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1 Approaches to improving consistency of interval fuzzy preference relations

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2

3 **Abstract**

4 This article introduces a consistency index for measuring the consistency level of an
5 interval fuzzy preference relation (IFPR). An approach is then proposed to construct an
6 additive consistent IFPR from a given inconsistent IFPR. By using a weighted averaging
7 method combining the original IFPR and the constructed consistent IFPR, a formula is
8 put forward to repair an inconsistent IFPR to generate an IFPR with acceptable
9 consistency. An iterative algorithm is subsequently developed to rectify an inconsistent
10 IFPR and derive one with acceptable consistency and weak transitivity. The proposed
11 approaches can not only improve consistency of IFPRs but also preserve the initial
12 interval uncertainty information as much as possible. Numerical examples are presented
13 to illustrate how to apply the proposed approaches.

14 *Keywords:* Interval fuzzy preference relation, Additive consistency, Acceptable
15 consistency, Weak transitivity, Decision making

16 **1. Introduction**

17 In decision analysis, a decision-maker (DM) is often asked to express his/her
18 preference ratings over objects in a pairwise comparison manner (Dong and Saaty 2014).
19 The pairwise comparison among criteria or alternatives in the analytic hierarchy process
20 (AHP) (Saaty 1980) yields multiplicative preference relations, which constitute the basis
21 to derive criteria weights and rank alternatives. To reflect vagueness in human judgment,
22 many researchers have been paying increasing attention to fuzzy preference relations in
23 recent years (Liu X. et al. 2012; Xia et al. 2013).

24 An important research topic in this area is to investigate consistency of preference
25 relations. For fuzzy preference relations, distinct transitivity definitions have been put
26 forward, such as additive transitivity, multiplicative transitivity, weak transitivity, max-
27 min transitivity, and max-max transitivity (Xu 2007). Let $R = (r_{ij})_{n \times n}$ be a fuzzy
28 preference relation, if $r_{ij} - r_{ji}$ is interpreted as the intensity of the DM's preference of the
29 object x_i over x_j , then additive consistency is a sensible vehicle to verify whether the

30 DM's judgments are contradiction-free; if the DM denotes r_{ij}/r_{ji} as its preference
31 intensity for x_i vs. x_j , then multiplicative consistency is an appropriate tool. The focus
32 of this paper is concerned with additive transitivity, which is regarded as a parallel
33 concept to the multiplicative consistent property in AHP (Herrera-Viedma et al. 2004;
34 Liu X. 2012; Xu et al. 2014) and is widely employed to characterize consistency of fuzzy
35 preference relations (Chen and Chao 2012; Chiclana et al., 2007; Herrera-Viedma et al.
36 2004, 2007; Liu X. et al. 2012; Ma et al. 2006; Xu et al. 2014). Based on additive
37 transitivity properties, some authors have proposed different approaches to improve
38 consistency of inconsistent fuzzy preference relations furnished by the DM. For example,
39 Herrera-Viedma et al. (2004) put forward an approach to construct a fuzzy preference
40 relation with additive consistency from a set of $n - 1$ preference values. Ma et al. (2006)
41 present two methods to examine weak transitivity of a fuzzy preference relation with
42 strict pairwise comparison judgments, and develop an algorithm to repair an inconsistent
43 fuzzy preference relation to reach weak transitivity. Herrera-Viedma et al. (2007)
44 introduce a consistency index to measure the consistency level (CL) of a fuzzy preference
45 relation and furnish a concept of fully additive consistency when $CL=1$. Liu X. et al.
46 (2012) consider incomplete fuzzy preference relations and develop a least square model
47 to complete an incomplete fuzzy preference relation and rectify its inconsistency based
48 on additive transitivity.

49 On the other hand, due to complexity and uncertainty in many decision problems, it is
50 hard for a DM to express his/her preference over objects with crisp values (Durbach and
51 Stewart 2012; Li and Chen 2014; Yu and Xu 2014). In this case, it is often more natural
52 to use interval fuzzy preference relations (IFPRs). The concept of IFPRs is introduced by
53 Xu (2004), in which judgment data are given as interval fuzzy numbers to characterize a
54 DM's preference degree or intensity of one object over another. In order to obtain
55 reasonable priority weights, consistency and acceptable consistency of IFPRs have been
56 studied and different methods have been designed for generating priority weights based
57 on IFPRs. For instance, Xu and Chen (2008) define additive and multiplicative consistent
58 IFPRs, in which the consistency conditions are established without accounting for
59 transitivity among three or more judgment data. Based on Xu and Chen's additive and
60 multiplicative consistency, some authors have devised different methods for generating

61 priority weights from IFPRs such as Genç et al. (2010), Lan et al. (2012), Xia and Xu
62 (2014), to name a few. Xu (2011) further proposes an approach to construct additive or
63 multiplicative consistent IFPRs by minimizing deviation between the initial and
64 constructed IFPRs. Xu et al. (2014) and Hu et al. (2014) propose revised definitions for
65 the additive consistency given by Xu and Chen (2008). Liu F. et al. (2012) adopt two
66 converted fuzzy preference relations to define an additive consistent IFPR, and develop
67 an algorithm for deriving priority weights from IFPRs. Wang and Li (2012) introduce
68 new additive and multiplicative consistency definitions for IFPRs based on interval
69 arithmetic. Wang and Li (2014) develop a multi-step goal programming method for group
70 decision making with incomplete IFPRs.

71 Consistency of preference relations plays an important role in reaching a reasonable
72 decision result. Nevertheless, it is often a challenge for a DM to provide a consistent
73 IFPR in many real-world decision situations. It is natural that highly inconsistent
74 judgment matrices may lead to misleading decision result. For IFPRs with low
75 consistency levels, they should be returned to the DMs for an update. If the DMs are
76 unavailable or unwilling to revise their original judgment information, it is helpful to
77 have an automated process to improve consistency of the original IFPRs furnished by the
78 DMs. In the case that the DMs are available to update their decision input, the results of
79 improved IFPRs can also serve as a valuable feedback and benchmark for the DMs in
80 updating their judgment. Although the approach in Xu (2011) is able to construct a
81 consistent IFPR, the consistency definitions are based on crisp weights and the derived
82 consistent IFPR may result in significant loss of information (for instance, the uncertainty
83 reflected in the interval width of the judgment may be substantially changed in the
84 conversion process). In addition, for the consistency definitions given by Xia and Xu
85 (2011) and Liu F. et al. (2012), Wang and Chen (2014) point out their technical
86 deficiency as the consistency status of an IFPR therein is sensitive to alternative
87 permutations. The fundamental motivation of this research is to address the aforesaid
88 issues. By adopting the additive consistency notion proposed by Wang and Li (2012), this
89 study focuses on improving consistency of IFPRs. The contributions of this article are
90 threefold: we first define a consistency index for IFPRs, then put forward a formula to
91 construct an additive consistent IFPR based on an inconsistent input, finally, we develop

92 a method and an algorithm to rectify an inconsistent IFPR. More specifically, a
 93 consistency index is first defined to measure the consistency level of an IFPR. For an
 94 inconsistent IFPR, an approach is then proposed to construct an additive consistent IFPR,
 95 which is employed as a reference to improve consistency of the given IFPR. By using a
 96 weighted averaging scheme combining the original IFPR and the constructed consistent
 97 IFPR, a method is put forward to repair an inconsistent IFPR to yield an IFPR with
 98 acceptable consistency. A further algorithm is developed to rectify an inconsistent IFPR
 99 to generate an IFPR with both acceptable consistency and weak transitivity.

100 The remainder of this paper is organized as follows. Section 2 provides preliminaries
 101 on consistent IFPRs and comparison of interval numbers. Section 3 defines a consistency
 102 index for IFPRs. In Section 4, an approach is proposed to construct an additive consistent
 103 IFPR based on any given IFPR. Section 5 presents two approaches to improving
 104 consistency of IFPRs. Finally, concluding remarks are furnished in Section 6.

105 2. Preliminaries

106 This section presents basic concepts of additive consistency and weak transitivity of
 107 IFPRs as well as comparison of interval numbers.

108 Consider a decision problem with a finite set of n objects, denoted by
 109 $X = \{x_1, x_2, \dots, x_n\}$, where the objects may be alternatives, criteria, attributes and so on.
 110 Let I be a real closed interval, $D(I) = \{[a^-, a^+] : a^- \leq a^+, a^-, a^+ \in I\}$. For any $x \in I$, define
 111 $x = [x, x]$.

112 Xu (2004) defines IFPRs where judgment data are expressed as interval fuzzy
 113 numbers to characterize a DM's preference degree of one object over another.

114 *Definition 2.1* (Xu 2004) An interval fuzzy preference relation (IFPR) \bar{R} on the set
 115 X is characterized by an interval fuzzy preference matrix $\bar{R} = (\bar{r}_{ij})_{n \times n} \subset X \times X$, where

$$116 \quad \bar{r}_{ij} = [r_{ij}^-, r_{ij}^+] \in D([0, 1]), \bar{r}_{ji} = 1 - \bar{r}_{ij} = [1 - r_{ij}^+, 1 - r_{ij}^-], \bar{r}_{ii} = [0.5, 0.5], \quad i, j = 1, 2, \dots, n \quad (2.1)$$

117 and \bar{r}_{ij} indicates the interval-valued fuzzy preference of x_i over x_j . r_{ij}^- and r_{ij}^+ are the
 118 lower and upper bounds of \bar{r}_{ij} , respectively.

119 As commented in Section 1, the additive consistency definitions introduced by Xu and
 120 Chen (2008), Xu et al. (2014) and Hu et al. (2014) are based on crisp weights, and the

121 consistency condition established therein fails to account for transitivity among three or
 122 more judgment data. To address this issue, Wang and Li (2012) put forward a new
 123 additive consistency notion for IFPRs by using interval arithmetic and the definition is
 124 furnished below.

125 *Definition 2.2* (Wang and Li 2012) An IFPR $\bar{R} = (\bar{r}_{ij})_{n \times n}$ is additive consistent if the
 126 following additive transitivity is satisfied

$$127 \quad \bar{r}_{ij} + \bar{r}_{jk} + \bar{r}_{ki} = \bar{r}_{kj} + \bar{r}_{ji} + \bar{r}_{ik} \quad \text{for all } i, j, k = 1, 2, \dots, n \quad (2.2)$$

128 To compare two interval numbers $\bar{a} = [a^-, a^+]$ and $\bar{b} = [b^-, b^+]$, where $a^-, b^- \geq 0$, the
 129 notion of likelihood is introduced. Let $\bar{a} \geq \bar{b}$ represent that \bar{a} is no smaller than \bar{b} . The
 130 likelihood of $\bar{a} \geq \bar{b}$ is defined as (Xu and Chen 2008)

$$131 \quad p(\bar{a} \geq \bar{b}) = \frac{\max\{0, a^+ - b^-\} - \max\{0, a^- - b^+\}}{a^+ - a^- + b^+ - b^-} \quad (2.3)$$

132 Some useful properties about likelihood $p(\bar{a} \geq \bar{b})$ are summarized as follows:

133 (a) $0 \leq p(\bar{a} \geq \bar{b}) \leq 1$;

134 (b) $p(\bar{a} \geq \bar{b}) + p(\bar{b} \geq \bar{a}) = 1$;

135 (c) $p(\bar{a} \geq \bar{b}) = 1$ if and only if $a^- \geq b^+$;

136 (d) $p(\bar{a} \geq \bar{b}) = 0$ if and only if $a^+ \leq b^-$;

137 (e) $p(\bar{a} \geq \bar{b}) \geq 0.5$ if and only if $\frac{a^- + a^+}{2} \geq \frac{b^- + b^+}{2}$. Especially, $p(\bar{a} \geq \bar{b}) = 0.5$ if and

138 only if $\frac{a^- + a^+}{2} = \frac{b^- + b^+}{2}$;

139 (f) For any interval numbers \bar{a}, \bar{b} and \bar{c} , if $p(\bar{a} \geq \bar{b}) \geq 0.5$ and $p(\bar{b} \geq \bar{c}) \geq 0.5$, then
 140 $p(\bar{a} \geq \bar{c}) \geq 0.5$.

141 According to the aforesaid properties of the likelihood concept, for an IFPR
 142 $\bar{R} = (\bar{r}_{ij})_{n \times n}$, $p(\bar{r}_{ij} \geq [0.5, 0.5]) = 0.5$ indicates a DM's indifference between x_i and x_j ,
 143 $p(\bar{r}_{ij} \geq [0.5, 0.5]) > 0.5$ signifies that x_i is preferred to x_j with a degree of $p(\bar{r}_{ij} \geq [0.5, 0.5])$,
 144 $p(\bar{r}_{ij} \geq [0.5, 0.5]) < 0.5$ describes that x_j is preferred to x_i with a degree of
 145 $1 - p(\bar{r}_{ij} \geq [0.5, 0.5])$, $p(\bar{r}_{ij} \geq [0.5, 0.5]) = 1$ means that x_i is absolutely preferred to x_j , and

146 $p(\bar{r}_{ij} \geq [0.5, 0.5]) = 0$ expresses that x_j is absolutely preferred to x_i .

147 Based on the likelihood definition and properties, Wang and Li (2012) introduce
 148 weak transitivity for IFPRs as follows.

149 *Definition 2.3* (Wang and Li 2012) An IFPR $\bar{R} = (\bar{r}_{ij})_{n \times n}$ is weakly transitive if
 150 $p(\bar{r}_{ik} \geq [0.5, 0.5]) \geq 0.5$ and $p(\bar{r}_{kj} \geq [0.5, 0.5]) \geq 0.5$ imply $p(\bar{r}_{ij} \geq [0.5, 0.5]) \geq 0.5$, for all
 151 $i, j, k = 1, 2, \dots, n$.

152 Base on Definition 2.2, Wang (2014) provides the following property to judge whether
 153 an IFPR is consistent.

154 *Lemma 2.1* (Wang 2014) An IFPR $\bar{R} = (\bar{r}_{ij})_{n \times n}$ is additive consistent if and only if

$$155 \quad r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+ = 3 \quad \forall i, j, k = 1, 2, \dots, n \quad (2.4)$$

156 3. Consistency measure

157 By Lemma 2.1, if $\bar{R} = (\bar{r}_{ij})_{n \times n}$ is an additive consistent IFPR, we have

158 $r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+ = 3 \quad \forall i, j, k = 1, 2, \dots, n$. However, if \bar{R} is inconsistent, the

159 preference values in \bar{R} will not satisfy (3.1). In other words, there exist some differences

160 between $r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+$ and 3 for some $i, j, k = 1, 2, \dots, n$. As $0 \leq r_{ij}^- \leq r_{ij}^+ \leq 1$ for

161 all $i, j = 1, 2, \dots, n$, one has $0 \leq |r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+ - 3| \leq 3$. Therefore, we can

162 define a consistency measure for an IFPR as follows.

163 *Definition 3.1* A consistency index of an IFPR $\bar{R} = (\bar{r}_{ij})_{n \times n}$ is defined as

$$164 \quad CI(\bar{R}) = 1 - \frac{1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n |r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+ - 3| \quad (3.1)$$

165 It is obvious that $0 \leq CI(\bar{R}) \leq 1$. If $CI(\bar{R}) = 1$, then the IFPR \bar{R} is additive consistent;

166 otherwise, \bar{R} is inconsistent, and the larger the $CI(\bar{R})$, the closer the \bar{R} is to a consistent

167 IFPR. According to the actual situation, if a DM can accept limited inconsistency in the

168 judgment, he/she may give a consistency threshold $0 < t < 1$ for $CI(\bar{R})$. This threshold

169 presumably reflects the DM's tolerance for inconsistency and should be furnished by the

170 DM upon examining the specific decision circumstances. If $CI(\bar{R}) \geq t$, the IFPR \bar{R} is

171 deemed acceptably consistent; otherwise, the consistency level of \bar{R} is considered
 172 unacceptable and, hence, \bar{R} should be rectified to ensure rationality of decisions.

173 If $r_{ij}^- = r_{ij}^+$, $i, j = 1, 2, \dots, n$, the IFPR \bar{R} is reduced to a fuzzy preference relation, and
 174 (3.1) is equivalent to the consistency index of a complete fuzzy preference relation
 175 proposed by Herrera-Viedma et al. (2007).

176 As per the additive consistency of IFPRs in Definition 2.2 and weak transitivity of
 177 IFPRs in Definition 2.3, we have the following theorem.

178 *Theorem 3.1* If an IFPR $\bar{R} = (\bar{r}_{ij})_{n \times n}$ is additive consistent, then \bar{R} is weakly transitive.

179 *Proof.* According to property (e) of the likelihood concept in Section 2, if
 180 $p(\bar{r}_{ik} \geq [0.5, 0.5]) \geq 0.5$ and $p(\bar{r}_{kj} \geq [0.5, 0.5]) \geq 0.5$, we have $r_{ik}^- + r_{ik}^+ \geq (0.5 + 0.5) = 1$ and
 181 $r_{kj}^- + r_{kj}^+ \geq (0.5 + 0.5) = 1$, $\forall i, j, k \in \{1, 2, \dots, n\}$. As per the reciprocal property of $\bar{r}_{ji} = 1 - \bar{r}_{ij}$,
 182 one can get $1 - r_{ki}^+ + 1 - r_{ki}^- \geq 1$ and $1 - r_{jk}^+ + 1 - r_{jk}^- \geq 1$, $\forall i, j, k \in \{1, 2, \dots, n\}$. It follows that
 183 $r_{ki}^- + r_{ki}^+ \leq 1$ and $r_{jk}^- + r_{jk}^+ \leq 1$, $\forall i, j, k \in \{1, 2, \dots, n\}$.

184 On the other hand, as \bar{R} is additive consistent, it follows from Lemma 2.1
 185 $r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+ = 3 \forall i, j, k = 1, 2, \dots, n$. Then, one has $r_{ij}^- + r_{ij}^+ \geq 1$, i.e.,

186 $\frac{r_{ij}^- + r_{ij}^+}{2} \geq 0.5$. As per the property (e) of the likelihood concept, we get

187 $p(\bar{r}_{ij} \geq [0.5, 0.5]) \geq 0.5$. Therefore, $\bar{R} = (\bar{r}_{ij})_{n \times n}$ is weakly transitive. ■

188 For any two IFPRs $\bar{R} = (\bar{r}_{ij})_{n \times n} = ([r_{ij}^-, r_{ij}^+])_{n \times n}$ and $\bar{R}' = (\bar{r}'_{ij})_{n \times n} = ([r'_{ij}^-, r'_{ij}^+])_{n \times n}$, let

$$189 \quad d(\bar{R}, \bar{R}') = \frac{1}{2n(n-1)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n (|r_{ij}^- - r'_{ij}^-| + |r_{ij}^+ - r'_{ij}^+|) \quad (3.2)$$

190 denote the mean absolute deviation for all off-diagonal intervals between \bar{R} and \bar{R}' . The
 191 smaller the value $d(\bar{R}, \bar{R}')$, the closer the \bar{R} is to \bar{R}' . Especially, if $d(\bar{R}, \bar{R}') = 0$, \bar{R} is
 192 the same as \bar{R}' .

193 4. An approach to constructing consistent IFPRs

194 This section develops a framework to construct an additive consistent IFPR based on
 195 any given inconsistent IFPR.

196 For a given IFPR $\bar{R} = (\bar{r}_{ij})_{n \times n}$, define the (i, j) entry of $\hat{\bar{R}} = (\hat{r}_{ij})_{n \times n} = ([\hat{r}_{ij}^-, \hat{r}_{ij}^+])_{n \times n}$ as
 197 follows

$$\begin{aligned} \hat{r}_{ij} = [\hat{r}_{ij}^-, \hat{r}_{ij}^+] = & \left[0.5 + \frac{1}{2n} \left(\sum_{l=1}^n r_{il}^- + \sum_{l=1}^n r_{il}^+ - \sum_{l=1}^n r_{jl}^- - \sum_{l=1}^n r_{jl}^+ \right) - \frac{r_{ij}^+ - r_{ij}^-}{2}, \right. \\ 198 & \left. 0.5 + \frac{1}{2n} \left(\sum_{l=1}^n r_{il}^- + \sum_{l=1}^n r_{il}^+ - \sum_{l=1}^n r_{jl}^- - \sum_{l=1}^n r_{jl}^+ \right) + \frac{r_{ij}^+ - r_{ij}^-}{2} \right] \end{aligned} \quad (4.1)$$

199 for all $i, j = 1, 2, \dots, n$. The following two theorems reveal some useful properties of \hat{r}_{ij} .

200 *Theorem 4.1* Let $\bar{R} = (\bar{r}_{ij})_{n \times n}$ be an IFPR and $\hat{r}_{ij} = [\hat{r}_{ij}^-, \hat{r}_{ij}^+]$ ($i, j = 1, 2, \dots, n$) be
 201 defined by (4.1), then

- 202 (i) $\hat{r}_{ij}^- \leq \hat{r}_{ij}^+$, $\hat{r}_{ii}^- = \hat{r}_{ii}^+ = 0.5 \quad \forall i, j = 1, 2, \dots, n$.
 203 (ii) $\hat{r}_{ji}^- = 1 - \hat{r}_{ij}^+$, i.e., $\hat{r}_{ji}^- = 1 - \hat{r}_{ij}^+$ and $\hat{r}_{ji}^+ = 1 - \hat{r}_{ij}^- \quad \forall i, j = 1, 2, \dots, n$.
 204 (iii) $\hat{r}_{ij}^+ - \hat{r}_{ij}^- = r_{ij}^+ - r_{ij}^- \quad \forall i, j = 1, 2, \dots, n$.
 205 (iv) $\hat{r}_{ij}^- + \hat{r}_{jk}^- + \hat{r}_{ki}^- = \hat{r}_{kj}^- + \hat{r}_{ji}^- + \hat{r}_{ik}^- \quad \forall i, j, k = 1, 2, \dots, n$.

206 *Proof.* (i) - (iii) can be immediately derived from (4.1) and, hence, the proof is only
 207 provided for (iv).

208 Since $\bar{R} = (\bar{r}_{ij})_{n \times n}$ is an IFPR, then we have $r_{ji}^- = 1 - r_{ij}^+$, $r_{ji}^+ = 1 - r_{ij}^- \quad \forall i, j = 1, 2, \dots, n$. It
 209 follows from (4.1) that

$$\begin{aligned} \hat{r}_{ij}^- + \hat{r}_{jk}^- + \hat{r}_{ki}^- &= \frac{3}{2} - \frac{r_{ij}^+ - r_{ij}^- + r_{jk}^+ - r_{jk}^- + r_{ki}^+ - r_{ki}^-}{2} = \frac{3}{2} - \frac{1 - r_{ji}^- - (1 - r_{ji}^+) + 1 - r_{kj}^- - (1 - r_{kj}^+) + 1 - r_{ik}^- - (1 - r_{ik}^+)}{2} \\ 210 &= \frac{3}{2} - \frac{r_{kj}^+ - r_{kj}^- + r_{ji}^+ - r_{ji}^- + r_{ik}^+ - r_{ik}^-}{2} = \hat{r}_{kj}^- + \hat{r}_{ji}^- + \hat{r}_{ik}^- \end{aligned}$$

211 Similarly, from (4.1), one can obtain $\hat{r}_{ij}^+ + \hat{r}_{jk}^+ + \hat{r}_{ki}^+ = \hat{r}_{kj}^+ + \hat{r}_{ji}^+ + \hat{r}_{ik}^+$. Therefore,

212 $\hat{r}_{ij}^- + \hat{r}_{jk}^- + \hat{r}_{ki}^- = \hat{r}_{kj}^- + \hat{r}_{ji}^- + \hat{r}_{ik}^-$. This completes the proof of Theorem 4.1. \blacksquare

213 *Theorem 4.2* If $\bar{R} = (\bar{r}_{ij})_{n \times n}$ is an additive consistent IFPR, then $\hat{r}_{ij}^- = \bar{r}_{ij}$
 214 $\forall i, j = 1, 2, \dots, n$.

215 *Proof.* Since \bar{R} is additive consistent, it follows from Lemma 2.1 that
 216

$$r_{il}^- + r_{il}^+ + r_{ij}^- + r_{ij}^+ = 3 - (r_{ji}^- + r_{ji}^+) \quad \forall i, j, l = 1, 2, \dots, n$$

217 Then,

$$\begin{aligned}
& \sum_{l=1}^n r_{il}^- + \sum_{l=1}^n r_{il}^+ - \sum_{l=1}^n r_{jl}^- - \sum_{l=1}^n r_{jl}^+ = \sum_{l=1}^n (r_{il}^- + r_{il}^+ - r_{jl}^- - r_{jl}^+) = \sum_{l=1}^n (r_{il}^- + r_{il}^+ - (1 - r_{ij}^+) - (1 - r_{ij}^-)) \\
& = \sum_{l=1}^n (r_{il}^- + r_{il}^+ + r_{ij}^- + r_{ij}^+ - 2) = \sum_{l=1}^n (3 - (r_{ji}^- + r_{ji}^+) - 2) = \sum_{l=1}^n (1 - (1 - r_{ij}^+ + 1 - r_{ij}^-)) \\
& = \sum_{l=1}^n (r_{ij}^- + r_{ij}^+ - 1) = n(r_{ij}^- + r_{ij}^+) - n
\end{aligned}$$

218

219 As per (4.1), we have $\hat{r}_{ij}^- = 0.5 + \frac{(r_{ij}^- + r_{ij}^+)}{2} - 0.5 - \frac{(r_{ij}^+ - r_{ij}^-)}{2} = r_{ij}^-$ and $\hat{r}_{ij}^+ = 0.5 + \frac{(r_{ij}^- + r_{ij}^+)}{2} -$

220 $0.5 + \frac{(r_{ij}^+ - r_{ij}^-)}{2} = r_{ij}^+$. It is verified that $\hat{r}_{ij}^- = \bar{r}_{ij}$. ■

221 Theorem 4.1 demonstrates that \hat{R} is an additive consistent IFPR if $0 \leq \hat{r}_{ij}^- \leq \hat{r}_{ij}^+ \leq 1$
222 $\forall i, j = 1, 2, \dots, n$ and the interval width of each element in \hat{R} remains the same for the
223 corresponding element in \bar{R} . Theorem 4.2 further confirms that $\hat{R} = \bar{R}$ if \bar{R} is additive
224 consistent. Therefore, a simple way to tell whether \bar{R} is additive consistent is to compute
225 $\hat{r}_{ij}^- \forall i < j = 1, 2, \dots, n$ and examine if $\hat{r}_{ij}^- = \bar{r}_{ij}$. This judgment method only needs to compute
226 $n \cdot (n-1) / 2$ values in contrast to the direct application of Definition 2.2 that has to
227 entertain n^3 data.

228 If \bar{R} is not additive consistent, we may obtain a matrix $\hat{R} = ([\hat{r}_{ij}^-, \hat{r}_{ij}^+])_{n \times n}$ with entries
229 outside $D([0,1])$. In this case, \hat{R} is not an IFPR. To construct an additive consistent
230 IFPR from \bar{R} , interval values $[\hat{r}_{ij}^-, \hat{r}_{ij}^+]$ have to be further converted to intervals on
231 $D([0,1])$. This conversion process should presumably preserve additive transitivity and
232 the complementary property in the sense of $\hat{r}_{ji}^- = 1 - \hat{r}_{ij}^+$.

233 Let

$$c = \begin{cases} 1, & \text{if } \hat{r}_{ij}^+ \leq 1, \forall i, j = 1, 2, \dots, n \\ \max \{ \hat{r}_{ij}^+ \mid \hat{r}_{ij}^+ > 1, i, j = 1, 2, \dots, n \}, & \text{Otherwise} \end{cases} \quad (4.2)$$

235 It is obvious that $c \geq 1$, and $\hat{r}_{ij}^- \leq \hat{r}_{ij}^+ \leq c \forall i, j = 1, 2, \dots, n$. According to Theorem 4.1, we
236 have $\hat{r}_{ij}^- = 1 - \hat{r}_{ji}^+ \forall i, j = 1, 2, \dots, n$. Thus, one can obtain $1 - c \leq 1 - \hat{r}_{ji}^+ = \hat{r}_{ij}^- \leq \hat{r}_{ij}^+ \leq c$

237 $\forall i, j = 1, 2, \dots, n$. Therefore, all elements in $\hat{R} = ([\hat{r}_{ij}^-, \hat{r}_{ij}^+])_{n \times n}$ should lie between
 238 $[1-c, 1-c]$ and $[c, c]$, i.e., $[\hat{r}_{ij}^-, \hat{r}_{ij}^+] \in D([1-c, c]) \quad \forall i, j = 1, 2, \dots, n$.

239 In order to convert \hat{R} into an additive consistent IFPR, an appropriate transformation
 240 function $\varphi: D([1-c, c]) \rightarrow D([0, 1])$ should possess the following properties:

241 (i) $\varphi([1-c, 1-c]) = [0, 0]$.

242 (ii) $\varphi([c, c]) = [1, 1]$.

243 (iii) $\varphi([0.5, 0.5]) = [0.5, 0.5]$

244 (iv) $\varphi(\bar{x}) = 1 - \varphi(1 - \bar{x}) \quad \forall \bar{x} \in D([1-c, c])$.

245 (v) $\forall \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}_1, \bar{y}_2, \bar{y}_3 \in D([1-c, c])$, if $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = \bar{y}_1 + \bar{y}_2 + \bar{y}_3$, then $\varphi(\bar{x}_1) + \varphi(\bar{x}_2) +$
 246 $\varphi(\bar{x}_3) = \varphi(\bar{y}_1) + \varphi(\bar{y}_2) + \varphi(\bar{y}_3)$.

247 (i) and (ii) ensure that the transformation function should be able to convert the
 248 smallest interval $[1-c, 1-c]$ and the largest interval $[c, c]$ into $[0, 0]$ and $[1, 1]$ on
 249 $D([0, 1])$, respectively. (iii) expects that $\varphi(\cdot)$ maintain indifference to be $[0.5, 0.5]$ after
 250 conversion. (iv) requires that $\varphi(\cdot)$ keep the complementary property in the sense of
 251 interval arithmetic. The last desired property (v) guarantees that additive transitivity
 252 remains after $\varphi(\cdot)$ is applied. If a transformation function satisfies these five properties,
 253 the following theorem immediately follows.

254 *Theorem 4.3* Let $\bar{R} = (\bar{r}_{ij})_{n \times n}$ be an IFPR, then $\hat{R} = (\varphi([\hat{r}_{ij}^-, \hat{r}_{ij}^+]))_{n \times n}$ is an additive
 255 consistent IFPR.

256 Next, similar to the function furnished in Herrera-Viedma et al. (2004) for fuzzy
 257 preference relations, the following transformation function with the aforesaid desired
 258 properties is provided for handling IFPRs. Let

259
$$\varphi(\bar{x}) = \left[\frac{x^- + c - 1}{2c - 1}, \frac{x^+ + c - 1}{2c - 1} \right] \quad \forall \bar{x} = [x^-, x^+] \in D([1-c, c]) \quad (4.3)$$

260 It is apparent that this function satisfies (i), (ii) and (iii). As for (iv), since

261
$$\varphi(\bar{x}) = \left[\frac{x^- + c - 1}{2c - 1}, \frac{x^+ + c - 1}{2c - 1} \right] = 1 - \left[\frac{1 - x^+ + c - 1}{2c - 1}, \frac{1 - x^- + c - 1}{2c - 1} \right] = 1 - \varphi([1 - x^+, 1 - x^-])$$

$$= 1 - \varphi(1 - \bar{x})$$

262 (iv) is thus verified. Moreover, if $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = \bar{y}_1 + \bar{y}_2 + \bar{y}_3$, (v) is confirmed as

$$263 \quad \begin{aligned} \varphi(\bar{x}_1) + \varphi(\bar{x}_2) + \varphi(\bar{x}_3) &= \left[\frac{x_1^- + x_2^- + x_3^- + 3c - 3}{2c - 1}, \frac{x_1^+ + x_2^+ + x_3^+ + 3c - 3}{2c - 1} \right] \\ &= \left[\frac{y_1^- + y_2^- + y_3^- + 3c - 3}{2c - 1}, \frac{y_1^+ + y_2^+ + y_3^+ + 3c - 3}{2c - 1} \right] = \varphi(\bar{y}_1) + \varphi(\bar{y}_2) + \varphi(\bar{y}_3) \end{aligned}$$

264 After applying the transformation function (4.3), \hat{r}_{ij} is converted to \hat{r}'_{ij} as shown below

$$265 \quad \hat{r}'_{ij} = [\hat{r}'_{ij}, \hat{r}'_{ij}] = \varphi([\hat{r}_{ij}, \hat{r}_{ij}]) = \left[\frac{\hat{r}_{ij}^- + c - 1}{2c - 1}, \frac{\hat{r}_{ij}^+ + c - 1}{2c - 1} \right] \quad (4.4)$$

266 where c is defined by (4.2).

267 *Corollary 4.1* Assume that the elements of $\hat{R}' = \varphi(\hat{R}) = (\hat{r}'_{ij})_{n \times n}$ are defined by (4.4),

268 then \hat{R}' is an additive consistent IFPR and $\hat{r}'_{ij} + \hat{r}'_{ij} = \frac{r_{ij}^+ - r_{ij}^-}{2c - 1} \quad \forall i, j = 1, 2, \dots, n$.

269 *Proof.* It can be obtained from Theorem 4.3 that \hat{R}' is an additive consistent IFPR. By

270 Theorem 4.1, we have $\hat{r}'_{ij} + \hat{r}'_{ij} = r_{ij}^+ - r_{ij}^- \quad \forall i, j = 1, 2, \dots, n$. It follows that $\hat{r}'_{ij} + \hat{r}'_{ij} =$

$$271 \quad \frac{\hat{r}'_{ij} + \hat{r}'_{ij}}{2c - 1} = \frac{r_{ij}^+ - r_{ij}^-}{2c - 1} . \quad \blacksquare$$

272 Corollary 4.1 shows that an additive consistent IFPR can be constructed from any

273 given \bar{R} . If \bar{R} is additive consistent, the constructed IFPR $\hat{R}' = \bar{R}$. For a given additive

274 inconsistent IFPR \bar{R} , if $c = 1$, the interval widths of each element in the constructed

275 consistent IFPR \hat{R}' is equal to that of the corresponding element in the original IFPR \bar{R} ;

276 if $c > 1$, the proposed method scales down the interval widths of each element in \bar{R} by a

277 common factor $\frac{1}{2c - 1}$. As the width of an interval is a natural way to gauge interval

278 uncertainty, the constructed consistent IFPR \hat{R}' is able to keep the original interval

279 uncertainty in terms of their widths if all converted elements in \hat{R}' fall within $D([0, 1])$. In

280 the case that some elements in \hat{R}' have an upper bound above 1 or a lower bound below 0,

281 this conversion process yields an \hat{R}' that proportionally scales down the largest upper

282 bound to 1 and scales up the smallest negative lower bound to 0, thereby preserving
 283 interval uncertainty as much as possible.

284 Next, a numerical example is presented to show how to apply the proposed method.

285 **Example 1.** Consider the following three IFPRs,

$$286 \quad \bar{R}_1 = \begin{bmatrix} [0.5, 0.5] & [0.4, 0.5] & [0.5, 0.6] & [0.4, 0.5] \\ [0.5, 0.6] & [0.5, 0.5] & [0.5, 0.6] & [0.6, 0.7] \\ [0.4, 0.5] & [0.4, 0.5] & [0.5, 0.5] & [0.6, 0.8] \\ [0.5, 0.6] & [0.3, 0.4] & [0.2, 0.4] & [0.5, 0.5] \end{bmatrix}$$

$$287 \quad \bar{R}_2 = \begin{bmatrix} [0.5, 0.5] & [0.1, 0.3] & [0.8, 0.9] & [0.5, 0.6] \\ [0.7, 0.9] & [0.5, 0.5] & [0.7, 0.9] & [0.9, 1] \\ [0.1, 0.2] & [0.1, 0.3] & [0.5, 0.5] & [0.8, 0.9] \\ [0.4, 0.5] & [0, 0.1] & [0.1, 0.2] & [0.5, 0.5] \end{bmatrix}$$

$$288 \quad \bar{R}_3 = \begin{bmatrix} [0.5, 0.5] & [0.4, 0.5] & [0.9, 1] & [0.9, 1] \\ [0.5, 0.6] & [0.5, 0.5] & [0.3, 0.4] & [0.95, 1] \\ [0, 0.1] & [0.6, 0.7] & [0.5, 0.5] & [0.95, 1] \\ [0, 0.1] & [0, 0.05] & [0, 0.05] & [0.5, 0.5] \end{bmatrix}$$

289 For \bar{R}_1 , by using (4.1), one obtains the following transformation matrix:

$$290 \quad \hat{\bar{R}}_1 = \begin{bmatrix} [0.5, 0.5] & [0.37500, 0.47500] & [0.41250, 0.51250] & [0.51250, 0.61250] \\ [0.52500, 0.62500] & [0.5, 0.5] & [0.48750, 0.58750] & [0.58750, 0.68750] \\ [0.48750, 0.58750] & [0.41250, 0.51250] & [0.5, 0.5] & [0.50000, 0.70000] \\ [0.38750, 0.48750] & [0.31250, 0.41250] & [0.30000, 0.50000] & [0.5, 0.5] \end{bmatrix}$$

291 Since all elements of $\hat{\bar{R}}_1$ are in $D([0, 1])$, as per (4.2), we have $c = 1$. Therefore, the
 292 constructed additive consistent IFPR $\hat{\bar{R}}_1 = \hat{\bar{R}}_1$. It can be easily verified that the widths of
 293 the intervals in $\hat{\bar{R}}_1$ are equal to the widths of the corresponding elements in \bar{R}_1 .

294 For \bar{R}_2 and \bar{R}_3 , by using (4.1), the following transformation matrices are derived:

$$295 \quad \hat{\bar{R}}_2 = \begin{bmatrix} [0.5, 0.5] & [0.16250, 0.36250] & [0.55000, 0.65000] & [0.68750, 0.78750] \\ [0.63750, 0.83750] & [0.5, 0.5] & [0.7375, 0.9375] & [0.92500, 1.02500] \\ [0.35000, 0.45000] & [0.06250, 0.26250] & [0.5, 0.5] & [0.58750, 0.68750] \\ [0.21250, 0.31250] & [-0.0250, 0.07500] & [0.31250, 0.41250] & [0.5, 0.5] \end{bmatrix}$$

$$\hat{\bar{R}}_3 = \begin{bmatrix} [0.5, 0.5] & [0.56875, 0.66875] & [0.61875, 0.71875] & [1.01250, 1.11250] \\ [0.33125, 0.43125] & [0.5, 0.5] & [0.50000, 0.60000] & [0.91875, 0.96875] \\ [0.28125, 0.38125] & [0.40000, 0.50000] & [0.5, 0.5] & [0.86875, 0.91875] \\ [-0.1125, -0.0125] & [0.03125, 0.08125] & [0.08125, 0.13125] & [0.5, 0.5] \end{bmatrix}$$

297 In $\hat{\bar{R}}_2$, the upper bound of \hat{r}_{24}^+ is greater than 1 (correspondingly, the lower bound of \hat{r}_{42}^-
 298 is less than 0). In $\hat{\bar{R}}_3$, both the upper and lower bounds of \hat{r}_{14}^+ are greater than 1
 299 (correspondingly, the upper and lower bounds of \hat{r}_{41}^- are both less than 0). Based on (4.2),
 300 their corresponding values of c are 1.025 and 1.1125, respectively. As such, the resulting
 301 transformation functions are as follows

$$\varphi([\hat{r}_{ij}^-, \hat{r}_{ij}^+]) = \left[\frac{\hat{r}_{ij}^- + 0.025}{1.05}, \frac{\hat{r}_{ij}^+ + 0.025}{1.05} \right] \quad \varphi([\hat{r}_{ij}^-, \hat{r}_{ij}^+]) = \left[\frac{\hat{r}_{ij}^- + 0.1125}{1.225}, \frac{\hat{r}_{ij}^+ + 0.1125}{1.225} \right]$$

303 Based on (4.4), the constructed consistent IFPRs based on \bar{R}_2 and \bar{R}_3 are obtained
 304 as

$$\hat{\bar{R}}_2 = \begin{bmatrix} [0.5, 0.5] & [0.17857, 0.36905] & [0.54762, 0.64286] & [0.67857, 0.77381] \\ [0.63095, 0.82143] & [0.5, 0.5] & [0.72619, 0.91667] & [0.90476, 1.00000] \\ [0.35714, 0.45238] & [0.08333, 0.27381] & [0.5, 0.5] & [0.58333, 0.67857] \\ [0.22619, 0.32143] & [0.00000, 0.09524] & [0.32143, 0.41667] & [0.5, 0.5] \end{bmatrix}$$

$$\hat{\bar{R}}_3 = \begin{bmatrix} [0.5, 0.5] & [0.55612, 0.63776] & [0.59694, 0.67857] & [0.91876, 1.00000] \\ [0.36224, 0.44388] & [0.5, 0.5] & [0.50000, 0.58163] & [0.84184, 0.88265] \\ [0.32143, 0.40306] & [0.41837, 0.50000] & [0.5, 0.5] & [0.80102, 0.84184] \\ [0.00000, 0.08163] & [0.11735, 0.15816] & [0.15816, 0.19898] & [0.5, 0.5] \end{bmatrix}$$

307 For the final constructed consistent IFPRs $\hat{\bar{R}}_2$ and $\hat{\bar{R}}_3$, computational results indicate that
 308 the widths of the original interval judgments in \bar{R}_2 and \bar{R}_3 have been scaled down by a
 309 factor of 1/1.05 and 1/1.225, respectively. By employing (3.2), one can determine the
 310 mean absolute deviations for all off-diagonal intervals between the original IFPRs and
 311 their corresponding constructed consistent IFPRs as follows:

$$312 \quad d(\bar{R}_1, \hat{\bar{R}}_1) = 0.05833, \quad d(\bar{R}_2, \hat{\bar{R}}_2) = 0.1246, \quad d(\bar{R}_3, \hat{\bar{R}}_3) = 0.15425$$

313 5. Approaches to improving consistency of IFPRs

314 The proposed approach in Section 4 is able to construct an additive consistent IFPR
315 $\hat{\bar{R}}$ based on any given inconsistent IFPR \bar{R} . However, this consistency comes at a cost
316 as the mean absolute deviation between \bar{R} and $\hat{\bar{R}}$ tends to be high. In many decision
317 situations, a DM may relax this consistency requirement as long as the inconsistency is
318 restricted to an acceptable level or the rectified IFPR possesses the weak transitivity
319 property. Presumably, this relaxation will result in an IFPR with a smaller mean absolute
320 deviation from the original IFPR \bar{R} . Similar to the treatment in Ma et al. (2006) for fuzzy
321 preference relations, a weighted averaging scheme combining $\hat{\bar{R}}$ and \bar{R} is proposed as
322 follows:

$$323 \quad \tilde{R}(\lambda) = (\tilde{r}_{ij}(\lambda))_{n \times n} = (1 - \lambda)\bar{R} + \lambda\hat{\bar{R}} \quad (5.1)$$

324 where λ is a weight with $\lambda \in [0, 1]$, $\hat{\bar{R}} = (\hat{r}_{ij}^+)$ is defined by (4.4) and
325 $\tilde{r}_{ij}(\lambda) = (1 - \lambda)\bar{r}_{ij} + \lambda\hat{r}_{ij}^+$ for all $i, j = 1, 2, \dots, n$.

326 As \bar{R} and $\hat{\bar{R}}$ are IFPRs, according to interval arithmetic and Definition 2.1 in
327 Section 2, it is easy to prove the following result.

328 *Theorem 5.1* Assume that $\tilde{R}(\lambda) = (\tilde{r}_{ij}(\lambda))_{n \times n} = \left(\left[\tilde{r}_{ij}^-(\lambda), \tilde{r}_{ij}^+(\lambda) \right] \right)_{n \times n}$ is defined by (5.1),

329 then for any $0 \leq \lambda \leq 1$, $\tilde{R}(\lambda)$ is an IFPR and $\tilde{r}_{ij}^+(\lambda) - \tilde{r}_{ij}^-(\lambda) = \left(1 - \frac{2(c-1)}{2c-1} \lambda \right) (r_{ij}^+ - r_{ij}^-)$.

330 If $c = 1$, it is apparent that $\tilde{r}_{ij}^+(\lambda) - \tilde{r}_{ij}^-(\lambda) = r_{ij}^+ - r_{ij}^-$, i.e., the interval width for any

331 element in the original IFPR \bar{R} (as well as the constructed consistent IFPR $\hat{\bar{R}}$) remains

332 the same after (5.1) is applied. If $c > 1$, it is easy to verify that $\frac{1}{2c-1} \leq 1 - \frac{2(c-1)}{2c-1} \lambda$.

333 Therefore, for any $0 \leq \lambda \leq 1$, $\hat{r}_{ij}^+ - \hat{r}_{ij}^- = \frac{r_{ij}^+ - r_{ij}^-}{2c-1} \leq \left(1 - \frac{2(c-1)}{2c-1} \lambda \right) (r_{ij}^+ - r_{ij}^-) = \tilde{r}_{ij}^+(\lambda) - \tilde{r}_{ij}^-(\lambda)$

334 $\leq r_{ij}^+ - r_{ij}^-$. This means that the interval width for an element in $\tilde{R}(\lambda)$ lies between that for

335 a corresponding element in the original IFPR \bar{R} and that for a corresponding element in

336 the constructed consistent IFPR $\hat{\bar{R}}$.

337 *Theorem 5.2* If $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, then $CI(\tilde{R}(\lambda_1)) \leq CI(\tilde{R}(\lambda_2))$.

338 *Proof.* As per (5.1), we have

$$\tilde{r}_{ij}^-(\lambda_1) + \tilde{r}_{ij}^+(\lambda_1) + \tilde{r}_{jk}^-(\lambda_1) + \tilde{r}_{jk}^+(\lambda_1) + \tilde{r}_{ki}^-(\lambda_1) + \tilde{r}_{ki}^+(\lambda_1) =$$

339 $(1 - \lambda_1)(r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+) + \lambda_1(\hat{r}_{ij}^- + \hat{r}_{ij}^+ + \hat{r}_{jk}^- + \hat{r}_{jk}^+ + \hat{r}_{ki}^- + \hat{r}_{ki}^+) \quad \forall i, j, k = 1, 2, \dots, n$

340 Since \hat{R} is additive consistent, it follows from Lemma 2.1 that $\hat{r}_{ij}^- + \hat{r}_{ij}^+ + \hat{r}_{jk}^- +$

341 $\hat{r}_{jk}^+ + \hat{r}_{ki}^- + \hat{r}_{ki}^+ = 3 \quad \forall i, j, k = 1, 2, \dots, n$. Then

342
$$CI(\tilde{R}(\lambda_1)) = 1 - \frac{1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n (|\tilde{r}_{ij}^-(\lambda_1) + \tilde{r}_{ij}^+(\lambda_1) + \tilde{r}_{jk}^-(\lambda_1) + \tilde{r}_{jk}^+(\lambda_1) + \tilde{r}_{ki}^-(\lambda_1) + \tilde{r}_{ki}^+(\lambda_1) - 3|)$$

$$= 1 - \frac{1 - \lambda_1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n (|r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+ - 3|)$$

343 Similarly,

344
$$CI(\tilde{R}(\lambda_2)) = 1 - \frac{1 - \lambda_2}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n (|r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+ - 3|)$$

345 As $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, one can obtain that $CI(\tilde{R}(\lambda_1)) \leq CI(\tilde{R}(\lambda_2))$. ■

346 Theorem 5.2 indicates that $CI(\tilde{R}(\lambda))$ is an increasing function in $\lambda \in [0, 1]$.

347 *Theorem 5.3* Let $\bar{R} = (\bar{r}_{ij})_{n \times n}$ be an IFPR with an unacceptable consistency level, and

348 t be an acceptable consistency threshold. If $\frac{t - CI(\bar{R})}{1 - CI(\bar{R})} \leq \lambda \leq 1$, then $\tilde{R}(\lambda)$ is an IFPR with

349 acceptable consistency.

350 *Proof.* Since

351
$$CI(\bar{R}) = 1 - \frac{1}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n (|r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+ - 3|),$$

352 and

353
$$CI(\tilde{R}(\lambda)) = 1 - \frac{1 - \lambda}{3n(n-1)(n-2)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1, k \neq i, j}^n (|r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+ - 3|)$$

354 we have $CI(\tilde{R}(\lambda)) = CI(\bar{R}) + \lambda(1 - CI(\bar{R}))$. Therefore, if $\frac{t - CI(\bar{R})}{1 - CI(\bar{R})} \leq \lambda \leq 1$, one can

355 ascertain that $CI(\tilde{R}(\lambda)) \geq t$. ■

356 By (5.1) and Theorem 5.2, one can see that if $\lambda \rightarrow 0$, $\tilde{R}(\lambda) \rightarrow \bar{R}$, indicating that the

357 closer the repaired IFPR $\tilde{R}(\lambda)$ reflects the original preference relation \bar{R} . However, the

358 consistency level of $\tilde{R}(\lambda)$ will be lower. On the other hand, if $\lambda \rightarrow 1$, $\tilde{R}(\lambda) \rightarrow \hat{R}$,
 359 implying that the closer $\tilde{R}(\lambda)$ is to the constructed additive consistent IFPR \hat{R} .
 360 Similarly, in this case, $\tilde{R}(\lambda)$ deviates further from the original IFPR \bar{R} . Therefore,
 361 according to Theorem 5.3, for a given IFPR \bar{R} and an acceptable consistency threshold t ,
 362 a sensible way to repair \bar{R} is to apply (5.1) by setting

$$363 \quad \lambda = \frac{t - CI(\bar{R})}{1 - CI(\bar{R})} \quad (5.2)$$

364 In this case, it is guaranteed that the modified IFPR $\tilde{R}(\lambda)$ has an acceptable consistency
 365 level and reflects the DM's original preference relation in \bar{R} as much as possible.

366 **Example 2.** For the three original IFPRs $\bar{R}_1, \bar{R}_2, \bar{R}_3$ in Example 1, assume that an
 367 acceptable consistency threshold is established as $t = 0.85$. By Definition 3.1, one has
 368 $CI(\bar{R}_1) = 0.9 > t$, $CI(\bar{R}_2) = 0.78333 < t$ and $CI(\bar{R}_3) = 0.78333 < t$. Example 1 indicates
 369 that \bar{R}_1 is additive inconsistent. However, if the DM can accept certain inconsistency as
 370 reflected in the threshold $t = 0.85$, the consistency level of \bar{R}_1 is deemed acceptable, but
 371 \bar{R}_2 and \bar{R}_3 are deemed to have unacceptable consistency. In this case, their consistency
 372 levels have to be improved to reach the acceptable threshold by using (5.1) where \hat{R}_2 and
 373 \hat{R}_3 are the corresponding consistent IFPR obtained in Example 1 and λ is determined by
 374 (5.2).

375 Given that the \bar{R}_2 and \bar{R}_3 have the same consistency index, by using (5.2), we have
 376 $\lambda = 0.3077$ for both IFPRs. As per (5.1), one can obtain $\tilde{R}_2(0.3077) = 0.6923\bar{R}_2 +$
 377 $0.3077\hat{R}_2$ and $\tilde{R}_3(0.3077) = 0.6923\bar{R}_3 + 0.3077\hat{R}_3$ as follows

$$378 \quad \tilde{R}_2(0.3077) = \begin{bmatrix} [0.5, 0.5] & [0.12418, 0.32125] & [0.72234, 0.82088] & [0.55495, 0.65348] \\ [0.67875, 0.87582] & [0.5, 0.5] & [0.70806, 0.90513] & [0.90146, 1.00000] \\ [0.17912, 0.27766] & [0.09487, 0.29194] & [0.5, 0.5] & [0.73333, 0.83187] \\ [0.34652, 0.44505] & [0.00000, 0.09854] & [0.16813, 0.26667] & [0.5, 0.5] \end{bmatrix}$$

379

$$380 \quad \tilde{R}_3(0.3077) = \begin{bmatrix} [0.5, 0.5] & [0.44804, 0.54239] & [0.80675, 0.90110] & [0.90565, 1.00000] \\ [0.45761, 0.55196] & [0.5, 0.5] & [0.36154, 0.45589] & [0.91672, 0.96389] \\ [0.09890, 0.19325] & [0.54411, 0.63846] & [0.5, 0.5] & [0.90416, 0.95133] \\ [0.00000, 0.09435] & [0.03611, 0.08328] & [0.04867, 0.09584] & [0.5, 0.5] \end{bmatrix}$$

381 One can verify that $CI(\tilde{R}_2(0.3077)) = CI(\tilde{R}_3(0.3077)) = 0.85 \geq t$. Therefore, after
 382 applying (5.1), the resulting $\tilde{R}_2(0.3077)$ and $\tilde{R}_3(0.3077)$ are two rectified IFPRs with
 383 acceptable consistency.

384 It should be noted that, if the DM is willing to accept limited inconsistency in a
 385 rectified IFPR \tilde{R} , its mean absolute deviation from the original IFPR \bar{R} should be
 386 smaller than that between a constructed consistent IFPR \hat{R} and \bar{R} . For instance, by
 387 using (3.2), one can verify that $d(\bar{R}_2, \tilde{R}_2(0.3077)) = 0.03834 < d(\bar{R}_2, \hat{R}_2) = 0.1246$ and
 388 $d(\bar{R}_3, \tilde{R}_3(0.3077)) = 0.04746 < d(\bar{R}_3, \hat{R}_3) = 0.15425$. Furthermore, computational results
 389 confirm a reduction ratio of 0.98535 between the interval width of an element in \bar{R}_2 and
 390 that of the corresponding element in $\tilde{R}_2(0.3077)$. Similarly, the reduction ratio is 0.94348
 391 between the interval width of each element in \bar{R}_3 and that of the corresponding element
 392 in $\tilde{R}_3(0.3077)$. On the other hand, the corresponding ratios are $\frac{1}{1.05} = 0.95238$ and

393 $\frac{1}{1.225} = 0.81633$ for the additive consistent IFPRs \hat{R}_2 and \hat{R}_3 , respectively. This result
 394 indicates that, if the consistency requirement can be relaxed to an acceptable consistency
 395 threshold, one can obtain a modified IFPR that is closer to the original IFPR in terms of
 396 both the mean absolute deviation and the interval uncertainty as reflected in the interval
 397 width.

398 According to Definition 2.3, one can verify that IFPRs \bar{R}_1, \bar{R}_3 and $\tilde{R}_3(0.3077)$ are not
 399 weakly transitive, but \bar{R}_2 and $\tilde{R}_2(0.3077)$ are. This indicates that there does not exist
 400 definite inclusion relationship between weak transitivity and acceptable consistency. For
 401 instance, \bar{R}_2 is weakly transitive, but at $t = 0.85$, its consistency level is unacceptable. On
 402 the other hand, as long as $t < 1$, an IFPR with acceptable consistency is not necessarily

403 weakly transitive. For example, $\tilde{R}_3(0.3077)$ is an IFPR with acceptable consistency at $t =$
404 0.85, but it is not weakly transitive due to $p(\tilde{r}_{21} \geq [0.5, 0.5]) = 0.5507 > 0.5$,
405 $p(\tilde{r}_{13} \geq [0.5, 0.5]) = 1 > 0.5$, but $p(\tilde{r}_{23} \geq [0.5, 0.5]) = 0 < 0.5$. Given that the consistency
406 index increases in λ (Theorem 5.2) and, if $t = 1$ or $\lambda = 1$, $\tilde{R} = \hat{R}$ is additive consistent
407 (Theorem 4.3) and, hence, weakly transitive (Theorem 3.1), it is possible to obtain a
408 rectified IFPR with both acceptable consistency and weak transitivity by increasing the
409 value of λ . Next, we shall turn our attention to put forward a framework to repair an
410 inconsistent IFPR, thereby obtaining a rectified IFPR with both acceptable consistency
411 and weak transitivity.

412 In the following, assume that a DM provides his/her IFPR with strict comparison
413 information, i.e., $p(\bar{r}_{ij} \geq [0.5, 0.5]) \neq 0.5$ for all $i, j = 1, 2, \dots, n$ and $i \neq j$. Then, for every
414 IFPR, its associated preference matrix can be defined as follows.

415 *Definition 5.1* The preference matrix $Q = (q_{ij})_{n \times n}$ of an IFPR $\bar{R} = (\bar{r}_{ij})_{n \times n}$ is defined as

$$416 \quad q_{ij} = \begin{cases} 1, & p(\bar{r}_{ij} \geq [0.5, 0.5]) > 0.5 \\ 0, & \text{otherwise} \end{cases} \quad i, j = 1, 2, \dots, n \quad (5.3)$$

417 The preference matrix Q expresses the DM's strict preference relations on X
418 without considering preference degrees. $q_{ij} = 1$ indicates that the DM prefers x_i to x_j ,
419 while $q_{ij} = 0$ means that the DM prefers x_j to x_i . As $p(\bar{r}_{ij} \geq [0.5, 0.5]) \neq 0.5$, the
420 elements in Q satisfy $q_{ij} + q_{ji} = 1$ for all $i, j = 1, 2, \dots, n, i \neq j$.

421 Let

$$422 \quad q_i = \sum_{j=1}^n q_{ij}, \quad i = 1, 2, \dots, n. \quad (5.4)$$

423 It is obvious that $0 \leq q_i \leq n-1$ for any $i = 1, 2, \dots, n$, and $\sum_{i=1}^n q_i = \frac{n(n-1)}{2}$.

424 Preference matrices here can model team tournaments with q_{ij} characterizing
425 whether team x_i defeats x_j , where $q_{ij} = 1$ indicates that x_i defeats x_j and $q_{ij} = 0$
426 describes that x_j defeats x_i , and no ties are allowed. According to the likelihood

427 property (e) in Section 2, $p(\bar{r}_{ij} \geq [0.5, 0.5]) > 0.5$ implies $\frac{r_{ij}^- + r_{ij}^+}{2} > 0.5$. Therefore, the
428 preference matrix Q here is equivalent to a preference matrix under the fuzzy preference
429 relation $P = (p_{ij})_{n \times n}$ in Ma et al. (2006), where $p_{ij} = \frac{r_{ij}^- + r_{ij}^+}{2}$. According to Corollary 3.3
430 and Proposition 1 in Ma et al. (2006), the following judgment methods can be established
431 for weak transitivity.

432 *Corollary 5.1* Let $\bar{R} = (\bar{r}_{ij})_{n \times n}$ be an IFPR and $Q = (q_{ij})_{n \times n}$ be its associated
433 preference matrix with $q_{ij} + q_{ji} = 1$ for all $i, j = 1, 2, \dots, n, i \neq j$. Then \bar{R} is weakly
434 transitive if and only if $S = 0$, where S is defined as (Ma et al. 2006)

$$435 \quad S = \frac{n(n-1)(n-2)}{6} - \frac{1}{2} \sum_{i=1}^n q_i(q_i - 1). \quad (5.5)$$

436 Alternatively, another method is furnished below to tell whether an IFPR is weakly
437 transitive based on its associated preference matrix.

438 *Corollary 5.2* Let $\bar{R} = (\bar{r}_{ij})_{n \times n}$ be an IFPR and $Q = (q_{ij})_{n \times n}$ be its associated
439 preference matrix. Then \bar{R} is weakly transitive if and only if n values q_i ($i = 1, 2, \dots, n$)
440 can be ordered as $\{n-1, n-2, \dots, 1, 0\}$.

441 *Proof.* First, we prove sufficiency. As q_i ($i = 1, 2, \dots, n$) can be ordered as

$$442 \quad \{n-1, n-2, \dots, 1, 0\}, \text{ we have } \sum_{i=1}^n (q_i)^2 = \sum_{l=0}^{n-1} l^2 = \frac{n(n-1)(2n-1)}{6} \text{ and } \sum_{i=1}^n q_i = \frac{n(n-1)}{2}.$$

443 Then, $S = \frac{n(n-1)(n-2)}{6} - \frac{1}{2} \sum_{i=1}^n q_i(q_i - 1) = 0$. By Corollary 5.1, \bar{R} is weakly transitive.

444 Next, we prove the necessary part. Since \bar{R} is weakly transitive, by Corollary 5.1, it

445 follows that $\sum_{i=1}^n (q_i)^2 = \frac{n(n-1)(n-2)}{3} + \sum_{i=1}^n q_i$. On the other hand, as per Definition 5.1 and

446 Eq. (5.4), one has $0 \leq q_i \leq n-1$ for any $i = 1, 2, \dots, n$, and $\sum_{i=1}^n q_i = \frac{n(n-1)}{2}$. Thus,

447 $\sum_{i=1}^n (q_i)^2 = \frac{n(n-1)(2n-1)}{6}$. Therefore, the n values q_i ($i=1,2,\dots,n$) can be arranged as
 448 $\{n-1, n-2, \dots, 1, 0\}$. ■

449 If \bar{R} is weakly transitive, then the rank order of the objects on X is the same as the
 450 ordering of q_i ($i=1,2,\dots,n$).

451 Based on Theorem 5.3, if an IFPR \bar{R} given by a DM is inconsistent, it can be
 452 converted to $\tilde{R}(\lambda)$ with acceptable consistency by using (5.1), where $\lambda = \frac{t - CI(\bar{R})}{1 - CI(\bar{R})}$. If
 453 $\tilde{R}(\lambda)$ is not weakly transitive, one can increase the value of λ to obtain a rectified IFPR
 454 with both acceptable consistency and weak transitivity.

455 Based on the aforesaid analyses, the following algorithm is formulated to improve
 456 the consistency of an IFPR.

457 **Algorithm:** Let $\bar{R} = (\bar{r}_{ij})_{n \times n}$ be an original IFPR and t be an acceptable consistency
 458 threshold given by a DM. The iteration procedure is described as follows.

459 Step 1. Establish the transformation matrix $\hat{R} = ([\hat{r}_{ij}^-, \hat{r}_{ij}^+])_{n \times n}$ as per (4.1).

460 Step 2. Compute the value c by using (4.2).

461 Step 3. Construct the additive consistent IFPR $\hat{R}' = (\hat{r}')_{n \times n}$ as per (4.4).

462 Step 4. Set $k = 0$ and calculate $\lambda_k = \max\left(\frac{t - CI(\bar{R})}{1 - CI(\bar{R})}, 0\right)$, where $CI(\bar{R})$ is determined
 463 by (3.1).

464 Step 5. Derive the weighted IFPR $\tilde{R}^{(k)}$ by $\tilde{R}^{(k)} = (1 - \lambda_k)\bar{R} + \lambda_k\hat{R}'$;

465 Step 6. Establish the preference matrix $Q^{(k)} = (q_{ij}^{(k)})_{n \times n}$ of $\tilde{R}^{(k)}$ by (5.3);

466 Step 7. Calculate $(q_1^{(k)}, q_2^{(k)}, \dots, q_n^{(k)})$ as per (5.4) or compute $S^{(k)}$ by (5.5);

467 Step 8. As per Corollary 5.1 or 5.2, if $\tilde{R}^{(k)}$ is weakly transitive, go to Step 11;
 468 otherwise, go to the next step.

469 Step 9. Let $\lambda_{k+1} = \lambda_k + \delta$ and $k = k + 1$, where δ is a given iteration step size, such
 470 that $0 < \lambda_k + \delta \leq 1$. To make the derived IFPR as close to the original IFPR as possible, a
 471 reasonably small value for δ is needed. Without loss of generality, set $0.01 \leq \delta \leq 0.1$.

472 Step 10. If $\lambda_k < 1$, go to step 5; otherwise, let $\tilde{R}^{(k)} = \hat{R}$, and go to step 6;

473 Step 11. Output $k, \tilde{R}^{(k)}, Q^{(k)}, (q_1^{(k)}, q_2^{(k)}, \dots, q_n^{(k)}), S^{(k)}$;

474 Step 12. End.

475 Next, it is ascertained that this algorithm will terminate after a finite number of
476 iterations.

477 *Theorem 5.4* Assume that $\bar{R} = (\bar{r}_{ij})_{n \times n}$ is an inconsistent IFPR, then a rectified IFPR
478 with acceptable consistency and weak transitivity will be obtained after applying the
479 above algorithm to \bar{R} for a finite number of iterations.

480 *Proof.* For a given consistency threshold t and step size δ , there exists a natural
481 number N such that $\lambda_0 + N \cdot \delta \geq 1$. Then after N iterations, we have $\tilde{R}^{(N)} = \hat{R}$. It follows
482 from Corollary 4.1 that $\tilde{R}^{(N)}$ is an additive consistent IFPR and, hence, weakly
483 transitive as per Theorem 3.1. On the other hand, if the algorithm terminates after k
484 iterations where $k < N$, then $\tilde{R}^{(k)}$ is an IFPR with weak transitivity. Moreover, as
485 $\lambda_k = \lambda_0 + k \cdot \delta \geq \lambda_0$, it follows from Theorem 5.3 that $\tilde{R}^{(k)}$ has acceptable consistency.
486 Therefore, it is ascertained that a rectified IFPR with acceptable consistency and weak
487 transitivity can always be obtained after a finite number of iterations. ■

488 **Example 3.** Assume that a DM conducts an exhaustive pairwise comparison on an
489 alternative set $X = \{x_1, x_2, x_3, x_4\}$, and the result is given as the following IFPR:

490
$$\bar{R} = \begin{bmatrix} [0.5, 0.5] & [0.8, 1] & [0.7, 0.9] & [0.5, 0.9] \\ [0, 0.2] & [0.5, 0.5] & [0.5, 0.7] & [0.7, 0.9] \\ [0.1, 0.3] & [0.3, 0.5] & [0.5, 0.5] & [0.6, 0.8] \\ [0.1, 0.5] & [0.1, 0.3] & [0.2, 0.4] & [0.5, 0.5] \end{bmatrix}$$

491 Without loss of generality, let $t = 0.85$. In the following, the proposed algorithm is
492 applied to improve the consistency level of \bar{R} .

493 As per (4.1), the transformation matrix $\hat{R} = ([\hat{r}_{ij}^-, \hat{r}_{ij}^+])_{4 \times 4}$ is established as

494
$$\hat{R} = \begin{bmatrix} [0.5, 0.5] & [0.6, 0.7] & [0.55, 0.65] & [0.65, 1.05] \\ [0.3, 0.4] & [0.5, 0.5] & [0.35, 0.55] & [0.65, 0.75] \\ [0.35, 0.45] & [0.45, 0.65] & [0.5, 0.5] & [0.7, 0.8] \\ [-0.05, 0.35] & [0.25, 0.35] & [0.2, 0.3] & [0.5, 0.5] \end{bmatrix}$$

495 Since $[\hat{r}_{14}^-, \hat{r}_{14}^+]$ and $[\hat{r}_{41}^-, \hat{r}_{41}^+]$ do not fall in $D([0,1])$, by (4.2), we have $c = 1.05$. Then,
 496 the following IFPR with additive consistency is constructed as per (4.4).

$$497 \quad \hat{R} = \begin{bmatrix} [0.5, 0.5] & [0.59091, 0.68182] & [0.54545, 0.63636] & [0.63636, 1.00000] \\ [0.31818, 0.40909] & [0.5, 0.5] & [0.36364, 0.54545] & [0.63636, 0.72727] \\ [0.36364, 0.45455] & [0.45455, 0.63636] & [0.5, 0.5] & [0.68182, 0.77273] \\ [0.00000, 0.36364] & [0.27273, 0.36364] & [0.22727, 0.31818] & [0.5, 0.5] \end{bmatrix}$$

498 As per (3.1), the consistency index of \bar{R} is computed as $CI(\bar{R}) = 0.8$, then we have

$$499 \quad \lambda_0 = \max\left(\frac{t - CI(\bar{R})}{1 - CI(\bar{R})}, 0\right) = 0.25. \quad \text{Therefore, an acceptably consistent IFPR } \tilde{R}(0.25) \text{ is}$$

500 constructed by (5.2) with $\lambda = \lambda_0$.

$$501 \quad \tilde{R}(0.25) = \begin{bmatrix} [0.5, 0.5] & [0.82273, 0.92046] & [0.43636, 0.53409] & [0.53409, 0.92500] \\ [0.07955, 0.17727] & [0.5, 0.5] & [0.46591, 0.66136] & [0.75909, 0.85682] \\ [0.46591, 0.56364] & [0.33864, 0.53409] & [0.5, 0.5] & [0.69546, 0.79318] \\ [0.07500, 0.46591] & [0.14318, 0.24091] & [0.20682, 0.30455] & [0.5, 0.5] \end{bmatrix}$$

502 Set $k = 0$ and $\tilde{R}^{(0)} = \tilde{R}(0.25)$, we can then examine whether $\tilde{R}^{(0)}$ is weakly transitive.

503 By (5.3), the preference matrix $Q^{(0)} = (q_{ij})_{4 \times 4}$ of the IFPR $\tilde{R}^{(0)}$ is derived as follows.

$$504 \quad Q^{(0)} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

505 Either Corollary 5.1 or 5.2 can be applied to judge if $\tilde{R}^{(0)}$ is weakly transitive. As
 506 per (5.4), we have $q_1^{(0)} = 2, q_2^{(0)} = 2, q_3^{(0)} = 2, q_4^{(0)} = 0$. By (5.5), one obtains $S^{(0)} = 1 \neq 0$.

507 From Corollary 5.1 or 5.2, one can see that $\tilde{R}^{(0)}$ is not weakly transitive. In fact, for
 508 preference matrix $Q^{(0)} = (q_{ij})_{4 \times 4}$, one can get a cyclic preference relation, $x_1 \succ x_2 \succ x_3 \succ x_1$.

509 Thus the weight λ in the rectifying formula (5.1) should be increased. Let $\delta = 0.02$, one
 510 can obtain a rectified IFPR $\tilde{R}^{(6)}$ with both weak transitivity and acceptable consistency.

511 The iterative process to improving consistency for the IFPR \bar{R} is described in Table 1.

512 It can be seen from Table 1 that $CI(\tilde{R}^{(0)}) < CI(\tilde{R}^{(1)}) < \dots < CI(\tilde{R}^{(6)})$ and
 513 $d(\bar{R}, \tilde{R}^{(0)}) > d(\bar{R}, \tilde{R}^{(1)}) > \dots > d(\bar{R}, \tilde{R}^{(6)})$. This is understandable: as the iteration process

514 continues, the consistency level of the resulting IFPR $\tilde{R}^{(k)}$ increases, but this comes at a
 515 cost with a greater deviation from the original IFPR \bar{R} . One can also see from the third
 516 last column that the rectified IFPR $\tilde{R}^{(k)}$ does not achieve weak transitivity until $k = 6$.

517 Table 1. The process to improving consistency of IFPR \bar{R}

518	Iteration k	Iterative preference relation $\tilde{R}^{(k)}$	$S^{(k)}$	$CI(\tilde{R}^{(k)})$	$d(\bar{R}, \tilde{R}^{(k)})$
520	$k=0$	$\begin{bmatrix} [0.5, 0.5] & [0.82273, 0.92046] & [0.43636, 0.53409] & [0.53409, 0.92500] \\ [0.07955, 0.17727] & [0.5, 0.5] & [0.46591, 0.66136] & [0.75909, 0.85682] \\ [0.46591, 0.56364] & [0.33864, 0.53409] & [0.5, 0.5] & [0.69546, 0.79318] \\ [0.07500, 0.46591] & [0.14318, 0.24091] & [0.20682, 0.30455] & [0.5, 0.5] \end{bmatrix}$	1	0.85	0.0379
521	$k=1$	$\begin{bmatrix} [0.5, 0.5] & [0.81655, 0.91409] & [0.43927, 0.53682] & [0.53682, 0.92700] \\ [0.08591, 0.18345] & [0.5, 0.5] & [0.46318, 0.65827] & [0.75582, 0.85336] \\ [0.46318, 0.56073] & [0.34173, 0.53682] & [0.5, 0.5] & [0.69509, 0.79264] \\ [0.07300, 0.46318] & [0.14664, 0.24418] & [0.20736, 0.30491] & [0.5, 0.5] \end{bmatrix}$	1	0.854	0.0409
522	$k=2$	$\begin{bmatrix} [0.5, 0.5] & [0.81036, 0.90773] & [0.44218, 0.53954] & [0.53954, 0.92900] \\ [0.09227, 0.18964] & [0.5, 0.5] & [0.46046, 0.65518] & [0.75254, 0.84991] \\ [0.46046, 0.55782] & [0.34482, 0.53954] & [0.5, 0.5] & [0.69473, 0.79209] \\ [0.07100, 0.46046] & [0.15009, 0.24746] & [0.20791, 0.30527] & [0.5, 0.5] \end{bmatrix}$	1	0.858	0.0439
523	$k=3$	$\begin{bmatrix} [0.5, 0.5] & [0.80418, 0.90136] & [0.44509, 0.54227] & [0.54227, 0.93100] \\ [0.09864, 0.19582] & [0.5, 0.5] & [0.45773, 0.65209] & [0.74927, 0.84645] \\ [0.45773, 0.55491] & [0.34791, 0.54227] & [0.5, 0.5] & [0.69436, 0.79155] \\ [0.06900, 0.45773] & [0.15355, 0.25073] & [0.20845, 0.30564] & [0.5, 0.5] \end{bmatrix}$	1	0.862	0.0470
524	$k=4$	$\begin{bmatrix} [0.5, 0.5] & [0.79800, 0.89500] & [0.44800, 0.54500] & [0.54500, 0.93300] \\ [0.10500, 0.20200] & [0.5, 0.5] & [0.45500, 0.64900] & [0.74600, 0.84300] \\ [0.45500, 0.55200] & [0.35100, 0.54500] & [0.5, 0.5] & [0.69400, 0.79100] \\ [0.06700, 0.45500] & [0.15700, 0.25400] & [0.20900, 0.30600] & [0.5, 0.5] \end{bmatrix}$	1	0.866	0.0500
525	$k=5$	$\begin{bmatrix} [0.5, 0.5] & [0.79182, 0.88864] & [0.45091, 0.54773] & [0.54773, 0.93500] \\ [0.11136, 0.20818] & [0.5, 0.5] & [0.45227, 0.64591] & [0.74273, 0.83954] \\ [0.45227, 0.54909] & [0.35409, 0.54773] & [0.5, 0.5] & [0.69364, 0.79046] \\ [0.06500, 0.45227] & [0.16046, 0.25727] & [0.20954, 0.30636] & [0.5, 0.5] \end{bmatrix}$	1	0.87	0.0530
526	$k=6$	$\begin{bmatrix} [0.5, 0.5] & [0.78564, 0.88227] & [0.45382, 0.55045] & [0.55045, 0.93700] \\ [0.11773, 0.21436] & [0.5, 0.5] & [0.44955, 0.64282] & [0.73945, 0.83609] \\ [0.44955, 0.54618] & [0.35718, 0.55045] & [0.5, 0.5] & [0.69327, 0.78991] \\ [0.06300, 0.44955] & [0.16391, 0.26055] & [0.21009, 0.30673] & [0.5, 0.5] \end{bmatrix}$	0	0.874	0.0561

528 Next, the approach proposed by Xu (2011) is employed to improve consistency for the
 529 same IFPR \bar{R} as a comparison.

530 Based on the additive consistency definition (Xu and Chen 2008), Xu (2011) develops
 531 a linear program (see (M-2) on page 3901) to construct a consistent IFPR from an
 532 inconsistent IFPR. Plugging \bar{R} into (M-2) in Xu (2011) and solving this model, one can
 533 get the optimal nonzero deviation values $\dot{d}_{24}^- = 0.1333, \dot{d}_{34}^- = 0.0333, \dot{d}_{42}^+ = 0.1333$ and
 534 $\dot{d}_{43}^+ = 0.0333$. By (21) in Xu (2011), we obtain the following constructed IFPR with Xu
 535 and Chen (2008)'s additive consistency.

$$536 \quad \hat{R}^{Xu} = \begin{bmatrix} [0.5, 0.5] & [0.8, 1] & [0.7, 0.9] & [0.5, 0.9] \\ [0, 0.2] & [0.5, 0.5] & [0.5, 0.7] & [0.5667, 0.9] \\ [0.1, 0.3] & [0.3, 0.5] & [0.5, 0.5] & [0.5667, 0.8] \\ [0.1, 0.5] & [0.1, 0.4333] & [0.2, 0.4333] & [0.5, 0.5] \end{bmatrix}$$

537 By Definition 2.3, one can easily verify that \hat{R}^{Xu} has weak transitivity. On the other
 538 hand, as per (3.1) and (3.2), one has $CI(\hat{R}^{Xu}) = 0.8389$ and $d(\bar{R}, \hat{R}^{Xu}) = 0.0139$.
 539 Although \hat{R}^{Xu} is weakly transitive and $d(\bar{R}, \hat{R}^{Xu}) < d(\bar{R}, \tilde{R}^{(k)})$ for all $k = 0, 1, \dots, 6$, \hat{R}^{Xu}
 540 does not possess acceptable consistency under (3.1) as $CI(\hat{R}^{Xu}) < t = 0.85$. This
 541 difference is resulted from the fact that the two rectification approaches employ different
 542 additive consistency constraints. The constraint in Xu (2011) is established by the
 543 feasible region model and the proposed method herein is based on interval arithmetic.

544 6. Conclusion

545 Based on the additive consistency definition proposed by Wang and Li (2012), this
 546 article begins with presenting new properties for additive consistent IFPRs. Then, a
 547 consistency index is defined to measure the level of consistency for IFPRs, which can be
 548 conveniently applied to check whether an IFPR is consistent. Subsequently, an innovative
 549 approach is developed to construct an additive consistent IFPR from any inconsistent
 550 IFPR. By introducing a weighted averaging scheme that integrates the original and the
 551 constructed consistent IFPRs, a novel approach is put forward to improve consistency of
 552 IFPRs. An iterative algorithm is then established to repair an inconsistent IFPR to derive
 553 a rectified IFPR with both acceptable consistency and weak transitivity.

554 The basic modeling principle is to ensure that the derived IFPRs can improve
555 consistency and, simultaneously, retain as much the initial interval uncertainty (measured
556 by interval widths) as possible. Numerical examples are presented to demonstrate how to
557 apply the proposed approaches. Further research is required to accommodate the cases
558 when IFPRs contain missing judgment data and induced preference matrix Q includes
559 indifference relations.

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