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Abstract

This article introduces a consistency index for measuring the consistency level of an interval fuzzy preference relation (IFPR). An approach is then proposed to construct an additive consistent IFPR from a given inconsistent IFPR. By using a weighted averaging method combining the original IFPR and the constructed consistent IFPR, a formula is put forward to repair an inconsistent IFPR to generate an IFPR with acceptable consistency. An iterative algorithm is subsequently developed to rectify an inconsistent IFPR and derive one with acceptable consistency and weak transitivity. The proposed approaches can not only improve consistency of IFPRs but also preserve the initial interval uncertainty information as much as possible. Numerical examples are presented to illustrate how to apply the proposed approaches.

Keywords: Interval fuzzy preference relation, Additive consistency, Acceptable consistency, Weak transitivity, Decision making

1. Introduction

In decision analysis, a decision-maker (DM) is often asked to express his/her preference ratings over objects in a pairwise comparison manner (Dong and Saaty 2014). The pairwise comparison among criteria or alternatives in the analytic hierarchy process (AHP) (Saaty 1980) yields multiplicative preference relations, which constitute the basis to derive criteria weights and rank alternatives. To reflect vagueness in human judgment, many researchers have been paying increasing attention to fuzzy preference relations in recent years (Liu X. et al. 2012; Xia et al. 2013).

An important research topic in this area is to investigate consistency of preference relations. For fuzzy preference relations, distinct transitivity definitions have been put forward, such as additive transitivity, multiplicative transitivity, weak transitivity, max-min transitivity, and max-max transitivity (Xu 2007). Let $R=(r_{ij})_{n\times n}$ be a fuzzy preference relation, if $r_{ij}-r_{ji}$ is interpreted as the intensity of the DM's preference of the object x_i over x_j, then additive consistency is a sensible vehicle to verify whether the

DM's judgments are contradiction-free; if the DM denotes r_{ij}/r_{ji} as its preference intensity for x_i vs. x_j, then multiplicative consistency is an appropriate tool. The focus of this paper is concerned with additive transitivity, which is regarded as a parallel concept to the multiplicative consistent property in AHP (Herrera-Viedma et al. 2004; Liu X. 2012; Xu et al. 2014) and is widely employed to characterize consistency of fuzzy preference relations (Chen and Chao 2012; Chiclana et al., 2007; Herrera-Viedma et al. 2004, 2007; Liu X. et al. 2012; Ma et al. 2006; Xu et al. 2014). Based on additive transitivity properties, some authors have proposed different approaches to improve consistency of inconsistent fuzzy preference relations furnished by the DM. For example, Herrera-Viedma et al. (2004) put forward an approach to construct a fuzzy preference relation with additive consistency from a set of n -1 preference values. Ma et al. (2006) present two methods to examine weak transitivity of a fuzzy preference relation with strict pairwise comparison judgments, and develop an algorithm to repair an inconsistent fuzzy preference relation to reach weak transitivity. Herrera-Viedma et al. (2007) introduce a consistency index to measure the consistency level (*CL*) of a fuzzy preference relation and furnish a concept of fully additive consistency when *CL*=1. Liu X. et al. (2012) consider incomplete fuzzy preference relations and develop a least square model to complete an incomplete fuzzy preference relation and rectify its inconsistency based on additive transitivity.

On the other hand, due to complexity and uncertainty in many decision problems, it is hard for a DM to express his/her preference over objects with crisp values (Durbach and Stewart 2012; Li and Chen 2014; Yu and Xu 2014). In this case, it is often more natural to use interval fuzzy preference relations (IFPRs). The concept of IFPRs is introduced by Xu (2004), in which judgment data are given as interval fuzzy numbers to characterize a DM's preference degree or intensity of one object over another. In order to obtain reasonable priority weights, consistency and acceptable consistency of IFPRs have been studied and different methods have been designed for generating priority weights based on IFPRs. For instance, Xu and Chen (2008) define additive and multiplicative consistent IFPRs, in which the consistency conditions are established without accounting for transitivity among three or more judgment data. Based on Xu and Chen's additive and multiplicative consistency, some authors have devised different methods for generating priority weights from IFPRs such as Genç et al. (2010), Lan et al. (2012), Xia and Xu (2014), to name a few. Xu (2011) further proposes an approach to construct additive or multiplicative consistent IFPRs by minimizing deviation between the initial and constructed IFPRs. Xu et al. (2014) and Hu et al. (2014) propose revised definitions for the additive consistency given by Xu and Chen (2008). Liu F. et al. (2012) adopt two converted fuzzy preference relations to define an additive consistent IFPR, and develop an algorithm for deriving priority weights from IFPRs. Wang and Li (2012) introduce new additive and multiplicative consistency definitions for IFPRs based on interval arithmetic. Wang and Li (2014) develop a multi-step goal programming method for group decision making with incomplete IFPRs.

Consistency of preference relations plays an important role in reaching a reasonable decision result. Nevertheless, it is often a challenge for a DM to provide a consistent IFPR in many real-world decision situations. It is natural that highly inconsistent judgment matrices may lead to misleading decision result. For IFPRs with low consistency levels, they should be returned to the DMs for an update. If the DMs are unavailable or unwilling to revise their original judgment information, it is helpful to have an automated process to improve consistency of the original IFPRs furnished by the DMs. In the case that the DMs are available to update their decision input, the results of improved IFPRs can also serve as a valuable feedback and benchmark for the DMs in updating their judgment. Although the approach in Xu (2011) is able to construct a consistent IFPR, the consistency definitions are based on crisp weights and the derived consistent IFPR may result in significant loss of information (for instance, the uncertainty reflected in the interval width of the judgment may be substantially changed in the conversion process). In addition, for the consistency definitions given by Xia and Xu (2011) and Liu F. et al. (2012), Wang and Chen (2014) point out their technical deficiency as the consistency status of an IFPR therein is sensitive to alternative permutations. The fundamental motivation of this research is to address the aforesaid issues. By adopting the additive consistency notion proposed by Wang and Li (2012), this study focuses on improving consistency of IFPRs. The contributions of this article are threefold: we first define a consistency index for IFPRs, then put forward a formula to construct an additive consistent IFPR based on an inconsistent input, finally, we develop a method and an algorithm to rectify an inconsistent IFPR. More specifically, a consistency index is first defined to measure the consistency level of an IFPR. For an inconsistent IFPR, an approach is then proposed to construct an additive consistent IFPR, which is employed as a reference to improve consistency of the given IFPR. By using a weighted averaging scheme combining the original IFPR and the constructed consistent IFPR, a method is put forward to repair an inconsistent IFPR to yield an IFPR with acceptable consistency. A further algorithm is developed to rectify an inconsistent IFPR to generate an IFPR with both acceptable consistency and weak transitivity.

The remainder of this paper is organized as follows. Section 2 provides preliminaries on consistent IFPRs and comparison of interval numbers. Section 3 defines a consistency index for IFPRs. In Section 4, an approach is proposed to construct an additive consistent IFPR based on any given IFPR. Section 5 presents two approaches to improving consistency of IFPRs. Finally, concluding remarks are furnished in Section 6.

2. Preliminaries

This section presents basic concepts of additive consistency and weak transitivity of IFPRs as well as comparison of interval numbers.

Consider a decision problem with a finite set of n objects, denoted by $X=\{x_1,x_2,...,x_n\}$, where the objects may be alternatives, criteria, attributes and so on. Let I be a real closed interval, $D(I)=\{[a^-,a^+]:a^-\le a^+,a^-,a^+\in I\}$. For any $x\in I$, define $x=[x,x]$.

Xu (2004) defines IFPRs where judgment data are expressed as interval fuzzy numbers to characterize a DM’s preference degree of one object over another.

Definition 2.1 (Xu 2004) An interval fuzzy preference relation (IFPR) $\overline{R}$ on the set X is characterized by an interval fuzzy preference matrix $\overline{R}=(\overline{r}_{ij})_{n\times n}\subset X\times X$, where

$$\overline{r}_{ij}=[r_{ij}^-,r_{ij}^+]\in D([0,1]),\overline{r}_{ji}=1-\overline{r}_{ij}=[1-r_{ij}^+,1-r_{ij}^-],\overline{r}_{ii}=[0.5,0.5],\ i,j=1,2,...,n \qquad (2.1)$$

and $\overline{r}_{ij}$ indicates the interval-valued fuzzy preference of x_i over x_j. r_{ij}^- and r_{ij}^+ are the lower and upper bounds of $\overline{r}_{ij}$, respectively.

As commented in Section 1, the additive consistency definitions introduced by Xu and Chen (2008), Xu et al. (2014) and Hu et al. (2014) are based on crisp weights, and the consistency condition established therein fails to account for transitivity among three or more judgment data. To address this issue, Wang and Li (2012) put forward a new additive consistency notion for IFPRs by using interval arithmetic and the definition is furnished below.

Definition 2.2 (Wang and Li 2012) An IFPR $\overline{R}=(\overline{r}_{ij})_{n\times n}$ is additive consistent if the following additive transitivity is satisfied

$$\overline{r}_{ij}+\overline{r}_{jk}+\overline{r}_{ki}=\overline{r}_{kj}+\overline{r}_{ji}+\overline{r}_{ik} \quad \text{for all } i,j,k=1,2,...,n \tag{2.2}$$

To compare two interval numbers $\overline{a}=[a^-,a^+]$ and $\overline{b}=[b^-,b^+]$, where $a^-,b^-\geq 0$, the notion of likelihood is introduced. Let $\overline{a}\geq\overline{b}$ represent that $\overline{a}$ is no smaller than $\overline{b}$. The likelihood of $\overline{a}\geq\overline{b}$ is defined as (Xu and Chen 2008)

$$p(\overline{a}\geq\overline{b})=\frac{\max\{0,a^+-b^-\}-\max\{0,a^--b^+\}}{a^+-a^-+b^+-b^-} \tag{2.3}$$

Some useful properties about likelihood $p(\overline{a}\geq\overline{b})$ are summarized as follows:

(a) $0\leq p(\overline{a}\geq\overline{b})\leq 1$;

(b) $p(\overline{a}\geq\overline{b})+p(\overline{b}\geq\overline{a})=1$;

(c) $p(\overline{a}\geq\overline{b})=1$ if and only if $a^-\geq b^+$;

(d) $p(\overline{a}\geq\overline{b})=0$ if and only if $a^+\leq b^-$;

(e) $p(\overline{a}\geq\overline{b})\geq 0.5$ if and only if $\frac{a^-+a^+}{2}\geq\frac{b^-+b^+}{2}$. Especially, $p(\overline{a}\geq\overline{b})=0.5$ if and only if $\frac{a^-+a^+}{2}=\frac{b^-+b^+}{2}$;

(f) For any interval numbers $\overline{a},\overline{b}$ and $\overline{c}$, if $p(\overline{a}\geq\overline{b})\geq 0.5$ and $p(\overline{b}\geq\overline{c})\geq 0.5$, then $p(\overline{a}\geq\overline{c})\geq 0.5$.

According to the aforesaid properties of the likelihood concept, for an IFPR $\overline{R}=(\overline{r}_{ij})_{n\times n}$, $p(\overline{r}_{ij}\geq[0.5,0.5])=0.5$ indicates a DM's indifference between x_i and x_j, $p(\overline{r}_{ij}\geq[0.5,0.5])>0.5$ signifies that x_i is preferred to x_j with a degree of $p(\overline{r}_{ij}\geq[0.5,0.5])$, $p(\overline{r}_{ij}\geq[0.5,0.5])<0.5$ describes that x_j is preferred to x_i with a degree of $1-p(\overline{r}_{ij}\geq[0.5,0.5])$, $p(\overline{r}_{ij}\geq[0.5,0.5])=1$ means that x_i is absolutely preferred to x_j, and

$p(\overline{r}_{ij} \geq [0.5, 0.5]) = 0$ expresses that x_j is absolutely preferred to x_i.

Based on the likelihood definition and properties, Wang and Li (2012) introduce weak transitivity for IFPRs as follows.

Definition 2.3 (Wang and Li 2012) An IFPR $\overline{R} = (\overline{r}_{ij})_{n\times n}$ is weakly transitive if $p(\overline{r}_{ik} \geq [0.5, 0.5]) \geq 0.5$ and $p(\overline{r}_{kj} \geq [0.5, 0.5]) \geq 0.5$ imply $p(\overline{r}_{ij} \geq [0.5, 0.5]) \geq 0.5$, for all $i, j, k = 1, 2, ..., n$.

Base on Definition 2.2, Wang (2014) provides the following property to judge whether an IFPR is consistent.

Lemma 2.1 (Wang 2014) An IFPR $\overline{R} = (\overline{r}_{ij})_{n\times n}$ is additive consistent if and only if

$$r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+ = 3 \quad \forall i, j, k = 1, 2, ..., n \tag{2.4}$$

3. Consistency measure

By Lemma 2.1, if $\overline{R} = (\overline{r}_{ij})_{n\times n}$ is an additive consistent IFPR, we have $r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+ = 3 \ \forall i, j, k = 1, 2, ..., n$. However, if $\overline{R}$ is inconsistent, the preference values in $\overline{R}$ will not satisfy (3.1). In other words, there exist some differences between $r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+$ and 3 for some $i, j, k = 1, 2, ..., n$. As $0 \leq r_{ij}^- \leq r_{ij}^+ \leq 1$ for all i, j=1, 2, …, n, one has $0 \leq \left| r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+ - 3 \right| \leq 3$. Therefore, we can define a consistency measure for an IFPR as follows.

Definition 3.1 A consistency index of an IFPR $\overline{R} = (\overline{r}_{ij})_{n\times n}$ is defined as

$$CI(\overline{R}) = 1 - \frac{1}{3n(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \sum_{k=1, k\neq i, j}^{n} | r_{ij}^- + r_{ij}^+ + r_{jk}^- + r_{jk}^+ + r_{ki}^- + r_{ki}^+ - 3 | \tag{3.1}$$

It is obvious that $0 \leq CI(\overline{R}) \leq 1$. If $CI(\overline{R}) = 1$, then the IFPR $\overline{R}$ is additive consistent; otherwise, $\overline{R}$ is inconsistent, and the larger the $CI(\overline{R})$, the closer the $\overline{R}$ is to a consistent IFPR. According to the actual situation, if a DM can accept limited inconsistency in the judgment, he/she may give a consistency threshold $0 < t < 1$ for $CI(\overline{R})$. This threshold presumably reflects the DM's tolerance for inconsistency and should be furnished by the DM upon examining the specific decision circumstances. If $CI(\overline{R}) \geq t$, the IFPR $\overline{R}$ is deemed acceptably consistent; otherwise, the consistency level of $\bar{R}$ is considered unacceptable and, hence, $\bar{R}$ should be rectified to ensure rationality of decisions.

If $r_{ij}^{-} = r_{ij}^{+}$, $i, j = 1, 2, ..., n$, the IFPR $\bar{R}$ is reduced to a fuzzy preference relation, and (3.1) is equivalent to the consistency index of a complete fuzzy preference relation proposed by Herrera-Viedma et al. (2007).

As per the additive consistency of IFPRs in Definition 2.2 and weak transitivity of IFPRs in Definition 2.3, we have the following theorem.

Theorem 3.1 If an IFPR $\bar{R} = (\bar{r}_{ij})_{n\times n}$ is additive consistent, then $\bar{R}$ is weakly transitive.

Proof. According to property (e) of the likelihood concept in Section 2, if $p(\bar{r}_{ik} \geq [0.5, 0.5]) \geq 0.5$ and $p(\bar{r}_{kj} \geq [0.5, 0.5]) \geq 0.5$, we have $r_{ik}^{-} + r_{ik}^{+} \geq (0.5 + 0.5) = 1$ and $r_{kj}^{-} + r_{kj}^{+} \geq (0.5 + 0.5) = 1$, $\forall i, j, k \in \{1, 2, ..., n\}$. As per the reciprocal property of $\bar{r}_{ji} = 1 - \bar{r}_{ij}$, one can get $1 - r_{ki}^{+} + 1 - r_{ki}^{-} \geq 1$ and $1 - r_{jk}^{+} + 1 - r_{jk}^{-} \geq 1$, $\forall i, j, k \in \{1, 2, ..., n\}$. It follows that $r_{ki}^{-} + r_{ki}^{+} \leq 1$ and $r_{jk}^{-} + r_{jk}^{+} \leq 1$, $\forall i, j, k \in \{1, 2, ..., n\}$.

On the other hand, as $\bar{R}$ is additive consistent, it follows from Lemma 2.1 $r_{ij}^{-} + r_{ij}^{+} + r_{jk}^{-} + r_{jk}^{+} + r_{ki}^{-} + r_{ki}^{+} = 3$ $\forall i, j, k = 1, 2, ..., n$. Then, one has $r_{ij}^{-} + r_{ij}^{+} \geq 1$, i.e., $\frac{r_{ij}^{-} + r_{ij}^{+}}{2} \geq 0.5$. As per the property (e) of the likelihood concept, we get $p(\bar{r}_{ij} \geq [0.5, 0.5]) \geq 0.5$. Therefore, $\bar{R} = (\bar{r}_{ij})_{n\times n}$ is weakly transitive. ■

For any two IFPRs $\bar{R} = (\bar{r}_{ij})_{n\times n} = ([r_{ij}^{-}, r_{ij}^{+}])_{n\times n}$ and $\bar{R}' = (\bar{r}'_{ij})_{n\times n} = ([r_{ij}'^{-}, r_{ij}'^{+}])_{n\times n}$, let

$$d(\bar{R}, \bar{R}') = \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \left(\left| r_{ij}^{-} - r_{ij}'^{-} \right| + \left| r_{ij}^{+} - r_{ij}'^{+} \right| \right) \tag{3.2}$$

denote the mean absolute deviation for all off-diagonal intervals between $\bar{R}$ and $\bar{R}'$. The smaller the value $d(\bar{R}, \bar{R}')$, the closer the $\bar{R}$ is to $\bar{R}'$. Especially, if $d(\bar{R}, \bar{R}') = 0$, $\bar{R}$ is the same as $\bar{R}'$.

4. An approach to constructing consistent IFPRs

This section develops a framework to construct an additive consistent IFPR based on any given inconsistent IFPR.

For a given IFPR $\bar{R}=(\bar{r}_{ij})_{n\times n}$, define the $(i,\ j)$ entry of $\hat{\bar{R}}=(\hat{\bar{r}}_{ij})_{n\times n}=([\hat{r}_{ij}^{-},\hat{r}_{ij}^{+}])_{n\times n}$ as follows

$$\hat{\bar{r}}_{ij}=[\hat{r}_{ij}^{-},\hat{r}_{ij}^{+}]=\left[0.5+\frac{1}{2n}\left(\sum_{l=1}^{n}r_{il}^{-}+\sum_{l=1}^{n}r_{il}^{+}-\sum_{l=1}^{n}r_{jl}^{-}-\sum_{l=1}^{n}r_{jl}^{+}\right)-\frac{r_{ij}^{+}-r_{ij}^{-}}{2},\right.$$
$$\left.0.5+\frac{1}{2n}\left(\sum_{l=1}^{n}r_{il}^{-}+\sum_{l=1}^{n}r_{il}^{+}-\sum_{l=1}^{n}r_{jl}^{-}-\sum_{l=1}^{n}r_{jl}^{+}\right)+\frac{r_{ij}^{+}-r_{ij}^{-}}{2}\right] \tag{4.1}$$

for all $i,j=1,2,...,n$. The following two theorems reveal some useful properties of $\hat{\bar{r}}_{ij}$.

Theorem 4.1 Let $\bar{R}=(\bar{r}_{ij})_{n\times n}$ be an IFPR and $\hat{\bar{r}}_{ij}=\left[\hat{r}_{ij}^{-},\hat{r}_{ij}^{+}\right]$ $(i,j=1,2,...,n)$ be defined by (4.1), then

(i) $\hat{r}_{ij}^{-}\le\hat{r}_{ij}^{+}$, $\hat{r}_{ii}^{-}=\hat{r}_{ii}^{+}=0.5\ \ \forall i,j=1,2,...,n$.

(ii) $\hat{\bar{r}}_{ji}=1-\hat{\bar{r}}_{ij}$, i.e., $\hat{r}_{ji}^{-}=1-\hat{r}_{ij}^{+}$ and $\hat{r}_{ji}^{+}=1-\hat{r}_{ij}^{-}\ \forall i,j=1,2,...,n$.

(iii) $\hat{r}_{ij}^{+}-\hat{r}_{ij}^{-}=r_{ij}^{+}-r_{ij}^{-}\ \ \forall i,j=1,2,...,n$.

(iv) $\hat{\bar{r}}_{ij}+\hat{\bar{r}}_{jk}+\hat{\bar{r}}_{ki}=\hat{\bar{r}}_{kj}+\hat{\bar{r}}_{ji}+\hat{\bar{r}}_{ik}\ \ \forall i,j,k=1,2,...,n$.

Proof. (i) - (iii) can be immediately derived from (4.1) and, hence, the proof is only provided for (iv).

Since $\bar{R}=(\bar{r}_{ij})_{n\times n}$ is an IFPR, then we have $r_{ji}^{-}=1-r_{ij}^{+},r_{ji}^{+}=1-r_{ij}^{-}\ \ \forall i,j=1,2,...,n$. It follows from (4.1) that

$$\hat{r}_{ij}^{-}+\hat{r}_{jk}^{-}+\hat{r}_{ki}^{-}=\frac{3}{2}-\frac{r_{ij}^{+}-r_{ij}^{-}+r_{jk}^{+}-r_{jk}^{-}+r_{ki}^{+}-r_{ki}^{-}}{2}=\frac{3}{2}-\frac{1-r_{ji}^{-}-(1-r_{ji}^{+})+1-r_{kj}^{-}-(1-r_{kj}^{+})+1-r_{ik}^{-}-(1-r_{ik}^{+})}{2}$$
$$=\frac{3}{2}-\frac{r_{kj}^{+}-r_{kj}^{-}+r_{ji}^{+}-r_{ji}^{-}+r_{ik}^{+}-r_{ik}^{-}}{2}=\hat{r}_{kj}^{-}+\hat{r}_{ji}^{-}+\hat{r}_{ik}^{-}$$

Similarly, from (4.1), one can obtain $\hat{r}_{ij}^{+}+\hat{r}_{jk}^{+}+\hat{r}_{ki}^{+}=\hat{r}_{kj}^{+}+\hat{r}_{ji}^{+}+\hat{r}_{ik}^{+}$. Therefore, $\hat{\bar{r}}_{ij}+\hat{\bar{r}}_{jk}+\hat{\bar{r}}_{ki}=\hat{\bar{r}}_{kj}+\hat{\bar{r}}_{ji}+\hat{\bar{r}}_{ik}$. This completes the proof of Theorem 4.1. ■

Theorem 4.2 If $\bar{R}=(\bar{r}_{ij})_{n\times n}$ is an additive consistent IFPR, then $\hat{\bar{r}}_{ij}=\bar{r}_{ij}$ $\forall i,j=1,2,...,n$.

Proof. Since $\bar{R}$ is additive consistent, it follows from Lemma 2.1 that

$$r_{il}^{-}+r_{il}^{+}+r_{lj}^{-}+r_{lj}^{+}=3-(r_{ji}^{-}+r_{ji}^{+})\ \forall i,j,l=1,2,...,n$$

Then,

$$\begin{aligned}
&\sum_{l=1}^{n} r_{il}^{-}+\sum_{l=1}^{n} r_{il}^{+}-\sum_{l=1}^{n} r_{jl}^{-}-\sum_{l=1}^{n} r_{jl}^{+}=\sum_{l=1}^{n}(r_{il}^{-}+r_{il}^{+}-r_{jl}^{-}-r_{jl}^{+})=\sum_{l=1}^{n}(r_{il}^{-}+r_{il}^{+}-(1-r_{lj}^{+})-(1-r_{lj}^{-}))\\
&=\sum_{l=1}^{n}(r_{il}^{-}+r_{il}^{+}+r_{lj}^{-}+r_{lj}^{+}-2)=\sum_{l=1}^{n}(3-(r_{ji}^{-}+r_{ji}^{+})-2)=\sum_{l=1}^{n}(1-(1-r_{ij}^{+}+1-r_{ij}^{-}))\\
&=\sum_{l=1}^{n}(r_{ij}^{-}+r_{ij}^{+}-1)=n(r_{ij}^{-}+r_{ij}^{+})-n
\end{aligned}$$

As per (4.1), we have $\hat{r}_{ij}^{-}=0.5+\frac{(r_{ij}^{-}+r_{ij}^{+})}{2}-0.5-\frac{(r_{ij}^{+}-r_{ij}^{-})}{2}=r_{ij}^{-}$ and $\hat{r}_{ij}^{+}=0.5+\frac{(r_{ij}^{-}+r_{ij}^{+})}{2}-0.5+\frac{(r_{ij}^{+}-r_{ij}^{-})}{2}=r_{ij}^{+}$. It is verified that $\hat{\bar{r}}_{ij}=\bar{r}_{ij}$. ■

Theorem 4.1 demonstrates that $\hat{\bar{R}}$ is an additive consistent IFPR if $0\le \hat{r}_{ij}^{-}\le \hat{r}_{ij}^{+}\le 1$ $\forall i,j=1,2,...,n$ and the interval width of each element in $\hat{\bar{R}}$ remains the same for the corresponding element in $\bar{R}$. Theorem 4.2 further confirms that $\hat{\bar{R}}=\bar{R}$ if $\bar{R}$ is additive consistent. Therefore, a simple way to tell whether $\bar{R}$ is additive consistent is to compute $\hat{\bar{r}}_{ij}$ $\forall i<j=1,2,...,n$ and examine if $\hat{\bar{r}}_{ij}=\bar{r}_{ij}$. This judgment method only needs to compute $n\cdot(n-1)/2$ values in contrast to the direct application of Definition 2.2 that has to entertain n^3 data.

If $\bar{R}$ is not additive consistent, we may obtain a matrix $\hat{\bar{R}}=([\hat{r}_{ij}^{-},\hat{r}_{ij}^{+}])_{n\times n}$ with entries outside $D([0,1])$. In this case, $\hat{\bar{R}}$ is not an IFPR. To construct an additive consistent IFPR from $\bar{R}$, interval values $[\hat{r}_{ij}^{-},\hat{r}_{ij}^{+}]$ have to be further converted to intervals on $D([0,1])$. This conversion process should presumably preserve additive transitivity and the complementary property in the sense of $\hat{\bar{r}}_{ji}=1-\hat{\bar{r}}_{ij}$.

Let

$$c=\begin{cases}1, & \text{if } \hat{r}_{ij}^{+}\le 1, \forall i,j=1,2,...,n\\ \max\{\hat{r}_{ij}^{+}\mid \hat{r}_{ij}^{+}>1, i,j=1,2,...,n\}, & \text{Otherwise}\end{cases} \tag{4.2}$$

It is obvious that $c\ge 1$, and $\hat{r}_{ij}^{-}\le \hat{r}_{ij}^{+}\le c$ $\forall i,j=1,2,...,n$. According to Theorem 4.1, we have $\hat{r}_{ij}^{-}=1-\hat{r}_{ji}^{+}$ $\forall i,j=1,2,...,n$. Thus, one can obtain $1-c\le 1-\hat{r}_{ji}^{+}=\hat{r}_{ij}^{-}\le \hat{r}_{ij}^{+}\le c$

$\forall i, j = 1,2,...,n$. Therefore, all elements in $\hat{\bar{R}} = ([\hat{r}_{ij}^-, \hat{r}_{ij}^+])_{n\times n}$ should lie between $[1-c, 1-c]$ and $[c,c]$, i.e., $[\hat{r}_{ij}^-, \hat{r}_{ij}^+] \in D([1-c,c])$ $\forall i, j = 1,2,...,n$.

In order to convert $\hat{\bar{R}}$ into an additive consistent IFPR, an appropriate transformation function $\varphi : D([1-c,c]) \to D([0,1])$ should possess the following properties:

(i) $\varphi([1-c,1-c]) = [0,0]$.

(ii) $\varphi([c,c]) = [1,1]$.

(iii) $\varphi([0.5,0.5]) = [0.5,0.5]$

(iv) $\varphi(\bar{x}) = 1 - \varphi(1-\bar{x})$ $\quad \forall \bar{x} \in D([1-c,c])$.

(v) $\forall \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}_1, \bar{y}_2, \bar{y}_3 \in D([1-c,c])$, if $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = \bar{y}_1 + \bar{y}_2 + \bar{y}_3$, then $\varphi(\bar{x}_1) + \varphi(\bar{x}_2) + \varphi(\bar{x}_3) = \varphi(\bar{y}_1) + \varphi(\bar{y}_2) + \varphi(\bar{y}_3)$.

(i) and (ii) ensure that the transformation function should be able to convert the smallest interval $[1-c,1-c]$ and the largest interval $[c,c]$ into [0, 0] and [1, 1] on $D([0,1])$, respectively. (iii) expects that $\varphi(.)$ maintain indifference to be [0.5,0.5] after conversion. (iv) requires that $\varphi(.)$ keep the complementary property in the sense of interval arithmetic. The last desired property (v) guarantees that additive transitivity remains after $\varphi(.)$ is applied. If a transformation function satisfies these five properties, the following theorem immediately follows.

Theorem 4.3 Let $\bar{R} = (\bar{r}_{ij})_{n\times n}$ be an IFPR, then $\hat{\bar{R}}' = (\varphi([\hat{r}_{ij}^-, \hat{r}_{ij}^+]))_{n\times n}$ is an additive consistent IFPR.

Next, similar to the function furnished in Herrera-Viedma et al. (2004) for fuzzy preference relations, the following transformation function with the aforesaid desired properties is provided for handling IFPRs. Let

$$\varphi(\bar{x}) = \left[\frac{x^- + c - 1}{2c-1}, \frac{x^+ + c - 1}{2c-1}\right] \quad \forall \bar{x} = [x^-, x^+] \in D([1-c,c]) \tag{4.3}$$

It is apparent that this function satisfies (i), (ii) and (iii). As for (iv), since

$$\begin{aligned}\varphi(\bar{x}) &= \left[\frac{x^- + c - 1}{2c-1}, \frac{x^+ + c - 1}{2c-1}\right] = 1 - \left[\frac{1 - x^+ + c - 1}{2c-1}, \frac{1 - x^- + c - 1}{2c-1}\right] = 1 - \varphi([1-x^+, 1-x^-]) \\ &= 1 - \varphi(1-\bar{x})\end{aligned}$$

(iv) is thus verified. Moreover, if $\bar{x}_1+\bar{x}_2+\bar{x}_3=\bar{y}_1+\bar{y}_2+\bar{y}_3$, (v) is confirmed as

$$\begin{aligned}\varphi(\bar{x}_1)+\varphi(\bar{x}_2)+\varphi(\bar{x}_3)&=\left[\frac{x_1^-+x_2^-+x_3^-+3c-3}{2c-1},\frac{x_1^++x_2^++x_3^++3c-3}{2c-1}\right]\\&=\left[\frac{y_1^-+y_2^-+y_3^-+3c-3}{2c-1},\frac{y_1^++y_2^++y_3^++3c-3}{2c-1}\right]=\varphi(\bar{y}_1)+\varphi(\bar{y}_2)+\varphi(\bar{y}_3)\end{aligned}$$

After applying the transformation function (4.3), $\hat{\bar{r}}_{ij}$ is converted to $\hat{\bar{r}}'_{ij}$ as shown below

$$\hat{\bar{r}}'_{ij}=[\hat{r}'^-_{ij},\hat{r}'^+_{ij}]=\varphi([\hat{r}^-_{ij},\hat{r}^+_{ij}])=\left[\frac{\hat{r}^-_{ij}+c-1}{2c-1},\frac{\hat{r}^+_{ij}+c-1}{2c-1}\right] \tag{4.4}$$

where c is defined by (4.2).

Corollary 4.1 Assume that the elements of $\hat{\bar{R}}'=\varphi(\hat{\bar{R}})=(\hat{\bar{r}}'_{ij})_{n\times n}$ are defined by (4.4), then $\hat{\bar{R}}'$ is an additive consistent IFPR and $\hat{r}'^+_{ij}-\hat{r}'^-_{ij}=\dfrac{r^+_{ij}-r^-_{ij}}{2c-1}$ $\forall i,j=1,2,...,n$.

Proof. It can be obtained from Theorem 4.3 that $\hat{\bar{R}}'$ is an additive consistent IFPR. By Theorem 4.1, we have $\hat{r}^+_{ij}-\hat{r}^-_{ij}=r^+_{ij}-r^-_{ij}$ $\forall i,j=1,2,...,n$. It follows that $\hat{r}'^+_{ij}-\hat{r}'^-_{ij}=\dfrac{\hat{r}^+_{ij}-\hat{r}^-_{ij}}{2c-1}=\dfrac{r^+_{ij}-r^-_{ij}}{2c-1}$. ■

Corollary 4.1 shows that an additive consistent IFPR can be constructed from any given $\bar{R}$. If $\bar{R}$ is additive consistent, the constructed IFPR $\hat{\bar{R}}'=\bar{R}$. For a given additive inconsistent IFPR $\bar{R}$, if c = 1, the interval widths of each element in the constructed consistent IFPR $\hat{\bar{R}}'$ is equal to that of the corresponding element in the original IFPR $\bar{R}$; if c > 1, the proposed method scales down the interval widths of each element in $\bar{R}$ by a common factor $\dfrac{1}{2c-1}$. As the width of an interval is a natural way to gauge interval uncertainty, the constructed consistent IFPR $\hat{\bar{R}}'$ is able to keep the original interval uncertainty in terms of their widths if all converted elements in $\hat{\bar{R}}$ fall within *D*([0, 1]). In the case that some elements in $\hat{\bar{R}}$ have an upper bound above 1 or a lower bound below 0, this conversion process yields an $\hat{\bar{R}}'$ that proportionally scales down the largest upper bound to 1 and scales up the smallest negative lower bound to 0, thereby preserving interval uncertainty as much as possible.

Next, a numerical example is presented to show how to apply the proposed method.

Example 1. Consider the following three IFPRs,

$$\bar{R}_1 = \begin{bmatrix} [0.5,0.5] & [0.4,0.5] & [0.5,0.6] & [0.4,0.5] \\ [0.5,0.6] & [0.5,0.5] & [0.5,0.6] & [0.6,0.7] \\ [0.4,0.5] & [0.4,0.5] & [0.5,0.5] & [0.6,0.8] \\ [0.5,0.6] & [0.3,0.4] & [0.2,0.4] & [0.5,0.5] \end{bmatrix}$$

$$\bar{R}_2 = \begin{bmatrix} [0.5,0.5] & [0.1,0.3] & [0.8,0.9] & [0.5,0.6] \\ [0.7,0.9] & [0.5,0.5] & [0.7,0.9] & [0.9,1] \\ [0.1,0.2] & [0.1,0.3] & [0.5,0.5] & [0.8,0.9] \\ [0.4,0.5] & [0,0.1] & [0.1,0.2] & [0.5,0.5] \end{bmatrix}$$

$$\bar{R}_3 = \begin{bmatrix} [0.5,0.5] & [0.4,0.5] & [0.9,1] & [0.9,1] \\ [0.5,0.6] & [0.5,0.5] & [0.3,0.4] & [0.95,1] \\ [0,0.1] & [0.6,0.7] & [0.5,0.5] & [0.95,1] \\ [0,0.1] & [0,0.05] & [0,0.05] & [0.5,0.5] \end{bmatrix}$$

For $\bar{R}_1$, by using (4.1), one obtains the following transformation matrix:

$$\hat{\bar{R}}_1 = \begin{bmatrix} [0.5,0.5] & [0.37500,0.47500] & [0.41250,0.51250] & [0.51250,0.61250] \\ [0.52500,0.62500] & [0.5,0.5] & [0.48750,0.58750] & [0.58750,0.68750] \\ [0.48750,0.58750] & [0.41250,0.51250] & [0.5,0.5] & [0.50000,0.70000] \\ [0.38750,0.48750] & [0.31250,0.41250] & [0.30000,0.50000] & [0.5,0.5] \end{bmatrix}$$

Since all elements of $\hat{\bar{R}}_1$ are in $D([0,1])$, as per (4.2), we have c = 1. Therefore, the constructed additive consistent IFPR $\hat{\bar{R}}_1^{'}$ = $\hat{\bar{R}}_1$. It can be easily verified that the widths of the intervals in $\hat{\bar{R}}_1$ are equal to the widths of the corresponding elements in $\bar{R}_1$.

For $\bar{R}_2$ and $\bar{R}_3$, by using (4.1), the following transformation matrices are derived:

$$\hat{\bar{R}}_2 = \begin{bmatrix} [0.5,0.5] & [0.16250,0.36250] & [0.55000,0.65000] & [0.68750,0.78750] \\ [0.63750,0.83750] & [0.5,0.5] & [0.7375,0.9375] & [0.92500,1.02500] \\ [0.35000,0.45000] & [0.06250,0.26250] & [0.5,0.5] & [0.58750,0.68750] \\ [0.21250,0.31250] & [-0.0250,0.07500] & [0.31250,0.41250] & [0.5,0.5] \end{bmatrix}$$

$$\hat{\bar{R}}_3 = \begin{bmatrix} [0.5, 0.5] & [0.56875, 0.66875] & [0.61875, 0.71875] & [1.01250, 1.11250] \\ [0.33125, 0.43125] & [0.5, 0.5] & [0.50000, 0.60000] & [0.91875, 0.96875] \\ [0.28125, 0.38125] & [0.40000, 0.50000] & [0.5, 0.5] & [0.86875, 0.91875] \\ [-0.1125, -0.0125] & [0.03125, 0.08125] & [0.08125, 0.13125] & [0.5, 0.5] \end{bmatrix}$$

In $\hat{\bar{R}}_2$, the upper bound of $\hat{\bar{r}}_{24}$ is greater than 1 (correspondingly, the lower bound of $\hat{\bar{r}}_{42}$ is less than 0). In $\hat{\bar{R}}_3$, both the upper and lower bounds of $\hat{\bar{r}}_{14}$ are greater than 1 (correspondingly, the upper and lower bounds of $\hat{\bar{r}}_{41}$ are both less than 0). Based on (4.2), their corresponding values of *c* are 1.025 and 1.1125, respectively. As such, the resulting transformation functions are as follows

$$\varphi([\hat{r}_{ij}^-, \hat{r}_{ij}^+]) = \left[\frac{\hat{r}_{ij}^- + 0.025}{1.05}, \frac{\hat{r}_{ij}^+ + 0.025}{1.05}\right] \qquad \varphi([\hat{r}_{ij}^-, \hat{r}_{ij}^+]) = \left[\frac{\hat{r}_{ij}^- + 0.1125}{1.225}, \frac{\hat{r}_{ij}^+ + 0.1125}{1.225}\right]$$

Based on (4.4), the constructed consistent IFPRs based on $\bar{R}_2$ and $\bar{R}_3$ are obtained as

$$\hat{\bar{R}}_2^{'} = \begin{bmatrix} [0.5, 0.5] & [0.17857, 0.36905] & [0.54762, 0.64286] & [0.67857, 0.77381] \\ [0.63095, 0.82143] & [0.5, 0.5] & [0.72619, 0.91667] & [0.90476, 1.00000] \\ [0.35714, 0.45238] & [0.08333, 0.27381] & [0.5, 0.5] & [0.58333, 0.67857] \\ [0.22619, 0.32143] & [0.00000, 0.09524] & [0.32143, 0.41667] & [0.5, 0.5] \end{bmatrix}$$

$$\hat{\bar{R}}_3^{'} = \begin{bmatrix} [0.5, 0.5] & [0.55612, 0.63776] & [0.59694, 0.67857] & [0.91876, 1.00000] \\ [0.36224, 0.44388] & [0.5, 0.5] & [0.50000, 0.58163] & [0.84184, 0.88265] \\ [0.32143, 0.40306] & [0.41837, 0.50000] & [0.5, 0.5] & [0.80102, 0.84184] \\ [0.00000, 0.08163] & [0.11735, 0.15816] & [0.15816, 0.19898] & [0.5, 0.5] \end{bmatrix}$$

For the final constructed consistent IFPRs $\hat{\bar{R}}_2^{'}$ and $\hat{\bar{R}}_3^{'}$, computational results indicate that the widths of the original interval judgments in $\bar{R}_2$ and $\bar{R}_3$ have been scaled down by a factor of 1/1.05 and 1/1.225, respectively. By employing (3.2), one can determine the mean absolute deviations for all off-diagonal intervals between the original IFPRs and their corresponding constructed consistent IFPRs as follows:

$$d(\bar{R}_1, \hat{\bar{R}}_1^{'}) = 0.05833,\ d(\bar{R}_2, \hat{\bar{R}}_2^{'}) = 0.1246,\ d(\bar{R}_3, \hat{\bar{R}}_3^{'}) = 0.15425$$

5. Approaches to improving consistency of IFPRs

The proposed approach in Section 4 is able to construct an additive consistent IFPR $\hat{\bar{R}}'$ based on any given inconsistent IFPR $\bar{R}$. However, this consistency comes at a cost as the mean absolute deviation between $\bar{R}$ and $\hat{\bar{R}}'$ tends to be high. In many decision situations, a DM may relax this consistency requirement as long as the inconsistency is restricted to an acceptable level or the rectified IFPR possesses the weak transitivity property. Presumably, this relaxation will result in an IFPR with a smaller mean absolute deviation from the original IFPR $\bar{R}$. Similar to the treatment in Ma et al. (2006) for fuzzy preference relations, a weighted averaging scheme combining $\hat{\bar{R}}'$ and $\bar{R}$ is proposed as follows:

$$\tilde{R}(\lambda) = (\tilde{r}_{ij}(\lambda))_{n\times n} = (1-\lambda)\bar{R} + \lambda\hat{\bar{R}}' \tag{5.1}$$

where λ is a weight with $\lambda \in [0,1]$, $\hat{\bar{R}}' = (\hat{\bar{r}}'_{ij})_{n\times n}$ is defined by (4.4) and $\tilde{r}_{ij}(\lambda) = (1-\lambda)\bar{r}_{ij} + \lambda\hat{\bar{r}}'_{ij}$ for all $i, j = 1, 2, \ldots, n$.

As $\bar{R}$ and $\hat{\bar{R}}'$ are IFPRs, according to interval arithmetic and Definition 2.1 in Section 2, it is easy to prove the following result.

Theorem 5.1 Assume that $\tilde{R}(\lambda) = \left(\tilde{r}_{ij}(\lambda)\right)_{n\times n} = \left(\left[\tilde{r}_{ij}^-(\lambda), \tilde{r}_{ij}^+(\lambda)\right]\right)_{n\times n}$ is defined by (5.1), then for any $0 \le \lambda \le 1$, $\tilde{R}(\lambda)$ is an IFPR and $\tilde{r}_{ij}^+(\lambda) - \tilde{r}_{ij}^-(\lambda) = \left(1 - \frac{2(c-1)}{2c-1}\lambda\right)\left(r_{ij}^+ - r_{ij}^-\right)$.

If c = 1, it is apparent that $\tilde{r}_{ij}^+(\lambda) - \tilde{r}_{ij}^-(\lambda) = r_{ij}^+ - r_{ij}^-$, i.e., the interval width for any element in the original IFPR $\bar{R}$ (as well as the constructed consistent IFPR $\hat{\bar{R}}'$) remains the same after (5.1) is applied. If c > 1, it is easy to verify that $\frac{1}{2c-1} \le 1 - \frac{2(c-1)}{2c-1}\lambda$. Therefore, for any $0 \le \lambda \le 1$, $\hat{r}_{ij}^{'+} - \hat{r}_{ij}^{'-} = \frac{r_{ij}^+ - r_{ij}^-}{2c-1} \le \left(1 - \frac{2(c-1)}{2c-1}\lambda\right)\left(r_{ij}^+ - r_{ij}^-\right) = \tilde{r}_{ij}^+(\lambda) - \tilde{r}_{ij}^-(\lambda)$ $\le r_{ij}^+ - r_{ij}^-$. This means that the interval width for an element in $\tilde{R}(\lambda)$ lies between that for a corresponding element in the original IFPR $\bar{R}$ and that for a corresponding element in the constructed consistent IFPR $\hat{\bar{R}}'$.

Theorem 5.2 If $0 \le \lambda_1 \le \lambda_2 \le 1$, then $CI(\tilde{R}(\lambda_1)) \le CI(\tilde{R}(\lambda_2))$.

*Proof***.** As per (5.1), we have

$$\tilde{r}_{ij}^{-}(\lambda_1)+\tilde{r}_{ij}^{+}(\lambda_1)+\tilde{r}_{jk}^{-}(\lambda_1)+\tilde{r}_{jk}^{+}(\lambda_1)+\tilde{r}_{ki}^{-}(\lambda_1)+\tilde{r}_{ki}^{+}(\lambda_1)=$$

$$(1-\lambda_1)(r_{ij}^{-}+r_{ij}^{+}+r_{jk}^{-}+r_{jk}^{+}+r_{ki}^{-}+r_{ki}^{+})+\lambda_1(\hat{r}_{ij}^{'-}+\hat{r}_{ij}^{'+}+\hat{r}_{jk}^{'-}+\hat{r}_{jk}^{'+}+\hat{r}_{ki}^{'-}+\hat{r}_{ki}^{'+})\ \forall i,j,k=1,2,...,n$$

Since $\hat{\bar{R}}'$ is additive consistent, it follows from Lemma 2.1 that $\hat{r}_{ij}^{'-}+\hat{r}_{ij}^{'+}+\hat{r}_{jk}^{'-}+\hat{r}_{jk}^{'+}+\hat{r}_{ki}^{'-}+\hat{r}_{ki}^{'+}=3\ \forall i,j,k=1,2,...,n$. Then

$$CI(\tilde{R}(\lambda_1))=1-\frac{1}{3n(n-1)(n-2)}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{n}(|\tilde{r}_{ij}^{-}(\lambda_1)+\tilde{r}_{ij}^{+}(\lambda_1)+\tilde{r}_{jk}^{-}(\lambda_1)+\tilde{r}_{jk}^{+}(\lambda_1)+\tilde{r}_{ki}^{-}(\lambda_1)+ \tilde{r}_{ki}^{+}(\lambda_1)-3|)=1-\frac{1-\lambda_1}{3n(n-1)(n-2)}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{n}(|r_{ij}^{-}+r_{ij}^{+}+r_{jk}^{-}+r_{jk}^{+}+r_{ki}^{-}+r_{ki}^{+}-3|)$$

Similarly,

$$CI(\tilde{R}(\lambda_2))=1-\frac{1-\lambda_2}{3n(n-1)(n-2)}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{n}(|r_{ij}^{-}+r_{ij}^{+}+r_{jk}^{-}+r_{jk}^{+}+r_{ki}^{-}+r_{ki}^{+}-3|)$$

As $0\le\lambda_1\le\lambda_2\le1$, one can obtain that $CI(\tilde{R}(\lambda_1))\le CI(\tilde{R}(\lambda_2))$. ■

Theorem 5.2 indicates that $CI(\tilde{R}(\lambda))$ is an increasing function in $\lambda\in[0,1]$.

Theorem 5.3 Let $\bar{R}=(\bar{r}_{ij})_{n\times n}$ be an IFPR with an unacceptable consistency level, and t be an acceptable consistency threshold. If $\frac{t-CI(\bar{R})}{1-CI(\bar{R})}\le\lambda\le1$, then $\tilde{R}(\lambda)$ is an IFPR with acceptable consistency.

*Proof***.** Since

$$CI(\bar{R})=1-\frac{1}{3n(n-1)(n-2)}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{n}(|r_{ij}^{-}+r_{ij}^{+}+r_{jk}^{-}+r_{jk}^{+}+r_{ki}^{-}+r_{ki}^{+}-3|),$$

and

$$CI(\tilde{R}(\lambda))=1-\frac{1-\lambda}{3n(n-1)(n-2)}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{n}(|r_{ij}^{-}+r_{ij}^{+}+r_{jk}^{-}+r_{jk}^{+}+r_{ki}^{-}+r_{ki}^{+}-3|)$$

we have $CI(\tilde{R}(\lambda))=CI(\bar{R})+\lambda(1-CI(\bar{R}))$. Therefore, if $\frac{t-CI(\bar{R})}{1-CI(\bar{R})}\le\lambda\le1$, one can ascertain that $CI(\tilde{R}(\lambda))\ge t$. ■

By (5.1) and Theorem 5.2, one can see that if $\lambda\to0$, $\tilde{R}(\lambda)\to\bar{R}$, indicating that the closer the repaired IFPR $\tilde{R}(\lambda)$ reflects the original preference relation $\bar{R}$. However, the consistency level of $\tilde{R}(\lambda)$ will be lower. On the other hand, if $\lambda \to 1$, $\tilde{R}(\lambda) \to \hat{\bar{R}}$, implying that the closer $\tilde{R}(\lambda)$ is to the constructed additive consistent IFPR $\hat{\bar{R}}'$. Similarly, in this case, $\tilde{R}(\lambda)$ deviates further from the original IFPR $\bar{R}$. Therefore, according to Theorem 5.3, for a given IFPR $\bar{R}$ and an acceptable consistency threshold *t*, a sensible way to repair $\bar{R}$ is to apply (5.1) by setting

$$\lambda = \frac{t - CI(\bar{R})}{1 - CI(\bar{R})} \tag{5.2}$$

In this case, it is guaranteed that the modified IFPR $\tilde{R}(\lambda)$ has an acceptable consistency level and reflects the DM's original preference relation in $\bar{R}$ as much as possible.

Example 2. For the three original IFPRs $\bar{R}_1, \bar{R}_2, \bar{R}_3$ in Example 1, assume that an acceptable consistency threshold is established as *t* = 0.85. By Definition 3.1, one has $CI(\bar{R}_1) = 0.9 > t$, $CI(\bar{R}_2) = 0.78333 < t$ and $CI(\bar{R}_3) = 0.78333 < t$. Example 1 indicates that $\bar{R}_1$ is additive inconsistent. However, if the DM can accept certain inconsistency as reflected in the threshold *t* = 0.85, the consistency level of $\bar{R}_1$ is deemed acceptable, but $\bar{R}_2$ and $\bar{R}_3$ are deemed to have unacceptable consistency. In this case, their consistency levels have to be improved to reach the acceptable threshold by using (5.1) where $\hat{\bar{R}}_2'$ and $\hat{\bar{R}}_3'$ are the corresponding consistent IFPR obtained in Example 1 and λ is determined by (5.2).

Given that the $\bar{R}_2$ and $\bar{R}_3$ have the same consistency index, by using (5.2), we have $\lambda = 0.3077$ for both IFPRs. As per (5.1), one can obtain $\tilde{R}_2(0.3077) = 0.6923\bar{R}_2 + 0.3077\hat{\bar{R}}_2'$ and $\tilde{R}_3(0.3077) = 0.6923\bar{R}_3 + 0.3077\hat{\bar{R}}_3'$ as follows

$$\tilde{R}_2(0.3077) = \begin{bmatrix} [0.5, 0.5] & [0.12418, 0.32125] & [0.72234, 0.82088] & [0.55495, 0.65348] \\ [0.67875, 0.87582] & [0.5, 0.5] & [0.70806, 0.90513] & [0.90146, 1.00000] \\ [0.17912, 0.27766] & [0.09487, 0.29194] & [0.5, 0.5] & [0.73333, 0.83187] \\ [0.34652, 0.44505] & [0.00000, 0.09854] & [0.16813, 0.26667] & [0.5, 0.5] \end{bmatrix}$$

$$\tilde{R}_3(0.3077)=\begin{bmatrix} [0.5,0.5] & [0.44804,0.54239] & [0.80675,0.90110] & [0.90565,1.00000] \\ [0.45761,0.55196] & [0.5,0.5] & [0.36154,0.45589] & [0.91672,0.96389] \\ [0.09890,0.19325] & [0.54411,0.63846] & [0.5,0.5] & [0.90416,0.95133] \\ [0.00000,0.09435] & [0.03611,0.08328] & [0.04867,0.09584] & [0.5,0.5] \end{bmatrix}$$

One can verify that $CI(\tilde{R}_2(0.3077)) = CI(\tilde{R}_3(0.3077)) = 0.85 \geq t$. Therefore, after applying (5.1), the resulting $\tilde{R}_2(0.3077)$ and $\tilde{R}_3(0.3077)$ are two rectified IFPRs with acceptable consistency.

It should be noted that, if the DM is willing to accept limited inconsistency in a rectified IFPR $\tilde{R}$, its mean absolute deviation from the original IFPR $\bar{R}$ should be smaller than that between a constructed consistent IFPR $\hat{\bar{R}}'$ and $\bar{R}$. For instance, by using (3.2), one can verify that $d(\bar{R}_2, \tilde{R}_2(0.3077)) = 0.03834 < d(\bar{R}_2, \hat{\bar{R}}_2') = 0.1246$ and $d(\bar{R}_3, \tilde{R}_3(0.3077)) = 0.04746 < d(\bar{R}_3, \hat{\bar{R}}_3') = 0.15425$. Furthermore, computational results confirm a reduction ratio of 0.98535 between the interval width of an element in $\bar{R}_2$ and that of the corresponding element in $\tilde{R}_2(0.3077)$. Similarly, the reduction ratio is 0.94348 between the interval width of each element in $\bar{R}_3$ and that of the corresponding element in $\tilde{R}_3(0.3077)$. On the other hand, the corresponding ratios are $\frac{1}{1.05} = 0.95238$ and $\frac{1}{1.225} = 0.81633$ for the additive consistent IFPRs $\hat{\bar{R}}_2'$ and $\hat{\bar{R}}_3'$, respectively. This result indicates that, if the consistency requirement can be relaxed to an acceptable consistency threshold, one can obtain a modified IFPR that is closer to the original IFPR in terms of both the mean absolute deviation and the interval uncertainty as reflected in the interval width.

According to Definition 2.3, one can verify that IFPRs $\bar{R}_1, \bar{R}_3$ and $\tilde{R}_3(0.3077)$ are not weakly transitive, but $\bar{R}_2$ and $\tilde{R}_2(0.3077)$ are. This indicates that there does not exist definite inclusion relationship between weak transitivity and acceptable consistency. For instance, $\bar{R}_2$ is weakly transitive, but at t = 0.85, its consistency level is unacceptable. On the other hand, as long as t < 1, an IFPR with acceptable consistency is not necessarily weakly transitive. For example, $\tilde{R}_3(0.3077)$ is an IFPR with acceptable consistency at t = 0.85, but it is not weakly transitive due to $p(\tilde{r}_{21} \geq [0.5,0.5]) = 0.5507 > 0.5$, $p(\tilde{r}_{13} \geq [0.5,0.5]) = 1 > 0.5$, but $p(\tilde{r}_{23} \geq [0.5,0.5]) = 0 < 0.5$. Given that the consistency index increases in λ (Theorem 5.2) and, if t = 1 or $\lambda = 1$, $\tilde{R} = \hat{\tilde{R}}'$ is additive consistent (Theorem 4.3) and, hence, weakly transitive (Theorem 3.1), it is possible to obtain a rectified IFPR with both acceptable consistency and weak transitivity by increasing the value of λ. Next, we shall turn our attention to put forward a framework to repair an inconsistent IFPR, thereby obtaining a rectified IFPR with both acceptable consistency and weak transitivity.

In the following, assume that a DM provides his/her IFPR with strict comparison information, i.e., $p(\overline{r}_{ij} \geq [0.5,0.5]) \neq 0.5$ for all $i, j = 1,2,...,n$ and $i \neq j$. Then, for every IFPR, its associated preference matrix can be defined as follows.

Definition 5.1 The preference matrix $Q = (q_{ij})_{n\times n}$ of an IFPR $\overline{R} = (\overline{r}_{ij})_{n\times n}$ is defined as

$$q_{ij} = \begin{cases} 1, & p(\overline{r}_{ij} \geq [0.5,0.5]) > 0.5 \\ 0, & \text{otherwise} \end{cases} \qquad i, j=1,2,\ldots,n \tag{5.3}$$

The preference matrix Q expresses the DM's strict preference relations on X without considering preference degrees. $q_{ij} = 1$ indicates that the DM prefers x_i to x_j, while $q_{ij} = 0$ means that the DM prefers x_j to x_i. As $p(\overline{r}_{ij} \geq [0.5,0.5]) \neq 0.5$, the elements in Q satisfy $q_{ij} + q_{ji} = 1$ for all $i, j = 1,2,...,n, i \neq j$.

Let

$$q_i = \sum_{j=1}^{n} q_{ij}, \quad i = 1,2,...,n. \tag{5.4}$$

It is obvious that $0 \leq q_i \leq n-1$ for any i = 1 ,2 ,…, n, and $\sum_{i=1}^{n} q_i = \frac{n(n-1)}{2}$.

Preference matrices here can model team tournaments with q_{ij} characterizing whether team x_i defeats x_j, where $q_{ij} = 1$ indicates that x_i defeats x_j and $q_{ij} = 0$ describes that x_j defeats x_i, and no ties are allowed. According to the likelihood property (e) in Section 2, $p(\bar{r}_{ij} \geq [0.5, 0.5]) > 0.5$ implies $\frac{r_{ij}^{-} + r_{ij}^{+}}{2} > 0.5$. Therefore, the preference matrix Q here is equivalent to a preference matrix under the fuzzy preference relation $P = (p_{ij})_{n \times n}$ in Ma et al. (2006), where $p_{ij} = \frac{r_{ij}^{-} + r_{ij}^{+}}{2}$. According to Corollary 3.3 and Proposition 1 in Ma et al. (2006), the following judgment methods can be established for weak transitivity.

Corollary 5.1 Let $\bar{R} = (\bar{r}_{ij})_{n \times n}$ be an IFPR and $Q = (q_{ij})_{n \times n}$ be its associated preference matrix with $q_{ij} + q_{ji} = 1$ for all $i, j = 1, 2, ..., n, i \neq j$. Then $\bar{R}$ is weakly transitive if and only if $S = 0$, where S is defined as (Ma et al. 2006)

$$S = \frac{n(n-1)(n-2)}{6} - \frac{1}{2} \sum_{i=1}^{n} q_i (q_i - 1). \tag{5.5}$$

Alternatively, another method is furnished below to tell whether an IFPR is weakly transitive based on its associated preference matrix.

Corollary 5.2 Let $\bar{R} = (\bar{r}_{ij})_{n \times n}$ be an IFPR and $Q = (q_{ij})_{n \times n}$ be its associated preference matrix. Then $\bar{R}$ is weakly transitive if and only if n values q_i $(i = 1, 2, ..., n)$ can be ordered as $\{n-1, n-2, ..., 1, 0\}$.

Proof. First, we prove sufficiency. As q_i $(i = 1, 2, ..., n)$ can be ordered as $\{n-1, n-2, ..., 1, 0\}$, we have $\sum_{i=1}^{n} (q_i)^2 = \sum_{l=0}^{n-1} l^2 = \frac{n(n-1)(2n-1)}{6}$ and $\sum_{i=1}^{n} q_i = \frac{n(n-1)}{2}$.

Then, $S = \frac{n(n-1)(n-2)}{6} - \frac{1}{2} \sum_{i=1}^{n} q_i (q_i - 1) = 0$. By Corollary 5.1, $\bar{R}$ is weakly transitive.

Next, we prove the necessary part. Since $\bar{R}$ is weakly transitive, by Corollary 5.1, it follows that $\sum_{i=1}^{n} (q_i)^2 = \frac{n(n-1)(n-2)}{3} + \sum_{i=1}^{n} q_i$. On the other hand, as per Definition 5.1 and Eq. (5.4), one has $0 \leq q_i \leq n-1$ for any i = 1, 2,…, n, and $\sum_{i=1}^{n} q_i = \frac{n(n-1)}{2}$. Thus,

$\sum_{i=1}^{n}(q_i)^2=\frac{n(n-1)(2n-1)}{6}$. Therefore, the n values $q_i\ (i=1,2,...,n)$ can be arranged as $\{n-1,n-2,...,1,0\}$. ■

If $\bar{R}$ is weakly transitive, then the rank order of the objects on X is the same as the ordering of $q_i\ (i=1,2,...,n)$.

Based on Theorem 5.3, if an IFPR $\bar{R}$ given by a DM is inconsistent, it can be converted to $\tilde{R}(\lambda)$ with acceptable consistency by using (5.1), where $\lambda=\frac{t-CI(\bar{R})}{1-CI(\bar{R})}$. If $\tilde{R}(\lambda)$ is not weakly transitive, one can increase the value of λ to obtain a rectified IFPR with both acceptable consistency and weak transitivity.

Based on the aforesaid analyses, the following algorithm is formulated to improve the consistency of an IFPR.

***Algorithm*:** Let $\bar{R}=(\bar{r}_{ij})_{n\times n}$ be an original IFPR and t be an acceptable consistency threshold given by a DM. The iteration procedure is described as follows.

Step 1. Establish the transformation matrix $\hat{\bar{R}}=([\hat{r}_{ij}^{-},\hat{r}_{ij}^{+}])_{n\times n}$ as per (4.1).

Step 2. Compute the value c by using (4.2).

Step 3. Construct the additive consistent IFPR $\hat{\bar{R}}'=(\hat{\bar{r}}')_{n\times n}$ as per (4.4).

Step 4. Set $k = 0$ and calculate $\lambda_k=\max\left(\frac{t-CI(\bar{R})}{1-CI(\bar{R})},0\right)$, where $CI(\bar{R})$ is determined by (3.1).

Step 5. Derive the weighted IFPR $\tilde{R}^{(k)}$ by $\tilde{R}^{(k)}=(1-\lambda_k)\bar{R}+\lambda_k\hat{\bar{R}}'$;

Step 6. Establish the preference matrix $Q^{(k)}=(q_{ij}^{(k)})_{n\times n}$ of $\tilde{R}^{(k)}$ by (5.3);

Step 7. Calculate $(q_1^{(k)},q_2^{(k)},...,q_n^{(k)})$ as per (5.4) or compute $S^{(k)}$ by (5.5);

Step 8. As per Corollary 5.1 or 5.2, if $\tilde{R}^{(k)}$ is weakly transitive, go to Step 11; otherwise, go to the next step.

Step 9. Let $\lambda_{k+1}=\lambda_k+\delta$ and $k=k+1$, where δ is a given iteration step size, such that $0<\lambda_k+\delta\le 1$. To make the derived IFPR as close to the original IFPR as possible, a reasonably small value for δ is needed. Without loss of generality, set $0.01\le\delta\le 0.1$.

Step 10. If $\lambda_k < 1$, go to step 5; otherwise, let $\tilde{R}^{(k)} = \hat{\hat{R}}'$, and go to step 6;

Step 11. Output $k, \tilde{R}^{(k)}, Q^{(k)}, (q_1^{(k)}, q_2^{(k)}, \ldots, q_n^{(k)}), S^{(k)}$;

Step 12. End.

Next, it is ascertained that this algorithm will terminate after a finite number of iterations.

Theorem 5.4 Assume that $\bar{R} = (\bar{r}_{ij})_{n\times n}$ is an inconsistent IFPR, then a rectified IFPR with acceptable consistency and weak transitivity will be obtained after applying the above algorithm to $\bar{R}$ for a finite number of iterations.

Proof. For a given consistency threshold t and step size δ, there exists a natural number N such that $\lambda_0 + N\cdot\delta \geq 1$. Then after N iterations, we have $\tilde{R}^{(N)} = \hat{\hat{R}}'$. It follows from Corollary 4.1 that $\tilde{R}^{(N)}$ is an additive consistent IFPR and, hence, weakly transitive as per Theorem 3.1. On the other hand, if the algorithm terminates after k iterations where $k < N$, then $\tilde{R}^{(k)}$ is an IFPR with weak transitivity. Moreover, as $\lambda_k = \lambda_0 + k\cdot\delta \geq \lambda_0$, it follows from Theorem 5.3 that $\tilde{R}^{(k)}$ has acceptable consistency. Therefore, it is ascertained that a rectified IFPR with acceptable consistency and weak transitivity can always be obtained after a finite number of iterations. ■

Example 3. Assume that a DM conducts an exhaustive pairwise comparison on an alternative set $X = \{x_1, x_2, x_3, x_4\}$, and the result is given as the following IFPR:

$$\bar{R} = \begin{bmatrix} [0.5,0.5] & [0.8,1] & [0.7,0.9] & [0.5,0.9] \\ [0,0.2] & [0.5,0.5] & [0.5,0.7] & [0.7,0.9] \\ [0.1,0.3] & [0.3,0.5] & [0.5,0.5] & [0.6,0.8] \\ [0.1,0.5] & [0.1,0.3] & [0.2,0.4] & [0.5,0.5] \end{bmatrix}$$

Without loss of generality, let t = 0.85. In the following, the proposed algorithm is applied to improve the consistency level of $\bar{R}$.

As per (4.1), the transformation matrix $\hat{\hat{R}} = ([\hat{r}_{ij}^-, \hat{r}_{ij}^+])_{4\times 4}$ is established as

$$\hat{\hat{R}} = \begin{bmatrix} [0.5,0.5] & [0.6,0.7] & [0.55,0.65] & [0.65,1.05] \\ [0.3,0.4] & [0.5,0.5] & [0.35,0.55] & [0.65,0.75] \\ [0.35,0.45] & [0.45,0.65] & [0.5,0.5] & [0.7,0.8] \\ [-0.05,0.35] & [0.25,0.35] & [0.2,0.3] & [0.5,0.5] \end{bmatrix}$$

Since $[\hat{r}_{14}^{-},\hat{r}_{14}^{+}]$ and $[\hat{r}_{41}^{-},\hat{r}_{41}^{+}]$ do not fall in $D([0,1])$, by (4.2), we have c = 1.05. Then, the following IFPR with additive consistency is constructed as per (4.4).

$$\hat{\bar{R}}' = \begin{bmatrix} [0.5,0.5] & [0.59091,0.68182] & [0.54545,0.63636] & [0.63636,1.00000] \\ [0.31818,0.40909] & [0.5,0.5] & [0.36364,0.54545] & [0.63636,0.72727] \\ [0.36364,0.45455] & [0.45455,0.63636] & [0.5,0.5] & [0.68182,0.77273] \\ [0.00000,0.36364] & [0.27273,0.36364] & [0.22727,0.31818] & [0.5,0.5] \end{bmatrix}$$

As per (3.1), the consistency index of $\bar{R}$ is computed as $CI(\bar{R}) = 0.8$, then we have $\lambda_0 = \max\left(\dfrac{t - CI(\bar{R})}{1 - CI(\bar{R})}, 0\right) = 0.25$. Therefore, an acceptably consistent IFPR $\tilde{R}(0.25)$ is constructed by (5.2) with $\lambda = \lambda_0$.

$$\tilde{R}(0.25) = \begin{bmatrix} [0.5,0.5] & [0.82273,0.92046] & [0.43636,0.53409] & [0.53409,0.92500] \\ [0.07955,0.17727] & [0.5,0.5] & [0.46591,0.66136] & [0.75909,0.85682] \\ [0.46591,0.56364] & [0.33864,0.53409] & [0.5,0.5] & [0.69546,0.79318] \\ [0.07500,0.46591] & [0.14318,0.24091] & [0.20682,0.30455] & [0.5,0.5] \end{bmatrix}$$

Set k = 0 and $\tilde{R}^{(0)} = \tilde{R}(0.25)$, we can then examine whether $\tilde{R}^{(0)}$ is weakly transitive. By (5.3), the preference matrix $Q^{(0)} = (q_{ij})_{4\times 4}$ of the IFPR $\tilde{R}^{(0)}$ is derived as follows.

$$Q^{(0)} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Either Corollary 5.1 or 5.2 can be applied to judge if $\tilde{R}^{(0)}$ is weakly transitive. As per (5.4), we have $q_1^{(0)} = 2, q_2^{(0)} = 2, q_3^{(0)} = 2, q_4^{(0)} = 0$. By (5.5), one obtains $S^{(0)} = 1 \neq 0$.

From Corollary 5.1 or 5.2, one can see that $\tilde{R}^{(0)}$ is not weakly transitive. In fact, for preference matrix $Q^{(0)} = (q_{ij})_{4\times 4}$, one can get a cyclic preference relation, $x_1 \succ x_2 \succ x_3 \succ x_1$. Thus the weight λ in the rectifying formula (5.1) should be increased. Let $\delta = 0.02$, one can obtain a rectified IFPR $\tilde{R}^{(6)}$ with both weak transitivity and acceptable consistency. The iterative process to improving consistency for the IFPR $\bar{R}$ is described in Table 1.

It can be seen from Table 1 that $CI(\tilde{R}^{(0)}) < CI(\tilde{R}^{(1)}) < \cdots < CI(\tilde{R}^{(6)})$ and $d(\bar{R},\tilde{R}^{(0)}) > d(\bar{R},\tilde{R}^{(1)}) > \cdots > d(\bar{R},\tilde{R}^{(6)})$. This is understandable: as the iteration process continues, the consistency level of the resulting IFPR $\tilde{R}^{(k)}$ increases, but this comes at a cost with a greater deviation from the original IFPR $\bar{R}$. One can also see from the third last column that the rectified IFPR $\tilde{R}^{(k)}$ does not achieve weak transitivity until $k = 6$.

Table 1. The process to improving consistency of IFPR $\bar{R}$

Iteration k	Iterative preference relation $\tilde{R}^{(k)}$	$S^{(k)}$	$CI(\tilde{R}^{(k)})$	$d(\bar{R},\tilde{R}^{(k)})$
k=0	$\begin{bmatrix} [0.5,0.5] & [0.82273,0.92046] & [0.43636,0.53409] & [0.53409,0.92500] \\ [0.07955,0.17727] & [0.5,0.5] & [0.46591,0.66136] & [0.75909,0.85682] \\ [0.46591,0.56364] & [0.33864,0.53409] & [0.5,0.5] & [0.69546,0.79318] \\ [0.07500,0.46591] & [0.14318,0.24091] & [0.20682,0.30455] & [0.5,0.5] \end{bmatrix}$	1	0.85	0.0379
k=1	$\begin{bmatrix} [0.5,0.5] & [0.81655,0.91409] & [0.43927,0.53682] & [0.53682,0.92700] \\ [0.08591,0.18345] & [0.5,0.5] & [0.46318,0.65827] & [0.75582,0.85336] \\ [0.46318,0.56073] & [0.34173,0.53682] & [0.5,0.5] & [0.69509,0.79264] \\ [0.07300,0.46318] & [0.14664,0.24418] & [0.20736,0.30491] & [0.5,0.5] \end{bmatrix}$	1	0.854	0.0409
k=2	$\begin{bmatrix} [0.5,0.5] & [0.81036,0.90773] & [0.44218,0.53954] & [0.53954,0.92900] \\ [0.09227,0.18964] & [0.5,0.5] & [0.46046,0.65518] & [0.75254,0.84991] \\ [0.46046,0.55782] & [0.34482,0.53954] & [0.5,0.5] & [0.69473,0.79209] \\ [0.07100,0.46046] & [0.15009,0.24746] & [0.20791,0.30527] & [0.5,0.5] \end{bmatrix}$	1	0.858	0.0439
k=3	$\begin{bmatrix} [0.5,0.5] & [0.80418,0.90136] & [0.44509,0.54227] & [0.54227,0.93100] \\ [0.09864,0.19582] & [0.5,0.5] & [0.45773,0.65209] & [0.74927,0.84645] \\ [0.45773,0.55491] & [0.34791,0.54227] & [0.5,0.5] & [0.69436,0.79155] \\ [0.06900,0.45773] & [0.15355,0.25073] & [0.20845,0.30564] & [0.5,0.5] \end{bmatrix}$	1	0.862	0.0470
k=4	$\begin{bmatrix} [0.5,0.5] & [0.79800,0.89500] & [0.44800,0.54500] & [0.54500,0.93300] \\ [0.10500,0.20200] & [0.5,0.5] & [0.45500,0.64900] & [0.74600,0.84300] \\ [0.45500,0.55200] & [0.35100,0.54500] & [0.5,0.5] & [0.69400,0.79100] \\ [0.06700,0.45500] & [0.15700,0.25400] & [0.20900,0.30600] & [0.5,0.5] \end{bmatrix}$	1	0.866	0.0500
k=5	$\begin{bmatrix} [0.5,0.5] & [0.79182,0.88864] & [0.45091,0.54773] & [0.54773,0.93500] \\ [0.11136,0.20818] & [0.5,0.5] & [0.45227,0.64591] & [0.74273,0.83954] \\ [0.45227,0.54909] & [0.35409,0.54773] & [0.5,0.5] & [0.69364,0.79046] \\ [0.06500,0.45227] & [0.16046,0.25727] & [0.20954,0.30636] & [0.5,0.5] \end{bmatrix}$	1	0.87	0.0530
k=6	$\begin{bmatrix} [0.5,0.5] & [0.78564,0.88227] & [0.45382,0.55045] & [0.55045,0.93700] \\ [0.11773,0.21436] & [0.5,0.5] & [0.44955,0.64282] & [0.73945,0.83609] \\ [0.44955,0.54618] & [0.35718,0.55045] & [0.5,0.5] & [0.69327,0.78991] \\ [0.06300,0.44955] & [0.16391,0.26055] & [0.21009,0.30673] & [0.5,0.5] \end{bmatrix}$	0	0.874	0.0561

Next, the approach proposed by Xu (2011) is employed to improve consistency for the same IFPR $\bar{R}$ as a comparison.

Based on the additive consistency definition (Xu and Chen 2008), Xu (2011) develops a linear program (see (M-2) on page 3901) to construct a consistent IFPR from an inconsistent IFPR. Plugging $\bar{R}$ into (M-2) in Xu (2011) and solving this model, one can get the optimal nonzero deviation values $\dot{d}_{24}^- = 0.1333, \dot{d}_{34}^- = 0.0333, \dot{d}_{42}^+ = 0.1333$ and $\dot{d}_{43}^+ = 0.0333$. By (21) in Xu (2011), we obtain the following constructed IFPR with Xu and Chen (2008)'s additive consistency.

$$\hat{\bar{R}}^{Xu} = \begin{bmatrix} [0.5, 0.5] & [0.8, 1] & [0.7, 0.9] & [0.5, 0.9] \\ [0, 0.2] & [0.5, 0.5] & [0.5, 0.7] & [0.5667, 0.9] \\ [0.1, 0.3] & [0.3, 0.5] & [0.5, 0.5] & [0.5667, 0.8] \\ [0.1, 0.5] & [0.1, 0.4333] & [0.2, 0.4333] & [0.5, 0.5] \end{bmatrix}$$

By Definition 2.3, one can easily verify that $\hat{\bar{R}}^{Xu}$ has weak transitivity. On the other hand, as per (3.1) and (3.2), one has $CI(\hat{\bar{R}}^{Xu}) = 0.8389$ and $d(\bar{R}, \hat{\bar{R}}^{Xu}) = 0.0139$. Although $\hat{\bar{R}}^{Xu}$ is weakly transitive and $d(\bar{R}, \hat{\bar{R}}^{Xu}) < d(\bar{R}, \tilde{R}^{(k)})$ for all k = 0, 1, …, 6, $\hat{\bar{R}}^{Xu}$ does not possess acceptable consistency under (3.1) as $CI(\hat{\bar{R}}^{Xu}) < t = 0.85$. This difference is resulted from the fact that the two rectification approaches employ different additive consistency constraints. The constraint in Xu (2011) is established by the feasible region model and the proposed method herein is based on interval arithmetic.

6. Conclusion

Based on the additive consistency definition proposed by Wang and Li (2012), this article begins with presenting new properties for additive consistent IFPRs. Then, a consistency index is defined to measure the level of consistency for IFPRs, which can be conveniently applied to check whether an IFPR is consistent. Subsequently, an innovative approach is developed to construct an additive consistent IFPR from any inconsistent IFPR. By introducing a weighted averaging scheme that integrates the original and the constructed consistent IFPRs, a novel approach is put forward to improve consistency of IFPRs. An iterative algorithm is then established to repair an inconsistent IFPR to derive a rectified IFPR with both acceptable consistency and weak transitivity.

The basic modeling principle is to ensure that the derived IFPRs can improve consistency and, simultaneously, retain as much the initial interval uncertainty (measured by interval widths) as possible. Numerical examples are presented to demonstrate how to apply the proposed approaches. Further research is required to accommodate the cases when IFPRs contain missing judgment data and induced preference matrix Q includes indifference relations.

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