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Deuterium Lamb shift via quenching-radiation anisotropy measurements
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The Lamb shift of a hydrogenic ion can be deduced from the anisotropy in the angular distribution of the 2s
\( \rightarrow \) 2p electric field quenching radiation. The accuracy of our previous anisotropy measurement for deuterium is improved to about \( \pm 150 \) ppm. The derived Lamb shift is \( (1059.36 \pm 0.16) \) MHz. The sources of error are carefully analyzed and the prospects for further improvements in the accuracy are discussed.

I. INTRODUCTION

The purpose of this paper is to report further progress in a program to develop an alternative method of measuring the 2s \( \rightarrow \) 2p Lamb shift in hydrogen and hydrogenic ions. The method is based on the observation that when a hydrogenic atom in the metastable 2s \( \text{state is quenched by an electric field, the induced Lyman-}\( \alpha \) (Ly-\( \alpha \)) radiation intensity possesses an anisotropy in its angular distribution which is (nearly) proportional to the Lamb shift. The complete angular distribution pattern need not be measured—only the ratio \( I_\alpha /I_\beta \) of the intensities emitted parallel and perpendicular to the applied field direction. The corresponding anisotropy \( R = (I_\alpha - I_\beta) / (I_\alpha + I_\beta) \) is a maximum for these two directions.

We have demonstrated previously that the anisotropy for H and D can be measured to an accuracy of \( \pm 0.1\% \). In this paper, the accuracy for D is improved by about an order of magnitude to \( \pm 0.014\% \) (\( \pm 140 \) ppm). We use D rather than H because time-dependent effects resulting from hyperfine structure are smaller.

There are several reasons why a measurement of improved accuracy is of interest. First, the method can be extended in a straightforward way to heavy hydrogenic ions. No other technique (such as quench-rate measurements or laser resonance measurements) presently offers comparable accuracy in the range \( 3 \leq Z \leq 10 \). It is therefore important to verify that all phenomena are thoroughly understood for an atom such as D, where the Lamb shift is already accurately known (\( \pm 60 \) ppm). Lamb shift measurements in hydrogenic ions have recently been reviewed by Kugel and Murnick. Second, the present measurement provides a precise test of theories describing the time-dependent interaction between atoms and external electric fields. Third, the prospects are good that the accuracy can be improved still further, making the anisotropy method competitive with microwave resonance techniques for H, D, and He*. The separated-oscillatory-fields measurement of Lundeen and Pipkin for H is accurate to \( \pm 20 \) ppm; other resonance-type measurements are accurate to \( \pm 60 \) ppm for H (Refs. 11 and 12) and D (Refs. 11 and 13), and \( \pm 80 \) ppm for He (Ref. 14).

The remainder of the paper is divided into the following sections: The experimental apparatus and procedure is discussed in Sec. II, followed by a description of the theory used to interpret the results in Sec. III. The experimental results are presented in Sec. IV, together with a careful analysis of the measurement statistics. The statistical analysis of our large number of data points is an interesting study in itself. The systematic corrections are discussed in Sec. V, and the final results presented in Sec. VI. We discuss here also the problems associated with further improvements in the accuracy of such measurements.

II. EXPERIMENTAL APPARATUS AND PROCEDURE

A. Description of the experiment

A schematic diagram of the apparatus is shown in Fig. 1. The experiment is basically the same as the one described previously, but a number of modifications have been made to improve the accuracy. Briefly, a deuterium beam containing a large fraction of metastable atoms is formed by passing 6.36-KeV D* ions through a cell containing Cs vapor. To reduce background noise, a small electric field (6 V/cm) is applied between the quenching plates to remove charged particles from the beam. The remaining beam is collimated and then enters the quadrupole quenching field region. Here, the induced photons emitted in two perpendicular directions are counted simultaneously by a double-counter system, and the beam current is recorded by a neutral-particle detector. Simultaneous counting is essential in a precision experiment to avoid systematic errors due to beam fluctuations.

In the present experiment, all the data are taken at a single field strength of about 82 V/cm, rather than using a range of field strengths and extrapo-
lating to zero as in our previous work. Since random fluctuations in the counting statistics are the dominant source of error, this strategy gives the greatest possible statistical accuracy while avoiding the errors inherent in taking data at very low field strengths (such as low counting rate, stray fields, contact potentials, etc.). The disadvantage is that the field strength along the beam axis must be accurately known in order to calculate reliably the time-dependent anisotropy at finite field strengths.

A new quadrupole system (see Fig. 1) was therefore precision machined to a tolerance of ±0.005 mm in all critical dimensions. The field at the beam axis is produced by connecting together the rods in two nearest-neighbor pairs with the pairs held at opposite potentials. The rods are enclosed in a box of square cross section held at ground potential to provide a well-defined boundary condition for the field inside. The end plates with small entrance and exit apertures are also grounded. The quadrupole rods 1.016 cm in diameter and 10.7 cm long are located on a circle of diameter 4.572 cm. The outer box is 9.65 cm on each side and 10.7 cm long.

The ultraviolet photon detectors (Galileo Electro-Optics model BX762) have a quantum efficiency of about 10% for Ly-α radiation.

B. Electric field calculation

The calculation of the electric field near the beam axis is in principle straightforward, but since the results must be accurate to ±0.1% or better for our somewhat complicated three-dimensional geometry, the computational problems are not trivial. The methods used will therefore be described in some detail.

\[
\begin{align*}
    &\frac{\partial^2 \phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \phi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0 \\
    \phi_0(\rho, \theta) &= V, \quad \theta_1 < \theta < \theta_2 \\
    \left. \frac{\partial \phi}{\partial \theta} \right|_{\rho = R} &= 0, \quad 0 < \theta < \theta_1 \quad \text{and} \quad \theta_2 < \theta < \frac{1}{2} \pi
\end{align*}
\]

where \( R = (2a^2 - b^2)^{1/2} \) (see Fig. 2) is the radius of the circle of inversion symmetry; i.e., the problem is invariant under the transformation \( \rho \rightarrow R^2/\rho \). Each term of (2) individually satisfies (1). The coefficients \( A_n \) are determined by the boundary conditions in the first quadrant:

\[
\phi_0(\rho, \theta) = V, \quad \theta_1 < \theta < \theta_2
\]

with

\[
\left. \frac{\partial \phi}{\partial \theta} \right|_{\rho = R} = 0, \quad 0 < \theta < \theta_1 \quad \text{and} \quad \theta_2 < \theta < \frac{1}{2} \pi
\]

where

\[
\begin{align*}
    \theta_1 &= \frac{1}{2} \sin^{-1}(1 - b^2/a^2), \\
    \theta_2 &= \frac{1}{2} \pi - \theta_1
\end{align*}
\]

are the values of \( \theta \) for the two radial lines which are tangent to the rod (see Fig. 2),

\[
\rho(\theta) = \sqrt{2} a \cos \beta - (b^2 - 2a^2 \sin^2 \beta)^{1/2},
\]

\[
\beta = \frac{1}{2} \pi - \theta
\]

is the equation of the arc defining the rod surface for \( \theta_1 < \theta < \theta_2 \), and \( V \) is the constant potential on the rod. Only the first quadrant need be considered since the symmetry conditions

\[
\phi_0(\rho, 0) = 0
\]

and

\[
\left. \frac{\partial \phi}{\partial \theta} \right|_{\theta = 1} = 0
\]

are automatically satisfied by (2).

To find the coefficients \( A_n \), the expansion (2) is truncated after the first \( N \) terms, and the functional

\[
J = c_1 I_1 + c_2 I_2,
\]
where

\[ J_1 = \int_{\theta_1}^{\theta_2} \left( \sum_{n=0}^{N} A_n \rho(\theta)^{2n+1} \sin(2n+1)\theta - V \right)^2 \left( \frac{d\alpha}{d\theta} \right)^2 d\theta, \]  

\[ J_2 = \int_{\theta_1}^{\theta_2} \left( \sum_{n=0}^{N} (2n+1)A_n R^{2n} \sin(2n+1)\theta \right)^2 d\theta \]

\[ + \int_{\theta_1}^{\theta_2} \left( \sum_{n=0}^{N} (2n+1)A_n R^{2n} \sin(2n+1)\theta \right) \frac{d\alpha}{d\theta} d\theta, \]

is formed to measure the mean-square deviation of the solution from the boundary conditions (3) and (4). The angle \( \alpha \) is the one shown in Fig. 2 and the factor \( d\alpha/d\theta \) is included in (9) so that points on the rod surface are weighted equally. The conditions

\[ \frac{\partial J}{\partial A_n} = 0, \quad n = 1, 2, \ldots, N \]

then determine \( N \) linear equations in \( N \) unknowns which are solved for the \( A_n \). The arbitrary weighting factors \( c_1 \) and \( c_2 \) in (8) can be adjusted to optimize the rate of convergence of the \( A_n \) with increasing \( N \). For \( N = 40 \), the first few \( A_n \)'s converge to the values

\[ A_0 = 0.91285 V/a, \]

\[ A_1 = 0.193 63 V/a^2, \]

\[ A_2 = -0.039 83 V/a^6, \]

for our geometry, with \( a = 1.6165 \pm 0.0005 \) cm.

If we define a coordinate system such that the \( y \) axis lies along the beam axis and the \( z \) axis points in the dominant field direction, then the electric field components near the beam axis are

\[ E_x = -\frac{\partial \Phi_0}{\partial x} = -A_0 - 3A_1(x^2 - y^2) + \cdots, \]

\[ E_z = -\frac{\partial \Phi_0}{\partial z} = -6A_2 xz + \cdots. \]

Since the beam radius is small compared to the dimensions of the quenching cell, the \( A_1 \) corrections to the field turn out to be negligible.

The small corrections to \( A_0 \) due to the finite box width and quenching-cell length are included by a superposition of solutions. The second step therefore consists of solving the two-dimensional problem with the boundary condition \( \Phi_1 = -\Phi_0 \) on the walls of the enclosing box, and \( \Phi_1 = 0 \) on the rods. This is done by the well-known numerical technique of subdividing the area with a square grid and demanding that the potential at each grid point be the average of its neighbors. A relatively coarse 48 \( \times \) 48 grid, together with the technique of "deferred approach to the limit,"\(^{16}\) is sufficient to determine the correction

\[ \Delta A_0^{(1)} = (-0.030 69 \pm 0.000 03)V/a. \]

The third step is to solve similarly the three-dimensional problem for the finite length correction. Here the boundary conditions are \( \Phi_2 = -(\Phi_1 + \Phi_0) \) on the grounded end plates, and \( \Phi_2 = 0 \) on the rods and outer box. This yields the further correction.
\[ \Delta A_0^{(2)} = (-0.00264 \pm 0.00001)V/a, \]

along with the field strength as a function of position along the beam axis. The dependence on position along the beam axis is well approximated by the formula

\[ E_\eta(y) = \begin{cases} \frac{2 \sin(\eta y)}{\frac{1}{2} + \sin^2(\frac{\eta y}{2})}, & \eta y < \frac{1}{2} \pi \\ E_{\text{max}} & \eta y \geq \frac{1}{2} \pi \end{cases} \]

with \( \eta = 0.356 \text{ cm}^{-1} \) and \( E_{\text{max}} = A_0 + \Delta A_0^{(1)} + \Delta A_0^{(2)} = 0.87952V/a. \)

In this experiment, \( V \) is fixed at 150.00 V. The field strength at the center of the cell is then \( E_{\text{max}} = 81.613 \pm 0.05 \text{ V/cm}. \) The uncertainty comes primarily from uncertainties in the dimensions of the quenching cell—particularly the rod diameters.

### III. Theory

We describe in this section the induced Ly-\( \alpha \) radiation which is emitted when a beam of metastable D atoms enters an asymptotically constant electric field with an initial fringing field extending over a substantial distance. The experimental conditions are such that field entry is nearly adiabatic, but nonadiabatic corrections are significant. We therefore integrate the full time-dependent Schrödinger equation in the finite basis set of 24 states with \( n = 2 \). These are the hyperfine states \( 2s_{1/2} (F = \frac{1}{2}, \frac{1}{2}), 2p_{1/2} (F = \frac{3}{2}, \frac{3}{2}) \) and \( 2p_{3/2} (F = \frac{3}{2}, \frac{3}{2}), \) each of which is \( (2F + 1) \)-fold degenerate. All calculations are done in the coupled \( (l, s, j, l, F, M_F) \) representation, except that the basis set is pre-diagonalized with respect to off-diagonal matrix elements of the hyperfine–structure interaction. This weakly mixes the states \( 2p_{1/2} (F = \frac{3}{2}) \) and \( 2p_{3/2} (F = \frac{3}{2}), \) and slightly alters the anisotropy. \(^{17} \)

The method of solution is basically the same as that described previously. \(^{18} \) Briefly, the time-dependent Schrödinger equation to be solved in the finite basis set is

\[ i \frac{d\psi}{dt} = U(t)\psi, \]

\[ U(t) = E + F(t)V, \]

in atomic units, where \( \psi \) is the 24-component column vector of time-dependent state amplitudes, \( E \) is the diagonal matrix of complex field-free eigenvalues, \( V \) is the interaction matrix with the external electric field, \( F(t) \) is the time-dependent field strength given by

\[ F(t) = E_x(vt), \]

where \( v \) is the beam velocity \((0.780 \times 10^4 \text{ cm/s}) \), and \( E_x \) is given by (15). The set of coupled equations (17) is solved numerically by first transforming to the representation of asymptotic field-perturbed states defined by

\[ \tilde{\psi}(t) = A^{-1}\tilde{\psi}(t), \]

where \( A \) is the matrix such that \( A^{-1}(\mathbf{E} + F(\infty)\mathbf{V})A \) is diagonal with complex eigenvalues \( \tilde{E}_j \). A separate solution must be found for the initial condition that the atoms are in each of the six hyperfine \( 2s_{1/2} (F = \frac{1}{2}, \frac{3}{2}) \) states at time \( t = 0 \). If these states are labeled by \( i = 1, \ldots, 6 \), then

\[ \psi_i(t) = \begin{cases} \mathbf{\Psi}_e^{-i\mathbf{F}t}, & t < 0 \\ \sum_i \mathbf{\Psi}_e \mathbf{A}^{-1} \mathbf{b}_i(t), & t > 0, \end{cases} \]

where \( \mathbf{b}_i(t) \) is the solution corresponding to the initial condition \( \psi_1(0) = \delta_{i,1} \). As \( F(t) \) approaches its limiting value \( F(\infty) \), the coefficients \( \mathbf{b}_i(t) \) tend to the limit functions

\[ \mathbf{b}_i(t) = \mathbf{b}_i(\infty)e^{\mathbf{F}t}. \]

The numerical integration can therefore be stopped when this limiting behavior is reached. The above representation yields less rapidly oscillating solutions in the fringing field region than the \( c_l(t) \) representation used in Ref. 18. The two differ by factors of \( e^{-2\mathbf{F}t}. \)

The field-induced electric dipole transitions to the ground-state manifold (labeled by \( f = 1, \ldots, 6 \)) proceed via mixing between the \( 2s \) and \( 2p \) states. The transition rate for a given polarization vector \( \mathbf{b} \) is proportional to the quantity

\[ \mathbf{I}(\tilde{\mathbf{b}},t) = \sum_l |\psi_i(t)|^2 \mathbf{b} \cdot \mathbf{F} |\psi_l(t)|^2, \]

assuming that the incident beam is an incoherent mixture of \( 2s_{1/2} \) hyperfine states with equal statistical weights. For our geometry with the beam along the \( y \) axis and the electric field along the \( z \) axis, the theoretical anisotropy uncorrected for the finite solid angle of observation is

\[ \mathbf{R}(t) = \frac{\mathbf{I}(\tilde{\mathbf{b}},t) - \mathbf{I}(\mathbf{\tilde{b}},t)}{\mathbf{I}(\tilde{\mathbf{b}},t) + 2\mathbf{I}(\mathbf{\tilde{b}},t) + \mathbf{R}(\tilde{\mathbf{b}},t)}. \]

The calculated \( \mathbf{R}(t) \) is plotted as a function of position along the beam axis in Fig. 3. Although the curve appears to be asymptotically flat, it is in fact slowly increasing at the rate of \( 2.09 \times 10^5 \text{ cm}^{-1} \). The observation region is as shown in the figure. The position of the observation region is such that the absolute intensity of the quenching radiation is approximately a maximum as a func-
tion of field strength at our working field of 81.613 V/cm. This reduces errors resulting from changes in absolute intensity due to small variations in field strength either across the beam diameter or as a function of time. Using the input data in Table I, the theoretical value of $R$ at the middle of the observation region is $0.144245$. We also list in Table I the rate of change of $R$ with respect to each of the input parameters.

### IV. EXPERIMENTAL DATA AND STATISTICAL ANALYSIS

In all, 2038 individual measurements of $R$ were made, each containing on the average $1.355 \times 10^6$ counts. The data were collected in 24 separate runs made on different days. Each measurement consisted of eight counter readings—one for each of the two counters at the four electric field directions obtained by rotating the potentials on the quadrupole rods in steps of $90^\circ$. The eight readings were combined into a single measurement as described previously to so as to eliminate the relative sensitivities of the two detector systems. The background noise was defined to be the isotropic part of the residual signal measured under identical conditions except that most of the metastables in the beam were quenched before entering the quenching-cell proper. With our signal-to-noise ratio of about 500 : 1, one noise measurement after every five signal measurements was sufficient to make the uncertainty due to noise counting statistics a small correction. The noise corrections to $R$ were typically about 0.2%.

The mean and standard deviation of all the measurements, weighted by the square root of the number of counts in each measurement, are defined by

$$R = \frac{\sum_{i=1}^{N} R_i n_i^{1/2}}{\sum_{i=1}^{N} n_i^{1/2}}$$

and

$$\sigma = \left( \sum_{i=1}^{N} \frac{(R_i - R)^2 n_i}{n_i (N - 1)} \right)^{1/2},$$

where $n_i$ is the number of counts in the $i$th measurement, $n_\tau$ is the total number of counts, and $N$ is the number of measurements. The experimental values are

$$R = 0.1440879, \quad \sigma = 0.0000199,$$

uncorrected for the finite solid angle of observation and electronic dead time. The theoretically expected standard deviation arising from counting statistics alone is

$$\sigma_R = \frac{1}{2} \left( \alpha^{1/2} + \alpha^{-1/2} \right) [1 - R^2] n_\tau^{1/2},$$

where $\alpha$ is the ratio of the sensitivities of the two-counter systems. Using $\alpha = 0.622$ and $n_\tau = 2.763 \times 10^6$, the above formula yields $\sigma_R = 0.000194$. Although this differs by only 3% from the experimental value, the chi-square test shows that the difference is statistically significant at the 98% confidence level. The difference is due to the statistical uncertainty in counting the noise. This effect introduces an additional uncertainty of

$$\sigma_{\text{noise}} = \sigma_R \sqrt{\frac{n_\text{noises}}{n_\tau}} \left( \frac{1 + R^2}{1 - R^2} \right)^{1/2},$$

for

### TABLE I. Input data ($x$) for the theoretical calculation of the $D(2s_{1/1})$ quenching anisotropy ($R$), and sensitivity coefficients ($s$)².

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\text{Value}$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(2s_{1/1}) - E(2p_{1/1})$</td>
<td>1059.241 MHz</td>
<td>0.95</td>
</tr>
<tr>
<td>$E(2p_{3/2}) - E(2p_{1/1})$</td>
<td>10972.02 MHz</td>
<td>-0.95</td>
</tr>
<tr>
<td>$\alpha_{\text{HFS}}$</td>
<td>40.932 MHz</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>$\Gamma(2p)$</td>
<td>6.265 x 10⁶ sec⁻¹</td>
<td>-0.0004</td>
</tr>
<tr>
<td>Field strength</td>
<td>81.613 V/cm</td>
<td>0.04</td>
</tr>
<tr>
<td>Beam energy</td>
<td>6335 keV</td>
<td>-0.0004</td>
</tr>
<tr>
<td>Distance to observation region</td>
<td>5.33 cm</td>
<td>0.0008</td>
</tr>
<tr>
<td>$\eta$ (Eq. 13)</td>
<td>0.356 cm⁻¹</td>
<td>0.0008</td>
</tr>
</tbody>
</table>

²The coefficient $s$ is defined by $\delta R/R = s \delta x/x$, where $\delta R$ is the change in $R$ induced by a change $\delta x$ in $x$.

²The hyperfine-structure energy shifts are given by

$$\Delta E_{\text{HFS}} = \frac{\alpha_{\text{HFS}}}{n^2} \left( \frac{E(F + 1) - E(j + 1) - I(j + 1)}{j(j + 1)\ell(\ell + \frac{3}{2})} \right).$$

FIG. 3. Calculated anisotropy as a function of position along the beam axis. The trapezium to the right is the geometrical slit function of the detector system showing its location along the beam axis.
where $n_{\text{noise}}/n_{e}$ is the noise-to-signal ratio and $F$ is the number of signal measurements between each noise measurement. Using $F = 5$ and $n_{\text{noise}}/n_{e} = 1/500$, the total theoretical standard deviation is

$$
\sigma_{\text{tot}} = (\sigma_{R}^{2} + \sigma_{\text{noise}}^{2})^{1/2} = 0.0000199, \tag{28}
$$
in statistical agreement with the experimental value. The standard deviation therefore shows no detectable fluctuations in the experimental results other than those arising from counting statistics. This conclusion is further supported by Fig. 4, which compares the histogram of the experimental data distribution about the mean (in units of the expected standard deviation for each point) with the theoretically expected histogram for a Gaussian distribution. The chi-square test of the fit, with the mean as the only adjustable parameter, yields $\chi^2 = 49.2$ for 49 degrees of freedom. Therefore, one could expect worse agreement about half the time.

One can also look for systematic variations in the data by means of runs tests. A high (low) run is defined to be a sequence of consecutive measurements which all fall above (below) the mean. The length of a run is simply the total number of elements contained in the sequence from beginning to end. The number of runs of a given length is asymptotically normally distributed. If $n$ measurements are drawn from a random population with probability $e_{1}$ of falling above the mean and probability $e_{2}$ of falling below the mean ($e_{1} + e_{2} = 1$), then the expected number of high runs of length $i$ is asymptotically

$$
s_{i} = n e_{1} e_{2}^{i+1}/(i+1), \tag{29}
$$
with variance

$$
\sigma_{i}^{2} = n e_{1} e_{2}^{i+1}/(i+1)^{2} e_{1} e_{2} - (2e_{2} - 2e_{1})e_{1} e_{2}^{i+1}/(i+1)^{2}. \tag{30}
$$
The observed numbers of runs are compared with the expected numbers in Table II assuming that the measurements are drawn from a symmetric distribution so that $e_{1} = e_{2} = 0.5$. Of the 2038 individual measurements, 49.0% fell above the mean and 51.0% fell below the mean. This is consistent with the expected fraction for a symmetric distribution of $(50.0 \pm 1.1)\%$ on each side. With the exception of one unusually long low run, the results in Table II are in reasonable agreement with the expected numbers of runs. This appears to be a valuable test for systematic variations in long sequences of repeated measurements.

<table>
<thead>
<tr>
<th>Run length</th>
<th>Low runs</th>
<th>High runs</th>
<th>Expected number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>245</td>
<td>254</td>
<td>255 ±14</td>
</tr>
<tr>
<td>2</td>
<td>133</td>
<td>131</td>
<td>127 ±10</td>
</tr>
<tr>
<td>3</td>
<td>60</td>
<td>61</td>
<td>64 ±7</td>
</tr>
<tr>
<td>4</td>
<td>33</td>
<td>33</td>
<td>32 ±5</td>
</tr>
<tr>
<td>5</td>
<td>21</td>
<td>13</td>
<td>16 ±4</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>5</td>
<td>8.0 ±2.7</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>7</td>
<td>4.0 ±1.9</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>2</td>
<td>2.0 ±1.4</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>1</td>
<td>1.0 ±1.0</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>0</td>
<td>0.5 ±0.7</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>0</td>
<td>0.25 ±0.5</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>0</td>
<td>0.12 ±0.35</td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>0</td>
<td>0.06 ±0.25</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>0</td>
<td>0.03 ±0.18</td>
</tr>
<tr>
<td>Total</td>
<td>506</td>
<td>507</td>
<td>510 ±11</td>
</tr>
</tbody>
</table>
V. SYSTEMATIC CORRECTIONS

The two systematic effects to be accounted for are the finite solid angle of observation of the detectors and the electronic dead time of the counting system.

The solid-angle effect is obtained by integrating the intensity as a function of direction over the source, and over the solid angle observed through the slit system shown in Fig. 4 for each point in the source. The intensity function is

\[ I(\theta) = A(1 - \cos^2\theta) + A'(1 + \cos^2\theta), \]

where \( A \) and \( A' \) are constants, and \( \theta \) is the angle between the direction of observation and the electric field direction. Equation (23) can then be rewritten

\[ R = \frac{(A' - A)}{(3A' + A)} \]

and the correction factor for the observed value \( R_{\text{obs}} \) can be expanded as

\[ R_{\text{obs}} = R \left( 1 - \frac{(\alpha^2 + \beta^2)}{12(\beta - \alpha)} \right) + \frac{\gamma^2}{6\beta} + \frac{\delta^2}{\beta^2} \]

up to second order in the corrections. Here, \( a, b, \alpha, \) and \( \beta \) are as shown in Fig. 5, \( \gamma \) is the slit height, and \( \delta \) is the beam diameter. Corrections for the off-axis field inhomogeneity due to the quadrupole geometry do not enter until fourth order, and are therefore negligibly small. Using the values

\[ \alpha = 0.3810 \pm 0.0012 \text{ cm}, \]
\[ \beta = 0.4445 \pm 0.0012 \text{ cm}, \]
\[ \gamma = 0.5639 \pm 0.0012 \text{ cm}, \]
\[ \delta = 0.152 \pm 0.012 \text{ cm}, \]
\[ a = 4.826 \pm 0.001 \text{ cm}, \]
\[ b = 10.922 \pm 0.001 \text{ cm}, \]

the correction factor in (33) is 1.001107 with an uncertainty of ±5 ppm. The dominant uncertainty is in the determination of the beam diameter 5.

The electronic dead time was estimated by measuring directly the duration of the pulses fed from the preamplifiers into the rate-meter counting system. Since the rest of the counting system was substantially faster, this should give a reasonable estimate. If the dead time is \( \tau \) and the count rate is \( n/T \), then to a first approximation the fractional loss of counts due to dead-time effects is

\[ \frac{\delta n}{n} = \tau n / T \]

and the fractional decrease in \( R \) is

\[ \frac{\delta R}{R} = \frac{1}{2} \tau (n/T)(1 - R^2), \]

where \( (n/T) \) is the total count rate for both counters averaged over a complete measurement. For our values of \( \tau = (3 \pm 1) \times 10^{-8} \text{ sec} \) and \( (n/T) = 5.4 \times 10^2 \text{ sec}^{-1} \), the fractional correction is \( 5R/R = (7.9 \pm 2.6) \times 10^{-5} \). The uncertainty of 26 ppm is the largest source of error other than counting statistics.

VI. RESULTS AND DISCUSSION

The final experimental value corrected for the finite solid angle of observation and electronic dead time is \( R = 0.144259 \pm 0.000021 \) (one standard deviation) in agreement with the theoretical value 0.144245 calculated from the data shown in Table I. The various sources of error which contribute to the uncertainty of ±143 ppm are summarized in Table III. A number of other effects, such as background magnetic fields, beam contamination by hydrogen, and relativistic effects, were carefully considered and found to be negligible.

If one assumes that the other data in Table I are correct, then the measured \( R \) value corresponds to a Lamb shift of 1059.36 ± 0.16 MHz (70% confidence level). This value is compared with other experimental and theoretical determinations in Table IV. Our value agrees with the measurement of Coens,\textsuperscript{19} but disagrees by two standard deviations with the earlier measurement of Triebwasser et al.\textsuperscript{11}

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( \delta R/R ) (ppm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counting statistics</td>
<td>138</td>
</tr>
<tr>
<td>Dead-time correction</td>
<td>26</td>
</tr>
<tr>
<td>Electric field strength</td>
<td>24</td>
</tr>
<tr>
<td>Fringing field effects</td>
<td>8</td>
</tr>
<tr>
<td>Solid-angle correction</td>
<td>5</td>
</tr>
<tr>
<td>( \left( \sum \delta R \right)^{1/2}/R )</td>
<td>143</td>
</tr>
</tbody>
</table>
TABLE IV. Comparison of theoretical and experimental Lamb shifts (MHz) in deuterium.

<table>
<thead>
<tr>
<th>Theory</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1059.271±0.023 $^a$</td>
<td>1059.24±0.06 $^c$</td>
</tr>
<tr>
<td>1059.241±0.027 $^b$</td>
<td>1059.00±0.06 $^d$</td>
</tr>
<tr>
<td>1059.36±0.16 $^e$</td>
<td>1059.36±0.16 $^e$</td>
</tr>
</tbody>
</table>

$^a$Erickson, Ref. 22.
$^b$Mohr, Ref. 23.
$^c$Cosens, Ref. 13.
$^d$Triebwasser, Ref. 11.
$^e$Present work.

Although the present experiment is about a factor of 2.7 less precise than the measurements of Triebwasser et al. and of Cosens, further substantial improvements should be possible. Of the sources of error listed in Table III, the dominant one due to counting statistics can be reduced simply by counting longer and/or faster. The difficulty with counting faster is that the dead-time correction, and its associated uncertainty, become larger. The dead time was estimated only roughly in the present work (±30%), and a more accurate determination (or faster electronics) will be necessary if a higher-precision experiment is attempted. The estimated error arising from the electric field calculation comes primarily from the ±0.005 mm uncertainty in the diameters of the quadrupole rods. More accurately machined rods are now available, which would reduce the uncertainty in $R$ from this source by about a factor of 2. An experiment at the 20 ppm level of accuracy should, therefore, be feasible.

For analogous experiments on heavier hydrogenic ions, the $R$ values would lie much closer to the zero field values at the field strengths one would conventionally use, making the electric field determination and fringing field effects less important. Also, the numerical integration of the time-dependent Schrödinger equation becomes unnecessary—one can safely assume adiabatic entry. Specifically, the ratios of the quench rate for a hydrogenic ion of nuclear charge $Z$ to the quench rate for deuterium is approximately $Z^2(E_z/E_p)^2(1-Z^2/4Z)$, where $E_z/E_p$ is the ratio of electric field strengths in the two cases and $1-Z^2/4Z$ is the ratio of Lamb shifts. Thus, the same quench rate for deuterium obtained at 80 V/cm in the present work would occur at about 500 V/cm for He$^+$ and about 20 kV/cm for O$^+$. The latter value is still within the range of field strengths that can be conveniently produced in the laboratory with the electrostatic quadrupole geometry. On the other hand, the finite electric field corrections to $R$ arising from higher-order perturbation effects decrease in proportion to $1/Z^4$ along the isoelectronic sequence when the electric field strength is adjusted as shown above to keep the quench rate constant.

The only major new problem one encounters for the heavier hydrogenic systems is a false asymmetry, which appears when electrons produced by collisions with the background gas are accelerated by the quenching field into the detectors. This effect can be suppressed by the application of a small axial magnetic field to deflect the electron trajectories. The quadrupole geometry is an advantage in this regard because the off-axis electric field inhomogeneity helps to magnify the deflection. The required magnetic field strength of about 100 G for O$^+$ at a quenching field of 20 kV/cm is sufficiently small so as to make the magnetic field correction to the measurements negligible. Anisotropy measurements on He$^+$ and O$^+$ are in progress.

Note added in proof. It has recently been pointed out that additional intensity asymmetries of measurable size may be present for the heavier hydrogenic ions due to interference between induced $E1$ and spontaneous $M1$ transitions to the ground state. For unpolarized beams, the asymmetry is with respect to mirror reflections in the plane perpendicular to the electric field direction. However, the effect averages to zero if both directions are counted, and hence does not influence the present results.

ACKNOWLEDGMENT

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19 Equation (26) is equivalent to Eq. (28) of Ref. 1 for \( \alpha = 1 \).