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Yejun Xu
Hohai University

Lei Chen
Hohai University

Kevin Li
University of Windsor

Huimin Wang
Hohai University

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A chi-square method for priority derivation in group decision making with incomplete reciprocal preference relations

Yejun Xu[1,2*], Lei Chen[1,2], Kevin W. Li[3], Huimin Wang[1,2]

[1] *State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering, Hohai University, Nanjing, 210098, PR China*

[2] *Research Institute of Management Science, Business School, Hohai University, Nanjing, 211100, PR China*

[3] *Odette School of Business, University of Windsor, Windsor, Ontario, Canada N9B 3P4*

Abstract: This paper proposes a chi-square method (CSM) to obtain a priority vector for group decision making (GDM) problems where decision-makers'(DMs') assessment on alternative is furnished as incomplete reciprocal preference relations with missing values. Relevant theorems and iterative algorithm about CSM are proposed. Satty's consistency ratio concept is adapted to judge whether an incomplete reciprocal preference relation provided by a DM is of acceptable consistency. If its consistency is unacceptable, an algorithm is proposed to repair it until its consistency ratio reaches a satisfactory threshold. The repairing algorithm aims to rectify an inconsistent incomplete reciprocal preference relation to one with acceptable consistency in addition to preserving the initial preference information as much as possible. Finally, four examples are examined to illustrate the applicability and validity of the proposed method, and comparative analyses are provided to show its advantages over existing approaches.

Keywords: Group decision making; chi-square method; incomplete reciprocal preference relation; priority; consistency.

1. Introduction

Group decision making (GDM) [9, 13, 14, 18, 21, 24, 35] is a procedure of drawing on the combined wisdom and experience of experts from different domains

* Corresponding author. Tel. +86-25-68514612;
E-mail address: xuyejohn@163.com(Y.-J.Xu).

to rank a finite number of alternatives. Reciprocal preference relations [21, 23, 27, 34, 39] are commonly used to represent decision-makers (DMs)' preferences over a set of possible alternative solutions, and have received considerable research attention in the past decades. However, owing to time pressure, lack of knowledge, and the DM's limited expertise in the specific problem domain [1, 4, 5, 33, 36, 37, 41-44], sometimes a DM can at best furnish his/her judgment on alternatives as a reciprocal preference relation with missing or incomplete entries. Therefore, the method to derive priorities from incomplete reciprocal preference relations [3, 10, 11, 15, 41] has presented itself as an important and promising research topic, and attracted considerable research interest.

For example, Xu and Da [32] put forward a normalizing ranking aggregation method (NRAM) to derive priorities from an incomplete reciprocal preference relation. Xu and Wang [40] extended the well-known eigenvector method (EM) for priority derivation for an incomplete reciprocal preference relation, and the improvement method therein not only increases the consistency level but also preserves the initial preference information as much as possible. It is worth noting that the aforementioned NRAM and EM can only be applied to a single incomplete reciprocal preference relation. Xu [41] proposed two goal programming models (GPM) to obtain a collective priority vector from several incomplete reciprocal preference relations. Gong [17] put forward a least-square method (LSM) to generate a collective priority vector from incomplete reciprocal preference relations furnished by multiple DMs. Gong's approach results in a simple equation. But it cannot be applied to obtain a priority vector when the matrix Q is singular or Q^{-1} does not exist. In contrast to LSM, which is only applicable to the case with at least one multiplicative inconsistent incomplete reciprocal preference relation, the logarithmic least squares method (LLSM) put forward by Xu *et al.* [38] can be used for all incomplete reciprocal preference relations regardless of their multiplicative consistency property. In real-world decision processes, different DMs often carry heterogeneous power in reaching the final recommendation. It is noted that the aforementioned methods did not take into account DMs' weights in the decision process.

This paper extends a chi-square method (CSM) to prioritize alternatives in a GDM context when DMs furnish their judgment as incomplete reciprocal preference relations. The CSM was initially developed for priorities by Jensen [19]，and was later cited by Blankmeyer [7]. The original approach is complicated and has rarely been used. Wang and Fu [28] developed a convergent and simple iterative algorithm to facilitate its application in practice. Due to its nonlinear property, this improved algorithm has many advantages such as ease in computer implementation. As such, the extended CSM has arisen as a simple but efficient approach to deal with incomplete reciprocal preference relations.

The key motivations to adopt the CSM can be summarized as follows: (1) The CSM can be used to obtain a collective priority vector from several incomplete reciprocal preference relations, while other methods such as EM and NRAM can only be applied to a single incomplete reciprocal preference relation. This advantage makes it a natural choice for handling GDM. (2) The CSM is convenient in considering different DMs' weights in the decision process while this issue has been largely omitted by other methods. (3) By properly setting model parameters, the CSM can be flexibly employed to handle both complete and incomplete reciprocal preference relations. (4) Compared with other methods, the CSM is known for its better fitting performance, rank preservation capability and discrimination power. After Wang and Fu [28]'s extension, the improved CSM has become an efficient and convenient tool to handle incomplete reciprocal preference relations. By exploiting CSM to derive priority weights from incomplete reciprocal preference relations in a GDM context, this article further enhances its applications and enriches the theory and methodology of priority derivation.

An important issue in GDM with incomplete reciprocal preference relations is consistency test and inconsistency repairing because consistency of the judgment given by DMs has a direct impact on the final decision result [22]. Xu and Wang [40] adapted Saaty [26]'s consistency ratio (*CR*) to a fuzzy context and introduced a so-called fuzzy consistency ratio (*FCR*), which can be applied to incomplete reciprocal preference relations. By adopting Saaty's suggested threshold, an incomplete reciprocal preference relation is deemed to be acceptably consistent if $FCR<0.1$ [24]. If an incomplete reciprocal preference relation given by the DM does not possess acceptable consistency, it has to be repaired so that its consistency reaches the acceptable threshold. This paper will put forward a CSM-based algorithm to accomplish this task.

The remainder of the paper is organized as follows. Section 2 provides a review on basic concepts of reciprocal preference relations, incomplete reciprocal preference relations and an acceptable FCR. An associated theorem is also presented. In Section 3, the CSM is extended to obtain a priority vector from incomplete reciprocal preference relations based on the multiplicative transitivity property, resulting in an iterative algorithm. Section 4 puts forward an approach to repair an unacceptably inconsistent incomplete reciprocal preference relation to derive one with acceptable consistency. In Section 5, four examples are examined to show how to apply the proposed CSM and its effectiveness in handling GDM problems. Comparative analyses with existing methods demonstrate its validity and advantages. Concluding remarks are furnished in Section 6.

2. Preliminaries

In this section, we will give the definitions of reciprocal preference relations, incomplete reciprocal preference relations and a FCR.

Denote $N=\{1,2,\dots,n\}$, $M=\{1,2,\dots,m\}$. Let $X=\{x_1,x_2,\dots,x_n\}$ $(n\geq 2)$ be a finite set of alternatives, where x_i denotes the i^{th} alternative. $E=\{e_1,\dots,e_m\}$ be a finite set of experts, where e_k stands for the k^{th} expert. $H=(h_1,\dots,h_m)^T$ be the weight vector of experts, where $\sum_{k=1}^{m}h_k=1$, $h_k\geq 0$ and h_k demonstrates the importance degree of expert e_k in the decision process.

A fuzzy preference relation is defined as follows [9, 16, 44]. The preference information on X is described by a fuzzy preference relation $R\subset X\times X$,

$R=(r_{ij})_{n\times n}$, with membership function $\mu_R : X\times X \to [0,1]$, where $\mu_R(x_i,x_j)=r_{ij}$ denotes the preference degree of alternative x_i over x_j. $r_{ij}=0.5$ indicates the DM's indifference between x_i and x_j. $r_{ij}=1$ signifies that x_i is definitely preferred to x_j. $0\le r_{ij}<0.5$ implies that x_j is preferred to x_i and the smaller r_{ij} the stronger the preference of alternative x_j over x_i. $0.5<r_{ij}<1$ means that x_i is preferred to x_j and the greater r_{ij} the stronger the preference of alternative x_i over x_j.

Definition 1 [21]. Let $R=(r_{ij})_{n\times n}$ be a fuzzy preference relation, then *R* is called a reciprocal preference relation if

$$r_{ij}\in[0,1],\ \ r_{ij}+r_{ji}=1,\ \ r_{ii}=0.5, \text{ for all } i,j\in N.$$

Definition 2 [12, 27]. Let $R=(r_{ij})_{n\times n}$ be a reciprocal preference relation, then *R* has multiplicative transitivity property if

$$\left(\frac{1}{r_{ij}}-1\right)\left(\frac{1}{r_{jk}}-1\right)=\frac{1}{r_{ik}}-1,\ \ i,j,k\in N.$$

It has been found [17] that a perfectly multiplicative transitivity reciprocal preference relation $R=(r_{ij})_{n\times n}$ can be precisely characterized by a priority vector $W=(w_1,w_2,\dots,w_n)^T$, where $r_{ij}=w_i/(w_i+w_j)$, $\sum_{i=1}^{n}w_i=1$ and $w_i>0$ for $i=1,2,\dots,n$.

Definition 3 [2]. A membership function $f: X \to Y$ is called partial if at least one element in the set *X* is not mapped to an element in the set *Y*. If every element from the set *X* is mapped to an element in *Y*, then we have a total function.

Definition 4 [2]. A reciprocal preference relation *P* on a set of alternatives *X* with a partial membership function is an incomplete reciprocal preference relation.

For any $i, j\in N$, let c_{ij} be the ij^{th} entry of an incomplete preference relation

$C=(c_{ij})_{n\times n}$, $\delta_{ij}=\begin{cases}1 & c_{ij}\neq -\\ 0 & c_{ij}=-\end{cases}$, and $c_{ij}=-$ indicates a missing element c_{ij} . According to Definition 3, $\delta_{ij}=1$ if and only if there exists $c_{ij}=\mu_C(x_i,x_j)$, $\delta_{ij}=0$ denotes that the preference value $c_{ij}=\mu_C(x_i,x_j)$ is not furnished or missing.

Theorem 1 [20]. *Let $C=(c_{ij})_{n\times n}$ be an incomplete reciprocal preference relation, Then C can be completed by the known elements if there exists at least one known non-diagonal element in each row or column of C. This implies that an incomplete reciprocal preference relation C which can be completed has at least (n-1) non-diagonal judgments.*

Let $C=(c_{ij})_{n\times n}$ be an incomplete reciprocal preference relation, its fuzzy consistency index and fuzzy consistency ratio are denoted by *FCI* and *FCR* for short, and their formulas are presented as follows [38].

$$\begin{cases} FCI=\dfrac{1}{n(n-1)}\displaystyle\sum_{1\le i<j\le n}\sigma_{ij}\delta_{ij}\left(\frac{c_{ij}}{c_{ji}}\frac{w_j}{w_i}+\frac{c_{ji}}{c_{ij}}\frac{w_i}{w_j}-2\right)\\ FCR=\dfrac{FCI}{RI}\end{cases} \tag{1}$$

where σ_{ij} and δ_{ij} are binary variables defined below and *RI* is the mean consistency index of randomly generated multiplicative preference matrices as given in Table 1.

$$\sigma_{ij}=\begin{cases}0, & \text{if } c_{ij}=0 \text{ or } 1,\\ 1, & \text{otherwise.}\end{cases},\ i,j\in N. \tag{2}$$

$$\delta_{ij}=\begin{cases}0, & \text{if } c_{ij}=-,\\ 1, & \text{if } c_{ij}\neq -,\end{cases},\ i,j\in N. \tag{3}$$

By adapting the acceptable consistency threshold 0.1 proposed by Saaty, we have

Definition 5 [38]. Let $C=(c_{ij})_{n\times n}$ be an incomplete reciprocal preference relation, if *FCR*<0.1, then *C* is of acceptable consistency, otherwise *C*'s consistency level is unacceptable.

Table 1. The mean consistency index of randomly generated matrix [26]

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
RI	0	0	0.52	0.89	1.12	1.26	1.36	1.41	1.46	1.49	1.52	1.54	1.56	1.58	1.59

3. A chi-square method for priority derivation from group incomplete reciprocal preference relations

Consider a GDM problem, where m DMs give their preferences in the form of reciprocal preference relations, i.e. expert e_k describes his/her preference information as $R^{(k)} = (r_{ij}^{(k)})_{n\times n}$.

Let $W = (w_1, w_2, \ldots, w_n)^T$ be the priority weight vector for the reciprocal preference relations $R^{(k)} = (r_{ij}^{(k)})_{n\times n}$, where $\sum_{i=1}^{n} w_i = 1$, $w_i > 0$, $i \in N$. If $R^{(k)} = (r_{ij}^{(k)})_{n\times n}$ is a complete reciprocal preference relation with multiplicative transitivity then it can be expessed as [17]

$$r_{ij}^{(k)} = \frac{w_i}{w_i + w_j}, \quad i, j \in N. \tag{4}$$

If some elements of $R^{(k)}$ are missing or not given by the DM, then $R^{(k)}$ is an incomplete reciprocal preference relation. We shall extend Eq. (4) to the case of incomplete reciprocal preference relations. For computational convenience, an indicator matrix $\Delta = (\delta_{ij})_{n\times n}$ is constructed for incomplete reciprocal preference relation $C^{(k)} = (c_{ij}^{(k)})_{n\times n}$ as follows

$$\delta_{ij}^{(k)} c_{ij}^{(k)} = \delta_{ij}^{(k)} \frac{w_i}{w_i + w_j}, \quad i, j \in N. \tag{5}$$

where $\delta_{ij}^{(k)}$ is a binary variable defined as [41]:

$$\delta_{ij}^{(k)} = \begin{cases} 0, & c_{ij}^{(k)} = -, \\ 1, & c_{ij}^{(k)} \neq -. \end{cases} \tag{6}$$

Due to additive reciprocity, it is easy to find that $\delta_{ij}^{(k)} = \delta_{ji}^{(k)}$.

Next, we turn to find a priority vector $W = (w_1, w_2, \ldots w_n)^T$ to satisfy Eq.(5), where $\sum_{i=1}^{n} w_i = 1,\ w_i \ge 0$. To accomplish this, the following chi-square optimization model is constructed:

$$Min \;\; F(W) = \sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n} h_k \delta_{ij}^{(k)} \left[\frac{\left(c_{ij}^{(k)} - w_i / (w_i + w_j)\right)^2}{w_i / (w_i + w_j)} \right] \tag{7}$$

$$\text{s. t.} \quad \begin{cases} \sum_{i=1}^{n} w_i = 1, \\ w_i > 0, \quad i \in N. \end{cases} \tag{8}$$

$$D_W = \left\{ W = (w_1, w_2, \ldots, w_n)^T \;\middle|\; \sum_{i=1}^{n} w_i = 1, \, w_i > 0, \; i \in N \right\}. \tag{9}$$

The idea is to minimize the overall deviation from Eq. (5). To solve this chi-square model, the following theorem is established.

Theorem 2. *F(W) has a unique minimum point* $W^* = (w_1, w_2, \ldots w_n)^T \in D_w$*, which is also the unique solution of the following system of equations in* D_W :

$$\sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{ji}^{(k)} c_{ji}^{2(k)} \frac{w_i}{w_j} = \sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{ij}^{(k)} c_{ij}^{2(k)} \frac{w_j}{w_i}. \tag{10}$$

Proof. As D_W is a bounded vector space and $F(W)$ is continuous function in D_W, for $F(W) \geq 0$, $w \in D_w$, $F(W)$ therefore has an infimum, namely there exists $w \in D_w$ such that function $F(W)$ reaches its minimum value.

In order to obtain the optimal priority vector $W^* = (w_1, w_2, \ldots w_n)^T \in D_w$, the following Lagrangian function is constructed.

$$L(W, \lambda) = F(W) + \lambda \left(\sum_{i=1}^{n} w_i - 1 \right). \tag{11}$$

where λ is the Lagrange multiplier. By setting the partial derivatives with respect to w_i to be zero, we obtain the following set of equations:

$$\frac{1}{w_i} \sum_{j=1}^{n}\sum_{k=1}^{m} h_k \left(\delta_{ji}^{(k)} c_{ji}^{2(k)} \frac{w_i}{w_j} - \delta_{ij}^{(k)} c_{ij}^{2(k)} \frac{w_j}{w_i} \right) + \sum_{j=1}^{n}\sum_{k=1}^{m} h_k \frac{w_j}{(w_i + w_j)^2} \left(\delta_{ij}^{(k)} - \delta_{ji}^{(k)} \right) + \lambda = 0,$$

$$i \in N \tag{12}$$

Given that $\delta_{ij}^{(k)} = \delta_{ji}^{(k)}$, Eq. (12) can be further simplified as follows

$$\frac{1}{w_i}\sum_{j=1}^{n}\sum_{k=1}^{m} h_k\left(\delta_{ji}^{(k)} c_{ji}^{2(k)} \frac{w_i}{w_j} - \delta_{ij}^{(k)} c_{ij}^{2(k)} \frac{w_j}{w_i}\right) + \lambda = 0\,,\quad i \in N \tag{13}$$

which is equivalent to

$$\sum_{j=1}^{n}\sum_{k=1}^{m} h_k\left(\delta_{ji}^{(k)} c_{ji}^{2(k)} \frac{w_i}{w_j} - \delta_{ij}^{(k)} c_{ij}^{2(k)} \frac{w_j}{w_i}\right) + \lambda w_i = 0\,,\quad i \in N \tag{14}$$

Summing up Eq. (14) with respect to w_i, $i \in N$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} h_k\left(\delta_{ji}^{(k)} c_{ji}^{2(k)} \frac{w_i}{w_j} - \delta_{ij}^{(k)} c_{ij}^{2(k)} \frac{w_j}{w_i}\right) + \lambda \sum_{i=1}^{n} w_i = 0 \tag{15}$$

Since $\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} h_k\left(\delta_{ji}^{(k)} c_{ji}^{2(k)} \frac{w_i}{w_j} - \delta_{ij}^{(k)} c_{ij}^{2(k)} \frac{w_j}{w_i}\right) \equiv 0$ and $\sum_{i=1}^{n} w_i = 1$, we have λ=0. Plugging λ=0 into Eq. (14), one has

$$\sum_{j=1}^{n}\sum_{k=1}^{m} h_k\left(\delta_{ji}^{(k)} c_{ji}^{2(k)} \frac{w_i}{w_j} - \delta_{ij}^{(k)} c_{ij}^{2(k)} \frac{w_j}{w_i}\right) = 0\,,\quad i \in N \tag{16}$$

That is

$$\sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{ji}^{(k)} c_{ji}^{2(k)} \frac{w_i}{w_j} = \sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{ij}^{(k)} c_{ij}^{2(k)} \frac{w_j}{w_i},\quad i \in N \tag{17}$$

It is clear that the minimum point W^* is a solution to Eq. (10), if the solution is unique in D_w, W^* can be uniquely determined. The uniqueness is proved by contradiction as follows.

Assume that $V = (v_1, v_2, \ldots, v_n)^T \in D_w$ and $W = (w_1, w_2, \ldots, w_n)^T \in D_w$ are two solutions to Eq. (10). Let $u_i = w_i / v_i$, $i \in N$ and $u_l = \max_{i \in N}\{u_i\}$. If there exists $j \in N$ such that $u_j < u_l$, then we have

$$\sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{lj}^{(k)} c_{lj}^{2(k)} \frac{v_j}{v_l} > \sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{lj}^{(k)} c_{lj}^{2(k)} \frac{v_j}{v_l}\cdot\frac{u_j}{u_l} = \sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{lj}^{(k)} c_{lj}^{2(k)} \frac{w_j}{w_l} \tag{18}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{jl}^{(k)} c_{jl}^{2(k)} \frac{v_l}{v_j} < \sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{jl}^{(k)} c_{jl}^{2(k)} \frac{v_l}{v_j}\cdot\frac{u_l}{u_j} = \sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{jl}^{(k)} c_{jl}^{2(k)} \frac{w_l}{w_j} \tag{19}$$

According to Eqs. (10), (18), (19), it can be deduced that

$$\sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{lj}^{(k)} c_{lj}^{2(k)} \frac{w_j}{w_l} < \sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{jl}^{(k)} c_{jl}^{2(k)} \frac{w_l}{w_j} \tag{20}$$

which contradicts Eq. (10), Thus, $u_j < u_l$ cannot hold. Therefore, for all $j \in N$, $u_j = u_l$, namely, $w_1 / v_1 = w_2 / v_2 = ... = w_n / v_n$. Due to the fact that $\sum_{i=1}^{n} v_i = 1$, $\sum_{i=1}^{n} w_i = 1$, we have $w_i \equiv v_i$, $\forall i \in N$. This proves the uniqueness of the solution to Eq. (10).

To solve Eq.(10), we put forward a simple convergent iterative algorithm as follows.

Algorithm 1.

Let $C_k = (c_{ij}^{(k)})_{n\times n}$ ($k \in M$) be the initial incomplete reciprocal preference relations provided by the DMs.

Step 1. Using Theorem 1 to judge whether an incomplete reciprocal preference relation $C_k (k \in M)$ given by the DM e_k can be completed. If not, it is returned to expert e_k for an updated reciprocal preference relation, otherwise, go to *Step 2*.

Step 2. Initiating the iteration by giving an initial priority vector $W(0) = \left(w_1(0), w_2(0), \ldots, w_n(0)\right)^T$ and specifying an error parameter ε ($0 < \varepsilon < 1$), for example, $\varepsilon = 0.0001$, and setting L=0.

Step 3. Calculating

$$\eta_i\left(W(L)\right) = \sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{ji}^{(k)} c_{ji}^{2(k)} \frac{w_i}{w_j} - \sum_{j=1}^{n}\sum_{k=1}^{m} h_k \delta_{ij}^{(k)} c_{ij}^{2(k)} \frac{w_j}{w_i},\ \ i \in N \tag{21}$$

If $\left|\eta_i\left(W(L)\right)\right| \le \varepsilon$ holds for all $i \in N$, then $W^* = W(L)$ and stop, otherwise, continue to *Step 4*.

Step 4. Determining *p* such that $\left|\eta_p\left(W(L)\right)\right| = \max_{i\in N}\left\{\left|\eta_i\left(W(L)\right)\right|\right\}$ and computing

$$T(L)=\sqrt{\frac{\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\delta_{pj}^{(k)}c_{pj}^{2(k)}\frac{w_j(L)}{w_p(L)}}{\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\delta_{jp}^{(k)}c_{jp}^{2(k)}\frac{w_p(L)}{w_j(L)}}} \tag{22}$$

$$f_i(L)=\begin{cases}T(L)w_p(L), & i=p,\\ w_i(L), & i\neq p,\end{cases} \tag{23}$$

$$w_i(L+1)=f_i(L)\Big/\sum_{i=1}^{n}f_i(L),\ \ i\in N. \tag{24}$$

Step 5. Let $L=L+1$ and go to *Step 3*.

For Algorithm 1, we can establish the following theorem.

Theorem 3. *Algorithm 1 is convergent for any* $\varepsilon>0$.

Proof. We shall examine how $F(W)$ changes, when $W(L)$ progresses to $W(L+1)$. Suppose that $t>0$ and $S(t)=F(f(L))=F(w_1(L),...,w_{p-1}(L),tw_p(L),w_{p+1}(L),...,$ $w_n(L))$.

Then we have

$$S(t)=\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\delta_{pj}^{(k)}\left[\left(c_{pj}^{(k)}-\frac{tw_p(L)}{tw_p(L)+w_j(L)}\right)^2\right]\frac{tw_p(L)+w_j(L)}{tw_p(L)}$$

$$+\sum_{i=1,i\neq p}^{n}\sum_{k=1}^{m}h_k\delta_{ip}^{(k)}\left[\left(c_{ip}^{(k)}-\frac{w_i(L)}{w_i(L)+tw_p(L)}\right)^2\right]\frac{w_i(L)+tw_p(L)}{w_i(L)}$$

$$+\sum_{i=1,i\neq p}^{n}\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\delta_{ij}^{(k)}\left[\left(c_{ij}^{(k)}-\frac{w_i(L)}{w_i(L)+w_j(L)}\right)^2\right]\frac{w_i(L)+w_j(L)}{w_i(L)} \tag{25}$$

which is equivalent to

$$S(t)=\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\delta_{pj}^{(k)}c_{pj}^{2(k)}\frac{w_j(L)}{w_p(L)}\cdot\frac{1}{t}+\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\delta_{jp}^{(k)}c_{jp}^{2(k)}\frac{w_p(L)}{w_j(L)}\cdot t$$

$$+\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\left[\delta_{jp}^{(k)}c_{jp}^{2(k)}+\delta_{pj}^{(k)}c_{pj}^{2(k)}-2\left(\delta_{jp}^{(k)}c_{jp}^{(k)}+\delta_{pj}^{(k)}c_{pj}^{(k)}\right)\right]$$

$$+\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\left(\delta_{jp}^{(k)}\frac{tw_p(L)}{tw_p(L)+w_j(L)}+\delta_{pj}^{(k)}\frac{w_j(L)}{tw_p(L)+w_j(L)}\right)$$

$$+\sum_{i=1,i\neq p}^{n}\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\delta_{ij}^{(k)}\left[\left(c_{ij}^{(k)}-\frac{w_i(L)}{w_i(L)+w_j(L)}\right)^2\right]\frac{w_i(L)+w_j(L)}{w_i(L)} \tag{26}$$

Let

$$q_1=\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\delta_{pj}^{(k)}c_{pj}^{2(k)}\frac{w_j(L)}{w_p(L)}, \tag{27}$$

$$q_2=\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\delta_{jp}^{(k)}c_{jp}^{2(k)}\frac{w_p(L)}{w_j(L)}, \tag{28}$$

$$q_3=\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\left[\delta_{jp}^{(k)}c_{jp}^{2(k)}+\delta_{pj}^{(k)}c_{pj}^{2(k)}-2\left(\delta_{jp}^{(k)}c_{jp}^{(k)}+\delta_{pj}^{(k)}c_{pj}^{(k)}\right)\right]$$

$$+\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\left(\delta_{jp}^{(k)}\frac{tw_p(L)}{tw_p(L)+w_j(L)}+\delta_{pj}^{(k)}\frac{w_j(L)}{tw_p(L)+w_j(L)}\right)$$

$$+\sum_{i=1,i\neq p}^{n}\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\delta_{ij}^{(k)}\left[\left(c_{ij}^{(k)}-\frac{w_i(L)}{w_i(L)+w_j(L)}\right)^2\right]\frac{w_i(L)+w_j(L)}{w_i(L)} \tag{29}$$

Since $\delta_{ij}^{(k)}=\delta_{ji}^{(k)}$, the second double summation term in Eq. (29) can be rewritten as

$$\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\left(\delta_{jp}^{(k)}\frac{tw_p(L)}{tw_p(L)+w_j(L)}+\delta_{pj}^{(k)}\frac{w_j(L)}{tw_p(L)+w_j(L)}\right)$$

$$=\begin{cases}(n-1) & \text{if } \delta_{jp}^{(k)}=\delta_{pj}^{(k)}=1\\ 0 & \text{if } \delta_{jp}^{(k)}=\delta_{pj}^{(k)}=0\end{cases} \tag{30}$$

Therefore, q_3 can be further simplified as

$$q_3=\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\left[\delta_{jp}^{(k)}c_{jp}^{2(k)}+\delta_{pj}^{(k)}c_{pj}^{2(k)}-2\left(\delta_{jp}^{(k)}c_{jp}^{(k)}+\delta_{pj}^{(k)}c_{pj}^{(k)}\right)\right]$$

$$+\sum_{i=1,i\neq p}^{n}\sum_{j=1,j\neq p}^{n}\sum_{k=1}^{m}h_k\delta_{ij}^{(k)}\left[\left(c_{ij}^{(k)}-\frac{w_i(L)}{w_i(L)+w_j(L)}\right)^2\right]\frac{w_i(L)+w_j(L)}{w_i(L)}$$

$$+\begin{cases} (n-1) & \text{if } \delta_{jp}^{(k)} = \delta_{pj}^{(k)} = 1, \\ 0 & \text{if } \delta_{jp}^{(k)} = \delta_{pj}^{(k)} = 0, \end{cases} \tag{31}$$

This indicates that q_3 is independent of t. Then Eq. (26) can be equivalently expressed as

$$S(t) = q_1 / t + q_2 \cdot t + q_3 . \tag{32}$$

By setting $\frac{dS(t)}{dt}$ to be zero, we have

$$t^* = \sqrt{q_1/q_2} = \sqrt{\frac{\sum_{j=1, j\neq p}^{n} \sum_{k=1}^{m} h_k \delta_{pj}^{(k)} c_{pj}^{2(k)} \frac{w_j(L)}{w_p(L)}}{\sum_{j=1, j\neq p}^{n} \sum_{k=1}^{m} h_k \delta_{jp}^{(k)} c_{jp}^{2(k)} \frac{w_p(L)}{w_j(L)}}}, \tag{33}$$

$$S(t^*) = 2\sqrt{q_1 q_2} + q_3 , \tag{34}$$

where t^* stands for the minimum point, and $S(t^*)$ gives the minimum value of $S(t)$.

If t^* =1, Eq. (33) is equivalent to

$$\sum_{j=1, j\neq p}^{n} \sum_{k=1}^{m} h_k \delta_{pj}^{(k)} c_{pj}^{2(k)} \frac{w_j(L)}{w_p(L)} = \sum_{j=1, j\neq p}^{n} \sum_{k=1}^{m} h_k \delta_{jp}^{(k)} c_{jp}^{2(k)} \frac{w_p(L)}{w_j(L)} . \tag{35}$$

which also holds for $j = p$, therefore, we have

$$\sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta_{pj}^{(k)} c_{pj}^{2(k)} \frac{w_j(L)}{w_p(L)} = \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta_{jp}^{(k)} c_{jp}^{2(k)} \frac{w_p(L)}{w_j(L)} . \tag{36}$$

That is

$$\eta_p\left(W(L)\right) = \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta_{jp}^{(k)} c_{jp}^{2(k)} \frac{w_p(L)}{w_j(L)} - \sum_{j=1}^{n} \sum_{k=1}^{m} h_k \delta_{pj}^{(k)} c_{pj}^{2(k)} \frac{w_j(L)}{w_p(L)} = 0 .$$

By the definition of p in *Step 3*, we have $\left|\eta_p\left(W(L)\right)\right| = 0$. Since p is the subscript such that $\left|\eta_i\left(W(L)\right)\right|$ is maximized, we thus have $\left|\eta_i\left(W(L)\right)\right| = 0$ for all $i \in N$. Therefore, the algorithm terminates and $W^* = W(L)$.

If $t^* \neq 1$, then

$$F\left(W(L)\right)-F\left(f(L)\right)=S(1)-S(t^*)=q_1+q_2-2\sqrt{q_1q_2}=\left(\sqrt{q_1}-\sqrt{q_2}\right)^2>0 \quad (37)$$

Since $F(W)$ is a homogenous function, $F\left(f(L)\right)=F\left(W(L+1)\right)$. Inequality (37) shows that $F\left(W(L+1)\right)<F\left(W(L)\right)$, for any $L\geq 0$. Therefore, $F\left(W(L)\right)$ is a monotonically decreasing sequence with an infimum in D_w and, hence, convergent.

4. A method for repairing inconsistency of incomplete reciprocal preference relations

If the consistency level of an incomplete reciprocal preference relation is too low and deemed unacceptable, it can be returned to the DM for a reassessment until the updated one reaches an acceptable consistency level. This approach is presumably more reliable and accurate, but it is often impracticable because the iteration process can be tedious and time-consuming. To facilitate the decision process, this section puts forwards an automated procedure to improve the consistency level of a given incomplete reciprocal preference relation with unacceptable consistency $FCR\geq 0.1$. Whenever possible, the DM's intervention should be called upon, but this procedure serves as a convenient tool and can be employed by the analyst to facilitate the DM in eliciting his/her preference expeditiously. We first introduce the consistency deviation variable for c_{ij} as follows.

$$d_{ij}=|c_{ij}-w_i/(w_i+w_j)| \quad (38)$$

If $d_{ij}\equiv 0$, for all $i,j\in N$ and $c_{ij}\neq -$, then C is a perfectly consistent incomplete reciprocal preference relation. The priority vector W is able to precisely represent *C*. The higher the deviation d_{ij}, the more likely c_{ij} should be updated.

Conceptually, a judgment c_{ij} should be as close to $w_i/(w_i+w_j)$ as possible to make it more consistent. In addition, the known element c_{ij} given by the expert often falls in the set $U=\{0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1\}$. To avoid excessive distortion of the DM's original judgment, the improved preference relation should not only increase the consistency level but also try to preserve the initial preference information. The procedure starts with identifying the unusual and false elements (UFEs) that are the most inconsistent element with the biggest d_{ij}. Once the UFEs are identified, the initial UFE (c_{ij}) will be updated with c'_{ij} , where $c'_{ij} = round(w_i / (w_i + w_j) \times 10) \times 10^{-1}$ and "*round*" is the usual round operation. This function ensures the updated judgment values are between 0 and 1 and have one decimal place. It is trivial to make adjustment to accommodate the case when the analyst or DM prefers to express the judgment in more decimal places.

Given the aforesaid discussion, the following algorithm is devised to repair inconsistency of an incomplete reciprocal preference relation.

Algorithm 2

Let $C = (c_{ij})_{n \times n}$ be an incomplete reciprocal preference relation given by the DM.

Step 1. Using the CSM algorithm in Section 3 to obtain the priority vector $W = (w_1, w_2, \ldots w_n)^T$.

Step 2. Determining the consistency ratio of the incomplete reciprocal preference relation as per Eq. (1), if *FCR*<0.1, go to *Step 5*, otherwise, go to *Step 3*.

Step 3. Computing deviations d_{ij}'s by using Eq. (38), and identifying the maximum deviation to find the corresponding UFEs.

Step 4. Updating the UFEs (c_{ij}) with c'_{ij}, where $c'_{ij} = round(w_i / (w_i + w_j) \times 10)$ $\times 10^{-1}$, and go to *Step 1*.

Step 5. Ranking the alternatives according the priority vector W^*.

Step 6. End.

5. Illustrative examples

In this section, four numerical examples are examined to demonstrate the applications and advantages of the proposed CSM framework. Example 1 is a GDM problem with incomplete reciprocal preference relations and a comparative analysis is conducted between CSM and three existing methods. Example 2 is a single incomplete reciprocal preference relation with unacceptable consistency and Algorithm 2 is utilized to repair it until its consistency becomes acceptable. Example 3 considers a single incomplete reciprocal preference relation with acceptable consistency. The purpose is to compare the result derived from CSM with those from EM, NRAM, GPM, LSM and LLSM on three performance evaluation criteria: *FCR*, MAD and MD. Example 4 discusses a GDM problem with incomplete reciprocal preference relations with a purpose to show the advantages of CSM.

Example 1. For a GDM problem with four decision alternatives x_i ($i = 1,2,3,4$) and three DMs e_k (k =1, 2, 3). The DMs provide their preferences over the four decision alternatives as three incomplete reciprocal preference relations [41].

$$C_1 = \begin{bmatrix} 0.5 & 0.6 & - & 0.7 \\ 0.4 & 0.5 & 0.2 & 0.8 \\ - & 0.8 & 0.5 & 0.4 \\ 0.3 & 0.2 & 0.6 & 0.5 \end{bmatrix}, C_2 = \begin{bmatrix} 0.5 & 0.8 & 0.4 & - \\ 0.2 & 0.5 & 0.3 & 0.6 \\ 0.6 & 0.7 & 0.5 & 0.3 \\ - & 0.4 & 0.7 & 0.5 \end{bmatrix}, C_3 = \begin{bmatrix} 0.5 & 0.3 & 0.4 & 0.6 \\ 0.7 & 0.5 & - & 0.5 \\ 0.6 & - & 0.5 & 0.7 \\ 0.4 & 0.5 & 0.3 & 0.5 \end{bmatrix}.$$

Xu [41] employed goal programming (GP) models to derive a priority vector $W^* = (0.265, 0.236, 0.276, 0.223)^T$ from the aforesaid three incomplete reciprocal preference relations. The research leads to a final ranking: $x_3 \succ x_1 \succ x_2 \succ x_4$, which is the same as the ranking generated by the logarithmic least square method (LLSM) [38] but slightly differs from the one obtained by the least-square method (LSM) [17] with the order between x_2 and x_4 being reversed. We now examine the problem using the CSM. In order to offer a fair comparison with Xu [41]'s method, we also set $h_1 = h_2 = h_3$ =1/3.

Step 1. According to Theorem 1, we know that C_k (k =1, 2, 3) can all be completed by known elements.

Step 2. Given an initial priority vector $W(0) = (0.25, 0.25, 0.25, 0.25)^T$, specify the parameter ε =0.1, and let L=0.

Step 3. Calculate $\eta_i\left(W(0)\right)$, we have

$$\left|\eta_1\left(W(0)\right)\right| = 0.2 > \varepsilon,\ \left|\eta_2\left(W(0)\right)\right| = 0.2 > \varepsilon,$$

$$\left|\eta_3\left(W(0)\right)\right| = 0.4 > \varepsilon,\ \left|\eta_4\left(W(0)\right)\right| = 0.4 > \varepsilon.$$

As $\left|\eta_i\left(W(0)\right)\right| > \varepsilon$ holds for all $i = 1, 2, 3, 4$, we continue to *Step 4*.

Step 4. Determine p such that $\left|\eta_p\left(W(L)\right)\right| = \max_{i\in N}\left\{\left|\eta_i\left(W(L)\right)\right|\right\}$, we can set $p = 3$, and compute $T(0)$, $f(0)$ and $W(1)$.

$$T(0) = 1.3650,\ f(0) = (0.2500, 0.2500, 0.3413, 0.2500)^T,$$

$$W(1) = (0.2291, 0.2291, 0.3127, 0.2291)^T.$$

Step 5. Let $L = L+1 = 1$ and go to *Step 3*.

The computation processes are detailed in Table 2. It is clear that iterations terminates at $L = 3$, when $|\eta_1| = 0.0296 < 0.1$, $|\eta_2| = 0.0072 < 0.1$, $|\eta_3| = 0.0227 < 0.1$, $|\eta_4| = 2.6357\times10^{-4} < 0.1$, indicating that the derived priority vector has reached an acceptable level of ε. The optimal priority vector is thus found to be $W^* = (0.2797, 0.2197, 0.3, 0.2007)^T$, resulting in a ranking of the four alternatives $x_3 \succ x_1 \succ x_2 \succ x_4$.

Remark 1. Computation results in Table 2 demonstrate that $F(W(L))$ decreases in iteration step L. However, for $\left|\eta_i(W(L))\right|$, this monotonicity does not hold any more and there may have ups and downs when L increases, but eventually $\left|\eta_i(W(L))\right|$ will decrease to a value below the threshold ε as ascertained by Theorem 3. As three of the four aforesaid methods derive an identical ranking with the other one yielding a slightly different order, this result demonstrates the robustness and credibility of the proposed CSM framework. To further compare the performance with the other three methods in fitting the three incomplete reciprocal preference relations, the following evaluation criteria are introduced:

Maximum deviation (MD) for incomplete reciprocal preference relations

$$\text{MD} = \max_{i,j,k}\left\{\delta_{ij}^{(k)}\left(\frac{c_{ij}}{c_{ji}}\frac{w_j}{w_i}+\frac{c_{ji}}{c_{ij}}\frac{w_i}{w_j}-2\right)\middle| i,j\in N,k\in M\right\} \tag{39}$$

Maximum absolute deviation (MAD) for incomplete reciprocal preference relations

$$\text{MAD} = \max_{i,j,k}\left\{\delta_{ij}^{(k)}\left|c_{ij}^{(k)}-\frac{w_i}{w_i+w_j}\right|\middle| i,j\in N,k\in M\right\}. \tag{40}$$

where $\delta_{ij}^{(k)}$ is defined by Eq. (6). $d_{ij}^{(k)} = c_{ij}^{(k)} - w_i/(w_i+w_j)$ is the consistency deviation for $c_{ij}^{(k)}$ in the incomplete reciprocal preference relation $C^{(k)} = (c_{ij}^{(k)})_{n\times n}$. If the priority vector $W = (w_1,\dots w_n)^T$ is able to precisely characterize the reciprocal preference relation $C^{(k)}$, then $\left|d_{ij}^{(k)}\right| \equiv 0$, otherwise, $\left|d_{ij}^{(k)}\right| > 0$.

Table 3 indicates that CSM results in an identical ranking as GPM and LLSM while the ranking derived by LSM is slightly different. CSM has a comparable MAD as GPM, which is smaller than both LSM and LLSM. In terms of MD, CSM outperforms all the other three methods as it yields the smallest deviation. This partly shows the advantage of the CSM.

Remark 2. To facilitate a comparative study with GPM, LSM and LLSM, the weights of three reciprocal preference relations were set to be equal ($h_1 = h_2 = h_3$ =1/3). However, CSM allows an analyst to set different weights as per the practical situation, to properly reflect different experts' varying influences in the GDM problem at hand. It is worth noting that if δ_{ij} =1, for all $i,j\in N$, then the proposed CSM can still be utilized to derive a priority vector from reciprocal preference relations. This means that CSM can be used for both complete and incomplete reciprocal preference relations. In addition, by setting h_1 =1 and h_k =0, for $k = 2,\dots,m$, the CSM can be conveniently applied to derive a priority vector from a single incomplete reciprocal preference relation. This allows CSM to be used for a single expert's decision making problems in Examples 2 and 3.

Furthermore, by using Algorithm 1, we can get the values of L, W, $F(W)$ and the ranking of alternatives for different ε's as listed in Table 4.

Table 2

The iterative processes for Example 1.

Iterative steps	$\lvert\eta_i(W(L))\rvert$				$W(L)$				$F(W)$
L	$\lvert\eta_1\rvert$	$\lvert\eta_2\rvert$	$\lvert\eta_3\rvert$	$\lvert\eta_4\rvert$	w_1	w_2	w_3	w_4	
0	0.2	0.2	0.4	0.4	0.25	0.25	0.25	0.25	0.7067
1	0.3031	0.0835	1.156×10^{-4}	0.2197	0.2291	0.2291	0.3127	0.2291	0.6449
2	6.197×10^{-5}	0.0571	0.0792	0.1362	0.2744	0.2256	0.2943	0.2156	0.6086
3	0.0296	0.0072	0.0227	2.6357×10^{-4}	0.2797	0.2197	0.3000	0.2007	0.6024

Table 3

Performance comparisons for Example 1.

Methods	W^*	Ranking	MD	MAD
CSM (this article)	$(0.2797, 0.2197, 0.3000, 0.2007)^T$	$x_3 \succ x_1 \succ x_2 \succ x_4$	1.9277	0.2992
LSM [17]	$(0.2822, 0.1968, 0.3202, 0.2009)^T$	$x_3 \succ x_1 \succ x_4 \succ x_2$	2.3282	0.3145
GPM [41]	$(0.265, 0.236, 0.276, 0.223)^T$	$x_3 \succ x_1 \succ x_2 \succ x_4$	2.0442	0.2858
LLSM [38]	$(0.2806, 0.2105, 0.3189, 0.1900)^T$	$x_3 \succ x_1 \succ x_2 \succ x_4$	2.1717	0.3266

Table 4

The values of L, W, $F(W)$ and ranking order for different ε in Example 1.

ε	L	W	Ranking	$F(W)$	$\lvert\eta_1\rvert$	$\lvert\eta_2\rvert$	$\lvert\eta_3\rvert$	$\lvert\eta_4\rvert$
10^{-1}	3	$(0.2797, 0.2197, 0.3000, 0.2007)^T$	$x_3 \succ x_1 \succ x_2 \succ x_4$	0.6024	0.0001	0.0728	0.0868	0.0141
10^{-2}	15	$(0.2753, 0.2204, 0.3032, 0.2011)^T$	$x_3 \succ x_1 \succ x_2 \succ x_4$	0.6020	0.0031	0	0.0083	0.0052
10^{-3}	24	$(0.2746, 0.2201, 0.3041, 0.2012)^T$	$x_3 \succ x_1 \succ x_2 \succ x_4$	0.6019	3.6858×10^{-4}	0	9.8063×10^{-4}	6.1205×10^{-4}
10^{-4}	34	$(0.2745, 0.2201, 0.3043, 0.2012)^T$	$x_3 \succ x_1 \succ x_2 \succ x_4$	0.6019	5.9215×10^{-5}	2.6310×10^{-5}	8.5525×10^{-5}	0
10^{-5}	44	$(0.2746, 0.2201, 0.3041, 0.2012)^T$	$x_3 \succ x_1 \succ x_2 \succ x_4$	0.6019	1.1102×10^{-16}	6.4088×10^{-6}	8.2368×10^{-6}	1.8281×10^{-6}
10^{-6}	53	$(0.2746, 0.2201, 0.3041, 0.2012)^T$	$x_3 \succ x_1 \succ x_2 \succ x_4$	0.6019	1.1102×10^{-16}	6.4088×10^{-6}	8.2368×10^{-6}	1.8281×10^{-6}

Example 2. Consider a single DM's decision problem with six alternatives x_i ($i = 1, 2, \ldots, 6$). The DM provides his/her preferences over the six decision alternatives, as an incomplete reciprocal preference relation which is shown below (adapted from [42]).

$$C = \begin{bmatrix} 0.5 & - & - & 0.3 & 0.8 & 0.3 \\ - & 0.5 & - & - & - & - \\ - & - & 0.5 & - & - & - \\ 0.7 & - & - & 0.5 & 0.4 & 0.8 \\ 0.2 & - & - & 0.6 & 0.5 & 0.7 \\ 0.7 & - & - & 0.2 & 0.3 & 0.5 \end{bmatrix}$$

Step 1. According to Theorem 1, it is easy to tell that *C* can be completed as no non-diagonal elements are furnished in the second or third row (column) of *C*. Therefore, the initial judgment matrix has to be returned to the DM for an update, resulting in the following incomplete reciprocal preference relation:

$$C = \begin{bmatrix} 0.5 & 0.3 & - & 0.3 & 0.8 & 0.3 \\ 0.7 & 0.5 & 0.7 & - & 0.6 & - \\ - & 0.3 & 0.5 & 0.4 & - & - \\ 0.7 & - & 0.6 & 0.5 & 0.4 & 0.8 \\ 0.2 & 0.4 & - & 0.6 & 0.5 & 0.7 \\ 0.7 & - & - & 0.2 & 0.3 & 0.5 \end{bmatrix}$$

Without loss of generality, let the original weight vector be $W(0) = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)^T$. Using Algorithm 1, one can get the values of *L*, *W*, *F*(*W*), *FCR*, $|\eta_i(W(L))|$ and ranking results by setting different ε values as listed in Table 5. When ε is sufficiently small, the weight vector approaches

$$W^* = (0.1301, 0.2714, 0.1281, 0.2090, 0.1509, 0.1106)^T,$$

Step 2. Computing *FCR* by Eq. (1).

$$FCI = 0.1870, FCR = FCI / RI = 0.1870 / 1.26 = 0.1484 > 0.1.$$

Since *FCR*>0.1, the incomplete reciprocal preference relation *C* does not possess satisfactory consistency. We need to find its UFEs to repair this preference relation.

Step 3. Calculating the deviations between original judgment c_{ij} and its corresponding consistent representation, we have

$$D=\begin{bmatrix} 0 & 0.0241 & 0 & 0.0837 & 0.3369 & 0.2405 \\ 0.0241 & 0 & 0.0206 & 0 & 0.0427 & 0 \\ 0 & 0.0206 & 0 & 0.0201 & 0 & 0 \\ 0.0837 & 0 & 0.0201 & 0 & 0.1808 & 0.1461 \\ 0.3369 & 0.0427 & 0 & 0.1808 & 0 & 0.1231 \\ 0.2405 & 0 & 0 & 0.1461 & 0.1231 & 0 \end{bmatrix}.$$

Obviously, the maximum deviations are d_{15} and d_{51}, so the UFEs are c_{15} and c_{51}.

Step 4. Updating the UFEs c_{ij} with $c'_{ij}=round(w_i/(w_i+w_j)\times 10)\times 10^{-1}$, one has c'_{15}=0.5 and c'_{51}=0.5.

Thus C is updated as

$$C'=\begin{bmatrix} 0.5 & 0.3 & - & 0.3 & 0.5 & 0.3 \\ 0.7 & 0.5 & 0.7 & - & 0.6 & - \\ - & 0.3 & 0.5 & 0.4 & - & - \\ 0.7 & - & 0.6 & 0.5 & 0.4 & 0.8 \\ 0.5 & 0.4 & - & 0.6 & 0.5 & 0.7 \\ 0.7 & - & - & 0.2 & 0.3 & 0.5 \end{bmatrix}$$

Using Algorithm 1, one can obtain the values of L, W, $F(W)$, FCR, $|\eta_i(W(L))|$ and ranking of alternatives with different ε's as listed in Table 6. When ε is sufficiently small, the final priority vector is obtained as

$$W^*=(0.0994, 0.2699, 0.1275, 0.2083, 0.1889, 0.1061)^T$$

Computing FCR by Eq. (1).

$$FCI=0.0661,\quad FCR=FCI/RI=0.0661/1.26=0.0525<0.1.$$

Thus, this updated C is deemed to have acceptable consistency.

Step 5. Using the final priority vector W^* to rank the alternatives as

$$W^*=(0.0994, 0.2699, 0.1275, 0.2083, 0.1889, 0.1061)^T.$$

$$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1.$$

By changing only c_{15} and c_{51}, we were able to rectify an incomplete reciprocal preference relation to derive one with acceptable consistency. This allows the analyst to avoid the hassle of returning the inconsistent preference relation to the DM for reconsideration.

Remark 3. Numerical results in Tables 4, 5 and 6 demonstrate that iteration step *L* increases when error parameter ε decreases. In general, *F*(*W*) and the consistency ratio of an incomplete reciprocal preference relation *C* gets smaller when ε decrease. When the error parameter ε is sufficiently small, *W*, *F*(*W*), *FCR*, and ranking results will converge to a set of values and remain unchanged.

In order to show the effectiveness of CSM, the other three methods EM [40], LSM [17], and LLSM [38] are also applied to the rectified C' and assessed in terms of the criteria FCR, MD and MAD. Table 7 lists the ranking results by the four methods. It is clear that CSM and LLSM yield the same ranking $x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$, but EM and LSM generate slightly different rankings. Most notably, the EM and LSM reverse the order of x_1 and x_6, while the DM's original judgment points to $x_6 \succ x_1$ because $c_{61} = c_{61}^{'} = 0.7$. It is apparent that this reverse is unwarranted and undesirable. Moreover, CSM produces the smallest MD and MAD among the four methods, and the FCR from CSM is marginally larger than that from LLSM, but is smaller than those derived from EM and LSM. Across the three metrics, FCR, MD and MAD, Table 7 shows that CSM overall performs better than the other three methods EM, LSM and LLSM.

Table 5. The values of L, W, $F(W)$, FCR and rankings for different ε of C in Example 2.

ε	L	W	Ranking	$F(W)$	FCR	$\lvert\eta_i(W(L))\rvert$
10^{-1}	26	$(0.1319, 0.2565, 0.1302, 0.2147, 0.1540, 0.1128)^T$	$x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_6$	1.0276	0.1490	4.4409×10^{-16}, 0.098, 0.0256, 0.0093, 0.0291, 0.034
10^{-2}	66	$(0.1305, 0.2699, 0.1280, 0.2094, 0.1511, 0.1110)^T$	$x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_6$	1.0239	0.1484	0.0041, 0.0095, 1.1102×10^{-16}, 0.0045, 2.2204×10^{-16}, 9.3303×10^{-4}
10^{-3}	107	$(0.1301, 0.2712, 0.1281, 0.2090, 0.1509, 0.1107)^T$	$x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_6$	1.0239	0.1484	4.2799×10^{-4}, 9.7803×10^{-4}, 2.2542×10^{-4}, 2.2980×10^{-4}, 2.2204×10^{-16}, 9.4821×10^{-5}
10^{-4}	146	$(0.1301, 0.2714, 0.1281, 0.2090, 0.1509, 0.1106)^T$	$x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_6$	1.0239	0.1484	2.2204×10^{-16}, 9.6242×10^{-5}, 2.2204×10^{-16}, 3.3698×10^{-5}, 2.8917×10^{-5}, 3.3627×10^{-5}
10^{-5}	187	$(0.1301, 0.2714, 0.1281, 0.2090, 0.1509, 0.1106)^T$	$x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_6$	1.0239	0.1484	2.2204×10^{-16}, 9.6242×10^{-5}, 2.2204×10^{-16}, 3.3698×10^{-5}, 2.8917×10^{-5}, 3.3627×10^{-5}
10^{-6}	227	$(0.1301, 0.2714, 0.1281, 0.2090, 0.1509, 0.1106)^T$	$x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_6$	1.0239	0.1484	2.2204×10^{-16}, 9.6242×10^{-5}, 2.2204×10^{-16}, 3.3698×10^{-5}, 2.8917×10^{-5}, 3.3627×10^{-5}

Table 6. The values of L, W, $F(W)$, FCR and rankings for different ε of C' in Example 2.

ε	L	W	Ranking	$F(W)$	FCR	$\lvert\eta_i(W(L))\rvert$
10^{-1}	22	$(0.1014, 0.2551, 0.1288, 0.2126, 0.1914, 0.1108)^T$	$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$	0.4193	0.0532	0.0143, 0.0943, 0.0127, 0.0406,0, 0.0267
10^{-2}	61	$(0.0994, 0.2685, 0.1278, 0.2086, 0.1892, 0.1064)^T$	$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$	0.4154	0.0525	2.2204×10^{-16}, 0.0092, 0.003, 0.003, 0.0022, 0.001
10^{-3}	97	$(0.0994, 0.2698, 0.1275, 0.2083, 0.1889, 0.1061)^T$	$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$	0.4154	0.0525	2.2345×10^{-4}, 8.7826×10^{-4}, 0, 2.2196×10^{-4}, 1.0941×10^{-4}, 3.2344×10^{-4}
10^{-4}	133	$(0.0994, 0.2699, 0.1275, 0.2083, 0.1889, 0.1061)^T$	$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$	0.4154	0.0525	2.1921×10^{-5}, 9.2656×10^{-5}, 1.1102×10^{-16}, 3.8488×10^{-5}, 2.2204×10^{-16}, 3.2246×10^{-5}
10^{-5}	170	$(0.0994, 0.2699, 0.1275, 0.2082, 0.1889, 0.1061)^T$	$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$	0.4154	0.0525	2.9494×10^{-6}, 9.6250×10^{-6}, 1.3882×10^{-6}, 1.5475×10^{-6},0, 3.7399×10^{-6}
10^{-6}	209	$(0.0994, 0.2699, 0.1275, 0.2082, 0.1889, 0.1061)^T$	$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$	0.4154	0.0525	0, 9.2517×10^{-7}, 2.4979×10^{-7}, 1.9570×10^{-7}, 3.6807×10^{-7}, 1.1160×10^{-7}

Table 7. Performance comparisons for Example 2.

Method	W^*	Ranking	FCR	MD	MAD
CSM	$(0.0994, 0.2699, 0.1275, 0.2083, 0.1889, 0.1061)^T$	$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$	0.0525	0.6434	0.1837
EM	$(0.1038, 0.2780, 0.1262, 0.2017, 0.1949, 0.0953)^T$	$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_1 \succ x_6$	0.054	0.9349	0.2213
LSM	$(0.1017, 0.3036, 0.1354, 0.1849, 0.1937, 0.0808)^T$	$x_2 \succ x_5 \succ x_4 \succ x_3 \succ x_1 \succ x_6$	0.0607	1.2774	0.2573
LLSM	$(0.0965, 0.2682, 0.1288, 0.2166, 0.1901, 0.0998)^T$	$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_6 \succ x_1$	0.0519	0.6994	0.1916

Example 3. Given a decision problem with six alternatives x_i ($i = 1, 2, \ldots, 6$), the DM provides his/her preferences over the six decision alternatives, as an incomplete reciprocal preference relation (adapted from [38])

$$C = \begin{bmatrix} 0.5 & 0.4 & - & 0.3 & 0.3 & 0.3 \\ 0.6 & 0.5 & 0.6 & 0.5 & - & 0.4 \\ - & 0.4 & 0.5 & 0.3 & 0.6 & - \\ 0.7 & 0.5 & 0.7 & 0.5 & 0.4 & 0.8 \\ 0.7 & - & 0.4 & 0.6 & 0.5 & 0.7 \\ 0.7 & 0.6 & - & 0.2 & 0.3 & 0.5 \end{bmatrix}$$

This incomplete reciprocal preference relation was investigated by Xu *et al.* [38], in which the optimal priority vector is derived by LLSM as $W^* = (0.0878, 0.1599, 0.1551, 0.2464, 0.2208, 0.1301)^T$. This yields a ranking of the six alternatives $x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$. We now examine the problem using CSM as follows.

According to Theorem 1, we know that C can be completed. Without loss of generality, we set the original weight vector as $W(0) = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)^T$. When ε is set to 10^{-3}, the values of W, $F(W)$, FCR and ranking of alternatives will stabilize and remain unchanged. At L=55, one has $F(W)$=0.5860, $|\eta_1|$=$1.1863\times10^{-5} < \varepsilon$, $|\eta_2|$=$0 < \varepsilon$, $|\eta_3|$=$1.2503\times10^{-5} < \varepsilon$, $|\eta_4|$= $9.6445\times10^{-5} < \varepsilon$, $|\eta_5|$=$2.9292\times10^{-5} < \varepsilon$, $|\eta_6|$=$4.2788\times10^{-5} < \varepsilon$, FCR=0.0728, $W^* = (0.0884, 0.1615, 0.1581, 0.2365, 0.2185, 0.1370)^T$, implying a ranking of these six alternatives as: $x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$.

For this single incomplete reciprocal preference relation, it can also be solved by EM [40], NRAM[32], LSM[17] , LLSM[38] and GPM [41]. The results are shown in Table 8, from which we can see that CSM achieves the same ranking as EM, NRAM, LSM and LLSM, $x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$, while GPM yields a slightly different ranking, $x_4 \sim x_5 \succ x_2 \succ x_3 \sim x_6 \sim x_1$, which fails to discriminate x_4 and x_5, as well as x_1, x_3 and x_6. Furthermore, both NRAM and GPM lead to unacceptable consistency ratio *FCR*>0.1, and have larger MD and MAD values than other methods. A further examination reveals that CSM results in the smallest MAD value among these six methods and outperforms NRAM, GPM and LSM in all the three criteria.

Example 4. Consider a GDM problem with three DMs providing the following incomplete reciprocal preference relations C_i $(i = 1,2,3)$ for a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$:

$$C_1 = \begin{bmatrix} 0.5 & 0.3 & - & 0.5 \\ 0.7 & 0.5 & 0.6 & 0.6 \\ - & 0.4 & 0.5 & 0.7 \\ 0.5 & 0.4 & 0.3 & 0.5 \end{bmatrix}, C_2 = \begin{bmatrix} 0.5 & 0.2 & 0.6 & 0.7 \\ 0.8 & 0.5 & 0.8 & - \\ 0.4 & 0.2 & 0.5 & 0.8 \\ 0.3 & - & 0.4 & 0.5 \end{bmatrix}, C_3 = \begin{bmatrix} 0.5 & 0.2 & 0.5 & 0.6 \\ 0.8 & 0.5 & - & 0.7 \\ 0.5 & - & 0.5 & 0.8 \\ 0.4 & 0.3 & 0.2 & 0.5 \end{bmatrix}.$$

Let $h_1 = h_2 = h_3 = 1/3$ and $\varepsilon = 0.0001$. After several iterations, $|\eta_1| = 5.7495 \times 10^{-5} < \varepsilon$, $|\eta_2| = 8.8394 \times 10^{-5} < \varepsilon$, $|\eta_3| = 3.0899 \times 10^{-5} < \varepsilon$, $|\eta_4| = 1.1102 \times 10^{-16} < \varepsilon$, indicating that the derived priority vector has reached an acceptable error level. Therefore, the optimal priority vector is found to be $W^* = (0.1954, 0.4386, 0.2316, 0.1344)^T$.

The comparative result is shown in Table 9. It is clear that CSM preforms the best in both MD and MAD. CSM obtains the same ranking as LLSM and LSM, $x_2 \succ x_3 \succ x_1 \succ x_4$, while GPM yields a slightly different ranking $x_2 \succ x_3 \sim x_1 \succ x_4$, as it fails to discriminate x_1 and x_3 and underperforms the proposed CSM and the other two methods in both MD and MAD.

Table 8. Performance comparisons for Example 3

Method	W^*	Ranking	*FCR*	MD	MAD
CSM (This article)	$(0.0884, 0.1615, 0.1581, 0.2365, 0.2185, 0.1370)^T$	$x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$	0.0728	0.7487	0.1802
EM [40]	$(0.0896, 0.1671, 0.1594, 0.2355, 0.2225, 0.1258)^T$	$x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$	0.0729	0.6047	0.1826
NRAM [32]	$(0.1204, 0.1681, 0.1648, 0.2000, 0.1931, 0.1537)^T$	$x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$	0.1014	1.3993	0.2345
GPM [41]	$(0.1091, 0.1636, 0.1091, 0.2545, 0.2545, 0.1091)^T$	$x_4 \approx x_5 \succ x_2 \succ x_3 \approx x_6 \approx x_1$	0.1033	1.7849	0.2999
LSM [17]	$(0.0978, 0.1765, 0.1591, 0.2263, 0.2220, 0.1183)^T$	$x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$	0.0778	0.6848	0.1987
LLSM [38]	$(0.0878, 0.1599, 0.1551, 0.2464, 0.2208, 0.1301)^T$	$x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_1$	0.0719	0.6037	0.1874

Table 9

Performance comparisons for Example 4

Methods	W^*	Ranking	MD	MAD
CSM(This article)	$(0.1954, 0.4386, 0.2316, 0.1344)^T$	$x_2 \succ x_3 \succ x_1 \succ x_4$	0.7520	0.1672
LSM[17]	$(0.1822, 0.4611, 0.2160, 0.1408)^T$	$x_2 \succ x_3 \succ x_1 \succ x_4$	0.9909	0.1946
GPM [41]	$(0.2000, 0.4667, 0.2000, 0.1333)^T$	$x_2 \succ x_3 \approx x_1 \succ x_4$	1.0411	0.1999
LLSM[38]	$(0.1864,0.4587,0.2274,0.1275)^T$	$x_2 \succ x_3 \succ x_1 \succ x_4$	0.8154	0.1825

6. Concluding remarks

This paper proposes a chi-square method to handle decision problems with incomplete reciprocal preference relations and develops a convergent iterative algorithm to determine a priority vector. An adapted acceptable consistency ratio is employed to judge whether an incomplete reciprocal preference relation is acceptably consistent. If its consistency is not acceptable, an algorithm is put forward to repair it until its consistency reaches Saaty's suggested threshold. This extended CSM not only improves the consistency level but also aims to preserve the initial preference information as much as possible.

Four numerical examples are examined to illustrate how to apply the proposed CSM and its effectiveness. Comparative studies with existing methods reveal the following features of the proposed CSM:

(1) In contrast to LSM, GPM and LLSM where DM's weights are not considered, the proposed CSM allows the analyst to assign proper weights to different experts to reflect their varying influences in GDM problems.

(2) By setting h_1 =1 and h_k =0 for $k=2,\ldots,m$, CSM can be conveniently applied to derive a priority vector from a single incomplete reciprocal preference relation. This implies that the proposed CSM model can be employed to handle both group and individual decision problems.

(3) By setting δ_{ij} =1, for all $i,j\in N$, CSM can be utilized to derive a priority vector from complete reciprocal preference relations. This indicates that it can be flexibly used to handle decision problems with both complete and incomplete reciprocal preference relations.

(4) Numerical experiments demonstrate that CSM often outperforms the other methods such as EM, GPM, LSM, LLSM, and NRAM in terms of FCR, MD, and MAD when handling incomplete reciprocal preference relations.

(5) As illustrated in Example 2, CSM tends to have better rank preservation capability and discrimination power.

Current research establishes CSM as a viable and effective tool to handle decision problems with incomplete reciprocal preference relations. In reality, DMs may provide their preference judgment in different formats of preference relations. As a worthy future research topic, it would be interesting to explore how the CSM framework can be extended to tackle other types of decision inputs such as incomplete intuitionistic fuzzy preference relations [29], incomplete linguistic preference relations [8, 25] and related consensus problems [6, 30, 31].

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