Finite element analysis of non-Darcy flow

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FINITE ELEMENT ANALYSIS
OF NON-DARCY FLOW

A Thesis
Submitted to the Faculty of Graduate Studies through the
Department of Civil Engineering in Partial Fulfillment
of the Requirements for the Degree of
Doctor of Philosophy at the
University of Windsor

by

John Alexander McCrorquodale

Windsor, Ontario
1970
This thesis contains the development of a finite element model for unsteady non-Darcian multiphase flow in a reservoir. The model is used to solve continuity equations for each phase in three-dimensional space for given initial conditions and boundary conditions. Simulations are performed for both moving and stationary phases.

The numerical solutions are obtained by steady flow methods and by applying finite element techniques. A study of the model's accuracy in order to assess the usefulness of the finite element approach is conducted. These results are in agreement with known equations and experimental results.

The results of the simulations are presented in tabular and graphical form. Graphical results include steady flow transient pressure and rate response. Each parallel case is confirmed by experimental data. The model is slightly more than ten times faster than the experimental setup checkers, by meter. Overall, the results indicate that this finite element model is applicable to real-world problems.

In the course of developing the model, it has been shown that the finite element method provides a powerful tool for solving complex multiphase flow problems in reservoir engineering.
ABSTRACT

This thesis presents the development of a finite element model for unsteady non-Darcy flow. The unsteady flow model is used to solve a free surface flow problem in which the initial conditions are given and the boundary conditions are specified functions of time.

The unsteady flow problem is solved in small time steps. At each time step the internal flow is solved by steady flow methods and a Lagrangian technique used to compute the new free surface. A transformation of the dependent variable is proposed in order to account for small inertial effects in the unsteady flow. The unsteady finite element model is applied to the solution of rapid drawdown in rockfill. There was good agreement between the experimental and computed drawdown profiles.

A theoretical evaluation of the unsteady inertial term indicates that this term is only of secondary importance for the material used in the experimental studies, i.e. 1.66 cm and 4.40 cm crushed rock. The finite element solutions, also, indicated that the unsteady inertial term was relatively small compared with frictional resistance.

The results of the experimental studies which were carried out to establish flow resistance equations for the verification studies, are also given. Both parallel and radial flow tests were made. A tendency, for the flow resistance of a material in a converging flow permeameter to be slightly less than for the same material in the parallel flow permeameter, is noted. General flow resistance equations based on the Ward equation and the Kovacs equation are proposed.

In the course of developing the unsteady flow finite element models, steady flow finite element models were developed. The steady
flow finite element models were also verified experimentally. A brief review of the steady flow development is presented.

The writer wishes to express his appreciation to his supervisor, Prof. S.P. Chen, for his helpful comments and suggestions during the project. Contributions of the members of his thesis committee at the time of the comprehensive examination were also appreciated. Thanks are also due to the laboratory technicians, Mr. C. Black, Mr. D. and Mr. P. Reiner.

The writer is also grateful to the Department of Civil Engineering and the Faculty of Medicine for the opportunity of carrying out this research.

Finally, the writer would like to thank his wife and family for their patience and moral support during the past four years.
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The writer wishes to express his appreciation to his supervisor, Dr. S.P. Chee, for his helpful comments and suggestions during the project. The comments of the members of his thesis committee at the time of the comprehensive examination were also appreciated. Thanks are also due to the laboratory technicians, Mr. G. Michalczuk and Mr. P. Feimer.

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Finally, the writer would like to thank his wife and family for their patience and moral support during the past four years.
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CHAPTER I

INTRODUCTION

1.1 Objective

The purpose of this thesis is the development of a finite element model to represent unsteady free surface non-Darcy flow. Steady flow techniques are developed and extended to the solution of problems with prescribed time dependent boundary conditions.

1.2 Definition of the Problem

A porous medium is defined by Collins (17) as: "a solid containing holes or voids either connected or disconnected, dispersed within it in either a regular or random manner provided that the holes occur relatively frequently within the solid."

Saturated fluid flowing in porous media can be classified in the following regimes:

a) Non-Newtonian or microseepage;

b) Laminar Newtonian flow with negligible inertial effects, i.e. Darcy flow;

c) Non-linear laminar flow, i.e. flow with stable streamlines in the pores but with significant inertial effects;

d) Turbulent transitional flow, i.e. flow in which some of the streamlines in the pores become unstable (turbulence) while steady inertial and viscous effects still exist and are significant;

e) Fully turbulent flow, i.e. flow in which inertial effects are much greater than viscous effects and practically no stable streamline flow exists in the pores.
1.3 Motivation

Very little literature exists on the solution of unsteady non-Darcy flow problems. Yet the writer has found a number of examples of the occurrence of unsteady non-Darcy flow. A few examples are:

a) Wave absorption and transmission in rubble-mound breakwaters;
b) Water level fluctuations, in the rockfill and filter layers of a dam, due to wave action on the dam;
c) Transmission of floods or tides through rockfill dams and causeways;
d) Unsteady non-Newtonian flow may occur in concrete and in some clay soils.

A challenging problem which can be solved by the proposed techniques of this thesis arises in the determination of the freeboard for the 'impervious' core of a rock protected dam. Freeboard allowances, as shown in figure 1.2, are provided so that the wave action and wind set up will not cause over-topping of either the embankment or the 'effective' portion of the 'impervious' core. Only the lower portion of the 'impervious' core can be relied upon to prevent seepage since the top portion (say the top three feet) would be subject to frost action and might be ineffective. If the conservative assumption, that the maximum water level on the core is the same as the maximum run-up on the rock face, is made the freeboard on the core will govern the maximum allowable reservoir level. A technique of computing the water level fluctuations within the rockfill would permit rational freeboards to be established. In some instances this would permit an engineer to justify either lower embankments or higher maximum reservoir levels with obvious economic benefits to the project.
Regimes (a), (c), (d) and (e) are referred to as non-Darcy flows. In this thesis the main emphasis is on the solution of unsteady non-Darcy flows in the turbulent transitional regime.

In particular a finite element technique is sought to determine the unsteady water surface profiles and internal piezometric heads in a rockfill section for given initial conditions and known time dependent boundary conditions. The solution although restricted in practice to a two dimensional medium should, theoretically, be applicable to a three dimensional medium. The solution technique should also account for the inertial term due to the macroscopic acceleration $\frac{\partial \mathbf{u}}{\partial t}$ in unsteady flow.

A typical unsteady flow problem is illustrated in figure 1.1.

![Figure 1.1 A Typical Unsteady Flow Problem](image_url)
Figure 1.2 Determination of the Freeboard on a Dam
1.4 The Approach in General

Even though the main theme of this thesis is the study of unsteady non-Darcy flow some preliminary work was required on steady non-Darcy flow. The characteristics of the media to be used in the study had to be established using steady flow tests. Steady flow experiments were carried out in permeameters with parallel, converging and diverging walls. Also in 1966 when this project was started no finite element method for non-Darcy flow was available; therefore the writer developed his own finite element model [McCorquodale and Ng (70) IAHR, Kyoto, Sept. 1969 and subsequent discussions (71)]. This steady flow model was similar to the one developed independently by Volker [(122), ASCE, November 1969]. A brief review of the writer's steady flow finite element method is presented as a background to the work on unsteady non-Darcy flow. The steady flow finite element method is verified by experimental data and analytical solutions.

The steady flow techniques are then extended to unsteady flow problems with given initial conditions and known time dependent boundary conditions. The proposed solution progresses in small time steps. At each time step the unsteady flow problem is reduced to one in which the internal flow field can be solved by a modification of the steady flow finite element method. The free surface is adjusted at each step using a Lagranian representation of the movement of the free surface particles.

The unsteady flow finite element model results are compared with the experimental results for rapid drawdown in coarse crushed rock. Models with and without the inertia term, \( c \frac{\partial q}{\partial t} \) are presented and compared to evaluate the importance of this term.
CHAPTER II
BACKGROUND AND REVIEW OF THE LITERATURE

2.1 Scope

In 1901 Forchheimer (32) published a paper in which he noted that Darcy's Law (19) is not universally valid for flow in porous media. Since that time several research papers have been presented on the hydrodynamics of this so-called non-Darcy flow. This chapter reviews some prominent contributions to the study of non-Darcy flow. The literature pertinent to steady one-dimensional flow (i.e., steady flow in which the macroscopic velocity or bulk velocity vectors are parallel) is treated first. Then the literature concerned with steady two-dimensional flow is considered. Finally, the existing researches on unsteady non-Darcy flow are reviewed. In the section on unsteady flow some of the Darcy flow techniques are re-examined in the light of the non-Darcy flow equations. Some modifications of these techniques, to include non-Darcy flow, are suggested.

2.2 Steady One-Dimensional Flow in Porous Media

2.2.1 The Nature of Flows in Porous Media

Stokes' Law (115) which dates from 1851 indicates that the drag on a sphere moving through a fluid is proportional to the first power of its velocity provided its Reynolds number is less than about 0.1. As the Reynolds number increases the linear relationship between drag and velocity breaks down; the exponent on the velocity increases with increasing Reynolds number approaching a value of 2.0 for very
turbulent flow (98). Some useful analogies between flow in a porous medium and flow past a single particle (e.g., a sphere) can be drawn. Of course in a porous medium there will be interference of flow patterns which will affect the drag on the particles of the medium; thus it will not be possible to obtain quantitative estimates of drag forces in a porous medium from the drag characteristics of the individual particles of the medium. However there is a correspondence between the flow regimes in a porous medium and the flow regimes for flow around a single particle.

In both cases a linear laminar or viscous regime exists at low Reynolds numbers i.e., drag is proportional to velocity. In a porous medium this is called Darcy flow (19); in the case of a single particle it is called Stokes flow (115). Viscous forces predominate over inertial 'forces' and the drag is primarily due to skin friction. The streamlines are steady and stable with a tendency for symmetry fore and aft of the particle (55), as shown in figure 2.1.

The differential equations for the linear laminar regime have the form

$$\nabla p = \mu \nabla^2 q_p$$

(2.1)

and

$$\nabla q_p = 0$$

(2.2)

where \( p \) = local pressure;

\( q_p \) = local velocity (pore velocity);
Figure 2.1  Flow in the Linear Laminar Regime

Figure 2.2  Flow in the Non-linear Laminar Regime
and \( \mu \) = dynamic viscosity.

Solving equations 2.1 and 2.2 for the stream functions for a sphere gives (55)

\[ \psi = \frac{3}{4} a \ U_\infty (r - \frac{1}{3} \frac{a^2}{r}) \sin^2 \theta \] (2.3)

where \( a \) = radius of sphere;
\((r, \theta)\) = polar coordinates;
\( U_\infty \) = free-stream velocity

The drag on the sphere is given by

\[ F_D = 6\pi a \mu U_\infty \] (2.4)

which is comparable to Darcy's Law

\[ i = \frac{\mu q}{k} \]

where \( i \) = hydraulic gradient
\( q \) = macroscopic or bulk velocity,
\( k \) = permeability of the medium.

As the inertial 'forces' (i.e. the local acceleration term) increase relative to the viscous forces the streamlines become more distorted, as shown schematically in figure 2.2, and the drag increases more rapidly than the first power of the velocity. However the streamlines remain stable i.e. they do not fluctuate. Some steady secondary
flow may appear at the higher velocities of this regime. This regime is sometimes referred to as Oseen flow in the case of flow past a sphere; in porous media it is called non-linear laminar or steady inertial flow.

Oseen [(84), 1910] approximated the equations of this regime (for a sphere) by

\[
\vec{U}_\infty \nabla \vec{q}_p = -\frac{\nabla \rho}{\rho} + \nu \nabla^2 \vec{q}_p \quad (2.6)
\]

and

\[
\nabla \cdot \vec{q}_p = 0 \quad (2.7)
\]

where \( \vec{q}_p \) is the local velocity. Lamb (55) gives a solution of these equations as

\[
\Psi = \frac{3}{2} \nu a (1 + \cos \theta) \left[ 1 - e^{-\frac{U \infty r}{2\nu}} (1 - \cos \theta) \right] \frac{1}{4} \frac{a^3}{r} U \infty \sin^2 \theta \quad (2.8)
\]

and

\[
F_D \sim 6\pi \mu a \infty (1 + \frac{3}{16} R_N) \quad (2.9)
\]

where \( R_N = \frac{2aU \infty}{\nu} \).

A plot of \( \Psi \) shows that the symmetry mentioned in the linear laminar case is lost as inertia becomes important. A similar phenomenon in porous media has been noted by Chauveteau and Thirriot (15).
For this regime the hydraulic gradient in the porous medium can be approximated by

\[ i = (a_1 + b_1 q)q \tag{2.10} \]

where \( q \) is the macro-velocity (also referred to as the bulk velocity); \( a_1 \) and \( b_1 \) are constants for a particular medium.

Increasing the Reynolds number still further (i.e. increasing the inertial 'forces' relative to the viscous forces) may cause the streamlines to become unstable and separate from the particle or particles. Thus the streamlines in at least some regions of the flow become unsteady and fluctuate either in a periodic or random manner \((24, 133)\), i.e. turbulence exists in the flow. This is illustrated in figure 2.3.

The drag in this regime, depends on both the viscous and the inertial 'forces'; however increasing the Reynold's number increases the relative importance of turbulence. Thus in this regime Oseen's solution is not applicable for the sphere and equation 2.10 is not applicable for the porous medium \((24)\). The hydraulic gradient for the porous medium could now be represented by

\[ i = (a_2 + b_2 q)q \tag{2.11} \]

where \( a_2 \) and \( b_2 \) are constants for a particular medium.
Figure 2.3  Flow in the Turbulent Transitional Regime

Figure 2.4  Flow in the Fully Turbulent Regime
The $a_2$ and $b_2$ would probably be different from the $a_1$ and $b_1$ for the non-linear laminar regime. It should also be noted that the drag on a particle is affected by the location of the point of flow separation. This may change with Reynolds number and is influenced by the particle shape.

The fully turbulent regime exists at very high Reynolds numbers when the skin friction can be ignored, turbulence exists in all regions of the flow, flow separation is stabilized, and the hydraulic gradient can be taken as

$$i = b q^2$$  \hspace{1cm} (2.12)

This is illustrated in figure 2.4.

There is still another non-Darcy flow regime but this is associated with extremely small velocities of the order of $10^{-5}$ cm/sec (39) at which the flow may become non-Newtonian. In this regime the forces of molecular attraction become important. Impurities in the fluid can affect the friction in the flow. Kovacs (52) refers to this regime as microseepage and represents it by

$$i = i_o + aq$$  \hspace{1cm} (2.13)

where $i_o$ is a threshold or adherence gradient.

More complex relationships have been proposed for the microseepage regime. For example, Swartzendruber (118) gives
\[ q = c_1 i + c_2 i^2 \]  \hspace{1cm} (2.14)

where \( c_1 \) and \( c_2 \) are constants, for microseepage through clays. Equation 2.14 is similar to the non-Darcy equations for high velocity flow, thus although this thesis is not primarily concerned with microseepage, some of the proposed finite element techniques could be extended to study two dimensional flow in the microseepage regime.

Figure 2.5, adapted from Kovacs (54), illustrates schematically the five flow regimes mentioned above.

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**Figure 2.5 Flow Regimes in Porous Media [(after Kovacs (54)]**
2.2.2 The Historical Development of Newtonian Non-Darcy Flow

The familiar linear laminar flow or Darcy's Flow Law was first proposed by Darcy (19) in 1856. In 1900 Allen (3) suggested that for relatively high flow velocities around a sphere the resistance is proportional to the velocity to approximately the power 1.5 rather than to the first power as given by Stokes. A year later Forchheimer (32) proposed an equation of the form

\[ i = (a + bq)q \]  

(2.15)

where \( a, b \) are constants, for the non-linear or non-Darcy flow regime in porous media. Forchheimer defended this equation on the basis of an analogy with tube flow. In 1924 he (33) also gave a revised equation

\[ i = aq + bq^2 + cq^3 \]  

(2.16)

which was found to fit his experimental data better. It would appear that Forchheimer's data were obtained from non-linear laminar flow (120).

There have emerged a number of representations for flow at high velocities in porous media:

a) the empirical form

\[ i = aq^M \]  

(2.17)

where \( 1 \leq M \leq 2 \),
b) the semi-theoretical series form

\[ i = aq + bq^2 + \ldots \]  \hspace{1cm} (2.18)

c) graphical representations,

c) statistical models.

**Exponential Form:** Missbach (74), (1937), suggested a more general form of equation 2.17, although White (127) in 1935 had earlier attempted to obtain values for \( \dot{M} \). Bakhmeteff and Feodoroff (7) in 1937 expressed equation 2.17 in the form of a friction factor as

\[ f_p = C R_p^{0.2} \]  \hspace{1cm} (2.19)

where \( R_p \) is a Reynolds number;

\[ C \] is a constant;

\[ i = \frac{f_p q^2}{2gD} \pi^5 m^3 \]  \hspace{1cm} (2.20)

\[ m \] = porosity;

and \( D \) = grain size.

The exponential equation was also used by Dudgeon in his studies (24, 25, 26). He expressed

\[ i = \frac{f_q}{2gD} q^2 \]  \hspace{1cm} (2.21)
where $f$ is a friction factor which is a function of the local Reynolds number, $D$ is a length characteristic of the medium. He suggested the approximate relationship

$$\begin{align*}
  f &\propto (\text{porosity})^5 \frac{R^M}{N} \\
  & (222)
\end{align*}$$

Dudgeon (26), Mott (75) Saunders and Ford (100) and Rose (96) have noted the importance of the wall effect in permeameter tests. Dudgeon indicates that the $i$ in the core is about 5% to 10% higher than the average $i$ for the whole cross-section of a permeameter.

Escande (29), Parkin et al (86) Curtis and Lawson (18) Ananda-krishnan and Varadarajulu (6) and Kirkham (50) have presented variations of the exponential relationship for non-Darcy flow. In 1967 Oliver (82) used an exponential equation similar in form to the Chezy equation, i.e.

$$q = Cr^{a'}i^{b'}$$

(2.23)

where  

- $i =$ hydraulic gradient,
- $c =$ a constant
- $r_m =$
- $e =$ void ratio
- $V =$ volume of rock
- $A_s =$ surface area of rock
- $a'$ and $b'$ are exponents,

...to compute seepage through a rockfill dam.
The main disadvantage of the exponential equation is that the exponent is usually a function of the local Reynolds number, thus making it difficult to treat two dimensional flow if large variations in the Reynolds number occurs.

Series Expressions: Besides the analogy used by Forchheimer (32) several approaches have been used to obtain equations of the form of equation 2.18. Dimensional analysis, capillaric models, hydraulic radius models, numerical solutions of the Navier-Stokes equations and other more intuitive approaches have been used and generally lead to equations of the form of equation 2.18.

Rose (94, 95, 96, 97) was one of the first to use dimensional analysis in the study of non-Darcy flow and proposed the relationship

\[
\frac{\Delta H}{D} = F \left[ \left( \frac{D_q}{u} \right)^{e_1} \left( \frac{D_g}{2} \right)^{e_2} \frac{\Delta}{} \gamma \left( \frac{D}{D} \right) \theta \left( \varepsilon \right) \eta \left( m \right) \lambda \left( SF \right) \omega \left( \sigma \right) \zeta \right]
\]  

(2.24)

where \( \Delta H \) = head loss
\( \Delta \) = flow distance
\( D \) = permeameter size
\( \varepsilon \) = particle roughness
\( SF \) = particle shape factor
\( \sigma \) = particle distribution

but on the basis of his experiments he gave (95)

\[
\lambda = \frac{C_1}{R_N} + \frac{C_2}{R_N} + C_3
\]  

(2.25)

where \( \lambda \) = a resistance coefficient
Rose's equation can easily be rewritten as

$$i = aq + bq^{3/2} + cq^2$$  \hfill (2.27)

Ward (123, 124, 125, 8, 58, 83, 109) also used dimensional analysis to obtain an equation

$$\frac{dp}{dl} = f(q, k, \rho, \mu)$$  \hfill (2.28)

or

$$i = \frac{bq}{k} + \frac{cq^2}{gk^{1/2}}$$  \hfill (2.29)

where $k$ is the permeability

$c$ is a constant

$\sim 0.55$

$p$ = pressure

Ward adapts the form of the Kozeny-Carman (105) equation

$$k \propto \frac{m^3}{KTS_o(1 - m)^2}$$  \hfill (2.30)

to obtain
where \( s = \text{specific surface}, \)
\( K = \text{constant (depends on shape of pore cross-section)}, \)
\( T = \text{tortuosity}, \)
\( M_g = \text{geometric mean particle size}, \)
\( \sigma_g = \text{geometric standard deviation}, \)
\( \alpha = \text{grain shape factor}. \)

Ward used \( k^{1/2} \) as a characteristic dimension of the medium.

Harleman et al. (40) also used \( k^{1/2} \) as a characteristic length for their study of dispersion in porous media.

Amhed and Sunada (2) developed Ward's equation by analyzing the Navier-Stokes equations. They neglected the presence of turbulence in the pores and limited their derivation to the case of one dimensional (mean) flow.

Lindquist (63) proposed a dimensionless equation

\[
f(R) = 2500 + 4R \tag{2.32}
\]

where \( f = \frac{2\rho D_i}{q^2} \tag{2.33} \)

and \( R = \frac{qD}{\nu} \)

which has a similar form to the Forchheimer equation 2.18.

Chaveteau and Thirriot (15) have also obtained an equation
similar to Lindquist's.

Engelund (27) on the basis of the observations of Lindquist (63), Muskat (76) and others (124, 58, 117) proposed the following equations for the constants in the Forchheimer equations 2.18:

\[
a = \alpha_0 \left(1 - \frac{m}{2}\right) \frac{\mu}{D^2}
\]  
and \[
b = \beta_0 \left(1 - \frac{m}{3}\right) \frac{\rho}{D}
\]  

where \(\alpha_0\) and \(\beta_0\) are shape factors.

For example:

a) spheres: \(\alpha_0 = 780\) and \(\beta_0 = 1.8\);

b) rounded sand: \(\alpha_0 = 1000\) and \(\beta_0 = 2.8\);

c) irregular angular grains: \(\alpha_0 = 1,500\) and \(\beta_0 = 3.6\)

Ng (79) has made studies on crushed rock in an attempt to adapt the Ward equation to angular materials. His results indicated that Ward's porosity function may not adequately describe the turbulent term in 2.18.

A capillaric model is one in which the medium is represented by a system of tubes. This analogy is reviewed by Scheidegger (105). The tube flow is assumed to be separable into a 'laminar' zone where \(i \propto q\) and a 'turbulent' zone where \(i \propto q^2\) which naturally leads to the Forchheimer equation 2.18.
Extending the Kozeny-Carman (105) theory to the capillaric model with turbulence making use of a tube flow equation proposed by Reynolds (93)

\[ \frac{\Delta P}{\Delta L} = aV + b \rho V^2 \]  

(2.36)

Ergun and Orning (28) derived

\[ |\nabla p| = 5 \alpha^* m (1-m)^2 \frac{\mu S^2}{q} + \frac{\beta}{3} (1-m) \rho \frac{S^2}{S_o} \]  

(2.37)

where \( \alpha \) is obtained from Carmen (12) as

\[ \alpha = 5 \alpha^* m (1-m)^2 \frac{\mu S^2}{S_o} \]  

(2.38)

\[ b = \frac{\beta}{2\zeta^*} \]  

(2.39)

\[ \zeta^* = \frac{4m}{S_o (1-m)} \]  

(2.40)

\( S_o \) = specific surface  
\[ = \left( \frac{\text{surface exposed to flow}}{\text{volume of solids}} \right) \]

\( \alpha^* \) and \( \beta^* \) indicated the distribution of laminar or turbulent flow respectively.

Recently Watson (126) and Stark (114) have numerically integrated the Navier-Stokes equations to obtain solutions for non-Darcy
flow. Watson and Stark solved an idealized two-dimensional medium by finite differences. Stark indicates how the a and b in equation 2.18 can be found from a computer analysis.

**Graphical Representations:** A number of researchers have presented their conductivity data in graphical form. Prinz (91, 60) in 1930 made a dimensional plot of 'i' verses 'q' for various materials. He placed the limit on Darcy's law at the point on each curve where the curve deviated from a straight line.

Mott (75) in 1950 presented a paper on the movement of particles through a fluid and the movement of a fluid through a bed of particles; similar wall effects and resistance curves were noted. He plotted $\lambda$ versus $R_N$, where

$$
\lambda = \frac{2g_i}{\rho q} \frac{3}{6(1-m)} + \frac{D_R}{4D/T} \quad (2.41)
$$

and

$$
R_N = \frac{qD_g}{\nu} \frac{4}{3.9 + 1.6 \frac{D_g}{D_T}}
$$

where $D_g$ is the grain size and $D_T$ is the permeameter size.

The experimental results of Bakhmeteff and Feodoroff (7), Burke and Plummer (11) Sanders and Ford (100) Chambers et al (13) and Chilton and Colburn (75) were utilized by Mott.

Dudgeon (24, 25) also plotted 'i' verses 'q' but on a log-log plot. He attempted to define various flow regimes by the straight line segments of his curves. Dudgeon's work included angular and rounded material with macroscopic velocities in the range $10^{-4}$ to 40 cm. per
second. This very extensive work will be examined in Chapter III in connection with the development of a general non-Darcy flow equation. Dudgeon has also presented his results in a dimensionless form as a friction factor,

\[ f = \frac{2g_D^{50_i}}{q^2} \]

versus a Reynolds number,

\[ \frac{qD_{50}}{\nu} \]

however the curves for the various materials did not plot on a single curve.

Lane (56) has also given some experimental data for crushed rock and rounded glacial gravels.

Wright (133) also used a friction factor versus Reynolds number plot. His friction factor had the form

\[ \lambda_g = \frac{g_D (1-m)^2}{R_g .1067 q^2 S(\text{shape, } \sigma_g, \varepsilon, D_T)} \]

where the subscript g refers to the grains;

\[ \varepsilon = \text{roughness of the grains.} \]

and his Reynolds number

\[ R_N = \frac{.6 q D_g}{\nu(1-m)} \]
Zampaglione (137) also used a friction factor

$$\lambda_m = \frac{i g D_\Delta}{2 q_m^2}$$  \hspace{1cm} (2.44)

versus

$$R_m = \frac{q_m D_\Delta}{\nu}$$  \hspace{1cm} (2.45)

to present his experimental results on a Fanning type diagram. He defined his terms as follows:

$$q_m = \frac{q}{m^{2/3}}$$  \hspace{1cm} (2.46)

$$D_\Delta = \text{reduced radius of the medium}$$

$$= \sqrt{\frac{q_m \nu}{1}}$$  \hspace{1cm} (2.47)

The reduced radius is similar to the characteristic dimension $\sqrt{K}$ used by Ward (124) and Massarani (68).

Kovacs (52, 53, 54) has presented some of his own research as well as a review of other research (mainly European) on Newtonian and non-Newtonian non-Darcy flow in porous media. In the case of Newtonian flows he uses the work of Zunker (141), Lindquist (63), and Nagy (77) and plots their data as well as his own on a dimensionless, log-log plot of

$$\frac{\lambda R_{es}}{s} \frac{m^3}{(1-m)^2} = f(R_{ep})$$  \hspace{1cm} (2.48)
where 

$$\lambda R_{es} = \frac{2g\eta D_h^2}{q\nu} \tag{2.49}$$

$$D_h = \text{effective diameter of the medium,}$$

$$R_{ep} = \frac{4\pi}{(1-m)} \frac{D_h q}{\alpha_s \nu} \tag{2.50}$$

$$\alpha_s = \text{shape factor}$$

$$= \frac{\text{Grain Surface Area}}{\text{Grain Volume}} D_h$$

This relationship is based on the Kozeny-Carman (105) equation.

Kovacs (53) studied the applicability of this relationship for various particle shapes and porosities and found that spherical particles packed at various porosities could adequately be represented but that disc shaped particles could not be represented. This would suggest a modification of the shape factor to take into account flat surfaces and angularity of the materials; this will be considered further in Chapter V.

Statistical Models: Scheidegger (103, 104, 105) has extended his own statistical treatment of Darcy flow which was based on Gibbs' work (37) to the case of non-Darcy flow. He introduces a probability - density function, \( P(x, t) \), for a fluid or particle to be at a point \( x \) at a time \( t \), (in an ensemble of porous media):

$$P(x, t) = \frac{1}{(4\pi D_1 t)^{3/2}} e^{-\frac{(x-x)^2}{4D_1 t}} \tag{2.51}$$
where $D_i$ is a dispersivity factor.

Further denoting, for the purely turbulent regime, the particle displacement during a small time step $\tau$, as

$$\frac{\vec{x}}{\tau} = -C\sqrt{\|\nabla p\|\cos \theta}$$

(2.52)

where $C$ = constant;

$|\nabla p|$ = magnitude of the gradient of the field of forces

$\theta$ = angle between $\xi$ and $\nabla p$.

and averaging (in an ensemble) he obtains the pore velocity

$$\vec{q}_p = -n \frac{\sqrt{\|\nabla p\|/\rho}\cos \theta}}{\tau}$$

(2.53)

where $\vec{n}$ is the direction of $\nabla p$,

since $\vec{q}_p = \frac{\vec{x}}{\tau}$

and $|\vec{q}_p| = \frac{|\vec{x}|}{\tau} \cos \theta$  (2.54)

Now $\vec{x} = \vec{q}_p \tau$ in the probability density equation 2.51. Apparently the factor $D_i$ can be found by observing the dispersion of a tracer in the medium.

Critical Reynolds Number: Since the pores in porous media are usually random in size, shape and orientation it is difficult to predict with any accuracy the limits of a particular regime, e.g.
estimates of the critical Reynolds number for the upper limit of Darcy flow vary from about 0.1 to 70 (105). However because of the random nature of porous media the change from one regime to another is gradual as contrasted with the abrupt changes associated with flow in straight prismatic channels or pipes (99, 17).

2.3 Steady Non-Darcy Flow in Two Dimensions

2.3.1 General

Some of the research on non-Darcy flow has been associated with the design of rockfill dams. Wilkins (129, 18) in 1956 appears to be the first to propose the use of a rockfill dam capable of passing flow both through and over the rockfill. Largely as a result of his work the Laughing Jack Dam in Tasmania was built as a through-and-overflow structure. This rockfill has an impervious core with a dropped central overflow section (in-built spillway) which is protected by a layer of concrete. The overflow section of the core is completely imbedded in the rockfill. One of the advantages of the inbuilt spillway is that the extensive energy dissipation works associated with conventional spillway are avoided. A typical arrangement of an inbuilt spillway is shown in figure 2.6.

Figure 2.6 A Sketch of an Inbuilt Spillway
Olivier (82) reports the results of various through-and-overflow spillways, such as the one shown in figure 2.7.

Figure 2.7 One of Olivier's Rockfill Dams

Parkin et al (85), Curtis and Lawson (18) and Fenton (30) have studied the hydrodynamics of flow through rockfill dams. Parkin and Fenton have discussed the stability of structures with in-built spillways. Curtis and Lawson consider the hydraulics of flow through and over rockfill structures; they also give a selected bibliography on rockfill dams.

The literature pertaining to the nature and computations of steady two dimensional non-Darcy flow will now be considered.

2.3.2 The Governing Equations for Steady Two Dimensional Non-Darcy Flow

Many investigators (see for example, Khristianovich (49) Sokolovskii (112) Engelund (27), Parkin et al (85)) have adapted the one dimensional non-Darcy equation
\[ i = F(q) q \]  

(2.55)

to describe two and three dimensional non-Darcy flow by putting

\[ \nabla \phi = -F(q) \vec{q} \]  

(2.56)

where \( \phi \) = piezometric head

\[ \phi = \frac{p}{\gamma} + y \]  

(2.57)

and \( q = |\vec{q}| \)

When equation 2.56 is combined with the continuity equation

\[ \nabla \cdot \vec{q} = 0 \]  

(2.58)

one obtains

\[ \nabla \cdot \left[ \frac{\nabla \phi}{F(q)} \right] = 0 \]  

(2.59)

or \[ \nabla \cdot [K(q) \nabla \phi] = 0 \]  

(2.60)

where \( K(q) = \) hydraulic conductivity

\[ = (F(q))^{-1} \]
Khristianovich (49) mentions both the modified Forchheimer equation

\[ F(q) = a + bq + cq^2 \quad (2.61) \]

and the exponential equation

\[ F(q) = \frac{q^{M-1}}{k_0} \quad (2.62) \]

where \(a, b, c\) and \(k_0\) are constants and the exponent \(M\) is slightly dependent on the local Reynolds number.

Engelund (27), Oka (81), Baturic-Rubcic (9, 10), Volker (122) and the writer (70, 71, 72, 73) have utilized the original Forchheimer equation

\[ F(q) = \left[ K(q) \right]^{-1} \]
\[ = a + bq \quad (2.63) \]

as well as the relations

\[ \vec{q} = u_i + v_j \quad (2.64) \]
\[ u = -K(q) \vec{\xi}_x \quad (2.65) \]
\[ v = -K(q) \vec{\xi}_y \quad (2.66) \]
and the continuity equation 2.58 in their derivation of equation 2.60.

From equations 2.63 to 2.65 it is possible to solve for $K(q)$ as a function of $|\nabla \Psi|$, viz.

$$K(|\nabla \Psi|) = \frac{a}{2b} \left( \sqrt{1 + c \frac{|\nabla \Psi|}{|\nabla \Phi|}} - 1 \right)$$  \hspace{1cm} (2.67)$$

where $c = \frac{4b}{a^2}$.

The writer (71) in discussing the work of Oka (81) suggested the following forms of the governing equations based on equation 2.60 and 2.67;

$$\nabla^2 \psi = -\frac{b}{F(q)|\nabla \psi|}Q(\psi)$$  \hspace{1cm} (2.68)$$

$$\nabla^2 \Phi = \frac{b}{|\nabla \psi|} \left[ \psi_x (\psi^2_y - \psi^2_x) + \psi_y \left( \psi_y \psi_y - \psi_{xx} \right) \right]$$  \hspace{1cm} (2.69)$$

$$\nabla^2 \Phi = -\frac{1}{|\nabla \Phi|} \left\{ \frac{F(|\nabla \Phi|)}{a \sqrt{1+c |\nabla \Phi|}} - 1 \right\} Q(\Phi)$$  \hspace{1cm} (2.70)$$

where $\psi$ is the stream function;

$$Q(p) = \left( p_x^2 p_{xx} + 2p_x p_y p_{xy} + p_y^2 p_{yy} \right)$$

and the subscripts $x$, $y$ denote partial differentiation.

These equations indicate the deviation from the Laplace equation due to non-linearity.

The Australian group (85, 18, 30) have preferred equation 2.62 to express $F(q)$; thus instead of equation 2.70 they obtain
where \( N = \frac{1}{M} \)

Kirkham (50) modified equation (2.71) using the total energy head, \( E \), instead of \( \phi \) to obtain

\[ \nabla^2 \phi = \frac{(N-1) \frac{Q(\phi)}{|\nabla \phi|}} {1} \tag{2.71} \]

where \( N = \frac{1}{M} \)

Equation 2.72 is based on the assumption that the flow will follow the maximum energy gradient.

The Russian investigator, Sokolovskii (112) found it useful from an analytical point of view to take

\[ F(q) = \frac{q}{k_o \sqrt{1-(q/N)^2}} \tag{2.74} \]

where \( k_o \) is a constant

and \( q \leq N \leq \infty \).

The value of \( N \) indicates the importance of the non-Darcy component of the flow.

Recently Wright (133) has questioned the validity of applying
equations of \( F(q) \), that have been derived from one dimensional studies, to two dimensional flow. In particular he made a study of flow resistance in parallel and converging (radial flow) permeameters. Rounded granular media were used in his studies. He found that the resistance factor \( \lambda \), was reduced up to 30 percent in the convergent flow relative to the parallel flow. The explanation for the phenomenon is not completely understood but it would appear that convergence (on a macroscopic scale) may affect the flow as follows:

a) there may be a slight change in the flow separation pattern thus reducing the drag;

b) convective acceleration on a macroscopic scale may delay the onset of turbulence;

c) the formation and decay of turbulent 'eddies' may be affected by convergence;

d) the average kinetic energy will vary from point to point in converging flow giving a relatively lower piezometric reading for the higher velocities;

Also experimental errors such as slight changes in porosity and the variable wall effect could contribute to the apparent effect.

2.3.3 Solution of Steady Two Dimensional Non-Darcy Flow

Since the partial differential equation for non-Darcy flow (equation 2.60) is non-linear, many of the analytical techniques for linear equations (e.g. superposition) are not applicable. However in some
instances the non-linear system can be transformed into a linear system and the available analytic techniques applied. Another method of solving non-linear systems is the numerical method of replacing the governing equations by a set of finite difference equations. If the non-linear system can be described in a variational form (e.g. as an extremal of energy functional) a finite element analysis might be attempted. Electrical analogues have been used to simulate non-Darcy flow. Hydraulic Models will at least give qualitative information about prototype non-Darcy flows. There are a number of other methods that can be used to solve non-Darcy flow. Trial and error flow nets can be used.

Linearization and simplification of the original equations lead to approximate solutions. The method of perturbations has not as yet been developed for non-Darcy flow. In a few special cases the non-Darcy flow equations can be solved exactly e.g. radial flow which is considered in Chapter III.

Method of Transformations: Kristianovich (49) in 1940 introduced a method of solving non-Darcy flow. His method involved a number of transformations and an analytic function. He considered

\[ \nabla \phi = -F(q) \, \vec{q} \]

\[ = -\frac{\vec{q}}{K(q)} \]  \hspace{1cm} (2.75)

from which he obtains

\[ \frac{u}{k} x - \frac{v}{k} y = 0 \] \hspace{1cm} (2.76)
and for incompressible flow

\[ \nabla \cdot \mathbf{q} = 0 \quad (2.58) \]

He introduces a stream function, \( \psi \), by

\[
\psi_y = u = q \cos \theta \quad (2.77)
\]

and

\[
\psi_x = -v = -q \sin \theta \quad (2.78)
\]

Now transforming to \( \phi \) and \( \psi \) as independent variables gives

\[ \theta \psi - \frac{F(q)}{q} q \phi = 0 \quad (2.79) \]

and

\[ \theta \phi \phi + \frac{f'(q)}{F(q)} q \psi = 0 \quad (2.80) \]

He then obtains a set of simultaneous equations for \( q, \theta, \psi \) and \( \phi \) as functions of new (auxiliary) independent variables \( (\mu, \nu) \) i.e.

\[ \psi_\nu + \frac{R(q)}{F(q)} \phi_\mu = 0 \quad (2.81) \]

\[ \psi_\mu - \frac{R(q)}{F(q)} \phi_\nu = 0 \quad (2.82) \]
\[ R(q) q_v - \theta_\mu = 0 \]  
(2.83)

and
\[ R(q) q_\mu + \theta_v = 0 \]  
(2.84)

where \( f(q) = \frac{F(q)}{q} \)

and \( R(q) = \frac{\sqrt{f''(q)}}{F(q)} \)

Defining a fictitious velocity

\[ S = \ln q = \int R(q) dq \]  
(2.85)

He finally obtains:

\[ S_v - \theta_\mu = 0 \]  
(2.86)

\[ S_\mu + \theta_v = 0 \]  
(2.87)

also

\[ \tilde{\theta}_\mu = -L(q) \psi_v \]  
(2.88)

\[ \tilde{\theta}_v = -L(q) \psi_\mu \]  
(2.89)

where \( L(q) = F(q) R(q) \)  
(2.90)

Therefore, once \((S, \theta)\) are found from 2.86 and 2.87, \( L(q) \) can be found in terms of \((\mu, v)\) and \( \tilde{\theta} \) and \( \psi \) solved in terms of \((\mu, v)\) from
It remains to relate the \((\mu, v)\) plane to the \((x, y)\) plane, which he assumes are related by an analytic function, \(Z\), viz.

\[
(\mu + iv) = Z(x + iy)
\]  

(2.91)

Sokolovshii (112) also proposed a method for obtaining analytical solutions to non-Darcy flow but his solution required that \(F(q)\) be defined by equation 2.74. He also used analytic functions in his solutions; in fact his method appears quite similar to the hodograph method used for solving compressible potential flow of gases (5, 14). The hodograph method is also suggested by Poritsky (90) as a possible means for solving saturated magnetic field problems (which have governing equations similar to non-Darcy flow).

Engelund (27) in 1953, transformed the independent variables \(x\) and \(y\) of equation 2.60 [with \(F(q) = a + bq\)] to \((q, \theta)\) thus linearizing the internal flow equation as

\[
\frac{\partial}{\partial q} \left( -\frac{q}{F(q)} \frac{\partial \delta}{\partial q} \right) + \frac{1}{F} \left( \frac{1}{q} + \frac{F'}{F} \right) \frac{\partial^2 \delta}{\partial \theta^2} + \frac{1}{F} \frac{F'}{F} \frac{\partial \delta}{\partial \theta} \left( 2.92 \right)
\]

and introducing a new variable \(\xi = \frac{b}{a}\) he writes

\[
\left( \frac{\xi}{F} \frac{\delta}{F} \right)_{\xi} + \frac{1}{F} \left( \xi^{-1} + \frac{F'}{F} \right) \xi_{\xi}^2 \xi_{\xi} = \left( 2.93 \right).
\]

He solves equation 2.93 for the special boundary conditions shown in
In solving equation 2.93 he assumes on the basis of experimental data that

\[ F = \frac{1}{K_0} \text{ for } \xi \leq 0.07 \] (2.94)

\[ F = a(1 + \xi) \text{ for } \xi > 0.07 \] (2.95)

i.e. he uses \( \xi \) as a form of Reynolds number.

Figure 2.8 Engelund's Boundary Conditions
Finite Difference Solution: Finite difference solutions to non-linear equations similar to equation 2.60 are to be found in other fields of engineering. A saturated magnetic field can be modelled (4, 5, 90, 121) by the equation

\[ \nabla \cdot [\mu (|\nabla \psi|) \nabla \psi] = 0 \]  \hspace{1cm} (2.96)

where \( \psi \) is a scalar magnetic potential,
\( \mu \) is the magnetic permeability which in a saturated ferromagnetic material depends on the magnetic intensity.

Poritsky (90) and Trutt et al (121) have applied the finite difference method (FDM) to solve equation 2.96. Poritsky used the regular computational grid shown in figure 2.9.

![Regular 5-Point Finite Difference Grid](image)

He then rewrites equation 2.96 as
\[ \psi_{xx} + \psi_{yy} + M_x \psi_x + M_y \psi_y = 0 \] (2.97)

where \( M = \ln \mu \)

and thus obtains the difference equation

\[ 4 \sum_{i=1}^{4} (\psi_i - \psi_o) + \sum_{i=1}^{4} M_i \psi_i - \sum_{i=1}^{4} M_{i+2} \psi_i = 0 \] (2.98)

Trutt et al used the irregular grid shown in figure 2.10

---

**Figure 2.10 Irregular Finite Difference Grid (After Trutt et al)**

They proposed a relaxation formula

\[ \psi_o = \frac{4, d}{\sum_{i=1}^{4, d} \mu_{a_1 i} \psi_i} \] (2.99)

where \( \alpha_i = \frac{2}{h_i (h_i + h_{i+2})} \) (2.100)

The case of
\[ \mu = (a + b|\nabla \psi|)^{-1} \quad (2.101) \]

was considered. An iterative procedure was used to solve 2.99.

An FDM was used by Young and Wheeler (136) to solve the problem of steady laminar non-Newtonian flow in a square duct. The governing equation for their problem was

\[ (F_{q_x})_x + (F_{q_y})_y + \frac{fR_N}{2} = 0 \quad (2.102) \]

where \( q \) = velocity in the z direction;
\[ F = \left[ (q_x^2 + q_y^2)^{1/2} \right]^{n-1}; \]
\( f \) = friction factor;
\( R_N \) = Reynolds number.

Their method of solution was similar to Poritsky's, except that the term \( \frac{fR_N}{2} \) was added to the difference equations. Their difference equations were

\[ \frac{1}{2} (2W_o + W_1 + W_3)U_o - \frac{1}{2}(W_o + W_1)U_1 - \frac{1}{2}(W_o + W_3)U_3 + \frac{1}{2}(2W_o + W_2 + W_4)U_4 \]

\[ - \frac{1}{2}(W_o + W_2)U_2 - \frac{1}{2}(W_o + W_4)U_4 = \frac{\nu}{2} \quad (2.103) \]

where \( U = u/c \)
\[ W = \left( U_x^2 + U_y^2 \right)^{(n-1)/2} \]
\( F = c^{n-1}W \)
\[
c \int W \, dx \, dy = \int u \, dx \, dy = 1
\]

and

\[
\frac{f_{\text{RN}}}{2} = c^n
\]

The system 2.103 was solved by a Peaceman-Rachford (88) inner iteration loop (to solve for \( U \)) with an outer iteration loop that updated the values of \( W \). Convergence was found to be good except for \( n = \frac{1}{2} \).

Curtis and Lawson (18) and Fenton (30) have applied the FDM to non-Darcy flow for the exponential conductivity equation 2.62. They used the regular 9-point computation molecule shown in figure 2.11.

Figure 2.11 Regular 9-Point Finite Difference Molecule

(After Curtis and Lawson)

Their relaxation equation (applied to equation 2.71) was
The case of an overtopped rectangular dyke was solved for boundary conditions obtained from a small scale laboratory model.

Oka (81) has recently applied the FDM to non-Darcy flow using a special anisotropic form of the Forchheimer equations. He used a 5-point computation molecule. The writer (71) has indicated how Oka's work could be extended to the more general case of non-Darcy flow represented by equation 2.60.

**Finite Element Analysis:** The possibility of applying variational calculus to steady non-Darcy flow was first suggested by Engelund (27) in 1953. The finite element method for non-Darcy flow was developed simultaneously at the University of Windsor by the writer (72) and in Australia by Fenton (30) and Volker (122). Fenton restricted his work to the use of the exponential form of the conductivity equation. Volker and the writer used equation 2.63 in their developments.

The basis of the finite element method, which is widely used in structural engineering, are outlined by Zienkiewicz (138), (139). Recently the finite element method has been applied to steady Darcy flow through porous media, by Zienkiewicz, Meyer and Cheung (140), Finn (31) and Taylor and Brown (119). Mauersberger (69) in 1968 applied variational
calculus and error distribution to the solution of Darcy flow problems.

The writer (72) following the work of Engelund (27) attempted to solve non-Darcy free surface flow problems by finite element analysis. Engelund gave the functional for turbulent and transitional two-dimensional flow as

\[ \chi = \int_A K(|\nabla \phi|)(\phi_x^2 + \phi_y^2) \, dx \, dy \] (2.105)

Engelund states that the minimization of Eq. 2.105 will yield the non-Darcy flow partial differential equation 2.60.

As pointed out by the writer (73) a finite element analysis based on Eq. 2.105 gives good results for:

(a) highly turbulent flow;
(b) laminar flow;
(c) parallel or nearly parallel transitional flow;
(d) certain forms of the functions, \( K(|\nabla \phi|) \), for example the exponential form used by Curtis and Lawson (18).

However, in the general case, it appears that a simple minimization of Eq. 2.105 could lead to flow discontinuity. This thesis presents a modification of the functional (Eq. 2.105) which will be suitable for laminar, turbulent and transitional two-dimensional flows. More details of the finite element approach are given in Chapter III.
**Electrical Analogues:** Poritsky (90) describes the use of a D.C. board to study saturated magnetic problems (equation 2.96); the variable \(\mu(|\nabla\psi|)\) was related to the inverse of the local plate resistance; the electric field was analogous to the magnetic field; voltage represented magnetic potential; current was analogous with the flux density. To reproduce non-linear behaviour, the local resistance of the board had to be adjusted manually so that it was a function of the observed local electric field.

Khristianovich (49) in 1940 suggested the use of an electrical analogue for studying non-Darcy flow. More recently Batric-Rubcic (9,10) developed an electrical analogue to represent

\[

\nabla \phi = - (a + bq) q

\]

(2.106)

by utilizing non-linear electrical elements, i.e. elements which do not conform to Ohm's Law. He found that the voltage-current relationship for certain electrical light bulbs could be approximated by

\[

V = (a_o + b_o I) I

\]

(2.107)

where \(V\) is voltage and \(I\) is current.

He installed the bulbs in a grid (see circuit shown in figure 2.12) that could be assumed to approximate a continuum.
Hydraulic Models: - Hydraulic models have been employed by Olivier (82), Curtis and Lawson (18), Parkin et al (85), Ng (79) Wong (132) and Sharma (107) in their studies of non-Darcy flow. Olivier investigated the feasibility of designing rockfill dams for both through and overflow. Curtis and Lawson used a model to establish the boundary conditions for their numerical analysis. Parkin et al used a flume model to show the development of a slip circle in a rockfill dam. Sharma
studied the critical exit gradients in the presence of non-Darcy flow.

Ng (79), Wong (132) and the writer (73) have used two dimensional flume models to study the boundary conditions for free surface non-Darcy flow. For example, experimental studies have been made to establish correlation equations to find the seepage height shown in figure 2.13.

\[ \theta_u = 90^\circ \]

Figure 2.13 Defining Sketch-Flow Through a Rockfill Dam

Ng proposed the seepage height equation for a rectangular section,

\[ (\theta_u = \theta_d = 90^\circ) \]
\[
\frac{H_s}{H_u} = (1 + R)^{-1.90} (1 - \frac{H_d}{H_u})^{-2.06} \frac{5.80(1 - \frac{L}{H})^{-0.611}}{(1 + \frac{L}{H_u})^{-2.06}}
\] (2.108)

This equation was later modified by Ng and the writer (73), using additional experimental data, to -

\[
\frac{H_s}{H_u} = (1 + \frac{R_N}{1000})^{-3.44} (1 - \frac{H_d}{H_u})^{-3.53} (1 + \frac{L}{H_u})^{-1.72}
\] (2.109)

A comparison of the empirical non-Darcy equation for seepage height with the analytical seepage height solutions presented by Muskat (76) and Polubarinova-Kochina (89) is shown in figure 2.14.

**Figure 2.14** Comparison Darcy and Non-Darcy Seepage Heights
Figure 2.15 Effect of Coarse Media on Entrance Angle (after Wong)

Experimental Data (Wong)

\[ M_g = 4.40 \text{ cm} \]
Wong (132) studied the more general case with various combinations of \( \theta_u \) and \( \theta_d \) and derived the correlation equation

\[
\frac{H_S}{H_u} = \frac{R_N}{100} \cdot 1463 \left(1 + \frac{R_N}{100} \right) \left(1 - \frac{d}{H_u} \right) \left(1 + \frac{L}{H_u} \right) ^{-1.256} \left(\sin \theta_d\right)^{-2.45}
\]

He also found that the entrance angle \( \beta \) deviated significantly from the theoretical value of \( 90^\circ \) as shown in figure 2.15. A possible explanation for this deviation is that the theory neglects the effect of grain size. In the model the radius curvature of the macroscopic streamlines at \( B \) is of the same order as the grain size; thus the flow around the grains may mask the macroscopic flow pattern in the neighbourhood of \( B \).

The theoretical tangency condition at the outcrop point \( C \) was also violated in the hydraulic models.

Other Approaches: Poritsky (90) suggested a graphical, flow net approach for obtaining the flux and potential lines in a non-linear field (i.e. for non-Darcy flow, stream functions and piezometric heads). Instead of the 'squares' of the Darcy flow nets, 'rectangles' are obtained in the non-Darcy case, as shown in figure 2.16.

Figure 2.16 Non-Darcy and Darcy Flow nets
The ratio of the sides of a rectangle in non-Darcy flow is

\[ \frac{\Delta s}{\Delta n} = K(q) \frac{\Delta \frac{s}{\Delta y}}{\Delta \gamma} \]  \hspace{1cm} (2.111)

The conductivity, \( K(q) \), can be found from a trial flow net, e.g. a Darcy flow net, and hence \( \frac{\Delta s}{\Delta n} \) recalculated and the flownet adjusted.

Thirriot (120) suggested an approximate solution, to non-Darcy flow through a rectangular section, in which the Dufour-Forschheimer assumption that the discharge/unit width,

\[ Q = -K(q) H \frac{dH}{dx} \]  \hspace{1cm} (2.112)

was utilized. The seepage height was neglected as shown in figure 2.17.

\[ \text{Figure 2.17 Defining Sketch-Flow Through a Rectangular Dyke} \]
He approximated $K(q)$ by

$$K(q) = \frac{K_0}{1 + \xi}$$

(2.113)

where $K_0 = 1/a$

and $\xi = b/a$

Substituting equation 2.113 into 2.112 and integrating gives

$$Q = \frac{K_0}{L} \left\{ \frac{H_1^2 - H_2^2}{2} - Q_\star (H_1 - H_2) + Q_\star^2 \ln \left( \frac{H_1/Q_\star + 1}{H_2/Q_\star + 1} \right) \right\}$$

(2.114)

where $Q_\star = \xi Q$

For the exponential resistance formula

$$\lambda \propto R^{\alpha - 2}$$

(2.115)

Wright (133) proposes a Poisson equation to replace equation 2.60. He gives

$$\nabla^2 \phi = \frac{1}{4} (1 - \frac{1}{a}) \nabla^2 \{ \ln \frac{1}{2} [\nabla^2 (\xi^2_\star)] \}$$

(2.116)

where $\xi_\star$ is the Darcy flow $\xi$.

Established numerical methods are available for solving the Poisson equation [see e.g. Smith (111)].

By using polar coordinates Kirkham was able to solve equation 2.72 for non-Darcy radial flow. Further considerations of analytical solutions of non-Darcy radial flow are given in Chapter III.
2.4 Unsteady Non-Darcy Flow

One of the practical applications of unsteady non-Darcy flow, that was described in Chapter I, is related to wave energy dissipation in rockfill hydraulic structures. Knowledge of the flow within the rockfill protection of a dam would enable an engineer to determine the possible fluctuations in water levels at the impervious core and thus establish a rational free board for the core. Also more insight into the influence of filter layers on the stability of armour blocks (protecting dykes and breakwaters) could be gained by studying the unsteady non-Darcy flow in the filters. A technique for solving unsteady non-Darcy flow would aid in computations of wave transmission and absorption by permeable breakwaters.

The following background to the unsteady non-Darcy flow developments of this thesis considers:

a) the literature on the governing equations;

b) the published solutions of these equations or techniques for studying unsteady non-Darcy flow;

c) some of the promising methods that have been used for similar linear and non-linear problems.

2.4.1 The Governing Equations

Polubarinova-Kochina (89) gives the equation for unsteady non-Darcy flow as

\[ i = aq + bq^2 + c \frac{\partial q}{\partial t} \] (2.117)
where $a$, $b$ and $c$ are constants.

This equation is similar to the unsteady Darcy flow equation (42, 43).

\[ - \nabla \phi = aq + \frac{1}{g_m} \frac{\partial q}{\partial t} \]  

(2.118)

Combining equation 2.118 with the continuity equation Hunt (42) shows that

\[ \nabla^2 \phi(x, y, t) = 0 \]  

(2.119)

It will be shown in Chapter III that the non-Darcy equations 2.117 cannot be reduced to the simple Laplacian. The term \( \frac{1}{g_m} \frac{\partial q}{\partial t} \) is commonly ignored in Darcy flow (see Liggett (62), Karadi et al (47)).

Proudman (92) and Lean (59) give the simplified equations for turbulent flow through a permeable wave absorber as

\[ \eta_t = - (hu)_x \]  

(2.120)

and

\[ u_t = -g\eta_x - \frac{ku}{h} \]  

(2.121)

where

- $\eta$ = water surface elevation above the mean level
- $u$ = horizontal velocity,
- $k$ = a resistance factor,
- $h$ = depth to the mean water level.
2.4.2 Solution of Non-Darcy Flow

Few analytical solutions have been published for unsteady non-Darcy flow. Lean (59) linearizes the equations 2.121 following a procedure suggested by Lorentz (64) and Proudman (92). The quadratic term is linearized by introducing a constant $f$, such that

$$f = \frac{8k}{3\nu \rho} U$$

where $U$ is the amplitude of the local wave particle velocity. The factor $f$ is chosen so that the linearized term will give the same energy loss per unit plan per cycle as the quadratic term. Thus equation 2.121 can be written in a linear form

$$u_t = -g \eta x - fu$$

which with equation 2.120 may be solved by conventional means. Lean computes reflection coefficients for high porosity wire mesh absorbers.

Experimental wave reflection coefficients have been determined by Straub et al (116), for highly permeable and impermeable wave absorbers.

Johnson et al (45) made an experimental study of scale effects on wave energy transmission through rockfill models. They used the Froude Law (57) for scaling three models which varied in length from 15 cm. to 60 cm, with particle sizes scaled from 0.90 cm to 3.60 cm. Their plot of transmission ratio $H_2/H_1$ against wave steepness is shown
in figure 2.18.

Figure 2.18 Scale Effects on Wave Transmission

(After Johnson et al)

2.4.3 Techniques for Solving Unsteady Flow Problems

Some of the techniques that have been used to solve unsteady Darcy flow may be extended to solve non-Darcy flow. A few of the promising methods will be described. They include:

a) Finite difference methods,

b) The method of characteristics,

c) The finite element method,
d) Others (e.g. perturbation methods and linearization).

**Finite Difference Methods:** Wigle (128) used the finite difference method to solve the problem of rapid drawdown in the two dimensional rectangular section shown in figure 2.19.

\[ \nabla^2 \hat{\phi} = 0 \]  \hspace{1cm} (2.119)

where \( \hat{\phi} = \hat{\phi}(x, y, t) \)  \hspace{1cm} (2.124)
Wigle considered that a particle once on the free surface would remain on it and since the free surface level equals the piezometric level,

\[ \frac{D}{Dt} \{ \xi(x, y, t) - y \} = 0 \]  \hspace{1cm} (2.125)

on the free surface. He then solves the unsteady flow problem in small time steps, \( \Delta t \). Equation 2.119 is solved first, assuming steady boundary conditions, using the extrapolated Liebmann (61) procedure

\[ \xi^{n+1}_0 = (1-\omega) \xi^n_0 + \frac{\omega}{4} \left( \sum_{i=1}^{4} \xi^n_i \right) \]  \hspace{1cm} (2.126)

where \( \omega \) is the relaxation factor.

Then, the time is varied by \( \Delta t \), and a new free surface is computed by the difference equation for 2.125.

\[ \Delta \xi = \frac{\Delta t \cdot K}{m} \left\{ \frac{\xi_x^2 + \xi_y^2}{4} - \xi_y \right\} \]  \hspace{1cm} (2.127)

where \( K \) is the hydraulic conductivity and \( m \) is the drainable porosity.

Hence a new internal flow solution is possible.

The finite difference method could be extended to non-Darcy flow problems if the inertia term \( c \frac{\partial q}{\partial t} \) is ignored and the difference equation 2.126 is replaced by Poritsky's equation or by the difference equation of Trutt et al (i.e. equation 2.98 or equation 2.99). The main difficulty with using this method lies in programming the boundary...
adjustments of the computational grid. Also the streamline boundary condition can be treated more easily by a finite element method than by the finite difference method.

The Method of Characteristics:- The method of characteristics is a useful method for solving hyperbolic differential equations. The method may be applied to linear or non-linear systems (1, 4). It has been used extensively to study the movement of flood waves in open channels (see for example Chow (16), Isaacson et al (44)).

Dracos (21, 22) applied the method of characteristics to determine wave energy dissipation in the Darcy regime. Mahdaviani (67) used this method to solve unsteady free surface Darcy flow to a well. He uses the Dupuis - Forchheimer approximation thus reducing the independent variables to \((r, t)\) in the axially symmetric system. His final solution is obtained from a set of difference equations.

If the Dupuis - Forchheimer approximation is applied to non-Darcy flow (cf. Lean (59)) one obtains the following system for flow in a rectangular section (figure 2.20) of mean depth \(h_o\):

\[
\begin{bmatrix}
  m(h_o+\eta) & 0 & \mu u & 1 \\
  u/m & 1 & \frac{\mu}{h_o} & 0 \\
  dx & dt & 0 & 0 \\
  0 & 0 & dx & dt \\
\end{bmatrix}
\begin{bmatrix}
  u_x \\
  u_t \\
  \eta_x \\
  \eta_t \\
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  -\frac{\mu}{h_o} F(u)u \\
  du \\
  d\eta \\
\end{bmatrix}
\]

\[(2.128)\]
Putting the determinant of the coefficient matrix equal to zero one obtains the characteristics:

\[
\alpha = \frac{dx}{dt} \bigg|_{\alpha} = uM + Mg(h_0 + \eta + \alpha x KE)
\]

and

\[
\beta = \frac{dx}{dt} \bigg|_{\beta} = uM - Mg(h_0 + \eta + \alpha x KE)
\]

where

\[
M = (1 + m)2m
\]

\[
\alpha x = \left(\frac{1 - m}{m}\right)^2/2\]

and

\[
h_{KE} = (u/m)^2/2g
\]

Since \(g(h_0 + \eta + \alpha x KE)\) is non-zero and positive the system is hyperbolic and may be solved by the method of characteristics (1, 4, 5).

In addition, to the two characteristic equations 2.129 and 2.130,
there are two more equations, i.e.

\[
\frac{d}{dt}(Mu + 2M \sqrt{g}(h_0 + \eta + \frac{\partial h_{KE}}{\partial x})) = -gm F(u) u \tag{2.131}
\]

and

\[
\frac{d}{dt}(Mu + 2M \sqrt{g}(h_0 + \eta + \frac{\partial h_{KE}}{\partial x})) = -gm F(u) u \tag{2.132}
\]

where \( F(u) = a + bu. \)

Equations 2.128 to 1.132 can be solved for \( u = u(x, t) \) and \( \eta = \eta(x, t) \) if the initial and boundary conditions are known. The velocity head \( h_{KE} \) is normally negligible compared with \( (h_0 + \eta) \). The solution is now obtained in the \( x-t \) plane as shown in figure 2.21.

![Figure 2.21 Advancing the Solution by the Method Characteristics](image)

The point \( P \) can be located from the characteristic equations and a finite difference scheme can be used to determine the conditions at \( P \) from the conditions on \( AMB. \)
In theory the method of characteristics could be extended to more than two independent variables as shown by Abbott (1), but most of the development has been for the two variable case.

The Finite Element Method: No reference in the literature was found concerning the application of the finite element method (FEM) to unsteady non-Darcy flow. However a number of transient flow problems have been solved by the FEM.

Wilson and Nickel (130) in 1966 following Gurtin's variational approach (38) set up a FEM to solve transient heat conduction problems. Zienkiewicz (138) outlines a finite element approach to solving time dependent field problems of the type described by the linear differential equations

\[ \nabla \cdot K_{x_i} \nabla \phi + Q - \frac{\partial \phi}{\partial t} = 0 \tag{2.133} \]

where \( K_{x_i} \) = hydraulic conductivity in direction \( x_i \).

He gives the functional for this problem as

\[ \chi = \iiint \left\{ \frac{1}{2} \left( K_{x} \frac{\partial \phi}{\partial x}^2 + K_{y} \frac{\partial \phi}{\partial y}^2 + K_{z} \frac{\partial \phi}{\partial z}^2 \right) - \left( Q - C \phi \right) \right\} \, dx \, dy \, dz \tag{1.134} \]

after discretization \( \chi \) is minimized to yield

\[ \frac{\partial \chi}{\partial \phi_i} = \sum_{j} S_{ij} \frac{\partial \phi}{\partial x_j} + \sum_{j} P_{ij} \frac{\partial \phi}{\partial t} + \sum_{i} F_i = 0 \tag{1.135} \]

where \( S_{ij} \) is a 'stiffness' matrix that includes \( K_{x_i} \); \( P_{ij} \) is similar to a
stiffness matrix except it is obtained from the term, $\hat{F}_i$; $\hat{F}_i$ is an equivalent 'load' associated with $Q$.

Witherspoon et al (131) also took advantage of the work of Gurtin (38) to solve the problem of confined unsteady Darcy flow in a radial flow system. Their governing equation was similar to equation 2.133; however they included the boundary conditions within their functional.

Recently, Sandhu and Wilson (101) made a FEM analysis of soil consolidation i.e. seepage in an elastic medium. They discretized the $x - t$ plane by triangular elements.

Further details of the finite element approach are given in Chapter III.

Some Other Methods:- A matrix method developed by Karadi et al (47) for the Dupuis - Forchheimer form of the unsteady Darcy flow equation could be applied to the linearized non-Darcy equations 2.120 and 2.121 given by Lean (59). Karadi et al solved the unsteady flow equation

$$\frac{m \frac{\partial h}{\partial t}}{K} = \frac{h}{\partial x} \left( h \frac{\partial h}{\partial x} \right)$$

(2.136)

in the dimensionless form

$$\frac{\bar{\omega} U}{\partial T} = \sqrt{U} \frac{\bar{\omega}^2 U}{\partial x^2}$$

(2.137)

by replacing equation 2.137 with the following differential difference equations (in matrix form):
\[
\frac{d\{u\}}{dT} = \frac{1}{\phi^2} \langle D \rangle \left[ A \right] \{u\}
\] (2.138)

where \( \phi \) is the interval in \( U \).

\[
\{u\} = \begin{cases} u_1 \\ u_2 \\ \vdots \\ u_n \end{cases}
\] (2.139)

\[
\langle D \rangle = \langle \sqrt{u_1}, \sqrt{u_2}, \ldots, \sqrt{u_n} \rangle
\] (2.140)

and

\[
\left[ A \right] = \begin{bmatrix}
-2 & 1 & 0 & 0 & \ldots \\
1 & -2 & 1 & 0 & \ldots \\
0 & 1 & -2 & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
0 & 1 & -2 & 1 & \ldots & \\
0 & 1 & -2 & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\] (2.141)

For the \( n \)th time interval the solution of 2.138 is
\[ \begin{align*}
\{U\} &= e^{B T} \{U\}^\text{Tr-1} \\
\text{where } &\Delta T \text{ is a small time step,}
\end{align*} \] (2.142)

\[ B_r = \frac{1}{\ell^2} \langle D \rangle \end{align*} \] (2.143)

and \( \langle D \rangle \) is the average of \( \langle D_r \rangle \) and \( \langle D_{r-1} \rangle \).

Liggett (62) used the Lagrangian form of the unsteady two dimensional Darcy flow equations to study initial motion during rapid drawdown in an earth embankment. The equations of motion and continuity in Lagrangian form are:

\[ \begin{align*}
\frac{1}{m} \frac{\partial^2 x}{\partial t^2} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{g}{K} \frac{\partial x}{\partial t} \\
\frac{1}{m} \frac{\partial^2 y}{\partial t^2} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} - \frac{g}{K} \frac{\partial y}{\partial t} - g \\
\text{and } \frac{1}{m} \left( \frac{\partial x}{\partial A} \frac{\partial y}{\partial B} - \frac{\partial y}{\partial A} \frac{\partial x}{\partial B} \right) &= 1
\end{align*} \] (2.144-2.146)

where \( x = x(A, B, t) \)

\( y = y(A, B, t) \)

and \( p = p(A, B, t) \).

Liggett represented \( x, y \) and \( p \) in time series as follows:
\[ x(A, B, t) = A + x^{(1)}(A, B)t + x^{(2)}(A, B)t^2 + \ldots \]

\[ y(A, B, t) = B + y^{(1)}(A, B)t + y^{(2)}(A, B)t^2 + \ldots \]

\[ p(A, B, t) = p^{(0)}(A, B) + p^{(1)}(A, B)t + p^{(2)}(A, B)t^2 + \ldots \] (2.147)

where the superscripts in parentheses represent the order of the approximation. Equations 2.147 are substituted into equations 2.144 to 2.145 and after some algebraic manipulation the first order solution is found and with it the second order solution can be found.

In non-Darcy flow the \( K \) would be given by

\[ K(q) = \frac{1}{a + bq} \] (2.148)

where

\[ q = \sqrt{x_t^2 + y_t^2} \] (2.149)

This of course makes equations 2.144 and 2.145 non-linear but this could be overcome by linearizing or at least simplifying the \( K(q) \) term for a small time interval. Although this method looked promising for the initial motion problem it appeared to be too cumbersome for the complete solution.
3.1 General Comments

The developments in this chapter are treated in three parts namely:

a) the derivation of the governing equations,

b) the expression of the problem in variational form,

c) the development of the finite element models

The general governing equations are developed for three dimensional unsteady flow in the turbulent transitional regime; however the finite element models are restricted practically (but not theoretically) to the two dimensional flow case. Finite element models are presented for steady confined radial flow, steady unconfined two dimensional flow and unsteady unconfined two dimensional flow. Although the case of non-Darcy flow is stressed the methods discussed are also applicable to the special case of Darcy flow.

3.2 The Governing Equations for Non-Darcy Flow

3.2.1 Derivation of the Differential Equations for Non-linear Flow in the Turbulent Transitional Regime

In Chapter II the existing developments pertaining to the equations that govern one and two dimensional steady non-Darcy flow in porous media were reviewed. No particular difficulty (Ref. (2), (105) and (124)) arose in the development of the one dimensional steady flow equation
\[ i = - \frac{\Delta \phi}{\alpha \phi} = (a + bq)q \]  

(3.1)

However the extension of this equation to unsteady two or three dimensional flow, as the Forchheimer type equation

\[ \dot{\phi} = -(a + bq)q - \frac{1}{\gamma m} \frac{\partial \phi}{\partial t} \]  

(3.2)

requires a number of assumptions. The Navier-Stokes equations cannot be reduced exactly to equation 3.2. Thus if the Forchheimer type of equation is to be used for two or three dimensional flows, one might expect the parameters 'b' and possibly 'a' to depend on the curvature, convergence or divergence of the macro-streamlines. Wright (133) has experimentally confirmed this dependence for the converging flow case.

The Navier-Stokes equations are now examined to show the limitations of equation 3.2 and the form of the parameters a and b. In applying the Navier-Stokes equations to the turbulent transitional regime one should be aware of the following effects:

a) viscous,

b) steady inertial,

c) turbulence,

d) dispersion (due to the randomness of the medium (105)).

In order to include all of these effects the equations of motion will be applied to: a single pore; an ensemble of identical pores; a represent-
ative control volume.

A Single Pore: Consider the flow in the single pore shown in figure 3.1. Note the presence of viscous, steady inertial and turbulence effects.

The Navier-Stokes equations (41) applicable to this pore are:

$$\rho \left( \frac{\partial \vec{U}}{\partial t} + \vec{U} \cdot \nabla \vec{U} \right) = -\rho g \frac{\partial \Pi}{\partial \vec{x}} + \frac{\partial}{\partial x_j} \left( \mu \frac{\partial \vec{U}}{\partial x_j} \right)$$

(3.3)

where $\vec{U}$ is the instantaneous local velocity

$\Pi$ is instantaneous local piezometric head

$$\Pi = \Pi^0 + \rho g r_i \vec{x}_i$$

(3.7)

$\Pi^0$ = instantaneous local pressure, and the continuity equation is:
Since turbulence (see Wright (133), also Appendix A) exists within the pore (in the turbulent transitional regime) the instantaneous velocity should be taken as

\[ \hat{\mathbf{u}} = \mathbf{\bar{u}} + \hat{\mathbf{u}} \]  \hspace{1cm} (3.5)

where \( \bar{\mathbf{u}} \) is an average point velocity taken over several geometrically identical pores (i.e. an identical ensemble average), and \( \hat{\mathbf{u}} \) is the random (turbulent) component of \( \hat{\mathbf{u}} \).

Similarly

\[ \hat{\phi} = \phi + \hat{\phi} \]  \hspace{1cm} (3.6)

Hinze (41) shows the substitution of equations 3.5 and 3.6 into equations 3.3 and 3.4, yields, after averaging, the Reynolds equations for flow in a single pore,

\[ \rho \left( \frac{\partial \hat{\mathbf{u}}}{\partial t} + \hat{\mathbf{u}} \frac{\partial \hat{\mathbf{u}}}{\partial x_j} \right) = -\rho g \frac{\partial \phi}{\partial x_i} + \frac{\partial \bar{\mathbf{u}}}{\partial x_j} \left( \mu \frac{\partial \hat{\mathbf{u}}}{\partial x_j} - \rho \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \right) \]  \hspace{1cm} (3.7)

\begin{tabular}{cccc}
  \hline
  pore & pore & pore & viscous & turb. \\
  time accel. & convective & piez. & effects & \\
  at pt. & accel. & grad. & & \\
  \hline
\end{tabular}
Equations For a System of Pores:— Now considering the flow through a system of pores which are random in orientation, shape and size i.e. random (about some mean pore characteristics for the medium), as shown in figure 3.2, it is noted that the fluid particle path fluc-

\[ \frac{\partial u_i}{\partial x_i} = 0 \]  

\[ \frac{\partial u_i}{\partial x_1} = 0 \]

In addition there are certain boundary conditions on the pore walls and at the entrance and exit to the pore which should be satisfied by the solution of equations 3.7, 3.8 and 3.9.

![Figure 3.2 Flow Through a Random System of Pores](image)
tuates about the mean direction of flow. The particle velocity changes in magnitude and orientation as it moves from one pore to the next. Scheidegger (105) describes this phenomenon as dispersion in his statistical model. Other investigators have accounted for this effect by a tortuosity term in the conductivity equation (see equation 2.31). In this thesis the 'dispersion' effect is incorporated in the Navier-Stokes equations in the same manner as used by Ahmed and Sunada (2) i.e. by defining the control volume as a statistically representative volume of the medium.

A mean or macroscopic flow within a medium may be defined (Hunt (42)) by the governing equations:

\[
\frac{\rho dq_i}{m dt} = - \rho g \frac{\partial \phi}{\partial x_i} + \rho X_i \tag{3.10}
\]

and

\[
\frac{\partial q_i}{\partial x_i} = 0 \tag{3.11}
\]

where \( q_i \) = macroscopic velocity or bulk velocity (i.e., in a statistical sense, the average of the point velocities taken from an infinite ensemble of random media having the same average pore characteristics);

\( \phi \) = the macroscopic piezometric head;

\( X_i \) = the apparent force per unit mass in the \( x_i \) direction due to viscosity and inertial terms.

The term \( X_i \) incorporates the viscous forces, as well as the turbulence...
and steady inertial terms that have not been accounted for by the macroscopic term, \( \frac{\partial q_i}{\partial t} \).

Now the actual point velocity varies from \( \frac{q_i}{m} \) due to local viscosity and inertial effects. For the purposes of this discussion the local velocity is represented by

\[
\dot{u} = \left( \frac{q_i}{m} \right) + \bar{u} + \bar{u} \tag{3.12}
\]

where \( \bar{u} = \) the average identical ensemble deviation from \( \frac{q_i}{m} \) at a point; and \( \bar{u} = \) the component due to turbulence.

and the local piezometric head is represented by

\[
\frac{\bar{h}}{\bar{h}} = \phi + \bar{\phi} + \bar{\phi} \tag{3.13}
\]

where \( \bar{\phi} = \) the average (identical ensemble) deviation from \( \phi \) at a point; \( \bar{\phi} = \) the deviation from \( \phi \) due to turbulence.

Substituting equations 3.12 and 3.13 into 3.3 and 3.4 and taking a point ensemble average over many identical models leads to an equation similar to the Reynolds equation:

\[
\frac{\rho}{m} \frac{dq_i}{dt} + \rho \frac{\partial \bar{u}_i}{\partial t} = - \rho g \frac{\partial \bar{\phi}}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \mu \frac{\partial (q_i + \bar{u}_j)}{\partial x_j} - \rho \bar{u}_i \bar{u}_j - \rho I \right] \tag{3.14}
\]

\( \bar{h} \) = micro piez.

\( \bar{\phi} \) = micro viscous

\( \bar{\phi} \) = micro turb.

\( \bar{u} \) = macro turbulent

\( \bar{u} \) = micro inertial

\( \bar{u} \) = piezometric
where \( \rho \tilde{T} = \rho \left[ \vec{u}_i \vec{u}_j + q_{ij} \vec{u}_k + \vec{u}_i q_{ij} \right] \)  

\[ (3.15) \]

= relative steady inertial effect.

From equation 3.15 it can be seen that 'I' depends on both the pore and mean flow patterns; thus the steady inertial term will be a function of the curvature and convergence or divergence of macroscopic streamlines.

Certain assumptions and approximations are now introduced in order to simplify equation 3.14:

a) The turbulence term \(( \vec{u}_i \vec{u}_j )\) is replaced by an eddy viscosity, \(\xi\) (Ref 23, 41) which is assumed to be related to the 'pore' inertial term, \((D_p q/m)\) where \(D_p =\) pore size; the grain roughness, \(\varepsilon\); the flow separation pattern.

b) The steady inertial term \((-I)\) is replaced by a velocity distribution coefficient, \(\xi_v\), which is similar to '\(\xi\)' but related to: the 'pore' inertia, \(D_p q/m\); the radius of curvature of the macroscopic streamlines, \(r_r\); the radius of divergence, \(r_r\) (or convergence), of the macroscopic streamlines; particle shape \(\propto\); flow separation.

Following Hinze (41) the stress tensor may now be written

\[
\sigma_{ij} = -\rho g (\dot{\phi} + \dot{\phi}') + \xi_{ij} + \mu + \rho (\dot{\varepsilon} + \dot{\varepsilon}_v) D_{ij} \]

\[ (3.16) \]

where \(\dot{\phi}' = \frac{1}{3g} \left[ \vec{u}_i \vec{u}_j + (\vec{u})^2 \right] \)

\[ (3.17) \]
and \( \ddot{D}_{ij} = \) the deformation tensor

\[
\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}
\]

and \( u_i^* = \dot{q}_i + \ddot{u}_i \)

Substitution of equation 3.16 into 3.14 gives

\[
\frac{\rho}{m} \frac{dq_i}{dt} + \frac{\partial u_i}{\partial t} = -\rho g \left( \alpha \left( \dot{\varepsilon}_i + \dot{\phi}_i + \dot{\theta}_i \right) + \frac{\partial}{\partial x_j} \left\{ [\mu + \rho(\varepsilon + \dot{\varepsilon}_v)] \frac{\partial u_i}{\partial x_j} \right\} \right)
\]

(3.18)

Following the procedure of Ahmed and Sunada (2) equation 3.18 is averaged (i.e. integrated) over a representative control volume, \( mV \). This control volume should account for the dispersion effect since its boundary conditions are representative of the medium in a statistical sense.

Therefore, if \( \varepsilon \) and \( \dot{\varepsilon}_v \) are considered to be constant within the control volume,

\[
\frac{\rho}{m} \left[ \iiint_{mV} \frac{dq_i}{dt} d(mV) + \int \frac{\partial u_i}{\partial t} d(mV) \right] \frac{d(mV)}{mV}
\]
\[
\begin{align*}
\frac{\partial \rho}{\partial t} &= - \rho \frac{\partial q_1}{\partial t} + \left[ \mu + \rho \left( \xi + \xi_v \right) \right] c_1 \frac{q_1}{d^2} \quad (3.20)
\end{align*}
\]

where \( d \) is a characteristic length for the medium;
\[
\frac{c_1 q_1}{d^2} = \frac{1}{mV} \iiint \frac{\partial^2 u_1}{\partial x^2_j} \, d(mV) \quad (3.21)
\]

\( c_1 \) is a constant for the medium.

The eddy coefficient, \( \xi \), was assumed to be
\[
\xi = f \left( D, q, m, \epsilon, \text{flow separation pattern} \right) \quad (3.22)
\]

and since the flow separation pattern depends on the particle shape, the macroscopic flow pattern and \( R_N \), dimensional analysis produced
\[
\xi = c_2 \frac{d^2}{q_1} f_2 \left( m, \frac{\xi}{q_1}, \frac{r_K}{d}, \frac{r}{d}, R_N \right) \quad (3.23)
\]
similarly
\[
\xi_v = c_3 \frac{d}{q_1} f_2 \left( m, \frac{\xi}{q_1}, \frac{r_K}{d}, \frac{r}{d}, R_N \right) \quad (3.24)
\]

where \( c_2 \) and \( c_3 \) are constants;
\[ q = \sqrt{\sum q_i^2} \] i.e. the magnitude of the macroscopic velocity; \( f_2 \) and \( f_3 \) are functions to be determined experimentally; and \( R_N \) is a Reynolds number.

Equation 3.20 can now be written

\[
\frac{\rho}{m} \frac{dq_i}{dt} = - \rho \frac{\partial \tilde{h}}{\partial x_i} + \frac{c_1 \mu q_i}{d} + \frac{c_1}{d} \left[ f_4(m, \frac{r}{d}, \frac{r_k}{d}, \alpha, R_N) q \right] q_i 
\]

\[ (3.25) \]

Ward (124) defines

\[ k = \frac{d^2}{c_1} \] \[ (3.26) \]

where \( k \) is the permeability of the medium.

The 'inertial' term \( \tilde{h} \), included in the piezometric term, is of the same order of magnitude as the local velocity head and therefore, quite small in most cases. Also for a stable separation pattern within a given flow regime, the term \( R_N \) can be dropped from equation 3.25. Therefore equation 3.25 becomes

\[
\frac{\rho}{m} \frac{dq_i}{dt} = - \rho \frac{\partial \tilde{h}}{\partial x_i} + \left[ \frac{\mu}{k} + \frac{c}{k^{1/2}} \right] f_5(m, \frac{r_k}{d}, \frac{r}{d}, \alpha) q \] \[ q_i \] \[ (3.27) \]

where \( c \) is a constant for a particular flow regime. Now equation 3.27 has the same form as equation 3.10 proposed by Hunt (42). Equation 3.27 can be written in the Forchheimer form as

\[
\frac{\rho}{m} \frac{dq_i}{dt} = - \rho \frac{\partial \tilde{h}}{\partial x_i} + \frac{\mu}{k} q_i + \frac{c}{k^{1/2}} q_i \]

\[ (3.28) \]
\[
\frac{1}{g_m} \frac{dq_i}{dt} = - \frac{\partial p}{\partial x_i} + (a + bq) q_i
\] (3.28)

where
\[
a = \frac{\mu}{\rho g k}
\] (3.29)

and
\[
b = \frac{c}{g k^{1/2}} f_5 (m, \frac{c}{d}, \frac{r_k}{d}, \frac{r}{g}, \infty)
\] (3.30)

Using Ward's (125) modification of the Kozeny-Carman permeability (equation 2.31) gives

\[
a = \frac{36 \mu K T (1-m)^2 \sigma_g}{\rho g x^2 m^3} 1 / 2 \ln c g
\] (3.31)

\[
b = \frac{c \sqrt{K T (1-m)^2 \sigma_g}}{g x^2 m g m^3 / 2} f(m, \frac{c}{d}, \frac{r_k}{d}, \frac{r}{g}, \infty)
\] (3.32)

Hunt (42) has shown that macroscopic vorticity, if it exists in a porous medium decays rapidly. Furthermore the macroscopic velocity head is small and except for regions of high convergence (or divergence) where \( r_x = 0(D) \) the rate \( \frac{\partial \phi}{\partial x} \), is also small compared with \( \frac{\partial p}{\partial x} \); therefore an approximate form of equation 3.28 is

\[
\frac{1}{g_m} \frac{\partial q_i}{\partial t} = - \frac{\partial p}{\partial x_i} + (a + bq) q_i
\] (3.33)

which has the same form as equation 3.2.

Combining equation 3.33 and the continuity equation 3.11 gives the unsteady non-Darcy flow equation
or in the case of steady flow

\[ \frac{\partial}{\partial x_1} \left[ \frac{1}{a + b q} \left( \frac{\partial \phi}{\partial x_1} + \frac{1}{g_m} \frac{\partial q_i}{\partial t} \right) \right] = 0 \]  

(3.35)

where \( a \) and \( b \) are given by equation 3.31 and 3.32 respectively.

### 3.2.2 Special Cases of the Governing Equations

In a few special cases analytical solutions may be obtained for equation 3.34. Two such cases which will later be utilized in this thesis are:

a) steady radial flow with constant \( a \) and \( b \);

b) unsteady one dimensional flow.

**Steady Radial Flow:** It has already been indicated that \( b \) is probably not constant in radial flow (see equation 3.32 and Ref. 133) however in order to verify this theory it will be useful to compare the analytical solution based on a constant \( b \) with the experimentally derived \( b \) for radial flow. Also the analytical solution will be used in the evaluation of the finite element method.

Transforming equation 3.35 into polar coordinates \((r, \theta)\) one obtains

\[ \frac{\partial}{\partial r} \left[ W r \theta \left( \frac{a + \sqrt{a^2 + 4b l_1^2}}{2b l_1} \right) \right] = 0 \]  

(3.36)

where \( W \) is the width of the radial permeameter; and the remaining notation
and boundary conditions are shown in figure 3.3.

Equation 3.36 is readily solved by two integrations to yield

\[ \phi = -\frac{c_1}{4b} \left[ -\frac{c_1}{r} + 2a \ln r \right] + c_2 \]  (3.37)

where \( c_1 = \frac{2bQ}{W\theta} \) \quad (3.38)

\[ \frac{1}{r^2} \left[ (\ln \frac{r_1}{r_2})^2 + \frac{4b}{a} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) (\phi_1 - \phi_2) \right] \]  (3.39)

or \[ c_1 = \frac{\ln \frac{r_1}{r_2}}{(\frac{1}{r_1} - \frac{1}{r_2})} \]

\[ c_2 = \phi_1 + \frac{c_1}{4b} \left( -\frac{c_1}{r} + 2a \ln r \right) \]  (3.40)

Equations 3.37 and 3.38 and the boundary conditions shown in figure 3.3 can be used to solve for \( b \) assuming \( a \) to be constant, viz.

\[ b = \frac{W\theta}{Q} (\phi_1 - \phi_2) \pm a \ln \left( \frac{r_1}{r_2} \right) \]  (3.41)

Equation 3.41 can be used with experimental data to estimate the effect of convergence or divergence on \( b \) since equations 3.31 and 3.32 indicate that convergence mainly affects the 'b' term.

Unsteady One Dimensional Flow: The one dimensional form of equation 3.33 is

\[ \frac{1}{gm} \frac{dq}{dt} + aq + bq^2 - i = 0 \]  (3.42)
where \( \lambda \) is the magnitude of the piezometric gradient which will be taken to be constant. Although equation 3.42 is non-linear it may be transformed to a linear form as follows.

Equation 3.42 may be written

\[
\frac{\partial u}{\partial t} + (q - \pi_1)(q - \pi_2) = 0
\]

(3.42)

where \( \pi_1 = \frac{n}{n+1}(\tau_o - 1) \) \( \pi_2 = -\frac{A}{2\pi} \frac{\ln(1 + \tau)}{\tau} \)

(3.43)

(3.44)

Defining a new dependent variable

\[ u = (q - \pi_1) \]

(3.45)

equation 3.43 becomes

\[
\frac{\partial u}{\partial t} + u^2 + \lambda u = 0
\]

(3.46)

where \( \lambda = \pi_1 - \pi_2 \)
where \( i \) is the magnitude of the piezometric gradient which will be taken to be constant. Although equation 3.42 is non-linear it may be transformed to a linear form as follows.

Equation 3.42 may be written

\[
\frac{dq}{d\tau} + (q - \pi_1)(q - \pi_2) = 0 \tag{3.43}
\]

where \( \tau = \frac{v}{mb} \) \tag{3.44}

\[
\pi_1 = \frac{a}{2b} (\sqrt{1+c^2} - 1) \tag{3.45}
\]

\[
\pi_2 = -\frac{a}{2b} (\sqrt{1+c^2} + 1) \tag{3.46}
\]

Defining a new dependent variable

\[
u = (q - \pi_1) \tag{3.47}
\]

equation 3.43 becomes

\[
\frac{du}{d\tau} + u^2 + \lambda u = 0 \tag{3.48}
\]

where \( \lambda = \pi_1 - \pi_2 \) \tag{3.49}
Equation 3.48 is the Bernoulli differential equation (Spiegel (113)) which may be linearized by the transformation \( v = u^{-1} \) and subsequently solved to obtain for, \( q(0) = 0 \) and \( i \neq 0 \),

\[
q(\tau) = q(\infty) + \left(1 - \frac{e^{\lambda \tau}}{\lambda} - \frac{e^{\lambda \tau}}{q(\infty)}\right)
\]

(3.50)

where \( q(\infty) \) is the steady state velocity; \( q(t) \) is the transient velocity at time \( t \).

For the special case in which \( i = 0 \) and \( q(t_o) \neq 0 \) the solution of equation 3.42 is

\[
q(t) = \frac{q(t_o)}{1 + b \frac{q(t_o)}{a}} \left\{ e^{-\frac{a}{q(t_o)} - \left(-\frac{a}{bq(t_o)} + 1\right)} \right\}^{-1}
\]

(3.51)

where \( q(t_o) \) is the initial velocity i.e. at time \( t = t_o \).

Equation 3.50 represents an accelerating or initial motion problem.

Figure 3.4 (for 1.6 cm crushed rock and \( i = 1.0 \)) indicates that the \( q(t) \) rapidly approaches the equilibrium velocity, \( q(\omega) \). Figure 3.4 also shows the fluid particle displacements obtained from the integration of equation 3.50. The error, in predicting fluid particle displacements due to neglecting the inertial term, \( \frac{1}{g_m \frac{\partial q}{\partial t}} \), is only 20 percent of the grain size of the medium as shown on figure 3.4.

Equation 3.51 represents decaying or decelerating motion. Figure 3.5 (for 1.6 cm crushed rock) indicates that the transient velocity decreased to zero in about 1 second. The fluid particle displacement error due to neglecting \( \frac{1}{g_m \frac{\partial q}{\partial t}} \), is also shown on figure 3.5. For the sample
Figure 3.4 The Effect of Inertia on Accelerating Flow (1.66 cm rock)

Figure 3.5 The Effect of Inertia on Decelerating Flow
particle displacement computation \( q(t_o) \) was taken to be 9.50 cm/sec which is the value of \( q(\omega) \) from figure 3.4 for \( i = 1.0 \). For decaying motion the fluid displacement error due to neglecting, \( \frac{1}{\Delta t} \frac{\partial q}{\partial t} \), is approximately one grain diameter, \( D \).

3.3 Variational Formulations for Non-Darcy Flow

3.3.1 The Case of Steady Flow

The Energy Functional: The energy dissipation rate per unit volume at a point in a porous medium is

\[
\dot{E} = (u \dot{\phi}_x + v \dot{\phi}_y + w \dot{\phi}_z)
\]  

(3.52)

The total energy dissipation rate for a region \( R \) is

\[
\tilde{\chi}' = - \iiint_R (u \dot{\phi}_x + v \dot{\phi}_y + w \dot{\phi}_z) \, dV
\]  

(3.53)

or

\[
\chi' = + \iiint_R K (\dot{\phi}_x^2 + \dot{\phi}_y^2 + \dot{\phi}_z^2) \, dV
\]  

(3.54)

since

\[
\begin{align*}
u &= - K \dot{\phi}_x \\
v &= - K \dot{\phi}_y \\
w &= - K \dot{\phi}_z
\end{align*}
\]

where \( K \) is the hydraulic conductivity. If \( K \) is a constant or a function of \( x \), \( y \) and \( z \) the extremal of \( \dot{\chi}' \), obtained by applying the Euler-Lagrange (87) equation is.
\[
\frac{\partial f_\phi}{\partial x} + \frac{\partial f_\phi}{\partial y} + \frac{\partial f_\phi}{\partial z} - \frac{\partial f}{\partial \psi} = 0
\] (3.55)

to the integrand \(f\) of equation 3.54, yields

\[\nabla' \cdot (K \nabla \psi) = 0\] (3.56)

which is the partial differential equation for Darcy flow. However if \(K\) is a function of \(\psi_x\) and \(\psi_y\) as it is for non-Darcy flow, the extremal of \(X\), gives, for the two dimensional case

\[
2\left[\frac{\partial}{\partial x} (K \psi_x) + \frac{\partial}{\partial y} (K \psi_y)\right] + \left[\frac{\partial}{\partial x} \left[ K \psi_x \cdot (|\nabla \psi|)^2\right]\right] + \left[\frac{\partial}{\partial y} \left[ K \psi_y \cdot (|\nabla \psi|)^2\right]\right] = 0
\]

(3.57)

where \(K = K(|\nabla \psi|)\)

which approaches equation 3.35 for the conditions listed in section 2.33; however for transitional flow when \(K\) is given by equation 2.67 the minimization of equation 3.54 does not exactly yield equation 3.35. Thus for (See Ref. 70)

\[
K = \frac{a}{2b} \left[ \sqrt{1 + \left(\frac{4b/a^2}{|\nabla \psi|}\right)^2} |\nabla \psi| - 1 \right]
\] (3.58)

equation 3.57 gives
\[ \nabla \cdot q = 0 \quad (2.58) \]

Therefore the direct minimization of the energy dissipation rate does not, in general, correspond to the true solution of equation 3.35. However if a term

\[ G(|\nabla \phi|) = \frac{a}{6bc} (2 - c |\nabla \phi|) \sqrt{1 + c |\nabla \phi|} \quad (3.60) \]

is added to the integrand of the functional \( \mathcal{X}' \), the continuity requirement can be satisfied and equations 3.35 obtained, exactly from the Euler-Lagrange equation (see appendix B). Now the functional can be written (for the two dimensional case for a region \( A \)).

\[ \mathcal{X} = \iint_A \left[ K(|\nabla \phi|) \left( \phi_x^2 + \phi_y^2 \right) + \frac{a}{6bc} (2 - c |\nabla \phi|) \sqrt{1 + c |\nabla \phi|} \right] \, dx \, dy \quad (3.61) \]

where \( c = \frac{4b}{a} \).

Equation 3.61 can be expressed as

\[ \mathcal{X} = \iint_A (E + G) \, dx \, dy \quad (3.62) \]
where $\dot{E}$ is the rate of energy dissipation; and $G$ is related to the rate of energy dissipation required to maintain continuity in the flow.

The solution of the non-linear elliptic partial differential equation 3.35 is now reduced to finding the distribution of $\xi$ which minimizes $\chi$ subject to certain boundary conditions, i.e. the solution is represented, in variational form, by

$$
\delta X = \int \int \delta (\dot{E} + G) \, dx \, dy = 0 \tag{3.63}
$$

This is equivalent to finding the function $\xi (x, y)$ which minimizes the rate of energy dissipation with enforced flow continuity.

The Second Variational: In order for $\delta X = 0$, to represent a true extremal, i.e. a maximum or minimum and not a saddle point, the second variational of $X$ is

$$
\delta^2 X \neq 0 \tag{3.64}
$$

For a minimum (102)

$$
\delta^2 X > 0 \tag{3.65}
$$

and for a maximum

$$
\delta^2 X < 0 \tag{3.66}
$$
It is shown in Appendix B that the second variation of $\chi$ leads to

$$\delta^2 \chi > 0$$

therefore

$$\delta \chi = 0$$

must represent a true minimum.

**Boundary Conditions:** The boundary conditions of the steady free surface non-Darcy flow problem are, (for example, see figure 3.6):

a) $\delta \phi \over \partial n = 0$ or $\psi = C_1$ along an impervious boundary;

b) $\delta \phi \over \partial n = 0$ or $\psi = C_2$ and $\phi = y + C_3$ along the free surface;

c) $\phi = y + C_3$ along the seepage face;

d) $\phi = C_4$ along the inflow face;

e) $\phi = C_5$ along the submerged outflow face.

It is shown in Appendix B that the variation formulation automatically takes into account the streamline boundary condition $\delta \phi \over \partial n = 0$.

The imposed boundary condition can also be satisfied as will be shown in the numerical analysis. However, the location of the free surface and the extent of the seepage face are unknown. There is a dual boundary
Figure 3.6 Steady Flow Boundary Conditions
condition at the free surface. If a solution exists then by trial and error it should be possible to obtain a free surface on which both boundary conditions are satisfied. For example if \( \bar{\phi} \) is unspecified on a trial free surface a solution would yield the streamline condition \( \frac{\partial \bar{\phi}}{\partial n} = 0 \) but unless this was the true free surface it would give \( \bar{\phi} \neq (y + C_3) \).

A new trial surface then could be chosen such that the error \( [\bar{\phi} - (y + C_3)] \) decreases.

### 3.3.2 A Variational Expression for Unsteady Flow

The unsteady non-Darcy flow equation was developed in section 3.2.1 as

\[
\frac{\partial}{\partial x_i} \left[ K(q) \left[ \frac{\partial \bar{\phi}}{\partial x_i} + \frac{1}{g^m} \frac{\partial q_i}{\partial t} \right] \right] = 0
\]

or

\[
\frac{\partial}{\partial x_i} (K(q) \bar{\phi}) \frac{\partial}{\partial x_i} + \frac{1}{g^m} \frac{\partial q_i}{\partial t} \cdot K(q) x_i = 0
\]

(3.67)

Only in the special case of \( K(q) x_i = 0 \) (i.e. \( K(q) = \text{constant} \)), does equation 3.67 reduce to an equation similar to the steady flow equation 3.35, i.e.

\[
\frac{\partial}{\partial x_i} (K \bar{\phi}) = 0
\]

(3.68)

where \( \bar{\phi} = \bar{\phi}(x, y, z, t) \).

In the general case of non-Darcy flow \( K(q) x_i \neq 0 \); however it was shown in section 3.2.2 that the term \( \frac{1}{g^m} \frac{\partial q_i}{\partial t} \) is relatively small compared with the friction effects therefore an approximate unsteady flow equation is
\[
\frac{\partial}{\partial x_1} \left[ K(q) \phi_{x_1} \right] = 0 \tag{3.69}
\]

where \( \phi = \phi(x, y, z, t) \)

which now has the same form as the steady flow equation 3.35.

If it is assumed for a given region \( A \) that the fluid and medium are incompressible then at any instant of time the flow within the porous media should satisfy simultaneously the requirements of a minimum energy dissipation rate and flow continuity (section 3.3.1). In a short interval of time, \( \Delta t \), the amount of energy dissipated should also be the minimum possible while maintaining continuity.

The rate of energy dissipation with enforced flow continuity at an instant, \( t \), in two dimensions, is

\[
L = \iint_A \left[ K(|\nabla \phi|)(\phi_x^2 + \phi_y^2) + G(|\nabla \phi|) \right] \, dx \, dy \tag{3.70}
\]

where \( \phi = \phi(x, y, t) \).

The total functional for an interval of time, \( \Delta t \), can be obtained by reference to the work of Luke (66) as

\[
\tilde{\chi} = \int_0^{\Delta t} L \, dt = \int_0^{\Delta t} \iint_A \left[ K(|\nabla \phi|)(\phi_x^2 + \phi_y^2) + G(|\nabla \phi|) \right] \, dx \, dy \, dt \tag{3.71}
\]
If \( A(t) \) is constant or a known function of \( t \) then the first variation

\[
\delta X = 0
\]

leads to equation 3.34 with the natural boundary conditions given by equation B-3 i.e. the same as for steady flow except \( \phi = \phi(x, y, t) \).

For example:

a) \( \delta \phi (x, y, t) = 0; \)

and/or

b) imposed boundary condition \( \phi = \phi(t) \) where \( \phi(t) \) is known,

are acceptable boundary conditions.

If \( A \) is not a known function, but rather is one of the unknowns itself, the first variational gives

\[
\frac{\Delta t}{\int_{A(t)} [\delta f] \, dx \, dy \, dt + \int_{0}^{f_A} \delta A \, dt} = 0
\]  

Two approaches were considered for the solution of this problem:

a) redefine the integrand \( f \)

such that

\[
\delta f = \frac{\partial}{\partial x_i} K(q) \phi_{x_i} = 0
\]

and

\[
f_A = 0;
\]
b) choose a very small time interval, $\Delta t$, over which $\delta A$ is very small such that an estimate of $A(t)$ may be made in terms of the initial condition e.g.

$$A(t) = A(0) + A'(0)t + A''(0) \frac{t^2}{2} + \ldots \quad (3.75)$$

The second approach was developed because of its simplicity. This approach is extended in section 3.4.3.

Further complications arise if the term $\frac{1}{g_m} \frac{\partial A}{\partial t}$ is not ignored; however a transformation of the dependent variable $\tilde{v}$ to $\zeta$ as defined by

$$\nabla \zeta = \nabla \tilde{v} + \frac{1}{g_m} \frac{\partial \tilde{v}}{\partial t}$$

reduces equation 3.34 to

$$\nabla \cdot K(|\nabla \zeta|) \nabla \zeta = 0$$

which has the same form as equation 3.69, thus the same form of the functional, i.e.

$$\chi = \int_0^\infty \int_\Omega A(t) \left[ K(|\nabla \zeta|) \left( \nabla_x^2 \zeta + \nabla_y^2 \zeta \right) + G(|\nabla \zeta|) \right] dx \, dy \, dt \quad (3.78)$$

can be used provided the boundary conditions on the new dependent variable are of the natural type.
From equation 3.76 it is evident that on a streamline (e.g. an impermeable boundary)

\[ \frac{\partial \phi}{\partial n} = \frac{\partial \zeta}{\partial n} = 0 \] (3.79)

On the boundaries where \( \phi \) is prescribed \( \zeta \) will also be prescribed. The transformation from \( \phi \) to \( \zeta \) is required only on these boundaries and is treated by a finite difference scheme (see section 3.4.3).

Thus the initial value unsteady non-Darcy flow problem, with small inertia, \( \left( \frac{1}{g_m \frac{\partial \zeta}{\partial t}} \right) \), has been reduced to a variational form subject to a set of transformed boundary conditions.

3.4 Development of the Finite Element Models

In this section a brief presentation of the development, of the steady flow non-Darcy finite element models, is given as a background to the development of an effective unsteady flow model for non-Darcy flow. The steady radial flow model is treated first; then the steady free surface model is considered. Finally a detailed development is presented for unsteady flow including the inertial term, \( \frac{1}{g_m \frac{\partial \zeta}{\partial t}} \).

Oden (80) describes the finite element representation of a function as a topological construction. In the finite element method (138, 139, 140, 119, 31, 130, 131) the flow field (i.e. the region of the solution) is divided into a number of finite subregions called finite elements. In two dimensional flow problems these elements are usually triangular in shape; although in structural problems other shapes, such as rectangles, have been used (139). A typical triangular discretiza-
tion of a flow field is shown in figure 3.7.

The next step in constructing a finite element model (FEM) is the specification of the variation of the function(s) within each element in terms of the boundary or nodal conditions on the element. In so doing the requirements of:

a) continuity of the function(s) and certain of its derivatives in the region of the solution,

and b) sufficient differentiability of the approximation to the function(s) within each element,

should be satisfied (139) to ensure monotonic convergence of the numerical solutions as the elements become very small.

For the variational approach with

\[ \chi = \int_{\Omega} f(x, \xi_x, \ldots, \xi_n) \, d\Omega \]  \hspace{1cm} (3.80)

Zienkiewicz (139) states that, "\( \phi \) and all its \((n-1)\) derivatives must be continuous and finite on the finite interface (element boundaries)." This implies that the \(n\)-th derivative must also be finite but may be discontinuous on the element boundaries. For \( \chi \) defined by equations 3.61 or 3.77 the highest derivatives are of first order, therefore a linear variation in the elemental \( \phi^e \) can be assumed for steady flow i.e.

\[ \phi^e = \alpha_1 + \alpha_2 x + \alpha_3 y \]  \hspace{1cm} (3.81)
where \( a_1 = a_2 \) and \( a_3 \). Equations three nodal values of \( h \) and the nodal values states (the nodes are located at the vertices of the triangles of Figure 3.6).

In the finite element procedure, each element is represented by a set of nodal points. The shape of each element is determined by connecting these nodes.

From the fact that the desired solution for the water flow problem is represented by equation (2.3)...

**Figure 3.7** Typical Triangular Discretization
where \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) form the three nodal values of \( \phi \) and the nodal coordinates (the nodes are located at the vertices of the triangles as shown in figure 3.8).

In the finite element approximation the total dissipation functional \( \chi \) is represented by

\[
\chi = \sum \chi^e
\]

(3.82)

where \( \chi^e \) is the functional for an element \( e \) with \( \phi^e \) represented by an approximate equation such as equation 3.81.

From the fact that the desired solution for \( \phi \) (in the non-Darcy flow problems) minimizes \( \chi \), differential calculus gives

\[
\frac{\partial \chi}{\partial \phi_i} = \sum \frac{\partial \chi^e}{\partial \phi_i} = 0
\]

(3.83)
where \( i = 1+n \) and \( n \) is the number of unknown nodal values of \( \phi \), i.e. a set of \( n \) simultaneous algebraic equations in the \( n \) unknown values of \( \phi \) is obtained. In general if the original partial differential equation is linear the system 3.83 will be linear and if the original partial differential equations is non-linear the system 3.83 will be non-linear. Equations 3.61 and 3.77 can be expected to yield sets of non-linear simultaneous equations in the finite element method.

3.4.1 The Steady Radial Flow Model

Since the equations for non-Darcy radial flow can readily be solved analytically, it would be useful to check the accuracy of the finite element method by comparing the numerical and analytical solutions for this special case.

The functional for non-Darcy flow in two dimensions was developed in section 3.3.1. Transforming to polar coordinates, for radial flow, equation 3.61 becomes

\[
\chi = W \int_{R_1}^{R_2} \int_{0}^{\theta} [K(|\phi_r|) \phi_r^2 + G(|\phi_r|)] rdrd\theta \quad (3.84)
\]

where \( \phi_r = \frac{\partial \phi}{\partial r} \);

\( W \) is the width of the section; and the remaining symbols are defined in figure 3.9.

Since \( \theta_1 \) is independent of \( r \) equation 3.84 can be simplified to

\[
\chi = W \int_{R_1}^{R_2} [K(|\phi_r|) \phi_r^2 + G(|\phi_r|)] r dr \quad (3.85)
\]
Equation 3.85 leads to a one dimensional variational problem.

Considering the radial flow problem shown in Figure 3.9 with the indicated boundary conditions, the FEM finite element model is a segment of an annulus as shown. Using a linear variation of $\phi$ within each element,

\[
\phi_r = \frac{\phi_1 - \phi_2}{r_2 - r_1} (r - r_1)
\]

where

\[
\frac{\phi_1}{r_1} = \frac{\phi_2}{r_2}
\]

and

\[
\phi_2 = \frac{r_2 - r_1}{r_2}
\]

From equation 3.86 the gradient within an element is

\[
\frac{\partial \phi}{\partial r} = \frac{\phi_1 - \phi_2}{r_2 - r_1}
\]

Now the functional (i.e., the energy functional) is to be minimized in order to obtain the nodal values of $\phi$ as specified by equation 3.83. For a particular element, $A$,

\[
\frac{d^2}{dr^2} = \frac{W_0}{r_2} \int_{r_1}^{r_2} \frac{1}{(r_1r_2)^2} \frac{d}{dr} \left( r \left[ \frac{d}{dr} \right] \right) r dr
\]

Substituting equation 3.90 for $\phi_r$ gives
Equation 3.85 leads to a one dimensional variational problem.

Considering the radial flow problem shown in figure 33 with the indicated boundary conditions, the obvious choice of a finite element is a segment of an annulus as shown. Using a linear variation of $\hat{\phi}$ within each element

$$\hat{\phi}^e = \alpha_1 + \alpha_2 r$$  \hspace{1cm} (3.86)

where

$$\alpha_1 = \frac{\hat{\phi}_j - \hat{\phi}_i}{r_j - r_i}$$  \hspace{1cm} (3.87)

and

$$\alpha_2 = \frac{\hat{\phi}_j - \hat{\phi}_i}{r_j - r_i}$$  \hspace{1cm} (3.88)

From equation 3.86 the gradient within an element is

$$\hat{\phi}_r = \alpha_2$$

$$= \frac{\hat{\phi}_j - \hat{\phi}_i}{r_j - r_i}$$  \hspace{1cm} (3.90)

Now the functional (i.e. the FEM) must be minimized in order to obtain the nodal values of $\hat{\phi}$. This is accomplished by equation 3.83. For a particular element, $e$,

$$\frac{\partial K}{\partial \hat{\phi}_i} = W \phi_1 J^j \int_{r_i}^{r_j} [K(|\hat{\phi}_r|)^e \hat{\phi}_r (\hat{\phi}_r) \frac{\partial \hat{\phi}_r}{\partial \hat{\phi}_i}] r dr$$  \hspace{1cm} (3.91)

Substituting equation 3.90 for $\hat{\phi}_r$ gives
\[
\frac{\partial x^e}{\partial \phi_i} = \frac{W \theta_1}{r_i} \int_{r_i}^r K(|\phi_r|)^\epsilon \left( \frac{\phi_i - \phi_j}{r_i - r_j} \right) \left( \frac{1}{r_i - r_j} \right) \, r \, dr
\] (3.92)

and after integration

\[
\frac{\partial x^e}{\partial \phi_i} = \frac{W \theta_1}{2} \left[ K(|\phi_r|)^\epsilon (\phi_i - \phi_j) \left( \frac{r_i + r_j}{(r_i - r_j)} \right) \right]
\] (3.93)

From equation 3.83 the finite element solution is represented by the system

\[
\frac{\partial x}{\partial \phi_i} = \sum \sum K(|\phi_r|)^\epsilon S_{ij} \phi_j = 0
\] (3.94)

where

\[
S_{ij} = \left( \frac{r_i + r_j}{r_i - r_j} \right)
\] (3.95)

and the summations are taken over all the elements and all the nodes.

**The Numerical Solution:** It can be shown that equation 3.94 is a symmetric tridiagonal non-linear system (see appendix C).

A number of methods have been presented for solving non-linear systems (4, 5, 34) of which two were considered for the solution of 3.94; i.e.,

a) A successive over-relaxation method (SOR) in which the system was linearized by holding the \( K(|\phi_r|)^\epsilon \) terms constant within an inner iteration loop then varying \( K(|\phi_r|)^\epsilon \) in an outer loop,

b) A modified Newton procedure which Ames (4) refers to as non-linear over-relaxation (NLOR).
Successive Over-relaxation:- If equation 3.94 is linearized by treating $K(\bar{\Phi}^{p-1})^e$ as constant (where $p$ is the number of outer iteration cycle) in each element a solution for $\bar{\Phi}^p$ can be attempted by the SOR procedure. If the linearized system can be solved a new value of $K(\bar{\Phi})^e$ can be computed in the outer loop and the whole procedure repeated until the change in $\bar{\Phi}$ with successive outer cycles is less than a small prescribed value. The flow chart for this computation is shown in figure 3.9 and the FORTRAN IV programme is given in Appendix C.

Will the solution converge? Very few mathematical criteria are available to apply to the convergence of non-linear simultaneous equations. The investigator can however be guided by criteria for similar linear systems and by the results of numerical experimentation on non-linear systems.

The successive over-relaxation (SOR) procedure for linear systems developed by Frankel (36) and Young (134, 4) is defined by

$$\bar{\Phi}_i^{n+1} = \bar{\Phi}_i^n (1 - \omega) + \omega \bar{\phi}_i$$  \hspace{1cm} (3.96)

where $\bar{\phi}_i$ is the Gauss-Seidel solution,

$$\bar{\phi}_i = \frac{1}{a_{ii}} \left( -w_i + \sum_{j=1}^{i-1} a_{ij} \bar{\Phi}_j^{n+1} + \sum_{j=i+1}^{N} a_{ij} \bar{\Phi}_j^n \right)$$  \hspace{1cm} (3.97)

where $\omega$ is the over-relaxation factor;

$N$ is the number of unknown values of $\bar{\Phi}$.

In the present special case the linearization (see appendix C) of
by holding $K|\varphi_r|e$ constant leads to a system which satisfies the convergence requirements for the SOR procedure as given by Young (135), i.e.,

a) $a_{ii} > 0$ \hspace{1cm} (3.98)

b) $\sum_{j=1}^{N} |a_{ij}| < \sum_{j=1}^{N} |a_{ij}|$; \hspace{1cm} (3.99)

c) Matrix $A$ is irreducible; \hspace{1cm} (3.100)

d) Matrix $A$ has property 'A'; \hspace{1cm} (3.101)

e) $0 > \omega > 2$ \hspace{1cm} (3.102)

Convergent solutions were obtained for the range of $\omega$'s investigated, i.e., $9 > \omega > 2$, using the method outlined in the flow chart shown in figure 3.9.

The Modified Newton Method: - A modified Newton method (4) solving simultaneous non-linear algebraic equations is

\[
\varphi_i^{n+1} = \varphi_i^n - \omega \frac{f_i(\varphi_1^{n+1}, \ldots, \varphi_2^{n+1}, \ldots, \varphi_i^{n+1}, \ldots, \varphi_K^{n+1})}{\sum_{j=1}^{K} \frac{\partial f_i}{\partial \varphi_i}(\varphi_1^{n+1}, \varphi_2^{n+1}, \ldots, \varphi_i^{n+1}, \ldots, \varphi_K^{n+1})}
\]

where

\[
f_i(\varphi_j) = \sum_{j=1}^{K} a_{ij} \varphi_j - w_i
\]
A direct application of this method to equations 3.94 failed to yield a convergent solution; therefore the method was modified slightly as shown below:

\[ \hat{\phi}_{i}^{n+1} = \hat{\phi}_{i}^{n}(1 - \omega) - \omega \left( \frac{f_{i}(\hat{\phi}_{1}^{n+1}, \ldots, \hat{\phi}_{i}^{n}, \ldots, \hat{\phi}_{k}^{n}) - a_{ii}\hat{\phi}_{i}^{n}}{-a_{ii} + s f_{i}(\hat{\phi}_{1}^{n}, \ldots, \hat{\phi}_{i}^{n}, \ldots, \hat{\phi}_{k}^{n})} \right) \]  

(3.105)

where \( s \) is called a stabilizing factor; \( s = 0 \) yields the SOR method thus showing it to be a special case of this modified Newton method.

The flow chart and programme of the modified Newton method are shown in Appendix C.

The convergence of the non-linear system for various values of \( \omega \) and \( s \) was studied and the results summarized in figure 3.10.

Although a slight decrease in the number of iterations, could be achieved by increasing \( s \) from 0 to about .4 the convergence was very sensitive to changes in \( s \) near the optimum \( \omega \) and \( s \). Thus the reliability of the method did not appear to be good.

3.4.2 The Finite Element Model for Steady Free Surface Flow

Since steady flow non-Darcy finite element models have been presented elsewhere (by Fenton (30), Volker (122) and the writer (70, 71)) only a brief description, of the writer's steady flow model, is presented here.

It has been established (section 3.3.1) that the determination of the \( \hat{\phi}(x, y) \) which minimizes the dissipation functional 3.61, is equivalent to solving the non-Darcy partial differential equation 3.35.

To apply the FEM, the flow field is divided into triangular elements
Figure 3.10 Optimizing $\omega$ and $\varepsilon$ in the NLOR Procedure
as illustrated in figure 3.7. The triangles are scaled so that the energy dissipation is approximately the same in each element. This can be accomplished by making the element boundaries coincide (approximately) with sides and diagonals of the rectangles of an approximate flownet as figure 3.11 indicates.

Figure 3.11 Scaling the Elemental Areas

Equation 3.81 is employed to approximate the variations of $\phi$ within each triangular element. From the three nodal values of $\phi$ one obtains

$$\{\alpha\}^e = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}^{-1} \{\phi\}^e$$

(3.106)
where \( e \) indicates an element.

Using equation 3.81 the elemental gradients are:

\[
\phi_x = \alpha_2
\]

and

\[
\phi_y = \alpha_3
\]

hence from equation 3.61

\[
\chi^e = \iint_{A^e} \{K(|\nabla \phi|)^e [\alpha_2^2 + \alpha_3^2] + G(|\nabla \phi|^e)\} \, dx \, dy
\]

where

\[|\nabla \phi| = \sqrt{\alpha_2^2 + \alpha_3^2}\]

From equations 3.83 and 3.109

\[
\frac{\Delta \chi}{\Delta \phi_1} = \sum \frac{\Delta \chi^e}{\Delta \phi_1^e} = \frac{\iint_{A^e} [K(|\nabla \phi|)^e [\alpha_2^2 \frac{\partial \phi_2}{\partial \phi_1} + \alpha_3^3 \frac{\partial \phi_3}{\partial \phi_1}] \, dx \, dy}{\iint_{A^e} K(|\nabla \phi|^e) \, dx \, dy} = 0
\]

Completing the integration in equation 3.110 gives

\[
\sum K(|\nabla \phi|^e) S_{ij} \phi_j = 0
\]
where \( S_{ij} = \frac{(b_i b_j + c_i c_j)}{A_e} \) (3.112)

\[ b_i = y_j - y_k; \] (3.113)

\[ c_i = x_k - x_j; \] (3.114)

\( A_e \) is the area of the triangular element; and \( i, j, k \) are the element node numbers and are interchanged in cyclic order. The summations in equation 3.111 are taken over all the elements and all the nodes (including the nodes with prescribed values of \( \phi \)). Equation 3.111 represents a set of non-linear simultaneous equations in the unknown values of \( \phi \).

The Numerical Solution:- A number of researchers (see for example Poritsky (90), Trutt, Erdelyi and Jackson (121), and Young and Wheeler (136)), have obtained finite difference solutions to nonlinear partial differential equations similar to equation 3.35. The method most commonly used involves an inner iteration loop based on a system of equations that are linearized by treating the 'conductivity' \( K \), as constant at each node of the grid and an outer iteration loop which varies \( K \) on the basis of the current inner solutions.

The convergence of iteration schemes for the solution of sets of equations arising from finite difference schemes, particularly the 5-point molecule, have been studied exhaustively (see for example Young (135)). However, the study of convergence of the iterative solutions for sets of equations arising from the variational approach is much less
advanced.

In general systems developed by the variational approach do not possess property 'A'. The linearization of the system 3.111 satisfies all of Young's conditions (equations 3.98 to 3.102) for the SOR procedure except that it does not possess property 'A'. However there are some instances in which the SOR procedure has proven effective for solving systems that do not possess property 'A'. For example, the SOR method converges for the 9-point computation molecule (135). Kahan (46, 135) has indicated that the symmetric positive definite matrices that are often obtained by the variational approach can be solved by SOR even though they do not possess property 'A'. Since no theoretical equation is available to predict the optimum over-relaxation factor for equation 3.111 the optimum $\omega$ was found by experimentation with the computer programme.

In this study, the nonlinear system 3.111 was solved by three iteration loops as shown in the flow chart figure 3.12 and the computer programme shown in Appendix D. The inner loop solved, by successive over-relaxation, the linearized system with K constant in each element; the second loop revised the K based on the current solution for $\bar{\phi}$; the outer loop adjusted the free surface profile in such a way that the surface $\bar{\phi}$ approached the surface $\gamma$.

The finite element method was applied to determine the flow rates and flow patterns for rockfill sections such as the one shown in figure 3.6. In order to start the numerical solution an initial approximate solution was required for $\hat{\phi}$ and the position of the free surface. The free surface was located arbitrarily and an approximate solution for $\hat{\phi}$
was obtained drawing an ordinary (Darcy Flow) flow net.

To complete the problem, the system 3.111 must be solved subject to boundary conditions such as those shown in figure 3.6. The known values of $\phi$ on AB and DE can be entered directly into equation 3.111 and yield the constant vector in the linearized system

$$\{w\} = [A]\{\phi\}$$  \hspace{1cm} (3.115)

The streamline condition along BC and AE is satisfied if the nodal values of $\phi$ along BC and AE are treated as unknowns. In this study the membrane analogy method was used to find the free surface which gives $\phi$ equal to $y$. However, the exit point C was established by the requirement that flow continuity must exist between the faces AB and EDC. In this case $\phi_c$ was initially equated to $y_c$ and streamline condition, $\psi = Q$, satisfied in the outer iteration by adjusting $y_c$.

3.4.3 The Finite Element Model For Unsteady Free Surface Flow

The Non-linear System: - For the general case where the inertia term $\frac{1}{g_m} \frac{\partial q}{\partial t}$ is included the functional, representing energy dissipation during a short period of time was developed in section 3.3.2 as

$$\chi = \int^t \int \int_{0}^{A(t)} (K(\nabla \xi))(\xi_x^2 + \xi_y^2) + G(|\nabla \xi|) \, dx \, dy \, dt = 0$$  \hspace{1cm} (3.78)

where

$$G(p) = \frac{a}{6bc} \left(2 - cp\right) \sqrt{1 + cp}$$  \hspace{1cm} (3.58)
where \( \vec{v} \) is a trial velocity field. The functional can be approximated by

\[
\nabla \phi = \nabla \phi_{\text{avg}} + \frac{1}{\gamma m} \vec{Q} \quad \text{(3.76)}
\]

and

\[
A(t) = A(0) + A'(0) \Delta t + \frac{1}{2} A''(0) \Delta t^2 + \ldots \quad \text{(3.75)}
\]

The function \( G \) is introduced to maintain flow continuity. The integrand of this functional has the natural boundary conditions:

a) \( \frac{\partial \phi}{\partial n} = 0 \);

b) prescribed boundary \( \zeta \);

where \( n \) is the direction normal to the boundary.

In order to deal with the difficulties associated with the time dependent domain in the finite element model, the system is solved in small time steps during which the domain \( A \) is assumed to be independent of \( t \). Therefore \( A(t) \) is replaced by its average value, \( \bar{A} \), for the interval \( t_0 \) to \( (t_0 + \Delta t) \). For very small \( \Delta t \),

\[
A(t) \approx A(0) + A'(0) \Delta t
\]

and

\[
A'(0) \approx \frac{A(\Delta t) - A(0)}{\Delta t}
\]

thus the average area of integration during \( \Delta t \) would be

\[
\bar{A} \approx A(0) + A'(0) \frac{\Delta t}{2} = \frac{A(0) + A(\Delta t)}{2} \quad \text{(3.116)}
\]
where \( A(\Delta t) \) is a trial value of \( A \) at \( \Delta t \).

For a very small \( \Delta t \), the functional can be approximated by

\[
\dot{\chi} = \int_0^\Delta t \int_A \left[ K(|\nabla \zeta|)(\zeta_x^2 + \zeta_y^2) + G(|\nabla \zeta|) \right] dt \, dx \, dy \tag{3.117}
\]

Equation 3.34 is now written

\[
\Delta t \quad \delta \chi = \int_0^{t+\Delta t} \int_A \int_0^{t_o} \delta \left[ K(|\nabla \zeta|)(\zeta_x^2 + \zeta_y^2) + G \right] dt \, dx \, dy = 0 \tag{3.118}
\]

where \( t_o \) is a time at which \( \zeta(x, y, t_o) \) is known, i.e. the initial conditions. For convenience set \( t_o = 0 \).

Now, the problem is to find \( \zeta(x, y, t_o + \Delta t) \) which minimizes \( \chi \) subject to the initial conditions and a set of boundary conditions of type (a), and/or (b).

The flow field in \((x-y-t)\) space can be discretized by elements of the type shown in figure 3.13 and 3.14. Using a linear distribution of \( \zeta \), within each element at any instant in time the elemental \( \zeta^e \) is

\[
\zeta^e = (\beta_1 + \beta_2 x + \beta_3 y)t + \beta_4 + \beta_5 x + \beta_6 y \tag{3.119}
\]

where \( t \) varies from \( t_o = 0 \) to \( t = \Delta t \) and the values of \( \beta \) may be obtained from the 6 nodal conditions shown in figure 3.13. The conditions are known at nodes 1, m, n and unknown at nodes i, j, k therefore
Figure 3.13: Element in (x-y) space

Figure 3.14: Steady Flow Discretization

Equation 3.120 is a reducible system and may be solved in two parts. Therefore solving Eq. (3.120) for 

\[ \frac{\partial^2 \theta}{\partial x^2} = 0 \]

\[ \frac{\partial^2 \theta}{\partial y^2} = 0 \]

\[ \frac{\partial \theta}{\partial x} = 0 \]

\[ \frac{\partial \theta}{\partial y} = 0 \]

\[ \theta(x,0) = 0 \]

\[ \theta(x,T) = 0 \]

\[ \theta(x,y) = 0 \text{ at } y = 0 \]

\[ \theta(x,y) = 0 \text{ at } y = L \]

\[ \theta(x,y) = 0 \text{ at } x = 0 \]

\[ \theta(x,y) = 0 \text{ at } x = L \]

\[ \theta(x,y) = \text{initial condition} \]

\[ \theta(x,y) = \text{boundary condition} \]
Equation 3.120 is a reducible system and may be solved in two parts.

Therefore solving first for $\beta_4$, $\beta_5$ and $\beta_6$ gives

$$
\beta_4 = \det \begin{bmatrix} 
\zeta_1 & x_1 & y_1 \\
\zeta_m & x_m & y_m \\
\zeta_n & x_n & y_n 
\end{bmatrix} / 2A_2
$$

(3.121)

$$
\beta_5 = \det \begin{bmatrix} 
1 & \zeta_1 & y_1 \\
1 & \zeta_m & y_m \\
1 & \zeta_n & y_n 
\end{bmatrix} / 2A_2
$$

(3.122)
\[ \beta_6 = \det \begin{bmatrix} 1 & x_1 & \zeta_1 \\ 1 & x_m & \zeta_m \\ 1 & x_n & \zeta_n \end{bmatrix} / 2A_2 \]  

(3.123)

Using the above equations for \( \beta_4, \beta_5 \) and \( \beta_6 \) permits the solution for \( \beta_1, \beta_2 \) and \( \beta_3 \) to be written as,

\[ \beta_1 = \det \begin{bmatrix} \Delta \zeta_i & x_i & y_i \\ \Delta \zeta_j & x_j & y_j \\ \Delta \zeta_k & x_k & y_k \end{bmatrix} / 2A_1 \]  

(3.124)

\[ \beta_2 = \det \begin{bmatrix} 1 & \Delta \zeta_i & y_i \\ 1 & \Delta \zeta_j & y_j \\ 1 & \Delta \zeta_k & y_k \end{bmatrix} / 2A_1 \]  

(3.125)

\[ \beta_3 = \det \begin{bmatrix} 1 & x_i & \Delta \zeta_i \\ 1 & x_j & \Delta \zeta_j \\ 1 & x_k & \Delta \zeta_k \end{bmatrix} / 2A_1 \]  

(3.126)
where \( A_1 \) = area of triangle ijk;
\[ A_2 = \text{area of triangle lmn}; \]
and
\[ \Delta \zeta_1 = \zeta_1 - \zeta_1^0 \quad (3.127) \]

\[ \zeta_1^0 = \beta_4 + \beta_5 \chi_1 + \beta_6 \psi_1 \quad (3.128) \]

The procedure for minimizing \( \chi \) for the unsteady flow case is almost identical to the steady flow case. Equation 3.119 is substituted into the functional equation 3.117 and the finite element approximation is applied as follows:

\[ \chi = \sum x^e \]

or
\[ \Delta t \int \int \int \delta \left[ K(|\nabla \zeta|) (\beta_2 + \beta_5)^2 + (\beta_3 + \beta_6)^2 \right] + G(|\nabla \zeta|) dt dx dy \]

\[ \bar{A}^e 0 \quad (3.129) \]

The first variation (equation 3.118) implies

\[ \frac{\partial \chi}{\partial \zeta_1} = \sum \int \int \int \frac{K(|\nabla \zeta|)^e}{\bar{A}^e 0} \left[ \left( \beta_2 + \beta_5 \right) \frac{\partial \beta_2}{\partial \zeta_1} + \left( \beta_3 + \beta_6 \right) \frac{\partial \beta_3}{\partial \zeta_1} \right] dt dx dy \quad (3.130) \]

\[ \Delta t \int \int \int \delta \left[ \beta_2^2 \frac{\partial \beta_2}{\partial \zeta_1} + \beta_3^2 \frac{\partial \beta_3}{\partial \zeta_1} \right] dt dx dy \]

\[ \sum K(|\nabla \zeta|) \bar{A}^e \left[ \left( \frac{\partial \beta_2}{\partial \zeta_1} \right)^2 + \left( \frac{\partial \beta_3}{\partial \zeta_1} \right)^2 \right] \]

\[ \Delta t \int \int \int \delta \left[ \beta_2^2 + \beta_3^2 \left( \frac{\partial \beta_2}{\partial \zeta_1} \right)^2 \right] + \left( \beta_5 \frac{\partial \beta_2}{\partial \zeta_1} + \beta_6 \frac{\partial \beta_3}{\partial \zeta_1} \right)^2 \]

\[ \sum \bar{A}^e 0 \quad (3.131) \]
where

\[ \bar{K}(\bar{V})^e = \text{average elemental conductivity during } \Delta t; \]

(3.132)

\[ \bar{A}^e = \text{average elemental area during } \Delta t \]

(3.133)

It is noted that \( K^e_{\Delta t} \) and \( A(\Delta t)^e \) represent, respectively, trial values of the elemental conductivity and element area at time \( \Delta t \).

Making use of equations 3.121 to 3.128, equation 3.131 can be simplified to give the following system of non-linear simultaneous equations in the unknown values of \( \zeta \):

\[ \frac{\partial \bar{K}}{\partial \zeta} = \sum \sum (\bar{A})^e \left[ 3(\beta_5 b_i + \beta_6 c_i) + S_{ij} (\zeta_j - \zeta_i^e) \right] = 0 \]  

(3.134)

where

\[ \bar{K} = \text{average elemental hydraulic conductivity during } \Delta t; \]

\[ b_i = y_j - y_k; \]

\[ c_i = x_k - x_j; \]

and

\[ S_{ij} = (b_i b_j + c_i c_j) A_{11}^e; \]

In order to complete the solution the boundary conditions must be expressed in terms of \( \zeta \).
Boundary Conditions: It is usual (128) to assume that once a particle is on the free surface it will remain there. Therefore, using a Lagrangian approach, if one computed the position of the free surface particles in time, the free surface profiles would be defined (62). Thus the free surface movement during each time step can be computed by

\[ \vec{s}(A, B, \Delta t) = \int_0^{\Delta t} \vec{q}(A, B, t) \, dt + \vec{s}(A, B, 0). \]

where \( \vec{s}(A, B, t) \) is the position of a fluid particle on the free surface at time \( t \); \( A \) and \( B \) are the rectangular coordinates of the particle at \( t = 0 \); \( \vec{q} \) is the average 'bulk' velocity during \( \Delta t \). Since \( \Delta t \) is taken to be very small, an initial estimate of \( \vec{q} \) can be obtained by putting the surface \( \zeta(\Delta t) = \zeta(0) \). To improve the estimate of the new free surface equations 3.134 and 3.135 are solved alternately until a stable free surface is obtained.

The boundary conditions used in this study are restricted to types (a) and (b), (see section 3.3.2) as illustrated in figures 3.14 and 3.15. On impervious boundaries, type (a) will exist; on the free surface and other permeable parts of the boundary, in general, \( \zeta \) will be prescribed i.e. type (b) boundary conditions. However it is usually \( \zeta \) that is known and the prescribed boundary \( \zeta \) must be obtained through equation 3.76. A finite difference scheme (see figure 3.16) is proposed.
to make the transformation from $\phi$ to $\zeta$. If the inertial term is assumed to be very small, equation 3.76 may be represented by the difference equation, (see appendix E),

$$\zeta_{111} = \zeta_{000} + \Delta\phi + \frac{1}{\text{gm}} (\bar{u}_t \Delta x + \bar{v}_t \Delta y) - (\bar{\phi}_t - \zeta_t) \Delta t$$  \hspace{1cm} (3.136)

where $\zeta_{000}$ refers to the initial position of the particle and $\zeta_{111}$ the final position. It was shown in section 3.2.2 that the inertial term is quite small for the media being studied in this thesis.

On the free surface

$$\Delta \phi = \Delta y$$  \hspace{1cm} (3.137)

Other approximate relationships to replace equation 3.136 are given in Appendix E.

In the unsteady flow system 3.134 the initial conditions on $\zeta$ and the prescribed boundary values of $\zeta$ give rise to the constant vector

$$[w] = [A] [\zeta]$$  \hspace{1cm} (1.138)

which is comparable to equation 3.115 for steady flow.

The Numerical Analysis:- In order to demonstrate the numerical solution of the system 3.134 and the accompanying boundary conditions the problem of rapid drawdown in a rockfill section is analysed.
The first step in the analysis is the specification of the initial conditions and the boundary conditions. Typical boundary and initial conditions are indicated in figure 3.14 and 3.15. The next step is the discretization of the flow field (also shown in figure 3.14). The node numbers shown in figure 3.14 are arranged, as suggested by Finn (31), in such a way as to minimize the band width of the matrix \( [A] \). The 'unknown' nodes are numbered from 1 to \( n \) where \( n \) is the total number of unknowns while the 'known' nodes are numbered from \( n+1 \) up.

The 'triangular' elements are of the type shown in figure 3.13, the ordinates of the elemental nodes, \( y_i \), are adjusted in direct proportion to the movement of the free surface ordinates \( y_s \), i.e.

\[
y_i(\Delta t) = y_i(0) \frac{y_s(\Delta t)}{y_s(0)}
\]  

(3.139)

In computing the new free surface position, using equation 3.135, the position of the 'tagged' free surface particles at \( \Delta t/2 \) was assumed to be directly over the centroid of the surface element as illustrated in figure 3.17. An approximate extrapolation formula

\[
\hat{\tau} = \hat{u} \hat{c} \hat{\tau} + \hat{v} \hat{c} \left( \frac{y_s}{\hat{y}_c} \right) \hat{\tau}
\]  

(3.140)

was used to obtain the surface particle velocities. The extrapolation accounts in part for the fact that magnitude of the vertical velocities tend to increase from the bed to the free surface.
Equation 3.133 gives the change in the $x$ and $y$ coordinates, ($dx$, $dy$), of a free surface (fluid) particle during a short interval $dt$. However, as we move along the vertical drop, $dy$ in the surface profile at a position $x$, may involve a relationship between $dx$ and $dy$ by the equation

$$\frac{dy}{dx} = \frac{\partial^2 y}{\partial x^2}$$

The slope correction factor, $S_c$, used in the programme was estimated from a profile, the actual and the computed $S_a$ and $S_c$ varied between 0.65 and 1.15, the programme $S_c = 1.0$ was used in the programme.

Other details of the finite-difference solution are best explained by reference to the flow chart (Figure 3.16), the outer loop (Loop 4) which time is varied. The programme then enters the outer computation loop (Loop 4) in which time is varied. The new tailwater elevation and bottom conditions are set along the seepage face and at the tailwater. On the outer iteration, only, an approximate solution including free surface boundary conditions is used.

**Figure 3.17 Free Surface Adjustments**
Equation 3.135 gives the change in the x and y coordinates, \((Δx, Δy)\), of a free surface (fluid) particle during a short interval \(Δt\). However it would be more useful to have the vertical drop, \(Δy_s\) in the surface profile at a particular value of \(x\) (say \(x_c\)). The relationship between \(Δy_s\) and \(Δy\) can be obtained from figure 3.17, i.e.

\[
Δy_s = Δy - S_a Δx
\]

or

\[
Δy_s = Δy - Δx \cdot S_a \cdot S_c
\] (3.141)

The slope correction factor, \(S_c\) used in the programme was estimated from a preliminary computation assuming \(S_c = 1.0\). It was found that \(S_c\) varied between 0.66 and 0.85; therefore a value of 0.75 was used in the programme.

Other details of the finite element solution are best explained by reference to the flow charts in figures 3.18 to 3.25 and the computer programme in appendix F.

Referring to the master flow chart (figure 3.18), the initial computations include non-repetitive computations e.g. generation of the \((1, m, n)^e\) array and initialization of time, \(t\). The programme then enters the outer computation loop (loop 4) in which time is varied. The new tailwater level is established and the prescribed boundary conditions are set along the seepage face and at the tailwater. On the outer iteration, only, an approximate solution including free surface boundary conditions is read.
**INITIAL CONDITIONS**
\[ (x, y, t)_l = (x_0, y_0, t_0) \]

**ALL NODES**

**STIFFNESS MATRIX**
\[ S_{ij} \]

**LINEAR TERMS**
\[ \text{EQ. 3.134} \]

**SOLVE** BY SOR FOR
\[ \xi_l \]

**ESTIMATE NEW FREE SURFACE**
\[ \xi_l \]

**ADJUST INTERNAL ELEMENTS**

**IS TRIAL SOLUTION STABLE?**

**PRINT** \( \xi(A, B, t) \)

**Figure 3.15** Master Flow Chart for the Unsteady Non-Linear Flow PDE

**NONLINEAR TERMS**

**ASSEMBLE EQUATIONS**

**Figure 2.21** Flow Chart-Assembly of Equations

**Figure 2.14** Flow Chart-Stiffness Matrix

**Figure 3.20** Flow Chart-Non-Linear Terms
NEW FREE SURFACE

SOLVE SOR

DO 1 IX=1, IEST

129

RETURN

Figure 3.22 Flow Chart-SOR

LINEAR INTERPOLATION TO OBTAIN SURFACE NODAL VALUES OF Δγ

INTERPOLATION AND AVERAGING

INTERPOLATION TO OBTAIN CORRECTION OF SURFACE

EXTRAPOLATION TO OBTAIN END POINT VALUES

AVERAGE INTERPOLATED AND EXTRAPOLATED VALUE FROM ALL TRIALS FOR A GIVEN TIME INTERVAL

Figure 3.25 Interpolation of Δγ

Figure 3.24 Interpolation of Δγₖ
On entering loop 3 the elemental areas and the stiffness matrix are computed. A detailed flow chart for these computations is shown in figure 3.19.

Loop 2 is shown in figure 3.18 and consists of:

a) a non-linear subroutine shown in figure 3.20 which computes:
   i) the non-linear terms required to assemble equation 3.134;
   ii) the velocities required for equations 3.135 and 3.136.
   iii) the inertial terms for equation 3.136.

b) a subroutine (shown in figure 3.21) which resembles equation 3.134;

c) the successive over-relaxation subroutine (loop 1, shown in figure 3.22) that solves the linearized form of equation 3.134 for the trial boundary conditions;

d) the free surface adjustments subroutines (shown in figures 3.23, 3.24 and 3.25) which compute:
   i) the movement of free surface particles (equation 3.135);
   ii) the drop in the free surface at \( x_c \) (equation 3.141);
   iii) the changes in \( \phi_s \) and \( \zeta_s \) at \( x_c \);
   iv) the changes in \( \phi \) and \( \zeta \) at the surface nodes;
   v) the average of the nodal changes in \( \Delta\phi, \Delta y \) and \( \Delta \zeta \) for successive trials and hence a new trial free surface;
   vi) the adjustments of the internal element geometries by equation 3.139.
Loop 2 is executed twice as indicated by 'ALT' in figure 3.18 before the 'stability' of the trial free surface is checked.

The 'stability' of the trial free surface is checked by determining the number of iterations of the SOR (loop 1) subroutine that are required to achieve the prescribed accuracy \( |\bar{\phi}^{n+1} - \bar{\phi}^n| \) after a trial free surface has been computed. If the prescribed accuracy can be obtained with only one iteration the trial solution is considered stable and a new time step is started. If more than two SOR iterations are required to achieve the prescribed accuracy then loop 3 is repeated.

At the beginning of each new time step the free surface and internal values of \( \zeta \) (or \( \bar{\phi} \)) that have just been computed become the initial conditions for the new computation i.e. the \( i, j, k \) nodes are relabelled \( 1, m, n \).

The inner loop (loop 1) used the SOR procedure defined by equation 3.96 to solve the linearized form of equation 3.134. The linearization is achieved by using the \( K(|\nabla \phi|) \) obtained from the values of \( \bar{\phi} \) computed in the preceding loop 2.

The effect of the inertial term, \( \frac{1}{\eta} \frac{\partial^2 \bar{\phi}}{\partial t} \), can easily be suppressed in the computer programme (Appendix F) by putting \( C_2 = 0.0 \) in the ADJUSY subroutine.

The development of the finite element method without inertia, is very similar to the case with inertia except that no transformation of the boundary conditions is required. The non-inertial model development is summarized in the form of a paper given in Appendix G along with the modified computer programme.
The stability of the unsteady flow solutions depends on the choice of $\Delta t$. After a number of trial solutions it was found that, without the 'averaging' procedure,

$$\Delta t \leq \frac{\Delta x}{\sqrt{2g_s}}$$

yielded stable solutions (where $\Delta x$ is the average horizontal spacing of the free surface nodes). The solution stability is further improved by averaging the trial free surface displacements, of the selected surface particles, for each time interval. This 'averaging procedure' is also applied to the computed changes in the free surface values of $\zeta$. 

a) parallel parameters,

b) converging and diverging parameters,

c) free surface models.

Once the resistance equations were determined and steady flow free surface models were operated to obtain data for comparison with the steady flow finite element solution.

Finally, unsteady flow free surface experiments were conducted on each of the selected materials. These consisted of captive cresten tests which could be compared with the finite element solution.

The geometric properties of the plates and the location, properties of the media were also determined.

4.4 Experimental Apparatus

All the experimental flow studies for data plates were performed in
CHAPTER IV
THE EXPERIMENTAL STUDIES

The main objective of the experimental programme was to aid in the evaluation and verification of the finite element models. Some preliminary work was required to establish the resistance characteristics of the media (namely 1.66 cm and 4.40 cm crushed Dolomite, see photographs 4.1 and 4.2) to be used in the verification studies. Three types of resistance tests were performed on each of the selected materials using:

a) parallel permeameters,
b) converging and diverging permeameters,
c) free surface models.

Once the resistance equations were determined steady flow free surface models were operated to obtain data for comparison with the steady flow finite element solution.

Finally unsteady flow free surface experiments were conducted on each of the selected materials. These consisted of rapid drawdown tests which could be compared with the finite element solutions.

The geometric properties of the grains and the institue porosities of the media were also determined.

4.1 Experimental Apparatus

All the experimental flow studies for this thesis were performed in
Photo 4.1 Sample of 1.66 cm Crushed Rock

Photo 4.2 Sample of 4.40 cm Crushed Rock
the 122 cm long test section of the 30.5 cm wide by 122 cm high flume shown in figure 4.1 and photograph 4.3. This flume had a 180 cm long plexiglass window for viewing the flow in the media. An array, of piezometric tappings, was installed in the window as shown in figure 4.1. Each piezometric tap was connected by plastic tubing to a piezometer on the vertical piezometer bank at the left hand side of the viewing window, as shown in photograph 4.3. The piezometer tubing could also be connected to a variable angle inclined manometer. Vertical grooves were provided at either end of the test section so that barriers or screens could be inserted. A rubberized fibre baffle was placed between the inflow pipe and the test section to dissipate large scale turbulence.

Water was supplied to the flume by a 240 US gpm centrifugal pump which could be operated to discharge, directly (to the flume) or through a 50 ft. constant head tank (to the flume). The discharge into the flume was regulated by a stem valve in the 6-inch diameter inflow pipe shown in figure 4.1. A venturimeter located in the inflow pipe and connected to a calibrated Merriam manometer was available to measure the higher discharges (10 to 240 gpm). The smaller discharges were determined volumetrically by using the flow from the drain valve (see figure 4.1).

The special adaptations to the flume, that were required for the various tests are described below:

4.1.1 Parallel Flow Set-up:

A typical setup for a parallel flow test is illustrated in figure 4.2. The section was sealed with plasticene and waterproof tape. A
Figure 4.1 Test Flume (a)
Figure 4.1 Test Flume
Photo 4.3 Test Flume
Figure 4.2 Typical Set-up for Parallel Flow

1.4 The Unsteady Flow set-up

The experimental arrangement for the unsteady flow (rapid, pseudoadiabatic) studies is shown in Figure 4.3. A sheet of plywood was placed against the upstream screen, and sealed with plasticene and tape. The rock was placed between this impermeable barrier and the downstream screen.
dye was used to check for possible leakage. An attempt was made to decrease the wall effect at the ceiling of the permeameter by 'roughening' the plywood surface with plasticene strips as indicated in figure 4.2. Also crushed rock of approximately half the normal size of the test material was placed in the larger voids along the upper rock surface. The piezometric taps were connected by plastic tubing to either the vertical manometer or the tilting manometer.

4.1.2 Radial Flow Set-up

Figure 4.3 and 4.4 respectively show test arrangements for the converging and diverging flow tests. Photograph 4.4 also shows a converging flow test in operation. The same techniques that were described for the parallel flow studies were employed to minimize the ceiling wall effect in the radial flow tests.

4.1.3 The Set-up For Steady Free Surface Flow

For the two dimensional steady flow tests, 'expanded' galvanized steel screens, with openings $\frac{1}{2}$ cm by $\frac{3}{4}$ cm, were inserted in the grooves at both ends of the test section as shown in figure 4.1. The test materials were placed between the screens as shown in photograph 4.3, and the piezometer tappings connected by plastic tubing to the vertical manometer.

4.1.4 The Unsteady Flow Set-up

The experimental arrangement for the unsteady flow (rapid drawdown) studies is shown in figure 4.5. A sheet of plywood was placed, against the upstream screen, and sealed with plasticene and tape. The rock was placed between this impermeable barrier and the downstream screen.
Figure 4.3 Typical Set-up for Converging Flow

Figure 4.4 Typical Set-up for Diverging Flow
Photo 4.4 Diverging Flow Test
The hinged tailgate shown in Figure 4.1 and 4.5 was used to control the tailwater level in the test section. A 25 cm long plastic gauge was available along with a timer (located over the photo graph 4.3).

4.2 The Experimental Procedure

The experiments were conducted in a separate flume. The following properties of the media were considered:

a) the size of the particle, its shape, and its density;

b) the size of the particle, its shape, and its density;

c) the distribution of grain size;

d) the distribution of grain size;

e) the density of the media.

Determining of Grain Size: All the materials used in this study were obtained by running a thoroughly washed quarry-run sample of crushed dolomite through a mechanical dolly screen. The selected
The hinged tailgate shown in figure 4.1 and 4.5 was used to control the tailwater level on the test section. A 35 mm Pentax camera was available along with a timer (located over the test section in photograph 4.5).

4.2 The Experimental Procedures

The experimental procedures are described under the headings:

a) Determination of grain and media characteristics,
b) Parallel flow studies,
c) Radial flow studies,
d) Steady free surface flow studies,
e) Unsteady free surface flow.

4.2.1 The Characteristics of the Media

The following properties of the media were considered:

a) the size of the individual grains, e.g. equivalent spherical diameter and sieve size,
b) the distribution of grain sizes,
c) the particle shape, e.g. specific surface,
d) the porosities of the media (both dry and drainable),
e) the density of the rock particles.

Determination of Grain Size: All the materials used in this study were obtained by running a thoroughly washed quarry-run sample of crushed dolomite through a mechanical sieve analyser. The selected
Photo 4.5 Unsteady Flow Test
materials were:

a) those which passed the 3/4 inch sieve but were retained on the 1/2 inch sieve;

b) Those which were retained on the 1-1/2 inch sieve (i.e. the 'oversized' rock).

The 'oversized' sample was found by inspection to have a maximum sieve size of 3 inches (the occasional larger grains were hand sorted and removed from the sample).

The equivalent spherical diameters of the individual rock particles were found from the measured particle masses and the density of the dolomite rock (2.60 gm/c.c.).

**Particle Size Distribution:** The geometric standard deviation was chosen to indicate the distribution of particle sizes within each of the selected media (i.e. the 1.66 cm and 4.40 cm crushed rock). In order to compute a geometric mean and a geometric deviation several particles (at least 30 from each sample) were chosen at random from each of the selected media and the particle size analysis was carried out on each of the particles to determine their equivalent diameters.

**Particle Shape:** Two measures of particle shape were determined:

a) the three mutually perpendicular axes that would describe the enclosing ellipsoid were measured (see figure 4.6 and photograph 4.6) for several particles from each of the selected media;
Photo 4.6 Measurement of SF

Photo 4.7 Measurement of Grain Surface Area
b) the surface area and equivalent area were calculated using the formula:

\[ SA = \frac{4}{3} \pi \cdot 9^2 \cdot 105 \cdot 9 \ cm^2 \]

The particles were enclosed in a layer of plasticine. The particles were then

\[ M_g = 4.40 \ \text{cm} \]

fully peeled off and flattened as shown in photograph A, B. The layers were

\[ M_g = 4.40 \ \text{cm} \]

measured with a planimeter (figure 4.7). The particles were placed in water before the coatings were applied, to prevent the plasticine

\[ M_g = 4.40 \ \text{cm} \]

from adhering to the rock surface.

The Porosity of the rocks: Two types of porosities were

\[ M_g = 4.40 \ \text{cm} \]

measured for each of the tests, i.e. dry porosity (negligible pore water) and

\[ M_g = 4.40 \ \text{cm} \]

detrinite porosity (porosity immediately after gravitational drainage has occurred).

The Test and the Test

\[ M_g = 4.40 \ \text{cm} \]

a) the tailgate in the basin (figure 4.1) was set in its normal position and sealed;

\[ M_g = 4.40 \ \text{cm} \]

b) with no rock in the tank section, volume versus stage curve

\[ M_g = 4.40 \ \text{cm} \]

was established for the basin by filling the basin with water;

\[ M_g = 4.40 \ \text{cm} \]

c) the basin was drained and the rocks were then placed in the

\[ M_g = 4.40 \ \text{cm} \]

flume and another volume versus stage curve was determined by

\[ M_g = 4.40 \ \text{cm} \]

filling in the flume:

\[ M_g = 4.40 \ \text{cm} \]

a) the bulk volume of rock versus stage was determined from the

\[ M_g = 4.40 \ \text{cm} \]
b) the surface areas and equivalent particle diameters were obtained for a number of particles from each of the selected media.

The length width and depth measurements for the enclosing ellipsoid were made with a micrometer as shown in photograph 4.6.

The surface areas of the particles were determined by coating the particles with a thin layer of plasticene. This coating was carefully peeled off and flattened as shown in photograph 4.7. The outline of the coating was then traced on a piece of paper and the outlined area measured with a planimeter (see figure 4.7). The particles were soaked in water before the coatings were applied, to prevent the plasticene from adhering to the rock surface.

The Porosity of the Media:-- Two types of porosities were measured for each of the tests, i.e. dry porosity (negligible pore water) and drainable porosity (porosity immediately after gravitational drainage has occurred).

The institute dry porosities were determined as follows:

a) the tailgate in the flume (figure 4.1) was set in its vertical position and sealed;

b) with no rock in the test section a volume versus stage curve was established for the flume by filling the flume with water;

c) the flume was drained and the rocks were then placed in the flume and another volume versus stage curve was determined by filling the flume again;

d) the bulk volume of rock versus stage was determined; thus the
dry porosity could be found from (b), (c) and (d);
e) the flume was again drained through valve B and the outflow
volume measured; thus with (b), (c), (d) and (e) the amount of
water retained in the voids could be estimated and hence the
drainable porosity obtained.

The porosity studies were repeated after the flow tests were
completed. Waiting periods of two to seven days were allowed for evapor­
ation of pore water.

The accuracy of the volumetric method of determining porosity was
checked in a few instances by weighing the material (rock) being placed
in a given volume.

Density of the Granular Particles:— The density of the dolo­
mite particles was determined by measuring the mass of a number of
particles on a scale balance and dividing this mass by the volume of
water displaced by the same particles when they were placed in a graduated
cylinder.

4.2.2 Parallel Flow Studies

The rockfill was placed between the screens shown in figure 4.2 and
levelled at a depth of approximately 40 cm. The larger voids along this
levelled surface were filled with smaller rock (approximately one half
the nominal grain size) to minimize the ceiling wall effect. The L­
shaped plywood cover shown in figure 4.2 was then placed on top of the
rock and compacted to eliminate any large gaps between the rock and the
ceiling. This cover was sealed and weighted down as indicated in figure
4.2.
The institute porosities of the media were determined as described in section 4.2.1.

After starting the pump, and setting valve A and the tailgate a period of time (5 to 30 minutes), was allowed for the flow in the flume to come to equilibrium, i.e., for the headwater and tailwater levels to become steady. For discharges greater than 10 gpm the flow was recorded from the venturi meter and the piezometric levels for the all piezometric taps on the front wall of the sample were read. For flows less than about 10 gpm the venturi meter manometer could not be read with sufficient accuracy; therefore the volumetric method was used to determine these flows. The free discharge from valve B was intercepted in a 5-gallon container and a stop watch used to measure the time of filling. The discharge was determined from the average of at least 4 volumetric measurements. For flows greater than 10 gpm the vertical piezometers were used while for smaller flows the tilting manometer was used.

The discharge was then changed and the above procedure repeated. Tests were performed on 1.66 cm. and 4.40 cm crushed rock.

The water temperature was recorded at the beginning and end of each day's testing.

4.2.3 Radial Flow Tests

The test procedures for the converging and diverging flow tests were almost identical to those for the parallel flow tests except the test arrangements were changed to those shown in figures 4.3 and 4.4. The sloping surfaces were graded and compacted, and the large voids filled
with smaller grains in the same manner as described for the parallel flow tests. The institute porosity was determined. For a range of discharges all the piezometers on the front wall of the sample were observed and recorded along with the radial positions of the tappings. Both 1.66 cm. and 4.40 cm. crushed rock were tested for converging and diverging flow. Other radial configurations are shown in fig. 4.8.

4.2.4 Steady Free Surface Flow

Steady free surface tests were carried out on both the 1.66 cm and the 4.40 cm crushed rock. The experimental procedure was similar to that described in section 4.2.2.

Rectangular rockfill dykes were retained in the test section by the two screens indicated in figure 4.1 and photograph 4.3. The procedure for discharge and piezometric measurements was the same as described in section 4.2.2. In addition to reading all the internal piezometric taps, the water levels were measured (to $\pm \frac{1}{2} \text{ mm}$) at several points along the free surface in the rockfill, at the headwater, at the outcrop point (seepage face) and at the tailwater.

A number of tailwater levels and discharges were used for each medium in order to obtain a representative sample of data for comparison with the finite element computations.

Additional readings were taken of headwater levels, tailwater levels and seepage heights for a number of different discharges. This data was collected to supplement the seepage height studies of Ng (79) and is not part of this thesis (see Ref. 73 and section 2.3.3).
Figure 4.5 Other Similar Flow Configurations
4.2.5 The Unsteady Flow Tests

Rapid drawdown tests were carried out on both the 1.66 cm and 4.40 cm crushed rock for the test arrangement shown in figure 4.5 and photograph 4.5. After the rock had been placed in the rectangular test section the tailgate was set in a vertical position and stayed by a guy-wire as shown in figure 4.5. The gate was sealed with plasticene on its upstream side.

Before each run the test section and tailwater section was filled with water (from the mains) to a depth of 56.0 ± 0.5 cm i.e. to approximately the top of the tailgate. A dye (KMnO₄) was then added to the water in the test section to aid in flow visualization (for the photographs). The timer was zeroed in preparation for the start of each run. A photograph was then taken just before the tailgate was released.

Three persons were required to complete each run. On a signal one assistant released the guy-wire allowing the tailgate to drop to a horizontal position and thus allowing the tailwater level to drop rapidly. At the instant the gate was released the timer was started by the second assistant. The third person operated the 35 mm camera taking photographs in rapid succession (lapse times of about 1, 2, 4, 6, 8, 10, 15, 20 and 30 seconds were used). A meter stick (for scale) and the timer were included in all the photographs. A number of runs of the rapid drawdown tests were made to obtain better mean experimental curves as well as to check the reproducibility of the experiments.

Also with the aid of five persons a series of manual observations were made of the falling waterlevel profiles. Three assistants marked
the free surface positions at selected horizontal stations for lapse times of 0, 1, 4, 6, 8, 12, 16, 20 and 24 seconds as indicated by a tapped signal from a timekeeper. In order to compensate in part for the reaction times of the 'markers', the tailgate was released by another assistant on the initial signal from the timekeeper.

4.3 Experimental Results

The experimental results are presented in the same order as the experimental procedures.

4.3.1 The Characteristics of the Crushed Rock

Particle Size and Size Distribution: - The results of mass and volumetric measurements of individual grains randomly selected from each of the two media are shown in tables 4.1 and 4.2. The accuracies of the readings are also indicated. The sieve sizes are given for each of the materials.

Particle Shape: - Tables 4.1, 4.2 and 4.3 show the micrometer measurements of the three mutually perpendicular distances representative of the length, width and depth of the grain (see figure 4.6). A range of materials sizes from about 1.5 cm to 5.0 cm equivalent diameter, are represented.

Tables 4.3 shows the surface areas of several particles varying in size from 1.5 cm to 4.5 cm. The mass and volume of each particle are also shown along with the length, width and depth dimensions.

Porosity: - Average dry and drainable porosities for the 1.66 cm and the 4.40 cm rock for the various test arrangements are shown in table 4.4. The method of measurement is also indicated.
Density of the Grains:— The density of the rock particles was found to be 2.60 ± 0.02 gm/c.c.

4.3.2 Parallel Flow Studies

Tables 4.5 to 4.7 give the discharge and piezometric readings for the parallel flow tests on the 1.66 cm and the 4.40 cm crushed rock. The tables also indicate the accuracy to which the data could be read as well as the methods of measurement.

4.3.3 Radial Flow Tests

Readings of the discharges and piezometric heads for the radial (converging and diverging) flow experiments are given in table 4.8 to 4.12.

4.3.4 Steady Free Surface Flow

Typical steady free surface data for the two selected materials are shown in figures 4.9 and 4.10.

4.3.5 Unsteady Free Surface Flow

The sequence of photographs 4.8(a) to 4.8(h) show a typical rapid drawdown test for 1.66 cm crushed rock. Photographs 4.9(a) to 4.9(h) are for a similar test with 4.40 cm crushed rock. The measured water levels taken from these as well as other photographs are summarized in table 4.13 and table 4.14. In addition, data obtained by manually marking the falling waterlevels, on the flume window at specified times, are presented in table 4.15 for the 1.66 cm rock and in table 4.16 for the 4.40 cm rock. The drainable porosity for the 1.66 cm rock fell between .38 and .42 thus an average value of 0.40 ± .015 is given. For 4.40 cm rock the drainable porosity estimates ranged from .39 to
.41, giving an average value of 0.40 \pm 0.01.
Figure 4.9 Verification Data (in cm).
FIGURE 4-10 SOME FREE SURFACE TESTS (D in cm).
Photos 4.8 Experimental Drawdown Profiles (1.66 cm Rock)
Photos 4.9 Experimental Drawdown Profiles (4.40 cm Rock)
Table 4.1  Mass Measurements for 1.66 cm Crushed Rock

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<th>A (inches)</th>
<th>B (inches)</th>
<th>C (inches)</th>
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Table 4.2 Mass Measurements for 4.4 cm Crushed Rock

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Table 4.3 Particle Shape Measurements

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Table 4.3 Particle Shape Measurements (contd.)

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<th>B inches</th>
<th>C inches</th>
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Table 4.4 Results of Porosity Tests

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*Errors shown are standard errors of the means.*
Table 4.5
Parallel Flow Data

Test No. 1.1

\( M_g = 1.66 \text{cm} \)

Cross-sectional Area = 1050

Manometer - Vertical; Flow - Venturi

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<th>( \Delta L ) (cm)</th>
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**Run No. 1**

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Table 4.5
Parallel Flow Result

Test No. 1.1

\[ M = 1.66 \text{ cm} \]

Cross-sectional Area = 1050 cm\(^2\)

Manometer - Vertical; Flow - Volumetric

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<th>(\Delta L) (cm)</th>
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Run No. 2

\[ T = 74^\circ F \]
Table 4.5
Parallel Flow Results

Test No. 1.1

Run No. 3

\[ M_g = 1.66 \text{ cm} \]

\[ T = 74^\circ \text{F} \]

Cross-sectional Area = 1050 cm²

Manometer - Inclined; Flow - Volumetric

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Table 4.7
Parallel Flow Data

Test No. 1.3

\( M = 4.40 \) g

Cross-sectional Area = 1167.0 cm\(^2\)

Manometer - vertical

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Table 4.8

Piezometric Readings-Convergent Flow

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Piezometric Readings-Convergent Flow

(\(M_g = 1.66 \text{ cm}; m = 0.42; \theta = 0.510\))

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**Piezometric Readings—Converging Flow**

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### Table 4.10 (cont.)

**Piezometric Readings—Convergent Flow**

\((M_g = 4.40 \text{ cm}; \ m = .42; \ \theta = 0.51^\circ)\)

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<th>R(cm)</th>
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\[ M = 1.66 \text{ cm}; \quad m = 0.435; \quad \theta = 0.328° \]

### Table 4.11

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<td><strong>R(cm)</strong></td>
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Table 4.12
Piezometric Readings-Divergent Flow

\( (M = 4.40 \text{ cm}; m = 0.425; \theta = 0.353^\circ) \)

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<th>7</th>
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Table 4.13
Unsteady Flow Profiles (Depths) Measured from Photographs

(M = 1.66 cm) Scale Corrections have been applied to all data.

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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>10.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>11.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>16.0</td>
</tr>
<tr>
<td>Test Run</td>
<td>M</td>
<td>Lapse</td>
<td>Horizontal Position cm</td>
<td>TWL</td>
</tr>
<tr>
<td>----------</td>
<td>---</td>
<td>-------</td>
<td>------------------------</td>
<td>-----</td>
</tr>
<tr>
<td>No.</td>
<td>No.</td>
<td>g</td>
<td>Time</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1.66</td>
<td>0</td>
<td>0</td>
<td>56.0</td>
</tr>
<tr>
<td>19.2</td>
<td></td>
<td></td>
<td>13.8</td>
<td>13.7</td>
</tr>
<tr>
<td>30.0</td>
<td></td>
<td></td>
<td>7.9</td>
<td>7.6</td>
</tr>
</tbody>
</table>
### Table 4.14

Unsteady Flow Profiles Measured From Photographs

<table>
<thead>
<tr>
<th>Test Run M</th>
<th>Lapse Time</th>
<th>Horizontal Position cm</th>
<th>TWL</th>
</tr>
</thead>
<tbody>
<tr>
<td>No.</td>
<td>No. cm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4.40</td>
<td>0</td>
<td>30.5</td>
</tr>
<tr>
<td>0.5</td>
<td>56.0</td>
<td>56.0</td>
<td>56.0</td>
</tr>
<tr>
<td>1.0</td>
<td>42.5 ± .5</td>
<td>42.0</td>
<td>41.0</td>
</tr>
<tr>
<td>2.0</td>
<td>39.0 ± .5</td>
<td>39.5</td>
<td>38.5</td>
</tr>
<tr>
<td>3.7</td>
<td>31.0</td>
<td>31.0</td>
<td>30.5</td>
</tr>
<tr>
<td>6.9</td>
<td>poor photo</td>
<td>23.0</td>
<td>---</td>
</tr>
<tr>
<td>8.85</td>
<td>14.5 ± .5</td>
<td>14.5</td>
<td>14.0</td>
</tr>
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<td>15.0</td>
<td>10.5 ± .5</td>
<td>poor photo</td>
<td>10.0</td>
</tr>
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<td>2</td>
<td>4.40</td>
<td>0</td>
<td>56.0</td>
</tr>
<tr>
<td>.75</td>
<td>52.0</td>
<td>51.2</td>
<td>51.0</td>
</tr>
<tr>
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<td>41.0</td>
<td>42.0</td>
<td>40.5</td>
</tr>
<tr>
<td>4.0</td>
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<td>---</td>
<td>30.0</td>
</tr>
<tr>
<td>7.0</td>
<td>24.0</td>
<td>23.5</td>
<td>23.0</td>
</tr>
<tr>
<td>10.5</td>
<td>15.0 ± .5</td>
<td>15.0</td>
<td>14.5</td>
</tr>
<tr>
<td>Test Run No.</td>
<td>M g cm</td>
<td>Lapse Time sec.</td>
<td>Horizontal Position cm</td>
</tr>
<tr>
<td>--------------</td>
<td>--------</td>
<td>-----------------</td>
<td>------------------------</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>30.5 61.0 91.5 106.6 122</td>
</tr>
<tr>
<td>2</td>
<td>4.4</td>
<td>13.5 12.2±.7</td>
<td>12.0 12.0 10.7±.5 10.5 8.0±.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>18.2 11.0</td>
<td>11.0 11.0 9.5 9.0±.5 8.0±.5</td>
</tr>
<tr>
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<td>4.4</td>
<td>0</td>
<td>56.0 56.0 56.0 56.0 56.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.9 36 ± 1</td>
<td>-- poor photo 30.0 24.5±1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7.15 20.5</td>
<td>-- 20.0 18.0 14.0±1 12.5±1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8.0 19.5</td>
<td>19.5 19.0 17.0 13.0 12.0±1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>13.0 11.5±.5</td>
<td>12.0 11.5 10.5±.5 9.0±1</td>
</tr>
</tbody>
</table>
Table 4.15  
Water Surface Profiles Obtained by Manual Measurements

(M = 1.66 cm)

<table>
<thead>
<tr>
<th>Test Run No.</th>
<th>M g cm</th>
<th>Lapse Time</th>
<th>Horizontal Position (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>1.66</td>
<td>0</td>
<td>33</td>
</tr>
<tr>
<td>1</td>
<td>50.3</td>
<td>0</td>
<td>33</td>
</tr>
<tr>
<td>2</td>
<td>44.8</td>
<td>0</td>
<td>64</td>
</tr>
<tr>
<td>4</td>
<td>32.2</td>
<td>0</td>
<td>64</td>
</tr>
<tr>
<td>6</td>
<td>26.3</td>
<td>0</td>
<td>120</td>
</tr>
<tr>
<td>8</td>
<td>20.6</td>
<td>0</td>
<td>120</td>
</tr>
<tr>
<td>12</td>
<td>16.2</td>
<td>0</td>
<td>TWL</td>
</tr>
<tr>
<td>16</td>
<td>13.4</td>
<td>0</td>
<td>TWL</td>
</tr>
<tr>
<td>20</td>
<td>11.0</td>
<td>0</td>
<td>TWL</td>
</tr>
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<td>55.5</td>
</tr>
<tr>
<td>1</td>
<td>53.5</td>
<td>0</td>
<td>55.2</td>
</tr>
<tr>
<td>2</td>
<td>48.3</td>
<td>0</td>
<td>55.2</td>
</tr>
<tr>
<td>4</td>
<td>42.1</td>
<td>0</td>
<td>56.1</td>
</tr>
<tr>
<td>6</td>
<td>33.9</td>
<td>0</td>
<td>56.1</td>
</tr>
</tbody>
</table>

TWL = Time Water Level
<table>
<thead>
<tr>
<th>Test Run No.</th>
<th>Magnesium Lapse No. cm</th>
<th>Time 0 cm</th>
<th>Horizontal Position (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.66</td>
<td>8</td>
<td>28.5</td>
<td>27.0</td>
</tr>
<tr>
<td>12</td>
<td>21.9</td>
<td>21.1</td>
<td>21.0</td>
</tr>
<tr>
<td>16</td>
<td>16.9</td>
<td>16.8</td>
<td>16.5</td>
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<tr>
<td>20</td>
<td>13.6</td>
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<td>13.5</td>
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</table>

Table 4.15 (cont.)
<table>
<thead>
<tr>
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<th>M</th>
<th>Lapse Time</th>
<th>Horizontal Position (cm)</th>
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</thead>
<tbody>
<tr>
<td></td>
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</tr>
<tr>
<td>No. No.</td>
<td>cm</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>56.2</td>
<td>55.3</td>
<td>55.5</td>
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<td>39.5</td>
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<td>4</td>
<td>30.5</td>
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<td>30.7</td>
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</tr>
<tr>
<td>20</td>
<td>10.3</td>
<td>9.9</td>
<td>10.8</td>
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</table>
Table 4.16
Water Surface Profiles Obtained by Manual Measurements

(M = 4.40 cm)

<table>
<thead>
<tr>
<th>Test Run</th>
<th>M</th>
<th>Lapse No.</th>
<th>cm</th>
<th>Time</th>
<th>Horizontal Position (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1 and 2</td>
<td>4.40</td>
<td>0</td>
<td>56.3</td>
<td>55.1</td>
<td>56.3</td>
</tr>
<tr>
<td>1</td>
<td></td>
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<td>55.4</td>
<td>56.3</td>
<td>55.8</td>
</tr>
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<td></td>
<td>56.6</td>
<td>55.5</td>
<td>56.3</td>
<td>55.8</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>56.7</td>
<td>55.6</td>
<td>56.3</td>
<td>55.8</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>56.8</td>
<td>55.7</td>
<td>56.3</td>
<td>55.8</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>56.9</td>
<td>55.8</td>
<td>56.3</td>
<td>55.8</td>
</tr>
<tr>
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<td>57.0</td>
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<td>56.3</td>
<td>55.8</td>
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</tbody>
</table>

200
<table>
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<tr>
<th>Test Run</th>
<th>Lapse</th>
<th>Horizontal Position (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mg No.</td>
<td>Time</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>sec</td>
<td>0</td>
</tr>
<tr>
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<tr>
<td>12</td>
<td></td>
<td>13.4</td>
</tr>
<tr>
<td>16</td>
<td></td>
<td>9.4</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>7.9</td>
</tr>
</tbody>
</table>
CHAPTER V  
ANALYSIS OF THE EXPERIMENTAL DATA

In this chapter the experimental resistance laws for crushed rock are determined, in dimensional and dimensionless forms. The free surface experimental data are reduced to convenient forms for comparison with the finite element solutions.

After establishing the characteristics of the crushed rock particles (e.g. mean size, standard deviation and shape) general resistance equations for crushed rock are derived using the experimental data of this study as well as data presented by other researchers. The radial flow data are examined to determine the influence of convergence or divergence of the macroscopic flow on hydraulic conductivity. Resistance equations are then recommended for use in the finite element models. Four considerations are used in arriving at these equations:

a) the results of the parallel flow tests of this study;
b) the general equation obtained from combining the results of a number of researchers;
c) the results of the radial flow tests;
d) a direct analysis of the measured gradients and flows in the steady free surface tests.

5.1 The Particle Characteristics

There are several ways of defining the grain size in a granular
medium. Some examples are:

a) median size based on sieve analysis, i.e. \( D_{50} \);
b) the arithmetic mean particle diameter (equivalent sphere);
c) the geometric mean particle diameter;
d) the 'fall' diameter;
e) the geometric mean sieve size.

In this thesis the measure of average grain size is taken to be the geometric mean diameter (after Ward (124)), i.e.

\[
M_g = \sqrt[n]{D_1 \times D_2 \cdots D_n}
\] (5.1)

Consequently the distribution of particle sizes is specified by the geometric standard deviation

\[
\sigma_g = e^{\frac{1}{n-1} \sum (\ln D_i - \ln M_g)^2}
\] (5.2)

Formulae 5.1 and 5.2 were applied to the particle mass data shown in tables 4.1 and 4.2 and the resulting geometric mean diameters and geometric deviations are shown in table 5.1.

Some other measures of particle size are also given in table 5.1.

The geometric mean sieve size for a class \( D_1 \) to \( D_2 \) is

\[
D_n = \sqrt{D_1 D_2}
\] (5.3)
Table 5.1

Characteristics of the Granular Media

<table>
<thead>
<tr>
<th>Material</th>
<th>Crushed Dolomite</th>
<th>Crushed Dolomite</th>
</tr>
</thead>
<tbody>
<tr>
<td>Property</td>
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<td></td>
</tr>
<tr>
<td>Density</td>
<td>2.60±.02</td>
<td>2.60±.02</td>
</tr>
<tr>
<td>Geometric Mean Diameter M cm</td>
<td>1.66±.01</td>
<td>4.40±.025</td>
</tr>
<tr>
<td>Geometric Standard Deviation σ</td>
<td>1.13</td>
<td>1.15</td>
</tr>
<tr>
<td>Geometric Mean Sieve Size D_n cm</td>
<td>1.56</td>
<td>5.40</td>
</tr>
<tr>
<td>Median Sieve Size D_50 cm</td>
<td>1.60</td>
<td>5.30</td>
</tr>
<tr>
<td>SF_1</td>
<td>2.15±.2</td>
<td>1.9±.2</td>
</tr>
<tr>
<td>SF_2</td>
<td>2.1±.1</td>
<td>1.7±.1</td>
</tr>
<tr>
<td>α_s</td>
<td>11.6±.3</td>
<td>10.3±.3</td>
</tr>
<tr>
<td>A_ng</td>
<td>~.9</td>
<td>~.9</td>
</tr>
</tbody>
</table>
A number of particle shape factors are presented in table 5.1. These values represented are the arithmetic averages of the shape factors presented in figures 5.1 to 5.4. The shape factors are computed from the data given in table 4.3.

Referring to figure 4.6 a shape factor can be defined as

\[ SF_1 = \frac{A}{\sqrt{BC}} \]  \hspace{1cm} (5.4) \]

A shape factor could also be defined in terms of the surface area of the particle by

\[ SF_2 = \frac{\text{particle surface area}}{\text{surface area of equivalent sphere}} \]

\[ = \frac{SA}{SA_e} \]  \hspace{1cm} (5.5) \]

Another commonly used shape factor (see Kovacs (53)) is

\[ \alpha_s = \frac{(\text{Surface area}) \times D_g}{(\text{Particle Volume})} \]

\[ = \frac{(SA) D_g}{V_g} \]  \hspace{1cm} (5.6) \]

where \( D_g \) is the equivalent diameter of the grain.

An angularity factor which indicates the fraction of the surface area composed of flat surfaces is defined by
Figure 3.1 Shape Factor ($SF_1$) vs Particle Size

Figure 3.2 Shape Factor ($SF_2$) vs Particle Size

Figure 3.3 Shape Factor ($\alpha_4$) vs Particle Size

Figure 3.4 Correlation of Shape Factors ($\alpha_4$ and $SF_1$)
A_{ng} = \frac{\text{Area of the six largest flat surfaces}}{\text{Total surface area}} \quad (5.7)

This factor is suggested as a means of accounting in a correlation equation, for the flat face packing effect described by Kovacs (53) as well as the effect of angularity on flow separation at high velocities. This factor was estimated to be approximately 0.9 for the crushed rock used in this study. The value of $A_{ng}$ for a sphere is zero.

5.2 Analysis of Parallel Flow

In this section the parallel flow data for the 1.66 cm and 4.40 cm crushed rock are treated separately and in combination with experimental data of other investigators (79, 24, 56, 53, 63, 77). The experimental data are analysed by an extension of Ward's analysis (124) and by Kovac's analysis (53).

5.2.1 Particular Equations

A check was made to determine whether the dimensional plot of $i/q$ vs $q$ could be represented by a Forchheimer type equation. Figures 5.5 and 5.6 show the writer's reduced experimental data for 1.66 cm and 4.40 cm rock respectively; the Forchheimer type equations, obtained by the method of least squares, are:

\[
\frac{i}{q} = 0.011 + 0.011 q \quad (5.8)
\]

with $C_r = 0.995$ for 1.66 cm rock;

and

\[
\frac{i}{q} = 0.0062 + 0.0037 q \quad (5.9)
\]
with $c = 0.952$ for 4.40 cm rock. Although reasonably good correlations were obtained for the Forchheimer type equations an apparent change is noted for low velocities for the 1.66 cm rock. Figure 5.5 also shows the experimental curve of eq. (179) for the same and other material with a different permeability and for a different sample for crushed rock.

### 3.2.2 General Equations

**Forchheimer Type Equations**

A simplification of the Wall (176) equation 2.79 is utilized to obtain a general Forchheimer type equation for crushed rock:

$$
Y = Y_0 e^{-K x}
$$

where $Y_0$ and $K$ are empirical constants for given flow conditions.

**Section 3.3:** $k$ is given by eq. (2.11) for a given material (e.g., sand and rock) the shape factor for 3.1 and 3.3) tortuosity and pore structure. The factor will remain approximately constant, therefore a simplification of the equation 3.10 can be written as:

$$
Y = Y_0 e^{-k x}
$$
with $C_r = 0.952$ for 4.40 cm rock. Although reasonably good correlations were obtained for the Forchheimer type equations an apparent regime change is noted for low velocities for the 1.66 cm rock. Figure 5.5 also shows the experimental curve of Ng (79) for the same 1.66 cm material but in a different permeameter and for a different porosity.

### 5.2.2 General Equations

**Forchheimer Type Equation:** A modification of the Ward (124) equation 2.29 is utilized to obtain a general Forchheimer type equation for crushed rock.

Rearranging equation 2.29 gives the dimensionless equation

$$Y = C_1 + C_2 X$$  \hspace{1cm} (5.10)

where

$$Y = \frac{gki}{\nu q}$$  \hspace{1cm} (5.11)

$$X = \frac{\sqrt{k} q}{\nu}$$  \hspace{1cm} (5.12)

$C_1$ and $C_2$ are dimensionless constants for given flow regime (see section 3.2); $k$ is given by equation 2.31.

For a given material (e.g. crushed rock) the shape factor (figure 5.1 and 5.3) tortuosity and pore cross-section factor will remain approximately constant, therefore a simplified form of equation 5.10 can be written as

$$Y = C_1 + C_2 X$$  \hspace{1cm} (5.13)
where

\[ Y = \frac{i M^2 C(m) \rho g}{q \mu E(\sigma_g)} \]  \hspace{1cm} (5.14)

\[ X = \frac{q M [C(m)]^{1/2}}{\nu \sqrt{g(\sigma_g)}} \]  \hspace{1cm} (5.15)

\[ C(m) = \frac{m^3}{(1 - m)^2} \]  \hspace{1cm} (5.16)

\[ E(\sigma_g) = \frac{1}{\sigma_g} \ln \sigma_g \]

and \( C_1 \) and \( C_2 \) are constants for a particular medium.

Two approaches were used to establish \( C_1 \) and \( C_2 \):

a) a least squares method using the full range of experimental data;

b) a determination of \( C_1 \) from the low flow data only (figure 5.7) and a least squares method using the full range of experimental data to determine \( C_2 \) (figure 5.8).

Method (a) was applied to the data of Ng (79) and the writer gave the correlation

\[ C_1 = 960 \pm 500, \]

\[ C_2 = 8.5 \pm 14 \]

\[ C_r = 0.977 \]

\[ \{ \] (5.17)
\[
Y = \frac{1.4 M g m^2 \rho}{q \mu c_g \ln c_g (1-m)^2} \quad \text{EQ. 5.14}
\]

\[
X = \frac{m^{3/2} q M_g}{c_g \ln c_g (1-m) \nu} \quad \text{EQ. 5.15}
\]

Figure 5.7 Dimensionless Friction Curve for Lower Flow (After Ward)

Figure 5.8 Dimensionless Friction Curve for Complete Range of Flows

\[Y = 0.80 + 0.0087X\]
and method (b) gave

\begin{align*}
C_1 &= 800 \pm 350, \\
C_2 &= 67 \pm 13
\end{align*} \tag{5.18}

Equation 5.17 should give better results for the high flows while
equation 5.18 would be better for low flows (see figures 5.7 and 5.8).

Equation 5.13 can be generalized to account for small variations
in grain shape factor by putting, \(124, 125\),

\begin{align*}
Y &= C_1 + C_2 X 
\text{where } Y &= \frac{i M^2 C(m) \rho g}{q E(\sigma_g) \mu_c \kappa_s^2} \tag{5.19} \\
X &= \frac{q M}{\sqrt{E(\sigma_g) \alpha}} \tag{5.20}
\end{align*}

Using the experimental data of Ng (79), Dudgeon (24), Lane (56)
and the writer, method(a) gave

\begin{align*}
C_1 &= 6.0 \pm 3.0 \\
C_2 &= 0.82 \pm 0.07 \\
C_r &= .987 \tag{5.22}
\end{align*}
and method (b) gave

\[ C_1 = 80 + 2. \quad (5.23) \]

\[ C_2 = 0.81 \pm 0.07 \]

The reduced experimental data are presented in figure 5.9 and computer input data are given in Appendix H.

A correction for wall effect based on the equation (See Appendix I),

\[ C_w = \frac{q_\infty}{q} \]

\[ = \left( \frac{L_e}{L_p} \frac{D}{\sqrt{A_T}} + 1 \right)^{-1} \quad (5.24) \]

where \( C = 2.0; \)

- \( L_e \) is the perimeter subject to wall effect;
- \( L_p \) is the total perimeter of the permeameter;
- \( D \) is the grain size;
- \( A_T \) is the cross-sectional area of the permeameter;
- \( q_\infty \) is the velocity in an infinitively large permeameter;

is included in equation 5.25, below:

\[ Y = C_1 + C_2 X \quad (5.25) \]

where

\[ Y = \frac{\mu M^2 C(m) q g}{\mu E(\sigma_g) \alpha q_s C_w} \quad (5.26) \]

\[ X = \frac{M [C(m)]^{1/2}}{\nu E(\sigma_g) \alpha q_s C_w} \quad (5.27) \]
Figure 5.9  Ward Friction Curve for Combined Data (No Wall Correction)
Using the same experimental data, that were used for equations 5.22 and 5.23, equation 5.25 yields

\[ c_1 = 2.0 \pm 1.0 \]
\[ c_2 = 1.18 \pm 0.10 \]
\[ c_r = 0.986 \]

by method (a)

\[ c_1 = 0.95 \pm 0.5 \]
\[ c_2 = 1.13 \pm 0.095 \]

and by method (b). The plot of equation 5.25 is shown in figure 5.10. The values of \( C_w \) for the various tests are given in Appendix H.

Kovac's Analysis:- Kovacs (53) derived the empirical equation

\[ \left( \frac{Y}{93} \right)^{3/4} - \left( \frac{X}{100} \right)^{3/4} = 1 \]  \hspace{1cm} (5.30)

where

\[ X = \frac{D}{\alpha_s} \left( \frac{4}{1 - m} \right) \frac{q}{\nu} \]  \hspace{1cm} (5.31)

\[ Y = \frac{q_i}{\nu q} \frac{D}{\alpha_s} \left( \frac{m}{1 - m} \right)^{2/3} \]  \hspace{1cm} (5.32)

\( D \) is the grain diameter;
\( \alpha_s \) is a shape factor given by equation 5.7;

for spherical materials. In addition to his own experimental data, he utilized data from Lindquist (63), Zunker (141) and Nagy (77), as shown in figure 5.11.

Reducing the experimental data for crushed rock by equations 5.31
Figure 5.10  Ward Friction Curve for Combined Data (With Wall Correction)
Figure 5.11 Kovacs' Friction Curve
and 5.32 and plotting a sample of these data on Kovacs' plot, figure 5.11, indicates that the angular crushed rock data have higher values of \( Y \) than the spherical materials. This was also noted by Kovacs (53) for disc-shaped particles that he had tested. The angularity factor \( A_{ng} \) is introduced into equation 5.31 and 5.32 to account for this apparent increase in resistance; thus (including the wall effect and a standard deviation effect),

\[
X = \frac{M}{g} \frac{4 q_o}{C_w \left( 1 + A_{ng} \right)^N} \left( \frac{\sigma}{\sqrt{E(\sigma)}} \right) (1 - m) \mu
\]

\[
Y = \frac{g}{C_w \nu q_o \left( 1 - n \right)^2} \frac{M}{E(\sigma) g}
\]

where \( N \) is an empirical exponent and

\[ E(\sigma)_g = \frac{\ln \sigma}{g}. \]

Applying equations 5.33 and 5.34 to the experimental data for crushed rock, with \( N \approx 0.5 \) determined by trial and error, gives reasonably good agreement with Kovacs equation 5.30 as figure 5.12 indicates. Thus equations 5.30, 5.33 and 5.34 define an approximate resistance equation which covers a range of grain shapes and a range of flow regimes from Darcy to fully turbulent flow.

**Weighted Average Values of 'a' and 'b':** The results of the parallel flow analysis are summarized in table 5.2 which shows the values of \( a \) and \( b \) computed by equations 5.8 to 5.29 for various porosities and both grain sizes. Also shown are the standard errors of the means.
Figure 5.12 Modified Kovacs' Analysis (With Wall Correction)
<table>
<thead>
<tr>
<th>Equation</th>
<th>( M_g )</th>
<th>( m )</th>
<th>( a^{(\text{sec}) \over cm} )</th>
<th>( b^{(\text{sec})^2 \over cm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.8</td>
<td>1.66</td>
<td>.43</td>
<td>.011 ± .0017</td>
<td>.011 ± .0004</td>
</tr>
<tr>
<td></td>
<td>1.66</td>
<td>.435*</td>
<td>.0104 ± .0017</td>
<td>.0107 ± .0004</td>
</tr>
<tr>
<td>5.9</td>
<td>4.40</td>
<td>.417</td>
<td>.0062 ± .001</td>
<td>.037 ± .0002</td>
</tr>
<tr>
<td></td>
<td>4.40</td>
<td>.42*</td>
<td>.0064 ± .001</td>
<td>.0038 ± .0002</td>
</tr>
<tr>
<td>Ng (79)</td>
<td>1.66</td>
<td>.435</td>
<td>.0120 ± .002</td>
<td>.0103 ± .0004</td>
</tr>
<tr>
<td></td>
<td>4.40</td>
<td>.42</td>
<td>.0020 ± .001</td>
<td>.0044 ± .0004</td>
</tr>
<tr>
<td>Ave. 5.17 and 5.18</td>
<td>1.66</td>
<td>.435</td>
<td>.0109 ± .006</td>
<td>.0106 ± .0009</td>
</tr>
<tr>
<td></td>
<td>4.40</td>
<td>.42</td>
<td>.0015 ± .001</td>
<td>.0041 ± .0004</td>
</tr>
<tr>
<td>Ave. 5.22 and 5.23</td>
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<td>.435</td>
<td>.0106 ± .005</td>
<td>.0135 ± .0011</td>
</tr>
<tr>
<td></td>
<td>4.40</td>
<td>.42</td>
<td>.0015 ± .001</td>
<td>.0041 ± .0004</td>
</tr>
<tr>
<td>Ave. 5.28 and 5.29</td>
<td>1.66</td>
<td>.435</td>
<td>.014 ± .006</td>
<td>.0112 ± .0009</td>
</tr>
<tr>
<td></td>
<td>1.66</td>
<td>.435</td>
<td>.0114 ± .001</td>
<td>.0035 ± .0003</td>
</tr>
<tr>
<td>Weighted Averages</td>
<td>1.66</td>
<td>.435</td>
<td>.011 ± .002</td>
<td>.0106 ± .0005</td>
</tr>
<tr>
<td></td>
<td>1.66</td>
<td>.42*</td>
<td>.0128 ± .002</td>
<td>.0115 ± .0005</td>
</tr>
<tr>
<td></td>
<td>4.40</td>
<td>.42</td>
<td>.0052 ± .001</td>
<td>.0039 ± .00025</td>
</tr>
<tr>
<td></td>
<td>4.40</td>
<td>.425*</td>
<td>.0050 ± .001</td>
<td>.0040 ± .00025</td>
</tr>
</tbody>
</table>

* Adjusted by Ward's porosity factor
for all the values of a and b. Average values of 'a' and 'b' were determined from results of the combined equations 5.17 to 5.29, the particular equations 5.8 and 5.9 and Ng's equation for 1.66 cm crushed rock (79). The average values were computed as a weighted average with the weights being the relative errors in the respective values of 'a' or 'b'.

5.3 Analysis of the Radial Flow Data

Assuming, as indicated in section 3.2, that convergence or divergence of the macroscopic streamlines mainly affects b, the non-Darcy term of the Forchheimer equation, the Darcy term 'a' can be treated as a constant and the effect of radial flow on 'b' can be studied by comparing values of 'b' for radial and parallel flow. Equation 3.41 can be used to compute the 'b' for radial flow (the '+' sign indicates diverging flow and the '-' sign indicates converging flow). Equation 2.31 can be used to correct slight differences in porosity between the parallel and radial flow tests. Since both parallel and radial flow tests were performed in the same flume, wall effects would tend to be approximately the same in all cases.

Equation 3.41 requires the data in tables 4.8 to 4.12 to be analysed in pairs i.e. \([\{r_1, \phi_1\} \text{ and } \{r_2, \phi_2\}]\). For each set of radial flow data, an 'a' is obtained by correcting the corresponding parallel flow 'a' for any difference in porosity. The data of each experiment are divided into pairs of \((r, \phi)\), as indicated in tables 4.8 to 4.12; the pairs of data are chosen such that they fall as nearly as possible along radial lines. The analysis was performed by the computer using the
programme shown in Appendix J, and the results are shown in table 5.3 along with the corresponding parallel flow results.

Figure 5.13 shows a typical plot of $b$ versus $r_c$ for 1.66 cm crushed rock. The large scatter shown in figure 5.13 was obtained for all the radial flow tests. Table 5.2 indicates a slight decrease in $b$ for converging flow and almost no difference between the $b$ for parallel and diverging flow. A statistical analysis is required to determine the significance of the apparent radial flow effects indicated in table 5.3. The average value of $b$ from each of the radial flow tests is compared with the corresponding value of $b$ for parallel flow obtained from table 5.2. The standard errors in the means are indicated in tables 5.2 and 5.3 for all values of $b$.

Since the radial flow and parallel flow means are computed from large samples, the normal distribution test (78) is used to determine the significance of the apparent differences shown in table 5.3.

Parallel and converging flows are compared in table 5.4. All data are corrected to a common porosity. The standard deviation of the differences in the means is computed from

$$ S_{md} = \sqrt{S_{m1}^2 + S_{m2}^2} $$

(5.35)

and the quantity

$$ u = \frac{|\bar{b}_1 - \bar{b}_2|}{S_{md}} $$

(5.36)
Figure 5.13 Comparison of Radial and Parallel Flow

\[ M = 1.66 \text{ cm}; \quad m = 0.42; \quad a = 0.0128 \]

AVERAGE CURVES:

- PARALLEL FLOW
- CONVERGING FLOW

Figure 5.13 Comparison of Radial and Parallel Flow
### Table 5.3

**Summary of Radial Flow Tests**

<table>
<thead>
<tr>
<th>Test</th>
<th>m</th>
<th>M cm</th>
<th>( \bar{r} ) cm</th>
<th>( \bar{r} ) cm</th>
<th>( \theta ) radians</th>
<th>a sec/cm</th>
<th>( b (sec/cm)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>.42</td>
<td>1.66</td>
<td>97.0</td>
<td>---</td>
<td>.328</td>
<td>.0128</td>
<td>.0090 ± .0015</td>
</tr>
<tr>
<td>2.2</td>
<td>.42</td>
<td>1.66</td>
<td>111.0</td>
<td>---</td>
<td>.518</td>
<td>.0128</td>
<td>.0109 ± .0040</td>
</tr>
<tr>
<td>Ave</td>
<td>.42</td>
<td>1.66</td>
<td>100</td>
<td>---</td>
<td>~.4</td>
<td>.0128</td>
<td>.00935 ± .0016</td>
</tr>
<tr>
<td>*</td>
<td>.435</td>
<td>1.66</td>
<td>100</td>
<td>---</td>
<td>~.4</td>
<td>.011</td>
<td>.0086 ± .0015</td>
</tr>
<tr>
<td>2.3</td>
<td>.42</td>
<td>4.40</td>
<td>100</td>
<td>88.0</td>
<td>.316</td>
<td>.011</td>
<td>.0096 ± .0015</td>
</tr>
<tr>
<td>*</td>
<td>.42</td>
<td>1.66</td>
<td>---</td>
<td>88.0</td>
<td>.316</td>
<td>.0128</td>
<td>.0104 ± .0015</td>
</tr>
<tr>
<td>3.1</td>
<td>.435</td>
<td>1.66</td>
<td>---</td>
<td>88.0</td>
<td>.316</td>
<td>.0050</td>
<td>.0038 ± .0002</td>
</tr>
<tr>
<td>*</td>
<td>.42</td>
<td>1.66</td>
<td>---</td>
<td>88.0</td>
<td>.316</td>
<td>.0052</td>
<td>.0039 ± .0002</td>
</tr>
<tr>
<td>3.2</td>
<td>.425</td>
<td>4.40</td>
<td>---</td>
<td>115.0</td>
<td>.353</td>
<td>.0052</td>
<td>.0039 ± .0002</td>
</tr>
<tr>
<td>*</td>
<td>.42</td>
<td>4.40</td>
<td>---</td>
<td>115.0</td>
<td>.353</td>
<td>.0052</td>
<td>.0039 ± .0002</td>
</tr>
<tr>
<td>Parallel Flow</td>
<td>.42</td>
<td>1.66</td>
<td>∞</td>
<td>∞</td>
<td>0</td>
<td>.0128</td>
<td>.0115 ± .0005</td>
</tr>
<tr>
<td>Flow</td>
<td>.42</td>
<td>4.40</td>
<td>∞</td>
<td>∞</td>
<td>0</td>
<td>.0052</td>
<td>.0039 ± .00025</td>
</tr>
</tbody>
</table>

* Corrected for Porosity
Table 5.4

Comparison of Parallel and Converging Flow
(For m = .42)

<table>
<thead>
<tr>
<th>M/cm²</th>
<th>Test</th>
<th>Parallel</th>
<th></th>
<th></th>
<th>Converging</th>
<th></th>
<th></th>
<th>Probability (p) that difference is due to chance alone</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>a</td>
<td>b(sect)²</td>
<td>v</td>
<td>a</td>
<td>b</td>
<td>u</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>~600</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.66</td>
<td>parallel + 2.1</td>
<td>0.0128 ± 0.002</td>
<td>0.0115 ± 0.005</td>
<td>~600</td>
<td>0.0128</td>
<td>0.0109 ± 0.0040</td>
<td>143</td>
<td>0.052</td>
</tr>
<tr>
<td>1.66</td>
<td>parallel + 2.2</td>
<td>0.0128 ± 0.002</td>
<td>0.0115 ± 0.005</td>
<td>~600</td>
<td>0.0128</td>
<td>0.0090 ± 0.0015</td>
<td>110</td>
<td>0.218</td>
</tr>
<tr>
<td>4.40</td>
<td>parallel + 2.3</td>
<td>0.0052 ± 0.001</td>
<td>0.0039 ± 0.0025</td>
<td>~600</td>
<td>0.0052</td>
<td>0.00343 ± 0.004</td>
<td>195</td>
<td>0.120</td>
</tr>
</tbody>
</table>

Average Effect: \[ D = \frac{d}{b_1} + \ldots + \frac{d}{b_n} \]/n; \[ \bar{s}_m = \frac{1}{n} \sqrt{\sum \frac{S_{m1}}{b_1}^2 + \ldots + \sum \frac{S_{mn}}{b_n}^2} \]; \[ D = .13; S_m = .132; \]

\[ \bar{u} = 0.98; p = 0.163 \]
is used as a measure of the significance of the difference between the means $\bar{b}_1$ and $\bar{b}_2$. The probability that the difference $\bar{b}_1 - \bar{b}_2$ is due to chance only can be found from a normal probability chart [e.g. see Neville and Kennedy's Table A-4 (78)]. The computations of $S_{md}$ and $u$ are summarized in table 5.4. Table 5.4 shows that probability of the apparent converging flow effect being due to chance is approximately 0.17 when all the results are pooled although the individual tests have probabilities between 0.05 and 0.44. Therefore there is some doubt concerning the existence of the converging flow effect; however the trend of the results indicate that the value of $b$ for converging flow is slightly less ($\sim 13\%$) than for parallel flow.

A similar statistical analysis for diverging flow is summarized in table 5.5. Table 5.5 indicates a high probability ($\sim 27\%$) that the apparent differences between diverging and parallel flow are due to chance.

5.4 Treatment of the Steady Flow Free Surface Data

Lines of equal piezometric head (equipiezometric lines) are constructed as shown in figures 4.9, 4.10, and 5.1(a) from the piezometric readings on the front walls of the free surface models. The tests were divided into two sets. The first set was used to check the friction equation for the medium and the second set was reserved for comparison with the finite element model.

Approximate estimates of the values of $a$ and $b$ for the free surface models can be obtained by a direct analysis of the observed discharge
Table 5.5
Comparison of Parallel and Diverging Flow
(For m = .42)

<table>
<thead>
<tr>
<th>M g cm</th>
<th>Tests</th>
<th>Parallel</th>
<th>Diverging</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>a</td>
<td>b (sec)²</td>
<td>v</td>
</tr>
<tr>
<td>1.66</td>
<td>+ 3.1</td>
<td>0.128±.002</td>
<td>0.0115±.0005</td>
</tr>
<tr>
<td>4.40</td>
<td>+ 3.2</td>
<td>0.0052±.001</td>
<td>0.0039±.00025</td>
</tr>
</tbody>
</table>

Average Effect: $D = 0.048$, $\bar{S} = 0.08$, $\bar{u} = 0.6$, $p = 0.27$
and piezometric levels in the rockfill section. A flow net, (see fig. 5.14) can be constructed on the basis of the experimental data. From this flow net estimates of the 'bulk' velocity and hydraulic gradient can be obtained.

Referring to figure 5.14 the bulk velocity, $u$, at a point $O$ on the rockfill is given by

$$ Q = 216.8 \text{ US gpm} $$

and the magnitude of the hydraulic gradient, $i$, is found from

$$ \frac{u}{v} $$

Figure 5.14 Typical Experimental Flow Net
and piezometric levels in the rockfill section. A flow net, (See figure 5.14) can be constructed on the basis of the experimental data. From this flow net estimates of the 'bulk' velocity and hydraulic gradient can be obtained.

Referring to figure 5.14 the bulk velocity, \( q \), at a point \( M \) in the rockfill is computed from

\[
q = \frac{\Delta \psi}{\Delta n} \quad (5.37)
\]

where

\[
\Delta \psi = \Delta Q / W
\]

\[
W = \text{flume width};
\]

\[
\Delta Q = \text{discharge through stream tube } C'D'
\]

and the magnitude of the hydraulic gradient, \( i \), is found from

\[
i = \frac{|\delta A' - \delta B'|}{\Delta s} \quad (5.38)
\]

where \( A' \) and \( B' \) are two points on a streamline as shown in figure 5.14.

A number of flow sections were analysed and the results \( (q/i \text{ vs } q) \) are shown in figures 5.15 and 5.16. The values of \( a \) and \( b \) determined by least squares analysis are shown in table 5.6.

5.5 Treatment of the Unsteady Flow Data

The unsteady flow data given in table 4.13 to 4.16 are plotted in figures 5.17 and 5.18 (water surface level versus time) and mean experimental curves are drawn. Estimates of standard deviations and standard
Figure 5.15 Friction Curve for 1.66 cm Rock from Direct Analysis

Figure 5.16 Friction Curve for 4.40 cm Rock from Direct Analysis
Table 5.6

Results of Direct Analysis

<table>
<thead>
<tr>
<th>Test</th>
<th>M (g/cm)</th>
<th>a (sec/cm)</th>
<th>b (sec/cm)^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>1.66</td>
<td>0.0138±0.006</td>
<td>0.0016±0.0016</td>
</tr>
<tr>
<td>4.2</td>
<td>4.40</td>
<td>0.0048±0.004</td>
<td>0.0037±0.0008</td>
</tr>
</tbody>
</table>

Table 5.7

Scatter in Unsteady Water Profile Measurements

<table>
<thead>
<tr>
<th>M (cm)</th>
<th>U/S</th>
<th>Middle</th>
<th>D/S</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>U/S</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.66</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>1.4</td>
<td>1.5</td>
</tr>
<tr>
<td>2</td>
<td>1.1</td>
<td>1.0</td>
<td>1.5</td>
</tr>
<tr>
<td>4</td>
<td>1.0</td>
<td>1.5</td>
<td>2.0</td>
</tr>
<tr>
<td>8</td>
<td>1.1</td>
<td>1.5</td>
<td>2.0</td>
</tr>
<tr>
<td>20</td>
<td>1.1</td>
<td>1.4</td>
<td>1.2</td>
</tr>
<tr>
<td>4.40</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.8</td>
<td>1.8</td>
<td>2.0</td>
</tr>
<tr>
<td>2</td>
<td>2.0</td>
<td>1.5</td>
<td>1.3</td>
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<td>1.4</td>
<td>1.0</td>
<td>1.3</td>
</tr>
<tr>
<td>8</td>
<td>1.0</td>
<td>1.4</td>
<td>1.4</td>
</tr>
<tr>
<td>20</td>
<td>1.1</td>
<td>1.0</td>
<td>1.2</td>
</tr>
</tbody>
</table>
deviations of the means were made from figures 5.17 and 5.18 and are summarized in table 5.7. The average experimental drawdown curves given in Chapter VI (figures 6.14 to 6.23) were derived from figures 5.17 and 5.18.

5.6 Recommended Resistance Equations for the Finite Element Models

On comparing the values of $a$ and $b$ from the direct analysis of the free surface models (table 5.6) with the values of $a$ and $b$ for parallel flow (table 5.2) and radial flow (table 5.3) it was noted that the values of $b$ in the free surface models were less than the corresponding parallel flow values but slightly greater than the corresponding convergent flow values. This tends to confirm the existence of a convergence effect on the flow resistance in the free surface models.

In the actual free surface models the flow varies from nearly parallel to highly convergent; therefore each of the finite element models was run with both the parallel and convergent flow resistance equations. The recommended values of $a$ and $b$ obtained from tables 5.2 and 5.3 are given in table 5.8.

Table 5.8 Recommended Resistance Equations for the Finite Element Analysis

<table>
<thead>
<tr>
<th>$M$</th>
<th>$g$</th>
<th>$cm$</th>
<th>$m$</th>
<th>Type of Flow</th>
<th>$a$ (sec/cm)</th>
<th>$b$ (sec/cm)$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.66</td>
<td>435</td>
<td></td>
<td></td>
<td>Parallel</td>
<td>0.011±.002</td>
<td>0.0106±.0005</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Converging</td>
<td>0.011±.002</td>
<td>0.0086±.0016</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Actual Free</td>
<td>0.0138±.006</td>
<td>0.0095±.0016</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Surface (Direct</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Analysis)</td>
<td></td>
<td></td>
</tr>
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<td>0.0039±.00025</td>
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<td>Converging</td>
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<td>0.0034±.00025</td>
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<tr>
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<td>Actual Free</td>
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<td>0.0037±.0008</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Analysis)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 5.17: Experimental Drawdown Curves

Figure 5.18: Experimental Drawdown Curves

Figure 5.19: Experimental Drawdown Curves

Figure 5.20: Experimental Drawdown Curves

\[ \frac{x}{cm} \quad 0 \quad 3.05 \quad 9.10 \quad 12.20 \]

\[ \frac{t}{second} \quad 0 \quad 5 \quad 10 \quad 15 \quad 20 \]

\[ \frac{\%}{cm} \quad 0 \quad 5 \quad 10 \quad 15 \quad 20 \]

\[ \frac{y}{cm} \quad 0 \quad 5 \quad 10 \quad 15 \quad 20 \]

\[ \frac{y}{cm} \quad 0 \quad 5 \quad 10 \quad 15 \quad 20 \]

\[ \frac{y}{cm} \quad 0 \quad 5 \quad 10 \quad 15 \quad 20 \]

\[ \frac{y}{cm} \quad 0 \quad 5 \quad 10 \quad 15 \quad 20 \]
CHAPTER VI
EVALUATION OF THE FINITE ELEMENT MODELS

This chapter compares the flows, waterlevels and piezometric heads computed by the finite element models, with the corresponding experimental values. In the case of radial flow an analytical solution exists for the governing equation; therefore the finite element solution is compared with this analytical solution.

The effect of element size on the FEM is also considered in the two dimensional steady flow case. Some considerations of convergence and stability are presented.

Finally experimental errors and theoretical limitations are discussed in relation to the comparisons of the finite element solutions and the experimental results.

6.1 Evaluation of the Radial Flow Finite Element Model

Equation 3.37 is the analytical solution of the steady, radial non-Darcy flow equation 3.35. If the finite element programme is executed with the parallel flow values of $a$ and $b$:

\[
\begin{align*}
    a & = 0.011 \text{ sec/cm} \\
    b & = 0.106 \text{ (sec/cm)}^2
\end{align*}
\]

and the boundary conditions:

\[
R_1 = 41.0 \quad ; \quad \delta_1 = 105.1 \text{ cm}
\]
the numerical results shown in figure 6.1 are obtained. Table 6.1 also shows the results obtained when the diverging flow values of \( a \) and \( b \) are used i.e. \( a = 0.011 \) and \( b = 0.0095 \). On comparison of the finite element solutions with the analytical solutions in table 6.1, a good agreement is noted. Only a slight difference exists between the solutions using the values of \( a \) and \( b \) obtained from parallel and diverging flow tests.

Table 6.1 and figure 6.1 indicate that the FE piezometric heads tend to be slightly higher than the corresponding experimental data. This may be attributed in part to:

a) small screen loss at the entrance to the permeameter,
b) a slightly increased flow resistance in the region of highly divergent flow near the inlet.
Figure 6.1 Comparison of FEM, Analytical and Experimental Solutions for Diverging Flow

LEGEND

\[ Q = 75.5 \text{ U.S.G./min.} \]
\[ a = 0.011 \text{ sec/cm} \]
\[ b = 0.0106 \text{ (sec/cm)}^2 \]

- - - - FINITE ELEMENT METHOD

- - - - EXACT THEORY

- - - - EXPERIMENTAL VALUES
Table 6.1 A Comparison of Analytical, Numerical and Experimental Solutions for Diverging Flow 
(Mg = 1.66).

<table>
<thead>
<tr>
<th>Method</th>
<th>( r_d ) cm</th>
<th>41.4</th>
<th>49.4</th>
<th>57.4</th>
<th>81.4</th>
<th>99.4</th>
<th>126.4</th>
<th>163.3</th>
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<tr>
<td>FEM 'Parallel a &amp;b'</td>
<td>105.1</td>
<td>94.38</td>
<td>86.50</td>
<td>71.78</td>
<td>65.15</td>
<td>58.51</td>
<td>52.7</td>
<td></td>
</tr>
<tr>
<td>FEM 'Diverging a &amp; b'</td>
<td>105.1</td>
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<td>71.82</td>
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<td>58.54</td>
<td>52.7</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>'Parallel a &amp; b'</td>
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<td>94.39</td>
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<td>65.20</td>
<td>58.55</td>
<td>52.7</td>
<td></td>
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<tr>
<td>Analytical</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diverging a &amp; b</td>
<td>105.1</td>
<td>94.41</td>
<td>86.58</td>
<td>71.88</td>
<td>65.25</td>
<td>58.57</td>
<td>52.7</td>
<td></td>
</tr>
<tr>
<td>Experimental</td>
<td>105.1±.05</td>
<td>93.8±5</td>
<td>83.9</td>
<td>71.3</td>
<td>64.6</td>
<td>58.3</td>
<td>52.7±.05</td>
<td></td>
</tr>
</tbody>
</table>

6.2 Evaluation of the Steady Flow Free Surface Finite Element Models

The steady free surface flow finite element model was verified by comparisons with physical models. The models were rectangular rockfill sections which were described in Chapters IV and V. Numerical results were obtained for both test media (1.66 cm and 4.40 cm rock) for values of 'a' and 'b' corresponding to:

a) parallel flow,
and b) converging flow.
The effect of element density was also considered by comparing the results of a fine and a relatively coarse grid.

6.2.1 The Numerical Solutions

Table 5.8 shows the recommended values of 'a' and 'b' that were used in the FE solutions.

The two discretizations, representing the fine and coarse element densities, are typified in figure 6.2 and 6.3. The fine grid has 59 elements and 29 unknown nodal values of $\frac{t}{4}$, while the coarse grid has 39 elements and 19 unknowns. Figure 6.2 and 6.3 indicate the node numbers. In general the numbering of the 'unknown' nodes is such as to minimize the band width of the coefficient matrix. No significance is attached to the numbering of the 'known' nodes except that they are numbered after all the 'unknown' nodes have been numbered. The numbering system for the elements is arbitrary.

The boundary conditions for flow through a rectangular rockfill section are given in section 3.3.1 and figure 3.6. Satisfaction of the double boundary condition on the free surface was accomplished by using the 'membrane analogy', i.e. the free surface (which is a streamline) is positioned so that the pressure across it is zero. A trial 'free' surface is chosen and the FE model used to solve for the pressure difference across the 'membrane'. On the basis of the computed pressure differences the surface ordinates are adjusted to decrease these differences. The location of the outcrop point was determined by adjusting the seepage height (in the programme) until the upstream and downstream discharges are equal. Referring to figures 6.2 and 6.3 point C is located so that the flows at sections 1 and 2 are
Figure 6.2 Fine Finite Element Grid

Figure 6.3 Coarse Finite Element Grid
equal. The values of $Q_1$ and $Q_2$ were computed for a number of values of $y_c$ and the correct $y_c$ determined from an error plot i.e.

$$\frac{Q_1 - Q_2}{Q_1} \text{ versus } y_c$$

(6.1)

as illustrated in figure 6.4(a). A typical horizontal velocity profile for section 2 is shown in figure 6.4(b).

A typical computer input is given in Appendix D where the various data are identified.

A few initial runs of the programme were made to establish the optimum value of the over-relaxation factor, $\omega$. Figure 6.5 shows the optimum $\omega$ (based on the minimum number of iterations) for the inner iteration loop is approximately 1.5. Figure 6.5 also shows that an $\omega$ of 1.5 yields the minimum number of iterations for the complete solution.

Figure 6.6(a), 6.7(a) and 6.8(a) show the discretizations and assumed free surfaces for three of the verification runs. The principle input data are summarized in table 6.2. The discharge error plots for the respective verification models are shown in figures 6.6(b), 6.7(b), and 6.8(b) and the final computed discharges and seepage heights are given in table 6.2. Using the recommended values of $a$ and $b$ from the parallel flow studies the numerical solutions shown in figures 6.6(c), 6.7(c) and 6.8(c) were obtained. Similarly using the converging flow values of $a$ and $b$ the results shown in figures 6.6(c) and 6.8(c) were derived. In figures 6.6(c) to 6.8(c) and figure 6.10 the equipiezometric lines obtained from the finite element solutions are compared with the corresponding experimental lines.
Figure 6.4(a) Error Plot

Figure 6.4(b) Horizontal Velocity Profile Near Downstream Face
Figure 6.5 The Over-Relaxation Factor
Table 6.1

Summary of Steady Flow Input-Output Data

<table>
<thead>
<tr>
<th>EXPERIMENTAL</th>
<th>a</th>
<th>b</th>
<th>Q</th>
<th>( \gamma_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>COARSE GRID</td>
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<td>-0.034</td>
<td>480</td>
<td>36.3</td>
</tr>
<tr>
<td>FINER GRID</td>
<td>-0.052</td>
<td>-0.034</td>
<td>490</td>
<td>36.4</td>
</tr>
<tr>
<td>FINER GRID</td>
<td>-0.052</td>
<td>-0.039</td>
<td>460</td>
<td>36.0</td>
</tr>
</tbody>
</table>

- **Figure 6.1a** Discretization

- **Figure 6.1b** Error Plot

- **Figure 6.1c** Solutions 4.0 cm Fracture

---

**OBSERVED RESULTS**

**THEORETICAL RESULTS**
Table 6.2

Summary of Steady Flow Input-Output Data

<table>
<thead>
<tr>
<th>Figure</th>
<th>a</th>
<th>b</th>
<th>No. of Elements</th>
<th>$Q^2$ cm^2 sec</th>
<th>$y_c$ cm</th>
<th>Q</th>
<th>$y_c$</th>
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<td>6.2</td>
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<td>.0106</td>
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<td>320</td>
<td>39.0</td>
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<td>37.3</td>
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<tr>
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<td>.011</td>
<td>.0086</td>
<td>59</td>
<td>355</td>
<td>39.0</td>
<td>360</td>
<td>37.3</td>
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<tr>
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<td>.0106</td>
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<td>---</td>
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<td>38.3</td>
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<tr>
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<td>.0039</td>
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<td>38.3</td>
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<tr>
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<td>.011</td>
<td>.0034</td>
<td>59</td>
<td>49</td>
<td>36.4</td>
<td>489</td>
<td>38.3</td>
</tr>
</tbody>
</table>

The experimental results vary somewhat from part to part due to local variations in quantity. The accuracy of the experimental...
The experimental discharges and seepage heights are also shown in table 6.2.

It is noted that the use of the converging flow values of 'a' and 'b' consistently gave better estimates of the total observed discharges than did the parallel flow values. The seepage height does not appear to be very sensitive to changes in a and b as table 6.2 indicates. Similarly the computed piezometric levels in the FE model were not changed very much by changing b from .011 to .0086 or from .0040 to .0034. In all the finite element solutions the free surface elevation tended to be underestimated (i.e. it fell below the experimental free surface), except in the neighbourhood of the outcrop point where there does not appear to be a significant trend in the estimates. Also, as figures 6.6 to 6.8 indicate the experimental hydraulic gradients tended to be greater than the computed gradients at the upstream and less than the computed gradients at the downstream end.

In summary the FEM gave excellent predictions of discharge and good predictions of the seepage height and fair predictions of the piezometric levels.

6.2.2 Errors and Limitations

The possible experimental error in reading the internal piezometric levels was ±.05 cm. Since in a number of cases the deviations of the computed values of $\phi$ exceed this error it is necessary to look elsewhere for an explanation of the difference between computed and experimental values of $\phi$.

The piezometric levels vary somewhat from pore to pore due to local variations in porosity. The standard deviation of the piezometric
head due to this effect was estimated by considering a number of piezometric readings for different pores at corresponding distances along a sample in the parallel flow tests. Some typical test data for 1.66 cm and 4.40 cm rock are given in Tables 4.5 and 4.6. The standard deviation due to pore variation was estimated to be .7 and for the 1.66 cm material and 1.0 cm for the 4.40 material. Assuming that the number of degrees of freedom for the mean experimental curves shown in figures 6.6 to 6.8 is two, the standard deviation of the mean curve for 1.66 cm rock is 0.5 and for 4.40 cm rock is 0.7 cm. These deviations cannot completely account for differences between the numerical and experimental curves.

The free surface of flow through coarse granular material tends to be irregular as illustrated in figure 6.9. Since the irregularity of the

![Flow](image)

Figure 6.9 Sketch Free Surface in Coarse Rock
surface was of the same order as the particle size, it was necessary to assign an error to the estimated average waterlevel reading. The errors given in figures 4.9 and 4.10 indicate the range of the fluctuations near the points of measurement. While the experimental errors due to interpreting this fluctuation could explain the difference between the numerical and experimental free surface curves for 4.40 cm rock, the differences for 1.66 cm rock are often too great to be explained by this error alone.

Since the values of a and b for the finite element solution were developed from studies carried out in the same flume as the free surface tests, the wall effects would tend to be incorporated in the values of a and b.

A possible error in the numerical solution could result from the discretization of the flow regime. In theory, the true minimizing distribution is only achieved as the element size approaches zero. The effect of element size was studied by comparing the results obtained from fine and relatively coarse grids shown in figures 6.2 and 6.3. The computed discharges, seepage heights and piezometric lines are compared in figure 6.10 and table 6.2. Apart from an improvement in the definition of the computed outcrop point and discharge as shown in figure 6.10 (a), increasing the density of elements did not significantly change the numerical solution.

Finally, one is led to question the validity of the Forchheimer type equation for non-parallel flow. The experimental hydraulic gradients in the downstream zones tend to be less than the numerically computed gradients indicating the possibility of a convergence effect i.e. a decrease
Figure 6.10(a) Error Plot

(a) Solutions from Coarse and Fine Grid

(b) Solutions from Coarse and Fine Grid
in the resistance due to convergence of the macroscopic streamlines. On the other hand the experimental gradients in the upstream zones are nearly the same or slightly greater than the computed gradients indicating no appreciable convergence effect. These effects can partially be attributed to the variation in the wall effect between the upstream and downstream zones. Based on equation 5.24, for the wall effect, the downstream resistance would be about 1.5% less than upstream resistance for the 1.66 cm crushed rock. Table 5.4 shows that convergence may decrease the value of \( b \) by about 13%. Thus the differences between the experimental and computed results are probably attributable to a combination of convergence effects, wall effects, discretization errors and experimental errors.

a) the values of \( a \) and \( b \) for the section,
b) the drainage capacity,
c) the initial conditions, i.e. the initial temperature, heads at all the nodes,
d) the boundary conditions, i.e., the boundary level as a function of time,
e) the nodal coordinates,
f) identification of the elements,
g) an approximate solution for the first time step (there can be very crude but should indicate the trend of the solution) and whether \( \Delta t \) is decreasing or not automatically,
h) the time steps, \( \Delta t \),
i) tolerances and criteria on the iterative solutions.
6.3 Evaluation of the Unsteady Flow Finite Element Models

No analytical solutions exist for two dimensional unsteady non-Darcy flow; therefore the numerical solutions were verified by comparisons with physical models. The experimental studies consisted of rapid drawdown tests in rectangular rockfill sections. These tests were described in Chapters IV and V.

6.3.1 Preliminary Considerations

The unsteady flow finite element model was described in Chapter III. In order to predict, numerically, the drawdown profiles for the physical models the following data must be supplied to the finite element programme:

a) the values of a and b, for the medium,
b) the drainable porosity,
c) the initial conditions, i.e. the initial piezometric heads at all the nodes,
d) the boundary conditions, e.g. the tailwater level as a function of time,
e) the nodal coordinates,
f) identification of the elements,
g) an approximate solution for the first time step (This can be very crude but should indicate the trend of the solution i.e. whether $\phi$ is increasing or decreasing.),
h) the time step, $\Delta t$,
i) tolerances and limits on the iterative solutions,
j) Certain factors, e.g. over-relaxation factor and slope correction factor.

A typical computer input is shown in Appendix F.

The recommended values of $a$ and $b$, shown in table 5.8, were used in the unsteady flow computations. Both 'parallel' and 'converging' values of $a$ and $b$ were tried to determine which gave the best agreement with the experimental results. Computer simulations were made for both test media, i.e. 1.66 cm and 4.40 cm crushed rock.

The drainable porosities of the test media are shown in table 4.4.

The initial piezometric heads at all the nodes were set at a constant value (56.0 cm).

Computations were made for an instantaneous drop in the tailwater level from 56.0 cm to 5.0 cm as well as for the observed tailwater recession curve which was approximated by

$$y(t) = \frac{60.0}{t + 1.0} - 2.0$$  \hspace{1cm} (6.2)

$$t \geq 0.4 \text{ seconds}.$$  

Other boundary conditions, which are automatically set within the programme are the piezometric heads along the free surface and seepage face (see figure 3.14 and 6.11). On both of these boundaries

$$\phi = y$$  \hspace{1cm} (6.3)

The conversion from $\phi$ to $\zeta$ is made within the programme. The streamline
boundary conditions shown in Figure 5.13 are natural boundary conditions for the variational formulation and therefore are automatically satisfied by simply treating the boundary values of $f$ or $g$ as unknowns.

The discretization of the flow field is shown in Figure 6.11. The element and node numbers are also shown. Both elements of the type shown in Figure 6.11 and the type shown in Figure 6.13 were used in the programme indicated that the use of a large time increment would lead to convergence difficulties in the solution.

a) equation (3.13) setting $A_1$.

b) the averaging procedure described in Section 3.4.3 were included in all programme.

For $T = 56$ cm and $A = 250$, equation 5.2 gives

\[ \hat{t} = 0.1 \text{ sec.} \]

Figure 6.11 Discretization for Unsteady Flow Problem

The initial water levels (in metres) were

\[ 2.4 [45^\circ], 35^\circ] \times 0.02 \text{ m.} \]

The trial free surfaces are computed and averaged until the change in the

Figure 6.12 Typical Assumed Starting Solution
boundary conditions shown in figure 3.15 are natural boundary conditions for the variational formulation and therefore are automatically satisfied by simply treating the boundary values of $\phi$ or $\zeta$ as unknowns.

The discretization of the flow field is shown in figure 6.11. The element and node numbers are also shown. Each element is of the type shown in figure 3.13.

A typical assumed 'starting' solution is illustrated in figure 6.12.

A few preliminary runs of the finite element programme indicated that the use of a large time step could lead to instabilities in the solution. Two procedures were used to prevent these instabilities:

a) equation 3.142 was used as a guide to setting $\Delta t$;

b) the averaging procedure described in section 3.4.3 was included in all programmes.

For $y_s = 56$ cm and $\Delta x \approx 30.0$, equation 3.142 gives

$$\Delta t \approx 0.12 \text{ seconds}$$

The initial motion drawdown computations ($t \leq 3.0$ minutes) were made with $\Delta t = 0.1$ seconds and thereafter the $\Delta t$ was increased (up to 0.5 in some programmes).

The inner iteration loop (LOOP 1) terminates when

$$\max |\phi_{n+1} - \phi_n| \leq 0.001 \text{ cm} \quad (6.3)$$

The trial free surfaces are computed and averaged until the change in the
computed free surface has a negligible effect on the inner iteration solution i.e. one cycle of the inner iteration solution re-establishes the 'tolerance' given by equation 6.3.

The method of determining the optimum $\omega$ was the same as described for the steady flow FEM i.e. a plot of $\omega$ versus the number of iterations to satisfy equation 6.3 was developed from a few preliminary runs. Figure 6.13 indicates that the optimum value of $\omega$ is about 1.65.

Figure 6.13 Optimum Over-Relaxation Factor for the Unsteady Flow FE Model

The determination of the slope correction factor was discussed in section 3.4.3.
6.3.2 Comparison of Numerical and Experimental Results

Table 6.3 shows the main combination of input data for which numerical solutions were obtained.

<table>
<thead>
<tr>
<th>Run No.</th>
<th>( M_g )</th>
<th>( a )</th>
<th>( b )</th>
<th>( m )</th>
<th>T.W.L. Slope</th>
<th>Remarks</th>
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<td>0.011</td>
<td>0.0106</td>
<td>40</td>
<td>No</td>
<td>5.0</td>
</tr>
<tr>
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<td>1.66</td>
<td>0.011</td>
<td>0.0086</td>
<td>40</td>
<td>Yes</td>
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<td>0.0086</td>
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<td>6.2</td>
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<td>0.0040</td>
<td>40</td>
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<td>Eq. 1.0</td>
</tr>
</tbody>
</table>

An example of a partial computer output is shown in Appendix E.

The computed water surface profiles for all the computer runs...
are compared with the corresponding experimental curves in figures 6.14 to 6.21. Figures 6.22 and 6.23 compare the computed water surface levels for various input data with corresponding experimental data at selected verticals.

Figures 6.14, 6.15 and 6.22 indicate that the inertial term is not very important in the case of the 1.66 cm rock; however, figures 6.16, 6.17 and 6.23 indicate that inertia is more important for the 4.40 cm rock although the effect is still not great.

Figures 6.14 to 6.17 indicate that the use of the values of a and b, from the converging flow tests rather than the parallel flow tests, results in better predictions of the experimental profiles.

Figures 6.18 and 6.19 show the effect of introducing a tailwater recession curve in the place of the assumed sudden drop in the tailwater level. As one would expect the receding tailwater level retards the drawdown, but since the recession is quite rapid the retarding effect is only slight as figures 6.18 and 6.19 indicate.

The effect of the slope correction factor is shown in figures 6.20 and 6.21. The inclusion of a slope correction factor mainly affects the shape of the computed drawdown profile. The profiles based on a slope correction factor of 0.75 were in better agreement with the experimental curves than the profiles that did not include a slope correction. It would appear that the factor 0.75 is probably too small, especially in the outflow region; this will be discussed further in the next Chapter.

In general the agreement between the computed and experimental results is good except for large t (t ≥ 8 seconds) where the computer programme tends to slightly underpredict the rate of drawdown.
EXPERIMENTAL VALUES

- FEM, NO INERTIA
- FEM, WITH INERTIA

Figure 6.15 Drawdown Profiles (Run 6.3, 6.4 and Exp.)

Figure 6.16 Drawdown Profiles (Run 6.5, 6.6 and Exp.)

Figure 6.17 Drawdown Profiles (Run 6.7, 6.8 and Exp.)
EXPERIMENTAL VALUES

FEM, NO INERTIA

FEM, WITH INERTIA

T.W. RECESSION EQ.

---

Figure 6.18 Drawdown Profiles (Run 6.11 (a) (b) and Expt)

Figure 6.19 Drawdown Profiles (Run 6.12 (a) (b) and Expt)

Figure 6.20 Drawdown Profiles (Run 6.9, 6.1 and Expt)

Figure 6.21 Drawdown Profiles (Run 6.10, 6.4 and Expt)
Figure 6.22 Water Surface Recession Curves (1.66 cm Rock)

Figure 6.23 Water Surface Recession Curves (4.40 cm Rock)
6.3.3 Experimental Errors

The experimental errors are treated in two groups:

a) those associated with the unsteady flow tests,
b) those associated with the experimentally derived input data for the FE programme.

Errors in the Unsteady Flow Data:— Estimates of the standard deviations and standard deviations of the means for the experimental drawdown profiles are given in table 5.7. The range of the experimental data measured from the photographs is shown in figures 6.22 to 6.23. In general the computed profiles (corrected for converging flow) fall within the range of the experimental data; however there is a tendency for the computer profiles to be higher than the mean experimental profiles during the latter stages of the drawdown.

In addition to the scatter in the experimental data there may be a number of other errors (some of them consistent) which could contribute to the differences between the computed and experimental curves. Some of these errors are:

a) errors in interpreting the irregularities of the free surface,
b) errors in time measurements due to human reaction times,
c) errors in interpreting the photographs.

A random error is introduced by averaging the free surface irregularities; however this would be reflected in the scatter of the measured data. There may be a consistent error introduced by assuming that the average of the irregularities represents the true free surface. Perhaps the true free surface lies slightly above or below the average. Since the
irregularity increased with increasing flow velocity and particle size. Any consistent error due to false interpretation of the irregular surface would mainly affect the upper profiles in the 4.40 cm rock.

The error in reading the time from the photographs was ± 0.1 seconds and would be reflected in the scatter in the experimental data. Besides the reading errors there would be errors, in the indicated lapsed times, because of the human response times involved in releasing the tailgate and starting the timer. From a study of the photographs, that were taken immediately after the start of the various runs, it would appear that the response time error had a random component of about 0.2 seconds (included in the estimated $S_m$) and a consistent component of about 0.1 second, i.e. there was a tendency for the average recorded times to be slightly less than the actual lapsed times. This 'delay' error would only be important during the initial drawdown period (say for $t < 2.0$).

The errors in the indicated lapsed times for the 'manual' measurements of the waterlevel profiles appeared to be slightly larger than in the case of the photographic method. The human response time error for these tests was estimated to vary between +.4 to -.2 seconds (or a random error of about 0.3 and a consistent error of about 0.1 second). Also the 'manual' measurements were subject to greater interpretation errors because of speed at which the markings of the profiles were made.

In order to keep the errors due to photographic interpretation to a minimum a meter scale was located in front of the flow section for each photograph. Scale variations from point to point in a photograph were determined by measuring the known regular grid intervals between the piezometric taps. A correction was applied to the measured water depths to account for the fact that the meter scale was located slightly
in front of the water surface (2.5 cm compared with a distance of 150 cm to the camera). A small error would result from neglecting the refraction of light through the plexiglass viewing window. The refraction error, as illustrated in figure 6.24, results in the upper profiles being over-estimated by less than 0.1 cm and the lower profiles being under-estimated by about 0.2 cm.

Experimental Errors in Input Data:- Much of the input data for the finite element programme were determined experimentally and thus are subject to experimental errors. The following are the most prominent experimental input errors:
a) errors in the determination of the drainable porosities,
b) errors in the recommended values of a and b,
c) errors in the determination of the dry porosities since these affect the computation of a and b,
d) calibration errors in the venturi meter (up to ± 2%).

Since the velocity of the free surface fluid particles is assumed to be

$$q_p = q/m_e$$  \hspace{1cm} (6.4)

where $m$ is the drainable porosity, any error in $m_e$ will produce a similar error in the fluid particle velocities and displacements. The standard error of the mean of $m$ is about 2.5% which would yield an error of about ±1.2 cm in the water surface profiles at $t = 20$ seconds.

The standard errors in the recommended values of a and b are shown in table 5.8. Figure 6.25 shows the effect of a change in $b$ on the prediction of the midpoint of the profile for $t = 20$ seconds.

![Figure 6.25 Effect of 'b' on $y_s$](image-url)
Additional errors in $a$ and $b$ could also result due to errors in the porosity adjustments since the dry porosities upon which these are based have errors of approximately $\pm 2.5\%$. Thus the standard errors in $a$ and $b$ should be increased as indicated in table 6.4.

Most of the discharge measurements on which $a$ and $b$ are based were obtained from the venturi meter which has a possible calibration error of about $\pm 2\%$. Thus the possible calibration error could produce an error in $'a'$ of about $\pm 2\%$ and an error in $'b'$ of about $\pm 4\%$.

All the experimental errors are combined into effective errors in table 6.5 for the purposes of checking the significance of the differences between the experimental and computed drawdown profiles. Table 6.5 shows that in general the difference between the computed and experimental profiles is not statistically significant.

6.3.4 Limitations of the Finite Element Model

The finite element model is subject to the following errors or limitations:

a) errors of idealization,
b) discretization errors,
c) limitations of the governing equations,
d) computation errors.

Idealization Errors: In developing the FE model it was assumed that the physical system was homogeneous, isotropic and two dimensional. Since the pore and grain sizes are significant the assumption of a homogeneous and isotropic medium is only true in a statistical sense and even then the wall and boundary zones must be excluded.
Table 6.5 Comparison of Computed and Experimental Data

<table>
<thead>
<tr>
<th>Computer Run No.</th>
<th>M g cm</th>
<th>t sec.</th>
<th>( \Delta ) cm</th>
<th>Consistent Errors Time Delay</th>
<th>Re却ed ( \Delta ) cm</th>
<th>Random Errors Unsteady Input Data</th>
<th>S e cm</th>
<th>u</th>
<th>Significance of ( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.11</td>
<td>1.66</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.2</td>
<td>0.2</td>
<td></td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>-6</td>
<td>-0.2</td>
<td>+1</td>
<td>-0.7</td>
<td>0.7</td>
<td>0.6</td>
<td>0.9</td>
</tr>
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<td></td>
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</tr>
<tr>
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<td></td>
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<td>+7</td>
<td>-0.1</td>
<td>-0.05</td>
<td>+0.55</td>
<td>0.8</td>
<td>1.2</td>
<td>1.4</td>
</tr>
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<td></td>
<td></td>
<td>20</td>
<td>+8</td>
<td>0</td>
<td>-0.1</td>
<td>+0.7</td>
<td>0.6</td>
<td>1.6</td>
<td>1.7</td>
</tr>
<tr>
<td>6.12</td>
<td>4.40</td>
<td>0</td>
<td>-2.0</td>
<td>-0.1</td>
<td>0</td>
<td>-2.1</td>
<td>0.9</td>
<td>1.7</td>
<td>2.0</td>
</tr>
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<td></td>
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<td>-0.1</td>
<td>0</td>
<td>+0.4</td>
<td>0.5</td>
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<td></td>
<td></td>
<td>20</td>
<td>+0.5</td>
<td>-0.0</td>
<td>-0.1</td>
<td>+0.4</td>
<td>0.7</td>
<td>1.3</td>
<td>1.5</td>
</tr>
</tbody>
</table>

\[ \Delta = \text{Max} (y_{\text{computed}} - y_{\text{expt}'1}) \]

\[ a = .0011; \ b = .0086; \ M_g = 1.66 \text{ cm} \]

\[ a = .0052; \ b = .0034; \ M_g = 4.40 \text{ cm} \]
The values of a and b recommended in table 5.8 include some wall effect i.e. the average wall effect for the parallel and converging flow tests; however in the unsteady flow tests the wall effect varies with time and is most significant in the latter stages of drawdown. The reason for this is that the portion of the total section in the 'wall' zone increases with drawdown (See Table 6.6). Equation 5.24 was used to compute the wall factors shown in table 6.6.

Table 6.6 Estimated Wall Factors, $C_w$

<table>
<thead>
<tr>
<th>M/cm</th>
<th>t sec</th>
<th>Steady Parallel</th>
<th>Steady Convergent</th>
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<tr>
<td>1.66</td>
<td>0</td>
<td>0.93</td>
<td>0.93</td>
</tr>
<tr>
<td>4.40</td>
<td>4</td>
<td>0.86</td>
<td>0.84</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.885</td>
<td>0.775</td>
</tr>
</tbody>
</table>

Thus the average velocities in the physical model will tend to be higher than the computed values, during the latter stages of drawdown. This is probably an important contributing factor to the apparent difference between the experimental and numerical results. Table 6.6 indicates that $C_w$ has decreased by about 6%, from the assumed average value, at the lapse time of 20 seconds; this would result in an increase in the velocity at t=20 seconds of about 6%. On the average an increase in velocity of about 2% over the 20 second period can be attributed to the variable wall effect; this would result in the '20 second' computed profiles being about 0.8 cm lower, if this effect had been included in the programme.

The screen losses (exit losses) were also neglected in the FE
model; however these losses would be quite small ($\lesssim 0.25$ cm)

**Discretization Errors:** Discretization errors due to element size were discussed in the steady flow section. In unsteady flow the discretization error may be affected by $\Delta t$; however preliminary runs with a number of $\Delta t$'s less than the unstable $\Delta t$ gave almost the same results. The $x$-$y$ discretization was probably adequate except at the free surface where an approximate extrapolation equation was introduced to obtain the free surface velocities from the computed centroidal velocities of the free surface elements. Obviously more accuracy could have been achieved by a higher element density but this accuracy was sacrificed to save computation time.

The free surface profile discretization necessitated a slope adjustment which is described by equation 3.141. Smaller surface elements would cause the slope correction factor, in equation 3.141, to approach unity thus decreasing the error due to estimating this factor.

**Limitation of the Governing Equations:** Equation 3.33, with a and b assumed to be constant, was used in the finite element model. This Forchheimer assumption is open to question. Equation 3.32 indicates that 'b' may be a function of the convergence of the macroscopic flow; this was also experimentally supported as shown in Chapter V. Also, figures 6.14 to 6.17 indicate that use of the converging flow values of a and b improves agreement between the numerical and experimental profiles. Macroscopic convective acceleration, in the unsteady flow models, results from the convergence of the flow and the 'spatially' varied nature of the flow. Since the convective acceleration effect varies throughout the flow as well as with time, it is probably not correct to use a constant a and b throughout the programme.
Computation Errors: - A detailed study of the roundoff and truncation errors in the computer programme, was not made. All of the results were computed using single precision accuracy (i.e. 8 significant figures). Reproducibility of the numerical results was checked by using different starting solutions, different values of the over-relaxation factor and different tolerances on the iterative solutions.

6.3.5 Summary

The proposed finite element model for unsteady non-Darcy flow predicts the experimental drawdown curves within the standard errors of the experimental data. The stability and convergence, of the proposed iterative method, are good.

The most serious limitations of the unsteady FE model appear to be:

a) neglect of wall effects,

b) discretization errors at the free surface,

c) questionable applicability of the Forchheimer type resistance equation to two dimensional flows.

The inertial term, \( \frac{1}{g_m} \frac{\partial q}{\partial t} \), is of secondary importance for the media used in this study.
CHAPTER VII

DISCUSSION OF MODIFICATIONS, APPLICATIONS
AND FUTURE RESEARCH

7.1 Suggested Modifications of the Finite Element Models

7.1.1 Steady Flow

It was pointed out in Chapter VI that steady two dimensional non-Darcy flow appears to be subject to a variable resistance due to variations in the convergence of the macroscopic streamlines. Therefore introducing a resistance equation, of the form

\[ b = b_0 (1.0 - 0.13e^{-\left(\frac{r_c}{100 \text{ Mg}}\right)} ) \]

where \( b_0 \) is the value of \( b \) for parallel flow and \( r_c \) is the radius of convergence, into the finite element model should improve the numerical simulation.

The use of an iterative acceleration scheme (20, 34) to speed up the convergence of the iterative solutions would be beneficial in saving computer time. These schemes make use of the solution at previous iteration cycles, thus requiring additional storage in the computer. Since the number of unknowns in the proposed finite element models is not great the additional storage requirement would not be a serious problem.

7.1.2 Suggested Modifications of the Unsteady Finite Element Models

Both of the modifications suggested for the steady flow FE models could be incorporated in the unsteady flow models.

The unsteady flow model could also be modified to compensate
for the variable wall effect if equation 5.24 is incorporated in the
programme.

In order to decrease the discretization errors at the free surface,
the modified discretization scheme shown in figure 7.1 is proposed. This
scheme increases the number of 'internal' unknowns from 20 to 29 while
doubling the number of surface elements.

7.2 Applications

Some applications of the steady and unsteady non-Darcy finite
element models were mentioned in Chapter I. A few completed sample appli-
cations are presented here.

7.2.1 Computation of Flow Through a Rockfill Dam

The steady flow FE model can easily be modified to solve the
problem of flow through more practical rockfill sections such as might
be used in causeways or dams. To illustrate this application the rockfill
section shown in photograph 7.1 and figure 7.2 is studied by the FEM.
The medium was 4.40 cm crushed rock. The discretization of the flow field
is shown in figure 7.2 (a). The numerical solution for this section,
shown on figure 7.2(b), is compared with the experimental results obtained
by Wong (132). Based on Wong's porosity measurements the values of a
and b were taken to be:

\[
\begin{align*}
  a &= 0.0039 \text{ sec/cm} \\
  b &= 0.0034 (\text{sec/cm})^2
\end{align*}
\]

The computed discharge was 370 cm²/sec compared to the experimental dis-
charge of 350 cm²/sec.
Figure 7.1 Modified Discretization Scheme
Photo 7.1

Steady Flow Through a Rockfill Dam
Figure 7.2(a) Discretization of a Rockfill Dam

Figure 7.2(b) Comparison of Experimental and FEM Results
7.2.2 Wave Energy Dissipation in Rockfill

The problem of computing waterlevel fluctuations in a rockfill section subject to wave action was discussed in Chapters I and II. Only a few minor changes in the unsteady flow finite element programme are required to solve this problem. The tailwater piezometric boundary condition is represented in the programme by a periodic function of time. As a first approximation, the wave motion on the face of the rockfill is taken as

\[ y_t = \bar{y}_o - A_o \sin \left( \frac{2\pi t}{T} \right) \]  

(7.2)

where \( \bar{y}_o \) is the mean tailwater level;

\( t \) is the lapsed time;

\( A_o \) is the amplitude of the fluctuation in waterlevel at the rock face;

\( T \) is the wave period.

Further, if the impact pressure is neglected, the boundary value of \( \phi \), is equal to the tailwater depth.

The slope correction factor was set at unity since in some instances it would tend to be greater than one while at other times it would be less than one.

The discretization of the flow field, along with the initial and boundary conditions, for the illustrative problem, are shown in figure 7.3. The following data were used in the programme:
Figure 7.1 The Problem of Wave Motion in a Rockfill Dyke

Figure 7.4 Some Computed Free Surface Profiles

Figure 7.5 Water Surface Movements at Various Sections
a = 0.0050 sec/cm;
b = 0.0040 (sec/cm)^2;
m = 0.40;
y_0 = 56.0 cm;
A_0 = 20.0 cm;
T = 2π seconds.

Some typical computed free surface profiles are shown in figure 7.4. Figure 7.5 shows a plot of the water level movement at the rock face and at the interior impervious barrier. The wave amplitude in this example decreases by about 35% in passing through the rockfill.

7.2.3 A Special Case-Unsteady Darcy Flow

The problem of unsteady unconfined Darcy flow may be solved by the non-Darcy flow models by placing

$$a = \frac{1}{K} \tag{7.3}$$

and

$$bq = 0 \tag{7.4}$$

in the non-Darcy flow programme. In equation 7.3, K represents the constant hydraulic conductivity of the medium. Also the programme may be simplified by omitting some of the non-linear terms such as the inertial term $\frac{1}{gm} \frac{\partial q}{\partial t}$.

The finite element model was applied to solve the problem of rapid drawdown in the rectangular earth section shown in figure 7.6. Drainage into a completely penetrating ditch is considered.
Using the discretization shown in Figure 7.7, with a hydraulic conductivity of 0.028 cm/sec and a transmissivity of 0.15 the drawdown curve shown in Figure 7.6 is obtained for the region in the surface water table shown in Figure 7.5. The parameters defined by the piezometric surface were required for the solution. The development of the surface water table in the surface water table system is shown in Figure 7.8.

Figure 7.6 compares the finite element solution with the results of a finite difference study carried out by Sorensen (1973) and with a finite difference study made by Migdal (1972). Sherry's results were corrected for density correction effects. Figure 7.6 shows reasonably good agreement between the Sorensen's and the finite element results; however, the finite element solution exhibits a better estimation of the drawdown over a larger distance from the injection point. The finite element model is more flexible than the finite difference model to represent the complex hydraulic conditions in part of the field. The finite element solution converges to the finite difference solution in the limit case of the coarse grid model, while the finite estimated drawdown is in the field.

7.3. Recommendations

On the basis of the previous discussion of the finite element solution, it is recommended that:
Using the discretization shown in figure 7.7 with a hydraulic conductivity of 0.028 cm/sec. and a drainable porosity of 0.23 the drawdown curves shown in figure 7.8 were computed. The dimensionless time parameter is defined by

\[ T' = \sqrt{\frac{4Kt}{mL^2}} \quad (7.5) \]

A finer grid was required for the Darcy flow problem than was required for the non-Darcy solutions because of the higher curvature in the surface profiles of the Darcy flow.

Figure 7.8 compares the finite element solution with the results of a Hele-Shaw model study carried out by Shery (108) and with a finite difference study made by Wigle (128). Shery's results were corrected for surface tension effects. Figure 7.8 shows reasonably good agreement between the Hele-Shaw and the finite element results; however the finite element solution slightly under-estimates the rate of drawdown. On the other hand the finite difference solution over-estimates the rate of drawdown; the agreement between the finite element solution and the finite difference solution is only fair. The tendency for the finite element model to underestimate the experimental drawdown rate can be attributed in part to the free surface discretization error.

7.3 Recommendations for Future Research

On the basis of the investigations of this thesis the following topics are suggested for future study;
a) The non-Darcy finite element models described herein could be extended to include steady and unsteady non-Newtonian flow in porous media.

b) With very little modification the proposed finite element models could be used to study similar non-linear problems in other fields of engineering, e.g. non-linear heat transfer, saturated magnetic field problems and non-Newtonian flows.

c) Further investigation of the effect of radial flow on the resistance equations is required to confirm or reject the apparent trends presented in this thesis.

d) An interesting study could be made of the non-Darcy flow regimes, in a two dimensionalized medium, using a bi-refringent fluid to indicate the shear patterns around the grains.

e) The effect of curvature of the macroscopic streamlines on the flow resistance should be studied experimentally.

f) Future investigations should include a wider range of particle shapes, sizes and institute porosities. Very little experimental data has been obtained for flow through very coarse material (say > 3 inches); a study of these large materials would require special attention to wall effects.

g) Experimental studies are required to perfect the non-Darcy finite element model for wave energy dissipation studies; it may be possible to simulate the entire wave action problem, including run-up and reflection, by a finite element model.

h) It would be useful to extend the proposed non-Darcy flow FEM
to include stratified media e.g. filter layers in a rock protected dam; this would also be useful for the special case of Darcy flow in tile drainage systems.
CHAPTER VIII

CONCLUSIONS

The conclusions of this thesis are considered in two parts:

a) those relating to the finite element models,

b) those relating to the experimental studies.

8.1 The Finite Element Models

Steady non-Darcy flow finite element models were developed and verified by comparison with analytical and experimental results. These models involved three iteration loops: the inner loop solved the linearized form of equation 3.111 by a successive over-relaxation procedure; the intermediate loop revised the conductivity matrix; the outer loop adjusted the free surface. The optimum rate of convergence was obtained for $\omega = 1.5$.

The steady flow non-Darcy model was extended and modified to solve unsteady non-Darcy flow. In particular the case of free surface flow was considered, although the case of confined flow may be solved by the proposed model. However the study was limited to steady flow problems with known initial conditions and time dependent boundary conditions of the natural and/or prescribed type. A finite difference scheme was used to account for small inertia effects, $\frac{1}{g m} \frac{\partial q}{\partial t}$.

The unsteady flow model involved four main computation loops: the inner loop solved the linearization of equation 3.134 by SOR; two
intermediate loops revised the conductivity and 'stiffness' matrices and computed a new trial free surface; the outer loop established the initial and boundary conditions for the next time step. Within a given time step several trial free surfaces were computed and averaged. The computations for a given time step were terminated when successive free surface predictions were so nearly the same that the inner iteration loop was unaffected. The final computed new free surface and the corresponding internal solution formed the initial conditions for the next time step. The programme automatically adjusts the geometry of the internal elements.

The unsteady flow FE models for rapid drawdown in rockfill were compared with physical models for the purposes of verification. Fairly good agreement was obtained between the numerical and experimental drawdown profiles.

Nevertheless the comparisons of both steady and unsteady F.E. simulations with the experimental data indicated the following limitations of the finite element models:

a) the Forchheimer equation derived from parallel flow studies may not be applicable to macroscopic radial flow;

b) the F.E. models do not exactly represent the wall effect of the physical models;

c) the discretization error affects the accuracy of the computed free surface velocities and slopes; a modified surface discretization is suggested in Chapter VII;

d) the inertia term $\frac{1}{gm} \frac{\partial q}{\partial t}$, was assumed to be small; however
in the case of a very porous coarse medium this term may become more important and the present model could not be expected to give good results.

8.2 Conclusions Based on the Experimental Studies

On the basis of the experimental analysis the following conclusions are drawn:

a) Equation 5.25 gives a general resistance equation for crushed rock which is applicable to the turbulent transitional regime.

b) The resistance relationships for parallel and converging crushed rock were found to be slightly different (at the 17% significant level) as shown in table 5.4; thus casting doubt on the general applicability of the Forchheimer type equation for two dimensional macroscopic flow. No significant difference could be detected between the parallel and diverging flows, as table 5.5 shows. It was found that converging macrostreamlines tend to increase the hydraulic conductivity i.e. decrease resistance relative to the parallel case. These conclusions are substantiated by the approximate, direct analysis of the steady free surface flow experiments as summarized in table 5.6. Also the average converging flow values of 'a' and 'b' in the steady finite element studies gave much better agreement with the observed discharges.
Ahmed and Sunada (2) neglected turbulence in their development of the non-Darcy flow equation. They estimated, on the basis of turbulence measurements made by Liu et al (65) in open channels, that the turbulent energy, $E'_t$, would be less than 7% of the total kinetic energy of the flow, $E_c$. However on the basis of the turbulent intensity measurements in the pores of a coarse granular medium, made by Wright (133, fig. 4 p. 858), it would appear that the turbulent intensity may reach 40% of the pore velocity for high Reynolds numbers ($R_N > 200$).

Therefore

$$\frac{E'_t}{E_c} = \frac{3}{2} \frac{\rho(q')^2}{\frac{1}{2} \rho q_{\text{pore}}^2} \quad (A.1)$$

$$= 3(0.4)^2$$

$$\approx 0.5$$

which certainly cannot easily be ignored.

Moreover the importance of turbulence in a flow is related to all the Reynolds stresses and not just the normal components considered by Ahmed and Sunada. One would not neglect turbulent shear stresses in an
open channel on the basis that the turbulent energy was only 7\% of the total kinetic energy.

APPENDIX A

Justification of the variational approach to steady flow

2.1 The Self-Adjoint Problem

Such of the theoretical treatment of the variational formulation has been concerned with linear differential equations which can be shown to be self-adjoint (6, 22). To show that an operator \( \mathcal{L} \) is self-adjoint if for two square integrable functions \( u \) and \( v \) the functional

\[
\int f(u, v) = \int \mathcal{L}(u) v
\]

is only a function of the boundary values of \( u \) and \( v \) and their boundary derivatives. For the linear operator

\[
\mathcal{L}(u) = a_1 u + a_2 \frac{\partial u}{\partial x} + a_3 \frac{\partial u}{\partial y} + a_4 v + a_5
\]

is the region \( D \), and the boundary conditions

\[
\eta = \phi_x, \quad \xi = \phi_y, \quad \phi = 0
\]

are self-adjoint if

\[
\int \mathcal{L}(u) v = \int u \frac{\partial \mathcal{L}(v)}{\partial x} + a_2 \frac{\partial u}{\partial x} v + a_3 \frac{\partial u}{\partial y} v + a_4 v^2 + a_5 v
\]

is only a function of the boundary values of \( u \) and \( v \) and their derivatives.
APPENDIX B

Justification of the Variation Approach to Non-Darcy Flow

B.1 The Self-Adjoint Problem

Much of the theoretical treatment of the variational formulation has been concerned with linear differential equations which can be shown to be self-adjoint (4, 35). Forsythe and Wason (35) state that an operator \( L(a) \) is self-adjoint if for two smooth functions \( u \) and \( v \) the integral

\[
I_R = \int_R \left[ u L(v) - v L(u) \right] dR \tag{B.1}
\]

is only a function of the boundary values of \( u \) and \( v \) and their boundary derivatives. Now the linear operator

\[
L(u) = a_1 u_{xx} + b_1 u_{xy} + c_1 u_{yy} + e_1 u_y + f_1 u
\]

in the region \( R \), and the boundary conditions

\[
\alpha_1 u + \gamma_1 + \beta_1 u_n + \delta_1 u_s = 0 \tag{B.3}
\]

are self-adjoint if

\[
I_R = 0 \tag{B.4}
\]
where \( a_1, b_1, c_1, d_1, e_1, f_1, g_1, \alpha_1, \beta_1, \gamma_1 \), and \( \delta_1 \) are constants or functions of the independent variable.

If the problem is self-adjoint equation B.3 is referred to as the natural boundary condition.

Since, according to Forsythe and Wasow, the Euler-Lagrange equation

\[
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} - \frac{\partial f}{\partial u} = 0 \quad \text{(B.5)}
\]

always gives a self-adjoint problem, the problem itself must be self-adjoint if it is to be represented by the variational formulation.

Younge (135) defines a linear self-adjoint system in \( u_x \) and \( u_y \) by

\[
\frac{\partial}{\partial x} (a' u_x) + \frac{\partial}{\partial y} (b' u_y) = 0 \quad \text{(B.6)}
\]

where \( a' \) and \( b' \) are functions of \( x \) and \( y \). The non-linear non-Darcy equation 3.35 has a similar form to equation B.6 except that \( a' \) and \( b' \) are functions of \( \phi_x \) and \( \phi_y \). However the non-Darcy equation is very similar to the non-linear Plateau problem which has been successfully represented in variation form (86). Moreover the boundary conditions for the non-Darcy flow problem of this thesis are:

a) \[ \frac{\partial \phi}{\partial n} = 0 \quad \text{(B.7)} \]

and/or b) \( \phi \) prescribed
which satisfy equation B.3. Therefore, using an heuristic approach, the non-Darcy problem is treated as self-adjoint.

B.2 The Variational Formulation

Extending the work of Engelund (27) the non-Darcy flow equation can be represented by an extremal of the functional \( \chi \) whose integrand is the sum of a point energy dissipation rate, \( \dot{E} \), and an unspecified continuity function, \( G \), i.e.

\[
\chi = \int \int (\dot{E} + G) \, dx \, dg \quad (B.8)
\]

where

\[
\dot{E} = K(|\nabla \phi|) (\frac{\phi_x^2}{\phi} + \frac{\phi_y^2}{\phi}) \quad (B.9)
\]

and

\[
G = G(|\nabla \phi|) \quad (B.10)
\]

The first variational of \( \chi \),

\[
\delta \chi = 0
\]

applied to equation B.8 yields (by the Euler-Lagrange Equation)

\[
\nabla \left[ K\nabla \phi + (K_x + K_y) (|\nabla \phi|^2) + (G_x + G_y) \right] = 0 \quad (B.11)
\]

where

\[
K = K(|\nabla \phi|) \quad (B.12)
\]

Using the governing non-Darcy flow equation
\[ \nabla \cdot K\nabla \Phi = 0 \] 

(B.13)

equation B.12 can be generalized, in tensor notation as

\[ [G(\Phi_x, \Phi_y)]_{x_i, x_i} = -[K(\Phi_x, \Phi_y)]_{x_i, x_i} \phi_{x_i}^2 + \lambda_1 K(\Phi_{x_i}) \phi_{x_i, x_i} \phi_{x_i} \]

(B.14)

where \( \lambda_1 \) is an arbitrary constant.

Integrating equation B.14 first with respect to \( x_i \) and then with respect to \( \Phi \) for \( \lambda = 1.0 \) and \( K(\Phi_{x_i}) \) defined by equation 3.58 gives

\[ G(\Phi) = \frac{a}{6bc} (2 - c \Phi) \sqrt{1 + c \Phi} + C' \]

(B.15)

where \( C' \) is an arbitrary constant and may be equated to zero.

In order for equation B.8 to have an extremal, i.e. a maximum or a minimum, the second, variational of \( \chi \) must be non-zero (106, 102).

If \( \chi \) has a minimum

\[ \delta^2 \chi > 0 \] 

(B.16)

From Schechter (102) the second variation for equation B.8 can be developed as

\[ \delta^2 \chi \geq \int \int_A e^2 \left( \delta_x^2 + \delta_y^2 \right) K(\Phi) \cdot 2 \left( \eta_x \delta_y - \eta_y \delta_x \right)^2 dx dy \] 

(B.18)
Since, \( \epsilon \neq 0; \)

\[ \eta_x \text{ and } \eta_y \text{ are non-zero at someplace in } A; \]

and \( K > 0; \)

then \( \delta^2 \chi > 0 \)

and \( \delta \chi = 0 \)

yields a minimum.

---

Figure 5.1: Sketch

For a problem with unknown values of \( \chi \) with \( K \) constant, our linearization of eq. 5.1 (a) is shown in figure 5.4.

By inspection, it can be seen that the system 5.7 has the following...
APPENDIX C

Some Properties of the Coefficient Matrix

Consider the radial flow system

\[ \sum K \left( \left| \frac{\partial \phi}{\partial r} \right| \right) e \cdot S_{ij} \phi_j = 0 \]  \hspace{1cm} (C.1)

with the boundary conditions

\[ r = r_A; \quad \phi = \phi_A \]
\[ r = r_B; \quad \phi = \phi_B \]

\[ r = r_i; \quad \phi = \phi_i \]  \hspace{1cm} (C.2)

For a problem with \( n \) unknown values of \( \phi \), with \( K \) constant, the
linearization of C.1 (a) is shown in figure C.2.

By inspection, it can be seen that the system C.2 has the following
where

\[ K_S = \left[ \begin{array}{cccc}
K_S & 0 & \cdots & 0 \\
0 & K_S + K_S & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & K_S + K_S \\
0 & 0 & \cdots & -K_S \\
\end{array} \right] \]

\[ e = \left[ \begin{array}{c}
e_1 \\
e_2 \\
\vdots \\
e_{n-1} \\
e_n \\
\end{array} \right] \]
properties:

a) The coefficient matrix is symmetric.

b) The system is irreducible.

c) The system possesses the property 'A', i.e. referring to figure C.1, any row of C.2 contains only one $x_i$ but one or more $x_0$'s, or one $x_0$ but one or more $x_i$'s.

d) $a_{ii} \leq \sum_{j \neq i} a_{ij}$ on any row and $a_{ii} < \frac{n}{\sum_{j \neq i} a_{ij}}$ on two rows.

e) The coefficient matrix is positive definite (all cofactors of [A] are positive or zero.)

Young (135) has shown that properties (a) to (e) ensure convergence of the successive over-relaxation procedure.

In the general two-dimensional case with triangular elements (figure C.1 (b)) the property 'A' is lost, although the remaining properties can be proven for the linearized system.
NLOR SUBROUTINE REPLACES SOR

FIG. C3

COMMON DIMENSION

D1 = 0.0

F(IN) = -W(IN)
DF(IN) = 0.0

DF(I) = DF(MA)

DO 50 IN = 1, MA

DO 40 IR = 1, MA

F(IN) = F(IN) + A(IN, IR) · PH1(IR)

DF(IN)

2

IN - IR

1

<

I

3

I

UT = 1.0

DF(IN)

D2 = (-ω) F(IN) / DF(IN) · S + A(IN, IN)

PHI(IN) = PHI(IN) + D2

DF(IN) ≠ A(IN, IR) - 0.0

121 - DI

DI = 121

RETURN
RADIAL FLOW SOLUTION BY FINITE ELEMENT METHOD

DIMENSION RH(1), A(1-9), B(1-9), C(1-9), D(1-9), E(1-9), F(1-9), G(1-9)

DIMENSION RI(1)

COMMON W,A,OMEGA,ITER,CH4K(0,0),CH4K(1,0),ME

READ 1,4A6,4EPS,M,6I(1),OMEGA+0.01

1 FORMAT (12I,F7.12,F8.5,F3A1)

READ 2, (R(I),I=1,10)

2 FORMAT (F*+2)

READ 3, (1E16+E16,1E16)

3 FORMAT (12E)

AS=1.0*AI

DO 995 I=1,9

SI=0.02

OMEXH(OMEG)+2.0

DO 985 I=1,10

SI=SI+0.1

ITER=ITER+1

READ 107, (PHI(I),11=1,10)

107 FORMAT (9F*+2)

AS=AI(1/2,4I)

BA(1,4)*/AI*1)

DO 20 I=1,9

IF(I=1) ME

IB=I(E)

JR=J(E)

RI(I)=S/RJ(I)-R(J)

IB(I)=RJ(I)(/RJ(I)-R(J))

ITER=ITER+1

IF(I=100) N=100

IF (ITER=90) N=100

CONTINUE

PRINT 5, ITER

CONTINUE

CALL NLOQ (PHI,EPS,MA,SI)

IF (101-EPS) 228,228,201

IF (ITER=90) 20,228,228

CONTINUE

PRINT 5, ITER

CONTINUE

CALL EXIT

STOP

C SOLVE FOR PHI(IN)

SUBROUTINE NLOQ (PHI,EPS,MA,SI)

DIMENSION PHI(1), EPS, MA(0,0)

DIMENSION PHI(1), EPS, MA(0,0)

COMMON W,A,OMEGA,ITER,CH4K(0,0),ME

DO 150 I=1,9

IF(I=1) ME

F(I)=W(I)

DOF(I)=0.0

DOF(I)=0.0

DOF(I)=-DOF(I)+CH4K(I,PHI+MA)+CH4K(I,PHI+MA)+CH4K(I,PHI+MA)

DO 40 1=1,9

DO 40 1=1,9

CONTINUE

RETURN

END

67 0/0001 8 0/011 0 0/005 14 9

490 49

57 40

81 4

99 40

128 40

41 40

163 32

7 1

2 3

3 4

5 5

6 6

89 21 82 50 78 71 69 00 63 00 57 50 105 10 52 70
In the solution of unsteady non-Darcy flow it was advantageous to introduce a new dependent variable $\zeta$ such that (for $\nabla X_{t} \neq 0$)

$$\nabla \zeta = \nabla \bar{\phi} + \frac{1}{\text{gm}} \frac{\partial \bar{g}}{\partial t} \tag{E.1}$$

or

$$\frac{\partial \zeta}{\partial x} = \frac{\partial \bar{\phi}}{\partial x} + \frac{1}{\text{gm}} \frac{\partial u}{\partial t} \tag{E.2}$$

$$\frac{\partial \zeta}{\partial y} = \frac{\partial \bar{\phi}}{\partial y} + \frac{1}{\text{gm}} \frac{\partial v}{\partial t}$$

The total differential of $\zeta$ is

$$d\zeta = \frac{\partial \zeta}{\partial x} dx + \frac{\partial \zeta}{\partial y} dy + \frac{\partial \zeta}{\partial t} dt \tag{E.3}$$

and the substitution of equations E.2 into E.3 gives

$$d\zeta = d\bar{\phi} + \frac{1}{\text{gm}} (u_{x} dx + v_{y} dy) + (\zeta - \bar{\phi}) dt \tag{E.4}$$

where $d\bar{\phi}$ is the total differential of $\bar{\phi}$.

Referring to figure 3.16 a finite difference equation can replace E.4 (if the inertial terms are small relative to the friction terms), i.e.
\[ \Delta \zeta = \Delta \phi + \frac{1}{\gamma m} (\bar{u}_t \Delta x + \bar{v}_t \Delta y) + (\zeta_r - \bar{\phi}_r) \Delta t \]  
(E.5)

Also, if \( \frac{1}{\gamma m} \) is small

\[ \frac{\partial \bar{q}^2}{\partial t} \approx -K \nabla (\zeta_r) \]  
(E.6)

and from equation E.1

\[ \zeta \approx \phi - \frac{\bar{K}}{\gamma m} \zeta_t \]  
(E.7)

where \( \bar{K} \) is the average conductivity during time \( \Delta t \).

Therefore equation E.7 can be used to establish the initial \( \zeta \).

\[ (\zeta_t - \bar{\phi}_t) = -\frac{\bar{K}}{\gamma m} \zeta_{tt} \]  
(E.8)

and equation E.5 becomes

\[ \Delta \zeta = \Delta \phi + \frac{1}{\gamma m} (\bar{u}_t \Delta x + \bar{v}_t \Delta y) - \frac{\bar{K}}{\gamma m} \zeta_{tt} \Delta t \]  
(E.9)

Now \( \zeta_{tt} \) can be approximated by \( \bar{q}_t \).

Considering only a vertical drop in the free surface equation E.8 can be simplified to

\[ \Delta \zeta \approx \Delta y \left[ 1 + \frac{\bar{v}_t}{\gamma m (1 - \bar{K}/\bar{v})} \right] \]  
(E.10)
$$\Delta \phi = \Delta y$$

(E.11)

on the free surface.
APPENDIX G

A Paper, entitled,

A Variational Approach to Non-Darcy Flow in Two Dimensions, accepted July, 1970 for publication in the Journal of the Hydraulics Division, ASCE.
A VARIATIONAL APPROACH TO NON-DARCY FLOW IN TWO DIMENSIONS

John A. McCorquodale, A.M. ASCE.

INTRODUCTION

The principles of variational calculus have been applied to many continuum mechanics problems and have led to the use of the finite element method in solving potential flow problems. The basic form of the finite element method, which is widely used in structural engineering, are outlined by Zienkiewicz (19). The finite element method has recently been applied to Darcy flow through porous media by Zienkiewicz, Neyer and Cheung (19). Fried (4), Taylor and Brown (12) and Witherspoon, Jawandel and Neuman (16).

McCorquodale and Ny (7) following the work of Engelund (3) attempted to solve non-Darcy free surface flow problems by finite element analysis. Engelund gave the functional for turbulent and transitional two dimensional flow as

\[ x = I \int I K |\mathbf{q}| \left( \frac{2}{a^2} \right) \, dx \, dy \]  

where:

\[ s = \frac{q}{K} \]  

\[ I = \frac{1}{a^2} \]  

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equations as the steady flow field. The solution of the unsteady flow problem is therefore reduced to the application of the steady flow techniques to a field with time dependent boundary conditions.

The energy functional can be written

\[ y = \int_0^t \frac{1}{2} \rho \left( \frac{\partial w}{\partial t} \right)^2 + \frac{1}{2} \mu \left( \nabla w \right)^2 + \rho g \left( \nabla w \right) \cdot w \, dt \]

where \( t_0 \) represents a time at which \( \delta(x,y,t_0) \) is known. The problem is now reduced to seeking the \( \delta(x,y,t_0 + \Delta t) \) which minimizes \( y \) subject to continuity at the free surface as well as the other boundary conditions such as those illustrated in Figure 2.

The flow field in the \((x - y - t)\) space can be discretized by elements of the type shown in Figure 3. Retaining the linear distribution of \( \delta \) within each element at any instant in time, the element \( \delta \) becomes

\[ \delta = \int_0^1 \left( a_1 + a_2 x + a_3 y + \frac{1}{2} a_4 x^2 + \frac{1}{2} a_5 y^2 \right) \, dx \, dy \]

where \( t \) varies from \( t_0 \) to \( t_0 + \Delta t \) and the values of \( a \) may be obtained from the nodal conditions shown in Figure 3. The conditions are known at nodes 1, m, n and unknown at nodes 1, j, k therefore

\[
\begin{bmatrix}
\delta_1 \\
\delta_m \\
\delta_n \\
\delta_j \\
\delta_k
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \Delta t \\
0 & 1 & 0 & 0 & 0 & \Delta t \\
0 & 0 & 1 & 0 & 0 & \Delta t \\
0 & 0 & 0 & 1 & 0 & \Delta t \\
0 & 0 & 0 & 0 & 1 & \Delta t
\end{bmatrix}
\begin{bmatrix}
\delta_1 \\
\delta_m \\
\delta_n \\
\delta_j \\
\delta_k
\end{bmatrix}
\]

where \( \lambda_1 \) is the area of triangle \( 1jk \) and

\[ \Delta \delta_i = \delta_i - \delta_i^0 \]

and

\[ \delta_i^0 = a_1 x_i + a_2 y_i + \frac{1}{2} a_4 x_i^2 + \frac{1}{2} a_5 y_i^2 \]

Substitution of Eq. 13 into Eq. 12 and minimizing \( y \) with respect to the unknown values of \( a \) gives, after integration, the following system of non-linear simultaneous equations in \( \delta_i \):

\[ \Delta \delta_i = \int_0^1 \left( b_{1i} \delta_i + b_{2i} \delta_j + b_{3i} \delta_k \right) \, dx \]

where \( K = \) average hydraulic conductivity during \( \Delta t \);

\( A = \) average elemental area during \( \Delta t \);

\( S_{ij} = \) stiffness matrix for triangle \( 1jk \) (see appendix II);

\( b_{1i}, b_{2i}, b_{3i} \) are defined in appendix II.

Eq. 23 is based on the fact that the variation in \( K \) and \( A \) during a small \( \Delta t \) is very small; therefore \( K \) and \( A \) have been replaced by their average values.

Solving for \( \delta_1, \delta_m \) and \( \delta_n \) gives

\[ \delta_i = \frac{\delta_i^0}{2} \]

The solution for \( \delta_1, \delta_m, \) and \( \delta_n \) can now be written

\[ \delta_i = \frac{\delta_i^0}{2} \]

where \( \lambda_1 \) is the area of triangle \( 1jk \).

The flow field in the \((x - y - t)\) space can be discretized by elements of the type shown in Figure 3. Retaining the linear distribution of \( \delta \) within each element at any instant in time, the element \( \delta \) becomes

\[ \delta = \int_0^1 \left( a_1 + a_2 x + a_3 y + \frac{1}{2} a_4 x^2 + \frac{1}{2} a_5 y^2 \right) \, dx \, dy \]

where \( t \) varies from \( t_0 \) to \( t_0 + \Delta t \) and the values of \( a \) may be obtained from the nodal conditions shown in Figure 3. The conditions are known at nodes 1, m, n and unknown at nodes 1, j, k therefore

\[
\begin{bmatrix}
\delta_1 \\
\delta_m \\
\delta_n \\
\delta_j \\
\delta_k
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \Delta t \\
0 & 1 & 0 & 0 & 0 & \Delta t \\
0 & 0 & 1 & 0 & 0 & \Delta t \\
0 & 0 & 0 & 1 & 0 & \Delta t \\
0 & 0 & 0 & 0 & 1 & \Delta t
\end{bmatrix}
\begin{bmatrix}
\delta_1 \\
\delta_m \\
\delta_n \\
\delta_j \\
\delta_k
\end{bmatrix}
\]

where \( \lambda_1 \) is the area of triangle \( 1jk \) and

\[ \Delta \delta_i = \delta_i - \delta_i^0 \]

and

\[ \delta_i^0 = a_1 x_i + a_2 y_i + \frac{1}{2} a_4 x_i^2 + \frac{1}{2} a_5 y_i^2 \]

Substitution of Eq. 13 into Eq. 12 and minimizing \( y \) with respect to the unknown values of \( a \) gives, after integration, the following system of non-linear simultaneous equations in \( \delta_i \):

\[ \Delta \delta_i = \int_0^1 \left( b_{1i} \delta_i + b_{2i} \delta_j + b_{3i} \delta_k \right) \, dx \]

where \( K = \) average hydraulic conductivity during \( \Delta t \);

\( A = \) average elemental area during \( \Delta t \);

\( S_{ij} = \) stiffness matrix for triangle \( 1jk \) (see appendix II);

\( b_{1i}, b_{2i}, b_{3i} \) are defined in appendix II.

Eq. 23 is based on the fact that the variation in \( K \) and \( A \) during a small \( \Delta t \) is very small; therefore \( K \) and \( A \) have been replaced by their average values.

Solving for \( \delta_1, \delta_m, \) and \( \delta_n \) gives

\[ \delta_i = \frac{\delta_i^0}{2} \]

The solution for \( \delta_1, \delta_m, \) and \( \delta_n \) can now be written

\[ \delta_i = \frac{\delta_i^0}{2} \]

where \( \lambda_1 \) is the area of triangle \( 1jk \).

The flow field in the \((x - y - t)\) space can be discretized by elements of the type shown in Figure 3. Retaining the linear distribution of \( \delta \) within each element at any instant in time, the element \( \delta \) becomes

\[ \delta = \int_0^1 \left( a_1 + a_2 x + a_3 y + \frac{1}{2} a_4 x^2 + \frac{1}{2} a_5 y^2 \right) \, dx \, dy \]

where \( t \) varies from \( t_0 \) to \( t_0 + \Delta t \) and the values of \( a \) may be obtained from the nodal conditions shown in Figure 3. The conditions are known at nodes 1, m, n and unknown at nodes 1, j, k therefore

\[
\begin{bmatrix}
\delta_1 \\
\delta_m \\
\delta_n \\
\delta_j \\
\delta_k
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \Delta t \\
0 & 1 & 0 & 0 & 0 & \Delta t \\
0 & 0 & 1 & 0 & 0 & \Delta t \\
0 & 0 & 0 & 1 & 0 & \Delta t \\
0 & 0 & 0 & 0 & 1 & \Delta t
\end{bmatrix}
\begin{bmatrix}
\delta_1 \\
\delta_m \\
\delta_n \\
\delta_j \\
\delta_k
\end{bmatrix}
\]

where \( \lambda_1 \) is the area of triangle \( 1jk \).

The solution for \( \delta_1, \delta_m, \) and \( \delta_n \) can now be written

\[ \delta_i = \frac{\delta_i^0}{2} \]

where \( \lambda_1 \) is the area of triangle \( 1jk \).

The free surface movement during each time step can be computed using Lagrangian coordinates,

\[ \frac{\partial \delta}{\partial t} = \frac{\partial \delta}{\partial t} + \frac{\partial \delta}{\partial t} \]

where \( \delta(A,B,t) \) is the position of the free surface at time \( t \);

\( A \) and \( B \) are the rectangular coordinates of the particle at \( t = \) or \( \delta \) is the average of the 'bulk' velocity of the particle during \( \Delta t \). Since \( \Delta t \) is assumed to be small \( \frac{\partial \delta}{\partial t} \) may initially be approximated from the first solution of Eq. 23 with the surface \( \delta(\Delta t) = \delta(\Delta t) \); Eq. 24 is then used to estimate the new free surface. To improve the estimated new free surface and to correct for non-linearities, Eqs. 23 and 24 are solved alternately until a stable solution for the new free surface is obtained.

**THE NUMERICAL SOLUTION**

**Steady Flow**

A number of researchers (see for example Politzky (10), Truett, Erdelyi and Jackson (13), and Young and Wheeler (18)) have obtained finite difference solutions to nonlinear partial differential equations similar to Eq. 2. The method most commonly used involves an inner iteration loop based on a system of equations that are linearized by treating the 'permeability' \( K \) as constant at each node of the grid and an outer iteration loop which varies \( K \) on the basis of the current inner solutions. Also it would appear that a nonlinear over-relaxation method similar to the Newton method could be used (see Ames (1)).
The stream line condition along BC and AE is satisfied by the approximate solution for the difference between the measured and calculated discharge. The data are in good agreement with the finite element solution. The average experimental curve is shown on Figure 8. Figure 8 indicates the computed free surface curves generally fall within the range of experimental data.

**Unsteady Flow**

The flow chart for the solution of an unsteady flow problem of the type illustrated in Figure 2, is shown in Figure 7. The method for solving the non-linear simultaneous equations was almost the same as for the steady flow case. An \( \nu = 1.5 \) was used in both cases. However, the stability of the unsteady flow solution depended on \( \Delta t \). It was found, after some mathematical experimentation, that

\[
\Delta t \leq \frac{2h}{\sqrt{\nu a}}
\]

yields stable solutions. For the case of rapid drawdown of the rockfill section shown in Figure 2 at \( \Delta t = 0.1 \) sec, was used to compute the free surface curves shown in Figure 8.

**EXPERIMENTAL VERIFICATION**

**Steady Flow**

The rectangular rockfill section shown in Figure 1 was set up in a 12" wide by 4 feet high flume with a plexiglass window as shown in Figure 9. Crushed rock with a sieve diameter of 1/2" was used as the porous medium. Piezometer taps were distributed over the rectangular section on a 6" x 6" grid. Head water, tail water, free surface, and piezometric heads were measured to ±1 mm. A venturimeter was used to measure discharge.

The characteristic constants, \( a \) and \( b \) of Eq. 4, for the 1/2" rock, were taken from a parallel flow study made by Eq. (8), \( a = 0.0103, \ b = 0.0995 \). The experimental and the finite element values of \( \eta \) are compared in Figure 6. The measured discharge through the section was 454 cm²/sec, compared with 436 cm²/sec from the finite element solution. The 4% difference between the measured and calculated discharges can be partly attributed to errors in \( a \) and \( b \) of the order of ± 5%.

**Unsteady Flow**

Several rapid drawdown tests were carried out for the rockfill section shown in Figures 2 and 8. The rock used for these tests was found to have: \( a = 0.013 \) sec/cm; \( b = 0.010 \) sec/cm³; and a drainable porosity \( m = 0.40 \). The tailwater level was dropped rapidly from 56 cm. to approximately 5 cm. The water surface profiles were measured from photographs such as Figure 10. The range of experimental data and the average experimental curves are shown on Figure 8. Figure 8 indicates that the computed free surface curves generally fall within the range of experimental data.

**CONCLUSIONS**

A converging successive over-relaxation solution for the non-linear elliptic partial differential equation for steady transitional free surface flow through porous media is possible when a variational approach is used to establish the difference equations. The experimental data are in good agreement with the finite element solution. The
Fig. 1 Definition of Nodes and Elements.

Fig. 3 Finite Element in x-y-t Space.

Fig. 4 Flow Chart For Finite Element Solution.

Fig. 2 Discretization and Boundary Conditions for Unsteady Flow.
Fig. 5 Boundary Conditions for Free Surface Flow.

\[ \nabla \cdot (K \nabla \phi) = 0 \n\]

\[ \frac{\partial \phi}{\partial n} = 0 \text{ or } \psi = 0 \]

REPLACE 'OLD' INITIAL CONDITIONS
(x, y, \phi)_i,m,n = (x, y, \phi)_{i,k}
ALL NODES

START

DIMENSION, COMMON, READ

INITIAL CALCULATION

\[ T = T + \Delta t \]

TWL = F(t);
SET \phi ALONG D/S FACE

A_i; A_n; S_i

NONLINEAR TERMS OF EQ. 23
\[ \omega, K; \text{ ALSO } p \]

ASSEMBLE EQ. 23

SOLVE BY SOR EQ. 23

ESTIMATE NEW FREE SURFACE
\[ \phi(A,B, t+ \Delta t) \]

SURFACE \[ \phi(\Delta t) = \phi(A,B, t+ \Delta t) \]

ADJUST INTERNAL ELEMENTS

PRINT \[ \phi(A,B, t) \]

STOP

Fig. 7 Flow Chart for the Unsteady Flow Solution.

Fig. 6 Comparison of Calculated and Measured Piesometric Heads.

Fig. 8 Comparison of Experimental and Computed Drawdown Curves.
APPENDIX I

The Permeameter Wall Effect

The following development of a wall correction is similar to that of Dudgeon (26) except that he used an exponential resistance equation and here a modification of the Ward equation is used. The wall correction factor is defined as

\[ C_w = \frac{q_\infty}{q_o} \]  

where \( q_\infty \) is the macroscopic (bulk) velocity in an infinitely large permeameter;

\( q_o \) is the apparent macroscopic velocity in the finite permeameter.

Consider the cross-section of the permeameter shown in figure 1.

Figure I.1 Definition of Wall Zone

According to Dudgeon (26) the porosity of the medium near the wall of the permeameter is less (by about 50\%) than in the core. This wall zone is restricted to more than a half a particle diameter from the
By continuity the total discharge, Q, is

\[ Q = q_o A_T \] (I.2)

\[ = q_w A_w + q_w A_w \] (I.3)

where \( A_T \) = the total area of the permeameter;
\( A_w \) = area of the wall zone;
\( q_w \) = the bulk velocity in the wall zone;
\( A_\infty \) = the core area,

\[ = A_T - A_w \] (I.4)

Now \( A_w \) can be expressed by

\[ A_w = C' L_e D_g \] (I.5)

where \( L_e \) is the effective length of the wall zone perimeter; \( C' \) is a constant; \( D_g \) is the grain diameter.

Assuming that the non-Darcy component of the flow resistance is predominate the Ward equation gives the approximate relationship

\[ \frac{q_w}{q_w} = \sqrt{\frac{C(m)}{C(m)_\infty}} \] (I.6)

where \( C(m) = m^3/(1-m)^2 \)

Substituting equation I.4, I.5 and I.6 into equation I.3 and equating I.2 and I.3 gives
\[
C_w = \left[ 1 + C'(\frac{\sqrt{C(m)_\infty}}{C(m)_{wall}} - 1) \frac{L_D}{A_T} \right]^{-1}
\]  

(I.7)

for a circular section this can be simplified to

\[
C_w = \left[ 1 + C'_1 \frac{L}{L_T} \frac{D_T}{T} \right]^{-1}
\]  

(I.8)

Or

\[
C_w = \left[ 1 + C'' \frac{L}{L_T} \frac{D_T}{T} \right]^{-1}
\]  

(I.9)

for a square section

where

\[
\frac{C''}{C'_1} = \frac{4}{2/\pi}
\]

(I.10)

\[L_T = \text{length of permeameter perimeter.}\]

On the basis of the wall studies made by Dudgeon (24) for crushed rock

\[C''_0 \approx 1.8\]

(I.11)

therefore \[C''_1 \approx 2.0\]

(I.12)
APPENDIX J

Radial Flow Analysis Programme

DIMENSION XARRAY(400*,18000*)
DIMENSION P(20,13*81000*,61300*,R1400*,8K1400*,9R1400*,10R1400*)
COMMON L,SB
L1=17
L2=11
READ 12(I1,J1){110,11}*J{11,L1}*J{11,L2}*
12 FORMAT(10F6*2/7F6*2)*
READ 12(I1,J1){110,11}*J{11,L1}*J{11,L2}*
13 FORMAT(11F6,1*),
READ 15N1,N2,N3,N4
15 FORMAT(16I2*,1*)
READ A1,A1*THETA,0,N
3 FORMAT(14F8,1*)
L=0
SB=0
DO 16 J=1,L2
16 QJ=QJ**6.3,1
XT=W*THETA**2,S*
NM1=N1,1
NP1=N1,1
NQ2+N2-1
NP2=N2,1
NM3=N3,1
NP3=N3,1
NQ4=N4,1
NP4=N4,1
DO 17 J=1,L2
DO 18 I=1,NP1
18 CONTINUE
PRINT 4, NP1,NM2
4 FORMAT(1H0, 3HNP1, 21A* *)
DO 19 I=NP1,NP2
19 CONTINUE
PRINT 29, B
29 FORMAT(1H0, 10F4, 8S )
STOP

SUBROUTINE BRB(1810*XT,P,R,A1,D,BK,RN,RM,J2,*
DIMENSION P(20,13*,81000*,61300*,R1400*,8K1400*,9R1400*,10R1400*)
COMMON L,SB
L=1
RQ=R1/R2,1*
B1=L10*XT{11,J},*P11,1*,1*J{11,*A1,1*,ALOG10**/*
11G1,J}10*R1-1*0/R11,1*,1*/XT*1
28 SN=S0+111*
RN1=Q1*J{10000,1*XT*R1}1,1*,1*930,1*
BK1=R1*YM1*RN1,R1*Q1
PRINT 27, B1{11,1*}9{11,1*}R11,10
27 FORMAT(1H0,412*X,9*F10,9*2*
SN=SN+111*
RN1=Q1*J{10000,1*XT*R1}1,1*,1*930,1*
BK1=R1*YM1*RN1,R1*Q1
PRINT 27, B1{11,1*}9{11,1*}R11,10
27 FORMAT(1H0,412*X,9*F10,9*2*
SN=SN+111*
RETURN
END
APPENDIX K
REFERENCES


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61. Liebmann, H., Sitzungsberichte der math.-phys. Klasse der Bayerische Akad, Munchen, 1918. (See Ref. 111)


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APPENDIX L
NOMENCLATURE

A
Area of a Flow Region

$A^e$
Element Area

$\bar{A}$
Average area during $\Delta t$

$A_1$
Area at time $t_1$

$A_2$
Area at time $t_2$

$A(t)$
Area as of function of time

$[A]$
Coefficient Matrix

$A_{ng}$
Angularity factor

$(A_0,B_0)$
Initial coordinates in a Lagrangian System

$a_{ij}$
A term of $[A]$

$a$
Darcy term in the Forchheimer resistance equation

$b$
Coefficient of the non-Darcy term in the Forchheimer equation

$b_i$
$(y_j - y_k)$

$C$
Constants

$c$
constants

$c$
$4b/a^2$

$c_i$
$(x_k - x_j)$

$c$
Element centroid

$c$
Subscript indicates outcrop point

$D_i$
Dispersivity factor

$D_{ij}$
Deformation Tensor

$D$
grain or pore size

$D_2$
grain diameter

$D_{50}$
Median Sieve Size
$D_T$ Permeameter size
$D_p$ Pore Size
$d$ Characteristic length
$\dot{E}$ Energy dissipation rate
$E(\sigma_g)$ $\ln \sigma_g$
$e_e$ superscript denotes an element
$e_e$ Base of natural logarithms
$F_D$ Drag force
$F(\ )$ Function
$f(\ )$ Function
$f$ Friction factor
$G(\ )$ Continuity function
$g$ Acceleration due to gravity
$H$ Depth of Flow; head
$h$ Depth of Water
$h_o$ Mean water depth
$I$ Inertial Effect
$i$ $|\nabla \phi|$ 
$i,j,k$ Summation indices; element node numbers
$j, \tilde{j}$ Unit vectors of (x,y)
$K$ Hydraulic Conductivity
$K(\ )$ Conductivity function
$k$ Permeability
$L$ Length
$l, m, n$ Node numbers for element at initial time
$M$ Exponent
$M_{g}$ Geometric mean diameter
m  
Porosity

N  
Constant; a number

n  
Normal direction (to boundary or streamline)

n  
Number of unknowns

n,p,q  
Superscripts indicating iteration cycles

p  
Pressure

p  
Dummy variable

p  
Probability

p  
Subscript indicates pore

Q  
Discharge

q  
Bulk or macroscopic velocity

\bar{q}  
Average velocity during \Delta t.

R_N  
Reynolds number

R  
Radius

r  
Radius

r_k  
Radius of curvature

r_c  
Radius of convergence

r_d  
Radius of divergence

S_o  
Specific Surface

S_{ij}  
Stiffness matrix

S_e  
Discretized Surface Slope

S_e  
Slope

SA  
Surface area of grain

SF  
Shape Factor

S_c  
Slope correction factor

S_m  
Standard deviation of the mean

S  
Distance along a streamline
Subscript indicates free surface

Stability factor in NLOR

Period

Dimensionless time

Tortuosity

Subscript denotes total

Time

Time step

Velocity tensor

velocity components

Volume

Width

Constant vector

Dimensionless velocity (Reynolds Number)

Dimensionless resistance

Cartesian coordinates

Tensor coordinate;

Free surface y

Outcrop y

Shape factor

Coefficient in \( \theta_e \)

Entrance angle

Coefficient in \( \zeta_e \)

Specific weight

Small change

Variation

Roughness; small quantity
\[ \zeta \quad \text{Transformed dependent variable} = \frac{\tilde{K}}{\mu_m \varepsilon_t} \]

\[ \eta \quad \text{Free surface perturbation height} \]

\[ \theta \quad \text{An angle} \]

\[ \lambda_i \quad \text{Arbitrary constant} \]

\[ \lambda = \frac{a}{b} \sqrt{1 + ci} \]

\[ \lambda \quad \text{Resistance coefficient} \]

\[ \mu \quad \text{Dynamic viscosity} \]

\[ \nu \quad \text{Kinematic viscosity} \]

\[ \xi \quad \text{Engelund's Reynolds number} \]

\[ \xi_e \quad \text{Eddy viscosity} \]

\[ \xi_{vv} \quad \text{Inertial coefficient, velocity distribution coefficient} \]

\[ \rho \quad \text{Density} \]

\[ \Sigma \quad \text{Summation} \]

\[ \sigma_g \quad \text{Geometric standard deviation} \]

\[ \tau \quad \text{Time factor} \]

\[ \phi \quad \text{Piezometric head (macroscopic)} \]

\[ \varphi \quad \text{Piezometric head due to turbulence} \]

\[ \chi \quad \text{Energy dissipation functional} \]

\[ \psi \quad \text{Stream function} \]

\[ \omega \quad \text{Over-relaxation factor} \]
VITA AUCTORIS

1938 | Born, Woodstock, Ontario.
1958 | Graduated, Ingersoll District Collegiate Institute, Ingersoll, Ontario.
1964 | Graduated, Master of Science, in Aeronautics and Fluid Mechanics, University of Glasgow, Glasgow, Scotland.
1966 | Enrolled in Doctor of Philosophy programme in the Department of Civil Engineering, University of Windsor, Windsor, Ontario.