t_{ii}^{(s+p)} \geq t_{ij}^{(s)} t_{ji}^{(p)} > 0$ and $t_{ii}^{(r+p)} \geq t_{ij}^{(r)} t_{ji}^{(p)} > 0$. Since d is the period of T, $d \big| (s+p)$ and $d \big| (r+p)$. Thus

$$d \big| s + p - r - p \implies d \big| s - r \implies s \equiv r (\text{mod } d). \tag{3.3.1}$$

WLOG, let $0 \leq r < d$. Hence for $s \in \mathbb{Z}^+$ such that $t_{ij}^{(s)} > 0$ implies that $s \equiv r(\text{ mod } d)$ and r clearly depends on i (so it can be written as r_i). Now we shall show the uniqueness of r_i. Let there exists $d > r' \geq 0$ such that $s \equiv r' (\text{mod } d)$. Then $r \equiv r' (\text{mod } d)$, using (3.3.1). Thus $d \big| r' - r$. Since $0 \leq r < d$ and $0 \leq r' < d$, $r' - r = 0$, proving the uniqueness of r_i.

(b) From part (a) for $s \in \mathbb{Z}^+$ such that $t_{ij}^{(s)} > 0$ we have $s \equiv r_i (\text{mod } d)$ or $s = md + r_i$ for some $m \in \mathbb{Z}$. Hence $t_{ij}^{(md+r_i)} > 0$. Since the period of T is d, by Lemma 3.3.1 there exists $N_0 > 0$ such that $t_{ii}^{(pd)} > 0$ for every $p \geq N_0$. Let $N(i) = N_0 + m$. Hence if $k \geq N(i) = N_0 + m$ implies $k = p + m$ for some $p \geq N_0$,

$$t_{ij}^{(kd+r_i)} = t_{ij}^{(pd+md+r_i)} \geq t_{ii}^{(pd)} t_{ij}^{(md+r_i)} > 0.$$

$\square$

Definition 3.3.4. Theorem 3.3.3 gives us a unique r_i for every index i and is called a *residue class modulo d.*

Definition 3.3.5. The set of indices i in $\{1, 2, \ldots, n\}$ corresponding to the same residue class modulo d is called a *subclass of the class* $\{1, 2, \ldots, n\}$ and is denoted by C_r $(0 \leq r < d)$.

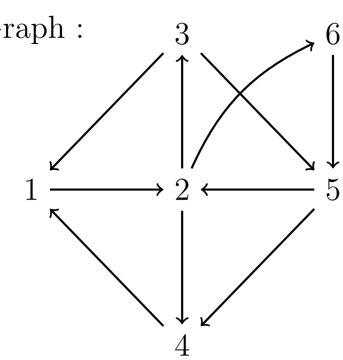
Remark 3.3.4. (a) As a conclusion of Theorem 3.3.3 we can actually define C_r for all non-negative integers r by putting $C_r = C_{r_j}$ if $r \equiv r_j (\text{mod } d)$.

(b) Clearly there are d numbers of subclasses C_r and are disjoint with union equal to $\{1, 2, \ldots, n\}$.

Example 3.3.5. Let us find the subclasses of the indices of the matrix

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Graph :



From the above graph we can easily see that $d = 3$. Now fix $j := 1$. Then using Theorem 3.3.3, we get

- For $i = 1$, $t_{11}^{(s)} > 0$, by Lemma 3.3.1 $s = kd$ for every sufficiently large k (as the period is 3). So $r_1 = 0$. i.e., $1 \in C_0$.
- Let $i = 2$. From the graph we can see that the index 1 is going to the index 2 through one edge, through a path of 4 edges, through a path of 7 edges and so on. This shows that $t_{21}^{(s)} > 0$ implies $s = 3n + 1$, so $r_2 = 1$. i.e., $2 \in C_1$. Similarly we can calculate the following:
- For $i = 3$, $t_{31}^{(s)} > 0$ implies $s = 3n + 2$, so $r_3 = 2$. i.e., $3 \in C_2$.
- For $i = 4$, $t_{41}^{(s)} > 0$ implies $s = 3n + 2$, so $r_4 = 2$. i.e., $4 \in C_2$.
- For $i = 5$, $t_{51}^{(s)} > 0$ implies $s = 3n + 3$, so $r_5 = 0$. i.e., $5 \in C_0$.
- For $i = 6$, $t_{61}^{(s)} > 0$ implies $s = 3n + 2$, so $r_6 = 2$. i.e., $6 \in C_2$.

Hence $C_0 = \{1, 5\}$, $C_1 = \{2\}$ and $C_2 = \{3, 4, 6\}$.

Now let's take $j := 2$. Then

- For $i = 1$, $t_{12}^{(s)} > 0$ implies $s = 3n + 2$, so $r_1 = 2$. i.e., $1 \in C_2$.
- For $i = 2$, $t_{22}^{(s)} > 0$ implies $s = 3k$ for every sufficiently large k (as period is 3), so $r_2 = 0$. i.e., $2 \in C_0$.
- For $i = 3$, $t_{32}^{(s)} > 0$ implies $s = 3n + 1$, so $r_3 = 1$. i.e., $3 \in C_1$.
- For $i = 4$, $t_{42}^{(s)} > 0$ implies $s = 3n + 1$, so $r_4 = 1$. i.e., $4 \in C_1$.
- For $i = 5$, $t_{52}^{(s)} > 0$ implies $s = 3n + 2$, so $r_5 = 2$. i.e., $5 \in C_2$.
- For $i = 6$, $t_{62}^{(s)} > 0$ implies $s = 3n + 1$, so $r_6 = 1$. i.e., $6 \in C_1$.

Hence $C_0 = \{2\}$, $C_1 = \{3, 4, 6\}$ and $C_2 = \{1, 5\}$.

Now let's take $j := 4$. Then

- For $i = 1$, $t_{14}^{(s)} > 0$ implies $s = 3n + 1$, so $r_1 = 1$. i.e., $1 \in C_1$.
- For $i = 2$, $t_{24}^{(s)} > 0$ implies $s = 3n + 2$, so $r_2 = 2$. i.e., $2 \in C_2$.
- For $i = 3$, $t_{34}^{(s)} > 0$ implies $s = 3n + 3$, so $r_3 = 0$. i.e., $3 \in C_0$.
- For $i = 4$, $t_{44}^{(s)} > 0$ implies $s = 3k$ for every sufficiently large k (as period is 3), so $r_4 = 0$. i.e., $4 \in C_0$.
- For $i = 5$, $t_{54}^{(s)} > 0$ implies $s = 3n + 4$, so $r_5 = 1$. i.e., $5 \in C_1$.

- For $i = 6$, $t_{64}^{(s)} > 0$ implies $s = 3n + 3$, so $r_6 = 0$. i.e., $6 \in C_0$.

Hence $C_0 = \{3, 4, 6\}$, $C_1 = \{1, 5\}$ and $C_2 = \{2\}$.

In Example 3.3.5 we see that changing the initial fixed index j does not actually change anything. In support of this we have the following lemma.

LEMMA 3.3.6. *The residue classes does not depend on the initial choice of fixed index j. An initial choice of another index merely subjects the subclass to a cyclic permutation.*

PROOF. Let $T = [t_{ij}]$ be an irreducible matrix with period d. Suppose j is the fixed index and we take a new fixed index j'. Let r'_i denote the residue class corresponding to i with respect to j'. Let $r_{j'}$ be the residue class corresponding to j' with respect to the fixed index j. Now

$$t_{ij}^{(md+r'_i+kd+r_{j'})} \geq t_{ij'}^{(kd+r'_i)} t_{j'j}^{(md+r_{j'})}.$$

By Theorem 3.3.3(b) for sufficiently large k and m, above inequality yields

$$t_{ij}^{(md+r'_i+kd+r_{j'})} \geq t_{ij'}^{(kd+r'_i)} t_{j'j}^{(md+r_{j'})} > 0.$$

So by Theorem 3.3.3(a), for i there exists r_i such that $0 \leq r_i < d$ and

$$md + r'_i + kd + r_{j'} \equiv r_i (\text{mod } d) \implies r'_i \equiv r_i - r_{j'} (\text{mod } d). \tag{3.3.2}$$

This shows that new residue classes r'_i are equal to $r_i - r_{j'}$ modulo d, i.e., the residue classes remains same but their order of occurrence changes by a cyclic permutation. □

NOTE 3.3.7. This property of composition of residue classes can also be referred as *the order of occurrence of subclasses remains constant up to a cyclic permutation.*

REMARK 3.3.8. With the help of Lemma 3.3.6 we can find different subclasses in Example 3.3.5 generated by changing the initially fixed j. Just for this example let's denote the subclasses with respect to a fixed j by C_r^j where $r = 0, 1, 2$. Initially the subclasses with respect to $j = 1$ are $C_0^1 = \{1, 5\}$, $C_1^1 = \{2\}$ and $C_2^1 = \{3, 4, 6\}$.

Let $j' = 2$. Since $2 \in C_1^1$, $r_{j'} = 1$. Then using (3.3.2) we have $r_i' \equiv r_i - r_{j'} (\mod d)$. So

$r_1' \equiv 0 - 1 \equiv 2(\text{mod } 3)$, $r_2' \equiv 1 - 1 \equiv 0(\text{mod } 3)$, $r_3' \equiv 2 - 1 \equiv 1(\text{mod } 3)$,
$r_4' \equiv 2 - 1 \equiv 1(\text{mod } 3)$, $r_5' \equiv 0 - 1 \equiv 2(\text{mod } 3)$, $r_6' \equiv 2 - 1 \equiv 1(\text{mod } 3)$.
Thus $C_0^2 = \{2\}$, $C_1^2 = \{3, 4, 6\}$ and $C_2^2 = \{1, 5\}$.

Now let $j' = 3$. Since $3 \in C_2^1$, $r_{j'} = 2$. Then using (3.3.2) we have $r_i' \equiv r_i - r_{j'}(\text{mod } d)$. So

$r_1' \equiv 0 - 2 \equiv 1(\text{mod } 3)$, $r_2' \equiv 1 - 2 \equiv 2(\text{mod } 3)$, $r_3' \equiv 2 - 2 \equiv 0(\text{mod } 3)$,
$r_4' \equiv 2 - 2 \equiv 0(\text{mod } 3)$, $r_5' \equiv 0 - 2 \equiv 1(\text{mod } 3)$, $r_6' \equiv 2 - 2 \equiv 0(\text{mod } 3)$.
Thus $C_0^3 = \{3, 4, 6\}$, $C_1^3 = \{1, 5\}$ and $C_2^3 = \{2\}$.

Now let $j' = 4$. Since $4 \in C_2^1$, $r_{j'} = 1$. Then using (3.3.2) we have $r_i' \equiv r_i - r_{j'}(\text{mod } d)$. So

$r_1' \equiv 0 - 2 \equiv 1(\text{mod } 3)$, $r_2' \equiv 1 - 2 \equiv 2(\text{mod } 3)$, $r_3' \equiv 2 - 2 \equiv 0(\text{mod } 3)$,
$r_4' \equiv 2 - 2 \equiv 0(\text{mod } 3)$, $r_5' \equiv 0 - 2 \equiv 1(\text{mod } 3)$, $r_6' \equiv 2 - 2 \equiv 0(\text{mod } 3)$.
Thus $C_0^4 = \{3, 4, 6\}$, $C_1^4 = \{1, 5\}$ and $C_2^4 = \{2\}$.

Observation 3.3.9. By Remark 3.3.8, we get the same subclasses. In fact the order of occurrence of these subclasses remains same, only notation changes with a cyclic permutation. Moreover, we observe that the index we fixes initially always falls under the subclass C_0, so if we know the order of occurrence of these subclasses then we don't really need to do any calculation to find every particular subclass. For example for the same matrix as in Example 3.3.5 if we fix $j = 5$, then the subclasses will be $C_0 = \{1, 5\}$, $C_1 = \{2\}$ and $C_2 = \{3, 4, 6\}$ or if we fix $j = 6$, the subclasses will be $C_0 = \{3, 4, 6\}$, $C_1 = \{1, 5\}$ and $C_2 = \{2\}$.

Canonical Form for Irreducible Matrices. Let m be a positive integer. Let i be an index such that $t_{ij}^{(m)} > 0$ for some j, note that such i exists. In fact, otherwise T^m would have j^{th} column entirely zero and so are its higher powers. This contradicts the irreducibility of T. Then by Theorem 3.3.3, $m \equiv r_i (\bmod\ d)$ implies $i \in C_{r_i}$. Now let k be any index such that $t_{kj}^{(m+1)} > 0$. Then similarly $m+1 \equiv r_i+1 (\bmod\ d)$ implies $k \in C_{r_{i+1}}$. Hence it follows that looking at the j^{th} column, the positive entries occur, for successive powers, in successive subclasses.

We can define a canonical form for irreducible matrices. If $d > 1$, (there are more that one subclasses exist), a canonical form of T is possible by relabeling the indices so that indices of C_0 comes first, then C_1 next and so on.

Example 3.3.10. Continuing Example 3.3.5, the subclasses of the matrix T are $C_0 = \{1, 5\}$, $C_1 = \{2\}$ and $C_2 = \{3, 4, 6\}$. The matrix and its canonical form are given as follows

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$T_c = \begin{array}{c} \\ 1 \\ 5 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 5 & 2 & 3 & 4 & 6 \end{array} \\ \left(\begin{array}{cc|c|ccc} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right) \end{array}$$

and the canonical form of above matrix can also be written as

$$T_c = \begin{array}{c} \\ C_0 \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{ccc} C_0 & C_1 & C_2 \end{array} \\ \begin{pmatrix} 0 & 0 & Q_{02} \\ Q_{10} & 0 & 0 \\ 0 & Q_{21} & 0 \end{pmatrix} \end{array}, \text{ where } Q_{02} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, Q_{10} = \begin{bmatrix} 1 & 1 \end{bmatrix} \text{ and } Q_{21} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

In view of the above example, we have the following definition:

DEFINITION 3.3.6. The canonical form for irreducible matrix is given as follows:

$$T_c = \begin{array}{c} \\ C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_{d-2} \\ C_{d-1} \end{array} \begin{array}{c} \begin{array}{cccccc} C_0 & C_1 & C_2 & \dots & C_{d-2} & C_{d-1} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & Q_{0,d-1} \\ Q_{10} & 0 & 0 & \dots & 0 & 0 \\ 0 & Q_{21} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & Q_{d-1,d-2} & 0 \end{pmatrix} \end{array},$$

where $Q_{r,r-1}$, for $r = 0, 1, 2, \dots, d-1$ (keeping in consideration that r, or subscript, is modulo d) is a non-negative matrix with some specific properties.

Note that all indices in the above definition are modulo d.

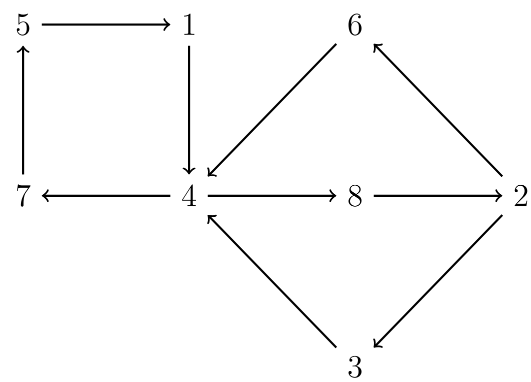
REMARK 3.3.11. If T is an irreducible matrix, then for $r, s = 0, 1, 2, \dots, d-1$ the elements of C_r leads to the elements of C_s. More specifically the elements of C_{r-1} leads to the elements of C_r through an edge, this connection between two consecutive subclasses is particularly represented by $Q_{r,r-1}$. In Example 3.3.5, we have $C_0 = \{1, 5\}$, $C_1 = \{2\}$ and $C_2 = \{3, 4, 6\}$. Thus

$$Q_{02} = \begin{array}{c} \\ 1 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 4 & 6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}, Q_{10} = 2 \begin{array}{c} \begin{array}{cc} 1 & 5 \end{array} \\ \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array} \text{ and } Q_{21} = \begin{array}{c} \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} 2 \\ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{array}.$$

Moreover, in this particular example, the entries in these matrices are particularly 1 and 0, because either the indices have an edge between them or there is no edge. However entries could be other than 1 or 0 in specific conditions, we will see that in following example.

EXAMPLE 3.3.12. Consider the following irreducible matrix T:

$$T = \begin{array}{c} \\ 2 \\ 5 \\ 1 \\ 3 \\ 6 \\ 4 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 2 & 5 & 1 & 3 & 6 & 4 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$



We can clearly see from the graph that $d = 4$ and the subclasses are $C_0 = \{2, 5\}$, $C_1 = \{1, 3, 6\}$, $C_2 = \{4\}$ and $C_3 = \{7, 8\}$. The canonical form of T is

$$T_c = \begin{array}{c} \\ C_0 \\ C_1 \\ C_2 \\ C_3 \end{array} \begin{array}{c} \begin{array}{cccc} C_0 & C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & Q_{03} \\ Q_{10} & 0 & 0 & 0 \\ 0 & Q_{21} & 0 & 0 \\ 0 & 0 & Q_{32} & 0 \end{pmatrix} \end{array},$$

where $Q_{03} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Q_{10} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$, $Q_{21} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ and $Q_{32} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Now we observe the powers of T_c:

$$T_c^2 = \begin{array}{c} \\ C_0 \\ C_1 \\ C_2 \\ C_3 \end{array} \begin{array}{c} \begin{array}{cccc} C_0 & C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} 0 & 0 & Q_{03}Q_{32} & 0 \\ 0 & 0 & 0 & Q_{10}Q_{03} \\ Q_{21}Q_{10} & 0 & 0 & 0 \\ 0 & Q_{32}Q_{21} & 0 & 0 \end{pmatrix} \end{array}.$$

For r modulo 4, $Q_{r,r-1}Q_{r-1,r-2}$ represents the path connection, more precisely, the number of paths of length two from the elements of subclass C_{r-2} to the elements of subclass C_r. For example $Q_{21}Q_{10} = \begin{matrix} & \begin{matrix} 2 & 5 \end{matrix} \\ 4 & \begin{pmatrix} 2 & 1 \end{pmatrix} \end{matrix}$ shows paths of length two from 2 to 4 are two in number and there is only one path of length two from 5 to 4. Similarly in

$$T_c^3 = \begin{matrix} & \begin{matrix} C_0 & C_1 & C_2 & C_3 \end{matrix} \\ \begin{matrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{matrix} & \begin{pmatrix} 0 & Q_{03}Q_{32}Q_{21} & 0 & 0 \\ 0 & 0 & Q_{10}Q_{03}Q_{32} & 0 \\ 0 & 0 & 0 & Q_{21}Q_{10}Q_{03} \\ Q_{32}Q_{21}Q_{10} & 0 & 0 & 0 \end{pmatrix} \end{matrix},$$

$Q_{r,r-1}Q_{r-1,r-2}Q_{r-2,r-3}$ represents the number of paths of length three from the elements of subclass C_{r-3} to the elements of subclass C_r.

Since $d = 4$, T_c^4 is of very special form

$$T_c^4 = \begin{matrix} & \begin{matrix} C_0 & C_1 & C_2 & C_3 \end{matrix} \\ \begin{matrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{matrix} & \begin{pmatrix} Q_{03}Q_{32}Q_{21}Q_{10} & 0 & 0 & 0 \\ 0 & Q_{10}Q_{03}Q_{32}Q_{21} & 0 & 0 \\ 0 & 0 & Q_{21}Q_{10}Q_{03}Q_{32} & 0 \\ 0 & 0 & 0 & Q_{32}Q_{21}Q_{10}Q_{03} \end{pmatrix} \end{matrix}.$$

That is, each diagonal matrix represents the number of paths of length four from elements of each subclass to the elements of itself and it also justifies the definition of *period*.

REMARK 3.3.13. Since matrix multiplication for the matrices of order 3×3 and above behaves likewise, we can conclude the general properties of $Q_{r,r-1}$ and their multiplications for any $d \geq 3$. As an entry of T_c^k, the entries of $Q_{r,r-1}Q_{r-1,r-2}\dots Q_{r-(k-1),r-k}$ represents the number of paths of length k from the elements of subclass C_{r-k} to the elements of subclass C_r. If $k = d$, then T_c^d becomes a diagonal matrix such that $Q_{r,r-1}Q_{r-1,r-2}\dots Q_{r-(d-1),r}$ is the $(r+1)^{th}$ diagonal block entry, for $r =$

$0, 1, 2, \ldots, d-1$, whose entries represents the number of paths of length d from the elements of subclass C_r to the elements of C_r itself, (thus $Q_{r,r-1}Q_{r-1,r-2}\ldots Q_{r-(d-1),r}$ is a square matrix). Note that these diagonal matrices may have zero entries in them, i.e., they are non-negative. However, the following theorem ensures that these diagonal block matrices are primitive.

THEOREM 3.3.14. *Let $T = [t_{ij}]$ be an irreducible cyclic matrix with period d. Then the diagonal block matrices $Q_{r,r-1}Q_{r-1,r-2}\ldots Q_{r-(d-1),r}$ for $r = 0, 1, 2, \ldots, d-1$ of the matrix T_c^d are primitive. Moreover, the powers of T may be studied in terms of powers of primitive matrices.*

PROOF. In view of Observation 3.3.9, for a specific selection of initial index any subclass can be named under C_0. So it is sufficient to show that

$$A := Q_{0,d-1}Q_{d-1,d-2}\ldots Q_{1,0}$$

is primitive. The entries of A are of the form $t_{ij}^{(d)}$ such that $i, j \in C_0$. Fix $j \in C_0$, from Theorem 3.3.3(b) $t_{ij}^{(kd)} > 0$ for every integer $k > N_j(i)$, where $N_j(i)$ depends on i and fixed j.

Let $N = \max\{N_j(i) : i, j \in C_0\}$. Then for every $i, j \in C_0$, $t_{ij}^{(kd)} > 0$ for every integer $k > N$. Hence $A^k \succ 0$. This shows that $A := Q_{0,d-1}Q_{d-1,d-2}\ldots Q_{1,0}$ is primitive.

Now the diagonal block matrices of

$$T_c^d = \begin{bmatrix} Q_{0,d-1}Q_{d-1,d-2}\ldots Q_{1,0} & 0 & \ldots & 0 \\ 0 & Q_{1,0}Q_{0,d-1}\ldots Q_{2,1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & Q_{d-1,d-2}Q_{d-2,d-3}\ldots Q_{0,d-1} \end{bmatrix}$$

are primitive. For $k \in \mathbb{Z}^+$,

$$T_c^{dk} = \begin{bmatrix} Q_{0,d-1}Q_{d-1,d-2}\dots Q_{1,0} & 0 & \dots & 0 \\ 0 & Q_{1,0}Q_{0,d-1}\dots Q_{2,1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_{d-1,d-2}Q_{d-2,d-3}\dots Q_{0,d-1} \end{bmatrix}^k$$

$$= \begin{bmatrix} (Q_{0,d-1}Q_{d-1,d-2}\dots Q_{1,0})^k & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (Q_{d-1,d-2}Q_{d-2,d-3}\dots Q_{0,d-1})^k \end{bmatrix}.$$

This implies the powers which are integral multiples of the period may be studied with the aid of primitive matrix theory. Also $T_c^{dk+1}, T_c^{dk+2}, \dots, T_c^{dk+(d-1)}$ can be considered as $T_c^{dk+1} = (T_c^{dk})T$, $T_c^{dk+2} = (T_c^{dk})T^2$, $\dots$, $T_c^{dk+(d-1)} = (T_c^{dk})T^{(d-1)}$. □

Notice that the period d is assumed to be strictly greater than 1 in the construction of the canonical form of an irreducible matrix. If the period $d = 1$, then only one subclass will exist and a canonical form will make no sense. However, we have the following theorem in support of irreducible matrices with period $d = 1$.

THEOREM 3.3.15. *An irreducible acyclic ($d = 1$) matrix $T = [t_{ij}]$ is primitive, and the converse is also true.*

PROOF. Clearly if the period of T is $d = 1$, then there is only one subclass of the index set, consisting of the index set itself. Thus the whole matrix represents the number of edges from the elements of the subclass to the elements of the subclass itself which is indeed the form of the diagonal block matrix of T^d. So by Theorem 3.3.14 T is primitive.

Conversely let T is primitive, we shall show that T is irreducible with period $d = 1$. Clearly T is irreducible. Let i be any index. Since T is primitive, there exists k such that $t_{ii}^{(k)} > 0$ and so $t_{ii}^{(k+1)} > 0$. Hence $\gcd(k, k+1) = 1$ implies $d = 1$. □

3.4. Perron-Frobenius Theorem for Irreducible matrices

We see that a primitive matrix is an irreducible matrix with period $d = 1$. In this section we state and give a precise proof of Perron-Frobenius theorem for irreducible matrices. Later we observe that number of highest modulo eigenvalues of an irreducible matrix is dependent on its period.

THEOREM 3.4.1. ***(The Perron-Frobenius theorem for irreducible matrices)*** *Suppose $T = [t_{ij}]$ is an irreducible matrix. Then all assertions (a) to (f) of Theorem 2.2.1 hold except that (b) is replaced by a weaker statement which is just $r = \rho(T)$.*

PROOF. In this proof we shall show that the condition of being primitive of T in Theorem 2.2.1 can be replaced by the condition of T being irreducible.

Part (a) of Theorem 2.2.1 holds up to (2.2.4). Beyond this we shall show that if $z = T\hat{x} - \hat{x}r \succeq 0$, then $z = 0$. Let us assume $z \gneqq 0$. Then there exists $i \in \{1, 2, \ldots, n\}$ such that $(z)_i > 0$, i.e., $(T\hat{x} - \hat{x}r)_i > 0$ implies

$$T\hat{x} - \hat{x}r \gneqq 0. \tag{3.4.1}$$

Clearly $(I + T)^k = T^k +$ (some non-negative matrices). This shows that $I + T$ is irreducible. Moreover, if $I + T = [\tilde{t}_{ij}]$, then $\tilde{t}_{ii} > 0$ for every $i \in \{1, 2, \ldots, n\}$ implies period of $I + T$ is 1. Hence by Theorem 3.3.15, $I + T$ is primitive. That is, for some $k \in \mathbb{Z}^+$, $(I + T)^k \succ 0$. Thus on multiplying $(I + T)^k$ to (3.4.1) we get, $T(I + T)^k\hat{x} - (I + T)^k\hat{x}r \succ 0 \implies T(I + T)^k\hat{x} \succ (I + T)^k\hat{x}r$, so for every $i \in \{1, 2, \ldots, n\}$,

$$\begin{aligned}
(T(I + T)^k\hat{x})_i >& ((I + T)^k\hat{x})_i r \\
\implies r <& \frac{(T(I + T)^k\hat{x})_i}{((I + T)^k\hat{x})_i} \\
\implies r <& \min_i \frac{(T(I + T)^k\hat{x})_i}{((I + T)^k\hat{x})_i}
\end{aligned}$$

$$\implies r < \min_i \frac{\left(T\frac{(I+T)^k\hat{x}}{||(I+T)^k\hat{x}||}\right)_i}{\left(\frac{(I+T)^k\hat{x}}{||(I+T)^k\hat{x}||}\right)_i}.$$

This is a contradiction to the definition of r as $\frac{(I+T)^k\hat{x}}{||(I+T)^k\hat{x}||} \in A$. Hence our assumption $z \gneq 0$ is not true which shows that $z = 0$, i.e., $T\hat{x} = \hat{x}r$.

Proof of part (b) is same as in Theorem 2.2.1 except for the "moreover part" as we don't have the uniqueness of highest modulo eigenvalue. Proof of (c), (e) and (f) are the same as in Theorem 2.2.1, whereas part (d) can be proved with the help of Theorem 3.4.3. □

NOTE 3.4.2. (1) Corollaries 2.2.2 and 2.2.3 also hold for irreducible matrices since the condition of being primitive used specifically in these corollaries can easily be replaced by the condition of being irreducible.

(2) The unimodular eigenvector $\hat{x}$ of T associated to r is also unique.

DEFINITION 3.4.1. In Theorem 3.4.1, the eigenvalue r of T is called the *Perron-Frobenius eigenvalue* or *PF eigenvalue* and its corresponding (left or right) unique unimodular positive eigenvector $\hat{x}$ is called the *unimodular Perron-Frobenius eigenvector*, in this paper we refer this eigenvector by *UPF eigenvector*.

THEOREM 3.4.3. ***(The Subinvariance Theorem)*** *Let $T = [t_{ij}] \in M_n(\mathbb{R})$ be an irreducible matrix, $s \in \mathbb{R}^+$ and $y \gneq 0 \in \mathbb{R}^n$ satisfying $Ty \preceq sy$. Then*

(a) $s \geq r$, where r is the PF eigenvalue of T. Moreover, $s = r$ if, and only if, $Ty = sy$; and

(b) $y \succ 0$.

PROOF. (a) We have $Ty \preceq sy$. Let $\hat{x}^t \succ 0$ be the left UPF eigenvector of T associated to the PF eigenvalue r. Then by Lemma 2.1.1,

$$\hat{x}^tTy \preceq s\hat{x}^ty \implies r\hat{x}^ty \preceq s\hat{x}^ty \implies r \leq s.$$

Now let $r = s$ and let us assume $Ty \preceq ry$ with strict inequality in at least one place. Then again by Lemma 2.1.1 we get, $\hat{x}^tTy \preceq r\hat{x}^ty \implies r\hat{x}^ty \preceq r\hat{x}^ty$, with strict inequality in at least one place, which implies $r < r$, which is absurd. Hence $Ty = ry$.

Now let $Ty = sy$. Then by Lemma 2.1.1,

$$\hat{x}^t Ty = s\hat{x}^t y \implies r\hat{x}^t y = s\hat{x}^t y \implies r = s.$$

(b) Let $y \not\succ 0$. Then there are at least one zero and at least one non zero entries in y. Let $i, j \in \{1, 2, \ldots, n\}$ such that $y_i = 0$ and $y_j > 0$. Since T is irreducible, there exists $m > 0$ such that $t_{ij}^{(m)} > 0$. Now

$$0 < t_{ij}^{(m)} y_j \leq \sum_{k=1}^{n} t_{ik}^{(m)} y_k = \big(T^m s\big)_i \leq s^m y_i \qquad \text{(as } Ts \preceq sy \text{ implies } T^m s \preceq s^m y\text{)},$$

i.e., $s^m y_i > 0$ implies $y_i > 0$, which is a contradiction. Hence $y \succ 0$. □

Note 3.4.4. (1) The vector y in Theorem 3.4.3 is called a *subinvariant* for T.

(2) If $y \succeq 0$ is an eigenvector of T and $Ty = sy$, for some positive number s, then by Theorem 3.4.3(a) $s = r$. This implies that every non-negative eigenvector of an irreducible matrix T is associated to its PF eigenvalue r.

The following theorem will emphasize the main difference between the Perron-Frobenius theorem for primitive matrices and for irreducible matrices.

Theorem 3.4.5. *For a cyclic matrix T with period $d > 1$, there are precisely d distinct eigenvalues λ with $|\lambda| = r$, where r is the PF eigenvalue of T. Moreover, these eigenvalues are: $r\,e^{i2\pi\frac{k}{d}}$, $k = 0, 1, \ldots, d-1$.*

Proof. Let us consider the canonical form T_c of T. From Theorem 3.3.15, for every $i = 0, 1, \ldots, d-1$, the diagonal block matrices $Q_{i,i+1}Q_{i+1,i+2}\ldots Q_{i+d-1,i}$ of T_c^d are primitive. Take an arbitrary i^{th} diagonal block matrix $Q_{i,i+1}Q_{i+1,i+2}\ldots Q_{i+d-1,i}$ of T_c^d. Let $r(i)$ be its PF eigenvalue and $y(i) \succ 0$ be its UPF eigenvector associated to $r(i)$. Then

$$Q_{i,i+1}Q_{i+1,i+2}\ldots Q_{i+d-1,i}\,y(i) = r(i)y(i).$$

Multiplying $Q_{i-1,i}$ from the left at both sides, we get

$$Q_{i-1,i}Q_{i,i+1}Q_{i+1,i+2}\dots Q_{i+d-1,i}y(i) = r(i)Q_{i-1,i}y(i).$$

Since subscripts are modulo d, $Q_{i+d-1,i} = Q_{i-1,i}$ and so we get

$$Q_{i-1,i}Q_{i,i+1}Q_{i+1,i+2}\dots Q_{i+d-2,i-1}\big(Q_{i-1,i}y(i)\big) = r(i)\big(Q_{i-1,i}y(i)\big). \tag{3.4.2}$$

Since $Q_{i-1,i} \gneqq 0$ and $y(i) \succ 0$, $Q_{i-1,i}y(i) \gneqq 0$. So $Q_{i-1,i}y(i)$ is a non-negative eigenvector of $Q_{i-1,i}Q_{i,i+1}Q_{i+1,i+2}\dots Q_{i+d-2,i-1}$ associated to $r(i)$. Hence by Note 3.4.4(2) $r(i-1) = r(i)$. Since i is arbitrary, for every $i = 0, 1, \dots, d-1$, $r(i)$ is constant , say equal to $\tilde{r}$. This shows that every primitive diagonal block matrix of T_c^d have the same PF eigenvalue. Moreover, by Theorem 2.2.1(b) every diagonal block matrix has a unique eigenvalue with highest modulus value. Also since eigenvalues of T_c^d consists of the eigenvalues of its diagonal block matrices, the PF eigenvalue of T_c^d is $\tilde{r}$ with algebraic multiplicity d.

Since eigenvalues of T_c^d are the d^{th} power of some eigenvalues of T_c, there must be d number of eigenvalues of T_c, so of T, with greatest modulo, having d^{th} power equals to $\tilde{r}$. Theorem 3.4.1 assures that the positive d^{th} root of $\tilde{r}$ has to be r. All other d^{th} roots of $\tilde{r}$, say λ, satisfies $|\lambda| = r$, are of the form $\lambda = r\,\mathrm{e}^{i2\pi\frac{k}{d}}$, $k = 1, 2, \dots, d-1$.

It only remains to show that for $k = 1, 2, \dots, d-1$ there exists an eigenvector of T_c associated to $r\,\mathrm{e}^{i2\pi\frac{k}{d}}$. Let $y = \begin{bmatrix} y_0 & y_1 & \dots & y_{d-1}\end{bmatrix}^t$ be a UPF eigenvector of T_c associated to r, where y_j's are the subvectors of components corresponding to the subclasses C_j. Then

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & Q_{0,d-1} \\ Q_{10} & 0 & 0 & \dots & 0 & 0 \\ 0 & Q_{21} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & Q_{d-2,d-1} & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{d-1} \end{bmatrix} = r \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{d-1} \end{bmatrix}. \tag{3.4.3}$$

Let $\tilde{y}_k = \begin{bmatrix} \mathrm{e}^{2\pi(0)\frac{k}{d}}\, y_0 \\ e^{2\pi(1)\frac{k}{d}} y_1 \\ e^{2\pi(2)\frac{k}{d}} y_2 \\ \vdots \\ e^{2\pi(d-1)\frac{k}{d}} y_{d-1} \end{bmatrix}$. Clearly $\tilde{y}_0 = y$. Now using (3.4.3) we have

$$Q_{j,j+1} e^{2\pi(j+1)\frac{k}{d}} y_{j+1} = e^{2\pi(j+1)\frac{k}{d}} Q_{j,j+1} y_{j+1} = e^{2\pi(j+1)\frac{k}{d}} r y_j = r e^{2\pi\frac{k}{d}} e^{2\pi j\frac{k}{d}} y_j,$$

which implies

$$T_c \tilde{y}_k = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & Q_{0,d-1} \\ Q_{10} & 0 & 0 & \dots & 0 & 0 \\ 0 & Q_{21} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & Q_{d-2,d-1} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{2\pi(0)\frac{k}{d}}\, y_0 \\ e^{2\pi(1)\frac{k}{d}} y_1 \\ e^{2\pi(2)\frac{k}{d}} y_2 \\ \vdots \\ e^{2\pi(d-1)\frac{k}{d}} y_{d-1}. \end{bmatrix} = r e^{2\pi\frac{k}{d}} \tilde{y}_k.$$

Hence $\tilde{y}_k$ is an eigenvector of T_c, so of T, associated to the eigenvalue $re^{2\pi\frac{k}{d}}$. □

NOTE 3.4.6. If $\lambda \neq 0$ is any eigenvalue of T, then the numbers $\lambda e^{2\pi\frac{k}{d}}$, $k = 0, 1, \dots, d-1$ are also eigenvalues of T.

Theorem 3.4.5 also holds for $d = 1$. In fact, for $d = 1$, the irreducible matrix T is a primitive matrix by Theorem 3.3.15. So Theorem 3.4.5 states that T has only one eigenvalue of highest modulo, which is the PF eigenvalue itself; and this statement coincides with part (b) of Theorem 2.2.1.

CHAPTER 4

Perron-Frobenius Theory for Strongly Connected k-Graphs

We first study the Perron-Frobenius theory for family of non-negative commuting matrices in this chapter. Later we introduce higher-rank graphs (or k-graphs) and study how a strongly connected k-graph is related to a family of commuting matrices. Then we state and prove a Perron-Frobenius theorem for strongly connected k-graphs.

4.1. Perron-Frobenius Theory For Commuting Matrices

LEMMA 4.1.1. *Let A and B be two commuting irreducible matrices. Then the UPF eigenvectors of A and B are equal.*

PROOF. Let x be the UPF eigenvector of A associated to the PF eigenvalue r. Then $ABx = BAx = rBx$. Hence Bx is an eigenvector of A. Also $B \succeq 0$ and $x \succ 0$ implies $Bx \succeq 0$. Hence by Theorem 3.4.1(c) Bx is a scalar multiple of x. i.e., there exists a positive number s such that $Bx = sx$. Now Theorem 3.4.3(a) ensures that s is the PF eigenvalue of B. Hence A and B have the same UPF eigenvector x. □

DEFINITION 4.1.1. Let $\{A_1, A_2, \ldots, A_k\} \subset M_n(\mathbb{R})$ be a family of non-negative commuting matrices, $m = (m_1, m_2, \ldots, m_k) \in \mathbb{N}^k$ and F be a finite subset of $\mathbb{N}^k$. We use the multi-index notation

$$A^m := \prod_{i=1}^{k} A_i^{m_i} \qquad \text{and} \qquad A_F := \sum_{m \in F} A^m.$$

We say that the family $\{A_1, A_2, \ldots, A_k\}$ is *irreducible* if each $A_i \neq 0$ and there exists a finite subset $F \in \mathbb{N}^k$ such that $A_F \succ 0$.

REMARK 4.1.2. (a) When $k = 1$ the above definition coincides with the definition of irreducible matrix, i.e., a matrix A is irreducible in the sense of Definition 3.3.1 if, and only if, $\{A\}$ is an irreducible family. Let $A = [a_{ij}] \in M_n(\mathbb{R})$ be an irreducible matrix. Then for every (i,j) there exits $s \in \mathbb{N}$ such that $a_{ij}^{(s)} > 0$. Let $m_{i,j} = \min\{s \in \mathbb{N} : a_{ij}^{(s)} > 0\}$. Let $F = \{m_{i,j} : i,j = 1,2,\ldots,k\}$. Clearly F is a non-empty finite subset of $\mathbb{N}$. Then $A_F = \sum\limits_{m\in F} A^m \succ 0$. Conversely, let $\{A\}$ be an irreducible family. Then there exists a finite subset $F \in \mathbb{N}$ such that $A_F \succ 0$. Now $0 \prec A_F = \sum\limits_{m\in F} A^m$ implies for every $(i,j)^{th}$ entry of A there exists some $m \in F$ such that $A^m(i,j) > 0$, i.e., $a_{ij}^{(m)} > 0$. This justifies that A is an irreducible matrix.

(b) In an irreducible family of matrices, the individual matrix A_i may not be irreducible. For example let $\{A_1, A_2\} \in M_2(\mathbb{R})$, where $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Take $F = \{(1,1),(1,2)\}$. Then $A_F = A_1A_2 + A_1A_2^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Hence $\{A_1, A_2\}$ is an irreducible family but A_1 is not irreducible. Therefore Definition 4.1.1 is a generalization of Definition 3.3.1.

The following lemma is the subinvariance theorem for non-negative matrices which is very useful in later results.

LEMMA 4.1.3. *Let $B \in M_n(\mathbb{R})$ be a non-negative matrix. Suppose that $0 \preceq x \in \mathbb{R}^n$ and $\lambda \geq 0$ satisfies $Bx \preceq \lambda x$. Then $\lambda \geq \rho(B)$. Moreover, if $Bx = \lambda x$, then $\lambda = \rho(B)$.*

PROOF. Let $B = [b_{ij}]$. If $B \succ 0$ (then it is primitive), then the proof is given by Theorem 3.4.3.

Let some entries in B be zero. Define a matrix B_k as $B_k(i,j) = \begin{cases} b_{ij}, & \text{if } b_{ij} \neq 0 \\ \frac{1}{k}, & \text{if } b_{ij} = 0 \end{cases}$.
Then clearly $\{B_k\}$ is a sequence of strictly positive matrices such that $B_k \to B$ as $k \to \infty$. Now fix $\epsilon > 0$. Since $Bx \preceq \lambda x$, for sufficiently large k one has $B_k x \prec (\lambda + \epsilon)x$. Hence by Theorem 3.4.3(a) $\lambda + \epsilon \geq \rho(B_k)$. Since $\epsilon > 0$ is arbitrary, we get $\lambda \geq \rho(B_k)$ for large k. Also we have a result from [**7**] which states that the roots of a polynomial vary continuously with its coefficients. One has $\rho(B_k) \to \rho(B)$ which implies $\lambda \geq \rho(B)$.

Now if $Bx = \lambda x$, then the above result also holds, i.e., $\lambda \geq \rho(B)$. Also by Definition 2.2.2 we have $\lambda \leq \rho(B)$. Hence $\lambda = \rho(B)$. □

NOTE 4.1.4. Lemma 4.1.3 clearly shows that if $x \succeq 0$ is an eigenvector of B and $By = \lambda y$, for some positive number λ, then $\lambda = \rho(B)$. This implies that for a non-negative matrix B every non-negative eigenvector is associated to $\rho(A)$.

In view of Definition 4.1.1, for an irreducible family of matrices $\{A_1, A_2, \ldots, A_k\}$, $A_F \succ 0$ which clearly implies that A_F is, in particular, a primitive matrix. So we can use the Perron-Frobenius theorem for primitive matrices (Theorem 2.2.1) to study the Perron-Frobenius theory for irreducible family of matrices as follows:

PROPOSITION 4.1.5. *Suppose that $\{A_1, A_2, \ldots, A_k\}$ is an irreducible family in $M_n(\mathbb{R})$. Let F be a finite subset of $\mathbb{N}^k$ such that $A_F \succ 0$ and let $\hat{x}$ be the UPF eigenvector of A_F. Then we have the following properties:*

- *(a) (i) The vector $\hat{x}$ is the unique strictly positive unimodular common eigenvector of all A_i's associated to $\rho(A_i)$ and $\rho(A_i) > 0$.*
  - *(ii) If $z \in \mathbb{C}^n$ and $A_i z = \rho(A_i) z$ for every $i = 1, 2, \ldots, k$, then $z \in \mathbb{C}\hat{x}$.*
- *(b) Suppose $y \in \mathbb{R}^n$, $y \succeq 0$ and $\lambda_i \in \mathbb{R}$ such that $\lambda_i \geq 0$ for $i = 1, 2, \ldots, k$ satisfies $A_i y \preceq \lambda_i y$ for $i = 1, 2, \ldots, k$. Then*
  - *(i) $y \succ 0$ and $\lambda_i \geq \rho(A_i)$ for every $i = 1, 2, \ldots, k$;*
  - *(ii) if $\lambda_i = \rho(A_i)$ and $||y|| = 1$, then $y = \hat{x}$.*
- *(c) $\rho(A^n) = \prod_{i=1}^{k} \rho(A_i)^{n_i} > 0$ for every $n \in \mathbb{N}^k$.*

PROOF. Let $\{A_1, A_2, \ldots, A_k\}$ be an irreducible family. So there exists a finite subset $F \in \mathbb{N}^k$ such that $A_F \succ 0$ (in particular A_F is primitive as well as irreducible).

(a)(i) For $i = 1, 2, \ldots, k$, we have

$$A_F(A_i\hat{x}) = A_i(A_F\hat{x}) = r_F A_i \hat{x} \qquad \text{(as the family } \{A_1, A_2, \ldots, A_k\} \text{ is commuting)},$$

where r_F is the PF eigenvalue of A_F. This shows that $A_i\hat{x}$ is a non-negative eigenvector of A_F associated to r_F. By Theorem 2.2.1(c) one can get $A_i\hat{x} = \lambda_i\hat{x}$ for some scalar λ_i. Now $A_i \succeq 0$ and $\hat{x} \succ 0$ implies $\lambda_i > 0$. So using Lemma 4.1.3 we get $\lambda_i = \rho(A_i)$. Hence for every $i = 1, 2, \ldots, k$, one has that $\hat{x}$ is an eigenvector of A_i associated to $\rho(A_i)$, i.e., $A_i\hat{x} = \rho(A_i)\hat{x}$. Since $A_i \succeq 0$ and $\hat{x} \succ 0$, $\rho(A_i) > 0$.

To prove uniqueness, let $y \in \mathbb{R}^n$ such that $y \succ 0$ and $||y|| = 1$ be a common eigenvector of all A_i. Then in view of Note 4.1.4, we have for every $i = 1, 2, \ldots, k$

$$A_i y = \rho(A_i)y. \tag{4.1.1}$$

Now using (4.1.1) one can have

(4.1.2)

$$A_F y = \Big(\sum_{n\in F}\prod_{i=1}^{k} A_i^{n_i}\Big)y = \sum_{n\in F}\Big(\prod_{i=1}^{k} A_i^{n_i} y\Big) = \sum_{n\in F}\prod_{i=1}^{k}\rho(A_i)^{n_i} y = \Big(\sum_{n\in F}\prod_{i=1}^{k}\rho(A_i)^{n_i}\Big)y.$$

Hence y is an eigenvector of A_F associated to $\sum\limits_{n\in F}\prod\limits_{i=1}^{k}\rho(A_i)^{n_i}$. Since A_F is irreducible, Note 3.4.4(2) guarantees that

$$r_F = \sum_{n\in F}\prod_{i=1}^{k}\rho(A_i)^{n_i}. \tag{4.1.3}$$

Thus y is an eigenvector of A_F associated to r_F such that $y \succ 0$ and $||y|| = 1$. From Theorem 2.2.1(c) we get $\hat{x} = y$.

(a)(ii) Suppose $A_i z = \rho(A_i)z$. Hence in view of arguments proving (4.1.1) and (4.1.2), we can get $A_F z = \Big(\sum\limits_{n\in F}\prod\limits_{i=1}^{k}\rho(A_i)^{n_i}\Big)z$ and (4.1.3) assures that $A_F z = r_F z$. Thus z is an eigenvector of A_F associated to r_F. So from Theorem 2.2.1(c) we get $z \in \mathbb{C}\hat{x}$.

(b)(i) Lemma 4.1.3 shows that $\lambda_i \geq \rho(A_i)$. Now we shall show that $y \succ 0$. Since $y \neq 0$, there exists $s \in \{1, 2, \ldots, n\}$ such that $y_s > 0$. Now fix $j \in \{1, 2, \ldots, n\}$. Also we have $A_F \succ 0$ for some $F \in \mathbb{N}^k$, so there exits $m \in F$ such that $A^m(j, s) > 0$. Now using the given condition $A_i y \preceq \lambda_i y$ for every $i = 1, 2, \ldots, n$ and the multi-index notation $\lambda^m = \prod_{i=1}^{k} \lambda_i^{m_i}$, we have

$$\left(\lambda^m y\right)_j = \left(\prod_{i=1}^{k} \lambda_i^{m_i} y\right)_j \geq \left(\prod_{i=1}^{k} A_i^{m_i} y\right)_j = (A^m y)_j \geq A^m(j,s) y_s > 0.$$

Hence $0 < (\lambda^m y)_j = \lambda^m y_j$ implies $y_j > 0$. This shows that $y \succ 0$.

(b)(ii) Now suppose $\lambda_i = \rho(A_i)$. So we get $A_i y \preceq \rho(A_i) y$ for $i = 1, 2, \ldots, k$ and

$$A_F y = \Big(\sum_{n\in F} \prod_{i=1}^{k} A_i^{n_i}\Big) y = \sum_{n\in F} \prod_{i=1}^{k} A_i^{n_i} y \preceq \sum_{n\in F} \prod_{i=1}^{k} \rho(A_i)^{n_i} y = \Big(\sum_{n\in F} \prod_{i=1}^{k} \rho(A_i)^{n_i}\Big) y$$

$$= r_F y \qquad \text{using (4.1.3).}$$

Hence Theorem 3.4.3(a) yields that $A_F y = r_F y$, i.e., y is a eigenvector of A_F associated to the PF eigenvalue r_F such that $||y|| = 1$. Then By Theorem 2.2.1(c) we get $y = \hat{x}$.

(c) We have

$$A^n \hat{x} = \prod_{i=1}^{k} A_i^{n_i} \hat{x} = \prod_{i=1}^{k} \rho(A_i)^{n_i} \hat{x} \qquad \text{(using (a)(i)).}$$

This implies that $\hat{x}$ is an eigenvector of A^n associated to $\prod_{i=1}^{k} \rho(A_i)^{n_i}$. Since from (a)(i) $\rho(A_i) > 0$ for every $i = 1, 2, \ldots, n$ and $\hat{x} \succ 0$, one has $A^n \succeq 0$. Hence Note 4.1.4 assures that $\prod_{i=1}^{k} \rho(A_i)^{n_i} = \rho(A^n)$. □

4.2. Category Theory

In order to define k-graphs we need some basic concepts from category theory.

Definition 4.2.1. A *category* $\mathcal{C}$ consists of two sets $\mathcal{C}^0$ and $\mathcal{C}^*$. The elements of $\mathcal{C}^0$ are called *objects* of $\mathcal{C}$ and the elements of $\mathcal{C}^*$ are called *morphisms* from one object of $\mathcal{C}$ to another. For $A, B \in \mathcal{C}^0$ we define $\hom_{\mathcal{C}}(A, B)$ is the set of all morphisms from A to B. For $f \in \hom_{\mathcal{C}}(A, B)$ and $g \in \hom_{\mathcal{C}}(B, C)$ there exists a morphism $g \circ f \in \hom_{\mathcal{C}}(A, C)$ called the *composition of the morphisms f and g* that satisfies:

(i) If $f \in \hom_{\mathcal{C}}(A, B)$, $g \in \hom_{\mathcal{C}}(B, C)$ and $h \in \hom_{\mathcal{C}}(C, D)$ are three morphisms, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(ii) For each object $B \in \mathcal{C}^0$ there exists a morphism $I_B : B \to B$ such that for any $f : A \to B$ and $g : B \to C$ we have $I_B \circ f = f$ and $g \circ I_B = g$. Such a morphism is called the *identity morphism* for $B \in \mathcal{C}^0$.

When there is no confusion, we denote $\hom_{\mathcal{C}}(A, B)$ by $\hom(A, B)$.

DEFINITION 4.2.2. In a category $\mathcal{C}$, for $f \in \hom(A, B)$, we say A is the *domain* or *source* of f, which is denoted by $s(f)$ and B is the *range* of f, which is denoted by $r(f)$.

DEFINITION 4.2.3. In a category $\mathcal{C}$, a morphism $f \in \hom(A, B)$ is called an *equivalence* if there is another morphism $g \in \hom(B, A)$ such that $g \circ f = I_A$ and $f \circ g = I_B$. If $f \in \hom(A, B)$ is an equivalence, then A, B are called *equivalent*.

EXAMPLE 4.2.1. (a) Let $\mathcal{S}^0$ be the class of all sets. For $A, B \in \mathcal{S}^0$, let $\hom(A, B)$ be the set of all functions from A to B. Then $\mathcal{S}$ is clearly a category. A morphism $f \in \hom(A, B)$ is an equivalence if, and only if, f is bijective.

(b) Let $\mathcal{G}^0$ be the class of all groups. For $A, B \in \mathcal{G}^0$, let $\hom(A, B)$ be the set of all group homomorphisms from A to B. Then $\mathcal{G}$ is clearly a category. A morphism $f \in \hom(A, B)$ is an equivalence if, and only if, f is an isomorphism.

(c) A group G can be considered as a category with one object G itself. By *Cayley's Theorem* every element of G can be considered as a bijection from G to itself. So we define $\hom(G, G)$ as the set of elements of G. Clearly the composition of morphisms is given by the group operation of G which is associative and the identity element of G is the identity morphism.

(d) Similarly a monoid S is a category with one object S itself where morphisms are the elements of the monoid S.

(e) Being a monoid, from (d) above $(\mathbb{N}^k, +)$ is a category with one object $\mathbb{N}^k$ itself. The morphisms are the elements of $\mathbb{N}^k$. The composition of morphisms is given by the addition defined on the elements and the identity morphism is $\mathbf{0} \in \mathbb{N}^k$. This category will play an important role in next section.

(f) Let $E = (E^0, E^1, r, s)$ be a directed graph. We define a category, named, the $\mathcal{P}ath\ \mathcal{C}ategory$ of E denoted by $\mathcal{P}(E)$. The objects in $\mathcal{P}(E)$ are the elements of E^0, i.e., the vertices in E, and the morphism are the elements of E^*, i.e., the finite paths in E. The composition of morphisms is the composition of paths and the identity morphism for $v \in E^0$ is the path from v to itself which is the vertex v itself.

DEFINITION 4.2.4. A category $\mathcal{C}$ is said to be *countable* if $\mathcal{C}^0$ and $\mathcal{C}^*$ are countable sets.

DEFINITION 4.2.5. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ is a pair of functions satisfying the following properties: the object part of $F : \mathcal{C}^0 \to \mathcal{D}^0$ maps each object C in $\mathcal{C}$ to an object $F(C)$ in $\mathcal{D}$, and the morphism part of $F : \mathcal{C}^* \to \mathcal{D}^*$ maps each morphism $f : C \to C'$ in $\mathcal{C}$ to a morphism $F(f) : F(C) \to F(C')$ in $\mathcal{D}$, such that

(a) the identity morphisms are preserved under the functors, i.e., for any $A \in \mathcal{C}$, $FI_A = I_{FA}$, and
(b) $F(g \circ f) = F(g) \circ F(f)$ whenever $g \circ f$ is defined $g, f \in \mathcal{C}^*$.

NOTE 4.2.2. The above defined functor is also known as *covariant functor*.

4.3. k-Graphs

In this section we consider $\mathbb{N}^k$ as a category (See Example 4.2.1(e)). Let $\{e_i\}_{i=1}^k$ be its standard generators.

DEFINITION 4.3.1. A *higher rank graph* or *k-graph* (Λ, d) is a countable category Λ together with a functor $d : \Lambda \to \mathbb{N}^k$, called the degree map, with the following unique factorization property: For every morphism $\lambda \in \Lambda$, with $d(\lambda) = m + n$ $(n, m \in \mathbb{N}^k)$, there exist unique morphisms $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

NOTE 4.3.1. We define Λ^n to be the set $d^{-1}(n)$. We call it the set of morphisms of Λ of degree n. Since we are regarding k-graphs as generalized graphs, we refer to the elements of Λ^n as paths of degree n.

The above definition of Λ^n is consistent when $n = \mathbf{0}$. The following lemma gives a precise proof of this consistency.

LEMMA 4.3.2. *Let (Λ, d) be a k-graph. Then the objects of Λ are the morphisms of Λ of degree zero.*

PROOF. Let v be an object in Λ. Then using Definition 4.2.5, we get

$$d(I_v) = I_{dv} = I_{\mathbb{N}^k} = 0. \tag{4.3.1}$$

Let λ be a morphism of degree zero with range v, i.e., $d(\lambda) = 0$ and $r(\lambda) = v$. So by factorization there exist unique morphisms $\mu, \nu \in \Lambda$ such that $d(\mu) = d(\nu) = 0$ and $\lambda = \mu\nu$. Using (4.3.1), for this factorization we can choose

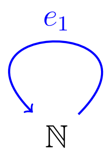
$$\mu = I_v, \nu = \lambda \qquad \textit{or} \qquad \mu = \lambda, \nu = I_{s(\lambda)}.$$

Hence by uniqueness $\lambda = I_v$. This shows that any morphism of degree zero is actually the identity morphism of its range, i.e., $\{\lambda \in \Lambda : d(\lambda) = 0\} = \{I_v : v \text{ is an object in } \Lambda\}$. Thus we can identify the objects of Λ with $\{\lambda \in \Lambda : d(\lambda) = 0\}$. Hence the objects of Λ are the morphisms of Λ of degree zero. □

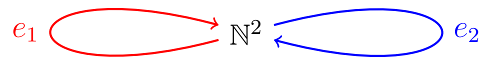
To visualize a k-graph Λ we draw its 1-skeleton, which is the directed graph $(\Lambda^0, \bigcup_{i=1}^{k} \Lambda^{e_i}, r, s)$ with edges in each Λ^{e_i}.

EXAMPLE 4.3.3. (1) A 1-graph is actually a path category $\mathcal{P}(E)$ of a directed graph E with the degree map $d : E^* \to \mathbb{N}$ defined by $d(\mu) = |\mu|$. In other words we can view 1-graph Λ as the path category of the directed graph $(\Lambda^0, \Lambda^1, r, s)$.

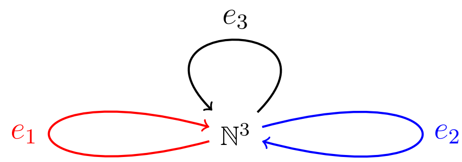
(2) $\mathbb{N}^k$ is also a k-graph. Let $\Lambda = \mathbb{N}^k$ with the identity map on $\mathbb{N}^k$. Then the edges in its 1-skeleton are $\{e_i\}_{i=1}^k$. The 1-skeletons of the 1-graph $\mathbb{N}$, 2-graph $\mathbb{N}^2$ and 3-graph $\mathbb{N}^3$ are given as follows:



$e_1 = 1$

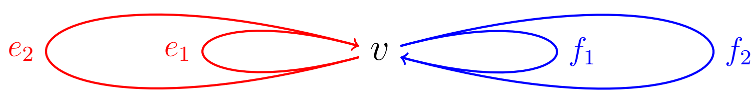


$e_1 = (1, 0)$, $e_2 = (0, 1)$



$e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$

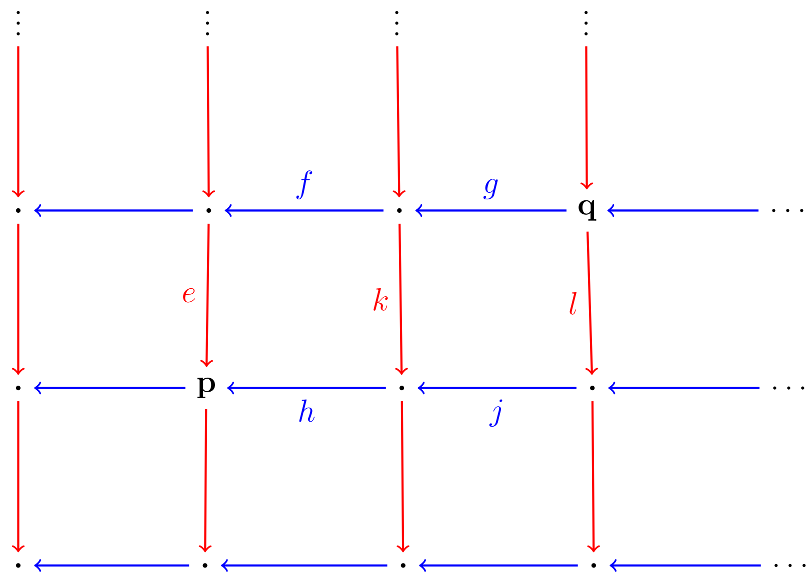
(3) Let Λ be a 2-graph whose 1-skeleton is given below:



Then $d(e_1) = d(e_2) = (1, 0)$ and $d(f_1) = d(f_2) = (0, 1)$.

(4) Define the k-graph Ω_k by setting the objects of Ω_k as $\Omega_k^0 = \mathbb{N}^k$, the morphisms $\Omega_k = \{(p, q) : p, q \in \mathbb{N}^k \text{ and } p \leq q\}$ and the degree map $d : \Omega_k \to \mathbb{N}^k$ defined as $d(p, q) = q - p$. Define the composition by $(p, q)(q, r) = (p, r)$ for $(p, q), (q, r) \in \Omega_k$.

The 1-skeleton of 2-graph Ω_2 looks like:



Here the edges in $\Omega_2^{e_1}$ are the horizontal arrows in blue color and the edges in $\Omega_2^{e_2}$ are the vertical arrows in red color, where $e_1 = (1,0)$ and $e_2 = (0,1)$. Clearly $d(p,q) = (2,1)$ and so it can be written as $d(p,q) = (1,0)+(1,0)+(0,1) = (1,0)+(0,1)+(1,0) = (0,1)+(1,0)+(1,0)$. So it has three factorizations $(p,q) = efg = hkg = hjl$, where $e,k,l \in \Omega_2^{e_1}$ and $f,g,h,j \in \Omega_2^{e_2}$.

Definition 4.3.2. We say that Λ is *finite* if Λ^n is finite for every $n \in \mathbb{N}^k$. Also Λ is said to be *row-finite* if $v\Lambda^{e_i}$ is a finite set for every $v \in \Lambda^0$ and $i = 1,2,\dots,k$.

For a k-graph Λ, we assume that $\Lambda^{e_i} \neq \emptyset$ for every $i = 1,2,\dots,k$. Otherwise we can consider Λ as a $(k-1)$-graph.

Notation 4.3.1. (a) For a k-graph Λ we use the convention that for $v \in \Lambda^0$ and $X \subseteq \Lambda$, $vX := \{\mu \in X : r(\mu) = v\}$, $Xv := \{\mu \in X : s(\mu) = v\}$ and $uXv := \{\mu \in X : s(\mu) = v, r(\mu) = u\}$.

(b) In a k-graph Λ. $v \in \Lambda^0$ is called a *source* if for every $i = 1,2,\dots,k$, $v\Lambda^{e_i} = \emptyset$.

In this thesis we always assume that every k-graph Λ is row-finite and has no source.

4.4. Perron-Frobenius Theory for Strongly Connected k-Graphs

In this section we relate a strongly connected k-graph with an irreducible family of matrices. Then we state and prove a Perron-Frobenius theorem for strongly connected k-graphs.

DEFINITION 4.4.1. A k-graph Λ is said to be *strongly connected* if for every $v, w \in \Lambda^0$, the set $v\Lambda w$ is non-empty.

DEFINITION 4.4.2. Let Λ be a finite k-graph. For $i = 1, 2, \ldots, k$ we define A_i be a matrix in $M_{\Lambda^0}(\mathbb{R})$ with entries $A_i(v, w) = |v\Lambda^{e_i} w|$ for $v, w \in \Lambda^0$. We call $A_1, A_2, \ldots, A_k$ the *coordinate matrices* of Λ.

REMARK 4.4.1. The set $v\Lambda^{e_i} w$ contains all i^{th} color edges μ such that $s(\mu) = w$ and $r(\mu) = v$.

The following lemma gives a precise relation between a strongly connected k-graph and irreducible family of matrices.

LEMMA 4.4.2. *Let Λ be a finite k-graph with coordinate matrices $A_1, A_2, \ldots, A_k$. Then A_i's are non-zero pairwise commuting matrices. Furthermore Λ is strongly connected if, and only if, $\{A_1, A_2, \ldots, A_k\}$ is an irreducible family of matrices.*

PROOF. Since for every $i = 1, 2, \ldots, k$ $\Lambda^{e_i} \neq \emptyset$, A_i's are non-zero matrices. For $v, w \in \Lambda^0$ and $i, j = 1, 2, \ldots, k$, we have

$$\begin{aligned} A_iA_j(v, w) &= \sum_{u\in\Lambda^0} A_i(v, u)A_j(u, w) = \sum_{u\in\Lambda^0} |v\Lambda^{e_i}u||u\Lambda^{e_j}w| \\ &= |v\Lambda^{e_i+e_j}w| = |v\Lambda^{e_j+e_i}w| = A_jA_i(v, w). \end{aligned} \tag{4.4.1}$$

So $A_iA_j = A_jA_i$. Hence A_i's are pairwise commuting.

Now suppose that Λ is strongly connected. Then for every $v, w \in \Lambda^0$, $v\Lambda w \neq \emptyset$. So there exists $n_{v,w} \in \mathbb{N}^k$ such that

$$v\Lambda^{n_{v,w}}w \neq \emptyset, \text{ i.e., } |v\Lambda^{n_{v,w}}w| > 0 \qquad \text{for every } v, w \in \Lambda^0. \tag{4.4.2}$$

Let $F := \{n_{v,w} : v, w \in \Lambda^0\}$. Clearly F is a non-empty finite subset of $\mathbb{N}^k$. Now using multi-index notation, for every $s, t \in \Lambda^0$ we have

$$A_F(s,t) = \sum_{v,w\in\Lambda^0} A^{n_{v,w}}(s,t) \geq A^{n_{s,t}}(s,t) = \prod_{i=1}^{k} A_i^{n_{s,t}^{(i)}}(s,t), \tag{4.4.3}$$

where $n_{v,w}^{(i)}$ is the i^{th} coordinate of $n_{v,w}$. From (4.4.1) we have $A_iA_j(v,w) = |v\Lambda^{e_i+e_j}w|$, so by induction and using (4.4.2), for every $t, s \in \Lambda^0$ we get

$$A_i^{n_{s,t}^{(i)}}(s,t) = |s\Lambda^{n_{s,t}^{(i)}e_i}t| \implies \prod_{i=1}^{k} A_i^{n_{s,t}^{(i)}}(s,t) = |s\Lambda^{\sum_{i=1}^{k} n_{s,t}^{(i)}e_i}t| = |s\Lambda^{n_{s,t}}t| > 0. \tag{4.4.4}$$

Hence (4.4.3) and (4.4.4) yields $A_F(s,t) > 0$ for every $s, t \in \Lambda^0$. This shows that $\{A_1, A_2, \dots, A_k\}$ is an irreducible family.

Conversely let us suppose $\{A_1, A_2, \dots, A_k\}$ be an irreducible family. So there exists $F \in \mathbb{N}^k$ such that $A_F \succ 0$. Now $A_F = \sum_{n\in F} A^n$ implies for $v, w \in \Lambda^0$ there exists $n \in F$ such that $A^n(v,w) \neq 0$. From (4.4.1) we have $A^n(v,w) = |v\Lambda^n w|$ implies $|v\Lambda^n w| \neq 0$. So $v\Lambda^n w \neq \emptyset$. Hence Λ is strongly connected. $\square$

Now we state a Perron-Frobenius theorem for strongly connected k-graphs and the proof can easily be obtained by Theorem 4.1.5. So we consider it as a corollary of Theorem 4.1.5.

COROLLARY 4.4.3. *Let Λ be a strongly connected finite k-graph. Let $\{A_1, A_2, \dots, A_k\}$ be the coordinate matrices of Λ. Then*

(a) each $\rho(A_i) > 0$ and for $n \in \mathbb{N}^k$ we have $\rho(A^n) = \prod_{i=1}^{k} \rho(A_i)^{n_i} > 0$;

(b) there exists a unique non-negative vector $x^\Lambda \in \mathbb{R}^{\Lambda^0}$ with unit norm such that $A_i x^\Lambda = \rho(A_i)x^\Lambda$ for every $i = 1, 2, \dots, k$; moreover, $x^\Lambda \succ 0$;

(c) if $z \in \mathbb{C}^{\Lambda^0}$ and $A_i z = \rho(A_i)z$ for every $i = 1, 2, \dots, k$, then $z \in \mathbb{C}x^\Lambda$; and

(d) if $y \in \mathbb{R}^{\Lambda^0}$ such that $y \succeq 0$, has unit norm and $A_i y = \rho(A_i)y$ for every $i = 1, 2, \dots, k$, then $y = x^\Lambda$.

PROOF. (a) Lemma 4.4.2 shows that the family $\{A_1, A_2, \dots, A_k\}$ is an irreducible and pairwise commuting family. So (a) follows from Proposition 4.1.5(a)(i) and (c).

(b) Proposition 4.1.5(a)(i) yields that the UPF eigenvector x^Λ of A_F is the unique non-negative unimodular common eigenvector of $A_i's$. Also by the definition of UPF eigenvector $x^\Lambda \succ 0$.

(c) This follows from Proposition 4.1.5(a)(ii).

(d) This follows from Proposition 4.1.5(b)(ii). □

DEFINITION 4.4.3. Let Λ be a strongly connected finite k-graph. We call the vector x^Λ of the above corollary, the *unimodular Perron-Frobenius (UPF) eigenvector of* Λ.

CHAPTER 5

Some Basics for C^*-Algebras and KMS States

In this chapter we provide some basics for C^*-algebras, C^*-dynamical systems and Kubo-Martin-Schwinger (KMS) states, which will be needed later. As an example, Gibbs states are discussed at the end of this chapter.

5.1. Basic Definitions and Some Results

The definitions and results in this section are taken from [**5**], unless otherwise stated. The first subsection consists of some basics of a C^*-algebra and positive elements in a C^*-algebra.

5.1.1. C^*-Algebras.

DEFINITION 5.1.1. Let $\mathfrak{A}$ be a vector space over $\mathbb{C}$. The space $\mathfrak{A}$ is called an *algebra* if it is equipped with a binary operation (usually called multiplication) from $\mathfrak{A} \times \mathfrak{A}$ to $\mathfrak{A}$ such that for every $A, B, C \in \mathfrak{A}$ and $\alpha \in \mathbb{C}$

(a) $A(BC) = (AB)C$, (b) $A(B + C) = AB + AC$ and

(c) $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

DEFINITION 5.1.2. Let $\mathfrak{A}$ be an algebra. A map $* : \mathfrak{A} \to \mathfrak{A}$ mapping an element $A \in \mathfrak{A}$ to some element $A^* \in \mathfrak{A}$ is called an *involution*, or *adjoint operator*, of the algebra $\mathfrak{A}$ if it has the following properties:

(a) $A^{**} = A$, (b) $(AB)^* = B^*A^*$ and (c) $(\alpha A + \beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^*$,

for $A, B \in \mathfrak{A}$ and $\alpha, \beta \in \mathbb{C}$. An algebra with an involution is called a $*$-*algebra*.

DEFINITION 5.1.3. An algebra $\mathfrak{A}$ is called a *normed algebra* if there is a norm $||.||$ defined on $\mathfrak{A}$ which satisfies $||AB|| \leq ||A||\,||B||$ for all $A, B \in \mathfrak{A}$.

Definition 5.1.4. (a) The topology induced by the metric defined by the norm on a normed algebra $\mathfrak{A}$ is called the *uniform topology.* If $\mathfrak{A}$ is complete with respect to the uniform topology then it is called a *Banach algebra.*

(b) A Banach algebra $\mathfrak{A}$ with involution and has property $||A|| = ||A^*||$ for all $A \in \mathfrak{A}$ is called a *Banach *-algebra.*

Definition 5.1.5. A C^**-algebra* is a Banach *-algebra $\mathfrak{A}$ with the C^*-identity $||A^*A|| = ||A||^2$ for every $A \in \mathfrak{A}$.

Note 5.1.1. The C^*-identity in Definition 5.1.5 combined with the submultiplicity of the norm in Definition 5.1.3 yields $||A|| = ||A^*||$ as $||A||^2 = ||A^*A|| \leq ||A^*||\,||A|| \implies ||A|| \leq ||A^*||$. Interchanging the roles of A and A^* we can get $||A|| = ||A^*||$.

Example 5.1.2. (1) Let H be a Hilbert space and $\mathcal{B}(H)$ be the set of all bounded linear operators on H. Define the sum and product for operators $f, g \in \mathcal{B}(H)$ as: $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(g(x))$. Define a norm on $A \in \mathcal{B}(H)$ as $||A|| = \sup\{||Ax|| : x \in H, ||x|| = 1\}$. Also the involution defined on $\mathcal{B}(H)$ maps $A \in \mathcal{B}(H)$ to $A^* \in \mathcal{B}(H)$, where A^* is the adjoint of A that satisfies $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for $x, y \in H$ and $\langle\cdot,\cdot\rangle$ is the inner product defined on H.

With respect to the operations and the norm defined above, $\mathcal{B}(H)$ is a Banach *-algebra. Moreover, using the Cauchy-Schwartz Inequality, we get

$$\begin{aligned}
||A||^2 &= \sup\{||Ax||^2 : x \in H, ||x|| = 1\} = \sup\{\langle Ax, Ax\rangle : x \in H, ||x|| = 1\} \\
&= \sup\{\langle x, A^*Ax\rangle : x \in H, ||x|| = 1\} \leq \sup\{||A^*Ax|| : x \in H, ||x|| = 1\} \\
&= ||A^*A|| \leq ||A^*||\,||A|| = ||A||^2 .
\end{aligned}$$

This show that $\mathcal{B}(H)$ is a C^*-algebra.

(2) For an n-dimensional Hilbert space $\mathcal{H}$ over $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ is isomorphic to $M_n(\mathbb{C})$. Thus $M_n(\mathbb{C})$ is also a C^*-algebra. Moreover, involution on $M_n(\mathbb{C})$ is given by the transpose conjugate (usually called the adjoint of a matrix).

DEFINITION 5.1.6. The *identity* 1 of an algebra $\mathfrak{A}$ is an element of $\mathfrak{A}$ such that $A = 1A = A1$ for every $A \in \mathfrak{A}$. An algebra with identity is also called a *unital algebra*.

In this thesis, all C^*-algebras are assumed to be unital.

DEFINITION 5.1.7. Let $\mathfrak{A}$ be a unital algebra. Then an element $A \in \mathfrak{A}$ is said to be *invertible* if there exists an element $A^{-1} \in \mathfrak{A}$, the *inverse* of A, such that $AA^{-1} = 1 = A^{-1}A$.

REMARK 5.1.3. Every invertible element has a unique inverse. Also if A and B are invertible, then
(a) $(A^{-1})^{-1} = A$, (b) $(AB)^{-1} = B^{-1}A^{-1}$, (c) $(A^*)^{-1} = (A^{-1})^*$.

DEFINITION 5.1.8. Let $\mathfrak{A}$ be a unital algebra over $\mathbb{C}$. The *resolvent* $r_{\mathfrak{A}}(A)$ of an element $A \in \mathfrak{A}$ is defined as $r_{\mathfrak{A}}(A) := \{\lambda \in \mathbb{C} : (\lambda 1 - A)^{-1} \text{exists}\}$. The *spectrum* $\sigma_{\mathfrak{A}}(A)$ of $A \in \mathfrak{A}$ is defined as $\sigma_{\mathfrak{A}}(A) := \{\lambda \in \mathbb{C} : (\lambda 1 - A)^{-1} \text{ does not exist}\} = \mathbb{C} \setminus r_{\mathfrak{A}}(A)$.

DEFINITION 5.1.9. An element A of a *-algebra $\mathfrak{A}$ is said to be *positive* if it is self-adjoint and its spectrum $\sigma_{\mathfrak{A}}(A)$ consists of non-negative real numbers. For a positive $A \in \mathfrak{A}$, we write $A \geq 0$. Also we can say $A \geq B$ if and and only if $A - B \geq 0$.

THEOREM 5.1.4. *Let $\mathfrak{A}$ be a C^*-algebra and $A \in \mathfrak{A}$ be a self-adjoint element. Then the following are equivalent.*

(a) A is positive.

(b) $A = B^2$ for a self-adjoint $B \in \mathfrak{A}$.

(c) $A = B^*B$ for some $B \in \mathfrak{A}$.

PROPOSITION 5.1.5. *Let $\mathfrak{A}$ be a C^*-algebra and $A, B \in \mathfrak{A}$. Then*
(a) $A \geq B \geq 0$ implies $||A|| \geq ||B||$ and (b) $A \geq 0$ implies $A\,||A|| \geq A^2$.

Now in the following subsection we define *-morphism between C^*-algebras in order to define C^*-dynamical system in further sections.

5.1.2. Representations.

DEFINITION 5.1.10. A **-morphism* between two *-algebras $\mathfrak{A}$ and $\mathfrak{B}$ is a mapping $\pi : \mathfrak{A} \to \mathfrak{B}$ such that for all $A, B \in \mathfrak{A}$ and $\alpha, \beta \in \mathbb{C}$

(a) $\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$, (b) $\pi(AB) = \pi(A)\pi(B)$, (c) $\pi(A^*) = \pi(A)^*$.

The following lemma ensures that all *-morphisms between C^*-algebras are automatically continuous.

LEMMA 5.1.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two C^*-algebras and π be a *-morphism of $\mathfrak{A}$ into $\mathfrak{B}$. Then*

(a) π is positivity preserving, i.e., $\pi(A^*A)$ is positive; and

(b) π is continuous, moreover, $||\pi(A)|| \leq ||A||$ for every $A \in \mathfrak{A}$.

PROOF. (a) Clearly we can write $\pi(A^*A) = \pi(A)^*\pi(A)$. So by Theorem 5.1.4 $\pi(A^*A)$ is positive.

(b) From Proposition 5.1.5(b) we have $0 \leq (A^*A)^2 \leq A^*A\,||A^*A||$. So Part (a) of this lemma implies that $0 \leq \pi(A^*A)^2 \leq \pi(A^*A)\,||A^*A||$. Now by Proposition 5.1.5(a) we get

$$||\pi(A)||^4 = ||\pi(A^*A)||^2 \leq ||\pi(A^*A)||\,||A^*A|| = ||\pi(A)||^2\,||A||^2 \implies ||\pi(A)|| \leq ||A||\,.$$

□

DEFINITION 5.1.11. A *-morphism π from $\mathfrak{A}$ to $\mathfrak{B}$ is called a **-isomorphism* if it is a bijection.

DEFINITION 5.1.12. A *representation* of a C^*-algebra $\mathfrak{A}$ is defined to be a pair (H, π), where H is a complex Hilbert space and π is a *-morphism of $\mathfrak{A}$ into $\mathcal{B}(H)$. A representation (H, π) is said to be *faithful* if π is a *-isomorphism. The space H is called the *representation space*, the operators $\pi(A)$ for $A \in \mathfrak{A}$ are called the *representatives* of $\mathfrak{A}$. We say that π is a *representation of* $\mathfrak{A}$ *on* H.

PROPOSITION 5.1.7. *Let (H, π) be a faithful representation of a C^*-algebra $\mathfrak{A}$. Then we have the following properties:*

(a) $\ker \pi = \{0\}$; *(b)* $||\pi(A)|| = ||A|| \ \forall \ A \in \mathfrak{A}$.

DEFINITION 5.1.13. A *-*automorphism* τ of a C^*-algebra $\mathfrak{A}$ is defined to be a *-isomorphism of $\mathfrak{A}$ onto itself.

COROLLARY 5.1.8. *A *-automorphism τ of a C^*-algebra $\mathfrak{A}$ is norm preserving, i.e., $||\tau(A)|| = ||A||$ for every $A \in \mathfrak{A}$.*

5.2. C^*-Dynamical Systems and KMS-States

This section consists of some basic definitions and results from [**5**] and [**6**] which will be used in Chapter 6.

Recall that a group with a topology defined on it which is locally compact is called a *locally compact group.*

DEFINITION 5.2.1. [**5**] A C^*-*dynamical system* is a triplet $(\mathfrak{A}, G, \tau)$, where $\mathfrak{A}$ is a C^*-algebra, G is a locally compact group, and τ is a strongly continuous representation of G in the automorphism group of $\mathfrak{A}$, i.e., $\tau : G \to \text{Aut}(\mathfrak{A})$ such that $\tau_e = I$, $\tau_{g_1}\tau_{g_2} = \tau_{g_1 g_2}$ for all $g_1, g_2 \in G$, and $g \mapsto \tau_g(A)$ is continuous in norm for each $A \in \mathfrak{A}$, where $e \in G$ is the identity and I is the identity map on $\mathfrak{A}$.

REMARK 5.2.1. In the above definition, we say $g \mapsto \tau_g(A)$ is continuous in norm, if there exists a net $\{g_k\}$ in G such that $g_k \to g$ for some $g \in G$, then for every $A \in \mathfrak{A}$ $||\tau_{g_k}(A) - \tau_g(A)|| \to 0$.

NOTE 5.2.2. In the following we only consider the one-parameter C^*-dynamical system $(\mathfrak{A}, \mathbb{R}, \tau)$. For simplicity we denote such a system simply by $(\mathfrak{A}, \tau)$.

DEFINITION 5.2.2. Let $(\mathfrak{A}, \tau)$ be a one-parameter C^*-dynamical system. An element $A \in \mathfrak{A}$ is said to be τ-*analytic* if the function $t \mapsto \tau_t(A)$ extends to an entire function on the complex plane. The set of all τ-analytic elements in $\mathfrak{A}$ is denoted by $\mathfrak{A}_\tau$.

NOTE 5.2.3. [**6**] (1) $\mathfrak{A}_\tau$ is a dense *-subalgebra of $\mathfrak{A}$ in the uniform topology defined on $\mathfrak{A}$.

(2) $\mathfrak{A}_\tau$ is τ-invariant, i.e., if $A \in \mathfrak{A}_\tau$, then $\tau_t(A) \in \mathfrak{A}_\tau$ for every $t \in \mathbb{R}$.

Now we define a state over a C^*-algebra, KMS condition and some important results.

Definition 5.2.3. A linear functional ω over a *-algebra $\mathfrak{A}$ is defined to be *positive* if $\omega(A^*A) \geq 0$ for every $A \in \mathfrak{A}$.

Definition 5.2.4. A positive linear functional ω over a C^*-algebra $\mathfrak{A}$ with $||\omega|| = 1$ is called a *state*.

Lemma 5.2.4. ***(Cauchy-Schwartz Inequality)*** *Let ω be a positive linear functional over a *-algebra $\mathfrak{A}$. It follows that for every $A, B \in \mathfrak{A}$*

(a) $\omega(A^*B) = \overline{\omega(B^*A)}$, *(b)* $|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B)$.

Proposition 5.2.5. *Let ω be a linear functional over a C^*-algebra $\mathfrak{A}$. If ω is positive, then for $A, B \in \mathfrak{A}$*

(a) $\omega(A^*) = \overline{\omega(A)}$, *(b)* $|\omega(A)|^2 \leq \omega(A^*A)\,||\omega||$,

(c) $|\omega(A^*BA)| \leq \omega(A^*A)\,||B||$, *(d)* $||\omega|| = \sup\{\omega(A^*A), ||A|| = 1\}$.

Proposition 5.2.6. *Let $\mathfrak{A}$ is a unital C^*-algebra and ω be a positive linear functional over $\mathfrak{A}$. Then $||\omega|| = \omega(1)$.*

Definition 5.2.5. Let $(\mathfrak{A}, \tau)$ be a C^*-dynamical system. A state ω over $\mathfrak{A}$ is defined to be *τ-KMS state at value $\beta \in \mathbb{R}$* or a *(τ, β)-KMS state* if

$$\omega(AB) = \omega(B\tau_{i\beta}(A)) \qquad \text{for every } A, B \in \mathfrak{A}_\tau.$$

Lemma 5.2.7. *Let ω be a (τ, β)-KMS state over a C^*-algebra $\mathfrak{A}$ and $\beta \in \mathbb{R} \setminus \{0\}$. Then ω is τ-invariant, i.e., $\omega(\tau_t(A)) = \omega(A)$ for all $A \in \mathfrak{A}$ and for every $t \in \mathbb{R}$.*

Proof. Let $\beta \in \mathbb{R} \setminus \{0\}$ and $A \in \mathfrak{A}_\tau$. Then by Definition 5.2.2 the function $f(t) = \tau_t(A)$ for $t \in \mathbb{R}$ extends to an entire function on the complex plane, i.e.,

$f(z) = \tau_z(A)$ for $z \in \mathbb{C}$ is an entire function. So

$$\lim_{h\to 0} \frac{\tau_{z+h}(A) - \tau_z(A)}{h} \text{ exists for every } z \in \mathbb{C}. \tag{5.2.1}$$

Define $F(z) = \omega(\tau_z(A))$. Since ω is a linear bounded (so is continuous) functional, we can get

$$\lim_{h\to 0} \frac{\omega(\tau_{z+h}(A)) - \omega(\tau_z(A))}{h} = \omega\Big(\lim_{h\to 0} \frac{\tau_{z+h}(A) - \tau_z(A)}{h}\Big).$$

Thus in view of (5.2.1), we can conclude that F is differentiable everywhere on $\mathbb{C}$. So F is an entire function. Furthermore

$$\begin{aligned} |F(z)| = |\omega(\tau_z(A))| &\le ||\omega||\,||\tau_z(A)|| \\ &= ||\tau_z(A)|| \qquad (\text{as } ||\omega|| = 1) \\ &= ||\tau_{\operatorname{Re} z}(\tau_{i \operatorname{Im} z}(A))|| \le ||\tau_{i \operatorname{Im} z}(A)|| \qquad (\text{by Lemma 5.1.6(b)}). \end{aligned} \tag{5.2.2}$$

Let $D := \{z \in \mathbb{C} : 0 \le \operatorname{Im} z \le |\beta|\}$ and $M := \sup\{||\tau_{ix}(A)|| : x \in [0, |\beta|]\}$. Then (5.2.2) yields that $|F(z)| \le M$ for every $z \in D$.

Now,

$$F(z + i\beta) = \omega(1\tau_{i\beta}(\tau_z(A))) = \omega(\tau_z(A)1) = F(z) \qquad (\text{as } \omega \text{ is a KMS state}).$$

This implies F is periodic with the period of $i\beta$. Hence $|F(z)| \le M$ for every $z \in \mathbb{C}$. Thus F is a bounded entire function. So by Lioville's theorem F has to be a constant function. Hence for any $z \in \mathbb{C}$

$$F(z) = F(0) \implies \omega(\tau_z(A)) = \omega(\tau_0(A)) = \omega(A).$$

In particular, $\omega(\tau_t(A)) = \omega(A)$ for all $A \in \mathfrak{A}$ and for every $t \in \mathbb{R}$. □

NOTE 5.2.8. In Lemma 5.2.7, we can observe that if the representation τ of $\mathbb{R}$ is extended to a representation of $\mathbb{C}$ in the automorphism group of $\mathfrak{A}$, then even ω is τ-invariant.

PROPOSITION 5.2.9. *Let $\mathfrak{A}$ be a C^*-algebra and ω be a state over $\mathfrak{A}$. Let $\beta \in \mathbb{R}$. Then the following are equivalent.*

(a) ω is a (τ, β)-KMS state over $\mathfrak{A}$.

(b) $\omega(\tau_{-i\frac{\beta}{2}}(B)\tau_{i\frac{\beta}{2}}(A)) = \omega(AB)$ for all $A, B \in \mathfrak{A}_\tau$.

PROOF. If $\beta = 0$, the result is trivial.

(a) $\implies$ (b): Let us suppose ω is a (τ, β)-KMS state over $\mathfrak{A}$ with $\beta \in \mathbb{R} \setminus \{0\}$. Let $A, B \in \mathfrak{A}_\tau$. Then we have

$$\begin{aligned}\omega(AB) = \omega(B\tau_{i\beta}(A)) &= \omega\Big(B\tau_{i\frac{\beta}{2}}\big(\tau_{i\frac{\beta}{2}}(A)\big)\Big)\\ &= \omega\Big(\tau_{i\frac{\beta}{2}}\big(\tau_{-i\frac{\beta}{2}}(B)\tau_{i\frac{\beta}{2}}(A)\big)\Big)\\ &= \omega\big(\tau_{-i\frac{\beta}{2}}(B)\tau_{i\frac{\beta}{2}}(A)\big) \qquad \text{(by Note 5.2.8).}\end{aligned}$$

(b) $\implies$ (a): Assume that

$$\omega(AB) = \omega(\tau_{-i\frac{\beta}{2}}(B)\tau_{i\frac{\beta}{2}}(A)) \qquad \text{for all } A, B \in \mathfrak{A}_\tau. \tag{5.2.3}$$

Let $A \in \mathfrak{A}_\tau$ and define $F(z) = \omega(\tau_z(A))$. Then the first part of Lemma 5.2.7 shows that $F(z)$ is an entire function and $|F(z)| \leq ||\tau_{i\,\mathrm{Im}\,z}(A)||$.

Let $D := \{z \in \mathbb{C} : 0 \leq \mathrm{Im}\, z \leq \left|\frac{\beta}{2}\right|\}$ and $M := \sup\{||\tau_{ix}(A)|| : x \in [0, \left|\frac{\beta}{2}\right|]\}$. Then $|F(z)| \leq M$ for every $z \in D$. Now

$$\begin{aligned}F(z + i\frac{\beta}{2}) = \omega(1\tau_{i\frac{\beta}{2}}(\tau_z(A))) &= \omega\Big(\tau_{-i\frac{\beta}{2}}\big(\tau_{i\frac{\beta}{2}}(\tau_z(A))\big)\tau_{i\frac{\beta}{2}}(1)\Big) \qquad \text{(using (5.2.3))}\\ &= \omega(\tau_z(A)) \qquad (\text{as } \tau_{i\frac{\beta}{2}}(1) = 1)\\ &= F(z).\end{aligned}$$

This implies F is periodic with the period of $i\frac{\beta}{2}$. Hence $|F(z)| \leq M$ for every $z \in \mathbb{C}$. Thus F is a bounded entire function. So by Lioville's theorem F has to be a constant function. Hence for any $z \in \mathbb{C}$

$$F(z) = F(0) \implies \omega(\tau_z(A)) = \omega(\tau_0(A)) = \omega(A).$$

Thus ω is τ-invariant. Hence from (5.2.3)

$$\omega(AB) = \omega(\tau_{-i\frac{\beta}{2}}(B)\tau_{i\frac{\beta}{2}}(A)) = \omega\big(\tau_{i\frac{\beta}{2}}(\tau_{-i\frac{\beta}{2}}(B)\tau_{i\frac{\beta}{2}}(A))\big) = \omega(B\tau_{i\beta}(A)).$$

This implies that ω is a (τ,β)-KMS state. ☐

5.3. An Example of KMS States: Gibbs States

In this section we will give an example of KMS states on the C^*-algebra $M_n(\mathbb{C})$.

LEMMA 5.3.1. *Let $H \in M_n(\mathbb{C})$. Then the function $f : \mathbb{C} \to M_n(\mathbb{C})$ defined as $f(a) = \mathrm{e}^{aH}$ is continuous.*

PROOF. WLOG, assume $H \neq 0$. Let $a, b \in \mathbb{C}$ and $A = \max\{|a|, |b|\}$. Then

$$\begin{aligned}
||f(a) - f(b)|| = \left|\left|\mathrm{e}^{aH} - \mathrm{e}^{bH}\right|\right| &= \left|\left|\sum_{m\geq 0} \frac{a^m}{m!}H^m - \sum_{m\geq 0}\frac{b^m}{m!}H^m\right|\right| \\
&= \left|\left|\sum_{m\geq 0} \frac{a^m - b^m}{m!}H^m\right|\right| \\
&= \left|\left|\sum_{m\geq 1} \frac{a^m - b^m}{m!}H^m\right|\right| \\
&= \left|\left|\sum_{m\geq 1} \frac{(a-b)(a^{m-1} + \cdots + b^{m-1})}{m!}H^m\right|\right| \\
&\leq |a-b| \sum_{m\geq 1} \frac{|a^{m-1}| + \cdots + |b^{m-1}|}{m!} \, ||H^m|| \\
&\leq |a-b| \sum_{m\geq 1} \frac{mA^{m-1}}{m!} \, ||H^m|| \\
&= |a-b| \, ||H|| \sum_{m\geq 1} \frac{A^{m-1}}{(m-1)!} \, ||H||^{m-1} \\
&= |a-b| \, ||H|| \, \mathrm{e}^{A||H||} .
\end{aligned}$$

Hence if $a \to b$ then $f(a) \to f(b)$. This shows that $f(a) = \mathrm{e}^{aH}$ is continuous. ☐

LEMMA 5.3.2. *Let $H \in M_n(\mathbb{C})$ be a self-adjoint matrix and $\tau : \mathbb{R} \to \operatorname{Aut}(M_n(\mathbb{C}))$ be defined by $\tau_t(A) = \mathrm{e}^{itH} A \, \mathrm{e}^{-itH}$ for $A \in M_n(\mathbb{C})$. Then $(M_n(\mathbb{C}), \tau)$ is a C^*-dynamical system.*

PROOF. Let $A, B \in M_n(\mathbb{C})$ and $a, b \in \mathbb{C}$. Then

(a) $\tau_t(aA + bB) = \mathrm{e}^{itH}(aA + bB)\,\mathrm{e}^{-itH} = a\tau_t(A) + b\tau_t(B)$
(b) $\tau_t(AB) = \mathrm{e}^{itH} AB\,\mathrm{e}^{-itH} = \mathrm{e}^{itH} Ae^{-itH}e^{itH} B\,\mathrm{e}^{-itH} = \tau_t(A)\tau_t(B)$
(c) $\tau_t(A^*) = \mathrm{e}^{itH} A^*\,\mathrm{e}^{-itH} = \left(\mathrm{e}^{-\bar{i}tH^*} A\,\mathrm{e}^{\bar{i}tH^*}\right)^* = \left(\mathrm{e}^{itH} A\,\mathrm{e}^{-itH}\right)^* = (\tau_t(A))^*$

This shows τ_t is a *-morphism.

(d) Since $\tau_t(A) = 0 \implies \mathrm{e}^{itH} A\,\mathrm{e}^{-itH} = 0 \implies A = 0 \implies \ker A = \{0\}$, i.e., τ_t is injective.
(e) Let $A \in M_n(\mathbb{C})$. Then $\tau_t(\mathrm{e}^{-itH} A\,\mathrm{e}^{itH}) = A$.

This shows that τ_t is a *-automorphism.

Now let $A \in M_n(\mathbb{C})$ and $s, t \in \mathbb{R}$. Then

(a) $\tau_0(A) = \mathrm{e}^{i0H} A\,\mathrm{e}^{-i0H} = A$ implies $\tau_0 = I$.
(b) $\tau_t\tau_s(A) = \tau_t(\mathrm{e}^{isH} A\,\mathrm{e}^{-isH}) = \mathrm{e}^{itH}\,\mathrm{e}^{isH} A\,\mathrm{e}^{-isH}\,\mathrm{e}^{-itH} = \mathrm{e}^{i(t+s)H} A\,\mathrm{e}^{-i(t+s)H} = \tau_{t+s}(A)$.
(c) Let $\{g_k\}$ be a net in $\mathbb{R}$ such that $g_k \to g$ for some $g \in \mathbb{R}$. Then using Lemma 5.3.1, we get

$$ig_k \to ig \implies \mathrm{e}^{ig_kH} \to \mathrm{e}^{igH} \implies \mathrm{e}^{ig_kH} Ae^{-ig_kH} \to \mathrm{e}^{igH} A\,\mathrm{e}^{-igH}.$$

So for every $A \in M_n(\mathbb{C})$, $t \mapsto \tau_t(A)$ is continuous in norm.

This shows that $(M_n(\mathbb{C}), \tau)$ is a C^*-dynamical system. □

THEOREM 5.3.3. *Let $H \in M_n(\mathbb{C})$ be a self-adjoint matrix. Let ω_β be a functional defined on $M_n(\mathbb{C})$ as*

$$\omega_\beta(A) = \frac{\operatorname{tr}(\mathrm{e}^{-\beta H} A)}{\operatorname{tr}(\mathrm{e}^{-\beta H})} \qquad \text{for all } A \in M_n(\mathbb{C}).$$

Then ω_β is the unique (τ, β)-KMS state for $(M_n(\mathbb{C}), \tau)$. This state is called a Gibbs state.

We need the following results to prove this theorem.

LEMMA 5.3.4. *If* $A \in M_n(\mathbb{C})$, *then* $\operatorname{tr}(A^*A) \geq 0$.

PROOF. Let $A \in M_n(\mathbb{C})$. Let $A(i,j) = a_{ij}$ and so $A^*(i,j) = \overline{a}_{ji}$. Then

$$A^*A(i,i) = \sum_{k=1}^{n} A^*(i,k)A(k,i) = \sum_{k=1}^{n} \overline{a}_{ki}a_{ki} = \sum_{k=1}^{n} |a_{ki}|^2.$$

That is, every diagonal entry of A^*A is positive. Hence $\operatorname{tr}(A^*A) \geq 0$. □

LEMMA 5.3.5. *A linear functional* f *on* $M_n(\mathbb{C})$ *is a scaler multiple of the trace functional if, and only if,* $f(AB) = f(BA)$ *for every* $A, B \in M_n(\mathbb{C})$. *Moreover,* $f(A) = \dfrac{f(I)}{\operatorname{tr}(I)} \operatorname{tr}(A)$.

PROOF. Let f be a functional on $M_n(\mathbb{C})$ such that $f(AB) = f(BA)$ for every $A, B \in M_n(\mathbb{C})$. Let $E_{ij} \in M_n(\mathbb{C})$ such that the $(i,j)^{th}$ entry is 1 and zero elsewhere. Then $E_{ij}E_{kl} = \begin{cases} E_{il}, & \text{if } j = k \\ \mathbf{0}, & \text{if } j \neq k \end{cases}$. Now for $i \neq j$

$$f(E_{ij}) = f(E_{i1}E_{1j}) = f(E_{1j}E_{i1}) = f(\mathbf{0}) = 0 \tag{5.3.1}$$

and for $i = j$

$$f(E_{ii}) = f(E_{i1}E_{1i}) = f(E_{1i}E_{i1}) = f(E_{11}). \tag{5.3.2}$$

Now let I be the identity matrix. Then $f(I) = f(\sum_{i=1}^{n} E_{ii}) = \sum_{i=1}^{n} f(E_{ii}) = \sum_{i=1}^{n} f(E_{11}) = nf(E_{11})$. So (5.3.2) yields

$$f(E_{ii}) = \frac{f(I)}{n}. \tag{5.3.3}$$

Let $A \in M_n(\mathbb{C})$. Then $A = \sum_{1 \leq i,j \leq n} a_{ij}E_{ij}$. So

$$\begin{aligned} f(A) = f\big(\sum_{1 \leq i,j \leq n} a_{ij}E_{ij}\big) &= \sum_{1 \leq i,j \leq n} a_{ij}f(E_{ij}) \\ &= \sum_{i=1}^{n} a_{ii}f(E_{ii}) \qquad \text{(using (5.3.1) and (5.3.2))} \end{aligned}$$

$$= \sum_{i=1}^{n} a_{ii} \frac{f(I)}{n} \qquad \text{(using (5.3.3))}$$

$$= \frac{f(I)}{n} \operatorname{tr}(A) = \frac{f(I)}{\operatorname{tr}(I)} \operatorname{tr}(A).$$

The Converse holds true trivially. □

Proof of Theorem 5.3.3. Let $H \in M_n(\mathbb{C})$ be a self-adjoint matrix and $\beta \in \mathbb{R}$. Let ω_β be a functional on $M_n(\mathbb{C})$ defined as

$$\omega_\beta(A) = \frac{\operatorname{tr}(\mathrm{e}^{-\beta H} A)}{\operatorname{tr}(\mathrm{e}^{-\beta H})}. \tag{5.3.4}$$

First we shall check that ω_β is a state of $M_n(\mathbb{C})$. Let $A \in M_n(\mathbb{C})$. Since H is self-adjoint, $\frac{1}{2}H$ is also self-adjoint. So

$$\mathrm{e}^{-\beta H} = \mathrm{e}^{-\beta \frac{1}{2} H} \mathrm{e}^{-\beta \frac{1}{2} H} = (\mathrm{e}^{-\beta \frac{1}{2} H})^* \mathrm{e}^{-\beta \frac{1}{2} H}. \tag{5.3.5}$$

Thus by Lemma 5.3.4 $\operatorname{tr}(\mathrm{e}^{-\beta H}) > 0$. Also

$$\omega_\beta(A^*A) = \frac{\operatorname{tr}(\mathrm{e}^{-\beta H} A^*A)}{\operatorname{tr}(\mathrm{e}^{-\beta H})} = \frac{\operatorname{tr}(A\, \mathrm{e}^{-\beta H} A^*)}{\operatorname{tr}(\mathrm{e}^{-\beta H})} \qquad (\text{as } \operatorname{tr}(AB) = \operatorname{tr}(BA))$$

$$= \frac{\operatorname{tr}(A(\mathrm{e}^{-\beta \frac{1}{2} H})^* \mathrm{e}^{-\beta \frac{1}{2} H} A^*)}{\operatorname{tr}(\mathrm{e}^{-\beta H})} \qquad \text{(by (5.3.5))}$$

$$= \frac{\operatorname{tr}\left(\left(\mathrm{e}^{-\beta \frac{1}{2} H} A^*\right)^* \left(\mathrm{e}^{-\beta \frac{1}{2} H} A^*\right)\right)}{\operatorname{tr}(\mathrm{e}^{-\beta H})} \geq 0 \qquad \text{(by Lemma 5.3.4).}$$

This shows that ω_β is a positive functional. Also by Proposition 5.2.6 we can conclude that

$$||\omega|| = \omega(I) = \frac{\operatorname{tr}(\mathrm{e}^{-\beta H} I)}{\operatorname{tr}(\mathrm{e}^{-\beta H})} = 1.$$

This proves that ω is a state.

Now we show that ω_β satisfies the KMS condition. For this, let $A, B \in M_n(\mathbb{C})$. Then

$$\omega_\beta(B\tau_{i\beta}(A)) = \frac{\operatorname{tr}\left(\mathrm{e}^{-\beta H} B\tau_{i\beta}(A)\right)}{\operatorname{tr}(\mathrm{e}^{-\beta H})}$$

$$
\begin{aligned}
&= \frac{\operatorname{tr}\left(\mathrm{e}^{-\beta H} B\, \mathrm{e}^{-\beta H} A\, \mathrm{e}^{\beta H}\right)}{\operatorname{tr}(\mathrm{e}^{-\beta H})} \\
&= \frac{\operatorname{tr}\left(\mathrm{e}^{-\beta H} A\, \mathrm{e}^{\beta H}\, \mathrm{e}^{-\beta H} B\right)}{\operatorname{tr}(\mathrm{e}^{-\beta H})} \qquad \left(\text{as} \operatorname{tr}(ST) = \operatorname{tr}(TS)\right) \\
&= \frac{\operatorname{tr}\left(\mathrm{e}^{-\beta H} AB\right)}{\operatorname{tr}(\mathrm{e}^{-\beta H})} = \omega_\beta(AB).
\end{aligned}
$$

Finally we show the uniqueness of ω_β. Let us assume ϕ be another state on $M_n(\mathbb{C})$ which satisfies the KMS condition. Then for $A, B \in M_n(\mathbb{C})$

$$
\phi(AB) = \phi(B\tau_{i\beta}(A)) = \phi(B\, \mathrm{e}^{-\beta H} A\, \mathrm{e}^{\beta H}) \tag{5.3.6}
$$

Let us define a map $\tilde{\phi}(A) = \phi(A\, \mathrm{e}^{\beta H})$. Then for $A, B \in \mathcal{B}(\mathcal{H})$ and using (5.3.6), we get

$$
\tilde{\phi}(AB) = \phi(AB\, \mathrm{e}^{\beta H}) = \phi(B\, \mathrm{e}^{\beta H}\, \mathrm{e}^{-\beta H} A\, \mathrm{e}^{\beta H}) = \phi(BA\, \mathrm{e}^{\beta H}) = \tilde{\phi}(BA).
$$

Hence from Lemma 5.3.5 $\tilde{\phi}$ is a scalar multiple of trace functional. In particular

$$
\tilde{\phi}(A) = \frac{\tilde{\phi}(I)}{\operatorname{tr}(I)} \operatorname{tr}(A)
$$

$$
\Longrightarrow \phi(A\, \mathrm{e}^{\beta H}) = \frac{\phi(\mathrm{e}^{\beta H})}{\operatorname{tr}(I)} \operatorname{tr}(A). \tag{5.3.7}
$$

Let $A = A\, \mathrm{e}^{-\beta H}$ in (5.3.7). Then

$$
\phi(A) = \frac{\phi(\mathrm{e}^{\beta H})}{\operatorname{tr}(I)} \operatorname{tr}(A\, \mathrm{e}^{-\beta H}). \tag{5.3.8}
$$

Now let $A = \mathrm{e}^{-\beta H}$ in (5.3.7). Then

$$
\phi(I) = \frac{\phi(\mathrm{e}^{\beta H})}{\operatorname{tr}(I)} \operatorname{tr}(\mathrm{e}^{-\beta H}) \implies \phi(\mathrm{e}^{\beta H}) = \frac{\operatorname{tr}(I)}{\operatorname{tr}(\mathrm{e}^{-\beta H})}\phi(I) = \frac{\operatorname{tr}(I)}{\operatorname{tr}(\mathrm{e}^{-\beta H})}. \tag{5.3.9}
$$

Substituting (5.3.9) in (5.3.8), we get

$$
\phi(A) = \frac{\operatorname{tr}(A\, \mathrm{e}^{-\beta H})}{\operatorname{tr}(\mathrm{e}^{-\beta H})} = \omega_\beta(A).
$$

□

CHAPTER 6

KMS States of the C^*-Algebras of k-Graphs

In this chapter, we first provide some necessary background for the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ and graph C^*-algebra $C^*(\Lambda)$ of a (row-finite) k-graph Λ (without sources). Then we apply the results from previous chapters to study their KMS states. The main sources of this chapter are [**8**], [**9**], [**10**] and [**6**].

6.1. Background

Definition 6.1.1. Let $\mathcal{H}$ be a Hilbert space. Let $M \subseteq \mathcal{H}$ be a closed subspace. Then we have the decomposition $\mathcal{H} = M \oplus M^\perp$. The *orthogonal projection* onto M, denoted by P_M, is a bounded linear operator on $\mathcal{H}$ such that for $x \in \mathcal{H}$, $x = m + m'$, where $m \in M$ and $m' \in M^\perp$, we have $P_M(x) = m$.

Note that $P \in \mathcal{B}(H)$ is a projection if, and only if, $P = P^* = P^2$.

Definition 6.1.2. A linear operator S on $\mathcal{H}$ is called a *partial isometry* if S is an isometry on $M = (\ker S)^\perp$. We call M the *initial space* and $N = SM$ the *final space* of S.

Proposition 6.1.1. *Let $\mathcal{H}$ be a Hilbert space. Let $S \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent:*

(a) S is a partial isometry;

(b) S^*S is an orthogonal projection P_M onto $M = (\ker S)^\perp$;

(c) $S = SS^*S$.

Proof. (a) $\Longrightarrow$ (b) Let S is a partial isometry. In order to prove S^*S is an orthogonal projection P_M onto $M = (\ker S)^\perp$.
Claim: $\langle (S^*S - P_M)h, h\rangle = 0$ for all $h \in \mathcal{H}$.

If $m \in (\ker S)^{\perp}$, then

$$\langle S^*Sm, m\rangle = ||Sm||^2 = ||m||^2 = \langle m, m\rangle = \langle P_M m, m\rangle$$
$$\Longrightarrow \langle (S^*S - P_M)m, m\rangle = 0. \tag{6.1.1}$$

Since S^*S and P_M are zero on $\ker S$ and both have range in $(\ker S)^{\perp}$, we can get

$$S^*S = P_M S^*S P_M. \tag{6.1.2}$$

Let $h \in \mathcal{H}$ such that $h = m + m'$, where $m \in (\ker S)^{\perp}$ and $m' \in \ker S$. Then $P_M h = m \in (\ker S)^{\perp}$. So we can get

$$\begin{aligned}
\langle (S^*S - P_M)h, h\rangle &= \langle P_M(S^*S - P_M)P_M h, h\rangle \qquad \text{(by (6.1.2))}\\
&= \langle (S^*S - P_M)P_M h, P_M h\rangle\\
&= \langle (S^*S - P_M)m, m\rangle = 0 \qquad \text{(by (6.1.1))}.
\end{aligned}$$

Hence in view of polarization identity

$$4\,\langle (S^*S - P_M)h, h'\rangle = \sum_{n=0}^{3} \langle (S^*S - P_M)(h + i^n h'), h + i^n h'\rangle \text{ for every } h, h' \in \mathcal{H},$$

we conclude that $\langle (S^*S - P_M)h, h'\rangle = 0$ for every $h, h' \in \mathcal{H}$. So $(S^*S - P_M)h = 0$ for every $h \in \mathcal{H}$.

(b) $\Longrightarrow$ (c) S^*S is an orthogonal projection P_M onto $M = (\ker S)^{\perp}$. Then $(S^*S)^2 = S^*S$. So we can get

$$\begin{aligned}
||S - SS^*S||^2 &= ||(S - SS^*S)^*(S - SS^*S)||\\
&= ||(S^* - S^*SS^*)(S - SS^*S)||\\
&= ||S^*S - S^*SS^*S - S^*SS^*S + S^*SS^*SS^*S||\\
&= \left|\left|S^*S - 2(S^*S)^2 + (S^*S)^3\right|\right| = 0.
\end{aligned}$$

Hence $S = SS^*S$.

(c) $\Longrightarrow$ (a) Let $S = SS^*S$. Recall that the projection P_M on a closed subspace M of $\mathcal{H}$ is the unique operator such that $P_M h \in M$ and $h - P_M h \perp M$ for all $h \in \mathcal{H}$. Now Let $h \in \mathcal{H}$. For $k \in \ker S$, we have $\langle S^*Sh, k\rangle = \langle Sh, Sk\rangle = 0$. So $S^*Sh \in (\ker S)^\perp$. Also for every $h \in \mathcal{H}$

$$\begin{aligned} S = SS^*S &\Longrightarrow S(h - S^*Sh) = (S - SS^*S)h = 0 \\ &\Longrightarrow h - S^*Sh \in \ker S \Longrightarrow h - S^*Sh \perp (\ker S)^\perp. \end{aligned}$$

Hence S^*S is the projection P_M onto $(\ker S)^\perp$. Now for $h \in (\ker S)^\perp$ we have

$$||Sh||^2 = \langle Sh, Sh\rangle = \langle S^*Sh, h\rangle = \langle h, h\rangle = ||h||^2.$$

This shows that S is a partial isometry. □

In general, in a C^*-algebra we have the following definition:

DEFINITION 6.1.3. In a C^*-algebra $\mathfrak{A}$, an element S in $\mathfrak{A}$ is called a *partial isometry* if it satisfies $S = SS^*S$ and an element P in $\mathfrak{A}$ is called a *projection* if it satisfies $P^2 = P = P^*$.

DEFINITION 6.1.4. Let S be a partial isometry, we call S^*S the *initial projection* and SS^* the *range projection* of S.

6.2. KMS States on the C^*-Algebras of Directed Graphs

In this section we define the Toeplitz algebras $\mathcal{T}C^*(E)$ for directed graphs and introduce KMS state on the C^*-dynamical system $(\mathcal{T}C^*(E), \alpha)$, where α is the representation induced by the gauge action of $\mathcal{T}C^*(E)$.

DEFINITION 6.2.1. Let E be a directed graph. A *Toeplitz-Cuntz-Krieger E-family* $\{S\}$ in a C^*-algebra A consists of projections $\{S_v \in A : v \in E^0\}$ satisfying $S_vS_w = 0$ for $v \neq w$ and partial isometries $\{S_e \in A : e \in E^1\}$ satisfying

(TCK1) $\qquad S_e^*S_e = S_{s(e)} \qquad$ for every $e \in E^1$; and

(TCK2) $$S_v \geq \sum_{\{e \in E^1 : r(e)=v\}} S_e S_e^*.$$

REMARK 6.2.1. The relation (TCK2) implies that the range projection $S_e S_e^*$ of S_e is dominated by $S_{r(e)}$, i.e., $S_e S_e^* \leq S_{r(e)}$. Thus we can conclude that range of $S_e S_e^*$ is contained in the range of $S_{r(e)}$ and we can write it as

(6.2.1) $$S_e = S_e S_{s(e)} = S_{r(e)} S_e.$$

The following example will ensure that for any row-finite directed graph there is at least one Toeplitz-Cuntz-Krieger E-family.

EXAMPLE 6.2.2. Let E be a row-finite directed graph. Let h_μ be the characteristic function for $\mu \in E^*$ on E^* and let $\mathcal{H} := \ell^2(E^*) = \overline{\text{span}}\{h_\mu : \mu \in E^*\}$. For $v \in E^0$ let S_v be the projection onto $\overline{\text{span}}\{h_\mu : r(\mu) = v\}$; more precisely S_v is given as

(6.2.2) $$S_v h_\mu = \begin{cases} h_\mu, & \text{if } r(\mu) = v \\ 0, & \text{otherwise} \end{cases}.$$

For $e \in E^1$ let S_e be the partial isometry in the C^*-algebra $\mathcal{B}(\mathcal{H})$ defined as

(6.2.3) $$S_e h_\mu = \begin{cases} h_{e\mu}, & \text{if } s(e) = r(\mu) \\ 0, & \text{otherwise.} \end{cases}$$

Then we claim that $\{S\}$ is a Toeplitz-Cuntz-Krieger E-family in the C^*-algebra $\mathcal{B}(\mathcal{H})$. In fact, in view of (6.2.3), S_e^* is given as:

(6.2.4) $$S_e^* h_\mu = \begin{cases} h_{\mu'}, & \text{if } \mu = e\mu' \\ 0, & \text{otherwise} \end{cases}.$$

Now let $\mu \in E^*$ and $e \in E^0$. Then

$$S_e^* S_e h_\mu = \begin{cases} S_e^* h_{e\mu}, & \text{if } s(e) = r(\mu) \\ 0, & \text{otherwise} \end{cases} \qquad \text{(by (6.2.3))}$$

$$= \begin{cases} h_\mu, & \text{if } s(e) = r(\mu) \\ 0, & \text{otherwise.} \end{cases} \qquad \text{(by (6.2.4))}$$

$$= S_{s(e)} h_\mu \qquad \text{(by (6.2.2))}.$$

Also,

$$S_e S_e^* h_\mu = \begin{cases} S_e h_{\mu'}, & \text{if } \mu = e\mu' \\ 0, & \text{otherwise} \end{cases} \qquad \text{(by (6.2.4))}$$

$$(6.2.5) \qquad = \begin{cases} h_{e\mu'}, & \text{if } \mu = e\mu' \\ 0, & \text{otherwise} \end{cases} \qquad (\text{as } \mu = e\mu' \implies r(\mu') = s(e))$$

$$= \begin{cases} h_\mu, & \text{if } \mu = e\mu' \\ 0, & \text{otherwise} \end{cases}.$$

This clearly shows that the range projection $S_e S_e^*$ is a projection onto $\overline{\text{span}}\{h_\mu : r(e) = r(\mu)\}$ and for $v \in E^0$, $\overline{\text{span}}\{h_\mu : r(e) = r(\mu) = v\}$ is a subset of $\overline{\text{span}}\{h_\mu : r(\mu) = v\}$. Hence for every $e \in E^1$ such that $r(e) = v$ the range of $S_e S_e^*$ is contained in the range of S_v. Thus

$$(6.2.6) \qquad S_v \geq S_e S_e^* \qquad \text{for every } e \in E^1 \text{ such that } r(e) = v.$$

Moreover, for $e, f \in vE^1$ such that $e \neq f$ and $h_\mu \in \mathcal{H}$, we have

$$\left\langle S_e S_e^* h_\mu, S_f S_f^* h_\mu \right\rangle = \begin{cases} \langle h_\mu, h_\mu \rangle, & \text{if } \mu = e\nu = f\sigma \\ 0, & \text{otherwise} \end{cases}$$

If $\mu = e\nu = f\sigma$, then e and f cannot be distinct edges, which is a contradiction. Hence $\left\langle S_e S_e^* h_\mu, S_f S_f^* h_\mu \right\rangle = 0$ and so $\{S_e S_e^* : e \in vE^1\}$ is a family of mutually orthogonal projections. This implies $\sum_{e \in vE^1} S_e S_e^*$ is a projection. Hence by (6.2.6), we get $S_v \geq \sum_{\{e \in E^1 : r(e) = v\}} S_e S_e^*$.

The following theorem from [**15**, Theorem 4.1] implies that there is a universal C^*-algebra generated by Toeplitz-Cuntz-Krieger E-families.

THEOREM 6.2.3. *Let E be a row-finite directed graph. Then there is a C^*-algebra, denoted by $\mathcal{T}C^*(E)$, generated by Toeplitz-Cuntz-Krieger E-family $\{s\}$ such that for every Toeplitz-Cuntz-Krieger E-family $\{Q\}$ in any C^*-algebra B, there is a homomorphism π_Q of $\mathcal{T}C^*(E)$ into B which satisfies $\pi_Q(s_e) = Q_e$ for every $e \in E^1$ and $\pi_Q(s_v) = Q_v$ for every $v \in E^0$.*

In view of the above theorem, we call the Toeplitz-Cuntz-Krieger E-family $\{s\}$ *having the universal property.* From now on we always denote the universal Toeplitz-Cuntz-Krieger E-family by the lowercase $\{s\}$ and a Toeplitz-Cuntz-Krieger E-family denoted by the uppercase $\{Q\}$.

DEFINITION 6.2.2. The C^*-algebra $\mathcal{T}C^*(E)$ generated by the universal Toeplitz-Cuntz-Krieger E-family $\{s\}$ is called the *Toeplitz algebra* of the graph E.

Let E be a row-finite directed graph. Let $\mu \in E^*$ be a path of length n. Then

$$\mu = \mu_1\mu_2\dots\mu_n, \qquad \text{where } \mu_i \in E^1 \text{ for every } i = 1, 2, \dots, n.$$

Let $\{S\}$ be a Toeplitz-Cuntz-Krieger E-family. Now we can extend the partial isometries for edges to the partial isometries for the paths by defining

$$S_\mu := S_{\mu_1}S_{\mu_2}\dots S_{\mu_n}.$$

In fact, repeated applications of (TCK1) give

$$\begin{aligned}
S_\mu^* S_\mu =& (S_{\mu_1}S_{\mu_2}\dots S_{\mu_n})^* S_{\mu_1}S_{\mu_2}\dots S_{\mu_n} \\
=& S_{\mu_n}^* S_{\mu_{n-1}}^* \dots S_{\mu_2}^*(S_{\mu_1}^* S_{\mu_1})S_{\mu_2}\dots S_{\mu_n} \\
=& S_{\mu_n}^* S_{\mu_{n-1}}^* \dots S_{\mu_2}^* S_{s(\mu_1)} S_{\mu_2}\dots S_{\mu_n} \\
=& S_{\mu_n}^* S_{\mu_{n-1}}^* \dots S_{\mu_2}^*(S_{r(\mu_2)} S_{\mu_2})\dots S_{\mu_n} && (\text{as } s(\mu_1) = r(\mu_2)) \\
=& S_{\mu_n}^* S_{\mu_{n-1}}^* \dots S_{\mu_2}^* S_{\mu_2}\dots S_{\mu_n} && (\text{using } (6.2.1))
\end{aligned}$$

$$\vdots$$

$$=S_{\mu_n}^* S_{\mu_n} = S_{s(\mu_n)} = S_{s(\mu)}$$

and then by Proposition 6.1.1 S_μ is a partial isometry.

Now we show that the range projections $S_e S_e^*$ are mutually orthogonal for any Toeplitz-Cuntz-Krieger E-family $\{S\}$. We need the following lemma in order to prove this.

LEMMA 6.2.4. *Let P and Q be two projections on a Hilbert space $\mathcal{H}$ such that $||P+Q|| \leq 1$. Then P and Q have orthogonal ranges.*

PROOF. Let $h \in P\mathcal{H}$. Then

$$\begin{aligned} 1 \geq ||P+Q|| \implies ||h||^2 &\geq ||Ph + Qh||^2 \\ &= ||h+Qh||^2 \\ &= \langle h+Qh, h+Qh\rangle \\ &= \langle h,h\rangle + \langle h,Qh\rangle + \langle Qh,Qh\rangle + \langle Qh,h\rangle \\ &= ||h||^2 + 3\langle Qh,Qh\rangle \qquad (\text{as } Q = Q^2 = Q^*) \\ &= ||h||^2 + 3\,||Qh||^2 . \end{aligned}$$

This implies $||Qh|| = 0$. So $Qh = 0$, and thus $QP = 0$. ☐

COROLLARY 6.2.5. *Let $\{S\}$ be a Toeplitz-Cuntz-Krieger E-family. Then the projections $\{S_e S_e^* : e \in E^1\}$ are mutually orthogonal.*

PROOF. Let $e, f \in E^1$ be such that $r(e) = r(f) = v$ for some $v \in E^0$. Then by (TCK2) we have $S_v \geq S_e S_e^* + S_f S_f^*$. Using Proposition 5.1.5, we get

$$||S_v|| \geq \left|\left|S_e S_e^* + S_f S_f^*\right|\right| \implies 1 \geq \left|\left|S_e S_e^* + S_f S_f^*\right|\right|.$$

Thus by Lemma 6.2.4, $S_e S_e^*$ and $S_f S_f^*$ are orthogonal.

Now if $e, f \in E^1$ such that $r(e) \neq r(f)$, then again using (TCK2) we have

$$S_{r(e)} \geq S_e S_e^* \qquad \text{and} \qquad S_{r(f)} \geq S_f S_f^*. \tag{6.2.7}$$

Since $S_{r(e)}$ and $S_{r(f)}$ are orthogonal, the inequalities in (6.2.7) shows that $S_e S_e^*$ and $S_f S_f^*$ are orthogonal. □

LEMMA 6.2.6. *Let $\{S\}$ be a Toeplitz-Cuntz-Krieger E-family. Then $S_e^* S_f = \delta_{e,f} S_{s(e)}$ for every $e, f \in E^1$, where $\delta_{e,f}$ is the Kronecker delta function.*

PROOF. Let $e, f \in E^1$. Then

$$\begin{aligned} S_e^* S_f &= S_e^* S_e S_e^* S_f S_f^* S_f \qquad (\text{as } S_e = S_e S_e^* S_e) \\ &= \begin{cases} S_e^* 0 S_f, & e \neq f \qquad (\text{using Corollary 6.2.5}) \\ S_e^* S_e, & e = f \qquad (\text{as } S_e = S_e S_e^* S_e). \end{cases} \\ &= \begin{cases} 0, & e \neq f \\ S_{s(e)}, & e = f \qquad (\text{using (TCK1)}). \end{cases} \end{aligned}$$

Hence $S_e^* S_f = \delta_{e,f} S_{s(e)}$. □

The following corollary gives us a product formula for range projections.

COROLLARY 6.2.7. *Let $\{S\}$ be a Toeplitz-Cuntz-Krieger E-family in a C^*-algebra B. Then for $\mu, \nu, \sigma, \tau \in E^*$, we have*

$$(S_\mu S_\nu^*)(S_\sigma S_\tau^*) = \begin{cases} S_{\mu\sigma'} S_\tau^*, & \text{if } \sigma = \nu\sigma' \\ S_\mu S_{\tau\nu'}^*, & \text{if } \nu = \sigma\nu' \\ 0, & \text{otherwise.} \end{cases} \tag{6.2.8}$$

This is called the product formula.

PROOF. Let $\mu, \nu, \sigma, \tau \in E^*$. Then

Case 1: $|\nu| \leq |\sigma|$.

Let $\sigma = \alpha\sigma'$, where $|\alpha| = |\nu|$. Then

$$(S_\mu S_\nu^*)(S_\sigma S_\tau^*) = (S_\mu S_\nu^*)(S_\alpha S_{\sigma'} S_\tau^*) \qquad (\text{as } \sigma = \alpha\sigma').$$

If $\nu = \alpha$, then

$$\begin{aligned}(S_\mu S_\nu^*)(S_\sigma S_\tau^*) =& S_\mu (S_\nu^* S_\nu) S_{\sigma'} S_\tau^* \\ =& S_\mu S_{s(\nu)} S_{\sigma'} S_\tau^* \\ =& S_\mu S_{r(\sigma')} S_{\sigma'} S_\tau^* \qquad (\text{as } \sigma = \alpha\sigma' \implies s(\nu) = s(\alpha) = r(\sigma')) \\ =& S_\mu S_{\sigma'} S_\tau^* \qquad (\text{using (6.2.1)}) \\ =& S_{\mu\sigma'} S_\tau^*.\end{aligned}$$

If $\nu \neq \alpha$, then suppose that $\nu = \nu_1\nu_2 \ldots \nu_n$ and $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$. Let i be the smallest integer such that $\nu_i \neq \alpha_i$. Now

$$\begin{aligned}S_\nu^* S_\alpha =& (S_{\nu_1} S_{\nu_2} \ldots S_{\nu_n})^* S_{\alpha_1} S_{\alpha_2} \ldots S_{\alpha_n} \\ =& S_{\nu_n}^* S_{\nu_{n-1}}^* \ldots S_{\nu_2}^* S_{\nu_1}^* S_{\alpha_1} S_{\alpha_2} \ldots S_{\alpha_n} \\ =& S_{\nu_n}^* S_{\nu_{n-1}}^* \ldots S_{\nu_i}^* S_{s(\nu_{i-1})} S_{\alpha_i} \ldots S_{\alpha_n} \qquad (\text{using (TCK1) and (6.2.1)}) \\ =& S_{\nu_n}^* S_{\nu_{n-1}}^* \ldots S_{\nu_i}^* S_{r(\nu_i)} S_{\alpha_i} \ldots S_{\alpha_n} \\ =& S_{\nu_n}^* S_{\nu_{n-1}}^* \ldots S_{\nu_i}^* S_{\alpha_i} \ldots S_{\alpha_n} \qquad (\text{using (6.2.1)}) \\ =& 0 \qquad (\text{using Lemma 6.2.6}).\end{aligned} \tag{6.2.9}$$

Case 2: $|\sigma| < |\nu|$.

Now take $\nu = \beta\nu'$, where $|\beta| = |\sigma|$ and repeat a similar argument as in Case 1. One can get $(S_\mu S_\nu^*)(S_\sigma S_\tau^*) = S_\mu S_{\tau\nu'}^*$ if $\beta = \sigma$ and $(S_\mu S_\nu^*)(S_\sigma S_\tau^*) = 0$ if $\beta \neq \sigma$.

Hence (6.2.8) is proved. □

The C^*-algebra generated by the universal Toeplitz-Cuntz-Krieger E-family $\{s\}$ is same as the C^*-algebra generated by the family $\{s_\mu s_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}$. We have the following lemma from [**18**, Lemma 3.3] which will give us a clear structure of the C^*-algebra $\mathcal{T}C^*(E)$ for any directed graph E.

LEMMA 6.2.8. *Let $\mathcal{A}$ be a C^*-algebra and $X \subseteq \mathcal{A}$. Then there is a C^*-algebra $C^*(X)$ generated by the X, which is the intersection of all C^*-subalgebras containing X. If X is closed under multiplication and involution, then $C^*(X) = \overline{\operatorname{span}} X$.*

REMARK 6.2.9. In view of the product formula in Corollary 6.2.7 we can see that the family $\{s_\mu s_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}$ is closed under multiplication; and clearly is closed under involution. Thus by Lemma 6.2.8 we can have

$$\mathcal{T}C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}.$$

Now we introduce Cuntz-Krieger E-family for a directed graph E and the C^*-algebra $C^*(E)$ generated by the universal Cuntz-Krieger E-family. Later we characterize the relation between $\mathcal{T}C^*(E)$ and $C^*(E)$.

DEFINITION 6.2.3. Let E be a directed graph. If a Toeplitz-Cuntz-Krieger E-family $\{S\}$ also satisfies

$$\text{(CK2)} \qquad S_v = \sum_{e \in vE^1} S_e S_e^* \qquad \text{for every } v \in E^0,$$

then $\{S\}$ is called a *Cuntz-Krieger E-family*.

As with the Toeplitz algebra $\mathcal{T}C^*(E)$ there is a C^*-algebra $C^*(E)$ generated by the universal Cuntz-Krieger E-family $\{s\}$.

The following result from [**8**, Lemma 2.6] tells us about the relation between $\mathcal{T}C^*(E)$ and $C^*(E)$.

PROPOSITION 6.2.10. *Let $\mathcal{T}C^*(E)$ be the Toeplitz algebra generated by the Toeplitz-Cuntz-Krieger E-family $\{s\}$. Let J be the closed two-sided ideal generated by $\{s_v - \sum_{e \in vE^1} s_e s_e^* : v \in E^0\}$ and $q : \mathcal{T}C^*(E) \to \mathcal{T}C^*(E)/J$ be the quotient map. Let $\bar{s}_e = q(s_e)$ for every $e \in E^1$ and $\bar{s}_v = q(s_v)$ for every $v \in E^0$. Then*

(i) $\{\bar{s}\}$ is a Cuntz-Krieger E-family which generates $\mathcal{T}C^*(E)/J$, and

(ii) if $\{S\}$ is a Cuntz-Krieger E-family in a C^*-algebra B, then there exists a homomorphism $\bar{\pi}_S : \mathcal{T}C^*(E)/J \to B$ which satisfies $\bar{\pi}_S(s_e) = S_e$ for every $e \in E^1$ and $\bar{\pi}_S(s_v) = S_v$ for every $v \in E^0$.

Remark 6.2.11. The above proposition shows that $C^*(E)$ is isomorphic to $\mathcal{T}C^*(E)/J$.

Definition 6.2.4. An action γ of $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ on the Toeplitz algebra $\mathcal{T}C^*(E)$ defined as $\gamma_z(s_\mu s_\nu^*) = z^{|\mu|-|\nu|} s_\mu s_\nu^*$ is called the *gauge action* of $\mathcal{T}C^*(E)$.

The γ defined in Definition 6.2.4 is an action because it satisfies (a) $\gamma_z \gamma_w = \gamma_{zw}$ for all $z, w \in \mathbb{T}$; (b) $\gamma_1 = I$; and (c) $z \mapsto \gamma_z(a)$ is continuous for each fixed $a \in \mathcal{T}C^*(E)$.

We define a representation $\alpha : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(E)$ such that $\alpha_t := \gamma_{\mathrm{e}^{it}}$, where γ is the gauge action of $\mathcal{T}C^*(E)$. Then $(\mathcal{T}C^*(E), \alpha)$ is a C^*-dynamical system. From now on we reserve the notation $(\mathcal{T}C^*(E), \alpha)$ for this C^*-dynamical system.

Note 6.2.12. Note that the representation $\alpha : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(E)$ can be extended from $\mathbb{R}$ to a representation on $\mathbb{C}$. In that case we define $\alpha_z := \gamma_{\mathrm{e}^{iz}}$ for every $z \in \mathbb{C}$.

Remark 6.2.13. From Note 5.2.8, we can conclude that if ω is a KMS$_\beta$ state on $(\mathcal{T}C^*(E), \alpha)$, then $\omega(\alpha_z(s_\mu s_\nu^*)) = \omega(s_\mu s_\nu^*)$ for every $\mu, \nu \in E^*$ and $z \in \mathbb{C}$. Now let $-i \ln z \in \mathbb{C}$. Then

$$\omega(s_\mu s_\nu^*) = \omega(\alpha_{-i \ln z}(s_\mu s_\nu^*)) = \omega(\gamma_{\mathrm{e}^{i(-i \ln z)}}(s_\mu s_\nu^*) = \omega(\gamma_z(s_\mu s_\nu^*)).$$

This shows that ω is γ-invariant. Note that for $z \in \mathbb{C}$, $\ln z = \ln |z| + i \operatorname{Arg}(z)$.

In what follows, we shall study two important results related to KMS state of $\mathcal{T}C^*(E)$. However Lemma 5.2.7 plays vital role in these results. So we assume $\beta \in \mathbb{R} \setminus \{0\}$ for the following two results.

Lemma 6.2.14. *A state ω on $(\mathcal{T}C^*(E), \alpha)$ is a KMS$_\beta$ state if, and only if,*

$$\omega\big((s_\mu s_\nu^*)(s_\sigma s_\tau^*)\big) = \mathrm{e}^{-\beta(|\mu|-|\nu|)}\, \omega\big((s_\sigma s_\tau^*)(s_\mu s_\nu^*)\big).$$

PROOF. In view of Proposition 5.2.9, it is sufficient to show that

$$e^{-\beta(|\mu|-|\nu|)}\,\omega\big((s_\sigma s_\tau^*)(s_\mu s_\nu^*)\big) = \omega\big(\alpha_{-i\frac{\beta}{2}}(s_\sigma s_\tau^*)\alpha_{i\frac{\beta}{2}}(s_\mu s_\nu^*)\big).$$

Now

$$\begin{aligned} e^{-\beta(|\mu|-|\nu|)}\,\omega\big((s_\sigma s_\tau^*)(s_\mu s_\nu^*)\big) =& \omega\big((s_\sigma s_\tau^*)\, e^{-\beta(|\mu|-|\nu|)}(s_\mu s_\nu^*)\big)\\ =&\omega\big((s_\sigma s_\tau^*)\gamma_{e^{-\beta}}(s_\mu s_\nu^*)\big)\\ =&\omega\big((s_\sigma s_\tau^*)\alpha_{i\beta}(s_\mu s_\nu^*)\big)\\ =&\omega\Big(\alpha_{i\frac{\beta}{2}}\big(\alpha_{-i\frac{\beta}{2}}(s_\sigma s_\tau^*)\alpha_{i\frac{\beta}{2}}(s_\mu s_\nu^*)\big)\Big)\\ =&\omega\big(\alpha_{-i\frac{\beta}{2}}(s_\sigma s_\tau^*)\alpha_{i\frac{\beta}{2}}(s_\mu s_\nu^*)\big) \qquad \text{(using Note 5.2.8)}. \end{aligned}$$

□

PROPOSITION 6.2.15. *A state ω is a KMS$_\beta$ state on $(\mathcal{T}C^*(E),\alpha)$ if, and only if,*

$$\omega(s_\mu s_\nu^*) = \delta_{\mu,\nu}\, e^{-\beta|\mu|}\,\omega(s_{s(\mu)}). \tag{6.2.10}$$

PROOF. First suppose that ω is a KMS$_\beta$ state on $(\mathcal{T}C^*(E),\alpha)$.
Case 1: Let $\mu=\nu$. Then using (TCK1), we get $s_\nu^* s_\mu = s_\mu^* s_\mu = s_{s(\mu)}$. Since ω satisfies KMS condition, we have

$$\omega(s_\mu s_\nu^*) = \omega(s_\nu^*\alpha_{i\beta}(s_\mu)) = e^{-\beta|\mu|}\,\omega(s_\nu^* s_\mu) = e^{-\beta|\mu|}\,\omega(s_{s(\mu)}).$$

Case 2: Let $\mu\neq\nu$.

Sub-case 2.1: Let $|\mu|=|\nu|$. Then repeating the same argument as in (6.2.9) one can get $s_\nu^* s_\mu = 0$. So

$$\omega(s_\mu s_\nu^*) = \omega(s_\nu^*\alpha_{i\beta}(s_\mu)) = e^{-\beta|\mu|}\,\omega(s_\nu^* s_\mu) = 0.$$

Sub-case 2.2: Let $|\mu|\neq|\nu|$. From Remark 6.2.13 we have that ω is γ-invariant. Then

$$\omega(s_\mu s_\nu^*) = \omega(\gamma_z(s_\mu s_\nu^*)) = z^{|\mu|-|\nu|}\omega(s_\mu s_\nu^*)$$

$$\implies \int_{\mathbb{T}} \omega(s_\mu s_\nu^*)dz = \int_{\mathbb{T}} z^{|\mu|-|\nu|}\omega(s_\mu s_\nu^*)dz$$

$$\implies \omega(s_\mu s_\nu^*)\int_{\mathbb{T}} dz = \omega(s_\mu s_\nu^*)\int_{\mathbb{T}} z^{|\mu|-|\nu|}dz$$

$$\implies \omega(s_\mu s_\nu^*) = 0 \qquad \text{(using Cauchy's Integral Theorem).}$$

Hence $\omega(s_\mu s_\nu^*) = \delta_{\mu,\nu}\, \mathrm{e}^{-\beta|\mu|}\, \omega(s_{s(\mu)})$.

Now suppose that ω is a state on $\mathcal{T}C^*(E)$ satisfying 6.2.10. In view of Lemma 6.2.14 it is sufficient to show that

$$\omega\big((s_\mu s_\nu^*)(s_\sigma s_\tau^*)\big) = \mathrm{e}^{-\beta(|\mu|-|\nu|)}\, \omega\big((s_\sigma s_\tau^*)(s_\mu s_\nu^*)\big)$$

Let $\mu, \nu, \sigma, \tau \in E^*$. Then using the product formula from (6.2.8) one can get

$$\omega((s_\mu s_\nu^*)(s_\sigma s_\tau^*)) = \begin{cases} \omega(s_{\mu\sigma'} s_\tau^*), & \text{if } \sigma = \nu\sigma' \\ \omega(s_\mu s_{\tau\nu'}^*), & \text{if } \nu = \sigma\nu' \\ 0, & \text{otherwise.} \end{cases}$$

Since $\omega(s_\mu s_\nu^*) = \delta_{\mu,\nu}\, \mathrm{e}^{-\beta|\mu|}\, \omega(s_{s(\mu)})$, we get

$$\omega((s_\mu s_\nu^*)(s_\sigma s_\tau^*)) = \begin{cases} e^{-\beta|\tau|}\omega(s_{s(\tau)}), & \text{if } \sigma = \nu\sigma' \text{ and } \tau = \mu\sigma' \\ e^{-\beta|\mu|}\omega(s_{s(\mu)}), & \text{if } \nu = \sigma\nu' \text{ and } \mu = \tau\nu' \\ 0, & \text{otherwise.} \end{cases} \tag{6.2.11}$$

Similarly,

$$\omega((s_\sigma s_\tau^*)(s_\mu s_\nu^*)) = \begin{cases} e^{-\beta|\sigma|}\omega(s_{s(\sigma)}), & \text{if } \tau = \mu\sigma' \text{ and } \sigma = \nu\sigma' \\ e^{-\beta|\nu|}\omega(s_{s(\nu)}), & \text{if } \mu = \tau\nu' \text{ and } \nu = \sigma\nu' \\ 0, & \text{otherwise.} \end{cases}$$

$$\implies \mathrm{e}^{-\beta(|\mu|-|\nu|)}\,\omega\big((s_\sigma s_\tau^*)(s_\mu s_\nu^*)\big) = \begin{cases} \mathrm{e}^{-\beta(|\mu|-|\nu|)}\, e^{-\beta|\sigma|}\omega\big(s_{s(\sigma)}\big), & \text{if } \tau=\mu\sigma' \text{ and } \sigma=\nu\sigma' \\ \mathrm{e}^{-\beta(|\mu|-|\nu|)}\, e^{-\beta|\nu|}\omega\big(s_{s(\nu)}\big), & \text{if } \mu=\tau\nu' \text{ and } \nu=\sigma\nu' \\ 0, & \text{otherwise.} \end{cases}$$

$$(6.2.12) \qquad = \begin{cases} \mathrm{e}^{-\beta(|\mu|-|\nu|+|\sigma|)}\,\omega\big(s_{s(\sigma)}\big), & \text{if } \tau=\mu\sigma' \text{ and } \sigma=\nu\sigma' \\ \mathrm{e}^{-\beta|\mu|}\,\omega\big(s_{s(\nu)}\big), & \text{if } \mu=\tau\nu' \text{ and } \nu=\sigma\nu' \\ 0, & \text{otherwise.} \end{cases}$$

Case 1: $\tau=\mu\sigma'$ and $\sigma=\nu\sigma'$, then $s(\tau)=s(\sigma')=s(\sigma)$ and

$$|\mu|-|\nu| = |\mu\sigma'|-|\nu\sigma'| = |\tau|-|\sigma| \implies |\tau| = |\mu|-|\nu|+|\sigma|.$$

So by (6.2.12), we get

$$\begin{aligned} \mathrm{e}^{-\beta(|\mu|-|\nu|)}\,\omega\big((s_\sigma s_\tau^*)(s_\mu s_\nu^*)\big) &= \mathrm{e}^{-\beta(|\mu|-|\nu|+|\sigma|)}\,\omega\big(s_{s(\sigma)}\big) \\ &= \mathrm{e}^{-\beta|\tau|}\,\omega\big(s_{s(\tau)}\big) = \omega\big((s_\mu s_\nu^*)(s_\sigma s_\tau^*)\big) \qquad \text{(by (6.2.11)).} \end{aligned}$$

Case 2: $\mu=\tau\nu'$ and $\nu=\sigma\nu'$, then $s(\mu)=s(\mu')=s(\nu)$. So by (6.2.12), we get

$$\mathrm{e}^{-\beta(|\mu|-|\nu|)}\,\omega\big((s_\sigma s_\tau^*)(s_\mu s_\nu^*)\big) = \mathrm{e}^{-\beta|\mu|}\,\omega\big(s_{s(\nu)}\big) = \mathrm{e}^{-\beta|\mu|}\,\omega\big(s_{s(\mu)}\big) \qquad \text{(by (6.2.11)).}$$

Case 3: $\omega\big((s_\mu s_\nu^*)(s_\sigma s_\tau^*)\big) = e^{-\beta(|\mu|-|\nu|)}\omega\big((s_\sigma s_\tau^*)(s_\mu s_\nu^*)\big) = 0.$

Hence $\omega\big((s_\mu s_\nu^*)(s_\sigma s_\tau^*)\big) = \mathrm{e}^{-\beta(|\mu|-|\nu|)}\,\omega\big((s_\sigma s_\tau^*)(s_\mu s_\nu^*)\big).$ □

6.3. KMS States on the C^*-Algebras of k-Graphs

In this section we introduce the Toeplitz algebras $\mathcal{T}C^*(\Lambda)$ for k-Graphs and then study KMS states on the C^*-dynamical system $(\mathcal{T}C^*(\Lambda), \alpha^r)$. This section is referred from [**3**], [**8**] and [**10**].

NOTATION 6.3.1. Let $\vec{m}=(m_1,m_2,\dots,m_k), \vec{n}=(n_1,n_2,\dots,n_k)\in\mathbb{R}^k$. Then we use the notation

$$\vec{m}\vee\vec{n} = \big(\max(m_1,n_1), \max(m_2,n_2), \dots, \max(m_k,n_k)\big),$$

$$\vec{m} \wedge \vec{n} = \big(\min(m_1, n_1), \min(m_2, n_2), \ldots, \min(m_k, n_k)\big).$$

DEFINITION 6.3.1. For $\mu, \nu \in \Lambda$, we define the *minimal extension* for the ordered pair (μ, ν) as

$$\Lambda^{\min}(\mu, \nu) = \{(\eta, \zeta) : \eta, \zeta \in \Lambda, \mu\eta = \nu\zeta \text{ and } d(\mu\eta) = d(\mu) \vee d(\nu)\}.$$

LEMMA 6.3.1. *For $m, n, p, q \in \mathbb{N}^k$ such that $m + p = n + q$. Then $m + p = m \vee n$ if, and only if, $p \wedge q = 0$.*

PROOF. Let us suppose $m + p = m \vee n$. Then for each $i = 1, 2, \ldots, k$

$$p_i = (m \vee n)_i - m_i = \max(m_i, n_i) - m_i = \begin{cases} 0, & \text{if } n_i < m_i \\ n_i - m_i, & \text{if } n_i \geq m_i \end{cases}.$$

Thus, $$q_i = m_i - n_i + p_i = \begin{cases} m_i - n_i, & \text{if } n_i < m_i \\ 0, & \text{if } n_i \geq m_i \end{cases}.$$

So $\min(p_i, q_i) = 0$ for every $i = 1, 2, \ldots, k$ which implies $p \wedge q = 0$.

Conversely, suppose that $p \wedge q = 0$, i.e., $\min(p_i, q_i) = 0$ for every $i = 1, 2, \ldots, k$. If $p_i \neq 0$, then $q_i = 0$. So $m_i + p_i = n_i$ implies $m_i + p_i = \max(m_i, n_i)$, i.e., $m + p = m \vee n$. If $p_i = 0$, then $m_i = m_i + p_i = n_i + q_i$ implies $m_i + p_i = \max(m_i, n_i)$, i.e., $m + p = m \vee n$. □

LEMMA 6.3.2. *Let $(\eta, \zeta) \in \Lambda^{\min}(\mu, \nu)$ for $\mu, \nu \in \Lambda$. Then $d(\eta) \wedge d(\zeta) = 0$.*

PROOF. Let $(\eta, \zeta) \in \Lambda^{\min}(\mu, \nu)$. Then $\mu\eta = \nu\zeta$ and $d(\mu\eta) = d(\nu\zeta) = d(\mu) \vee d(\nu)$. This implies

$$d(\mu) + d(\eta) = d(\nu) + d(\zeta) \qquad \text{and} \qquad d(\mu) + d(\eta) = d(\mu) \vee d(\nu).$$

Then by Lemma 6.3.1, we conclude $d(\eta) \wedge d(\zeta) = 0$. □

DEFINITION 6.3.2. Let Λ be a k-graph. A *Toeplitz-Cuntz-Krieger* Λ*-family* $\{T_\lambda\}$ consists of partial isometries $\{T_\lambda : \lambda \in \Lambda\}$ such that

(T1) $\{T_v : v \in \Lambda^0\}$ are mutually orthogonal projections;

(T2) $T_\lambda T_\mu = T_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;

(T3) $T_\lambda^* T_\lambda = T_{s(\lambda)}$ for every $\lambda \in \Lambda$;

(T4) for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k,$ we have $T_v \geq \sum_{\lambda \in v\Lambda^n} T_\lambda T_\lambda^*$;

(T5) for every $\mu, \nu \in \Lambda,$ we have $T_\mu^* T_\nu = \sum_{(\eta,\zeta)\in\Lambda^{\min}(\mu,\nu)} T_\eta T_\zeta^*.$

REMARK 6.3.3. (1) In (T5) we consider any empty sum as 0.

(2) The relation (T4) implies that the range projection $T_\lambda T_\lambda^*$ of T_λ is dominated by $T_{r(\lambda)}$, i.e., $T_\lambda T_\lambda^* \leq T_{r(\lambda)}$. Thus we can conclude that range of $T_\lambda T_\lambda^*$ is contained in the range of $T_{r(\lambda)}$, i.e.,

$$T_\lambda = T_\lambda T_{s(\lambda)} = T_{r(\lambda)} T_\lambda. \tag{6.3.1}$$

As Example 6.2.2 for directed graphs, we have a similar example for k-graphs (see [**16**, Example 7.4]).

PROPOSITION 6.3.4. *Let Λ be a k-graph. Let h_λ be the characteristic function on Λ for $\lambda \in \Lambda$ and $\mathcal{H} = \ell^2(\Lambda)$. Let T_μ be a partial isometry defined by*

$$T_\mu h_\lambda = \begin{cases} h_{\mu\lambda}, & s(\mu) = r(\lambda) \\ 0, & \textit{otherwise.} \end{cases}$$

Then $\{T_\mu : \mu \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger Λ-family.

The following proposition from [**17**, Theorem 3.1.5] implies that there is a universal C^*-algebra generated by Toeplitz-Cuntz-Krieger Λ-families.

PROPOSITION 6.3.5. *Let Λ be a k-graph. Then there exists a C^*-algebra denoted by $\mathcal{T}C^*(\Lambda)$ generated by a Toeplitz-Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ such that for* *every Toeplitz-Cuntz-Krieger Λ-family $\{T_\lambda : \lambda \in \Lambda\}$ in a C^*-algebra B, there exists a homomorphism π_T of $\mathcal{T}C^*(\Lambda)$ into B which satisfies $\pi_T(t_\lambda) = T_\lambda$ for every $\lambda \in \Lambda$.*

In view of the above proposition, we call the Toeplitz-Cuntz-Krieger Λ-family $\{t_\lambda\}$ *having the universal property*. From now on we always denote the universal Toeplitz-Cuntz-Krieger Λ-family by the lower case family $\{t_\lambda : \lambda \in \Lambda\}$.

Definition 6.3.3. In view of the above proposition we say that $\mathcal{T}C^*(\Lambda)$ has the *universal property*, and $\mathcal{T}C^*(\Lambda)$ is called the *Toeplitz algebra* of Λ.

Lemma 6.3.6. *Let $\{T_\lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. Then $\{T_\lambda T_\lambda^* : \lambda \in \Lambda^n\}$ are mutually orthogonal for each $n \in \mathbb{N}^k$.*

Proof. Let $\lambda, \mu \in \Lambda^n$. If $r(\lambda) = r(\mu) = v$ for some $v \in \Lambda^0$, then from (T4) we get

$$T_v \geq T_\lambda T_\lambda^* + T_\mu T_\mu^* \implies 1 \geq \left|\left|T_\lambda T_\lambda^* + T_\mu T_\mu^*\right|\right| \qquad \text{(by Proposition 5.1.5).}$$

Thus Lemma 6.2.4 implies that $T_\lambda T_\lambda^*$ and $T_\mu T_\mu^*$ are orthogonal.

Now if $r(\lambda) \neq r(\mu)$, then again using (T4) we have

$$T_{r(\lambda)} \geq T_\lambda T_\lambda^* \qquad \text{and} \qquad T_{r(\mu)} \geq T_\mu T_\mu^*.$$

Since $T_{r(\lambda)}$ and $T_{r(\mu)}$ are orthogonal, the above inequalities show that $T_\lambda T_\lambda^*$ and $T_\mu T_\mu^*$ are orthogonal. □

Lemma 6.3.7. *For $\mu, \nu \in \Lambda$ such that $d(\mu) = d(\nu)$, we have*

$$T_\mu^* T_\nu = \delta_{\mu,\nu} T_{s(\mu)}.$$

Proof. Let $\mu, \nu \in \Lambda$ such that $d(\mu) = d(\nu)$. Then

$$\begin{aligned} T_\mu^* T_\nu =& T_\mu^* T_\mu T_\mu^* T_\nu T_\nu^* T_\nu \qquad \text{(as } T_\mu = T_\mu T_\mu^* T_\mu) \\ =& \begin{cases} T_\mu^* 0 T_\nu, & \text{if } \mu \neq \nu \qquad \text{(by Lemma 6.3.6)} \\ T_\mu^* T_\mu, & \text{if } \mu = \nu \qquad \text{(as } T = T_\mu T_\mu^* T_\mu) \end{cases} \end{aligned}$$

$$= \begin{cases} 0, & \text{if } \mu \neq \nu \\ T_{s(\mu)}, & \text{if } \mu = \nu \qquad \text{(using (T3))}. \end{cases}$$

Hence $T_\mu^* T_\nu = \delta_{\mu,\nu} T_{s(\mu)}$. □

LEMMA 6.3.8. *Let $\{T_\lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family in a C^*-algebra B. Then for $\mu, \nu, \sigma, \tau \in \Lambda$ we have*

$$(T_\mu T_\nu^*)(T_\sigma T_\tau^*) = \sum_{(\eta,\zeta)\in\Lambda^{\min}(\nu,\sigma)} T_{\mu\eta} T_{\tau\zeta}^*.$$

PROOF. Let $\mu, \nu, \sigma, \tau \in \Lambda$. Then

$$\begin{aligned}(T_\mu T_\nu^*)(T_\sigma T_\tau^*) = T_\mu (T_\nu^* T_\sigma) T_\tau^* &= T_\mu \Big(\sum_{(\eta,\zeta)\in\Lambda^{\min}(\nu,\sigma)} T_\eta T_\zeta^* \Big) T_\tau^* \qquad \text{(using (T5))} \\ &= \sum_{(\eta,\zeta)\in\Lambda^{\min}(\nu,\sigma)} T_\mu T_\eta T_\zeta^* T_\tau^* \\ &= \sum_{(\eta,\zeta)\in\Lambda^{\min}(\nu,\sigma)} T_{\mu\eta} T_{\tau\zeta}^*.\end{aligned}$$

□

REMARK 6.3.9. In view of the product formula in the above lemma and using Lemma 6.2.8, we can conclude that

$$\mathcal{T}C^*(\Lambda) = \overline{span}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda\}.$$

Now we introduce Cuntz-Krieger Λ-family for a k-graph Λ and the C^*-algebra $C^*(\Lambda)$ generated by a universal Cuntz-Krieger Λ-family. Later we characterize the relation between $\mathcal{T}C^*(\Lambda)$ and $C^*(\Lambda)$.

DEFINITION 6.3.4. A Toeplitz-Cuntz-Krieger Λ-family is said to be a *Cuntz-Krieger Λ-family* if (T4) is replaced by

(CK) for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have $T_v = \sum_{\lambda\in v\Lambda^n} T_\lambda T_\lambda^*$.

As with the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ there is a C^*-algebra $C^*(\Lambda)$ generated by the universal Cuntz-Krieger Λ-family $\{t_\lambda\}$.

The following proposition proved in [**8**, Lemma 5.4] gives a relation between $\mathcal{T}C^*(\Lambda)$ and $C^*(\Lambda)$.

Proposition 6.3.10. *Let $\mathcal{T}C^*(\Lambda)$ be the Toeplitz algebra generated by the universal Toeplitz-Cuntz-Krieger Λ-family $\{t_\lambda\}$. Let J be the two-sided closed ideal generated by $\{t_v - \sum_{\lambda\in v\Lambda^{e_i}} t_\lambda t_\lambda^* : v \in \Lambda^0, i = 1, 2, \ldots, k\}$ and $q : \mathcal{T}C^*(\Lambda) \to \mathcal{T}C^*(\Lambda)/J$ be the quotient map. Let $\bar{t}_\lambda = q(t_\lambda)$. Then*

(i) $\{\bar{t}_\lambda\}$ is a Cuntz-Krieger Λ-family which generates $\mathcal{T}C^*(\Lambda)/J$, and

(ii) if $\{T_\lambda\}$ is a Cuntz-Krieger Λ-family in a C^*-algebra B, then there exists a homomorphism $\bar{\pi}_T : \mathcal{T}C^*(\Lambda)/J \to B$ which satisfies $\bar{\pi}_T(\bar{t}_\lambda) = T_\lambda$ for every $\lambda \in \Lambda$.

Remark 6.3.11. The above proposition shows that a $C^*(\Lambda)$ is isomorphic to a quotient of $\mathcal{T}C^*(\Lambda)$.

We introduce a representation α^r on $\mathcal{T}C^*(\Lambda)$ and then we study the KMS states of the C^*-dynamical system $(\mathcal{T}C^*(\Lambda), \alpha^r)$.

Definition 6.3.5. Define an action γ of $\mathbb{T}^k$ on $\mathcal{T}C^*(\Lambda)$ such that $\gamma_z \in \operatorname{Aut} \mathcal{T}C^*(\Lambda)$ defined as

$$\gamma_z(t_\mu t_\nu^*) = z^{d(\mu)-d(\nu)} t_\mu t_\nu^*,$$

where $z^{d(\mu)-d(\nu)} = \prod_{i=1}^{k} z_i^{d(\mu)_i - d(\nu)_i}$ (the multi-index notation is used). This action γ is called the *gauge action* of $\mathcal{T}C^*(\Lambda)$.

Definition 6.3.6. Let $r \in (0, \infty)^k$. We define a representation $\alpha^r : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(\Lambda)$ such that $\alpha_t^r := \gamma_{e^{itr}}$, where γ is the gauge action of $\mathcal{T}C^*(\Lambda)$. Then $(\mathcal{T}C^*(\Lambda), \alpha^r)$ is a C^*-dynamical system.

Note 6.3.12. (1) As defined in Definition 6.3.6,

$$\alpha_t^r(t_\mu t_\nu^*) = \gamma_{e^{itr}}(t_\mu t_\nu^*) = e^{itr\cdot(d(\mu)-d(\nu))}\, t_\mu t_\nu^*,$$

where $r \cdot (d(\mu) - d(\nu)) = \sum_{i=1}^{k} r_i(d(\mu)_i - d(\nu)_i)$.

(2) For $t \in \mathbb{R}$ the map $t \mapsto \mathrm{e}^{itr\cdot\left(d(\mu)-d(\nu)\right)}\, t_\mu t_\nu^*$ extends to an analytic function $z \mapsto \mathrm{e}^{izr\cdot\left(d(\mu)-d(\nu)\right)}\, t_\mu t_\nu^*$ for $z \in \mathbb{C}$.

From now on we use the notation $(\mathcal{T}C^*(\Lambda), \alpha^r)$ for the above defined C^*-dynamical system unless specified.

PROPOSITION 6.3.13. *Let $\beta \in [0, \infty)$, $r \in (0,\infty)^k$ and ω be a state on $\mathcal{T}C^*(\Lambda)$.*

(a) If ω is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$, then for every $\mu, \nu \in \Lambda$ with $d(\mu) = d(\nu)$

$$\omega(t_\mu t_\nu^*) = \delta_{\mu,\nu}\, \mathrm{e}^{-\beta r\cdot d(\mu)}\, \omega(t_{s(\mu)}).$$

(b) If

$$\omega(t_\mu t_\nu^*) = \delta_{\mu,\nu}\, \mathrm{e}^{-\beta r\cdot d(\mu)}\, \omega(t_{s(\mu)}) \text{ for every } \mu, \nu \in \Lambda, \tag{6.3.2}$$

then ω is a KMS$_\beta$ state of $\mathcal{T}C^*(\Lambda)$.

(c) If $r \in (0,\infty)^k$ has rationally independent coordinates, then ω is a KMS$_\beta$ of $(\mathcal{T}C^*(\Lambda), \alpha^r)$ if, and only if, (6.3.2) *holds.*

PROOF. (a) Let us suppose ω is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$. Let $\mu, \nu \in \Lambda$ with $d(\mu) = d(\nu)$. Then using Lemma 6.3.7, we get

$$\omega(t_\mu t_\nu^*) = \omega(t_\nu^* \alpha^r_{i\beta}(t_\mu)) = \mathrm{e}^{-\beta r\cdot d(\mu)}\, \omega(t_\nu^* t_\mu) = \delta_{\mu,\nu}\, \mathrm{e}^{-\beta r\cdot d(\mu)}\, \omega(t_{s(\mu)}).$$

(b) Now suppose that ω is a state of $\mathcal{T}C^*(\Lambda)$ which satisfies (6.3.2).
For $\mu, \nu \in \Lambda$ such that $s(\mu) \neq s(\nu)$ we have

$$t_\mu t_\nu^* = t_\mu t_\mu^* t_\mu t_\nu^* t_\nu t_\nu^* = t_\mu t_{s(\mu)} t_{s(\nu)} t_\nu^* = 0 \qquad \text{(by (T1))}. \tag{6.3.3}$$

Let $\mu, \nu, \sigma, \tau \in \Lambda$.

<u>Case 1:</u> $s(\mu) \neq s(\nu)$ or $s(\sigma) \neq s(\tau)$. Then by (6.3.3) either $t_\mu t_\nu^* = 0$ or $t_\sigma t_\tau^* = 0$. So $\omega(t_\mu t_\nu^* t_\sigma t_\tau^*) = 0$. This shows that $\omega(t_\mu t_\nu^* t_\sigma t_\tau^*) = \omega(t_\sigma t_\tau^* \alpha^r_{i\beta}(t_\mu t_\nu^*))$.

Case 2: $s(\mu)=s(\nu)$ and $s(\sigma)=s(\tau)$. Then from Lemma 6.3.8 we get

$$\begin{aligned}
\omega(t_\mu t_\nu^* t_\sigma t_\tau^*) &= \sum_{(\eta,\zeta)\in\Lambda^{\min}(\nu,\sigma)} \omega(t_{\mu\eta}t_{\tau\zeta}^*)\\
&= \sum_{(\eta,\zeta)\in\Lambda^{\min}(\nu,\sigma)} \delta_{\mu\eta,\tau\zeta}\, \mathrm{e}^{-\beta r\cdot d(\mu\eta)}\, \omega(t_{s(\mu\eta)}) \qquad \text{(using (6.3.2))} \qquad (6.3.4)\\
&= \sum_{\{(\eta,\zeta)\in\Lambda^{\min}(\nu,\sigma):\mu\eta=\tau\zeta\}} \mathrm{e}^{-\beta r\cdot d(\mu\eta)}\, \omega(t_{s(\eta)}).
\end{aligned}$$

Similarly,

$$\omega(t_\sigma t_\tau^* t_\mu t_\nu^*) = \sum_{\{(\gamma,\lambda)\in\Lambda^{\min}(\tau,\mu):\sigma\gamma=\nu\lambda\}} \mathrm{e}^{-\beta r\cdot d(\sigma\gamma)}\, \omega(t_{s(\gamma)}). \qquad (6.3.5)$$

In order to prove that ω is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda),\alpha^r)$, we need to show that

$$\omega(t_\mu t_\nu^* t_\sigma t_\tau^*) = \omega(t_\sigma t_\tau^* \alpha^r_{i\beta}(t_\mu t_\nu^*)) = \mathrm{e}^{-\beta r\cdot(d(\mu)-d(\nu))}\, \omega(t_\sigma t_\tau^* t_\mu t_\nu^*).$$

Thus in view of (6.3.4) and (6.3.5), we need to show that

$$\begin{aligned}
&\sum_{\{(\eta,\zeta)\in\Lambda^{\min}(\nu,\sigma):\mu\eta=\tau\zeta\}} \mathrm{e}^{-\beta r\cdot d(\mu\eta)}\omega(t_{s(\eta)})\\
&\qquad = \sum_{\{(\gamma,\lambda)\in\Lambda^{\min}(\tau,\mu):\sigma\gamma=\nu\lambda\}} \mathrm{e}^{-\beta r\cdot\left(d(\mu)-d(\nu)+d(\sigma\gamma)\right)}\, \omega(t_{s(\gamma)}).
\end{aligned} \qquad (6.3.6)$$

Let $(\eta,\zeta)\in\Lambda^{\min}(\nu,\sigma)$ such that $\mu\eta=\tau\zeta$. Since $(\eta,\zeta)\in\Lambda^{\min}(\nu,\sigma)$, Lemma 6.3.2 implies that

$$d(\eta)\wedge d(\zeta)=0. \qquad (6.3.7)$$

Also $\mu\eta=\tau\zeta$ implies $d(\mu)+d(\eta)=d(\tau)+d(\zeta)$. Thus from (6.3.7) and by Lemma 6.3.1 we get

$$d(\mu)+d(\eta)=d(\mu)\vee d(\tau), \text{ i.e, } d(\mu\eta)=d(\mu)\vee d(\tau).$$

Hence $(\eta,\zeta)\in\Lambda^{\min}(\mu,\tau)$ or $(\zeta,\eta)\in\Lambda^{\min}(\tau,\mu)$. Also $(\eta,\zeta)\in\Lambda^{\min}(\nu,\sigma)$ implies that $\sigma\zeta=\nu\eta$. Thus

$$(\zeta,\eta)\in\{(\gamma,\lambda)\in\Lambda^{\min}(\tau,\mu):\sigma\gamma=\nu\lambda\}.$$

Similarly for $(\gamma,\lambda)\in(\gamma,\lambda)\in\Lambda^{\min}(\tau,\mu)$ such that $\sigma\gamma=\nu\lambda$ one can get

$$(\lambda,\gamma)\in\{(\eta,\zeta)\in\Lambda^{\min}(\nu,\sigma):\mu\eta=\tau\zeta\}.$$

So we can conclude that the map $(\eta,\zeta)\mapsto(\zeta,\eta)$ is a bijection map from the indexing set of left hand sum of (6.3.6) onto the indexing set of right hand sum of (6.3.6). Thus to prove (6.3.6), it is sufficient to show that the $(\eta,\zeta)^{th}$ term in left hand sum of (6.3.6) is equal to the $(\zeta,\eta)^{th}$ term in the right hand sum of (6.3.6).

Now $(\zeta,\eta)\in\{(\gamma,\lambda)\in\Lambda^{\min}(\tau,\mu):\sigma\gamma=\nu\lambda\}$ implies that $\sigma\zeta=\nu\eta$ which clearly shows that $s(\eta)=s(\zeta)$ and $d(\sigma\zeta)=d(\nu\eta)$. Thus $(\zeta,\eta)^{th}$ term in right hand sum of (6.3.6) is

$$\begin{aligned}
\mathrm{e}^{-\beta r\cdot\left(d(\mu)-d(\nu)+d(\sigma\zeta)\right)}\,\omega(t_{s(\zeta)}) &= \mathrm{e}^{-\beta r\cdot\left(d(\mu)-d(\nu)+d(\nu\eta)\right)}\,\omega(t_{s(\eta)})\\
&= \mathrm{e}^{-\beta r\cdot\left(d(\mu)+d(\eta)\right)}\,\omega(t_{s(\eta)})=\mathrm{e}^{-\beta r\cdot d(\mu\eta)}\,\omega(t_{s(\eta)}),
\end{aligned}$$

which is the $(\eta,\zeta)^{th}$ term of the left hand sum of (6.3.6).

(c) Let $r\in(0,\infty)^k$ have rationally independent coordinates. We need to show that ω is a KMS$_\beta$ state if, and only if, (6.3.2) holds.

First let us suppose ω is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda),\alpha^r)$. Let $\mu,\nu\in\Lambda$ such that $s(\mu)=s(\nu)$ (otherwise $t_\mu t_\nu^*=0$ and a same argument as in Case 1 of part (a) gives the result).

If $d(\mu)=d(\nu)$, then the part (a) of this proposition gives the result.

If $d(\mu)\neq d(\nu)$, then

$$\begin{aligned}
\omega(t_\mu t_\nu^*)=\omega(t_\nu^*\alpha^r_{i\beta}(t_\mu)) &= \mathrm{e}^{-\beta r\cdot d(\mu)}\,\omega(t_\nu^* t_\mu)\\
&= \mathrm{e}^{-\beta r\cdot d(\mu)}\,\omega(t_\mu\alpha^r_{i\beta}(t_\nu^*)) \qquad \text{(using KMS condition)}\\
&= \mathrm{e}^{-\beta r\cdot d(\mu)}\,\omega(t_\mu(\alpha^r_{i\beta}(t_\nu))^*)
\end{aligned}$$

$$= \mathrm{e}^{-\beta r\cdot d(\mu)}\, \mathrm{e}^{\beta r\cdot d(\nu)}\, \omega(t_\mu t_\nu^*) = \mathrm{e}^{-\beta r\cdot(d(\mu)-d(\nu))}\, \omega(t_\mu t_\nu^*). \tag{6.3.8}$$

Since r has rationally independent coordinates,

$$d(\mu)\neq d(\nu) \implies r\cdot(d(\mu)-d(\nu))\neq 0 \implies \mathrm{e}^{-\beta r\cdot(d(\mu)-d(\nu))}\neq 1.$$

Thus (6.3.8) is valid if, and only if $\omega(t_\mu t_\nu^*)=0$. Hence we got

$$\omega(t_\mu t_\nu^*) = \begin{cases} \mathrm{e}^{-\beta r\cdot(d(\mu)-d(\nu))}\, \omega(t_\mu t_\nu^*), & \text{if } d(\mu)=d(\nu) \\ 0, & \text{otherwise} \end{cases},$$

which clearly states that $\omega(t_\mu t_\nu^*) = \delta_{\mu,\nu}\, \mathrm{e}^{-\beta r\cdot d(\mu)}\, \omega(t_{s(\mu)})$.

Conversely, let ω satisfies the equation (6.3.2). Then the proof is given by part (b) of this proposition. □

Since $C^*(\Lambda)$ is a quotient of $\mathcal{T}C^*(\Lambda)$, we can construct a representation $\bar{\alpha}^r : \mathbb{R}\to \operatorname{Aut} C^*(\Lambda)$ such that $\bar{\alpha}^r_t(\bar{a}) = \pi(\alpha^r_t(a))$ for every $a\in\mathcal{T}C^*(\Lambda)$, where α^r is the representation induced by the gauge action (see Definition 6.3.6) and π is the quotient map. Then $(C^*(\Lambda),\bar{\alpha}^r)$ is a C^*-dynamical system.

The following proposition is the subinvariance relation in $(\mathcal{T}C^*(\Lambda),\alpha^r)$.

Proposition 6.3.14. *Let Λ be a finite k-graph and A_i be its coordinate matrix for $i=1,2,\dots,k$. Let $r\in(0,\infty)^k$, $\beta\in[0,\infty)$ and ω be a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda),\alpha^r)$.*

(a) Define $m^\omega=(m^\omega_v)\in[0,1]^{\Lambda^0}$ by $m^\omega_v=\omega(t_v)$. Then $m^\omega\succeq 0$ and $\sum_{v\in\Lambda^0} m_v = 1$ (we call such m^ω a probability measure on Λ^0). Also for every $K\subseteq\{1,2,\dots,k\}$ we have $\prod_{i\in K}(1-\mathrm{e}^{-\beta r_i}A_i)m^\omega\succeq 0$.

(b) The KMS$_\beta$ state ω factors through a KMS$_\beta$ state of $(C^*(\Lambda),\bar{\alpha}^r)$ if, and only if, $A_i m^\omega = \mathrm{e}^{\beta r_i}\, m^\omega$ for every $i=1,2,\dots,k$.

To prove this proposition we need the following three lemmas.

Lemma 6.3.15. *Let A be a C^*-algebra. Let $a_i\in A$ for every $i=1,2,\dots,n$ and let $\emptyset\neq K\subseteq\{1,2,\dots,n\}$. Let $x\in A$ such that $x^2=x$, $xa_i=a_ix=a_i$ and write $a_\emptyset = x$.*

Then

$$\prod_{i\in K}(x-a_i)=\sum_{J\subseteq K}(-1)^{|J|}(\prod_{j\in J}a_j). \tag{6.3.9}$$

PROOF. We shall prove this by induction on n. For $n=1$, we have $K=\{1\}$. The right hand side of (6.3.9) is

$$(-1)^{|\emptyset|}(a_\emptyset)+(-1)^{|\{1\}|}a_1=x-a_1,$$

which is equal to left hand side of (6.3.9).

Let us assume the result is true for $n=m$, i.e., for $K\subseteq\{1,2,\dots,m\}$ (6.3.9) holds.

We need to prove that the result holds for $n=m+1$. Let $K\subseteq\{1,2,\dots,m\}$ and $K'=K\bigcup\{m+1\}$. Then

$$\begin{aligned}
\prod_{i\in K'}(x-a_i)&=\prod_{i\in K}(x-a_i)(x-a_{m+1})\\
&=\Big(\sum_{J\subseteq K}(-1)^{|J|}(\prod_{j\in J}a_j)\Big)(x-a_{m+1}) &&\text{(by the induction asumption)}\\
&=\sum_{J\subseteq K}(-1)^{|J|}(\prod_{j\in J}a_j)(x-a_{m+1})\\
&=\sum_{J\subseteq K}\Big((-1)^{|J|}(\prod_{j\in J}a_j)x-(-1)^{|J|}(\prod_{j\in J}a_j)a_{m+1}\Big)\\
&=\sum_{J\subseteq K}\Big((-1)^{|J|}(\prod_{j\in J}a_j)-(-1)^{|J|}(\prod_{j\in J}a_j)a_{m+1}\Big) &&\text{(as } a_jx=a_j)\\
&=\sum_{J\subseteq K'}(-1)^{|J|}(\prod_{j\in J}a_j).
\end{aligned}$$

□

LEMMA 6.3.16. *Let $K\subseteq\{1,2,\dots,k\}$. For $J\subseteq K$ and $v\in\Lambda^0$, we write $e_J:=\sum_{j\in J}e_j$, where $\{e_i\}$ are the standard generators of $\mathbb{N}^k$ and $t_J=\sum_{\mu\in v\Lambda^{e_J}}t_\mu t_\mu^*\in\mathcal{T}C^*(\Lambda)$; also we write $t_\emptyset:=t_v$ and $t_i=t_{\{i\}}$ for $i\in\{1,2,\dots,k\}$. Then $\prod_{i\in K}(t_v-t_i)=\sum_{J\subseteq K}(-1)^{|J|}t_J$.*

PROOF. The relation (T4) clearly infers that

$$t_i = \sum_{\mu \in v\Lambda^{e_i}} t_\mu t_\mu^* \leq t_v \implies t_v t_i = t_i \text{ and } t_i t_v = t_i.$$

Hence by Lemma 6.3.15, we get

$$\prod_{i \in K}(t_v - t_i) = \sum_{J \subseteq K}(-1)^{|J|}\big(\prod_{j \in J} t_j\big). \tag{6.3.10}$$

Suppose $\emptyset \neq L \subseteq J \setminus \{i\}$. Then

$$e_i \vee e_L = e_i \vee \sum_{j \in L} e_j = \sum_{j \in L \cup \{i\}} e_j = e_{L \bigcup \{i\}}. \tag{6.3.11}$$

Now

$$\begin{aligned}
t_i t_L &= \sum_{\mu \in \Lambda^{e_i}} t_\mu t_\mu^* \sum_{\lambda \in \Lambda^{e_L}} t_\lambda t_\lambda^* \\
&= \sum_{\mu \in \Lambda^{e_i}} \sum_{\lambda \in \Lambda^{e_L}} t_\mu t_\mu^* t_\lambda t_\lambda^* \\
&= \sum_{\mu \in \Lambda^{e_i}} \sum_{\lambda \in \Lambda^{e_L}} t_\mu \big(\sum_{\{(\eta,\zeta): \mu\eta = \lambda\zeta \text{ and } d(\mu\eta) = e_i \vee e_L\}} t_\eta t_\zeta^* \big) t_\lambda^* \qquad \text{(by (T5))} \\
&= \sum_{\mu \in \Lambda^{e_i}} \sum_{\lambda \in \Lambda^{e_L}} \sum_{\{(\eta,\zeta): \mu\eta = \lambda\zeta \text{ and } d(\mu\eta) = e_i \vee e_L\}} t_\mu t_\eta t_\zeta^* t_\lambda^* \qquad \text{(by (6.3.11))} \\
&= \sum_{\mu \in \Lambda^{e_i}} \sum_{\lambda \in \Lambda^{e_L}} \sum_{\{(\eta,\zeta): \mu\eta = \lambda\zeta \text{ and } d(\mu\eta) = e_{L \cup \{i\}}\}} t_{\mu\eta} t_{\lambda\zeta}^* \qquad \text{(by (T2))} \\
&= \sum_{\gamma \in v\Lambda^{e_{L \cup \{i\}}}} t_\gamma t_\gamma^* = t_{L \cup \{i\}}.
\end{aligned}$$

Thus we can conclude that $\prod_{j \in J} t_j = t_J$. So (6.3.10) yields

$$\prod_{i \in K}(t_v - t_i) = \sum_{J \subseteq K}(-1)^{|J|} t_J.$$

□

LEMMA 6.3.17. [**9**] *Suppose $(\mathcal{A}, \tau)$ is a C^*-dynamical system and $\mathcal{J}$ is a closed two-sided ideal in $\mathcal{A}$ generated by a set $\mathcal{P}$ of projections which are fixed by τ. Let $\mathcal{A}_\tau$ be* *the family of τ-analytic elements of $\mathcal{A}$ which is τ-invariant and dense in $\mathcal{A}$. Moreover, for every $a \in \mathcal{A}_\tau$, there is a scalar-valued function f_a satisfying $\tau_z(a) = f_a(z)a$. If ϕ is a KMS$_\beta$ state of $(\mathcal{A}, \tau)$ and $\phi(p) = 0$ for every $p \in \mathcal{P}$, then ϕ factors through a KMS state of $\mathcal{A}/\mathcal{J}$.*

Proof. We know that ϕ factors through a KMS state of $\mathcal{A}/\mathcal{J}$ if, and only if, there exists a KMS state $\tilde{\phi}$ of $\mathcal{A}/\mathcal{J}$ such that $\phi = \tilde{\phi} \circ \pi$, where π is the quotient map from $\mathcal{A}$ to $\mathcal{A}/\mathcal{J}$. In order to show this, it is sufficient to show that ϕ vanishes on $\mathcal{J}$ and $\tilde{\phi}$ is a KMS state. Let $a \in \mathcal{A}$, $p \in \mathcal{P}$. Then

$$\begin{aligned} |\phi(pap)|^2 \leq & \phi(p^*p)\phi((ap)^*ap) \qquad \text{(by Theorem 5.2.4(b))} \\ = & \phi(p)\phi(pa^*ap) = 0 \qquad \text{(as } p^* = p = p^2 \text{ and } \phi(p) = 0). \end{aligned}$$

This clearly shows that ϕ vanishes on $p\mathcal{A}p$.

Now fix $a, b \in \mathcal{A}_\tau$. Since the elements of $\mathcal{P}$ are fixed by τ, we have $\tau_t(ap) = \tau_t(a)p$ for every $t \in \mathbb{R}$. This shows that ap is τ-analytic with $\tau_z(ap) = \tau_z(a)p = f_a(z)ap$. Thus KMS$_\beta$ conditions give

$$\begin{aligned} \phi(apb) = & \phi((ap)(pb)) \\ = & \phi((pb)\tau_{i\beta}(ap)) = f_a(i\beta)\phi(pbap) = 0 \qquad \text{(as } \phi \text{ vanishes on } p\mathcal{A}p). \end{aligned}$$

Since $\mathcal{A}_\tau$ is dense in $\mathcal{A}$ and ϕ is continuous and linear, $\phi(apb) = 0$ for every $a, b \in \mathcal{A}$ and hence ϕ vanishes on $\mathcal{J}$.

Note that $\tilde{\phi}$ is a state of the C^*-dynamical system $(\mathcal{A}/\mathcal{J}, \bar{\tau})$, where $\bar{\tau}$ is the representation defined as $\bar{\tau}_t(\bar{a}) = \pi(\tau_t(a))$ for every $a \in \mathcal{A}$. Now we shall prove that $\tilde{\phi}$ satisfies KMS condition. Let $\bar{a}, \bar{b} \in \mathcal{A}_\tau/\mathcal{J}$. Then

$$\begin{aligned} \tilde{\phi}(\bar{a}\bar{b}) = \tilde{\phi} \circ \pi(ab) = \phi(ab) & \\ & = \phi(b\tau_{i\beta}(a)) \qquad \text{(as } \phi \text{ is a KMS)} \\ & = \tilde{\phi} \circ \pi(b\tau_{i\beta}(a)) = \tilde{\phi}(\bar{b}\bar{\tau}_{i\beta}(\bar{a})). \end{aligned}$$

Hence $\tilde{\phi}$ is a KMS state of $(\mathcal{A}/\mathcal{J}, \bar{\tau})$. □

Proof of Proposition 6.3.14. (a) Since projections are positive and ω is also positive, we can conclude that $m^\omega \succeq 0$. Moreover,

$$\sum_{v\in\Lambda^0} m_v^\omega = \sum_{v\in\Lambda^0} \omega(t_v) = \omega(\sum_{v\in\Lambda^0} t_v) = \omega(1) = 1.$$

Let $K \subseteq \{1,2,\dots,k\}$ and $v \in \Lambda^0$. Then using the notation as in Lemma 6.3.16, we get $t_i = \sum_{\mu\in v\Lambda^{e_i}} t_\mu t_\mu^* \le t_v$, for every $i = 1,2,\dots,k$. This implies $t_i = t_i t_v = t_v t_i$. Recall that the product of two commuting bounded positive operators is positive. Hence $\prod_{i\in K}(t_v - t_i) \ge 0$. Thus using positivity of ω and Lemma 6.3.16, we get

$$\begin{aligned}
0 \le \omega(\prod_{i\in K}(t_v - t_i)) &= \omega\big(\sum_{J\subseteq K}(-1)^{|J|} t_J\big) \\
&= \sum_{J\subseteq K}(-1)^{|J|}\omega(t_J) \\
&= \sum_{J\subseteq K}(-1)^{|J|}\big(\sum_{\mu\in v\Lambda^{e_J}} \omega(t_\mu t_\mu^*)\big) \\
&= \sum_{J\subseteq K}(-1)^{|J|}\big(\sum_{\mu\in v\Lambda^{e_J}} \mathrm{e}^{-\beta r.e_J}\,\omega(t_{s(\mu)})\big) \qquad \text{(by Proposition 6.3.13)} \\
&= \sum_{J\subseteq K}(-1)^{|J|}\,\mathrm{e}^{-\beta r.e_J}\,\big(\sum_{\mu\in v\Lambda^{e_J}} m_{s(\mu)}^\omega\big) \\
&= \sum_{J\subseteq K}(-1)^{|J|}\,\mathrm{e}^{-\beta r.e_J}\,\big(\sum_{w\in\Lambda^0} |v\Lambda^{e_J} w| m_w^\omega\big) \\
&= \sum_{J\subseteq K}(-1)^{|J|}\,\mathrm{e}^{-\beta r.e_J}\,\big(\sum_{w\in\Lambda^0} A^{e_J}(v,w) m_w^\omega\big) \\
&= \sum_{J\subseteq K}(-1)^{|J|}\,\mathrm{e}^{-\beta r.e_J}\,\big(A^{e_J} m^\omega\big)_v \\
&= \sum_{J\subseteq K}(-1)^{|J|}\prod_{j\in J}\mathrm{e}^{-\beta r_j}\,\big((\prod_{j\in J} A_j) m^\omega\big)_v \qquad (\text{as } e_J = \sum_{j\in J} e_j) \\
&= \sum_{J\subseteq K}(-1)^{|J|}\big((\prod_{j\in J}\mathrm{e}^{-\beta r_j}\,A_j) m^\omega\big)_v \\
&= \big(\sum_{J\subseteq K}(-1)^{|J|}(\prod_{j\in J}\mathrm{e}^{-\beta r_j}\,A_j) m^\omega\big)_v
\end{aligned}$$

$$= \Big(\Big(\sum_{J\subseteq K}(-1)^{|J|}\big(\prod_{j\in J} \mathrm{e}^{-\beta r_j}\, A_j\big)\Big)m^\omega\Big)_v$$
$$= \Big(\Big(\prod_{i\in K}(I - \mathrm{e}^{-\beta r_i}\, A_i)\Big)m^\omega\Big)_v \qquad \text{(by Lemma 6.3.15).}$$

(b) Proposition 6.3.10(b) states that $C^*(\Lambda)$ is the quotient of $\mathcal{T}C^*(\Lambda)$ by the ideal J generated by the projections $\{t_v - \sum_{\lambda\in v\Lambda^{e_i}} t_\lambda t_\lambda^* : v \in \Lambda^0, 1 \le i \le k\}$. Now

$$\begin{aligned}
\omega\big(t_v - \sum_{\lambda\in v\Lambda^{e_i}} t_\lambda t_\lambda^*\big) &= \omega(t_v) - \sum_{\lambda\in v\Lambda^{e_i}} \omega(t_\lambda t_\lambda^*)\\
&= m_v^\omega - \sum_{\lambda\in v\Lambda^{e_i}} \mathrm{e}^{-\beta r_i}\, \omega(t_{s(\lambda)}) \qquad \text{(by Proposition 6.3.13)}\\
&= m_v^\omega - \sum_{w\in\Lambda^0} \mathrm{e}^{-\beta r_i}\, |v\Lambda^{e_i}w|\omega(t_w)\\
&= m_v^\omega - \sum_{w\in\Lambda^0} \mathrm{e}^{-\beta r_i}\, A_i(v,w)m_w\\
&= m_v^\omega - \mathrm{e}^{-\beta r_i}(A_i m^\omega)_v. \qquad (6.3.12)
\end{aligned}$$

If ω factors through a state of $C^*(\Lambda)$, then ω vanishes on the generators of $\mathcal{J}$, i.e., $\omega\big(t_v - \sum_{\lambda\in v\Lambda^{e_i}} t_\lambda t_\lambda^*\big) = 0$. Thus the equation (6.3.12) yields $m^\omega = \mathrm{e}^{-\beta r_i}\, A_i m^\omega$.

Conversely, let us suppose $m^\omega = \mathrm{e}^{-\beta r_i}\, A_i m^\omega$. Then the equation (6.3.12) implies that ω vanishes on the generators of $\mathcal{J}$. Also

$$\begin{aligned}
\alpha_z^r(t_v - \sum_{\lambda\in v\Lambda^{e_i}} t_\lambda t_\lambda^*) &= \alpha_z^r(t_v) - \sum_{\lambda\in v\Lambda^{e_i}} \alpha_z^r(t_\lambda t_\lambda^*)\\
&= \mathrm{e}^{izr\cdot d(v)}\, t_v - \sum_{\lambda\in v\Lambda^{e_i}} \mathrm{e}^{izr.(d(\lambda)-d(\lambda))}(t_\lambda t_\lambda^*) = t_v - \sum_{\lambda\in v\Lambda^{e_i}} t_\lambda t_\lambda^*.
\end{aligned}$$

This shows that α^r fixes the generators of $\mathcal{J}$. Moreover, for all $t_\mu t_\nu^* \in \mathcal{T}C^*(\Lambda)$, the analytic function $f_a(z) = \mathrm{e}^{izr.(d(\mu)-d(\nu))}$ satisfies $\alpha_z^r(a) = f_a(z)a$. Thus Lemma 6.3.17 implies that ω factors through a KMS$_\beta$ state of $\mathcal{T}C^*(\Lambda)/\mathcal{J} \cong C^*(\Lambda)$. $\square$

Corollary 6.3.18. *Suppose Λ is a strongly connected finite k-graph. Let $r \in (0,\infty)^k$ and ω is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$. Then $\beta r_i \ge \ln\rho(A_i)$ for every $i = 1, 2, \ldots, k$.*

PROOF. Define $m^\omega = (m^\omega_v) \in [0,1]^{\Lambda^0}$ by $m^\omega_v = \omega(t_v)$. Since t_v is a projection so is positive and ω is positive, $\omega(t_v) \geq 0$. This implies $m^\omega \succeq 0$. Apply Proposition 6.3.14(a) to the singleton set $K = \{i\}$, we get $(1 - \mathrm{e}^{-\beta r_i} A_i)m^\omega \succeq 0$ for $i = 1, 2, \ldots, k$. This clearly implies that

$$A_i m^\omega \preceq \mathrm{e}^{\beta r_i} m^\omega. \tag{6.3.13}$$

Since Λ is a finite k-graph, A_i's are non-negative matrices for every $i = 1, 2, \ldots, k$ having at least one non zero-entry. Thus using Lemma 4.1.3 and the inequality (6.3.13) we conclude that $\mathrm{e}^{\beta r_i} \geq \rho(A_i)$. Moreover, since Λ is strongly connected, by Corollary 4.4.3(a) one has $\rho(A_i) > 0$. So $\beta r_i \geq \ln \rho(A_i)$. □

COROLLARY 6.3.19. *Suppose that Λ is a strongly connected finite k-graph. Let $r \in (0,\infty)^k$ and ω is a KMS$_\beta$ state of $(C^*(\Lambda), \bar{\alpha}^r)$. Then $\beta r_i = \ln \rho(A_i)$ for $i = 1, 2, \ldots, k$.*

PROOF. Define $m^\omega = (m^\omega_v) \in [0,1]^{\Lambda^0}$ by $m^\omega_v = \omega(t_v)$. Then by Proposition 6.3.14(b) m^ω satisfies $A_i m^\omega = \mathrm{e}^{\beta r_i} m^\omega$ for every $i = 1, 2, \ldots, k$. Since each A_i is non-negative, the moreover part of Lemma 4.1.3 implies that $\mathrm{e}^{\beta r_i} = \rho(A_i)$. Thus by Corollary 4.4.3(a), $\rho(A_i) > 0$. So $\beta r_i = \ln \rho(A_i)$. □

In part (c) of Proposition 6.3.13 it is assumed that r has rationally independent coordinates in order to show that $d(\mu) \neq d(\nu)$ implies $r \cdot d(\mu) \neq r \cdot d(\nu)$ which makes $\omega(t_\mu t_\nu^*) = 0$ in the equation (6.3.8). However, it is possible that if $d(\mu) \neq d(\nu)$, then $\omega(t_\mu t_\nu^*) = 0$ without assuming that r has rationally independent coordinates. For that we need to restrict the value of β in the following theorem from [**10**, Theorem 5.1].

THEOREM 6.3.20. *Let Λ be a finite k-graph and A_i's be the coordinate matrices of Λ for $i = 1, 2, \ldots, k$. Let $r \in (0,\infty)^k$ and $\beta \in (0,\infty)$ such that $\beta r_i > \ln \rho(A_i)$ for $i = 1, 2, \ldots, k$. Then a state ω of $(\mathcal{T}C^*(\Lambda), \alpha^r)$ is a KMS$_\beta$ state if, and only if,*

$$\omega(t_\mu t_\nu^*) = \delta_{\mu,\nu}\, \mathrm{e}^{-\beta r \cdot d(\mu)}\, \omega(t_{s(\mu)}) \qquad \text{for all } \mu, \nu \in \Lambda. \tag{6.3.14}$$

With the help of Theorem 6.3.20 we now give a precise method to find all possible KMS_β states of $(\mathcal{T}C^*(\Lambda), \alpha^r)$. For that we need the following lemma.

Lemma 6.3.21. *Suppose Λ is a finite k-graph and A_i's are the coordinate matrices of Λ for $i = 1, 2, \ldots, k$. Let $\beta \in [0, \infty)$ and $r \in (0, \infty)^k$ such that $\beta r_i > \ln \rho(A_i)$ for every $i = 1, 2, \ldots, k$. Then the series $\sum_{n \in \mathbb{N}^k} \mathrm{e}^{-\beta r \cdot n} A^n$ converges in the operator norm to $\prod_{i=1}^{k} (I - \mathrm{e}^{-\beta r_i} A_i)^{-1}$.*

Proof. The N^{th} partial sum is given by:

$$\sum_{0 \le n \le N} \mathrm{e}^{-\beta r \cdot n} A^n = \sum_{0 \le n \le N} \prod_{i=1}^{k} \mathrm{e}^{-\beta r_i \cdot n_i} A_i^{n_i} = \prod_{i=1}^{k} \Big(\sum_{n_i=0}^{N_i} \mathrm{e}^{-\beta r_i n_i} A_i^{n_i} \Big). \tag{6.3.15}$$

For every i we have $\beta r_i > \rho(A_i)$ and a result from [**11**, Lemma VII.3.4] implies that $\sum_{n_i=0}^{N_i} \mathrm{e}^{-\beta r_i \cdot n_i} A^{n_i}$ converges to $(I - \mathrm{e}^{-\beta r_i} A_i)^{-1}$ in operator norm as $N_i \to \infty$. Since $N \to \infty \iff N_i \to \infty$ for all i, the the sum in (6.3.15) converges to $\prod_{i=1}^{k} (I - \mathrm{e}^{-\beta r_i} A_i)^{-1}$ as $N \to \infty$. □

Theorem 6.3.22. *Let Λ be a finite k-graph and let A_i be the coordinate matrices of Λ for $i = 1, 2, \ldots, k$. Suppose $r \in (0, \infty)^k$ and $\beta \in (0, \infty)$ such that $\beta r_i > \ln \rho(A_i)$ for $i = 1, 2, \ldots, k$.*

(a) For $v \in \Lambda^0$, the series $\sum_{\mu \in \Lambda v} \mathrm{e}^{-\beta r \cdot d(\mu)}$ converges to a sum $y_v \ge 1$. Set $y = (y_v) \in [1, \infty)^{\Lambda^0}$, and consider $\epsilon \in [0, \infty)^{\Lambda^0}$. Then $m := \prod_{i=1}^{k} (I - \mathrm{e}^{-\beta r_i} A_i)^{-1} \epsilon$ satisfies $A_i m \preceq \mathrm{e}^{\beta r_i} m$ for $i = 1, 2, \ldots, k$; m is a probability measure on Λ^0 if, and only if, $\epsilon \cdot y = 1$.

(b) Suppose $\epsilon \in [1, \infty)^{\Lambda^0}$ such that $\epsilon \cdot y = 1$ and set $m := \prod_{i=1}^{k} (I - \mathrm{e}^{-\beta r_i} A_i)^{-1} \epsilon$. Then there is a KMS_β state ω_ϵ of $(\mathcal{T}C^*(\Lambda), \alpha^r)$ satisfying

$$\omega_\epsilon(t_\mu t_\nu^*) = \delta_{\mu,\nu} \mathrm{e}^{-\beta r \cdot d(\mu)} m_{s(\mu)}. \tag{6.3.16}$$

(c) The map $\epsilon \mapsto \omega_\epsilon$ is a bijection from $\Sigma_\beta = \{\epsilon \in [0, \infty)^{\Lambda^0} : \epsilon \cdot y = 1\}$ to the set of all KMS_β states of $(\mathcal{T}C^*(\Lambda), \alpha^r)$.

Proof. (a) Let $v \in \Lambda^0$. Now

$$\sum_{\mu\in\Lambda v} \mathrm{e}^{-\beta r\cdot d(\mu)} = \sum_{n\in\mathbb{N}^k}\sum_{\mu\in\Lambda^n v} \mathrm{e}^{-\beta r\cdot n} = \sum_{n\in\mathbb{N}^k}\sum_{w\in\Lambda^0} \mathrm{e}^{-\beta r\cdot n}\,|w\Lambda^n v| = \sum_{n\in\mathbb{N}^k}\sum_{w\in\Lambda^0} \mathrm{e}^{-\beta r\cdot n}\, A^n(w,v). \tag{6.3.17}$$

Lemma 6.3.21 implies that $\sum_{n\in\mathbb{N}^k} \mathrm{e}^{-\beta r\cdot n} A^n$ converges in operator norm. So for every fixed $w \in \Lambda^0$, the series $\sum_{n\in\mathbb{N}^k} \mathrm{e}^{-\beta r\cdot n} A^n(w,v)$ converges. Also since Λ is finite, the sum (6.3.17) converges. Moreover, the sum is at least 1 because $\mathrm{e}^{-\beta r\cdot n} A^n(w,v)$ are non-negative and $\mathrm{e}^0 A^0(v,v) = 1$.

Let $m := \prod_{i=1}^{k}(I - \mathrm{e}^{-\beta r_i} A_i)^{-1}\epsilon$. Then by Lemma 6.3.21, we can conclude that

$$m = \sum_{n\in\mathbb{N}^k} \mathrm{e}^{-\beta r\cdot n} A^n \epsilon \tag{6.3.18}$$

Then clearly $m \geq 0$ and

$$\begin{aligned}\sum_{v\in\Lambda^0} m_v = \sum_{v\in\Lambda^0}\sum_{n\in\mathbb{N}^k} (\mathrm{e}^{-\beta r\cdot n} A^n\epsilon)_v &= \sum_{v\in\Lambda^0}\Big(\sum_{n\in\mathbb{N}^k}\big(\mathrm{e}^{-\beta r\cdot n} A^n\big)\epsilon\Big)_v\\ &= \sum_{v\in\Lambda^0}\sum_{n\in\mathbb{N}^k}\sum_{w\in\Lambda^0} \mathrm{e}^{-\beta r\cdot n} A^n(v,w)\epsilon_w\\ &= \sum_{w\in\Lambda^0}\epsilon_w\Big(\sum_{v\in\Lambda^0}\sum_{n\in\mathbb{N}^k} \mathrm{e}^{-\beta r\cdot n}\,|v\Lambda^n w|\Big)\\ &= \sum_{w\in\Lambda^0}\epsilon_w\Big(\sum_{\mu\in\Lambda w} \mathrm{e}^{-\beta r\cdot d(\mu)}\Big) = \epsilon\cdot y.\end{aligned}$$

Hence m is a probability measure if, and only if, $\epsilon \cdot y = 1$.

(b) To construct ω_ϵ, we use Proposition 6.3.5 with the Toeplitz-Cuntz-Krieger Λ-family from Proposition 6.3.4. Define a linear functional ω_ϵ on $\mathcal{T}C^*(\Lambda)$ as:

$$\omega_\epsilon(a) = \sum_{\lambda\in\Lambda} \Delta_\lambda\langle \pi_T(a)h_\lambda, h_\lambda\rangle \qquad \text{for } a \in \mathcal{T}C^*(\Lambda),$$

where $\Delta_\lambda = \mathrm{e}^{-\beta r\cdot d(\lambda)}\,\epsilon_{s(\lambda)}$ for $\lambda \in \Lambda$.

Now, let $a^*a \in \mathcal{T}C^*(\Lambda)$ be a positive element, then

$$\begin{aligned}\omega_\epsilon(a^*a) = \sum_{\lambda\in\Lambda} \Delta_\lambda \langle \pi_T(a^*a)h_\lambda, h_\lambda\rangle &= \sum_{\lambda\in\Lambda} \Delta_\lambda \langle \pi_T(a)^*\pi_T(a)h_\lambda, h_\lambda\rangle \\ &= \sum_{\lambda\in\Lambda} \Delta_\lambda \langle \pi_T(a)h_\lambda, \pi_T(a)h_\lambda\rangle \\ &= \sum_{\lambda\in\Lambda} \Delta_\lambda \left|\left|\pi_T(a)h_\lambda\right|\right|^2 .\end{aligned}$$

Since $\Delta_\lambda \geq 0$, ω_ϵ is a positive functional.

Now fix $v \in \Lambda^0$, then

$$\begin{aligned}\sum_{\mu\in v\Lambda} \Lambda_\mu = \sum_{n\in\mathbb{N}^k}\sum_{\mu\in v\Lambda^n} \mathrm{e}^{-\beta r\cdot d(\mu)}\, \epsilon_{s(\mu)} &= \sum_{n\in\mathbb{N}^k} \mathrm{e}^{-\beta r\cdot d(\mu)} \Big(\sum_{w\in\mathbb{N}^k}\sum_{\mu\in v\Lambda^n w} \epsilon_w\Big) \\ &= \sum_{n\in\mathbb{N}^k} \mathrm{e}^{-\beta r\cdot d(\mu)} \Big(\sum_{w\in\mathbb{N}^k} |v\Lambda^n w|\epsilon_w\Big) \\ &= \sum_{n\in\mathbb{N}^k} \mathrm{e}^{-\beta r\cdot d(\mu)} \Big(\sum_{w\in\mathbb{N}^k} A^n(v,w)\epsilon_w\Big) \\ &= \sum_{n\in\mathbb{N}^k} \mathrm{e}^{-\beta r\cdot d(\mu)} \big(A^n\epsilon\big)_v \\ &= \Big(\sum_{n\in\mathbb{N}^k} \mathrm{e}^{-\beta r\cdot d(\mu)}\, A^n\epsilon\Big)_v = m_v \qquad (6.3.19)\end{aligned}$$

Since $\epsilon \cdot y = 1$, part (a) of this theorem implies that m is a probability measure. So

$$1 = \sum_{v\in\Lambda^0} m_v = \sum_{v\in\Lambda^0}\sum_{\mu\in v\Lambda} \Lambda_\mu = \sum_{\mu\in\Lambda} \Delta_\mu. \qquad (6.3.20)$$

Thus by Proposition 5.2.6, we get

$$\begin{aligned}||\omega_\epsilon|| = \omega_\epsilon(1) = \omega_\epsilon(\sum_{v\in\Lambda^0} t_v) &= \sum_{v\in\Lambda^0} \omega_\epsilon(t_v) \\ &= \sum_{v\in\Lambda^0}\sum_{\lambda\in\Lambda} \Delta_\lambda \langle \pi_T(t_v)h_\lambda, h_\lambda\rangle \\ &= \sum_{v\in\Lambda^0}\sum_{\lambda\in\Lambda} \Delta_\lambda \langle T_v h_\lambda, h_\lambda\rangle \\ &= \sum_{v\in\Lambda^0}\sum_{\lambda\in v\Lambda} \Delta_\lambda \langle h_\lambda, h_\lambda\rangle\end{aligned}$$

$$= \sum_{v \in \Lambda^0} \sum_{\lambda \in v\Lambda} \Delta_\lambda = \sum_{\mu \in \Lambda} \Delta_\mu = 1 \qquad \text{(by (6.3.20))}.$$

So ω_ϵ is a state. Now,

$$\omega_\epsilon(t_\mu t_\nu^*) = \sum_{\lambda \in \Lambda} \Delta_\lambda \langle T_\mu T_\nu^* h_\lambda, h_\lambda \rangle = \sum_{\lambda \in \Lambda} \Delta_\lambda \langle T_\nu^* h_\lambda, T_\mu^* h_\lambda \rangle.$$

But

$$\langle T_\nu^* h_\lambda, T_\mu^* h_\lambda \rangle = \begin{cases} 1, & \lambda = \mu\lambda' = \nu\lambda' \\ 0, & \textit{otherwise} \end{cases} \tag{6.3.21}$$

Also by the unique factorization, $\mu\lambda' = \nu\lambda'$ implies $\mu = \nu$. So $\langle T_\nu^* h_\lambda, T_\mu^* h_\lambda \rangle = 0$ if $\mu \neq \nu$ implies $\omega_\epsilon(t_\mu t_\nu^*) = 0$ if $\mu \neq \nu$. If $\mu = \nu$, then

$$\begin{aligned}
\omega_\epsilon(t_\mu t_\mu^*) &= \sum_{\lambda \in \Lambda} \Delta_\lambda \langle \pi_T(t_\mu t_\mu^*) h_\lambda, h_\lambda \rangle \\
&= \sum_{\lambda \in \Lambda} \Delta_\lambda \langle T_\mu^* h_\lambda, T_\mu^* h_\lambda \rangle \\
&= \begin{cases} \sum_{\lambda \in \Lambda} \Delta_\lambda, & \text{if } \lambda = \mu\lambda' \\ 0, & \textit{otherwise} \end{cases} \qquad \text{(by (6.3.21))} \\
&= \sum_{\mu\lambda' \in \Lambda} \Delta_{\mu\lambda'}.
\end{aligned}$$

Thus

$$\omega_\epsilon(t_\mu t_\nu^*) = \begin{cases} \sum_{\mu\lambda' \in \Lambda} \Delta_{\mu\lambda'}, & \text{if } \mu = \nu \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\begin{aligned}
\sum_{\mu\lambda' \in \Lambda} \Delta_{\mu\lambda'} &= \sum_{\mu\lambda' \in \Lambda} \mathrm{e}^{-\beta r \cdot d(\mu\lambda')} \, \epsilon_{s(\mu\lambda')} \\
&= \mathrm{e}^{-\beta r \cdot d(\mu)} \sum_{\lambda' \in s(\mu)\Lambda} \mathrm{e}^{-\beta r \cdot d(\lambda')} \, \epsilon_{s(\lambda')}
\end{aligned}$$

$$= \mathrm{e}^{-\beta r\cdot d(\mu)} \sum_{\lambda'\in s(\mu)\Lambda} \Delta_{\lambda'} = \mathrm{e}^{-\beta r\cdot d(\mu)}\, m_{s(\mu)} \qquad \text{(by (6.3.19))}.$$

Hence $\omega_\epsilon(t_\mu t_\nu^*) = \delta_{\mu,\nu}\,\mathrm{e}^{-\beta r\cdot d(\mu)}\, m_{s(\mu)}$.

Now in order to prove that ω_ϵ is a KMS$_\beta$ state, by Proposition 6.3.13(b) it is sufficient to prove that $m_{s(\mu)} = \omega_\epsilon(t_{s_\mu})$.

$$\begin{aligned}
\omega_\epsilon(t_{s(\mu)}) &= \sum_{\lambda\in\Lambda} \Delta_\lambda \langle \pi_T(t_{s(\mu)})h_\lambda, h_\lambda\rangle \\
&= \sum_{\lambda\in\Lambda} \Delta_\lambda \langle T_{s(\mu)}h_\lambda, h_\lambda\rangle \\
&= \sum_{\lambda\in\Lambda} \Delta_\lambda \langle T_{s(\mu)}h_\lambda, T_{s(\mu)}h_\lambda\rangle \qquad (\text{as } T_{s(\mu)} = T_{s(\mu)}^2 = T_{s(\mu)}^*) \\
&= \sum_{\lambda\in s(\mu)\Lambda} \Delta_\lambda \qquad \text{(by (6.3.21))} \\
&= m_{s(\mu)} \qquad \text{(by (6.3.19))}.
\end{aligned}$$

(c) Let $\epsilon \in \Sigma_\beta$. Then by part (b) of this Theorem ω_ϵ is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$. Moreover, if $\epsilon' \in \Sigma_\beta$ such that $\epsilon \neq \epsilon'$, then by part (b) of this theorem $\omega_\epsilon \neq \omega_{\epsilon'}$. So the map $\epsilon \mapsto \omega_\epsilon$ is injective.

Now let ω be a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$. Then by Proposition 6.3.14, $m^\omega = (m_v^\omega) \in [0,1]^{\Lambda^0}$, where $m_v^\omega = \omega(t_v)$, is a probability measure which satisfies $\epsilon := \prod_{i=1}^{k}(I - \mathrm{e}^{-\beta r_i} A_i)m^\omega \succeq 0$. So $m^\omega = \prod_{i=1}^{k}(I - \mathrm{e}^{-\beta r_i} A_i)^{-1}\epsilon$ and by part (a) of this theorem, $\epsilon \cdot y = 1$. Now compare the formula (6.3.14) in Theorem 6.3.20 with (6.3.16), we can conclude $\omega_\epsilon = \omega$. This shows that the map $\epsilon \mapsto \omega_\epsilon$ is surjective. □

Theorem 6.3.22 gives all possible KMS$_\beta$ states when $\beta r_i > \ln \rho(A_i)$. However if we create a condition when $\beta r_i = \ln \rho(A_i)$, then for a strongly connected k-graph Λ, we formulate a unique KMS state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$ for a specific value of r with the help of Theorem 6.3.22.

THEOREM 6.3.23. *Let Λ be a strongly connected finite k-graph. Suppose A_i's are the coordinate matrices of Λ and let $r = (\ln \rho(A_1), \ln \rho(A_2), \ldots, \ln \rho(A_k)) \in (0, \infty)^k$*

such that r has rationally independent coordinates. Let x^Λ be the UPF eigenvector for Λ.

(a) Then there is a unique KMS_1 state ω of $(\mathcal{T}C^*(\Lambda), \alpha^r)$ such that

$$\omega(t_\mu t_\nu^*) = \delta_{\mu,\nu}\, \mathrm{e}^{-r\cdot d(\mu)}\, x^\Lambda_{s(\mu)} \qquad \text{for all } \mu, \nu \in \Lambda.$$

(b) This state in (a) factors through a state $\tilde{\omega}$ of the quotient $C^*(\Lambda)$. The state $\tilde{\omega}$ is the only KMS_1 state for $(C^*(\Lambda), \bar{\alpha}^r)$.

Proof. Let $\{\beta_n\}$ be a decreasing sequence of real numbers such that $\beta_n \to 1$. Since Λ is a strongly connected, by Proposition 4.1.5(a)(i) x^Λ is a common eigenvector to all A_i's associated to the eigenvalue $\rho(A_i)$, i.e., $A_i x^\Lambda = \rho(A_i)x^\Lambda$ for all $i = 1, 2, \ldots, k$.

Moreover, x^Λ is strictly positive and a unimodular vector. These imply x^Λ is a probability measure satisfying

$$A_i x^\Lambda = \rho(A_i)x^\Lambda \preceq \mathrm{e}^{\beta_n \ln \rho(A_i)}\, x^\Lambda = \mathrm{e}^{\beta_n r_i}\, x^\Lambda \qquad \text{(as for every } n,\ \beta_n \geq 1).$$

Since $\beta_n r_i > \ln \rho(A_i)$, Theorem 6.3.22(b) with $\epsilon = \prod_{i=1}^{k}(I - \mathrm{e}^{-\beta_n}\, A_i)x^\Lambda$ gives a KMS$_{\beta_n}$ state ω_n satisfying

$$\omega_n(t_\mu t_\nu^*) = \delta_{\mu,\nu}\, \mathrm{e}^{-\beta_n r\cdot d(\mu)}\, x^\Lambda_{s(\mu)}. \tag{6.3.22}$$

Recall that the Banach-Alaoglu Theorem states that, for any normed space X, the closed unit ball of the dual space X^* is compact with respect to the weak* topology. So if we consider the state space of $\mathcal{T}C^*(\Lambda)$, Banach-Alaoglu Theorem implies that it is weak* compact. Hence there exist a subsequence $\{\omega_{n_l}\}$ of the sequence $\{\omega_n\}$ which is convergent. Let $\omega_{n_l} \to \omega$. Now from (6.3.22), we get

$$\omega(t_\mu t_\nu^*) = \delta_{\mu,\nu}\, \mathrm{e}^{-r\cdot d(\mu)}\, x^\Lambda_{s(\mu)}. \tag{6.3.23}$$

To show the uniqueness, let ψ be a KMS$_1$ state for $(\mathcal{T}C^*(\Lambda), \alpha^r)$. Then by Proposition 6.3.14(a) $m^\psi = (\psi(t_v))$ is a probability measure satisfying $(I - \mathrm{e}^{-\ln \rho(A_i)}\, A_i)m^\psi \succeq$

0 or $A_i m^\psi \preceq \rho(A_i) m^\psi$. Using Lemma 4.1.3, we get $A_i m^\psi = \rho(A_i) m^\psi$, and clearly $\big|\big|m^\psi\big|\big| = \psi\Big(\sum_{v\in\Lambda^0} t_v\Big) = \psi(1) = 1$. Thus we get that m^ψ is a unimodular eigenvector of A_i associated to $\rho(A_i)$. Hence Proposition 4.1.5(a) assures that $m^\psi = x^\Lambda$. Since r has rationally independent coordinates, Proposition 6.3.13(c) implies that

$$\psi(t_\mu t_\nu^*) = \delta_{\mu,\nu}\, \mathrm{e}^{-r\cdot d(\mu)}\, x^\Lambda_{s(\mu)} \qquad \text{for all } \mu, \nu \in \Lambda.$$

Thus the equation (6.3.23) yields that $\psi = \omega$.

(b) For $m^\omega = x^\Lambda$, we have $A_i m^\omega = \rho(A_i) m^\omega$. So Proposition 6.3.14(b) implies that ω factors through a KMS state $\tilde{\omega}$ of $(C^*(\Lambda), \bar{\alpha}^r)$, i.e.,

$$\omega = \tilde{\omega} \circ \pi. \tag{6.3.24}$$

To show the uniqueness, let $\tilde{\psi}$ be a KMS$_\beta$ state of $(C^*(\Lambda), \bar{\alpha}^r)$. Then $\tilde{\psi} \circ \pi$ is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$, where π is the quotient map from $\mathcal{T}C^*(\Lambda)$ to $C^*(\Lambda)$. Now Proposition 6.3.14(b) implies that

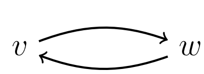
$$A_i m^{\tilde{\psi}\circ\pi} = \rho(A_i)^\beta m^{\tilde{\psi}\circ\pi}.$$

Using the same arguments of part (a), we can conclude that $m^{\tilde{\psi}\circ\pi} = x^\Lambda$. So $A_i x^\Lambda = \rho(A_i)^\beta x^\Lambda$ and thus Proposition 4.1.5(a) implies that

$$\rho(A_i)^\beta = \rho(A_i) \qquad \text{for all } i = 1, 2, \ldots, k.$$

Since r has rationally independent coordinates, there exists $\rho(A_i)$ which is not equal to one. We can conclude that $\beta = 1$. By the uniqueness of KMS$_1$ state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$ we get $\tilde{\psi} \circ \pi = \omega$ and (6.3.24) assures that $\tilde{\psi} = \tilde{\omega}$. □

Example 6.3.24. Let us consider a directed graph E, which is indeed a 1-graph with its coordinate matrix given as follows:



$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then the degree map $d : E^* \to \mathbb{N}$ is defined by $d(\mu) = |\mu|$.

Let $\beta = 1$ and $r = 1$. Then by Theorem 6.3.22 we can calculate KMS_1 states on $(\mathcal{T}C^*(E), \alpha^1)$. Let $y = (y_v) \in [1, \infty)^{E^0}$, where $y_v = \sum_{\mu \in E^* v} \mathrm{e}^{-|\mu|}$. Now

$$\sum_{\mu \in E^* v} \mathrm{e}^{-|\mu|} = \sum_{n \in \mathbb{N}} \mathrm{e}^{-n} = \frac{\mathrm{e}}{\mathrm{e} - 1}.$$

Then $y = \begin{bmatrix} \frac{\mathrm{e}}{\mathrm{e}-1} & \frac{\mathrm{e}}{\mathrm{e}-1} \end{bmatrix}^t$. Let $\epsilon \in [1, \infty)^{E^0}$ which satisfies $\epsilon \cdot y = 1$. If $\epsilon = \begin{bmatrix} u & v \end{bmatrix}^t$, then

$$\begin{aligned} \epsilon \cdot y = 1 &\implies u \frac{\mathrm{e}}{\mathrm{e}-1} + v \frac{\mathrm{e}}{\mathrm{e}-1} = 1 \\ &\implies u + v = 1 - \mathrm{e}^{-1}. \end{aligned} \tag{6.3.25}$$

Hence

(6.3.26)

$$\begin{aligned} m = (I - e^{-1} A)^{-1} \begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} 1 & -\mathrm{e}^{-1} \\ -\mathrm{e}^{-1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} \\ &= \frac{1}{1 - \mathrm{e}^{-2}} \begin{bmatrix} 1 & \mathrm{e}^{-1} \\ \mathrm{e}^{-1} & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{1 - \mathrm{e}^{-2}} \begin{bmatrix} u + v\,\mathrm{e}^{-1} \\ u\,\mathrm{e}^{-1} + v \end{bmatrix}. \end{aligned}$$

Note that $m = \frac{1}{1 - \mathrm{e}^{-2}} \begin{bmatrix} u + v\,\mathrm{e}^{-1} \\ u\,\mathrm{e}^{-1} + v \end{bmatrix}$ is a probability measure as $m_v \geq 0$ for every $v \in E^0$ and

$$\begin{aligned} \sum_{v \in E^0} m_v = \frac{1}{1 - \mathrm{e}^{-2}} \big(u + v\,\mathrm{e}^{-1} + u\,\mathrm{e}^{-1} + v\big) &= \frac{1}{(1 - \mathrm{e}^{-1})(1 + \mathrm{e}^{-1})} \big((u + v)(1 + \mathrm{e}^{-1})\big) \\ &= 1 \qquad \text{(by (6.3.25))}. \end{aligned}$$

Hence by Theorem 6.3.22, the KMS_1 states on $(\mathcal{T}C^*(E), \alpha^1)$ are given by:

$$\omega_\epsilon(t_\mu t_\nu^*) = \delta_{\mu,\nu}\, \mathrm{e}^{|\mu|}\, m_{s(\mu)},$$

where m is given by (6.3.26) and $u, v \in [1, \infty)$ satisfies (6.3.25). Moreover, part (c) of Theorem 6.3.22 assures that these are the only KMS_1 states.

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