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Bounded Rationality and Irreversible Network Change

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Abstract: A network change is said to be irreversible if the initial network equilibrium cannot be restored by revoking the change. The phenomenon of irreversible network change has been observed in reality. To model this phenomenon, we develop a day-to-day dynamic model whose fixed point is a boundedly rational user equilibrium (BRUE) flow. Our BRUE based approach to modeling irreversible network change has two advantages over other methods based on Wardrop user equilibrium (UE) or stochastic user equilibrium (SUE). First, the existence of multiple network equilibria is necessary for modeling irreversible network change. Unlike UE or SUE, the BRUE multiple equilibria do not rely on non-separable link cost functions, which makes our model applicable to real-world large-scale networks, where well-calibrated non-separable link cost functions are generally not available. Second, travelers' boundedly rational behavior in route choice is explicitly considered in our model. The proposed model is applied to the Twin Cities network to model the flow evolution during the collapse and reopening of the I-35W Bridge. The results show that our model can to a reasonable level reproduce the observed phenomenon of irreversible network change.

Key Words: Bounded rationality, user equilibrium, day-to-day dynamics, irreversible network change

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1. Introduction

Transportation systems are subject to all kinds of modifications or changes due to planned or incidental reasons. Most changes made to transportation networks (e.g., lane closure, capacity reduction, signal retiming) can be physically revoked, but their impacts on the network flow pattern may not be reversible. For example, when a link is closed for an extended period of time, the network flow will change to a new state; however, when the link is reopened, the network flow pattern may not completely change back. This phenomenon is observed in reality during the collapse and reopening of the I-35W Bridge over the Mississippi River in Minneapolis, Minnesota (e.g., Danczyk et al., 2010). *A network change is said to be irreversible if the initial network flow state cannot be restored by revoking the change, and is said to be reversible otherwise.* For the link closure case, if reopening the link can change the network flow back to its original pattern, then the link closure is a reversible network change; otherwise it is an irreversible one. Reversibility/irreversibility is an important issue for temporary network modifications, because irreversible changes, even if temporarily implemented, will permanently change the network flow pattern.

To discuss reversibility/irreversibility more rigorously, we need to refer to the concepts of network flow equilibrium and day-to-day dynamical system of flow evolution. Specifically, the whole process of changing a given network temporarily (and changing it back) can be described as follows: the network flow is originally at equilibrium state A; with a network change made, the flow pattern starts to evolve according to some dynamical mechanism, and finally achieves a new equilibrium state B[1]; with the network change revoked, the flow pattern evolves again and finally achieves equilibrium state C. If equilibrium state C is equal to equilibrium state A, then the original network flow pattern is restored, and thus the network change is a reversible one; otherwise it is irreversible. Note that in this context, the term "equilibrium" is rather general, i.e., an equilibrium state is defined as a fixed point of the day-to-day dynamical system of flow evolution. Depending on the model specification, an equilibrium state could be Wardrop user equilibrium (UE), stochastic user equilibrium (SUE), boundedly rational user equilibrium (BRUE), and so on. We should also emphasize that, in this paper, the network reversibility/irreversibility issue is discussed in the context of day-to-day traffic equilibration, therefore temporary network change and restoration within a day and its impact to traffic is beyond the scope of this paper, particularly when such network changes/restorations have no impact on travelers' decision-making on the next day.

[1] If the network is changed for a period not long enough, it could be the case that the new potential equilibrium state has not been attained yet so that the network flow pattern is at some disequilibrium state B.

With the above definition, assuming that the day-to-day dynamical system always converges to its fixed point (an equilibrium state), then, if the network equilibrium is unique, any network change is reversible. This is because the uniqueness of equilibrium guarantees that equilibrium state C is exactly equal to equilibrium state A. This observation implies that, to model the phenomenon of irreversible network change, we have to consider *multiple equilibria*. That is, equilibrium state C could be different from equilibrium state A only if the network has multiple equilibria.

The simplest way to have multiple equilibria is to consider *non-separable* link (or path) cost functions with traditional UE or SUE equilibrium definitions (UE and SUE both have unique solutions under separable link cost functions). Bie (2008) studied the reversibility/irreversibility issue of network modifications, adopting hypothetical non-separable link cost functions and a day-to-day dynamical system with logit-based SUE as network equilibrium. His work is based on the stability and attraction domain analyses of traffic equilibria in the context of multiple UE or SUE solutions (Bie and Lo, 2010). His work gives very elegant theoretical demonstration of the reversibility/irreversibility issue. Nevertheless, the setting of non-separable link cost functions makes his work inapplicable (or at least difficult to be applied) to real-world large-scale networks, because there are usually no well-calibrated non-separable link cost functions available for real-world large-scale networks.

In this paper we develop a boundedly rational day-to-day dynamical system, with BRUE as its fixed point, to model the phenomenon of irreversible network change. Because the BRUE flow is generally non-unique (Lou et al., 2010), it has multiple equilibria and thus can be used to model irreversible network change. Furthermore, unlike UE or SUE, the BRUE multiple equilibria do not rely on non-separable link cost functions, i.e., it has multiple equilibria under traditional separable link cost functions, which makes our approach applicable to real-world large-scale networks. Besides this technical advantage, the concept of bounded rationality is important in its own right. That is, in the definition of UE and SUE, road users are assumed to be perfectly or unboundedly rational[2], namely that they always choose the paths with the shortest (perceived) travel costs. In reality, however, users are boundedly rational in the sense that they may choose non-shortest paths if the travel time saving offered by switching to the shortest path is not big enough. In summary, by adopting a boundedly rational day-to-day dynamical system with BRUE as network equilibrium, our approach to modeling irreversible network change has two advantages over other UE or SUE based methods: 1) not relying on non-separable link cost

[2] The SUE definition arguably makes a weaker assumption on users' rationality than the UE definition does, as it allows the perceived travel time to be inaccurate, but the theoretical foundation of SUE is users' utility maximization behavior, which still assumes perfect rationality. While in the BRUE definition, users' boundedly rational behavior in route choice is explicitly formulated.

functions, which makes possible the application to real-world large-scale networks, and 2) users' bounded rationality being explicitly considered.

The concept of bounded rationality has been extensively studied in the economic and psychology literature, and it has been shown that bounded rationality is important in many contexts (see, e.g., Conlisk, 1996). In transportation field, there are only a small number of studies on bounded rationality. Mahmassani and Chang (1987) studied the existence, uniqueness, and stability properties of BRUE in the standard single-link bottleneck network. Many simulation and experimental studies have incorporated travelers' boundedly rational behaviors (e.g., Jayakrishnan et al., 1994; Hu and Mahamssani, 1997; Mahamssani and Liu, 1999; Mahamssani 2000). The simulation results of Nakayama et al. (2001) imply a need to examine the validity of the perfect rationality assumption in traffic equilibrium analysis. Szeto and Lo (2006) used the bounded rationality formulation in their dynamic traffic assignment problem. Lou et al. (2010) is the first to systematically examine the mathematical properties of BRUE in a network traffic assignment context. They studied some basic mathematical properties of the BRUE solution, e.g., nonuniqueness and non-convexity of the BRUE flow set.

Another branch of literature closely related to this paper is on day-to-day dynamics of network flow evolution. This branch of studies can be roughly categorized into two classes, deterministic models and stochastic models. Deterministic day-to-day models all provide explicit flow evolution trajectory (e.g., Smith, 1984; Friesz et al., 1994; Nagurney and Zhang, 1996; Yang, 2005). Stochastic day-to-day models may focus on the probability distribution of flow states and/or the expected flow state (e.g., Cascetta, 1989; Davis and Nihan, 1993; Hazelton and Watling, 2004; Yang and Liu, 2007). Several papers used the day-to-day dynamical system approach to study the stability of network equilibrium (e.g. Horowitz, 1984; Watling, 1999; Bie and Lo, 2010). Recently, He et al. (2010) pointed out that many earlier path-based deterministic day-to-day models have two shortcomings, namely the path-flow-nonuniqueness problem and the path-overlapping problem. They proposed a link-based model to overcome the two problems. In this paper, to model irreversible network change, we give a link-based deterministic day-to-day dynamic model, with users' bounded rationality explicitly formulated.

The remainder of the paper is organized as follows. Section 2 presents our boundedly rational day-to-day dynamic model and discusses its property. Section 3 applies the model to a simple yet illustrative network to demonstrate how irreversible network change can be modeled by our method. In Section 4, we apply our approach to the Twin Cities network to model the observed phenomenon of irreversible network change during the collapse and reopening of the I-35W Bridge. Finally, concluding remarks are given in Section 5.

2. A link-based day-to-day dynamic model with bounded rationality

Let a transportation network be a fully-connected directed graph denoted as $G(N,L)$, consisting of a set of nodes N and a set of links L. Let W be the set of OD pairs, d_w be the fixed travel demand between OD pair $w \in W$, P_w be the set of paths connecting OD pair $w \in W$, f_{pw}^t be the path flow on path $p \in P_w$ on day t, x_a^t be the link flow on link $a \in L$ on day t. Denote demand, path flow and link flow vectors as $\mathbf{d}$, $\mathbf{f}^t$, and $\mathbf{x}^t$, respectively. Let $\mathbf{A}$ be the link-path incidence matrix, then $\mathbf{x}^t = \mathbf{A}\mathbf{f}^t$. Let $\mathbf{\Phi}$ be the OD-path incidence matrix, then $\mathbf{d} = \mathbf{\Phi}\mathbf{f}^t$. Let $c_a(\mathbf{x})$ be the link cost function of link $a \in L$, which is assumed to be monotonically increasing, then $c_a(\mathbf{x}^t)$ is the link cost of link $a \in L$ on day t, and we denote $\mathbf{c}(\mathbf{x}^t)$ as the corresponding link cost vector. Let $\mathbf{F}^t$ denote the path cost vector on day t, with individual path cost F_{pw}^t, then it holds $\mathbf{F}^t = \mathbf{A}'\mathbf{c}(\mathbf{x}^t) = \mathbf{A}'\mathbf{c}(\mathbf{A}\mathbf{f}^t)$, where $\mathbf{A}'$ is the transpose of $\mathbf{A}$.

The above notations are sufficient for describing discrete-time day-to-day traffic dynamics. For continuous-time versions, we denote the day-to-day path flow dynamic as $\dot{\mathbf{f}}$, which is the derivative of path flow with respect to time, and denote the day-to-day link flow dynamic as $\dot{\mathbf{x}}$. It holds readily $\dot{\mathbf{x}} = \mathbf{A}\dot{\mathbf{f}}$.

Denote the feasible path flow set as $\Omega_f = \{\mathbf{f} : \mathbf{d} = \mathbf{\Phi}\mathbf{f}, \mathbf{f} \geq 0\}$, and the feasible link flow set as $\Omega_x = \{\mathbf{x} : \mathbf{x} = \mathbf{A}\mathbf{f}, \mathbf{f} \in \Omega_f\}$. We give a formal definition of BRUE as below.

Definition 1. A path flow pattern $\mathbf{f} \in \Omega_f$ is said to be a boundedly rational user equilibrium (BRUE) flow pattern if it holds that

$$F_{pw} \leq \mu_w + \varepsilon_w, \text{ if } f_{pw} > 0,\ w \in W \tag{1}$$

where μ_w is the shortest path cost between OD pair $w \in W$ under flow $\mathbf{f}$, and $\varepsilon_w \geq 0$ is the bounded rationality threshold of travelers between OD pair $w \in W$.

In the above definition, condition (1) simply states that, under a BRUE flow pattern, the travel cost of any used path can be higher than the shortest path, but within a threshold. Observe that, when the bounded rationality threshold is zero, i.e., $\varepsilon_w = 0$ for all $w \in W$, condition (1) reduces to $F_{pw} = \mu_w$ for all used paths, and thus the BRUE definition becomes the classic UE definition. Also note that, the UE flow pattern always satisfies condition (1) (due to $F_{pw} = \mu_w$ on all used paths), and thus is always one BRUE solution.

In some simulation studies (e.g., Hu and Mahamssani, 1997), the bounded rationality threshold ε_w is given as a percentage of the minimum OD cost μ_w rather than a constant value. For example, $\varepsilon_w = 0.1\mu_w$ means that the cost of any used path at BRUE should be not more than 10% higher than the minimum OD cost.

The BRUE link flow definition is given as follows.

Definition 2. A link flow pattern $\mathbf{x} \in \Omega_x$ is said to be a BRUE link flow pattern if there exists a BRUE path flow $\mathbf{f} \in \Omega_f$ such that $\mathbf{x} = \mathbf{Af}$.

Note that the above BRUE link flow definition does not put any mathematical restriction on the flow pattern beyond the original BRUE definition. Simply speaking, a link flow pattern is a BRUE link flow as long as its (or one of its) corresponding path flow pattern is a BRUE path flow[3].

To present our link-based day-to-day dynamic model with bounded rationality, we need to first define *acceptable flows* under a given cost vector.

Definition 3. *Acceptable flow under given cost*: For a given link cost vector $\mathbf{c}(\mathbf{x})$ and thereby a given path cost vector $\mathbf{F} = \mathbf{A}'\mathbf{c}(\mathbf{x})$, let $\mu_w(\mathbf{c}(\mathbf{x}))$ be the shortest path cost between OD pair $w \in W$, then we can define the *acceptable path set* under given cost $\mathbf{c}(\mathbf{x})$ as

$$P_w^{br}(\mathbf{c}(\mathbf{x})) = \left\{ p : F_{pw} \le \mu_w(\mathbf{c}(\mathbf{x})) + \varepsilon_w, p \in P_w \right\}, \ w \in W \tag{2}$$

Then the *acceptable path flow set* under given cost $\mathbf{c}(\mathbf{x})$, which allows positive path flows on acceptable paths only, is given by

$$\Omega_f^{br}(\mathbf{c}(\mathbf{x})) = \left\{ \mathbf{f} : \sum_{p \in P_w^{br}(\mathbf{c}(\mathbf{x}))} f_{pw} = d_w, w \in W, \mathbf{f} \in \Omega_f \right\} \tag{3}$$

[3] Thus we do not have the link-based vs. path-based BRUE issue as in Lou et al. (2010), i.e., a BRUE flow (link flow or path flow) in this paper corresponds to the path-based BRUE in their work.

and the *acceptable link flow set* under given cost $\mathbf{c}(\mathbf{x})$ is

$$\Omega_x^{br}(\mathbf{c}(\mathbf{x})) = \{\mathbf{x} : \mathbf{x} = \mathbf{Af}, \mathbf{f} \in \Omega_f^{br}(\mathbf{c}(\mathbf{x}))\} \tag{4}$$

The superscripts "br" in $P_w^{br}(\mathbf{c}(\mathbf{x}))$, $\Omega_f^{br}(\mathbf{c}(\mathbf{x}))$ and $\Omega_x^{br}(\mathbf{c}(\mathbf{x}))$ represent "boundedly rational", because the definitions of acceptable path and flow follow the concept of bounded rationality. In the above definition, condition (2) means that the cost of an acceptable path $p \in P_w^{br}(\mathbf{c}(\mathbf{x}))$ under given cost $\mathbf{c}(\mathbf{x})$ should be not higher than the minimum OD cost plus the bounded rationality threshold. Condition (3) means that an acceptable path flow pattern $\mathbf{f} \in \Omega_f^{br}(\mathbf{c}(\mathbf{x}))$ under given cost $\mathbf{c}(\mathbf{x})$ assigns positive flows on acceptable paths only. Condition (4) simply says that a link flow pattern is an acceptable flow pattern under given cost $\mathbf{c}(\mathbf{x})$ as long as its (or one of its) corresponding path flow is acceptable under given cost $\mathbf{c}(\mathbf{x})$.

Note that the above definition of "acceptable flow under given cost" is not directly related to BRUE. That is, link flow $\mathbf{y} \in \Omega_x^{br}(\mathbf{c}(\mathbf{x}))$ only means that $\mathbf{y}$ is acceptable under given cost $\mathbf{c}(\mathbf{x})$, without any implication of whether $\mathbf{y}$ is BRUE or not. Similarly, the acceptable flow set $\Omega_x^{br}(\mathbf{c}(\mathbf{x}))$ under given cost $\mathbf{c}(\mathbf{x})$ is not directly related to the BRUE flow set. The connection between the "acceptable flow under given cost" and the BRUE concept is given by the following lemma.

Lemma 1. A link flow pattern $\mathbf{x}$ is a BRUE flow if and only if $\mathbf{x} \in \Omega_x^{br}(\mathbf{c}(\mathbf{x}))$.

Proof: Let $\mathbf{f} \in \Omega_f$ be a path flow such that $\mathbf{x} = \mathbf{Af}$, it suffices to prove that $\mathbf{f}$ is a BRUE path flow if and only if $\mathbf{f} \in \Omega_f^{br}(\mathbf{c}(\mathbf{x}))$. We only need to prove that $\mathbf{f} \in \Omega_f^{br}(\mathbf{c}(\mathbf{x}))$ is equivalent to condition (1). From condition (3), we have

$$\mathbf{f} \in \Omega_f^{br}(\mathbf{c}(\mathbf{x})) \Leftrightarrow \sum_{p \in P_w^{br}(\mathbf{c}(\mathbf{x}))} f_{pw} = d_w, w \in W, \text{ and } \mathbf{f} \in \Omega_f \tag{5}$$

Because $\mathbf{f} \in \Omega_f$ represents flow conservation and nonnegativity, $\sum_{p \in P_w^{br}(\mathbf{c}(\mathbf{x}))} f_{pw} = d_w$ together with $\mathbf{f} \in \Omega_f$ guarantees that if $f_{pw} > 0$, then $p \in P_w^{br}(\mathbf{c}(\mathbf{x}))$. Thus (5) can be rewritten as

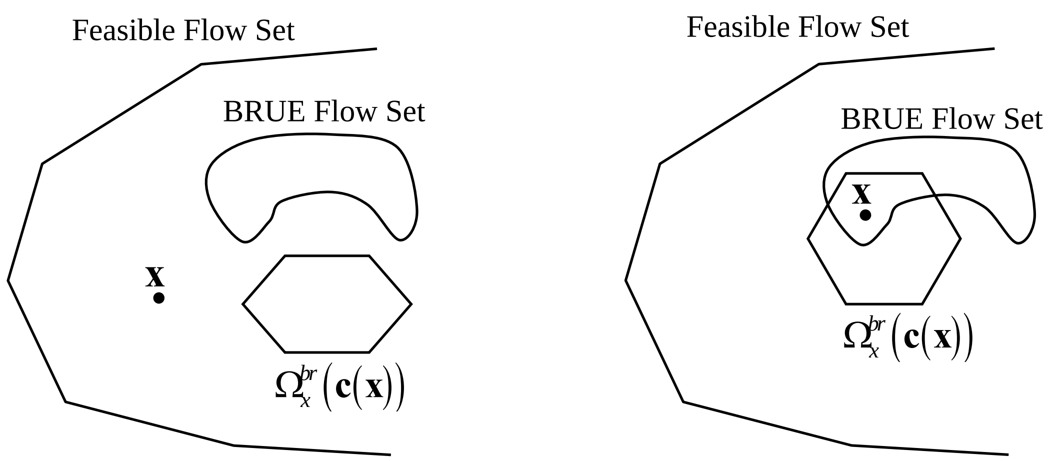
$$\mathbf{f} \in \Omega_f^{br}(\mathbf{c}(\mathbf{x})) \Leftrightarrow p \in P_w^{br}(\mathbf{c}(\mathbf{x})), \text{ if } f_{pw} > 0, \ w \in W \tag{6}$$

From condition (2), $p \in P_w^{br}\left(\mathbf{c}(\mathbf{x})\right)$ simply means $F_{pw} \le \mu_w\left(\mathbf{c}(\mathbf{x})\right) + \varepsilon_w$, thus (6) can be written as

$$\mathbf{f} \in \Omega_f^{br}\left(\mathbf{c}(\mathbf{x})\right) \Leftrightarrow F_{pw} \le \mu_w\left(\mathbf{c}(\mathbf{x})\right) + \varepsilon_w, \text{ if } f_{pw} > 0,\ w \in W \tag{7}$$

Comparing (7) and condition (1), it is readily seen that $\mathbf{f} \in \Omega_f^{br}\left(\mathbf{c}(\mathbf{x})\right)$ is equivalent to condition (1). This completes the proof. ♦

Lemma 1 simply states that a flow is a BRUE flow if and only if it is an acceptable flow under the cost *incurred by itself*. Lemma 1 holds because the definition of BRUE essentially means that a BRUE flow assigns positive flows on acceptable paths (under its own cost) only. Figure 1 gives a graphic illustration of Lemma 1. As demonstrated by Lou et al. (2010), the BRUE flow set is generally non-convex, thus we plot the BRUE flow set as a non-convex set in Figure 1. By contrast, the acceptable flow set $\Omega_f^{br}\left(\mathbf{c}(\mathbf{x})\right)$ under given cost $\mathbf{c}(\mathbf{x})$ is a convex polyhedron as can be easily seen from conditions (3) and (4). The difference between the BRUE flow set and the "acceptable flow set under a given cost" arises from the fact that, the definition of BRUE considers both flow and cost (cost is a function of flow), while the definition of "acceptable flow under a given cost" considers flow only (cost is already given).



(a) $\mathbf{x} \notin \Omega_x^{br}\left(\mathbf{c}(\mathbf{x})\right)$ when $\mathbf{x}$ is not a BRUE flow (b) $\mathbf{x} \in \Omega_x^{br}\left(\mathbf{c}(\mathbf{x})\right)$ when $\mathbf{x}$ is a BRUE flow

Figure 1. Graphic illustration of Lemma 1

Now we are ready to present our link-based day-to-day dynamic model with bounded rationality. In continuous time, our model is given by

$$\dot{\mathbf{x}} = \delta\left(\mathbf{y} - \mathbf{x}\right) \tag{8}$$

where δ is a positive constant parameter determining the flow changing rate, and $(\mathbf{y}-\mathbf{x})$ provides a flow changing direction. In other words, dynamic (8) means that, on any day, the (link) flow pattern tends to move from the current flow pattern $\mathbf{x}$ towards a "target" flow pattern $\mathbf{y}$ based on the current day situation. Thus the model is essentially determined by how the "target" flow pattern $\mathbf{y}$ is defined. Here we let $\mathbf{y}$ solve the following problem given current link flow $\mathbf{x}$:

$$\min_{\mathbf{y}\in\Omega_x^{br}(\mathbf{c}(\mathbf{x}))} D(\mathbf{x},\mathbf{y}) \tag{9}$$

where constraint $\mathbf{y}\in\Omega_x^{br}(\mathbf{c}(\mathbf{x}))$ means that $\mathbf{y}$ is an acceptable flow under the current cost $\mathbf{c}(\mathbf{x})$, and $D(\mathbf{x},\mathbf{y})$ is a measure of the distance between the target flow $\mathbf{y}$ and the current flow $\mathbf{x}$. For example, $D(\mathbf{x},\mathbf{y})$ could take the form of the Euclidean distance $D(\mathbf{x},\mathbf{y})=(\mathbf{x}-\mathbf{y})'(\mathbf{x}-\mathbf{y})$, or the formulation proposed by He et al. (2010), $D(\mathbf{x},\mathbf{y})=\sum_{a\in L}\int_{x_a}^{y_a}(c_a(w)-c_a(x_a))dw$. In words, the target flow $\mathbf{y}$ defined by problem (9) is *the acceptable flow under the current cost* $\mathbf{c}(\mathbf{x})$ *that is closest to the current flow* $\mathbf{x}$. Intuitively, dynamic system (8)-(9) means that, at any time, travelers tend to switch to the acceptable paths under the current cost situation, and in the meanwhile change as little as possible. In behavioral sense, this captures travelers' bounded rationality in route choice (choose acceptable paths instead of shortest paths) as well as their inertia (reluctance to change routes).

In the following we shall prove that the fixed point of dynamic system (8)-(9) corresponds to BRUE. To do so, we first give the flowing lemma.

Lemma 2. Let $D(\mathbf{x},\mathbf{y})$ in (9) be a distance measure between $\mathbf{y}$ and $\mathbf{x}$, and $\mathbf{y}=\mathbf{x}$ minimizes $D(\mathbf{x},\mathbf{y})$ globally, then, for a given link flow pattern $\mathbf{x}$, $\mathbf{y}=\mathbf{x}$ solves problem (9) if and only if it holds $\mathbf{x}\in\Omega_x^{br}(\mathbf{c}(\mathbf{x}))$.

Proof: If $\mathbf{y}=\mathbf{x}$ solves problem (9), then it holds $\mathbf{x}\in\Omega_x^{br}(\mathbf{c}(\mathbf{x}))$ in view of the constraint $\mathbf{y}\in\Omega_x^{br}(\mathbf{c}(\mathbf{x}))$ of problem (9). If it holds $\mathbf{x}\in\Omega_x^{br}(\mathbf{c}(\mathbf{x}))$, then $\mathbf{y}=\mathbf{x}$ is a feasible solution to problem (9), and because $\mathbf{y}=\mathbf{x}$ minimizes $D(\mathbf{x},\mathbf{y})$ globally, it is readily that $\mathbf{y}=\mathbf{x}$ solves problem (9). This completes the proof. ♦

The fixed point of dynamic system (8)-(9), i.e., the link flow $\mathbf{x}$ that gives $\dot{\mathbf{x}} = 0$, is clearly the $\mathbf{x}$ such that $\mathbf{y} = \mathbf{x}$ solves problem (9), which, from Lemma 2, means that $\mathbf{x} \in \Omega_x^{br}\left(\mathbf{c}\left(\mathbf{x}\right)\right)$. From Lemma 1, we know that $\mathbf{x} \in \Omega_x^{br}\left(\mathbf{c}\left(\mathbf{x}\right)\right)$ means that $\mathbf{x}$ is a BRUE flow. Therefore, combining Lemma 1 and Lemma 2, we have the following theorem.

Theorem 1. A link flow pattern $\mathbf{x}$ is a fixed point of the dynamic system (8)-(9) if and only if it is a BRUE flow pattern.

Recall that in modeling temporary network change, we define an equilibrium state as a fixed point of the day-to-day dynamical system of flow evolution, thus Theorem 1 simply states that the network equilibrium corresponding to the proposed dynamic system (8)-(9) is BRUE.

The discrete-time version of the dynamic system (8)-(9) is given by

$$\begin{cases} \mathbf{x}^{t+1} - \mathbf{x}^t = \alpha\left(\mathbf{y}^t - \mathbf{x}^t\right) \\ \mathbf{y}^t = \underset{\mathbf{y}^t \in \Omega_x^{br}(\mathbf{c}(\mathbf{x}^t))}{\operatorname{argmin}} \ D\left(\mathbf{x}^t, \mathbf{y}^t\right) \end{cases} \tag{10}$$

where $0 < \alpha \le 1$ is a step-size parameter in this discrete-time version, and $\mathbf{y}^t$ solves problem (9) for given current link flow $\mathbf{x}^t$. Figure 2 gives a graphic illustration of the discrete-time dynamic (10). Note that, if $\mathbf{x}^t \notin \Omega_x^{br}\left(\mathbf{c}\left(\mathbf{x}^t\right)\right)$, then $\mathbf{y}^t$ will be always on the boundary of $\Omega_x^{br}\left(\mathbf{c}\left(\mathbf{x}^t\right)\right)$, because problem (10) minimizes a distance measure (Euclidean or not) between $\mathbf{y}$ and $\mathbf{x}$.

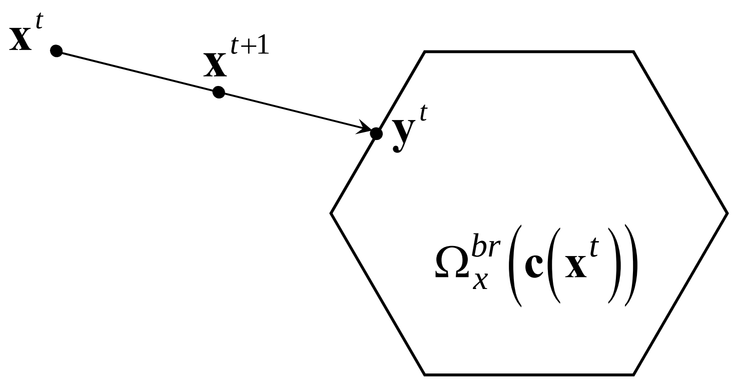


Figure 2. Graphic illustration of discrete-time dynamic (10)

From Figure 2, also in view of the definitions of acceptable flows given by (2)-(4), the implementation of our boundedly rational day-to-day dynamic model in discrete time is straightforward:

Step 1: on day t, with link flow $\mathbf{x}^t$, we immediately have link cost $\mathbf{c}\left(\mathbf{x}^t\right)$ and path cost $\mathbf{F}=\mathbf{A}'\mathbf{c}\left(\mathbf{x}\right)$, and then we can easily judge whether a path is acceptable or not;

Step 2: generate an "acceptable" link-path incidence matrix, which simply excludes the unacceptable paths;

Step 3: the "acceptable" link-path incidence matrix gives a virtual network such that constraint $\mathbf{y}\in\Omega_x^{br}\left(\mathbf{c}\left(\mathbf{x}\right)\right)$ simply means that $\mathbf{y}$ is a feasible link flow in this network, thus problem (9) reduces to a minimization problem with standard flow conservation and non-negativity constraints, and can be easily solved to give target flow $\mathbf{y}^t$;

Step 4: update link flow to get link flow $\mathbf{x}^{t+1}$ on day $t+1$ according to (10).

It should be mentioned here that the implementation of the above steps requires a predetermined path set between each OD pair, otherwise it is a NP-hard problem to enumerate all the acceptable paths. Later in Section 4, when we apply our model to Twin Cities network, the path set is pre-established by running gradient projection algorithm to solve for UE both before and after bridge collapse and recording all the paths that have been searched.

Our model in discrete time has two parameters, namely the bounded rationality threshold parameter ε and the step-size parameter α. The bounded rationality threshold parameter ε is not a new parameter introduced in this paper, but a traditional one associated with the concept of BRUE. This parameter has an explicit physical meaning at individual level, i.e., one will consider changing to a new route only if the new route is at least ε (say, 10%) shorter than her current route. In view of this physical meaning at individual level, the parameter value (or average value among a population) can be systematically estimated using surveys or experimental methods.

The step-size parameter α is a common one in deterministic discrete-time day-to-day dynamic models, which represents how drastically travelers change routes from day to day (at an aggregate level, no physical meaning at individual level). To the best of our knowledge, He and Liu (2010) is the first to apply a day-to-day model to a real-world scenario and thus is also the first to calibrate day-to-day model parameters against real world data. They used a mesh method to search for a two-parameter combination that best fits the observed flow evolution pattern. The mesh method can be straightforwardly applied to estimate the value of α in our model if ε is independently estimated using surveys or experimental methods, or we can use the mesh method to estimate an (ε , α) combination that best fits the observed data.

3. Irreversible network change under bounded rationality: a small illustrative example

In this section we apply our boundedly rational day-to-day dynamic to a small example to demonstrate how the phenomenon of irreversible network change can be modeled by our approach.

To be able to plot the flow pattern and flow sets in a two-dimensional space, we cannot use networks whose flow pattern has a degree of freedom of three or more, thus we consider a simple three-link network as shown in Figure 3. Note that for this small network with fixed demand, the flows on any two links fully depict the network flow state. We shall use the flows on Link 1 and Link 2 to represent the network flow state.

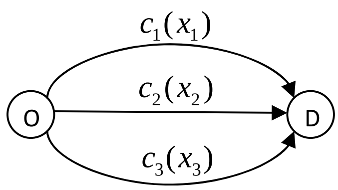


Figure 3. A small example network

Consider that the travel demand between Node O and Node D is $d = 50$, the bounded rationality threshold parameter is $\varepsilon = 10$, and the link cost functions are

$$c_1(x_1) = 30 + x_1,\ c_2(x_2) = 30 + 3x_2,\ c_3(x_3) = 30 + 3x_3$$

Consider that the initial network flow pattern is a BRUE flow $(x_1, x_2, x_3) = (31, 8, 11)$, which gives a cost structure $(c_1, c_2, c_3) = (61, 54, 63)$. It can be verified that this flow pattern satisfies the BRUE condition (1). The test scenario is that a temporary lane closure is conducted on Link 1. A lane is closed on Link 1 on day 0 such that the cost function of Link 1 becomes

$$\tilde{c}_1(x_1) = 30 + 6x_1$$

and after a long enough period (20 days in this example), the closed lane on Link 1 is reopened so that the Link 1 cost function is changed back. We are going to show that this lane closure is an irreversible network change, i.e., the equilibrium states before lane closure and after lane reopening are different.

With this simple network setting, we are able to manually derive the BRUE flow sets based on condition (1) for both the initial network and the degraded network. Figure 4 plots the BRUE flow sets of the original and the degraded networks within the general feasible flow set in the (x_1, x_2) two-dimensional space.

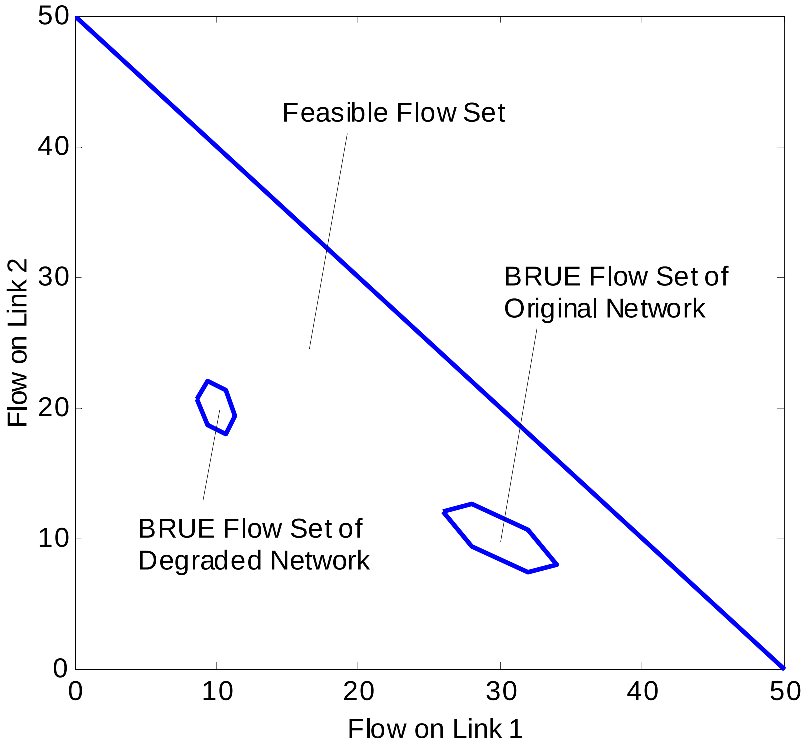


Figure 4. Flow sets of both the original and the degraded networks

Applying the discrete-time version dynamic (10) to this test scenario, with a step-size parameter value $\alpha = 0.1$, and adopting the Euclidean distance $D(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y})'(\mathbf{x} - \mathbf{y})$ in problem (9)[4], we can calculate the network flow evolution during the whole process of the network degradation and restoration. Figure 5 plots the network flow state evolution trajectory in the (x_1, x_2) two-dimensional space. As shown in Figure 5, after the lane closure takes place, the flow pattern starts to evolve from the initial BRUE flow and finally attains a new equilibrium flow state, which is a BRUE flow under the degraded network. When the lane is reopened, the flow pattern starts to evolve again and finally attains another equilibrium state, which is a BRUE flow under the restored network. It can be clearly seen that the final equilibrium flow pattern after network

[4] Although the Euclidean distance measure has the deficiency of being not robust to the "dummy node" effect (He et al., 2010), it performs well in real-world applications (better than the integral formulation does), thus in this paper we use the Euclidean distance in both the small example here and the real-world application in the next section.

fully restored is different from the initial equilibrium flow pattern before the lane closure happened. Thus the lane closure in this test scenario is an irreversible network change.

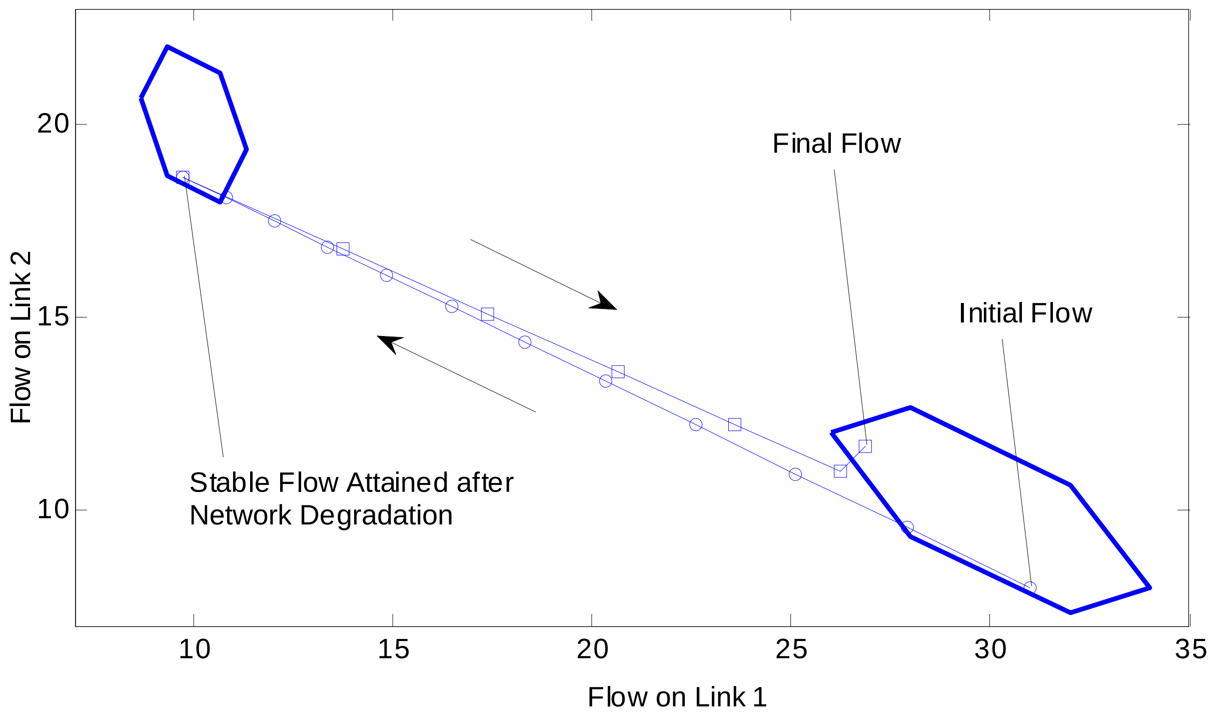


Figure 5. The flow pattern evolution trajectory in the (x_1, x_2) two-dimensional space

Figure 6 plots the day-to-day link flow evolution for all three links. To facilitate illustration, the day-to-day link flow of each link is normalized by its initial link flow, thus all link flow evolutions start with value 1. We can see from Figure 6 that the impact of an irreversible network change (lane closure in this test scenario) on individual links could be significant in terms of percentage flow change: the equilibrium flow on Link 2 increased almost 50% after the whole process of lane closure and reopening.

It is also interesting to compare the system performance of the initial and the final equilibrium states. The initial equilibrium state $(x_1, x_2, x_3) = (31, 8, 11)$ with cost $(c_1, c_2, c_3) = (61, 54, 63)$ gives a total system cost 3016, and the final equilibrium state $(x_1, x_2, x_3) = (26.86, 11.64, 11.50)$ with cost $(c_1, c_2, c_3) = (56.86, 64.92, 64.50)$ gives a total system cost 3024.7. Therefore, in this testing scenario, the lane closure and reopening on Link 1 reduces the equilibrium flow and cost on Link 1, increases the equilibrium flow and cost on Link 2 and 3, and increases the total system travel cost. This observation implies that, with travelers' bounded rationality considered, a temporary network change can be intentionally made to change the flow pattern from the status quo to a new pattern. Therefore careful study and design is needed to ensure that the new pattern is a more desirable one.

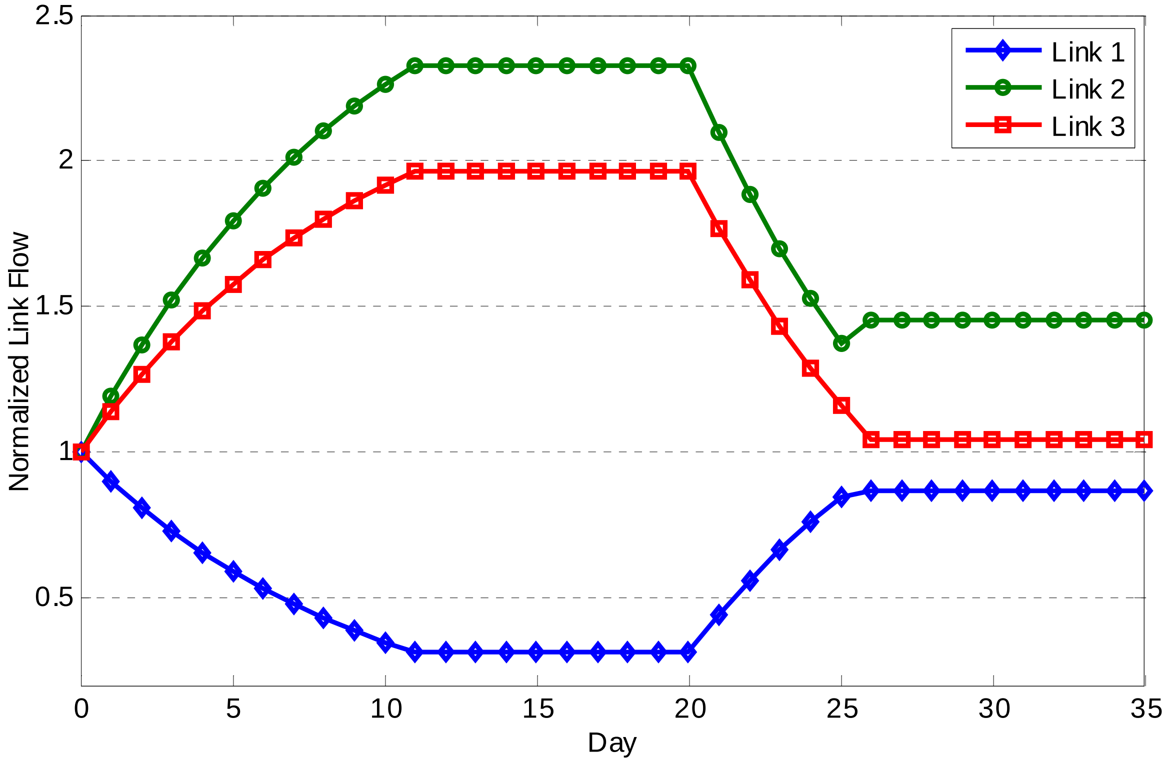


Figure 6. Link flow evolution from day to day

Now we set different values to the two model parameters ε and α to see their impacts on the flow evolution pattern. For graph simplicity, we only show the flow evolution of Link 1. In each subfigure of Figure 7, the x-axis value is time "day" and the y-axis value is the flow on Link 1 normalized by its initial flow. Thus each curve in Figure 7 represents a day-to-day flow evolution of Link 1 with specific ε and α values. Each subfigure of Figure 7 corresponds to one ε value, which is changed from 1% to 5%, 15% and 30% of the minimum OD cost. Here we use percentage values for ε because comparisons among percentage values of ε are more sensible. For example, $\varepsilon = 1\%$ represents that travelers are sensitive to a 36-second travel time difference for a 60 minute trip, while $\varepsilon = 30\%$ means that travelers are insensitive to any travel time difference less than 18 minutes for a 60 minute trip. To ensure that the initial flow pattern is always BRUE even under very small ε values, the initial flow pattern is selected to be the UE flow pattern, $(x_1, x_2, x_3) = (30,10,10)$, which gives a cost structure $(c_1, c_2, c_3) = (60,60,60)$. For each ε value (each subfigure of Figure 7), the step-size parameter α is changed from 0.1 to 0.3 and 0.5.

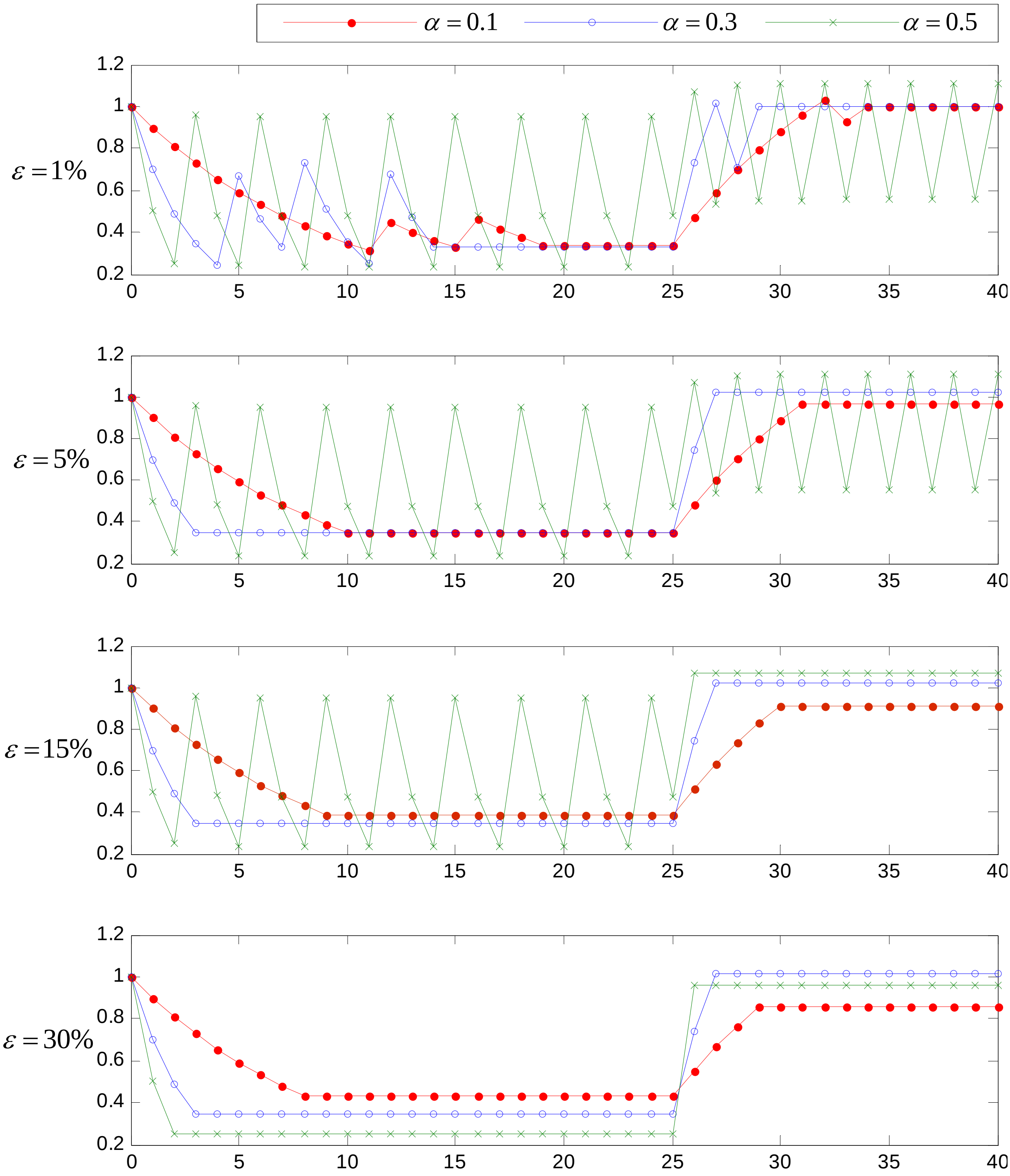


Figure 7. Link 1 flow evolution with different model parameters

We can immediately see from Figure 7 that when the step-size parameter α is big (in our case $\alpha = 0.5$), the day-to-day dynamic may not converge. This is a standard result for discrete-time day-to-day dynamics, i.e., to ensure convergence of the flow evolution, the step-size parameter cannot be too big. In our example, the flow evolution can still converge when $\alpha = 0.5$ but only with an unrealistically large $\varepsilon = 30\%$ (when $\varepsilon = 15\%$, the flow evolution with $\alpha = 0.5$ does not converge after network degradation although it converges finally after network restoration). This leads to another observation: by comparing the four subfigures of Figure 7, it can be seen that a larger ε value makes the flow evolution pattern smoother and easier to converge. This is because a larger ε value gives a larger BRUE flow set (easier to converge) and also gives a larger acceptable path set on any day (smoother evolution from day to day).

Another interesting observation is that, if the step-size parameter α is within the range that the dynamic will converge ($\alpha = 0.1$ and $\alpha = 0.3$ in the first three subfigures; all the three α values in the last subfigure), then a larger α gives a shorter disequilibrium evolution time, or in other words, the day-to-day dynamic converges faster with a larger step-size parameter. Intuitively, this is because a larger step-size makes the flow pattern evolves quicker towards the new BRUE flow set.

We conclude this section by pointing out that network changes are generally irreversible under the assumption of bounded rationality. This is sensible from our small example and in particular can be seen from Figure 5: the initial flow is just one point within the initial BRUE flow set, when the network change is revoked, we should only expect the final flow to be back in the initial *BRUE flow set* rather than be exactly the initial *equilibrium flow*. Simply speaking, under the assumption of bounded rationality, a network change is generally irreversible (perfect reversibility may happen only as special cases). This result is consistent with our intuition about bounded rationality: some travelers who get used to new routes after network change simply would not change back to their initial routes after network restoration (even if their initial routes are a bit shorter) due to bounded rationality.

4. Application to a real-world large-scale network

In this section we apply our boundedly rational day-to-day dynamic model to a real-world large-scale network to model an observed phenomenon of irreversible network change.

The I-35W Mississippi River Bridge in Minneapolis, Minnesota collapsed on August 1, 2007, and was reopened on September 18, 2008. As we will demonstrate in the following, an irreversible network change was observed due to the bridge collapse and reopening. Figure 8 plots the daily morning peak hour trip (6 a.m. to 10 a.m.) crossing the I-35W bridge from September to December in the year 2006 (before collapse) and 2008 (after reopening). Both curves show stable traffic counts for non-holiday weekdays, thus each corresponds to an equilibrium flow pattern. As can be seen from Figure 8, before collapse, about 36,000 vehicles crossed the I-35W Bridge daily during the morning peak hour, while after reopening, the number is about 28,000. Thus the daily morning peak hour traffic crossing the bridge was reduced about 22% because of the bridge collapse and reopening. The empirical study of Danczyk et al. (2010) reported that the travel demand around the Twin Cities area remains the same between 2006 and 2008 (or, at least there is no evidence suggesting otherwise). Also, the network supply after the bridge reopening is the same as that before the bridge collapse. Moreover, the only major network change during this period is the I-35W Bridge collapse and reopening. Thus we can conclude that *the bridge closure in this case is an irreversible network change*: with the same travel demand and network supply, the final equilibrium state is different from the initial equilibrium state.

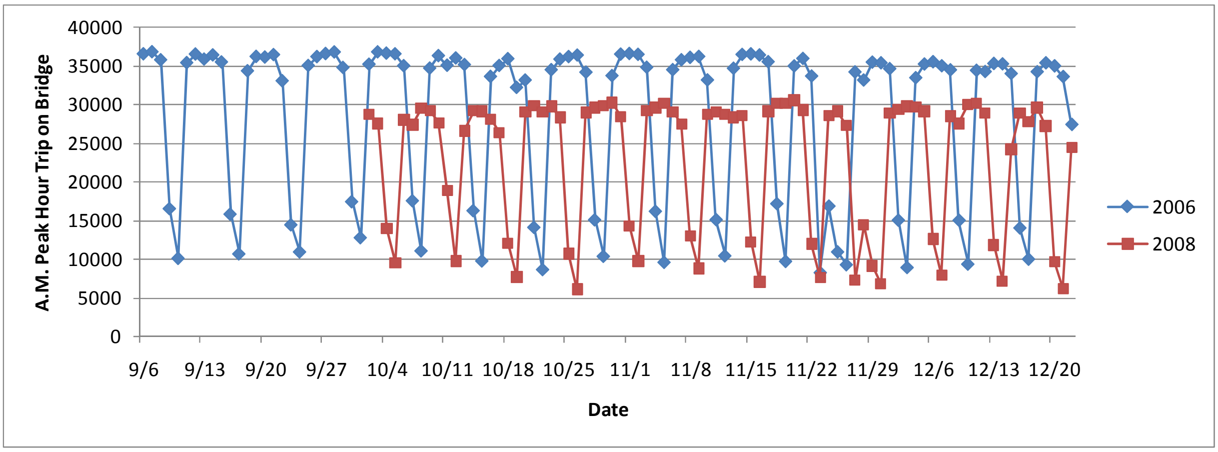


Figure 8. Morning peak hour trip on the bridge before collapse vs. after reopening

To model the flow pattern evolution during the whole process of bridge collapse and reopening, we shall use the Twin Cities Seven-County conflated planning network, which adopts separable link cost functions (BPR-type link cost functions). If we use traditional day-to-day dynamics with UE or SUE as network equilibrium, we cannot model the observed phenomenon of irreversible network change, because the final equilibrium will be definitely equal to the initial equilibrium due to the uniqueness of UE or SUE in this network. In the meanwhile, it is unrealistic to develop well-calibrated non-separable link cost functions for such a large-scale network. What is more important, as argued earlier, travelers' bounded rationality in route choice is an important issue in its own right. Therefore, we shall apply our boundedly rational day-to-day dynamic model to this scenario to model the flow evolution during the whole process of bridge collapse and reopening, and thereby model the observed phenomenon of irreversible network change.

The network used in this study, the Twin Cities Seven-County conflated planning network, contains 22,476 links, and 8,618 nodes, of which 1201 are traffic analysis zones (TAZs) generating and absorbing trips. A trip table, derived from 2006 Longitudinal Employer-Household Dynamics (LEHD) database, is adopted as the origin-destination demand data. The projected daily demands in 2006 LEHD were scaled to reflect trip-desires during morning peak hours. A demand multiplier was determined to make the initial traffic assignment result as close as possible to the observed data. We choose two study periods, from July 10, 2007 to November 2, 2007, and from August 8, 2008 to October 31, 2008, only non-holiday weekdays are considered. The first period covers the flow equilibration process after bridge collapse, and the second period covers the flow equilibration process after bridge reopening.

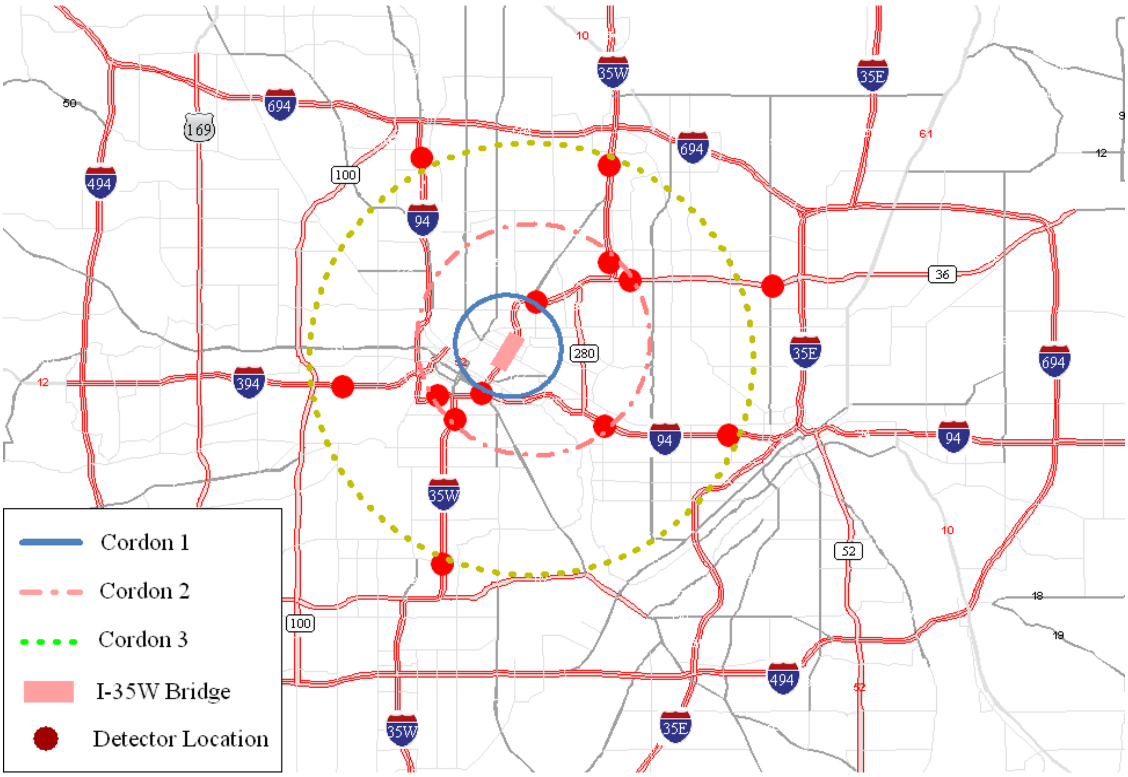


Figure 9. Three cordon circles around the I-35W Bridge

When showing the observed flow data as well as the modeled flow results, following He and Liu (2010), we use three cordons as shown in Figure 9. Specifically, we aggregate the inbound traffic volumes for each cordon and use the inbound flow evolutions of the three cordons to represent the flow evolution of the network. The three cordons represent three studying areas around the I-35W Bridge. The first cordon, shown as a blue solid circle, has a radius about half-mile, and covers the immediate adjacent area of I-35W Bridge. The second cordon, shown as a red dash-dot circle, has a radius of 2.5 miles. This cordon includes I-94 and Trunk Highway 280 (TH 280), which was designated as the detour route after the bridge collapse. The third cordon, shown as a 5.5-mile-radius green dot circle, covers the Minneapolis Central Business District and the City of Minneapolis. Major freeway routes accessing into the Minneapolis Central Business District are included. Loop detectors around the cordon lines are shown as red dots in Figure 9.

Figure 10 shows the observed flow evolution by plotting the average hourly inbound flow of the three cordons during morning peak period. For Cordon 1, the initial equilibrium inbound flow (before bridge collapse) and the final equilibrium inbound flow (after bridge reopening) are obviously different, which reflects the same phenomenon of irreversible network change as demonstrated by Figure 8. Detailed discussions on the characteristics of the flow evolutions caused by the bridge collapse and reopening were given in the empirical study of Danczyk et al. (2010). In a network traffic assignment context, He and Liu (2010) studied the bridge collapse scenario in particular (bridge reopening was not studied in their paper).

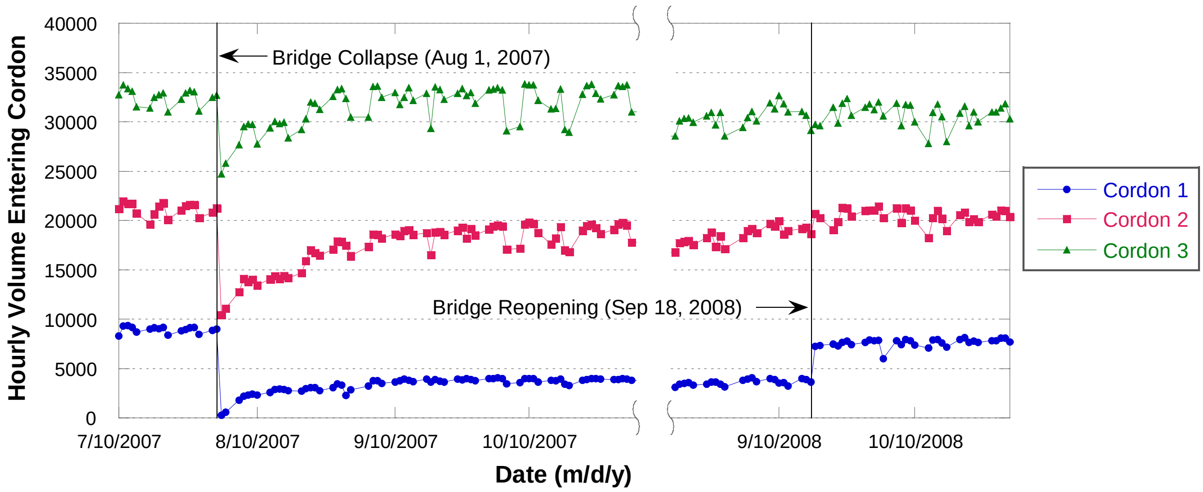


Figure 10. Flow evolution during bridge collapse and reopening (from detector data)

To apply our model to this real-world scenario, two issues need to be addressed. First, assuming the network equilibrium is initially a BRUE flow pattern, we have the non-uniqueness problem of the initial link flow, because the BRUE solution is not unique. Ideally, we should observe all link flows (or observe as many as possible) to give a best estimation of the initial link flow. In this study, however, we only have flow data on freeway links, while flow data on local arterial links are not available. Consequently, we shall run a UE traffic assignment and use the UE link flow as the initial link flow pattern. Because the UE link flow is always a BRUE flow pattern, and it has been widely used to represent long-term network equilibrium in the literature and in practice, this issue is reasonably solved. Nevertheless, it should be mentioned here that the estimation of the most-likely BRUE solution based on incomplete link counts may be an interesting future research topic and may deserve another paper.

The second issue is that, as demonstrated by He and Liu (2010), to correctly model the flow evolution after the bridge collapse, a prediction component needs to be included to capture travelers' forward-looking behavior after an unexpected network disruption. Here in our study, the bridge reopening scenario is not affected by this issue, but we need to add the prediction component when we model the equilibration process after the bridge collapse. Therefore, we shall apply our boundedly rational day-to-day dynamic in the "prediction-correction" framework proposed by He and Liu (2010). Simply speaking, to model the equilibration process after the bridge collapse, the perceived link cost in the network involves not only driver's past experiences, but also driver's anticipated congestion resulting from the loss of the bridge. Note that our definitions of bounded rationality and acceptable paths and flows directly apply in this context, i.e., we simply need to use the perceived link cost instead of real cost. We adopt the same prediction damping function as in He and Liu (2010), so that as time goes on, the initial prediction effect (on the bridge-collapse day) vanishes, and the model reduces to our boundedly rational day-to-day dynamic and thereby converges to BRUE.

With the above two issues addressed, we can apply the discrete-time version of our boundedly rational dynamic (10) to this bridge collapse and reopening scenario. The step-size parameter in (10) is synthesized into the "prediction-correction" framework[5], while the bounded rationality threshold is a unique parameter of our own model. We set the bounded rationality threshold to be 10% of the minimum OD cost, and, as mentioned earlier, let the distance measure in (9) have the Euclidean formulation. Finally, we can calculate the network flow evolution during the whole process of bridge collapse and reopening. Figure 11 shows the flow evolution pattern based on our boundedly rational day-to-day dynamic system. Comparing the modeled flow evolution shown in Figure 11 and the real-world observation shown in Figure 10, we can say that our model - as an aggregate-level traffic assignment model with only a few parameters - to a reasonable level reproduces the observed phenomenon of irreversible network change. In particular, our model gives a 29% reduction of the morning peak hour trip on the bridge after the whole process of bridge collapse and reopening, which is comparable to (although a bit larger

[5] Specifically, we set the cost updating weight in the "prediction-correction" process to be 0.6 together with the step-size parameter in (10) being 0.9. On the bridge-collapse day, the step-size parameter is set to be 1 to ensure no flow remains on the collapsed bridge on the next day.

than) the number given by detector data (22%). This result could be improved if a systematic parameter estimation process is conducted. For example, the bounded rationality threshold parameter could be systematically estimated using some form of survey method. In the current study, because our aim is to give a real-world application example of our model, rather than to provide a comprehensive case study, we just manually tried out several parameter combinations and picked one which has the best fit, which could be regarded as a simplified version of the mesh method.

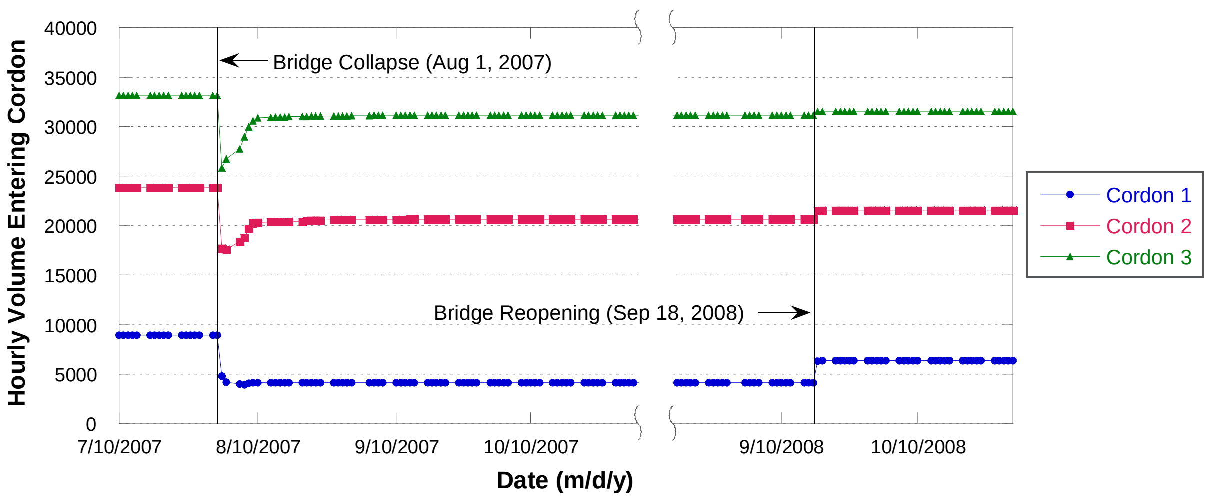


Figure 11. Flow evolution during bridge collapse and reopening (from our model)

5. Conclusions

With the objective of modeling the phenomenon of irreversible network change, we have developed a boundedly rational day-to-day dynamic model, which is a link-based model with BRUE as its fixed point. Because BRUE generally has multiple equilibrium solutions, our dynamic model can be used to model irreversible network change. Our BRUE based approach has two advantages over other UE or SUE based methods in modeling irreversible network change. First, unlike UE or SUE, the BRUE multiple equilibria do not rely on non-separable link cost functions, which makes our model applicable to real-world large-scale networks. Second, travelers' boundedly rational behavior in route choice is explicitly considered in our model.

To demonstrate the applicability of our approach, we applied our model to the I-35W Bridge collapse and reopening scenario in the Twin Cities network, where a real-world irreversible network change was observed. The results show that our method to a reasonable level reproduces the observed phenomenon of irreversible network change.

Because in this paper we just proposed a new dynamic system and demonstrated that it could model an interesting phenomenon, there remain many intriguing problems on this topic. For example, in both the small example in Section 3 and the real-world example of the I-35W Bridge case, we have observed that, a temporary capacity reduction on a link (Link 1 in the small example and the bridge in the I-35W example) can cause a permanent flow reduction on that link. This observation is somehow consistent with our intuition about bounded rationality: after being forced to change route by the link capacity reduction, some travelers may get used to new routes, and, due to bounded rationality, will not change back if the travel time saving from route switching is not big enough. If this is the case, we can have the following "proposition": *a temporary link capacity reduction will always result in a permanent link flow reduction, or in other words, the flow of a degraded link will not fully recover after link restoration*. However, we have observed counterexamples in Figure 7 that the flow of the degraded link (Link 1) can be increased after network restoration if the step-size parameter is not small. Thus one technical condition required by the "proposition" would be a small enough step-size parameter. Intuitively, when the step-size parameter is small, the flow evolution will not suddenly jump deep into the interior of the BRUE flow set, and thus the flow evolution will always converge to a point close to the boundary of the BRUE flow set. For the network degradation and restoration scenario, this means that the final equilibrium state will be close to a boundary BRUE solution (as illustrated in Figure 5), and in particular, will be closer to the degraded network equilibrium compared to the initial interior BRUE solution (as shown in Figure 5), which simply means that the flow of the degraded link will not fully recover. Intuitive and numerically easy to be verified as it is, it is not so straightforward to prove this "proposition" theoretically. To do so, we may need to look into the theoretical stability properties of the proposed dynamic system, which is an interesting topic in its own right. If the "proposition" can be theoretically proved, we still need future empirical work to verify whether the small step-size condition holds (or conditionally holds) in real world scenarios.

Another future research topic is to study how to guide the network flow pattern to move from one BRUE state to a more desirable one. Generally speaking, because the BRUE solution is not unique, from a static network equilibrium viewpoint, there is no guarantee of attainability of any specific target flow pattern by implementing any network changes (e.g. tolls, capacity changes); nevertheless, from a disequilibrium flow evolution perspective, it is possible to design certain network design strategies (e.g., toll implementation sequence) to guide the network flow pattern to evolve towards a target pattern. For example, if our conjecture is correct that the flow of a degraded link will not fully recover (under certain technical conditions), then this property can be used to guide flow evolution. These topics are left for future research.

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