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### A multi-step goal programming approach for group decision making with incomplete interval additive reciprocal comparison matrices

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4 A multi-step goal programming approach for group decision making with incomplete interval  
5 additive reciprocal comparison matrices

6  
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## 14 15 **Abstract**

16  
17 This article presents a goal programming framework to solve group decision making  
18 problems where decision-makers' judgments are provided as incomplete interval additive  
19 reciprocal comparison matrices (IARCMs). New properties of multiplicative consistent  
20 IARCMs are put forward and used to define consistent incomplete IARCMs. A two-step goal  
21 programming method is developed to estimate missing values for an incomplete IARCM. The  
22 first step minimizes the inconsistency of the completed IARCMs and controls uncertainty  
23 ratios of the estimated judgments within an acceptable threshold, and the second step finds the  
24 most appropriate estimated missing values among the optimal solutions obtained from the  
25 previous step. A weighted geometric mean approach is proposed to aggregate individual  
26 IARCMs into a group IARCM by employing the lower bounds of the interval additive  
27 reciprocal judgments. A two-step procedure consisting of two goal programming models is  
28 established to derive interval weights from the group IARCM. The first model is devised to  
29 minimize the absolute difference between the logarithm of the group preference and that of  
30 the constructed multiplicative consistent judgment. The second model is developed to  
31 generate an interval-valued priority vector by maximizing the uncertainty ratio of the  
32 constructed consistent IARCM and incorporating the optimal objective value of the first  
33 model as a constraint. Two numerical examples are furnished to demonstrate validity and  
34 applicability of the proposed approach.

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51 *Keywords:* Goal programming, Interval additive reciprocal comparison matrices,  
52 Multiplicative consistency, Uncertainty, Group decision making

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## 53 54 55 **1. Introduction**

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57 The pairwise comparison method and hierarchy analysis technology have been widely used

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62 Email: wangzj@xmu.edu.cn (Z.J. Wang).

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4 to decompose a complex multi-criteria decision making (MCDM) into a series of more  
5 tractable and simpler sub-problems. In a conventional analytic hierarchy process (AHP)  
6 (Saaty, 1980), a decision problem is structured as a hierarchy of criteria, sub-criteria and  
7 alternatives, and a multiplicative reciprocal comparison matrix is employed to express a  
8 decision-maker's pairwise comparison results, where the judgments are provided as crisp  
9 values. However, in many real-life decision problems, a decision-maker's judgments may  
10 contain vagueness and uncertainty and, hence, cannot be represented as crisp data (Dubois &  
11 Prade, 2012; Durbach & Stewart, 2012; Entani & Sugihara, 2012; Guo & Tanaka, 2010; Saaty  
12 & Vargas, 1987; Wan & Li, 2013; Xia & Chen, 2014; Xu & Chen, 2008; Zhu & Xu, 2014). As  
13 such, other forms of pairwise comparison matrices have been developed to deal with  
14 imprecise and uncertain judgment information, such as interval multiplicative reciprocal  
15 comparison matrices (Saaty & Vargas, 1987) and interval additive reciprocal comparison  
16 matrices (IARCM) (also called interval fuzzy preference relations (Xu & Chen, 2008)).

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20 In a complete  $n \times n$  comparison matrix, all judgment values are totally known. Given the  
21 reciprocity of a comparison matrix, it implies that the decision-maker should provide either  
22 the upper or lower diagonal  $n(n-1)/2$  elements on a level with  $n$  alternatives or criteria. In  
23 reality, the decision-maker is sometimes unable or unwilling to provide his/her opinions over  
24 some alternatives due to insufficient information or limited expertise, especially in face of a  
25 large number of criteria or alternatives. In this situation, an incomplete comparison matrix is  
26 resulted (Alonso et al., 2008, 2010; Chiclana et al., 2008, 2009a; Fedrizzi & Giove, 2007;  
27 Gong, 2008; Herrera-Viedma et al., 2007; Liu, Zhang, & Wang, 2012; Liu, Pan, Xu, & Yu,  
28 2012; Xu, 2004, 2012; Xu, Li, & Wang, 2014). MCDM with incomplete comparison matrices  
29 have been receiving increasing attention and many different methods have been developed to  
30 estimate missing or unknown values for incomplete additive reciprocal comparison matrices  
31 (Alonso et al., 2008, 2010; Chiclana et al., 2009a; Gong, 2008; Herrera-Viedma et al., 2007;  
32 Liu, Pan, Xu, & Yu, 2012; Xu, 2004). For instance, Xu (2004) introduced the concept of  
33 incomplete additive reciprocal comparison matrices (or referred to as incomplete fuzzy  
34 preference relations), and proposed two goal programming models for obtaining priority  
35 weights of incomplete additive reciprocal comparison matrices from the viewpoints of  
36 additive transitivity and multiplicative consistency, respectively. An iterative procedure for  
37 estimating missing values was put forward by Herrera-Viedma et al. (2007) and applied to  
38 handle group decision making (GDM) problems with incomplete additive reciprocal  
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4 comparison matrices based on additive transitivity. Liu, Pan, Xu, and Yu (2012) put forward a  
5 completion method by establishing a least squares model. Based on multiplicative consistency,  
6 Alonso et al. (2010) furnished a procedure to estimate missing values and developed a  
7 web-based consensus support system for GDM with incomplete additive reciprocal  
8 comparison matrices.  
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12 Genç et al. (2010) employed the feasible-region-based multiplicative transitivity (Xu &  
13 Chen, 2008) to develop two estimation approaches for incomplete IARCMs. Xia and Xu  
14 (2011) extended the functional equation proposed by Chiclana et al. (2009b) to define perfect  
15 multiplicative consistent IARCMs and calculate missing values for incomplete IARCMs.  
16 From a multiplicative perspective, an interval additive reciprocal judgment can be  
17 transformed to an equivalent interval multiplicative reciprocal judgment (Liu, Zhang, &  
18 Zhang, 2013). After the conversion, the uncertainty level of the interval additive reciprocal  
19 judgment can be measured by the quotient of the upper and lower bounds of the  
20 corresponding interval multiplicative reciprocal judgment. Under this notion, a quotient of 1  
21 indicates a crisp judgment without any uncertainty and the larger the ratio, the more uncertain  
22 the interval judgment. For the foresaid estimation methods in (Genç et al., 2010; Xia & Xu,  
23 2011), no mechanism is designed to consider the acceptability of the uncertainty levels of the  
24 estimated interval additive reciprocal judgments. As such, they sometimes yield highly  
25 uncertain estimated values. To obtain rational and reliable decision results, it is crucial to  
26 adapt the acceptable uncertainty levels of the estimated values as highly uncertain data  
27 contains less beneficial decision information.  
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42 In a GDM process, once all individual incomplete comparison matrices are completed and  
43 a group comparison matrix is obtained from the completed individual comparison matrices, a  
44 pivotal remaining issue is to derive a priority vector from the group comparison matrix.  
45 According to additive or multiplicative transitivity, different prioritization methods have been  
46 developed for obtaining an interval-valued priority vector from a complete interval reciprocal  
47 comparison matrices such as linear programs (Arbel, 1989; Gou & Wang, 2012; Hu, Ren, Lan,  
48 Wang, & Zheng, 2014; Kress, 1991; Wang, Lan, Ren, & Luo, 2012; Xu & Chen, 2008),  
49 nonlinear programs (Xia & Xu, 2014), and goal programs (Wang & Elhag, 2007; Wang & Li,  
50 2012; Wang, Yang, & Xu, 2005).  
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58 Current research reveals that consistency properties are fundamental bases for estimating  
59 missing values and generating priority weights of pairwise comparison matrices. When  
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4 decision-makers' pairwise comparisons are represented as incomplete IARCMs in a GDM  
5 problem, it is important to evaluate missing values first before a group priority vector is  
6 derived. Based on the multiplicative consistency concept proposed by Wang and Li (2012),  
7 new properties of consistent IARCMs are presented and employed to define multiplicative  
8 consistent incomplete IARCMs. A two-step framework consisting of two goal programs is  
9 developed to estimate missing values for incomplete IARCMs. The first step aims to estimate  
10 missing values such that the resulting complete IARCM possesses either multiplicative  
11 consistency or minimal inconsistency, and uncertainty ratios of the estimated values are  
12 controlled to be within an acceptable threshold specified by the decision-maker. This is  
13 accomplished by minimizing the absolute difference between the two sides of the logarithmic  
14 expression of the multiplicative transitivity equation and imposing acceptable uncertainty  
15 ratio constraints. The second step is established to find the most appropriate estimated missing  
16 values among the optimal solutions obtained from the first model. The modeling idea is that  
17 the missing values in an incomplete IARCM reflect the decision-maker's uncertainty about  
18 the pairwise comparison. Therefore, by incorporating the optimal solutions in the first model  
19 into its constraints, the second model maximizes the uncertainty ratio for the estimated  
20 interval additive reciprocal judgments to retain the decision-maker's inherent uncertainty in  
21 the original missing values. Subsequently, a weighted geometric mean approach is put  
22 forward to aggregate individual preferences into a group IARCM by directly employing the  
23 lower bounds of the interval additive reciprocal judgments (upper bounds are indirectly  
24 utilized due to reciprocity). It is shown that the group IARCM has multiplicative consistency  
25 if all individual IFPRs have multiplicative consistency. Next, a two-step procedure  
26 comprising two goal programs is established to derive interval weights from the aggregated  
27 group IARCM. By employing a parameterized transformation relation between multiplicative  
28 consistent IARCMs and interval weights, the first model minimizes the absolute difference  
29 between the logarithm of the group preference and that of the transformed consistent  
30 judgment such that the constructed multiplicative consistent IARCMs are the closest to the  
31 group IARCM. The second model determines the most appropriate interval-valued priority  
32 vector by maximizing the uncertainty ratio of the constructed consistent IARCM and  
33 employing the optimal objective value of the first model as a constraint. The optimal  
34 interval-valued priority vector derived from the second model is able to be transformed to an  
35 IARCM with multiplicative consistency that is closest to that obtained by interval arithmetic

and the group IARCM. Finally, by putting the foresaid models together, an algorithm is proposed for solving GDM problems with incomplete IARCMs.

The remainder of the paper is organized as follows. Section 2 reviews some basic concepts related to additive reciprocal comparison matrices and IARCMs. New properties of multiplicative consistent IARCMs and the multiplicative consistency definition of incomplete IARCMs are introduced in Section 3. Section 4 develops two goal programs for estimating missing values in an incomplete IARCM. A goal programming approach is presented for generating an interval-valued priority vector of the group IARCM and a procedure is further put forward to solve GDM problems with incomplete IARCMs in Section 5. Section 6 provides concluding remarks.

## 2. Preliminaries

Let  $X = \{x_1, x_2, \dots, x_n\}$  be a set of  $n$  alternatives, if a pairwise comparison matrix  $R = (r_{ij})_{n \times n}$  on  $X$  satisfies

$$r_{ij} \in [0, 1], r_{ij} + r_{ji} = 1, r_{ii} = 0.5, \quad \forall i, j = 1, 2, \dots, n, \quad (2.1)$$

then  $R = (r_{ij})_{n \times n}$  is called an additive reciprocal comparison matrix (or referred to as an additive reciprocal preference relation (De Baets & De Meyer, 2005; De Baets, De Meyer, & De Loof, 2010)).

Element  $r_{ij}$  in  $R$  denotes the  $[0, 1]$ -valued preference or importance degree of  $x_i$  over  $x_j$ . The larger the value of  $r_{ij}$ , the smaller the value of  $r_{ji} = 1 - r_{ij}$  and the stronger the preference ratio  $\frac{r_{ij}}{r_{ji}}$  of  $x_i$  over  $x_j$ .  $r_{ij} > 0.5$  indicates that  $\frac{r_{ij}}{r_{ji}} > 1$  and  $x_i$  is superior to

$x_j$  with the preference ratio  $\frac{r_{ij}}{r_{ji}}$ .  $r_{ij} < 0.5$  shows that  $\frac{r_{ij}}{r_{ji}} < 1$  and  $x_i$  is non-preferred to  $x_j$

with the preference ratio  $\frac{r_{ij}}{r_{ji}}$ . Especially, if  $r_{ij} = 0.5$ , then  $\frac{r_{ij}}{r_{ji}} = 1$ , implying that  $x_i$  and  $x_j$

are equally preferred.

*Definition 2.1* (Tanino, 1984) Let  $R = (r_{ij})_{n \times n}$  be an additive reciprocal comparison matrix with  $0 < r_{ij} < 1, \forall i, j = 1, 2, \dots, n$ . If  $R$  satisfies the following transitivity condition:

$$\frac{r_{ik}}{r_{ki}} = \frac{r_{ij}}{r_{ji}} \frac{r_{jk}}{r_{kj}}, \forall i, j, k = 1, 2, \dots, n. \quad (2.2)$$

then  $R$  is said to have multiplicative consistency.

By the additive reciprocal property of  $r_{ij} + r_{ji} = 1$ , (2.2) can be equivalently expressed as the following functional equation (Chiclana et al., 2009b):

$$r_{ik} = \frac{r_{ij} r_{jk}}{r_{ij} r_{jk} + (1 - r_{ij})(1 - r_{jk})}, \forall i, j, k = 1, 2, \dots, n. \quad (2.3)$$

After examining the property of (2.3), Chiclana et al. (2009b) pointed out that the multiplicative consistency by Tanino (1984) is the most appropriate vehicle to model transitivity of additive reciprocal comparison matrices.

Due to increasing complexity of many decision problems, it is often hard for decision-makers to provide exact preferences over decision alternatives. To better characterize decision-makers' vague and uncertain preferences, Xu and Chen (2008) introduced the concept of IARCMs.

*Definition 2.2* (Xu & Chen, 2008) An IARCM  $\bar{R}$  on  $X$  is denoted by an interval-valued pairwise comparison matrix  $\bar{R} = (\bar{r}_{ij})_{n \times n}$  with the condition:

$$\bar{r}_{ij} = [r_{ij}^-, r_{ij}^+], 0 \leq r_{ij}^- \leq r_{ij}^+ \leq 1, r_{ij}^- + r_{ji}^+ = 1, r_{ij}^+ + r_{ji}^- = 1, r_{ii}^- = r_{ii}^+ = 0.5, \forall i, j = 1, 2, \dots, n. \quad (2.4)$$

where  $\bar{r}_{ij}$  gives an interval preference or importance degree of  $x_i$  over  $x_j$ .

The multiplicative consistency definition of IARCMs is given by Wang and Li (2012) as follows.

*Definition 2.3* (Wang & Li, 2012) Let  $\bar{R} = (\bar{r}_{ij})_{n \times n} = ([r_{ij}^-, r_{ij}^+])_{n \times n}$  be an IARCM with  $0 < r_{ij}^- \leq r_{ij}^+ < 1, \forall i, j = 1, 2, \dots, n$ . If  $\bar{R}$  satisfies multiplicative transitivity:

$$\frac{\bar{r}_{ij}}{\bar{r}_{ji}} \otimes \frac{\bar{r}_{jk}}{\bar{r}_{kj}} \otimes \frac{\bar{r}_{ki}}{\bar{r}_{ik}} = \frac{\bar{r}_{ik}}{\bar{r}_{ki}} \otimes \frac{\bar{r}_{kj}}{\bar{r}_{jk}} \otimes \frac{\bar{r}_{ji}}{\bar{r}_{ij}}, \forall i, j, k = 1, 2, \dots, n, \quad (2.5)$$

where “ $\otimes$ ” and “ $-$ ” indicate the interval multiplication and division operations, respectively,

then  $\bar{R}$  is said to have multiplicative consistency.

### 3. Multiplicative consistency

In this section, we first introduce new properties for multiplicative consistent IARCMs and, then employ these properties to define multiplicative consistency for incomplete

IARCMs.

Based on Definition 2.3, we have the following theorem.

**Theorem 3.1.** Let  $\bar{R} = (\bar{r}_{ij})_{n \times n} = ([r_{ij}^-, r_{ij}^+])_{n \times n}$  be a complete IARCM with  $0 < r_{ij}^- \leq r_{ij}^+ < 1, \forall i, j = 1, 2, \dots, n$ .  $\bar{R}$  has multiplicative consistency if and only if

$$r_{ij}^- r_{ij}^+ r_{jk}^- r_{jk}^+ r_{ki}^- r_{ki}^+ = r_{ik}^- r_{ik}^+ r_{kj}^- r_{kj}^+ r_{ji}^- r_{ji}^+, \forall i, j, k = 1, 2, \dots, n. \quad (3.1)$$

*Proof.* First, we prove the sufficiency. As per (3.1), one gets  $\frac{r_{ij}^- r_{jk}^- r_{ki}^-}{r_{ji}^+ r_{kj}^+ r_{ik}^+} = \frac{r_{ik}^- r_{kj}^- r_{ji}^-}{r_{ki}^+ r_{jk}^+ r_{ij}^+}$  and

$\frac{r_{ij}^+ r_{jk}^+ r_{ki}^+}{r_{ji}^- r_{kj}^- r_{ik}^-} = \frac{r_{ik}^+ r_{kj}^+ r_{ji}^+}{r_{ki}^- r_{jk}^- r_{ij}^-}$ . According to interval arithmetic, we have

$$\frac{\bar{r}_{ij}^-}{\bar{r}_{ji}^+} \otimes \frac{\bar{r}_{jk}^-}{\bar{r}_{kj}^+} \otimes \frac{\bar{r}_{ki}^-}{\bar{r}_{ik}^+} = \left[ \frac{r_{ij}^- r_{jk}^- r_{ki}^-}{r_{ji}^+ r_{kj}^+ r_{ik}^+}, \frac{r_{ij}^+ r_{jk}^+ r_{ki}^+}{r_{ji}^- r_{kj}^- r_{ik}^-} \right] = \left[ \frac{r_{ik}^- r_{kj}^- r_{ji}^-}{r_{ki}^+ r_{jk}^+ r_{ij}^+}, \frac{r_{ik}^+ r_{kj}^+ r_{ji}^+}{r_{ki}^- r_{jk}^- r_{ij}^-} \right] = \frac{\bar{r}_{ik}^-}{\bar{r}_{ki}^+} \otimes \frac{\bar{r}_{kj}^-}{\bar{r}_{jk}^+} \otimes \frac{\bar{r}_{ji}^-}{\bar{r}_{ij}^+}. \text{ By Definition 2.3,}$$

$\bar{R}$  is an IARCM with multiplicative consistency.

Next, we prove the necessary part. As per (2.5) and interval arithmetic, one has

$$\frac{r_{ij}^- r_{jk}^- r_{ki}^-}{r_{ji}^+ r_{kj}^+ r_{ik}^+} = \frac{r_{ik}^- r_{kj}^- r_{ji}^-}{r_{ki}^+ r_{jk}^+ r_{ij}^+} \quad \forall i, j, k = 1, 2, \dots, n. \text{ Thus, (3.1) holds true. } \blacksquare$$

As (3.1) is equivalent to  $\sqrt{r_{ij}^- r_{ij}^+} \sqrt{r_{jk}^- r_{jk}^+} \sqrt{r_{ki}^- r_{ki}^+} = \sqrt{r_{ik}^- r_{ik}^+} \sqrt{r_{kj}^- r_{kj}^+} \sqrt{r_{ji}^- r_{ji}^+}, \forall i, j, k = 1, 2, \dots, n$ , the multiplicative consistency can be also called geometric consistency from the viewpoint of the geometric mean of interval endpoints.

As per the additive reciprocal property of  $r_{ji}^- = 1 - r_{ij}^+$  and  $r_{ji}^+ = 1 - r_{ij}^-$ , (3.1) can be equivalently rewritten as any of the following equations:

$$r_{ij}^- r_{jk}^- r_{ki}^- (1 - r_{ik}^-) (1 - r_{kj}^-) (1 - r_{ji}^-) = r_{ik}^- r_{kj}^- r_{ji}^- (1 - r_{ij}^-) (1 - r_{jk}^-) (1 - r_{ki}^-), \forall i, j, k = 1, 2, \dots, n. \quad (3.2)$$

$$r_{ij}^+ r_{jk}^+ r_{ki}^+ (1 - r_{ik}^+) (1 - r_{kj}^+) (1 - r_{ji}^+) = r_{ik}^+ r_{kj}^+ r_{ji}^+ (1 - r_{ij}^+) (1 - r_{jk}^+) (1 - r_{ki}^+), \forall i, j, k = 1, 2, \dots, n. \quad (3.3)$$

**Theorem 3.2.** Let  $\bar{R} = (\bar{r}_{ij})_{n \times n} = ([r_{ij}^-, r_{ij}^+])_{n \times n}$  be a complete IARCM with  $0 < r_{ij}^- \leq r_{ij}^+ < 1, \forall i, j = 1, 2, \dots, n$ , then the following statements are equivalent:

(i)  $r_{ij}^- r_{ij}^+ r_{jk}^- r_{jk}^+ r_{ki}^- r_{ki}^+ = r_{ik}^- r_{ik}^+ r_{kj}^- r_{kj}^+ r_{ji}^- r_{ji}^+, \forall i, j, k = 1, 2, \dots, n.$

(ii)  $r_{ij}^- r_{ij}^+ r_{jk}^- r_{jk}^+ r_{ki}^- r_{ki}^+ = r_{ik}^- r_{ik}^+ r_{kj}^- r_{kj}^+ r_{ji}^- r_{ji}^+, \forall i, j, k = 1, 2, \dots, n, i < j < k. \quad (3.4)$

*Proof.* Obviously, if (i) holds, (ii) follows.



(ii)  $\Rightarrow$  (i). As per (2.4), we have  $r_{ii}^- = r_{ii}^+ = 0.5$  for all  $i = 1, 2, \dots, n$ . Thus, (i) always holds if all or any two of the indices  $i, j, k$  are equal.

For  $i \neq j \neq k$ , there exist six possible cases:

(1)  $i < j < k$ . In this case, (i) is reduced to (ii). Thus, (i) holds.

(2)  $i < k < j$ . As per (3.4), we have  $r_{ik}^- r_{ik}^+ r_{kj}^- r_{kj}^+ r_{ji}^- r_{ji}^+ = r_{ij}^- r_{ij}^+ r_{jk}^- r_{jk}^+ r_{ki}^- r_{ki}^+$ . Then,  $r_{ij}^- r_{ij}^+ r_{jk}^- r_{jk}^+ r_{ki}^- r_{ki}^+ = r_{ik}^- r_{ik}^+ r_{kj}^- r_{kj}^+ r_{ji}^- r_{ji}^+$ .

(3)  $j < i < k$ . By (3.4), we obtain  $r_{ji}^- r_{ji}^+ r_{ik}^- r_{ik}^+ r_{kj}^- r_{kj}^+ = r_{jk}^- r_{jk}^+ r_{ki}^- r_{ki}^+ r_{ij}^- r_{ij}^+$ . Thus,  $r_{ij}^- r_{ij}^+ r_{jk}^- r_{jk}^+ r_{ki}^- r_{ki}^+ = r_{ik}^- r_{ik}^+ r_{kj}^- r_{kj}^+ r_{ji}^- r_{ji}^+$ .

Similarly, by shuffling the order of the indices  $i, j, k$ , (i) holds for the remaining three cases: (4)  $j < k < i$ , (5)  $k < i < j$  and (6)  $k < j < i$ . The proof is thus completed.  $\blacksquare$

As per the reciprocal property of  $r_{ji}^- = 1 - r_{ij}^+$  and  $r_{ji}^+ = 1 - r_{ij}^-$ , (3.4) can be equivalently expressed as any of the following equations:

$$r_{ij}^- r_{jk}^- r_{ki}^- (1 - r_{ik}^-) (1 - r_{kj}^-) (1 - r_{ji}^-) = r_{ik}^- r_{kj}^- r_{ji}^- (1 - r_{ij}^-) (1 - r_{jk}^-) (1 - r_{ki}^-), \forall i < j < k. \quad (3.5)$$

$$r_{ij}^+ r_{jk}^+ r_{ki}^+ (1 - r_{ik}^+) (1 - r_{kj}^+) (1 - r_{ji}^+) = r_{ik}^+ r_{kj}^+ r_{ji}^+ (1 - r_{ij}^+) (1 - r_{jk}^+) (1 - r_{ki}^+), \forall i < j < k. \quad (3.6)$$

Based on the foresaid theorems and analysis, the following corollary can be directly obtained.

**Corollary 3.1** Let  $\bar{R} = (\bar{r}_{ij})_{n \times n} = ([r_{ij}^-, r_{ij}^+])_{n \times n}$  be a complete IARCM with  $0 < r_{ij}^- \leq r_{ij}^+ < 1, \forall i, j = 1, 2, \dots, n$ , then the following statements are equivalent:

- (a)  $\bar{R}$  is multiplicative consistent;
- (b)  $r_{ij}^- r_{ij}^+ r_{jk}^- r_{jk}^+ r_{ki}^- r_{ki}^+ = r_{ik}^- r_{ik}^+ r_{kj}^- r_{kj}^+ r_{ji}^- r_{ji}^+, \forall i, j, k = 1, 2, \dots, n$ ;
- (c)  $r_{ij}^- r_{jk}^- r_{ki}^- (1 - r_{ik}^-) (1 - r_{kj}^-) (1 - r_{ji}^-) = r_{ik}^- r_{kj}^- r_{ji}^- (1 - r_{ij}^-) (1 - r_{jk}^-) (1 - r_{ki}^-), \forall i, j, k = 1, 2, \dots, n$ ;
- (d)  $r_{ij}^+ r_{jk}^+ r_{ki}^+ (1 - r_{ik}^+) (1 - r_{kj}^+) (1 - r_{ji}^+) = r_{ik}^+ r_{kj}^+ r_{ji}^+ (1 - r_{ij}^+) (1 - r_{jk}^+) (1 - r_{ki}^+), \forall i, j, k = 1, 2, \dots, n$ ;
- (e)  $r_{ij}^- r_{ij}^+ r_{jk}^- r_{jk}^+ r_{ki}^- r_{ki}^+ = r_{ik}^- r_{ik}^+ r_{kj}^- r_{kj}^+ r_{ji}^- r_{ji}^+, \forall i < j < k$ ;
- (f)  $r_{ij}^- r_{jk}^- r_{ki}^- (1 - r_{ik}^-) (1 - r_{kj}^-) (1 - r_{ji}^-) = r_{ik}^- r_{kj}^- r_{ji}^- (1 - r_{ij}^-) (1 - r_{jk}^-) (1 - r_{ki}^-), \forall i < j < k$ ;
- (g)  $r_{ij}^+ r_{jk}^+ r_{ki}^+ (1 - r_{ik}^+) (1 - r_{kj}^+) (1 - r_{ji}^+) = r_{ik}^+ r_{kj}^+ r_{ji}^+ (1 - r_{ij}^+) (1 - r_{jk}^+) (1 - r_{ki}^+), \forall i < j < k$ .

If  $r_{ij}^- r_{jk}^- r_{ki}^- = r_{ik}^- r_{kj}^- r_{ji}^-$  and  $r_{ij}^+ r_{jk}^+ r_{ki}^+ = r_{ik}^+ r_{kj}^+ r_{ji}^+$ , then  $r_{ij}^- r_{ij}^+ r_{jk}^- r_{jk}^+ r_{ki}^- r_{ki}^+ = r_{ik}^- r_{ik}^+ r_{kj}^- r_{kj}^+ r_{ji}^- r_{ji}^+$ . As per (a)

and (e) in Corollary 3.1, the following corollary can be derived.

**Corollary 3.2** Let  $\bar{R} = (\bar{r}_{ij})_{n \times n} = ([r_{ij}^-, r_{ij}^+])_{n \times n}$  be a complete IARCM with  $0 < r_{ij}^- \leq r_{ij}^+ < 1, \forall i, j = 1, 2, \dots, n$ , if  $r_{ij}^- r_{jk}^- r_{ki}^+ = r_{ik}^- r_{kj}^+ r_{ji}^+$  and  $r_{ij}^+ r_{jk}^+ r_{ki}^- = r_{ik}^+ r_{kj}^- r_{ji}^-$  for all  $i < j < k$ , then  $\bar{R}$  has multiplicative consistency.

It is worth noting that we cannot remove the constraint  $i < j < k$  in Corollary 3.2. If  $r_{ij}^- r_{jk}^- r_{ki}^+ = r_{ik}^- r_{kj}^+ r_{ji}^+$  and  $r_{ij}^+ r_{jk}^+ r_{ki}^- = r_{ik}^+ r_{kj}^- r_{ji}^-$  for all  $i, j, k = 1, 2, \dots, n$ , then let  $k = i$ , we have  $r_{ij}^- r_{ji}^- r_{ii}^+ = r_{ii}^- r_{ij}^+ r_{ji}^+ \Rightarrow 0.5 r_{ij}^- r_{ji}^- = 0.5 r_{ij}^+ r_{ji}^+ \Rightarrow r_{ij}^- r_{ji}^- = r_{ij}^+ r_{ji}^+$ . As per the reciprocal property of  $r_{ji}^- = 1 - r_{ij}^+$  and  $r_{ji}^+ = 1 - r_{ij}^-$ , one can obtain  $r_{ij}^- = r_{ij}^+$  for all  $i, j = 1, 2, \dots, n$ . In this case,  $\bar{R}$  is only an additive reciprocal comparison matrix. The implication of the restriction  $i < j < k$  is that the order of alternative indices matters for this consistency condition.

From the viewpoint of pairwise comparison, consistency conditions should be independent of alternative labels. Therefore, it is inappropriate to use  $r_{ij}^- r_{jk}^- r_{ki}^+ = r_{ik}^- r_{kj}^+ r_{ji}^+, r_{ij}^+ r_{jk}^+ r_{ki}^- = r_{ik}^+ r_{kj}^- r_{ji}^- \forall i < j < k$  to define consistent IARCM as the inverse of Corollary 3.2 does not hold.

Xia and Xu (2011) extended the functional equation (2.3) to define perfect multiplicative consistent IARCMs. It is easy to prove that the functional equation therein (See Eq. (11) on page 1048 in (Xia & Xu, 2011)) is equivalent to  $r_{ij}^- r_{jk}^- r_{ki}^+ = r_{ik}^- r_{kj}^+ r_{ji}^+, r_{ij}^+ r_{jk}^+ r_{ki}^- = r_{ik}^+ r_{kj}^- r_{ji}^- \forall i < j < k$ . Therefore, the perfect multiplicative consistency is dependent on alternative labels. One can verify that the perfect multiplicative consistency definition may yield contradictory results for the same pairwise comparisons when alternatives are relabeled in a different order.

For a complete IARCM  $\bar{R}$ , a decision-maker need provide  $n(n-1)/2$  upper (or lower) triangular interval additive reciprocal judgments. If the decision-maker is unable or unwilling to furnish his/her judgments over some pairs of alternatives for some reason, an incomplete IARCM is resulted and missing or unknown values may be the lower, upper or both bounds of additive reciprocal judgments.

*Definition 3.1* An IARCM  $\bar{R}$  is called incomplete if some lower, upper or both bounds of its interval additive reciprocal judgments are not provided by the decision-maker.

Note that Definition 3.1 slightly differs from the concept of incomplete IARCMs in Genç et al. (2010), where both the lower and upper bounds of a missing element in  $\bar{R}$  are required to

be unknown.

Due to reciprocity, an IARCM  $\bar{R}$  can be determined by  $n(n-1)$  lower or upper bounds of additive reciprocal judgments. Therefore, based on Corollary 3.1, the multiplicative consistency of incomplete IARCMs can be defined as follows by using lower bounds only.

*Definition 3.2* Let  $\bar{R} = (\bar{r}_{ij})_{n \times n} = ([r_{ij}^-, r_{ij}^+])_{n \times n}$  be an incomplete IARCM with  $0 < r_{ij}^- < 1, 0 < r_{ij}^+ < 1, \forall (i, j) \in K_{\bar{R}}^L$ .  $\bar{R}$  is multiplicative consistent if there exists  $\hat{r}_{ij}^-$  for all  $i, j = 1, 2, \dots, n$  such that

$$\hat{r}_{ij}^- \hat{r}_{jk}^- \hat{r}_{ki}^- (1 - \hat{r}_{ik}^-) (1 - \hat{r}_{kj}^-) (1 - \hat{r}_{ji}^-) = \hat{r}_{ik}^- \hat{r}_{kj}^- \hat{r}_{ji}^- (1 - \hat{r}_{ij}^-) (1 - \hat{r}_{jk}^-) (1 - \hat{r}_{ki}^-), \quad \forall i, j, k = 1, 2, \dots, n \quad (3.7)$$

$$0 < \hat{r}_{ij}^- < 1, \hat{r}_{ij}^{-L} \leq \hat{r}_{ij}^- \leq \hat{r}_{ij}^{-U}, \quad \forall i, j = 1, 2, \dots, n \quad (3.8)$$

$$\hat{r}_{ij}^- + \hat{r}_{ji}^- \leq 1, \quad i, j = 1, 2, \dots, n, i \neq j, (i, j) \notin K_{\bar{R}}^L, (j, i) \notin K_{\bar{R}}^L \quad (3.9)$$

where  $\hat{r}_{ij}^{-L}$  and  $\hat{r}_{ij}^{-U}$  are obtained by the following formulae:

$$\hat{r}_{ij}^{-L} = \begin{cases} r_{ij}^- & (i, j) \in K_{\bar{R}}^L \\ 0.5 & i = j \\ 0 & (i, j) \notin K_{\bar{R}}^L \end{cases}, \quad \hat{r}_{ij}^{-U} = \begin{cases} r_{ij}^- & (i, j) \in K_{\bar{R}}^L \\ 0.5 & i = j \\ 1 - r_{ji}^- & (i, j) \notin K_{\bar{R}}^L, (j, i) \in K_{\bar{R}}^L \\ 1 & (i, j) \notin K_{\bar{R}}^L, (j, i) \notin K_{\bar{R}}^L \end{cases} \quad (3.10)$$

#### 4. Goal programming models for estimating missing values

This section develops goal programming models to estimate missing values for incomplete IARCMs.

Eq. (3.2) can be equivalently rewritten as the following logarithmic expression:

$$\begin{aligned} \ln r_{ij}^- + \ln r_{jk}^- + \ln r_{ki}^- + \ln(1 - r_{ik}^-) + \ln(1 - r_{kj}^-) + \ln(1 - r_{ji}^-) = \\ \ln r_{ik}^- + \ln r_{kj}^- + \ln r_{ji}^- + \ln(1 - r_{ij}^-) + \ln(1 - r_{jk}^-) + \ln(1 - r_{ki}^-), \quad \forall i, j, k = 1, 2, \dots, n. \end{aligned} \quad (4.1)$$

Eq. (3.2) or (4.1) holds for multiplicative consistent IARCMs. However, if  $\bar{R} = (\bar{r}_{ij})_{n \times n} = ([r_{ij}^-, r_{ij}^+])_{n \times n}$  is inconsistent, then the elements in  $\bar{R}$  do not satisfy (3.2) or (4.1).

To estimate missing values in an inconsistent incomplete IARCM  $\bar{R}$ , some deviations are allowed by relaxing the relation in (3.2) or (4.1) for all  $(i, j) \in MV_{\bar{R}}^L$  and  $k = 1, 2, \dots, n, k \neq i, k \neq j$ , where

$$MV_{\bar{R}}^L = \{(i, j) \mid i, j = 1, 2, \dots, n, i \neq j\} - K_{\bar{R}}^L \quad (4.2)$$

For an interval additive reciprocal judgment  $\bar{r}_{ij} = [r_{ij}^-, r_{ij}^+] = [r_{ij}^-, 1 - r_{ji}^-]$  with  $0 < r_{ij}^- \leq r_{ij}^+ < 1$ , its equivalent interval multiplicative reciprocal judgment is computed as

$$\left[ \frac{r_{ij}^-}{r_{ji}^+}, \frac{r_{ij}^+}{r_{ji}^-} \right] = \left[ \frac{r_{ij}^-}{1 - r_{ij}^-}, \frac{1 - r_{ji}^-}{r_{ji}^-} \right]$$

whose uncertainty ratio is the quotient of its upper and lower bounds. The larger the uncertainty ratio  $\frac{(1 - r_{ji}^-)(1 - r_{ij}^-)}{r_{ji}^- r_{ij}^-}$ , the more uncertain the interval

judgment  $\bar{r}_{ij}$ . It is widely accepted that extremely uncertain judgment information has little or no use in reaching final decision results (Dubois & Prade, 2012; Entani & Sugihara, 2012; Guo & Tanaka, 2012). Therefore, it is sensible to consider acceptable uncertainty levels (as reflected by uncertainty ratios) of the estimated interval additive reciprocal judgments. Presumably, this uncertainty threshold should be solicited from the decision-maker. Based on this modelling idea, the following multiple objective programming models are established to estimate missing values for an incomplete IARCM  $\bar{R} = (\bar{r}_{ij})_{n \times n} = ([r_{ij}^-, r_{ij}^+])_{n \times n}$ .

$$\begin{aligned} \min J_{ij} &= \sum_{\substack{k=1,2,\dots,n \\ k \neq i, k \neq j}} \left| \frac{\ln r_{ij}^- + \ln r_{jk}^- + \ln r_{ki}^- + \ln(1 - r_{ik}^-) + \ln(1 - r_{kj}^-) + \ln(1 - r_{ji}^-)}{\ln r_{ik}^- + \ln r_{kj}^- + \ln r_{ji}^- + \ln(1 - r_{ij}^-) + \ln(1 - r_{jk}^-) + \ln(1 - r_{ki}^-)} \right| & (i, j) \in MV_{\bar{R}}^L \\ \text{s.t.} & \begin{cases} 0 < r_{ij}^- < 1, & \forall (i, j) \in MV_{\bar{R}}^L \\ r_{ij}^- + r_{ji}^- \leq 1, & \forall (i, j) \in MV_{\bar{R}}^L \\ \frac{(1 - r_{ij}^-)(1 - r_{ji}^-)}{r_{ij}^- r_{ji}^-} \leq t, & \forall (i, j) \in MV_{\bar{R}}^L \end{cases} \end{aligned} \quad (4.3)$$

where  $t (t \geq 1)$  is an acceptable uncertainty ratio threshold for the estimated interval additive reciprocal judgments, the first line of inequalities ensures that the estimated values are (0, 1)-valued, the next line of constraints requires that the completed value  $r_{ij}^-$  together with  $1 - r_{ji}^-$  constitute an interval additive reciprocal judgment  $[r_{ij}^-, 1 - r_{ji}^-]$ , i.e.,  $r_{ij}^- \leq 1 - r_{ji}^-$ , the last group of inequalities guarantees that the estimated interval judgments  $[r_{ij}^-, 1 - r_{ji}^-]$  and  $[r_{ji}^-, 1 - r_{ij}^-]$  possess acceptable uncertainty ratios, and  $r_{ij}^- ((i, j) \in MV_{\bar{R}}^L)$  are decision variables, specifying the lower bounds of the missing interval additive reciprocal judgments that are to be estimated.

Let

$$\delta_{ij}^k = \ln r_{ij}^- + \ln r_{jk}^- + \ln r_{ki}^- + \ln(1-r_{ik}^-) + \ln(1-r_{kj}^-) + \ln(1-r_{ji}^-) - \left( \ln r_{ik}^- + \ln r_{kj}^- + \ln r_{ji}^- + \ln(1-r_{ij}^-) + \ln(1-r_{jk}^-) + \ln(1-r_{ki}^-) \right) \quad (4.4)$$

$$\delta_{ij}^{k+} = \frac{|\delta_{ij}^k| + \delta_{ij}^k}{2}, \quad \delta_{ij}^{k-} = \frac{|\delta_{ij}^k| - \delta_{ij}^k}{2} \quad (4.5)$$

for all  $(i, j) \in MV_{\bar{R}}^L$  and  $k = 1, 2, \dots, n, k \neq i, k \neq j$ .

As per (4.4) and (4.5), we have  $\delta_{ij}^k = \delta_{ij}^{k+} - \delta_{ij}^{k-}$ ,  $|\delta_{ij}^k| = \delta_{ij}^{k+} + \delta_{ij}^{k-}$  and  $\delta_{ij}^{k+} \delta_{ij}^{k-} = 0$  for all  $(i, j) \in MV_{\bar{R}}^L$  and  $k = 1, 2, \dots, n, k \neq i, k \neq j$ . Consequently, (4.3) is equivalently transformed to the following goal programming model:

$$\begin{aligned} \min \quad & J = \sum_{(i,j) \in MV_{\bar{R}}^L} \sum_{\substack{k=1, \\ k \neq i, k \neq j}}^n \alpha_{ij} (\delta_{ij}^{k+} + \delta_{ij}^{k-}) \\ \text{s.t.} \quad & \begin{cases} \ln r_{ij}^- + \ln r_{jk}^- + \ln r_{ki}^- + \ln(1-r_{ik}^-) + \ln(1-r_{kj}^-) + \ln(1-r_{ji}^-) - \\ \left( \ln r_{ik}^- + \ln r_{kj}^- + \ln r_{ji}^- + \ln(1-r_{ij}^-) + \ln(1-r_{jk}^-) + \ln(1-r_{ki}^-) \right) & \forall (i, j) \in MV_{\bar{R}}^L, \\ & k = 1, 2, \dots, n, \\ & k \neq i, k \neq j \\ -\delta_{ij}^{k+} + \delta_{ij}^{k-} = 0, & \\ 0 < r_{ij}^- < 1, & \forall (i, j) \in MV_{\bar{R}}^L \\ r_{ij}^- + r_{ji}^- \leq 1, (1-r_{ij}^-)(1-r_{ji}^-) \leq tr_{ij}^- r_{ji}^-, & \forall (i, j) \in MV_{\bar{R}}^L \\ \delta_{ij}^{k+} \geq 0, \delta_{ij}^{k-} \geq 0. & \forall (i, j) \in MV_{\bar{R}}^L, \\ & k = 1, 2, \dots, n, k \neq i, k \neq j \end{cases} \end{aligned} \quad (4.6)$$

where  $\alpha_{ij}$  is the weight of the objective function  $J_{ij}$  ( $(i, j) \in MV_{\bar{R}}^L$ ) in (4.3).

If all of the foresaid objective functions are uniformly weighted, one can set  $\alpha_{ij} = 1$   $(i, j) \in MV_{\bar{R}}^L$ , and (4.6) is rewritten as

$$\begin{aligned} \min \quad & J = \sum_{(i,j) \in MV_{\bar{R}}^L} \sum_{\substack{k=1, \\ k \neq i, k \neq j}}^n (\delta_{ij}^{k+} + \delta_{ij}^{k-}) \\ \text{s.t.} \quad & \begin{cases} \ln r_{ij}^- + \ln r_{jk}^- + \ln r_{ki}^- + \ln(1-r_{ik}^-) + \ln(1-r_{kj}^-) + \ln(1-r_{ji}^-) - \\ \left( \ln r_{ik}^- + \ln r_{kj}^- + \ln r_{ji}^- + \ln(1-r_{ij}^-) + \ln(1-r_{jk}^-) + \ln(1-r_{ki}^-) \right) & \forall (i, j) \in MV_{\bar{R}}^L, \\ & k = 1, 2, \dots, n, \\ & k \neq i, k \neq j \\ -\delta_{ij}^{k+} + \delta_{ij}^{k-} = 0, & \\ 0 < r_{ij}^- < 1, & \forall (i, j) \in MV_{\bar{R}}^L \\ r_{ij}^- + r_{ji}^- \leq 1, (1-r_{ij}^-)(1-r_{ji}^-) \leq tr_{ij}^- r_{ji}^-, & \forall (i, j) \in MV_{\bar{R}}^L \\ \delta_{ij}^{k+} \geq 0, \delta_{ij}^{k-} \geq 0. & \forall (i, j) \in MV_{\bar{R}}^L, \\ & k = 1, 2, \dots, n, k \neq i, k \neq j \end{cases} \end{aligned} \quad (4.7)$$

Alternative optimal solutions may exist for model (4.7) under a particular threshold  $t$ . As the missing values are inherently uncertain, it is logical to expect that the corresponding

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4 estimated interval additive reciprocal judgments properly reflect this uncertainty. In the  
5 context of multiplicative consistency, this uncertainty is captured by the uncertainty ratio,  
6 which is effectively contained by the threshold  $t$  in (4.7). To eventually estimate missing  
7 values, the following nonlinear program is established, which takes the optimal solution to  
8 (4.7) as its constraints and maximizes the uncertainty ratios. The aim is to retain the  
9 uncertainty inherent in the original missing values without sacrificing the consistency level  
10 and acceptable uncertain threshold achieved in (4.7).  
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$$\begin{aligned}
\max \quad & J' = \sum_{(i,j) \in MV_{\bar{R}}^L} \sum_{\substack{k=1, \\ k \neq i, k \neq j}}^n \frac{(1-r_{ij}^-)(1-r_{ji}^-)}{r_{ij}^- r_{ji}^-} \\
s.t. \quad & \begin{cases} \ln r_{ij}^- + \ln r_{jk}^- + \ln r_{ki}^- + \ln(1-r_{ik}^-) + \ln(1-r_{kj}^-) + \ln(1-r_{ji}^-) - \\ \left( \ln r_{ik}^- + \ln r_{kj}^- + \ln r_{ji}^- + \ln(1-r_{ij}^-) + \ln(1-r_{jk}^-) + \ln(1-r_{ki}^-) \right) \\ -\delta_{ij}^{k+} + \delta_{ij}^{k-} = 0, & \forall (i,j) \in MV_{\bar{R}}^L, \\ & k=1,2,\dots,n, \\ & k \neq i, k \neq j \\ \sum_{(i,j) \in MV_{\bar{R}}^L} \sum_{\substack{k=1, \\ k \neq i, k \neq j}}^n (\delta_{ij}^{k+} + \delta_{ij}^{k-}) = J^* \\ 0 < r_{ij}^- < 1, r_{ij}^- + r_{ji}^- \leq 1, (1-r_{ij}^-)(1-r_{ji}^-) \leq t_0 r_{ij}^- r_{ji}^-, & \forall (i,j) \in MV_{\bar{R}}^L \\ \delta_{ij}^{k+} \geq 0, \delta_{ij}^{k-} \geq 0. & \forall (i,j) \in MV_{\bar{R}}^L, \\ & k=1,2,\dots,n, k \neq i, k \neq j \end{cases} \quad (4.8)
\end{aligned}$$

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34 where  $J^*$  is the optimal objective value for model (4.7),  $t_0$  is the acceptable uncertainty  
35 ratio threshold therein, and  $r_{ij}^-$  ( $(i,j) \in MV_{\bar{R}}^L$ ) are decision variables.  
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39 By setting a threshold value  $t$  and solving (4.7), we obtain an optimal objective value  $J^*$ .  
40 Solving (4.8) yields its optimal solutions  $r_{ij}^{-*}$  ( $(i,j) \in MV_{\bar{R}}^L$ ) for the incomplete IARCM  
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$$\begin{aligned}
\bar{R} &= (\bar{r}_{ij})_{n \times n} = ([r_{ij}^-, r_{ij}^{+}])_{n \times n}, \quad \text{and a complete IARCM is determined as} \\
\bar{R}^c &= (\bar{r}_{ij}^c)_{n \times n} = ([r_{ij}^{-c}, r_{ij}^{+c}])_{n \times n}, \quad \text{where}
\end{aligned}$$

$$r_{ij}^{-c} = \begin{cases} r_{ij}^- & (i,j) \in K_{\bar{R}}^L \\ 0.5 & i = j \\ r_{ij}^{-*} & (i,j) \in MV_{\bar{R}}^L \end{cases}, \quad r_{ij}^{+c} = \begin{cases} r_{ij}^+ & (j,i) \in K_{\bar{R}}^L \\ 0.5 & i = j \\ 1-r_{ji}^{-*} & (j,i) \notin K_{\bar{R}}^L \end{cases} \quad (4.9)$$

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54 It is noted that if the objective value of (4.7)  $J^* = 0$  and the incomplete IARCM  $\bar{R}$   
55 has multiplicative consistency, then the completed IARCM  $\bar{R}^c$  satisfies (3.2), implying that  
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60  $\bar{R}^c$  has multiplicative consistency.

61 **Example 1.** Consider an MCDM problem with a set of four alternatives  $x_1, x_2, x_3, x_4$ . A  
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4 decision-maker employs the pairwise comparison method to elicit his/her judgment  
5 information and furnishes the following incomplete IARCM.  
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$$7 \quad \bar{R} = (\bar{r}_{ij})_{4 \times 4} = ([r_{ij}^-, r_{ij}^+])_{4 \times 4} = \begin{bmatrix} [0.5, 0.5] & [2/3, 3/4] & [3/4, 6/7] & [1/2, 3/4] \\ [1/4, 1/3] & [0.5, 0.5] & [3/5, 2/3] & - \\ [1/7, 1/4] & [1/3, 2/5] & [0.5, 0.5] & [1/4, 1/3] \\ [1/4, 1/2] & - & [2/3, 3/4] & [0.5, 0.5] \end{bmatrix}$$

14 where a “-” denotes a missing value.

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16 By Definition 3.2, one can easily verify that the incomplete IARCM  $\bar{R}$  has multiplicative  
17 consistency.  
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20 Plugging the incomplete IARCM  $\bar{R} = (\bar{r}_{ij})_{4 \times 4}$  into (4.7) and solving this model under  
21 different threshold  $t$  values by the optimization modelling software Lingo 11, we obtain their  
22 corresponding objective value  $J^*$  as shown in the last column of Table 1. Subsequently, this  
23 information is fed into model (4.8) to estimate the missing interval additive reciprocal  
24 judgments  $[r_{24}^-, r_{24}^+]$  and  $[r_{42}^-, r_{42}^+]$  as shown in Table 1.  
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31 **Table 1.** Estimated interval additive reciprocal judgments based on  $\bar{R}$

$t$	$[r_{24}^-, r_{24}^+]$	$[r_{42}^-, r_{42}^+]$	Objective value $J^*$ of (4.7)
1	[0.4142,0.4142]	[0.5858,0.5858]	$0.4 \times 10^{-9}$
1.5	[0.3660,0.4641]	[0.5359,0.6340]	$0.4 \times 10^{-9}$
2	[0.3333,0.5000]	[0.5000,0.6667]	$0.4 \times 10^{-9}$
2.5	[0.3090,0.5279]	[0.4721,0.6910]	$0.4173409 \times 10^{-8}$
3	[0.2899,0.5505]	[0.4495,0.7101]	$0.4173409 \times 10^{-8}$
3.5	[0.2743,0.5695]	[0.4305,0.7257]	$0.4173409 \times 10^{-8}$
4.0	[0.2612,0.5858]	[0.4142,0.7388]	$0.4173409 \times 10^{-8}$
4.5	[0.2500,0.6000]	[0.4000,0.75000]	$0.4173409 \times 10^{-8}$
5	[0.2403,0.6162]	[0.3874,0.7597]	$0.4173409 \times 10^{-8}$
5.5	[0.2317,0.6238]	[0.3762,0.7683]	$0.4173409 \times 10^{-8}$
6	[0.2240,0.6340]	[0.3660,0.7760]	$0.4173409 \times 10^{-8}$

Next, the estimation methods proposed by Genç et al. (2010) and Xia and Xu (2011) will be used to determine the completed interval additive reciprocal judgments based on the same input  $\bar{R}$ .

For a missing element  $\bar{r}_{ij}$  ( $(i, j) \in MV_{\bar{R}}^L, (j, i) \in MV_{\bar{R}}^L$ ), the estimation method by Genç et al. (2010) firstly identifies possible values of the missing element by a formula (See Eq. (28) in Genç et al. (2010)). The formula can be rewritten as per the notation in this article as:

$$\bar{r}_{ij}^{(k)} = [r_{ij}^{-(k)}, r_{ij}^{+(k)}] = \left[ \frac{r_{ik}^- r_{kj}^-}{r_{ik}^- r_{kj}^- + (1 - r_{ik}^-)(1 - r_{kj}^-)}, \frac{r_{ik}^+ r_{kj}^+}{r_{ik}^+ r_{kj}^+ + (1 - r_{ik}^+)(1 - r_{kj}^+)} \right] \quad (4.10)$$

where  $k$  satisfies  $(i, k) \notin MV_{\bar{R}}^L, (k, i) \notin MV_{\bar{R}}^L, (j, k) \notin MV_{\bar{R}}^L, (k, j) \notin MV_{\bar{R}}^L$ . These possible values are then aggregated by a weighted geometric operator (See Eq. (30) in Genç et al. (2010)) to determine the missing element as

$$\bar{r}_{ij} = \left[ \left( \prod r_{ij}^{-(k)} \right)^{1/\#K}, \left( \prod r_{ij}^{+(k)} \right)^{1/\#K} \right] \quad (4.11)$$

where  $\#K$  is the number of possible values.

By (4.10) and (4.11), one estimates missing interval additive judgments as  $\bar{r}_{24} = \left[ \sqrt{\frac{1}{12}}, \sqrt{\frac{3}{10}} \right] = [0.2887, 0.5477]$  and  $\bar{r}_{42} = [0.4523, 0.7113]$ .

Xia and Xu (2011) proposed another formula (See Eq. (45) in Xia and Xu (2011)) to estimate missing elements for incomplete IARCMs, which can be expressed by using the notation in this article as:

$$\bar{r}_{ij} = \left[ \min\{\xi_{ij}, \eta_{ij}\}, \max\{\xi_{ij}, \eta_{ij}\} \right], \quad (i, j) \in MV_{\bar{R}}^L, (j, i) \in MV_{\bar{R}}^L \quad (4.12)$$

where

$$\xi_{ij} = \frac{\left( \prod r_{ik}^- r_{kj}^- \right)^{1/\#K}}{\left( \prod r_{ik}^- r_{kj}^- \right)^{1/\#K} + \left( \prod (1 - r_{ik}^-)(1 - r_{kj}^-) \right)^{1/\#K}}, \quad \eta_{ij} = \frac{\left( \prod r_{ik}^+ r_{kj}^+ \right)^{1/\#K}}{\left( \prod r_{ik}^+ r_{kj}^+ \right)^{1/\#K} + \left( \prod (1 - r_{ik}^+)(1 - r_{kj}^+) \right)^{1/\#K}}. \quad (4.13)$$

As per (4.12), the completed missing values are determined as  $\bar{r}_{24} = [1/3, 0.5]$  and  $\bar{r}_{42} = [0.5, 2/3]$ .

Computation results indicate that the completed values obtained from the three different approaches are overall consistent. For this particular incomplete IARCM  $\bar{R}$ , the completed interval additive judgments obtained based on the method in Xia and Xu (2011) are identical to the results derived from the method here by setting  $t=2$  and Genç et al. (2010)'s approach



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4 yields the completed information that is very close to the result at  $t=3$  in Table 1. Generally  
5 speaking, it appears that the proposed approach here is able to generate the results obtained by  
6 the methods given by Genç et al. (2010) and Xia and Xu (2011) by properly setting the value  
7 of  $t$ . On the other hand, the models in Genç et al. (2010) and Xia and Xu (2011) do not  
8 possess a mechanism to address the acceptable uncertainty ratio issue for the estimated  
9 missing values. In addition, a decision-maker may sometimes provide the lower or upper  
10 bound of an interval judgment based on a pessimistic or optimistic scenario. In this case, a  
11 missing value in  $\bar{R}$  is not entirely unknown but only its lower or upper bound is unknown,  
12 such as the incomplete IARCMs in Example 2 in Section 5. It is worth noting that the two  
13 estimation models in (Genç et al., 2010; Xia & Xu, 2011) cannot handle such missing values,  
14 but our approach is convenient in tackling these cases.

## 24 5. Group decision making with incomplete IARCMs

25 Group decisions often occur when multiple stakeholders are involved in a decision situation  
26 and the final choice has to account for all stakeholders' input. Consider a GDM problem with  
27 a decision alternative set  $X = \{x_1, x_2, \dots, x_n\}$ . Assume that  $D = \{d_1, d_2, \dots, d_m\}$  is a set of  
28 decision-makers, and the importance weights of  $m$  decision-makers are  $(\lambda_1, \lambda_2, \dots, \lambda_m)^T$  with  
29

30  $\sum_{l=1}^m \lambda_l = 1$  and  $\lambda_l \geq 0$  for  $l=1, 2, \dots, m$ . Each DM  $d_l$  ( $l=1, 2, \dots, m$ ) provides his/her  
31 judgment over each pair of alternatives as an incomplete IARCM

$$32 \bar{R}^{(l)} = (\bar{r}_{ij}^{(l)})_{n \times n} = ([r_{ij}^{-(l)}, r_{ij}^{+(l)}])_{n \times n}.$$

33  
34 By (4.7) and (4.8), missing values in  $\bar{R}^{(l)}$  can be estimated to yield a corresponding  
35 complete IARCM  $\bar{R}^{(l)c} = (\bar{r}_{ij}^{(l)c})_{n \times n} = ([r_{ij}^{-(l)c}, r_{ij}^{+(l)c}])_{n \times n}$  ( $l=1, 2, \dots, m$ ). Next, a key issue is to  
36 aggregate the completed IARCMs  $\bar{R}^{(l)c}$  ( $l=1, 2, \dots, m$ ) into a group IARCM. The following  
37 discussion takes the same multiplicative consistency line of thinking.

38 Let

$$39 r_{ij}^{-G} = \frac{\prod_{l=1}^m (r_{ij}^{-(l)c})^{\lambda_l}}{\prod_{l=1}^m (r_{ij}^{-(l)c})^{\lambda_l} + \prod_{l=1}^m (1 - r_{ij}^{-(l)c})^{\lambda_l}}, \quad \forall i, j = 1, 2, \dots, n \quad (5.1)$$

40 where  $\sum_{l=1}^m \lambda_l = 1$  and  $\lambda_l \geq 0$  for all  $l=1, 2, \dots, m$ . Eq. (5.1) is an aggregation method based  
41

on the weighted geometric mean.

**Theorem 5.1.** Let  $\bar{R}^{(l)c} = (\bar{r}_{ij}^{(l)c})_{n \times n} = ([r_{ij}^{-(l)c}, r_{ij}^{+(l)c}])_{n \times n}$  ( $l=1,2,\dots,m$ ) be complete IARCMs with  $0 < r_{ij}^{-(l)c} \leq r_{ij}^{+(l)c} < 1, \forall i, j=1,2,\dots,n$ , and  $r_{ij}^{-G}$  be defined by (5.1), then

$\bar{R}^G = (\bar{r}_{ij}^G)_{n \times n} = ([r_{ij}^{-G}, 1 - r_{ji}^{-G}])_{n \times n}$  is an IARCM.

*Proof.* Obviously,  $r_{ii}^{-G} = 1 - r_{ii}^{-G} = 0.5, 0 < r_{ij}^{-G} < 1, 0 < 1 - r_{ji}^{-G} < 1$ , for all  $i, j=1,2,\dots,n$ .

On the other hand,

$$r_{ij}^{-(l)c} \leq r_{ij}^{+(l)c} = 1 - r_{ji}^{-(l)c} \quad \forall l=1,2,\dots,m \Rightarrow \frac{1}{r_{ij}^{-(l)c}} \geq \frac{1}{1 - r_{ji}^{-(l)c}} \quad \forall l=1,2,\dots,m \Rightarrow$$

$$\left( \frac{1}{r_{ij}^{-(l)c}} - 1 \right)^{\lambda_l} \geq \left( \frac{1}{1 - r_{ji}^{-(l)c}} - 1 \right)^{\lambda_l} \quad \forall l=1,2,\dots,m \Rightarrow \prod_{l=1}^m \left( \frac{1 - r_{ij}^{-(l)c}}{r_{ij}^{-(l)c}} \right)^{\lambda_l} + 1 \geq$$

$$\prod_{l=1}^m \left( \frac{r_{ji}^{-(l)c}}{1 - r_{ji}^{-(l)c}} \right)^{\lambda_l} + 1 \Rightarrow \frac{1}{\prod_{l=1}^m \left( \frac{1 - r_{ij}^{-(l)c}}{r_{ij}^{-(l)c}} \right)^{\lambda_l} + 1} \leq \frac{1}{\prod_{l=1}^m \left( \frac{r_{ji}^{-(l)c}}{1 - r_{ji}^{-(l)c}} \right)^{\lambda_l} + 1} \Rightarrow$$

$$\frac{\prod_{l=1}^m (r_{ij}^{-(l)c})^{\lambda_l}}{\prod_{l=1}^m (r_{ij}^{-(l)c})^{\lambda_l} + \prod_{l=1}^m (1 - r_{ij}^{-(l)c})^{\lambda_l}} = r_{ij}^{-G} \leq \frac{\prod_{l=1}^m (1 - r_{ji}^{-(l)c})^{\lambda_l}}{\prod_{l=1}^m (r_{ji}^{-(l)c})^{\lambda_l} + \prod_{l=1}^m (1 - r_{ji}^{-(l)c})^{\lambda_l}} = 1 - r_{ji}^{-G}$$

As per Definition 2.2,  $\bar{R}^G$  is an IARCM. ■

**Theorem 5.2.** Let  $\bar{R}^{(l)c} = (\bar{r}_{ij}^{(l)c})_{n \times n} = ([r_{ij}^{-(l)c}, r_{ij}^{+(l)c}])_{n \times n}$  ( $l=1,2,\dots,m$ ) be complete IARCMs with  $0 < r_{ij}^{-(l)c} \leq r_{ij}^{+(l)c} < 1$ , and  $r_{ij}^{-G}$  be defined by (5.1). If all  $\bar{R}^{(l)c}$  ( $l=1,2,\dots,m$ ) have multiplicative consistency, then  $\bar{R}^G = (\bar{r}_{ij}^G)_{n \times n} = ([r_{ij}^{-G}, 1 - r_{ji}^{-G}])_{n \times n}$  has multiplicative consistency.

*Proof.* As per (5.1), we have

$$\begin{aligned} \frac{r_{ij}^{-G}}{1 - r_{ij}^{-G}} \frac{r_{jk}^{-G}}{1 - r_{jk}^{-G}} \frac{r_{ki}^{-G}}{1 - r_{ki}^{-G}} &= \frac{\prod_{l=1}^m (r_{ij}^{-(l)c})^{\lambda_l}}{\prod_{l=1}^m (1 - r_{ij}^{-(l)c})^{\lambda_l}} \frac{\prod_{l=1}^m (r_{jk}^{-(l)c})^{\lambda_l}}{\prod_{l=1}^m (1 - r_{jk}^{-(l)c})^{\lambda_l}} \frac{\prod_{l=1}^m (r_{ki}^{-(l)c})^{\lambda_l}}{\prod_{l=1}^m (1 - r_{ki}^{-(l)c})^{\lambda_l}} \\ &= \prod_{l=1}^m \left( \frac{r_{ij}^{-(l)c} r_{jk}^{-(l)c} r_{ki}^{-(l)c}}{(1 - r_{ij}^{-(l)c})(1 - r_{jk}^{-(l)c})(1 - r_{ki}^{-(l)c})} \right)^{\lambda_l} \end{aligned}$$

and

$$\begin{aligned} \frac{r_{ik}^{-G}}{1-r_{ik}^{-G}} \frac{r_{kj}^{-G}}{1-r_{kj}^{-G}} \frac{r_{ji}^{-G}}{1-r_{ji}^{-G}} &= \frac{\prod_{l=1}^m (r_{ik}^{-(l)c})^{\lambda_l}}{\prod_{l=1}^m (1-r_{ik}^{-(l)c})^{\lambda_l}} \frac{\prod_{l=1}^m (r_{kj}^{-(l)c})^{\lambda_l}}{\prod_{l=1}^m (1-r_{kj}^{-(l)c})^{\lambda_l}} \frac{\prod_{l=1}^m (r_{ji}^{-(l)c})^{\lambda_l}}{\prod_{l=1}^m (1-r_{ji}^{-(l)c})^{\lambda_l}} \\ &= \prod_{l=1}^m \left( \frac{r_{ik}^{-(l)c} r_{kj}^{-(l)c} r_{ji}^{-(l)c}}{(1-r_{ik}^{-(l)c})(1-r_{kj}^{-(l)c})(1-r_{ji}^{-(l)c})} \right)^{\lambda_l} \end{aligned}$$

As  $\bar{R}^{(l)c}$  ( $l=1,2,\dots,m$ ) are all multiplicative consistent, by (3.2), one can obtain

$$\frac{r_{ij}^{-(l)c} r_{jk}^{-(l)c} r_{ki}^{-(l)c}}{(1-r_{ij}^{-(l)c})(1-r_{jk}^{-(l)c})(1-r_{ki}^{-(l)c})} = \frac{r_{ik}^{-(l)c} r_{kj}^{-(l)c} r_{ji}^{-(l)c}}{(1-r_{ik}^{-(l)c})(1-r_{kj}^{-(l)c})(1-r_{ji}^{-(l)c})},$$

$$\forall i, j, k = 1, 2, \dots, n, l = 1, 2, \dots, m.$$

Thus,  $\frac{r_{ij}^{-G}}{1-r_{ij}^{-G}} \frac{r_{jk}^{-G}}{1-r_{jk}^{-G}} \frac{r_{ki}^{-G}}{1-r_{ki}^{-G}} = \frac{r_{ik}^{-G}}{1-r_{ik}^{-G}} \frac{r_{kj}^{-G}}{1-r_{kj}^{-G}} \frac{r_{ji}^{-G}}{1-r_{ji}^{-G}}, \forall i, j, k = 1, 2, \dots, n$ . By Corollary 3.1,  $\bar{R}^G$

has multiplicative consistency.  $\blacksquare$

Theorem 5.1 indicates that a group complete IARCM  $\bar{R}^G$  is obtained by aggregating individual IFPRs as per (5.1). Theorem 5.2 further reveals that  $\bar{R}^G$  has multiplicative consistency if all individual complete IARCMs possess such a property.

Once the group complete IARCM  $\bar{R}^G$  is determined, the next issue for GDM is to derive a priority vector from  $\bar{R}^G$ .

Let  $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_n)^T = ([\omega_1^-, \omega_1^+], [\omega_2^-, \omega_2^+], \dots, [\omega_n^-, \omega_n^+])^T$  be an interval-valued weight vector satisfying the following normalization condition (Sugihara, Ishii, & Tanaka, 2004):

$$0 < \omega_i^- \leq \omega_i^+ \leq 1, \sum_{j=1}^n \omega_j^- + \omega_i^+ \leq 1, \omega_i^- + \sum_{j=1}^n \omega_j^+ \geq 1 \quad i = 1, 2, \dots, n \quad (5.2)$$

then we define the interval multiplicative reciprocal preference or importance intensity of  $x_i$

over  $x_j$  ( $i \neq j$ ),  $\bar{a}_{ij} = [a_{ij}^-, a_{ij}^+]$  as  $\left[ \frac{\omega_i^-}{\beta_{ij} \omega_j^+}, \frac{\beta_{ij} \omega_i^+}{\omega_j^-} \right]$ , where  $\beta_{ij}$  is a parameter such that

$$\sqrt{\frac{\omega_i^- \omega_j^-}{\omega_i^+ \omega_j^+}} \leq \beta_{ij} \leq 1 \quad \text{and} \quad \beta_{ji} = \beta_{ij} \quad \text{for all } i, j = 1, 2, \dots, n, i \neq j.$$

Let  $\bar{a}_{ii} = [a_{ii}^-, a_{ii}^+] = [1, 1]$ , one can verify that  $a_{ij}^- a_{ji}^+ = 1$ , i.e.,  $\bar{a}_{ij} = \frac{1}{\bar{a}_{ji}}$  for all

$i, j = 1, 2, \dots, n$ . Therefore,  $A = (\bar{a}_{ij})_{n \times n}$  is an interval multiplicative comparison matrix

introduced by Saaty and Vargas (1987).

It should be noted that interval arithmetic is quite different from crisp arithmetic. Normally,  $\frac{\bar{\omega}_i}{\bar{\omega}_j} \neq [1,1]$  and  $\frac{\bar{\omega}_i}{\bar{\omega}_j} \otimes \frac{\bar{\omega}_j}{\bar{\omega}_i} \neq [1,1]$ . For instance,  $\frac{[0.1,0.15]}{[0.1,0.15]} = \left[ \frac{2}{3}, \frac{3}{2} \right] \neq [1,1]$  and

$\frac{[0.1,0.15]}{[0.2,0.3]} \otimes \frac{[0.2,0.3]}{[0.1,0.15]} = \left[ \frac{4}{9}, \frac{9}{4} \right] \neq [1,1]$ . This indicates that a difference exists in the

uncertainty ratio obtained from the parameterized pairwise comparison interval  $\bar{a}_{ij}$  and that

derived by interval arithmetic  $\frac{\bar{\omega}_i}{\bar{\omega}_j} = \left[ \frac{\omega_i^-}{\omega_j^+}, \frac{\omega_i^+}{\omega_j^-} \right]$ .

Obviously, for any parameter value  $\beta_{ij}$ , the geometric means of the endpoints of all parameterized intervals  $\bar{a}_{ij} = \left[ \frac{\omega_i^-}{\beta_{ij}\omega_j^+}, \frac{\beta_{ij}\omega_i^+}{\omega_j^-} \right]$  are identical to that of  $\frac{\bar{\omega}_i}{\bar{\omega}_j}$  as

$\sqrt{\frac{\omega_i^- \omega_j^-}{\omega_i^+ \omega_j^+}} \leq \beta_{ij} \leq 1$ , but the uncertainty ratio differs between  $\left[ \frac{\omega_i^-}{\beta_{ij}\omega_j^+}, \frac{\beta_{ij}\omega_i^+}{\omega_j^-} \right]$  and  $\frac{\bar{\omega}_i}{\bar{\omega}_j}$ . If

$\beta_{ij} = \sqrt{\frac{\omega_i^- \omega_j^-}{\omega_i^+ \omega_j^+}}$ , one has  $a_{ij}^- = a_{ij}^+ = \sqrt{\frac{\omega_i^- \omega_i^+}{\omega_j^- \omega_j^+}}$ , indicating that the pairwise comparison between

$x_i$  and  $x_j$  is reduced to a crisp judgment without any uncertainty. In this case, the maximal

difference is achieved in the uncertainty ratio of  $\bar{a}_{ij}$  and  $\frac{\bar{\omega}_i}{\bar{\omega}_j}$ . If  $\beta_{ij} = 1$ , then

$\bar{a}_{ij} = \left[ \frac{\omega_i^-}{\omega_j^+}, \frac{\omega_i^+}{\omega_j^-} \right] = \frac{\bar{\omega}_i}{\bar{\omega}_j}$ , implying that the pairwise comparison  $\bar{a}_{ij}$  between  $x_i$  and  $x_j$  is

strictly based on interval arithmetic and, hence, there is no difference in uncertainty ratio of

$\bar{a}_{ij}$  and  $\frac{\bar{\omega}_i}{\bar{\omega}_j}$ . If  $\sqrt{\frac{\omega_i^- \omega_j^-}{\omega_i^+ \omega_j^+}} < \beta_{ij} < 1$ , then  $a_{ij}^- = \frac{\omega_i^-}{\beta_{ij}\omega_j^+} < \frac{\beta_{ij}\omega_i^+}{\omega_j^-} = a_{ij}^+$  and  $1 < \frac{(\beta_{ij})^2 \omega_i^+ \omega_j^+}{\omega_i^- \omega_j^-}$

$< \frac{\omega_i^+ \omega_j^+}{\omega_i^- \omega_j^-}$ , indicating that the pairwise comparison between  $x_i$  and  $x_j$ ,  $\bar{a}_{ij}$ , is not strictly

based on interval arithmetic, and a difference exists in the uncertainty ratio of  $\bar{a}_{ij}$  and  $\frac{\bar{\omega}_i}{\bar{\omega}_j}$ .

The larger the  $\beta_{ij}$ , the smaller the difference in the uncertainty ratio. Therefore,  $\beta_{ij}$  is a

parameter that characterizes the difference in the uncertainty ratio of the pairwise comparison  $\bar{a}_{ij}$  and the result determined by interval arithmetic  $\frac{\bar{\omega}_i}{\bar{\omega}_j}$ .

On the other hand, for any interval multiplicative reciprocal judgment  $[a_{ij}^-, a_{ij}^+]$ , its corresponding interval additive judgment can be determined as  $\left[ \frac{a_{ij}^-}{1+a_{ij}^-}, \frac{a_{ij}^+}{1+a_{ij}^+} \right]$ . As per the

multiplicative reciprocal property of  $a_{ij}^- a_{ji}^+ = 1, i, j = 1, 2, \dots, n$ , one can obtain

$$0 < \frac{a_{ij}^-}{1+a_{ij}^-} \leq \frac{a_{ij}^+}{1+a_{ij}^+} < 1 \quad \text{and} \quad \frac{a_{ij}^-}{1+a_{ij}^-} + \frac{a_{ji}^+}{1+a_{ji}^+} = \frac{a_{ij}^-}{1+a_{ij}^-} + \frac{1}{a_{ij}^- + 1} = 1 \quad \text{for all } i, j = 1, 2, \dots, n.$$

Therefore, for a given interval-valued priority weight vector  $\bar{\omega}$ , the interval additive reciprocal preference or importance intensity of  $x_i$  over  $x_j$ ,  $\bar{t}_{ij} = [t_{ij}^-, t_{ij}^+]$ , can be denoted by the following parameterized transformation function:

$$\bar{t}_{ij} = [t_{ij}^-, t_{ij}^+] = \begin{cases} [0.5, 0.5] & i = j \\ \left[ \frac{\omega_i^-}{\omega_i^- + \beta_{ij} \omega_j^+}, \frac{\beta_{ij} \omega_i^+}{\omega_j^- + \beta_{ij} \omega_i^+} \right] & i \neq j \end{cases} \quad (5.3)$$

where  $\beta_{ij}$  is a parameter such that  $\sqrt{\frac{\omega_i^- \omega_j^-}{\omega_i^+ \omega_j^+}} \leq \beta_{ij} \leq 1$  and  $\beta_{ji} = \beta_{ij}$  for all

$i, j = 1, 2, \dots, n, i \neq j$ .

**Theorem 5.3** Let  $\bar{t}_{ij}$  ( $i, j = 1, 2, \dots, n$ ) be defined by (5.3). If  $0 < \omega_i^- \leq \omega_i^+ \leq 1$  for all  $i = 1, 2, \dots, n$ , then  $\bar{T} = (\bar{t}_{ij})_{n \times n}$  is an IARCM with multiplicative consistency.

*Proof.* Obviously,  $0 \leq t_{ij}^- \leq 1$ ,  $0 \leq t_{ij}^+ \leq 1$ ,  $t_{ii}^- = t_{ii}^+ = 0.5$  and  $t_{ii}^- + t_{ii}^+ = 1$  for

$i, j = 1, 2, \dots, n$ . Since  $\beta_{ji} = \beta_{ij}$ , we have  $t_{ij}^- + t_{ji}^+ = \frac{\omega_i^-}{\omega_i^- + \beta_{ij} \omega_j^+} + \frac{\beta_{ji} \omega_j^+}{\omega_i^- + \beta_{ji} \omega_j^+} = 1$  and

$t_{ij}^+ + t_{ji}^- = \frac{\beta_{ij} \omega_i^+}{\omega_j^- + \beta_{ij} \omega_i^+} + \frac{\omega_j^-}{\omega_j^- + \beta_{ji} \omega_i^+} = 1$  for  $i, j = 1, 2, \dots, n, i \neq j$ . Thus, by Definition 2.2,

$\bar{T} = (\bar{t}_{ij})_{n \times n}$  is an IARCM.

As  $\beta_{ji} = \beta_{ij}$  for all  $i, j = 1, 2, \dots, n, i \neq j$ , by (5.3), one gets

$$\frac{t_{ij}^- t_{jk}^- t_{ki}^-}{t_{ji}^+ t_{kj}^+ t_{ik}^+} = \frac{\omega_i^- \omega_j^- \omega_k^-}{\beta_{ji} \omega_j^+ \beta_{kj} \omega_k^+ \beta_{ik} \omega_i^+} = \frac{\omega_i^- \omega_k^- \omega_j^-}{\beta_{ki} \omega_k^+ \beta_{jk} \omega_j^+ \beta_{ij} \omega_i^+} = \frac{t_{ik}^- t_{kj}^- t_{ji}^-}{t_{ki}^+ t_{jk}^+ t_{ij}^+}.$$

By Corollary 3.1,  $\bar{R}^G$  has multiplicative consistency. ■

Theorem 5.3 reveals that  $t_{ij}^-$  ( $i \neq j$ ) reflects the interval additive reciprocal preference intensity of  $x_i$  over  $x_j$ . By setting  $\beta_{ij}$  at different values, numerous multiplicative consistent IARCMs are obtained for a given normality interval-valued weight vector.

As per Theorem 5.3, if  $\bar{R}^G = \bar{T}$ , then there exists a normality interval-valued weight vector  $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_n)^T$  and  $\beta_{ij}$  ( $i, j = 1, 2, \dots, n, i \neq j$ ),  $\beta_{ji} = \beta_{ij}$  such that  $r_{ij}^{-G} = t_{ij}^-$  and  $1 - r_{ji}^{-G} = t_{ij}^+$  for all  $i, j = 1, 2, \dots, n$ . Apparently, such an  $\bar{R}^G$  is an IARCM with multiplicative consistency and  $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_n)^T$  is a normality interval-valued priority vector of  $\bar{R}^G$ .

However, in many group decision situations,  $\bar{R}^G$  has no multiplicative consistency. In this case, we turn around to find a normality interval-valued priority vector  $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_n)^T$  and  $\beta_{ij}$  ( $i, j = 1, 2, \dots, n, i \neq j$ ),  $\beta_{ji} = \beta_{ij}$  such that  $\bar{R}^G \approx \bar{T}$ . The closer  $\bar{R}^G$  and  $\bar{T}$  is, the better the interval-valued priority vector  $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_n)^T$  is. As per the additive reciprocal property of IARCMs, if  $r_{ij}^{-G} = \frac{\omega_i^-}{\omega_i^- + \beta_{ij} \omega_j^+}$  for  $i, j = 1, 2, \dots, n, i \neq j$ , then by

$$\beta_{ji} = \beta_{ij} \quad (i, j = 1, 2, \dots, n, i \neq j), \quad \text{one has } r_{ij}^{-G} = 1 - r_{ji}^{-G} = \frac{\beta_{ij} \omega_i^+}{\omega_j^- + \beta_{ij} \omega_i^+} \quad \text{for all}$$

$i, j = 1, 2, \dots, n, i \neq j$ . Thus, it is equivalent to find an interval-valued priority vector  $\bar{\omega}$  and  $\beta_{ij}$  ( $i, j = 1, 2, \dots, n, i \neq j$ ) such that

$$r_{ij}^{-G} \approx \frac{\omega_i^-}{\omega_i^- + \beta_{ij} \omega_j^+}, \quad i \neq j \quad (5.4)$$

From (5.4), we have

$$\frac{r_{ij}^{-G}}{1 - r_{ij}^{-G}} \approx \frac{\omega_i^-}{\beta_{ij} \omega_j^+}, \quad i \neq j \quad (5.5)$$

Eq. (5.5) can be equivalently expressed as

$$\ln r_{ij}^{-G} - \ln(1 - r_{ij}^{-G}) \approx \ln \omega_i^- - \ln \omega_j^+ - \ln \beta_{ij}, \quad i \neq j \quad (5.6)$$

Therefore, the following logarithmic goal programming model is established to find a group

interval-valued priority vector for  $\bar{R}^G$ .

$$\begin{aligned} \min \quad & J = \sum_{i=1}^n \sum_{j=1, j \neq i}^n \left| \ln r_{ij}^{-G} - \ln(1 - r_{ij}^{-G}) + \ln \beta_{ij} + \ln \omega_j^+ - \ln \omega_i^- \right| \\ \text{s.t.} \quad & \begin{cases} \ln \omega_i^- + \ln \omega_j^- - \ln \omega_i^+ - \ln \omega_j^+ \leq 2 \ln \beta_{ij} \leq 0, \ln \beta_{ji} = \ln \beta_{ij}, & i, j = 1, 2, \dots, n, i \neq j \\ 0 < \omega_i^- \leq \omega_i^+ \leq 1, \sum_{\substack{j=1 \\ j \neq i}}^n \omega_j^- + \omega_i^+ \leq 1, \omega_i^- + \sum_{\substack{j=1 \\ j \neq i}}^n \omega_j^+ \geq 1. & i = 1, 2, \dots, n \end{cases} \end{aligned} \quad (5.7)$$

where the first group of inequalities are the logarithmic expressions of  $\sqrt{\frac{\omega_i^- \omega_j^-}{\omega_i^+ \omega_j^+}} \leq \beta_{ij} \leq 1$  and  $\beta_{ij} = \beta_{ji}$ , the remaining constraints are the normalized conditions corresponding to (5.2), and  $\omega_i^-, \omega_i^+$  ( $i = 1, 2, \dots, n$ ) and  $\beta_{ij}$  ( $i, j = 1, 2, \dots, n, i \neq j$ ) are decision variables.

Let

$$\varepsilon_{ij} = \ln r_{ij}^{-G} - \ln(1 - r_{ij}^{-G}) + \ln \beta_{ij} + \ln \omega_j^+ - \ln \omega_i^- \quad (5.8)$$

$$\varepsilon_{ij}^+ = \frac{|\varepsilon_{ij}| + \varepsilon_{ij}}{2}, \quad \varepsilon_{ij}^- = \frac{|\varepsilon_{ij}| - \varepsilon_{ij}}{2} \quad (5.9)$$

for  $i, j = 1, 2, \dots, n, i \neq j$ .

Thus, we have  $\varepsilon_{ij} = \varepsilon_{ij}^+ - \varepsilon_{ij}^-$ ,  $|\varepsilon_{ij}| = \varepsilon_{ij}^+ + \varepsilon_{ij}^-$  and  $\varepsilon_{ij}^+ \varepsilon_{ij}^- = 0$  for  $i, j = 1, 2, \dots, n, i \neq j$ .

Therefore, model (5.7) is equivalently transformed to the following model:

$$\begin{aligned} \min \quad & J = \sum_{i=1}^n \sum_{j=1, j \neq i}^n (\varepsilon_{ij}^+ + \varepsilon_{ij}^-) \\ \text{s.t.} \quad & \begin{cases} \ln r_{ij}^{-G} - \ln(1 - r_{ij}^{-G}) + \ln \beta_{ij} + \ln \omega_j^+ - \ln \omega_i^- - \varepsilon_{ij}^+ + \varepsilon_{ij}^- = 0, & i < j = 1, 2, \dots, n \\ \ln r_{ij}^{-G} - \ln(1 - r_{ij}^{-G}) + \ln \beta_{ji} + \ln \omega_j^+ - \ln \omega_i^- - \varepsilon_{ij}^+ + \varepsilon_{ij}^- = 0, & i > j = 1, 2, \dots, n \\ \ln \omega_i^- + \ln \omega_j^- - \ln \omega_i^+ - \ln \omega_j^+ \leq 2 \ln \beta_{ij} \leq 0, & i < j = 1, 2, \dots, n \\ 0 < \omega_i^- \leq \omega_i^+ \leq 1, \sum_{\substack{j=1 \\ j \neq i}}^n \omega_j^- + \omega_i^+ \leq 1, \omega_i^- + \sum_{\substack{j=1 \\ j \neq i}}^n \omega_j^+ \geq 1, & i = 1, 2, \dots, n \\ \varepsilon_{ij}^+ \geq 0, \varepsilon_{ij}^- \geq 0. & i, j = 1, 2, \dots, n, i \neq j \end{cases} \end{aligned} \quad (5.10)$$

where  $\omega_i^-, \omega_i^+$  ( $i = 1, 2, \dots, n$ ),  $\beta_{ij}$  ( $j > i = 1, 2, \dots, n$ ) and  $\varepsilon_{ij}^+, \varepsilon_{ij}^-$  ( $i, j = 1, 2, \dots, n, i \neq j$ ) are decision variables.

Multiple solutions may exist for model (5.10). In order to obtain a reasonable decision result, it is natural to expect that the group opinions in  $\bar{R}^G$  be sufficiently reflected by the

1  
2  
3  
4  
5 final interval-valued priority vector as per interval arithmetic. As  $\sqrt{\frac{\omega_i^- \omega_j^-}{\omega_i^+ \omega_j^+}} \leq \beta_{ij} \leq 1$ , and the  
6  
7  
8 larger the  $\beta_{ij}$ , the closer  $\bar{T}$  is to the result of interval arithmetic operations, it is sensible to  
9  
10 select a solution of (5.10) that maximizes  $\beta_{ij}$  ( $j > i = 1, 2, \dots, n$ ) without sacrificing the  
11 consistency level. Based on this idea, we establish the following goal programming model,  
12  
13 which takes the optimal objective value  $J^*$  of (5.10) as a constraint.  
14  
15

$$\begin{aligned}
16 \quad & \max \quad J' = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \beta_{ij} \\
17 \quad & \left\{ \begin{array}{l}
18 \quad \ln r_{ij}^{-G} - \ln(1 - r_{ij}^{-G}) + \ln \beta_{ij} + \ln \omega_j^+ - \ln \omega_i^- - \varepsilon_{ij}^+ + \varepsilon_{ij}^- = 0, \quad j > i = 1, 2, \dots, n \\
19 \quad \ln r_{ij}^{-G} - \ln(1 - r_{ij}^{-G}) + \ln \beta_{ji} + \ln \omega_j^+ - \ln \omega_i^- - \varepsilon_{ij}^+ + \varepsilon_{ij}^- = 0, \quad i > j = 1, 2, \dots, n \\
20 \quad \ln \omega_i^- + \ln \omega_j^- - \ln \omega_i^+ - \ln \omega_j^+ \leq 2 \ln \beta_{ij} \leq 0, \quad j > i = 1, 2, \dots, n \\
21 \quad \sum_{i=1}^n \sum_{j=1, j \neq i}^n (\varepsilon_{ij}^+ + \varepsilon_{ij}^-) = J^* \\
22 \quad 0 < \omega_i^- \leq \omega_i^+ \leq 1, \sum_{\substack{j=1 \\ j \neq i}}^n \omega_j^- + \omega_i^+ \leq 1, \omega_i^- + \sum_{\substack{j=1 \\ j \neq i}}^n \omega_j^+ \geq 1, \quad i = 1, 2, \dots, n \\
23 \quad \varepsilon_{ij}^+ \geq 0, \varepsilon_{ij}^- \geq 0. \quad i, j = 1, 2, \dots, n, i \neq j
\end{array} \right. \quad (5.11)
\end{aligned}$$

24  
25  
26 By solving (5.11), we obtain an optimal group interval-valued priority vector denoted by  
27  
28  $\bar{\omega}^* = (\bar{\omega}_1^*, \bar{\omega}_2^*, \dots, \bar{\omega}_n^*)^T = ([\omega_1^-, \omega_1^+], [\omega_2^-, \omega_2^+], \dots, [\omega_n^-, \omega_n^+])^T$  for  $\bar{R}^G$ .  
29

30  
31 Based on the foresaid analyses, the following algorithm for GDM with incomplete  
32  
33 IARCMs is now developed and graphically illustrated in Figure 1.  
34

### 35 **Algorithm 1**

36  
37 *Step 1.* Consider a GDM problem with a set of decision alternatives  $X = \{x_1, x_2, \dots, x_n\}$  and  
38  
39 a group of decision-makers  $D = \{d_1, d_2, \dots, d_m\}$ . The decision-makers' importance weight  
40  
41 vector is  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T$  with  $\sum_{l=1}^m \lambda_l = 1$  and  $\lambda_l \geq 0$  ( $l = 1, 2, \dots, m$ ). The  
42  
43 decision-makers furnish their pairwise comparisons on  $X$  by means of incomplete IARCMs  
44  
45  $\bar{R}^{(l)} = (\bar{r}_{ij}^{(l)})_{n \times n} = ([r_{ij}^{-(l)}, r_{ij}^{+(l)}])_{n \times n}$  ( $l = 1, 2, \dots, m$ ).  
46  
47

48  
49 *Step 2.* Solicit an acceptable uncertainty ratio threshold  $t$  from the decision-makers and  
50  
51 estimate missing values for each  $\bar{R}^{(l)}$  ( $l = 1, 2, \dots, m$ ) by solving the models (4.7) and (4.8),  
52  
53 thereby deriving the individual complete IARCMs  $\bar{R}^{(l)c} = (\bar{r}_{ij}^{(l)c})_{n \times n} = ([r_{ij}^{-(l)c}, r_{ij}^{+(l)c}])_{n \times n}$   
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( $l=1,2,\dots,m$ ) as per (4.9).

*Step 3.* Aggregate individual complete IARCMs  $\bar{R}^{(l)c}$  ( $l=1,2,\dots,m$ ) together with the decision-maker weights  $\lambda_l$  ( $l=1,2,\dots,m$ ) into a group opinion

$$\bar{R}^G = (\bar{r}_{ij}^G)_{n \times n} = ([r_{ij}^{-G}, 1 - r_{ji}^{-G}])_{n \times n} \text{ as per (5.1).}$$

*Step 4.* Determine the optimal objective value  $J^*$  by solving (5.10).

*Step 5.* Solve model (5.11) and, then obtain an optimal group interval-valued priority vector

$$\bar{\omega}^* = (\bar{\omega}_1^*, \bar{\omega}_2^*, \dots, \bar{\omega}_n^*)^T = ([\omega_1^-, \omega_1^+], [\omega_2^-, \omega_2^+], \dots, [\omega_n^-, \omega_n^+])^T \text{ for } \bar{R}^G.$$

*Step 6.* Establish the possibility matrix  $P = (p_{ij})_{n \times n} = (P(\bar{\omega}_i^* \geq \bar{\omega}_j^*))_{n \times n}$  as per the following possibility formula (Wang, Yang, & Xu, 2005; Xu & Chen, 2008).

$$P(\bar{a} \geq \bar{b}) = \frac{\max\{0, a^+ - b^-\} - \max\{0, a^- - b^+\}}{a^+ - a^- + b^+ - b^-} \quad (5.12)$$

where  $\bar{a} = [a^-, a^+]$  and  $\bar{b} = [b^-, b^+]$  are two positive interval numbers.

*Step 7.* Add up all values in each row of  $P$ , we get  $\phi_i = \sum_{j=1}^n p_{ij}$  ( $i=1,2,\dots,n$ ).

*Step 8.* As per the decreasing order of the values  $\phi_i$  ( $i=1,2,\dots,n$ ), a ranking order of all decision alternatives is obtained, and “ $x_i$  being preferred to  $x_j$ ” is expressed as

$$x_i \succeq_{P(\bar{\omega}_i^* \geq \bar{\omega}_j^*)} x_j.$$

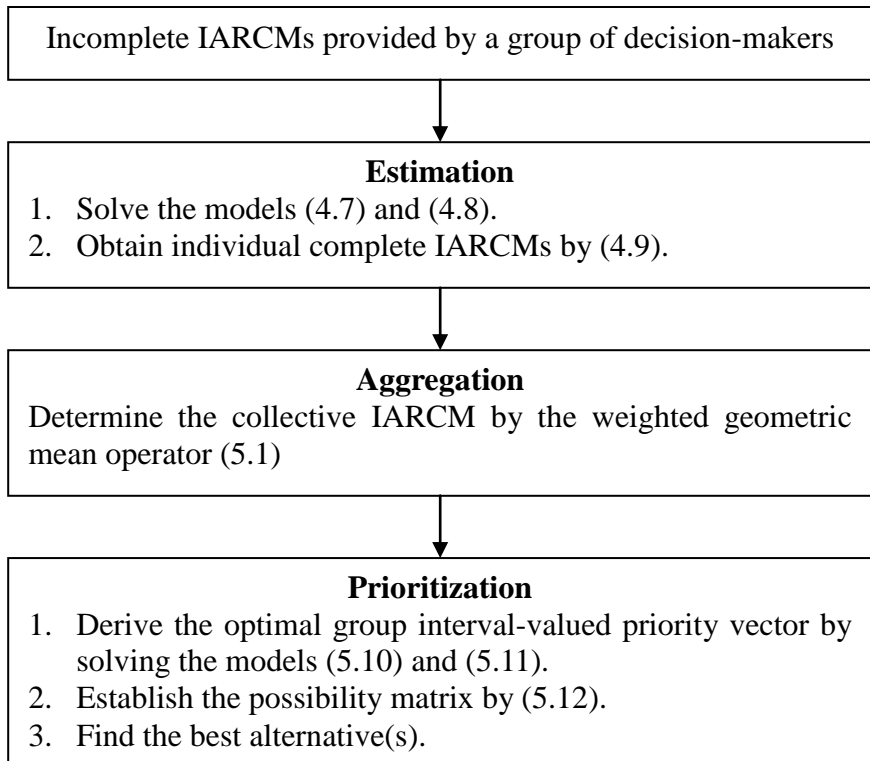


Figure 1. Resolution process of GDM with incomplete IARCMs

Next, we apply a GDM problem concerning selecting a supplier for a mobile phone manufacturing firm (adapted from Wan and Li (2013)) to illustrate the proposed decision models.

**Example 2.** With the growing trend of economic globalization, efficient supply chain management becomes critical for a firm to improve its competitive advantage in a global market. This example examines a supplier selection problem, where four suppliers  $X = \{x_1, x_2, x_3, x_4\}$  are determined as potential candidates and a selection committee is called to evaluate the four suppliers. Assume that the committee comprises three decision-makers  $d_l$  ( $l=1, 2, 3$ ), with varying importance weights  $\lambda = (\lambda_1, \lambda_2, \lambda_3)^T = (0.35, 0.4, 0.25)^T$ . Each decision-maker  $d_l$  ( $l=1, 2, 3$ ) conducts pairwise comparison on the four suppliers and furnishes his/her judgments by means of an incomplete IARCM

$$\bar{R}^{(l)} = (\bar{r}_{ij}^{(l)})_{4 \times 4} = ([r_{ij}^{- (l)}, r_{ij}^{+ (l)}])_{4 \times 4}.$$

$$\bar{R}^{(1)} = \begin{bmatrix} [0.5, 0.5] & [0.6, 0.8] & [-, 0.75] & [0.4, 0.7] \\ [0.2, 0.4] & [0.5, 0.5] & - & [0.35, 0.55] \\ [0.25, -] & - & [0.5, 0.5] & [0.3, -] \\ [0.3, 0.6] & [0.45, 0.65] & [-, 0.7] & [0.5, 0.5] \end{bmatrix}$$

$$\bar{R}^{(2)} = \begin{bmatrix} [0.5, 0.5] & [-, 0.7] & [0.5, 0.75] & - \\ [0.3, -] & [0.5, 0.5] & [0.3, -] & [0.55, 0.8] \\ [0.25, 0.5] & [-, 0.7] & [0.5, 0.5] & [0.7, 0.8] \\ - & [0.2, 0.45] & [0.2, 0.3] & [0.5, 0.5] \end{bmatrix}$$

$$\bar{R}^{(3)} = \begin{bmatrix} [0.5, 0.5] & [0.1, 0.3] & - & [0.3, -] \\ [0.7, 0.9] & [0.5, 0.5] & [0.2, 0.4] & - \\ - & [0.6, 0.8] & [0.5, 0.5] & [0.7, 0.8] \\ [-, 0.7] & - & [0.2, 0.3] & [0.5, 0.5] \end{bmatrix}$$

For the missing values in  $\bar{R}^{(l)}$  ( $l=1, 2, 3$ ), if the acceptable uncertainty ratios of estimated interval additive reciprocal judgments are expected to be less than or equal to 4, then we can set  $t=4$  for model (4.7). In this case, by solving (4.7), their corresponding optimal objective values are obtained as:

$$J^{(1)*} = 1.863116, J^{(2)*} = 0.1519312, J^{(3)*} = 3.632376.$$

Plugging  $J^* = J^{(l)*}$  and  $t_0 = 4$  into (4.8), we obtain the following optimal solutions.

$$r_{13}^{-(1)*} = 0.6418, r_{23}^{-(1)*} = 0.3212, r_{32}^{-(1)*} = 0.3457, r_{43}^{-(1)*} = 0.3684.$$

$$r_{12}^{-(2)*} = 0.7000, r_{14}^{-(2)*} = 0.7206, r_{32}^{-(2)*} = 0.4375, r_{41}^{-(2)*} = 0.0884.$$

$$r_{13}^{-(3)*} = 0.0655, r_{24}^{-(3)*} = 0.4955, r_{31}^{-(3)*} = 0.7809, r_{41}^{-(3)*} = 0.7000, r_{42}^{-(3)*} = 0.2029.$$

As per (4.9), the completed IARCMs  $\bar{R}^{(l)c} = (\bar{r}_{ij}^{(l)c})_{4 \times 4} = ([r_{ij}^{-(l)c}, r_{ij}^{+(l)c}])_{4 \times 4}$  ( $l=1,2,3$ ) are

determined as follows.

$$\bar{R}^{(1)c} = \begin{bmatrix} [0.5, 0.5] & [0.6, 0.8] & [0.6418, 0.75] & [0.4, 0.7] \\ [0.2, 0.4] & [0.5, 0.5] & [0.3212, 0.6543] & [0.35, 0.55] \\ [0.25, 0.3582] & [0.3457, 0.6788] & [0.5, 0.5] & [0.3, 0.6316] \\ [0.3, 0.6] & [0.45, 0.65] & [0.3684, 0.7] & [0.5, 0.5] \end{bmatrix}$$

$$\bar{R}^{(2)c} = \begin{bmatrix} [0.5, 0.5] & [0.7, 0.7] & [0.5, 0.75] & [0.7206, 0.9116] \\ [0.3, 0.3] & [0.5, 0.5] & [0.3, 0.5625] & [0.55, 0.8] \\ [0.25, 0.5] & [0.4375, 0.7] & [0.5, 0.5] & [0.7, 0.8] \\ [0.0884, 0.2794] & [0.2, 0.45] & [0.2, 0.3] & [0.5, 0.5] \end{bmatrix}$$

$$\bar{R}^{(3)c} = \begin{bmatrix} [0.5, 0.5] & [0.1, 0.3] & [0.0655, 0.2129] & [0.3, 0.3] \\ [0.7, 0.9] & [0.5, 0.5] & [0.2, 0.4] & [0.4955, 0.7971] \\ [0.7809, 0.9345] & [0.6, 0.8] & [0.5, 0.5] & [0.7, 0.8] \\ [0.7, 0.7] & [0.2029, 0.5045] & [0.2, 0.3] & [0.5, 0.5] \end{bmatrix}$$

By (5.1), the group IARCM is obtained as

$$\bar{R}^G = (\bar{r}_{ij}^G)_{4 \times 4} = ([r_{ij}^{-G}, 1 - r_{ji}^{-G}])_{4 \times 4} = \begin{bmatrix} [0.5, 0.5] & [0.4829, 0.6485] & [0.3869, 0.6239] & [0.5063, 0.7346] \\ [0.3515, 0.5171] & [0.5, 0.5] & [0.2794, 0.5554] & [0.4648, 0.7245] \\ [0.3761, 0.6131] & [0.4446, 0.7206] & [0.5, 0.5] & [0.5632, 0.7483] \\ [0.2654, 0.4937] & [0.2755, 0.5352] & [0.2517, 0.4368] & [0.5, 0.5] \end{bmatrix}$$

Plugging  $\bar{R}^G$  into (5.10) and, then solving this model yields its optimal objective value  $J^* = 0.5532709$ .

By solving (5.11), we obtain the optimal group interval-valued priority vector  $\bar{\omega}^* = (\bar{\omega}_1^*, \bar{\omega}_2^*, \bar{\omega}_3^*, \bar{\omega}_4^*)^T = ([0.2021, 0.3647], [0.1519, 0.2817], [0.2255, 0.3917], [0.1318, 0.1749])^T$ .

As per the possibility formula (5.12), the possibility matrix is determined as

$$P = \begin{bmatrix} 0.5 & 0.7278 & 0.4234 & 1 \\ 0.2722 & 0.5 & 0.1899 & 0.8670 \\ 0.5766 & 0.8101 & 0.5 & 1 \\ 0 & 0.1330 & 0 & 0.5 \end{bmatrix}$$

Adding up all values in each row, we obtain  $\phi_1 = 2.6512, \phi_2 = 1.8291, \phi_3 = 2.8867$  and  $\phi_4 = 0.6330$ .

As  $\phi_3 > \phi_1 > \phi_2 > \phi_4$ , the four suppliers are ranked as  $x_3 \succ^{57.66\%} x_1 \succ^{72.78\%} x_2 \succ^{86.70\%} x_4$ .

## 6. Conclusions

A goal programming framework is developed to solve GDM problems with incomplete IARCMs. A key characteristic of this research is to take an integrative approach to addressing uncertainty and inconsistency of decision-makers' pairwise judgments. Based on the multiplicative consistency concept (Wang & Li, 2012), new properties of consistent IARCMs are first investigated and employed to define multiplicative consistent incomplete IARCMs. A two-step goal programming method is then established to estimate missing values for an individual incomplete IARCM. By employing the lower bounds of the interval additive reciprocal judgments, a weighted geometric mean approach is subsequently proposed to aggregate individual IARCMs into a group IARCM. By analyzing the inherent link among normality interval-valued weights, multiplicative consistent IARCMs and their uncertainty levels, a two-step procedure comprising two goal programming models is eventually developed to derive an interval-valued priority vector from the group IARCM. Two numerical examples are furnished to illustrate the proposed models.

Further research is needed to address some significant issues. For instance, it is unclear how to judge and deal with extremely uncertain or/and inconsistent information in the original incomplete IARCMs provided by decision-makers. It is contemplated that the notion of acceptable consistency and uncertainty ratios has to be further explored and an interactive decision mechanism may have to be introduced to gauge the acceptance of the input data given by decision-makers. After these issues are properly addressed, it would be worthwhile to investigate how the current framework can be adapted to handle these cases.

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