A review of P L C analyses and evaluation of some unconventional coupling arrangements.

D. Kochhar

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A REVIEW OF PLC ANALYSES AND EVALUATION OF SOME UNCONVENTIONAL COUPLING ARRANGEMENTS

by

D. KUCHEL

A Thesis

Submitted to the Faculty of Graduate Studies through the Department of Electrical Engineering in Partial Fulfillment of the Requirements for the Degree of Master of Applied Science at the University of Windsor.

Windsor, Ontario
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ABSTRACT

This thesis investigates the techniques of modal analysis applied to propagation of power line carrier (PLC) signals on overhead transmission lines. Lossless and lossy systems are reviewed and analyzed. Natural modes of propagation are applied to the phase currents and voltages of PLC signals. Conductor voltages and currents are represented as vectors and resolved into modes of the propagation matrix.

The application of directional couplers to PLC is analyzed. The application of two types of parallel wire coupler for carrier coupling on high voltage lines, as possible substitutes for capacitive couplers and line traps, is examined. It is shown by applying the theory of coupled transmission lines that it is possible to couple carrier frequency power onto and off high voltage lines with high efficiency using a parallel conductor.

These concepts are applied to a novel coupling arrangement. PLC communication using insulated bundle subconductors (IBS) and its recent developments are also analyzed. The effect of coupling wire size, spacing and coupling factor on bandwidth is determined.
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CHAPTER 1
INTRODUCTION

1.1. Background Material

Carrier current transmission is the technique by which low frequency radio currents are propagated with the utilization of wire line, cables or power lines which are capable of guiding this type of energy. The use of carrier frequency currents for the transmission of information on power transmission lines is called power line-carrier (PLC).

If we first study the question of what makes PLC systems so suitable for the solution of communication problems involved in the operation of Extra High Voltage (EHV) systems in spite of other systems being available (wire, cable, radio relay etc), the following main reasons may be given:

1. The long transmission distances that are usually encountered in EHV networks, amounting to as much as 400 miles, can be covered by direct PLC links, in contrast to the other means, which all require relay or repeater stations.

2. The PLC, which, considered purely from the mechanical aspect, is an extraordinarily robust and reliable transmission medium, although not designed in conformity with telecommunication standards, ensures very stable transmission of carrier signals with a minimum of attenuation.

3. The construction of efficient, reliable PLC sets has now reached such a high standard that even in situations where
other radios such as radio are available for these technical
cases which are vital to the transport of energy, preference.
has been shown for PLC links.

The basic difference in the operation of PLC transmission and power transmission is the frequency. In both cases
the energy is linked by electric and magnetic fields. Carrier
frequencies normally used on this continent lie in the range
from 30 to 300 kHz.

The attraction of power line carrier these days is due
to many practical advantages. It is capable of transmitting
energy economically and also provides a small number of
communications channels on power transmission lines.
Cost per station does not depend on the distance over
which it operates.

PLC systems should operate within a limited frequency
spectrum due to the reason that it is susceptible to power
line noise. Compared with open-wire communication lines,
it has a better insulation and an appreciably higher
mechanical strength. Damage caused by adverse weather
conditions is therefore a very rare occurrence. Besides,
a P.C. communication system saves the cost of constructing a separate line.

While feeding and tapping carrier frequency currents on high voltage transmission lines, special care should be taken to avoid danger to the communication equipment and to the operating personnel. The transmission of carrier frequency currents to the desired point, special paths should be formed in the power line network.

Communication is achieved by a P.C. signal that is coupled from a grounded transmitter to the line through a high voltage capacitor; in addition to the capacitor a tuned wave trap is provided at those points where the communication currents might be dissipated in switching stations or line sections outside the transmission path, and coupling media where communication equipment is to be connected to the power line. Although the cost of line coupling material and the carrier line traps may assume considerable proportions at the high service voltage and phase currents involved, the installation of PLC links may often prove much cheaper than systems using other media, especially where long distances have to be considered. PLC
offers rapid and dependable communication for interoffice business, and for load dispatching. Carrier relaying permits high speed clearing of many types of faults.

1.2. **Historical Review**

Power line carrier is an outgrowth of developments made during the first world war. It was in France in 1918 that General Squiers of the U.S. signal corps conducted experiments with carrier currents. Later in 1920-1921, he successfully used carrier currents on power transmission lines. PLC equipment has been applied to the power lines in this country since the early twenties. In 1919 the General Electric Company demonstrated the feasibility of communication by means of PLC, and in 1921 the first such system was placed in service.

Telephone communication was originally the most important application of PLC. Modern conditions of power system operation and the necessity of increased utilization of carrier installations have led to the transmission of signals for remote control, protection and tele-metering. Values of voltage, current, active and reactive power, frequency, power factor, water levels, and temperatures are transmitted for system operation and often for remote control of generating, transformer and switching stations.

In 1953 Brown Boveri installed their first PLC systems in Canada. The enormous construction programs for power plants and high voltage transmission lines across the vast Canadian territory offered the best prospects for this undertaking, considering the
fact that the construction of a power system automatically calls
for communication facilities. Nevertheless it was essential to
consider that practically all Canadian power utilities had
been using PLC equipment of American origin over the previous
twenty-five years and had thus adopted North American methods
which differ appreciably from European practice.

In contrast to the American and Canadian practice of
almost exclusively using PLC equipment with double side band
amplitude (DSB) or frequency modulation (FM), Brown Boveri
adopted more than ten years ago the single-sideband (SSB)
amplitude modulation technique which implies the transmission
of only one of the two sidebands generated during the modula-
tion process. Compared with double-sideband operation only
half the bandwidth is required. The receiver picks up less
noise energy because of the reduced bandwidth and since no
carrier is transmitted, the total power sent is available
exclusively for the message. Assuming equal power sent, the
transmission range is therefore greater than in the double-
sideband case. For these reasons single sideband terminals
are today generally preferred in PLC communication.

For telesprinter and telemetering applications the
bandwidth is less than 500 Hz. In a band sufficient for
transmitting speech frequencies (500 - 2500 Hz), it is there-
fore possible to accommodate a number of telegraph channels
for telemetering purposes. In PLC networks, it is only the
5 kHz allocation scheme for double side band (DSB) transmission as well as 2.5 kHz for SSB transmission which is of interest. Wide variation in the technology of signal generation and modulation using various transducers are described in manufacturer's literature and will not be discussed here.

In the last twenty five years, microwave systems have been added as a new tool for providing communication channels for the various functions required predominantly on the high traffic density routes. Where microwave is used, PLC has also been added for the vital point to point functions such as protective relaying and important control circuits as well as for feeder links to the microwave in peripheral parts of the system.

At Hydro-Quebec, the use of microwave radio to provide protective relaying channels has become an accepted practice. A recent policy governing the medium of transmission to be used on all future Hydro-Quebec lines states that for the protective relaying of 120,230 and 300 kV lines, one medium of transmission is to be used. This can be either microwave or PLC, for the 735 kV lines, both microwave and PLC channels are required.

A most important consideration in connecting to a high voltage three phase power transmission line is the protection of personnel and communication apparatus from the hazards of high operating line voltages, switching and lightning surges.
In the early years PLC transmission was accomplished with the help of antennas due to the reason that capacitors of sufficient rating were not designed to withstand high voltages. Eventually it was realized that the energy which found its way into the power line was transferred mainly through the capacitance between the antenna and the line. This led to the development of coupling capacitor units for coupling. Today a value of 2200pF has been generally accepted as a minimum for lines up to 230kV. Associated with each coupling capacitor, certain auxiliary apparatus generally known as line tuning equipment located in the circuit between the coupling capacitor and the carrier terminal equipment, performs the primary function of tuning or resonating with the reactance of the coupling capacitance to provide a low loss path for one or more carrier frequencies between the terminal equipment and the power line.

There are a number of different ways of utilizing one or more conductors of the three phase power line as conductors for carrier frequency currents. Common ones are phase to ground and phase to phase coupling. The result of the analysis of transmitter coupling arrangements lead to some conclusions of practical importance.

A new PLC method has recently being investigated in Germany and the U.S.S.R. with the aim of increasing the number of communication channels on bundled conductor lines. Results
of these investigations are encouraging and hence, a modal analysis study of various practical aspects of the new communication method has been carried out here. It was found that the insulated bundle Subconductor (IBS) communication system presents some attractive features not to be expected in conventional PLC installations as discussed in chapter four.

The intrabundle communication system proposed uses a simple inexpensive but highly efficient broadband coupling method which permits signal isolation between subconductors without inserting additional impedance at the power frequency or reducing the power transmission capability of the line.

Nakamura proposed that PLC coupling be applied at a quarter wavelength from the bus end of the bundle. This eliminates the need for a costly line trap since the bus may be connected directly. From this idea most recently P.D. Pullen and his associates in the Central Electricity Research Laboratories in England describe a coupling method for use with intrabundle communication that is cheaper and more efficient than the conventional method.

1.3 Current Work

There has always been a great need for both analytical and experimental work on PLC communication due to its great demand and practical applications to power system communication.

The main objectives of this thesis are to provide a summary
of the analysis of the transmission of PLC signals and terminal coupling arrangements and to apply these concepts to a novel coupling arrangement. Originally, it was thought that the carrier is transmitted along the phase conductors to which the transmitter and receiver are connected. A main problem was to analyze the physical nature and properties of the power line carrier frequency signals along a system of parallel conductors.

Chapter two gives a review and critical analysis of the theory of PLC propagation on power transmission lines. A brief history of modal analysis is made. For a lossless system characteristics of the natural modes for PLC are calculated for a three phase horizontal line with two grounded sky wires having no transposition. The effect of transposition and of transmitter and receiver couplings are discussed. Propagation analysis of carrier currents and voltages on lossy power lines is also reviewed. The effect of earth conductivity and resistivity on modal parameters at PLC frequencies are discussed. Carson's correction factors and the recent work by Professor Perz and Dr. Baghuveer are discussed. The application of lossless modal analysis is applied to intra-bundle subconductors and calculations of modal impedance and modal transformation matrices for two, three and four bundles are presented. Experimental results by Ontario Hydro on the Pinard Hanmer 500kV, three phase line are discussed.
Chapter three deals with unconventional coupling arrangements. Conventional methods of coupling PLC signals use coupling capacitors and traps. Due to the increase in cost of the coupling, unconventional coupling arrangement which uses a parallel conductor close to the phase of transmission line is of great interest due mainly to economics and its wide bandwidth capability. The characteristic equation, coupling and directivity as derived by Firestone is reviewed. Carrier coupling to transmission line by a parallel conductor, its principle of operation and the theory of two coupled transmission lines are analyzed. Equations for transmission and coupling factor are developed with the help of boundary conditions. Results gave the same equations as derived by Oliver using the concept of reflections. Carrier current coupling using an insulated sky wire coupler is examined.

In chapter four, recent developments on PLC communication systems using insulated bundle subconductors are reviewed. Input impedance and power coupled in a quarter wave length of transmission line short circuited at one end are calculated for two cases, one neglected the effects of distributed coupling and the second, treated the signal conductor as parallel wire coupler. Lastly effects of coupling factor, coupling wire diameter and spacing ratio on the bandwidth are discussed.

Conclusions and recommendations are made in chapter five.
CHAPTER 2

CONVENTIONAL PLC TRANSMISSION AND COUPLING ARRANGEMENTS

2.1. General Considerations and Description of PLC Systems

The rapid developments over the last three decades of high voltage lines for long distance transmission of electric power has caused an increasing demand for PLC facilities. The applications of this important and dependable means of communication for telecontrol, protective relaying and telemetering has called for an investigation of characteristics inherent in the transmission of HF signals over a system of parallel conductors. The HF signals are superimposed on the 60 cycle frequency of the power lines.

Essentially a power system consists of three distinct parts:

1. The transmitters and receivers,
2. The coupling and tuning equipment,
3. The power transmission line in order to provide a suitable path for transmission of PLC signals.

A block diagram of a phase to ground carrier current communication system is illustrated in figure 1 below. The signal is to be transmitted from station A to C via the intermediate station B.
The devices or tools available and actually used in establishing carrier circuits on power lines are relatively simple and few in number. The physical and economical restrictions imposed by a high voltage power system limit the devices that can be inserted directly in the power line to essentially only two, the coupling capacitor and the line trap. Depending on the number of carrier channels to be transmitted over an EHV line, a coupling capacitor of sufficient capacitance combined with a trap of corresponding inductance produces the required blocking range.
High-Voltage Coupling Capacitor

A high voltage coupling capacitor whose reactive impedance is very high at the power frequency but quite small at the carrier frequencies is connected directly to the transmission line conductor.

A typical carrier coupling capacitor \(^4, 5\) is shown in figure 2.

![Diagram of Carrier Coupling Capacitor]

**Fig. 2 CARRIER COUPLING CAPACITOR.**

Coupling capacitors are physically composed of several paper capacitor elements connected in series and immersed in oil. The capacitors usually are made up of a large number of sections, connected in series to attain the desired voltage rating. The voltage rating of a coupling capacitor is the nominal phase to phase voltage of the transmission line on which it is to
be used, although the actual potential across the capacitor is the phase to ground voltage. In addition to continuously withstanding this voltage, the capacitor insulation must withstand high voltage impulses caused by lightning and switching surges. The purpose of the drain coil is to ground the capacitor terminal opposite the line terminal at 60 hertz and at the same time offer a high impedance at the carrier frequency.

**Line Traps**

The resonant traps in common use on PLC system consists of air core inductor coils with conductors sufficiently large to carry line current and tuned with capacitance to parallel resonance at the frequency of operation of the carrier equipment. Carrier line traps are inserted in series with the high voltage line and they restrict the high frequency currents to the desired sections of line, and in so doing substantially improved transmission.

![Diagram of a single frequency line trap](image-url)
Since the line trap is directly connected to the power line, it must have an insulation resistance rated for the power voltage. Above all, it must be rated for the full operating current. Since line traps must be strong enough to accommodate, during line fault, power frequency currents of several thousand amperes, they are large. Hence they add appreciably to the cost of PLC installations.

**Line Tuning Equipment**

Associated with each coupling capacitor in its function of connecting carrier circuits into the power lines, certain auxiliary apparatus generally known as line tuning equipment, located in the circuit between the coupling capacitor and the carrier terminal equipment, performs the primary function of tuning or resonating with the capacitive reactance of the coupling capacitor to provide a low loss path for one or more carrier frequencies between the terminal equipment and the power line.

The combination of line tuning equipment and coupling capacitor may be thought of as forming a filter of the band pass type, passing or offering a low loss path for the desired carrier frequencies or channels and rejecting or offering a high loss path for the power frequency and undesired frequencies in the carrier band.

For a single frequency or channel the line tuning equipment...
may consist of a single variable inductor connected in series with the coupling capacitor to form a series resonant circuit for the carrier frequency used.

![Diagram of single frequency line tuning unit]

**Fig. 4** Single Frequency Line Tuning Unit.

The complete single frequency line tuning assembly also include a protective unit and an impedance matching transformer.
Coupling Methods for Power Line Carrier Systems

Transfer of power line carrier energy to and from a transmission line takes place by one of two methods: Line to Ground coupling and Line to Line coupling. Line to Ground coupling is favored on this continent whereas Line to Line coupling is often used in Europe.

Ground return coupling utilizes only one phase wire of the transmission line and ground. The principal advantage of ground return coupling is that it generally requires only half the number of coupling capacitors and line traps resulting in lower initial cost and simpler maintenance. In addition, the line tuning equipment required at terminals and bypasses is usually less complicated. Its main disadvantages lie in its somewhat higher attenuation and higher noise level as compared with phase to phase coupling. For many types of carrier channels, the disadvantages of the ground return circuit are greatly outweighed by its simplicity and lower cost. It is almost universally used for all pilot relay channels.

Line to Line carrier transmission is less susceptible to interference than line to ground because the two conductors form a balanced line. The major portion of the energy transmitted will be found between the two coupled conductors in the form of an electromagnetic wave. Some of the advantages of line to line coupling are lower attenuation, less variation of attenuation with weather, less pick up of interference and greater service reliability.
(a) PHASE TO GROUND COUPLING

1. WAVE TRAP
2. COUPLING CAPACITOR
3. PROTECTIVE DEVICE
4. COUPLING FILTER

(b) PHASE TO PHASE COUPLING

FIG. 5: COUPLING METHODS FOR PLC
2.2. Transmission Line Equations for Multiconductor Lines

Electromagnetic energy propagating along parallel line conductors is guided by the conductors and ground. Currents and voltages in the line are directly associated with the electromagnetic field in the surrounding space. It may be shown that the guided electromagnetic field forms equal phase plane waves perpendicular to the lossless line and that all conductor currents are in phase with the voltages. In a lossless system, currents do not penetrate into the conductors and earth plane but remain on the perfectly conducting surfaces and, hence, no electromagnetic energy can penetrate and be lost or trapped inside the conducting media.

In consequence of these properties, the ground plane can be removed and perfect image conductors introduced for both the voltages and currents. This in turn allows the electric and magnetic fields to be calculated from classical equations for static fields applied to the system composed of the conductors and their images.

This analysis of propagation of signals on multiconductor transmission lines with ground return is based on the concept of vector spaces discussed in reference 31. Conductor voltages and currents are represented as vectors.

Assuming a fixed plane wave propagating along a system of parallel conductors, the relation between conductor currents and voltages is defined by the Telegraphist's Equation:
\[-\frac{\partial I}{\partial x} = [L]\frac{\partial I}{\partial t} + [R] I \]  
\[-\frac{\partial V}{\partial x} = [\mathcal{C} \frac{\partial V}{\partial t} + [G] V \]  

where \( V \) and \( I \) are column matrices with \( n \) elements, \( G \) and \( \mathcal{C} \) are square symmetrical matrices of inductance and resistances per unit length and \( C \) and \( g \) are square symmetrical matrices of capacitance and shunt conductance per unit length.

For sinusoidal signals:
\[
[V] = [V] e^{j\omega t} \]  
\[
[I] = [I] e^{j\omega t} \]

where \( V \) and \( I \) are matrices with complex elements whose values depend only on the distance \( x \) along the line.

Differentiating (2) with respect to \( t \) and \( x \):
\[
\frac{\partial [V]}{\partial t} = [V] e^{j\omega t}, \quad j\omega = j\omega [V] \]
\[
\frac{\partial [I]}{\partial t} = [I] e^{j\omega t}, \quad j\omega = j\omega [I] \]
\[
\frac{\partial [V]}{\partial x} = \frac{\partial [V]}{\partial x} e^{j\omega t}, \quad \frac{\partial [I]}{\partial x} \frac{\partial [I]}{\partial x} e^{j\omega t}, \frac{\partial [I]}{\partial x} e^{j\omega t} \]

Substituting (2) (3) into (1):
\[-\frac{\partial [V]}{\partial x} = -\frac{\partial [V]}{\partial x} e^{j\omega t} = (j\omega[L] + [R]) [I] e^{j\omega t} \]  
\[-\frac{\partial [I]}{\partial x} = -\frac{\partial [I]}{\partial x} e^{j\omega t} = (j\omega[C] + [G]) [V] e^{j\omega t} \]
Eliminating $e^{j\omega t}$ from (4) and (5):

$$\frac{\partial [V]}{\partial x} = (j\omega [\ell] + [\gamma]) [I] = [z] [I]$$

$$\frac{\partial [I]}{\partial x} = (j\omega [\gamma] + [\gamma]) [V] = [y] [V]$$


Hence, the fundamental differential equations of a system of "n" parallel conductors above ground, at frequency at which the wavelength is much larger than any transverse spacing in the physical system, reduce to:

$$\frac{d}{dx} [V] = -[z] [I],$$

$$\frac{d}{dx} [I] = -[y] [V].$$

where $[V]$ and $[I]$ are column matrices whose entries are the $n$ conductor-to-ground voltage phasors and the $n$ conductor current phasors at any distance $x$ from the reference point $x=0$, and $[z]$ and $[y]$ are the square symmetrical matrices of self and mutual series impedances and shunt admittances per unit length of the conductors.

Differentiating both sides of equation (7) and substituting for the first derivative of $I$ and $V$:

$$\frac{d^2 [V]}{dx^2} = [z][y] [y] = \gamma^2 [V] = [p] [V]$$

$$\frac{d^2 [I]}{dx^2} = [y][z] [I] = \gamma^2 [I] = [P] [I]$$

where subscript $t$ denotes the transpose, $y$ and $z$ are symmetric,

$$[P] = [z] [y]$$

and $[P]_t = [y] [z]$. 
The solution in exponential form of Eq. (8) is:

\[
\begin{bmatrix}
V \\
I
\end{bmatrix} = (e^{-\gamma_1 x}) A + (e^{-\gamma_2 x}) B \\
= (e^{-\gamma_2 x}) C + (e^{-\gamma_2 x}) D
\]

(9)

(10)

where \(A, B, C, D, \gamma_1, \gamma_2\) are matrices with constant elements.

Defined by boundary conditions and the line geometry, and \(\gamma_1^2, \gamma_2^2\) are obtained after differentiating 9 and 10 twice and substituting the result in (8):

\[
\frac{dM}{dx} = -\gamma_1 e^{-\gamma_1 x} A - \gamma_2 e^{-\gamma_2 x} B
\]

\[
\frac{d^2M}{dx^2} = -\gamma_2^2 e^{-\gamma_1 x} A + \gamma_2 e^{-\gamma_2 x} B
\]

\[
\gamma_1^2 = [Z] [y] [V] \\
\gamma_2^2 = [Z] [y] [z]
\]

(11)

From Eq. (8), \(\frac{d^2M}{dx^2} = [Z] [y] [V]\).

For a reflection-free line (infinitely long or properly terminated) \(B = 0\) in (9) and (10):

\[
\begin{bmatrix}
V \\
I
\end{bmatrix} = (e^{-\gamma_1 x}) A \\
= (e^{-\gamma_2 x}) C
\]

(12)

From (12) and (6), \(\frac{d[I]}{dx} = [y] [V] = (\gamma_2 e^{-\gamma_2 x}) C = -\gamma_2 [I]\).

Premultiplying (13) by \(y^{-1}\) and from (11),

\[
[V] = [y]^{-1} [y] [Z] [I]
\]

where \([Z] = [y]^{-1} ([y] [Z]) [Z])^2\).
For \( V = V_0 \) and \( I = I_0 \) at \( x = 0 \) the following fundamental equations are utilized in any analysis of propagation problems:

\[
\begin{align*}
\frac{-d[v]}{dx} &= [z][v], \\
\frac{-d[i]}{dx} &= [y][v], \\
[v] &= \left(\frac{\gamma_1}{\epsilon_1}\right)[v_0], \\
[i] &= \left(\frac{\gamma_2}{\epsilon_2}\right)[i_0],
\end{align*}
\]

\( \gamma_1^2 = [z][y][y]^T \), \( \gamma_2^2 = [y][z] \)

\[
[v] = [z][i] \text{ where } [z] = [y]^{-1}([z][z]^T)^{-1/2}
\]

\( [z] = j\omega[l] + [r], \) \( [y] = j\omega[c] + [s] \)

\([z]\) is the characteristic impedance matrix.

Equation(14) directly relates the conductor voltages with the conductor currents at any position \( x \) on the system. The characteristic impedance of the transmission line can be determined from transmission line parameters.

**Loss-Less System**:

![Diagram of a system of conductors above a lossless ground plane]

**Fig.6. System of Conductors Above Lossless Ground**
The behaviour of a lossless system is determined by the geometry of the system. All the energy has to be guided along the three conductors.

For a system of \( n \) conductors with charges \( q_1, q_2, q_3, \ldots, q_n \):

The ratio of rise in potential \( V_m \) of the conductor \( m \) to the charge \( q_n \) placed on conductor \( n \) to produce this rise is called the co-efficient of potential \( p_{mn} \)

\[
V_1 = p_{11}q_1 + p_{12}q_2 + p_{13}q_3 + \cdots + p_{1n}q_n \\
V_2 = p_{21}q_1 + p_{22}q_2 + p_{23}q_3 + \cdots + p_{2n}q_n \\
\vdots \\
V_n = p_{n1}q_1 + p_{n2}q_2 + p_{n3}q_3 + \cdots + p_{nn}q_n
\]
In matrix form

\[
[V] = [P] [Q]
\]

Equation (15)

[\text{[P]}] \text{is called Maxwell's Potential Co-efficient matrix}

\[
[P] = \frac{1}{2 \pi \epsilon_0} [G] = 1.8 \times 10^{10} \text{ G m/farad, } \text{G=Geometry matrix}
\]

\[
G_{ll} = \frac{\lambda_{2H}}{r_l}, \quad G_{lj} = G_{jl} = \frac{\lambda_{d_{lj}}}{d_{lj}}
\]

From (15)

\[
[Q] = [P]^{-1} [V] = [C] [V]
\]

\[
[C] = [P]^{-1} \text{ capacitance matrix} = 2 \pi \epsilon_0 [G]^{-1} = 5.556 \times 10^{-11} [G]^{-1}
\]

Also

\[
[C] = \frac{1}{\nu_0^2} [L]^{-1} \text{ F/m}
\]

\[
\nu_0 = \text{velocity of light}
\]

\[
(16)
\]
\[ [L] = \frac{\mu_0}{2\pi} [G] = 2.10^{-7} [\text{G}] \text{ henry/m} \]

Lossless line parameter matrices are

\[ [z_0] = j \omega [L] = j 2.10^{-7} [G] \]
\[ [y_0] = j \omega [C] = j 5.55 \cdot 10^{-11} [G]^{-1} \]
\[ [p] = [P_0] = [z_0] [y] = -\frac{2\pi f}{v_0} x [1] \]
\[ \xi^x [p] = -j \frac{2\pi f}{v_0} x [1] = 1 - j2\frac{y f}{v_1} [1] + \frac{12\pi f^2}{v_0^2} x^2 [1] \ldots \ldots \ldots \ldots \ldots \ldots \text{(17)} \]

From Eq. (6), (7) and (13)

\[ [v] = \frac{-j}{\epsilon} \frac{2\pi f x}{v_0} [1] [v_0] \]
\[ [i] = \frac{-j}{\epsilon} \frac{2\pi f x}{v_0} [1] [i_0] \]
\[ \frac{d[v]}{dx} = \frac{-j2\pi f}{v_0} [v] = -[z_0] [i] \]
\[ \frac{d[i]}{dx} = \frac{-j2\pi f}{v_0} [i] = -[y_0] [v] \]
\[ [v] = -\frac{v_0}{2\pi f j} [z_0] [i] = \frac{j\omega v_0}{2\pi f j} 2.10^{-7} [G] [i] \]
\[ [v] = 60 [G] [i] , [v] = [z_0] [i] \]

\text{(19)}
Thus the fundamental solution for conductor voltages and currents of a lossy or lossless line can be written as

\[ V = Z I \]

In order to explain this equation explicitly, a two conductor line semi-infinite in length and perfectly uniform is considered.

The relations between voltages and currents can be written as

\[ V_1 = Z_{11} Z_{12} I_1 \]
\[ V_2 = Z_{12} Z_{22} I_2 \]

The matrix Z is always symmetrical; the entries of matrix Z for a lossless line are purely resistive and independent of frequency. For a lossy line the entries vary with frequency and are complex impedances.

2.3. History of Modal Analysis

One of the first analyses of a PLC transmission system can be attributed to Fallou\textsuperscript{13} who has applied symmetrical components to the resolution of carrier signals. H.F. currents and voltages can be resolved into a zero sequence component in phase on all the three conductors and positive and negative sequence components, for which one conductor forms the return path for currents in phase in the remaining two conductors. Each system of component is a combination of incident and reflected wave,
the amplitudes of which can be determined by the boundary conditions. The characteristic impedance and propagation constant are different for both waves and according to Fallou depend on the self and mutual inductances and capacitances of the circuits formed between the individual phase conductors and earth.

Chevallier\textsuperscript{13,14} has shown that the positive and negative sequence components have the same magnitude and can be reduced to a single phase wave between the phase conductors.

A three phase set of unbalanced voltage vectors, $V_a, V_b$, and $V_c$ can be resolved into three balanced or symmetrical sets of vectors $V_0, V_1$ and $V_2$ by the use of the following equation:

$$\begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix}$$

In a more concise form:

$$\begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix} = [a] [V_{abc}]$$

where $[a]$ is the transformation matrix. The symmetrical component transformation is power invariant — that is, the sum of the sequence powers is equal to the total power expressed in phase quantities. It should be noted that these symmetrical components are the "natural modes" of a continuously transposed three phase lines.
In general the problems are much too difficult or impossible to tackle using the classical mathematical method of solving the differential equations describing the system and then introducing the boundary conditions to evaluate the various constants. This method is only considered when simplifying assumptions are made. These assumptions however can lead to very misleading and incorrect results, as most power lines are not completely or, in some cases, even partially transposed. The well understood method of symmetrical components must be ruled out as an instrument to give accurate solutions to these problems.

Because of the above mentioned difficulties, the concept of natural modes of propagation along multiconductor lines was developed. For the case of a single frequency signal application J.R. Carson and R.S. Hoyt formulated a solution in 1972 by applying the generalized telegraph equations to a multiwire system.

The system of a equations, which constitutes the generalized telegraph equations will be written as equation 11 of the above reference paper.

\[
\begin{align*}
    m_{11} I_1 + m_{12} I_2 + m_{13} I_3 + \cdots + m_{1n} I_n &= 0 \\
    m_{21} I_1 + m_{22} I_2 + m_{23} I_3 + \cdots + m_{2n} I_n &= 0 \\
    m_{31} I_1 + m_{32} I_2 + m_{33} I_3 + \cdots + m_{3n} I_n &= 0 \\
    \cdots & \\
    m_{n1} I_1 + m_{n2} I_2 + m_{n3} I_3 + \cdots + m_{nn} I_n &= 0
\end{align*}
\]
The solution indicates the existence of natural modes of propagation derived from the roots of the characteristic determinant of the set of equations.

In 1941 S. O. Rice applying matrix algebra\textsuperscript{16} developed a solution of transmission line equations as an extension of Carson's and Hoyt's work. A relatively simple and practical method of analyzing PLC propagation on a three-phase line by means of symmetrical components was published by A. Chevallier\textsuperscript{14} in 1945.

The techniques used in the mathematical analysis of propagation of high frequency signals on multiphase power lines is commonly called modal analysis. When a signal is applied to a multiphase transmission line, it breaks up into different components called modes which are equal in number to the number of conductors present in a system. The propagation of pulses, such as those encountered in surge phenomena, is more complicated than the propagation of a single frequency signal and this problem is systematically treated by L. V. Bewley.\textsuperscript{17} He discusses the physical significance of the concept of multivelocity waves originated by Sekku in 1923 in Japan\textsuperscript{18} and illustrates it by test results.

The original and extensive work of Bewley was followed by that of L. A. Pipes who in 1937 published the paper, "Matrix Theory of Multiconductor Transmission Lines."\textsuperscript{19} Pipes showed
how a matrix algebra is used to solve the difficult problem of multiconductor transmission lines in an easy way.

A more general application of natural modes, derived from characteristic properties of matrices, was initiated by Adams in his analysis of radio noise on power lines. He considered a three phase line and developed six main differential equations relating phase voltages and currents. To simplify the solution of equations a system of orthogonal components was applied. On a multiconductor line, the actual voltages and currents with their mutual interactions can be resolved into as many independent components or modes of propagation as there are conductors. These modes can be allowed to propagate independently, theoretically through properly developed equations, but not in real practice on the lines except in hypothetical ideal cases. For radio interference work, this concept is extremely important as it allows the calculation of electric field intensity or the H field in the vicinity of the line. The basic difficulty in this problem starts in the beginning, namely resolving the voltages and currents due to the corona pulses into independent modes such that they propagate independently without mutual interaction. Dr. G.E. Adams assumed an infinitely conducting ground surface, neglected all losses and resolved the voltages into independent modes by evaluating the eigenvectors of the matrix of the Maxwell's potential coefficients used in the
calculation of inductance and capacitance elements of the multiconductor line.

Having obtained the eigenvalues (the impedances) and the eigenvectors, the voltages actually propagating on the conductors can be resolved into several independent modes, all of them travelling at light velocity. They can be excited on the three phase line at one end and allowed to propagate independently, in theory. This has permitted the design of a great many experiments, for determining attenuation factors of the different modes, the lateral profiles of RI etc.

Adam demonstrated the effects of power line parameters on the natural mode components or eigenvectors calculated from the conductor system matrix. Several curves were derived showing the effect of conductor size, phase spacing and conductor height above ground for horizontal and vertical 300 kV to 400 kV transmission lines. The main work of Adam is related to theoretical and experimental investigations on many aspects of corona noise generation, propagation and field strength of radio interference (RI).

Barthold 22 presented an original and inspiring method leading to the formulation of a closed form of solution by which a PLC link including faults, transpositions and terminations is resolved into three independent natural mode systems. The system is assumed to be lossless. Practical lines are not lossless, but if the effects of imperfectly
conducting ground and conductors are small the lossless line method is very helpful in analyzing line terminations, faults and transpositions.

As long as the depth of penetration in the soil is small compared with interphase spacing and the line height, the method leads to results confirmed by measurements. Barthold derived the result that the modal attenuation from skin effect in the conductors and earth on the assumptions that the conductor portion of radio frequency losses varies as the square root of frequency and that the earth component of attenuation varies directly with frequency in the PLC range and that for high frequencies, it varies as the square root of frequency.

The problem of how a signal travels down a horizontal three phase lossless line has been solved independently by Perz, by separating the signal into three modes or transmission paths. He has done a genuine service to the industry by taking modal concepts previously presented in somewhat sophisticated and abstract terms and converting portions of this analysis to terms and methods readily understood by power engineers.

A method of analyzing the performance of a PLC link on a long horizontal power line is developed in reference 23. Natural modes of propagation are applied to the phase currents and voltages of PLC signals between two line discontinuities.
Fierz's papers are very clear expositions of the methods of
matrix eigenvalue analysis as applied to multiconductor trans-
mission lines and constitute a major advance in the practical
application of modal techniques in the application of PLC to
power transmission lines. The lossless line modal analysis
finds many useful applications mainly because of its simplic-
ity. However real line conductors have resistance and the
ground near the line also absorbs high frequency energy.
Hence the development of modal analysis for lossy systems was
well justified particularly for PLC transmission on long EHV
power lines.

L.M. Wedepohl in the U.K. and D.E. Hedman in the
U.S.A. independently developed quite general modal analyses
of lossy power lines. Using the theory of natural modes the
results of Wedepohl's work show that over all carrier perfor-
mance on different multiconductor systems can be successfully
studied. It was also shown that in the case of PLC coupling,
optimum arrangements may be predicted by considering the
relative proportions of modes for various forms of energisa-
tion. It was emphasized that the low attenuation modes should
be dominant for most efficient propagation.

The propagation of carrier frequency signals along
multiconductor transmission lines can be understood with
the help of modal propagation theory. This fruitful theory
which was first put forward some forty-five years ago was
for a long time used as a means of estimating radio interference caused by HV power lines. However, it subsequently proved its value as a means of studying PLC transmission conditions. The main features of modal analysis²,³ are described below.

Every transmission of HF energy over an n phase system can be resolved into n components. As the term "natural modes of propagation" indicates, these components are closely related to the properties of the system used, in particular to the impedance matrix of the system.

Each node is identifiable through a characteristic propagation constant (Eigenvalue), a definite voltage and current distribution (Eigenvector), and a characteristic impedance. For each of the n modes there is a set of phase voltages and currents which bear a constant relationship to one another along the length of line. The fact that no energy is transferred between individual modes is most important for practical application.

In the light of the above mentioned properties, if characteristic modal parameters are known, one can analyze a homogeneous multiconductor line by applying conventional transmission line theory and the principle of superposition.

**Basic Modes**

For a single circuit three phase horizontal line, there are three modes with the following properties.
Mode 1: The currents flow in the same direction in all three phases and are approximately equal in magnitude. Owing to the earth return path, attenuation is highest with this mode.

Mode 2: The currents in the outer phases are equal and opposed, while the middle phase does not contribute anything to the transmission. Attenuation is between that of mode 1 and 3.

Mode 3: The current in the middle phase is roughly twice that in the outer phases and flows in the opposite direction. The attenuation is least with this mode.

Modes numbered one and three are interchanged from the earlier designation as it is consistent with the designation used in other countries and it is more consistent with the general modal theory.
2.4. Nodal Analysis of Lossless System

2.4.1. Three Phase Horizontal Line:

In order to analyse the propagation of PLC signals simplifying assumptions, are made.

1. The conductors have a circular cross section and their radii are very small compared with other dimensions. The conductors are ideally parallel to each other and to the ground.

2. Electromagnetic energy propagates essentially in the form of plane waves.

3. Height and spacing of the conductors are very small compared with the wavelength of the plane waves. With these assumptions the differential line equations will reduce to a simple set of linear equations relating the amplitudes of conductor currents and voltages referred to ground.

The natural node analysis allows one to estimate the propagation performance of PLC signals on power lines.

[Diagram of three-phase horizontal line with labels for conductors, sky-wires, and ground plane.]
Voltage current relations for a lossless system are uniquely defined from the system geometry. For the above system with well grounded sky wires, there are five equations:

\[
\begin{align*}
Z_{11} I_1 + Z_{12} I_2 + Z_{13} I_3 + Z_{14} I_4 + Z_{15} I_5 &= V_1 \\
Z_{21} I_1 + Z_{22} I_2 + Z_{23} I_3 + Z_{24} I_4 + Z_{25} I_5 &= V_2 \\
Z_{31} I_1 + Z_{32} I_2 + Z_{33} I_3 + Z_{34} I_4 + Z_{35} I_5 &= V_3 \\
Z_{41} I_1 + Z_{42} I_2 + Z_{43} I_3 + Z_{44} I_4 + Z_{45} I_5 &= 0 \\
Z_{51} I_1 + Z_{52} I_2 + Z_{53} I_3 + Z_{54} I_4 + Z_{55} I_5 &= 0
\end{align*}
\]

In matrix notation \([Z] [I] = [V]\)

The elements of the square matrix \(Z\) are the self and mutual impedances which are determined by the geometry of the line, and can be expressed as:

\[
\begin{align*}
Z_{kk} &= 60 \ln \frac{2h_k}{\Gamma_k} \\
Z_{ij} &= Z_{ji} = 60 \ln \frac{D_{ij}}{d_{ij}}
\end{align*}
\]

(21)

Where \(h_k\) is the height above ground of conductor \(k\),

\(\Gamma_k\) is the geometric mean radius (G.M.R.) of conductor \(k\),

\(D_{ij}\) is the distance between conductor \(i\) and image of conductor \(j\),

\(d_{ij}\) is the distance between conductor \(i\) and \(j\).

Reducing the five equations to a set of three equations, by solving for \(I_4\) and \(I_5\) from the last two equations of (20) and substituting into the first three equations results in the following, as given in Appendix 'A'.
The reduced equations can be written as Eq. (22)

\[
AI_1 + BI_2 + CI_3 = V_1
\]
\[
BI_1 + BI_2 + BI_3 = V_2
\]
\[
CI_1 + BI_2 + AI_3 = V_3
\]

or

\[
\begin{bmatrix}
A & B & C \\
B & D & B \\
C & B & A \\
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\end{bmatrix}
\]

(22)

Relations are established between the phase mode currents and actual phase currents as shown in Appendix A, Eq. 42.

\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 \\
p & o & q \\
1 & -1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
I^{(1)}_1 \\
I^{(2)}_1 \\
I^{(3)}_1 \\
\end{bmatrix}
\]

In general:

\[
\begin{bmatrix}
I_k \\
\end{bmatrix}
= 
\begin{bmatrix}
m \\
\end{bmatrix}
\begin{bmatrix}
I^{(n)}_1 \\
\end{bmatrix},
\]

\[
M = 
\begin{bmatrix}
1 & 1 & 1 \\
p & o & q \\
1 & -1 & 1 \\
\end{bmatrix}
\]

By substituting \( V \) for \( I \), similar relations are obtained for voltage components,

\[
\begin{bmatrix}
V_k \\
\end{bmatrix}
= 
\begin{bmatrix}
m \\
\end{bmatrix}
\begin{bmatrix}
V^{(n)}_1 \\
\end{bmatrix}
\]

Usually the problem consists of finding the mode components from known values of phase components,

\[
\begin{bmatrix}
I^{(n)}_1 \\
\end{bmatrix}
= 
\begin{bmatrix}
m \\
\end{bmatrix}^{-1}
\begin{bmatrix}
I_k \\
\end{bmatrix},
\]

\[
\begin{bmatrix}
V^{(n)}_1 \\
\end{bmatrix}
= 
\begin{bmatrix}
m \\
\end{bmatrix}^{-1}
\begin{bmatrix}
V_k \\
\end{bmatrix}
\]
where,

\[
M^{-1} = \frac{1}{2(q-p)} \begin{bmatrix}
q, & -2, & q \\
q-p, & 0, & -q+p \\
-p, & 2, & -p
\end{bmatrix}
\]

The columns of matrix \([M]\) are non-normalized eigenvectors.

The normalized modal transformation matrix \([N]\), with columns equal to eigenvectors of unity length is:

\[
[N] = \begin{bmatrix}
\frac{1}{\sqrt{2+p^2}}, & \frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{2+q^2}} \\
\frac{0}{\sqrt{2+p^2}}, & 0, & \frac{0}{\sqrt{2+q^2}} \\
\frac{1}{\sqrt{2+p^2}}, & \frac{-1}{\sqrt{2}}, & \frac{1}{\sqrt{2+q^2}}
\end{bmatrix}
\]

The transformation matrix \([M]\) and its inverse are the key matrices in the calculation of modal powers and conductor modal components of either voltages or currents. The resolution of phase currents and voltages into natural modes allows one to solve the propagation problem between two points of a uniform three-phase line as if the line were composed of three independent single conductor lines.

There are \(k\) equations relating \(k\) conductor voltages, \(V\), with currents, \(I\), that can be written in a matrix form similar to equation (19):

\[
[V] = [Z_0] [I]
\]
Since the mutual impedance between conductor i and j is the same as that between conductor j and i, the corresponding off-diagonal entries in matrix $[Z_0]$ are identical and, hence, $[Z_0]$ is symmetrical or $[Z_0]^T$ and its transpose are identical and, therefore only one set of eigenvectors or mode phase components exist. This means that on a lossless horizontal line, the mode current components, columns of matrix $M$ are not only linearly independent but also orthogonal. Hence, there is no exchange of energy between natural modes on a lossless line, and for each mode the ratio of voltage to current is the same on each phase conductor. Furthermore this ratio is equal to the modal surge impedance. This means that for each natural mode fed into an untransposed line a reflection-free receiving end termination can be composed of three identical resistors connected from each phase to ground and equal in value to the mode surge impedance.

**Summary 8.32 of Properties of Lossless Lines:**

1) Conductor currents on a system of k conductors are in phase with conductor voltages.

2) Voltages are linearly related to currents through a symmetric matrix $[Z]$ whose elements are the real self and mutual surge impedances.

3) There is only one set of k orthogonal modes, with real components for both the voltages and currents. There is no coupling between voltage or current components of one mode and corresponding current or voltage components of any of the remaining modes.
4) Voltage components of one node are in phase with and proportional to the corresponding current components. The co-efficients of proportionality are the modal surge impedances, \( Z^m \), equal to the eigenvalues of matrix \( [Z] \).

5) Modal resolution is power-invariant.

6) The modes are numbered according to the sequence of decreasing modal surge impedances.

7) The system behaviour is independent of frequency.

5) All physical characteristics of lossless lines are directly associated with simple mathematical properties of the real symmetric surge impedance matrix \( [Z] \) derived from the system geometry.

9) The conductor voltage and current waves travel along the line at a single velocity (that of light) and with no attenuation.

10) The reflection-free termination consists of pure resistances.

2.4.2. Analysis of PLC Transmitter Couplings:

High frequency power fed from the transmitter into the line propagates in independent modes, each subjected to different attenuation. The resolution of transmitted power into modes depends on the transmitting end terminations and the choice of coupled phase conductors.

There are two conventional methods of coupling a PLC transmitter to a line, phase to ground, and balanced phase to phase. To simplify the analysis, it is assumed
that the line is long enough to neglect reflections arriving at the transmitting end from the receiving end termination. Hence the analysis essentially consists of the derivation of conductor voltages or currents at the transmitting end, their resolution into modal components and the calculation of modal powers. The input impedance presented to the transmitter coupled to one or two phase conductors, conductor currents and voltages are generally defined from the so called "boundary conditions".

Phase components of a transmitter connected to a line can be calculated directly from equation (22):

\[
\begin{align*}
AI_1 + BI_2 + CI_3 &= V_1 \\
SI_1 + DI_2 + BI_3 &= V_2 \\
CI_1 + SI_2 + AI_3 &= V_3
\end{align*}
\]

For example, in one case let the centre phase be connected to a transmitter, supplying a voltage \( V_2 \) and the unused phases 1 and 3 are left open.

Then, the following relations can be written:

\[
\begin{align*}
I_1 = I_3 &= 0 \, , \, V_1 = V_2 \\
I_2 &= \frac{V_2}{D} \, , \, F_1 = \frac{V_2}{D} \\
1 & 2 \\
0 & 0 \\
3 & 0 \\
V_1 &= V_2
\end{align*}
\]

Fig. (a). ARRANGEMENT OF TRANSMITTER COUPLING
Substituting equation (44) into (42) and solving,

\[ I_1^{(2)} = 0, \quad I_1^{(1)} = -I_1^{(3)} = -\frac{V_2}{D(q-p)} \quad (45) \]

The mode current components of the outer phases are calculated from equation (43):

\[ I_2^{(2)} = I_3^{(2)} = 0, \quad I_2^{(1)} = pI_1^{(1)}, \quad I_3^{(1)} = I_1^{(1)}, \quad (46) \]

\[ I_2^{(3)} = qI_1^{(1)}, \quad I_3^{(3)} = I_1^{(3)} \]

For calculations of mode voltage components for each phase, the current components are multiplied by the corresponding mode impedances.

\[ V_k^{(n)} = Z_k I_k^{(n)}, \quad n \text{ designates the modes, } k \text{ the phases} \]

The power mode components expressed in terms of input power \( P_1 \) are \( P^{(n)} = \frac{p(n)}{P_1}, \quad P_1 = \eta P_1 \)

\[ \eta = \text{coupling efficiency of mode } (n) \]

\[ \frac{1}{\eta} = \left\{ I_1^{(1)}^2 + I_2^{(1)}^2 + I_3^{(1)}^2 \right\} \frac{1}{Z_1} \]

Substituting \( P_1 \) from equation (44), (45) and (46):

\[ \frac{1}{\eta} = \frac{Z^{(1)}}{(1 + p^2 + 1) I_1^{(1)}^2} \frac{1}{P_1} = \frac{[2 + p^2]Z(1)}{(q-p)^2 D} \quad (47) \]
\[ \eta(2) = \frac{\Delta(2)\Delta(2)^2}{\Delta_1} = 0 \]

\[ \eta(3) = \frac{\Delta(3)(1+q+1)^2}{\Delta_1} = \frac{2+q^2}{(q-p)^2} \cdot \frac{\Delta(3)}{p} \]  

For known attenuation coefficients \( \alpha(1), \alpha(2), \alpha(3) \), the incident power at a distance \( \ell \) from the transmitter is

\[ \mathcal{P} = \mathcal{P}(1) + \mathcal{P}(2) + \mathcal{P}(3) = \gamma(1)e^{-2\alpha(1)\ell} + \gamma(2)e^{-2\alpha(2)\ell} + \gamma(3)e^{-2\alpha(3)\ell} \quad \text{Eq. (48)} \]

The phase mode components at a distance \( \ell \) are

\[ I_{k}^{(n)}(\ell) = I_{k}^{(n)}e^{-\alpha(n)\ell} \quad \text{Eq. (49)} \]

\[ V_{k}^{(n)}(\ell) = 2_{k}^{(n)}I_{k}^{(n)}(\ell) = V_{k}^{(n)}e^{-\alpha(n)\ell} \]

Conversely, values of equation (49) can be computed from Eqs. (47) and (48). We can apply a similar procedure for analyzing various transmitter coupling arrangements.

It will be seen that for a long PLC link it is important that most of the input power appears in mode 3. This can be achieved with the center phase to outer phase coupling of the push-pull type figure 9(d) or the center phase-to-ground coupling figure 9(b). Outer phases grounded at carrier frequencies, can provide an economical solution. Analysis of various couplings for the Pinard-Hammar Ontario Hydro 500kV line indicates that this coupling converts nearly 70% of the input power into mode 3, and 30% into mode 2 and practically no input power is converted into mode 1 power.
2.4.3. Effect of Transposition:

In connection with PLC communication it is well known that under certain circumstances like transpositions can lead to undesirably high additional line attenuation. When compared with untransposed lines, for which the optimum coupling methods are known, transposed lines present some difficulties in arriving at a valid conclusion regarding the best possible coupling arrangements.

A method of analyzing the performance of a power line
carrier link on horizontal power lines was suggested by Perz using the modal propagation theory for analyzing discontinuities, faults and transpositions. The step-by-step analysis presented taking into consideration the incident, reflected and refracted components, was applied to a PLC link of a three phase, 500kV, 230 mile Pinard-Hanmer Ontario Hydro line with a transposition near the centre, between the three phases.

![Diagram of PLC link with transmitter, transposition, and receiver with labels for incident, reflected, and refracted signals.]

Fig. 10. PINARD-HANMER PLC LINK.

The signals from the transmitter G resolve into modes and propagate toward the transposition T. Natural modes of propagation are applied to the phase currents and voltages of PLC signals between the two line discontinuities, under the assumptions that:

1. Only two modes are present since they are of significant contribution to the transmission of PLC signals along a
three phase line (because of the excessive attenuation, the third or ground node can be disregarded).

2. The line has a reflection free termination. (Although the coupled phases of a line are normally terminated in an unknown station impedance and are therefore mismatched, the reflections due to this mismatch at the line ends result in an additional attenuation).

3. The transpositions do not introduce additional reflections.

---

![Diagram](image)

**Fig. 11. GENERAL NODE NETWORK**

The line discontinuities are considered as node networks connected to the phase conductors. The phase signals are resolved into mode components and their attenuated values are recombined at the next discontinuity. At the transposition the attenuated incident mode signals are recombined into phase quantities. The recombination in phase quantities allows the actual phase
voltages and currents at the point of discontinuity to be found.

Finally the refracted phase values are resolved into nodes of the transposed line. The attenuated node components arriving at the receiving end R are again recombined into incident phase quantities. From these the currents and voltages of the receiving end terminating network are calculated. From the received power the overall attenuation of the link can be calculated for normal operating conditions.

**Transposition**

For a single transposition located near the centre of a long line, the mode 1 (the ground node) vanishes and mode 2 may be in or out of phase with mode 3.

This method is based on evaluating phase quantities at points of discontinuity and resolving refracted quantities into incident and reflected values. The mode components are used only between the points of discontinuity.

The sum of the incident and reflected phase quantities must be equal to the refracted quantities.

\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix} + \begin{bmatrix}
I'_1 \\
I'_2 \\
I'_3
\end{bmatrix} = \begin{bmatrix}
I''_1 \\
I''_1 \\
I''_3
\end{bmatrix}
\]

Or

\[
\begin{bmatrix}
I_k \\
I_k
\end{bmatrix} + \begin{bmatrix}
I'_k \\
I'_k
\end{bmatrix} = \begin{bmatrix}
I''_k \\
I''_k
\end{bmatrix}
\]
Similarly for voltages the relation can be written as

\[
\begin{bmatrix}
V_k \\
V_k' \\
V_k''
\end{bmatrix} + \begin{bmatrix}
V_k' \\
V_k''
\end{bmatrix} = \begin{bmatrix}
V_{tk}
\end{bmatrix}
\]

\(k = 1, 2, 3\) designates the phase sequence to the left of transposition, no prime, single prime and double prime indicates incident, reflected and refracted quantities, respectively; and \(t\) indicates that the phase sequence is transposed.

The relations between phase voltages and current are:

\[
\begin{bmatrix}
I_k \\
I_k'
\end{bmatrix} = \begin{bmatrix}
Y_k \\
Y_k'
\end{bmatrix} \begin{bmatrix}
V_k \\
V_k'
\end{bmatrix}
\]

\[
\begin{bmatrix}
I_k''
\end{bmatrix} = \begin{bmatrix}
Y_{tk}
\end{bmatrix} \begin{bmatrix}
V_k''
\end{bmatrix}
\]

Substituting into equation (50)

\[
\begin{bmatrix}
Y_k \\
Y_k'
\end{bmatrix} \begin{bmatrix}
V_k \\
V_k'
\end{bmatrix} - \begin{bmatrix}
Y_k \\
Y_k'
\end{bmatrix} \begin{bmatrix}
V_k \\
V_k'
\end{bmatrix} = \begin{bmatrix}
Y_{tk}
\end{bmatrix} \begin{bmatrix}
V_{tk}
\end{bmatrix}
\]

incident + reflected = transmitted

\[
\begin{bmatrix}
Y_k \\
Y_k'
\end{bmatrix} = \begin{bmatrix}
Y_k \\
Y_k'
\end{bmatrix} \text{ since line is same}
\]

\[
\begin{bmatrix}
Y_k \\
Y_k'
\end{bmatrix} \begin{bmatrix}
V_k \\
V_k'
\end{bmatrix} = \begin{bmatrix}
Y_{tk}
\end{bmatrix} \begin{bmatrix}
V_{tk}
\end{bmatrix}
\]

Or

\[
(\begin{bmatrix}
Y_k \\
Y_k'
\end{bmatrix} + \begin{bmatrix}
I_{tk}
\end{bmatrix}) \begin{bmatrix}
V_{tk}
\end{bmatrix} = 2 \begin{bmatrix}
I_k \\
I_k'
\end{bmatrix} \begin{bmatrix}
V_k
\end{bmatrix}
\]

Eliminating \(V_{tk}\)

\[
\begin{bmatrix}
Y_k \\
Y_k'
\end{bmatrix} \begin{bmatrix}
V_k \\
V_k'
\end{bmatrix} = \left( \begin{bmatrix}
V_k \\
V_k'
\end{bmatrix} + \begin{bmatrix}
V_k \\
V_k'
\end{bmatrix} \right) \begin{bmatrix}
Y_{tk}
\end{bmatrix}
\]

(51)
\[
\begin{align*}
\left( [y_k] - [y_{tk}] \right) [v_k] &= [v_k'] \left( [y_{tk}] + [y_k] \right) \\
\left( [y_k] + [y_{tk}] \right) [v_{tk}] &= 2 [y_k] [v_k] \\
\left( [y_k] + [y_{tk}] \right) [v_{tk}'] &= \left( [y_k'] - [y_{tk}] \right) [v_{tk}]
\end{align*}
\]

By expanding these relations,

\[
\begin{bmatrix}
2y_{11} & y_{12} + y_{13} & y_{12} + y_{13} \\
y_{12} + y_{13} & y_{11} + y_{22} & 2y_{12} \\
y_{12} + y_{13} & 2y_{12} & y_{11} + y_{22}
\end{bmatrix}
\begin{bmatrix}
v_1' \\
v_2' \\
v_3'
\end{bmatrix}
= 2
\begin{bmatrix}
y_{11} & y_{12} & y_{13} \\
y_{12} & y_{22} & y_{12} \\
y_{13} & y_{12} & y_{11}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
\]

From these relations, refracted and reflected quantities in terms of incident values are calculated and incident, reflected and refracted powers can be found.

Barthold suggested the application of natural mode components and thus his analysis consists of resolving a three-phase line into three independent single wire lines. However, a discontinuity on a uniform line, such as unmatched termination or transposition introduces a cross-coupling between the independent natural modes. Moreover in the instance of a single mode transmitted along a line, the point of discontinuity is not only a point of reflection for this mode but it also generates other modes.

Barthold overcomes this difficulty by developing a conversion of three-phase networks on a line into modal
networks defined by the normalized eigenvector matrices of
the line, and then applying general equations of two terminal
networks to the new modal system for the radio frequency
problems on polyphase lines.

![Diagram A](image1)

(A)

![Diagram B](image2)

(B)

Fig. 12. THREE PHASE TRANSMISSION LINE AND
CORRESPONDING MODAL NETWORKS

Figure 12 represents an actual three-phase transmission
line with radio frequency discontinuities and corresponding
uncoupled modal networks each consisting of two-conductor transmission lines.

Modal networks are numbered in order of decreasing modal surge impedance and each mode is characterised by its own propagation constant.

(A)

(B) Three conductor transmission line with two intermediate discontinuities.

(C) Transmitter terminal.

(C) Receiver terminal.

Fig. 13. PHASE DIAGRAM OF THREE PHASE TRANSMISSION LINE
**Method of A,B,C,D Constants:**

A,B,C,D constants are commonly used to analyse simple two-terminal networks. The constants are defined such that

\[ V_1 = AV_2 + BI_2 \]
\[ I_1 = CV_2 + DI_2 \]  \hspace{1cm} (52)

The problem of figure 13(A) can be described with matrix quantities.

\[
\begin{bmatrix} V_{ds} \\ V_{ms} \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} V_{mR} \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} I_{mR} \end{bmatrix} \]  \hspace{1cm} (53)

(m stands for modal values, s stands for sending end, R stands for receiving end.)

The sending end current and voltage has been expressed in terms of receiving end quantities with the help of A,B,C,D constants.

The voltage and current at the receiving terminal are related as

\[
\begin{bmatrix} V_{pR} \\ I_{pR} \end{bmatrix} = \begin{bmatrix} Z_{pR} \end{bmatrix} \begin{bmatrix} I_{pR} \end{bmatrix} \]  \hspace{1cm} (54)

Since phase quantities and modal quantities are related by the modal matrix \( \Phi \), therefore

\[
\begin{bmatrix} V_{mR} \\ I_{mR} \end{bmatrix} = \begin{bmatrix} \Phi \end{bmatrix} \begin{bmatrix} V_{pR} \\ I_{pR} \end{bmatrix} \]  \hspace{1cm} (55)
\[
\begin{bmatrix}
\mathbf{v}_m \\
\mathbf{Z}_m
\end{bmatrix} = \begin{bmatrix}
\mathbf{Z}_m \\
\mathbf{Z}_p
\end{bmatrix}^{-1}
\]

Where
\[
\begin{bmatrix}
\mathbf{Z}_m \\
\mathbf{Z}_p
\end{bmatrix} = \begin{bmatrix}
\mathbf{N} \\
\mathbf{Z}_p
\end{bmatrix}^{-1}
\]\n
From fundamental transmission line equations

\[
\begin{align*}
\mathbf{V}_s &= \mathbf{V}_R \cosh \gamma l + \mathbf{I}_R \mathbf{Z}_c \sinh \gamma l \\
\mathbf{I}_s &= \mathbf{I}_R \cosh \gamma l + \frac{\mathbf{V}_R}{\mathbf{Z}_c} \sinh \gamma l
\end{align*}
\]

\[
\mathbf{V}_s = AV_R + BI_R
\]

\[
\mathbf{I}_s = CV_R + DI_R
\]

The A, B, C, D constants for any of the line sections can be written in matrix form.

\[
\mathbf{A}_m = \begin{bmatrix}
\cosh \gamma_1 & 0 & 0 \\
0 & \cosh \gamma_2 & 0 \\
0 & 0 & \cosh \gamma_3
\end{bmatrix}
\]

\[
\mathbf{B}_m = \begin{bmatrix}
\mathbf{Z}_1 \sinh \gamma_1 & 0 & 0 \\
0 & \mathbf{Z}_2 \sinh \gamma_2 & 0 \\
0 & 0 & \mathbf{Z}_3 \sinh \gamma_3
\end{bmatrix}
\]

\[
\mathbf{C}_m = \begin{bmatrix}
\mathbf{Y}_1 \sinh \gamma_1 & 0 & 0 \\
0 & \mathbf{Y}_2 \sinh \gamma_2 & 0 \\
0 & 0 & \mathbf{Y}_3 \sinh \gamma_3
\end{bmatrix}
\]

\[
\mathbf{D}_m = \mathbf{A}_m
\]

At point U of the figure (13), the voltage and current are

\[
\mathbf{V}_m U = \mathbf{A}_m \mathbf{V}_R + \mathbf{B}_m \mathbf{I}_R
\]

\[
\mathbf{I}_m U = \mathbf{C}_m \mathbf{V}_R + \mathbf{D}_m \mathbf{I}_R
\]

\[
\mathbf{V}_m U \text{ and } \mathbf{I}_m U \text{ indicates modal voltage and current at terminal U}
\]
Substituting \( v_{mR} = Z_{mR}i_{mR} \)

\[
\begin{align*}
\dot{v}_{mU} &= \left\{ A_m + B_m v_{mR} \right\} v_{mR} \\
\dot{i}_{mU} &= \left\{ C_m + D_m v_{mR} \right\} v_{mR}
\end{align*}
\]

(59)

(A) -- Modal voltage and current designation for network of figure 13A.

(B) -- Modal voltage and current designation for network of figure 13B.

Fig. 14. MODAL QUANTITIES FOR THE NETWORK OF FIG. 13
It can be seen from figure (13)A, that transposition introduces a change in position of conductors and the voltage on either side of the transposition can be equated as:

\[
\begin{bmatrix}
v_{aT} \\
v_{bT} \\
v_{cT}
\end{bmatrix} =
\begin{bmatrix}
v_{bU} \\
v_{cU} \\
v_{aU}
\end{bmatrix}
\]

Writing modal voltages in terms of phase voltages at point T

\[
\begin{align*}
\begin{bmatrix}
v_T(1) \\
v_T(2) \\
v_T(3)
\end{bmatrix} &=
\begin{bmatrix}
M_a(1) & M_b(1) & M_c(1) \\
M_a(2) & M_b(2) & M_c(2) \\
M_a(3) & M_b(3) & M_c(3)
\end{bmatrix}
\begin{bmatrix}
v_{aT} \\
v_{bT} \\
v_{cT}
\end{bmatrix}
\end{align*}
\]

By use of equation \((60)\):

\[
\begin{align*}
\begin{bmatrix}
v_T(1) \\
v_T(2) \\
v_T(3)
\end{bmatrix} &=
\begin{bmatrix}
M_a(1) & M_b(1) & M_c(1) \\
M_a(2) & M_b(2) & M_c(2) \\
M_a(3) & M_b(3) & M_c(3)
\end{bmatrix}
\begin{bmatrix}
v_{bU} \\
v_{cU} \\
v_{aU}
\end{bmatrix}
\end{align*}
\]

Writing phase voltages at U in terms of mode voltages

\[
\begin{align*}
\begin{bmatrix}
v_{bU} \\
v_{cU} \\
v_{aU}
\end{bmatrix} &=
\begin{bmatrix}
N_a(1) & N_b(2) & N_c(3) \\
N_a(1) & N_b(2) & N_c(3) \\
N_a(1) & N_b(2) & N_c(3)
\end{bmatrix}
\begin{bmatrix}
v_U(1) \\
v_U(2) \\
v_U(3)
\end{bmatrix}
\end{align*}
\]
Elements of the \( A \) matrix in equation (63) have shifted upward one row to express the phase voltages in the same order as they occur in equation (62).

A row cycling matrix is defined such that

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
N_a(1) & N_a(2) & N_a(3) \\
N_b(1) & N_b(2) & N_b(3) \\
N_c(1) & N_c(2) & N_c(3)
\end{bmatrix}
= 
\begin{bmatrix}
N_b(1) & N_b(2) & N_b(3) \\
N_c(1) & N_c(2) & N_c(3) \\
N_a(1) & N_a(2) & N_a(3)
\end{bmatrix}
\]

(64)

Eq. (62) can be written as

\[
\begin{bmatrix}
V_{mT}
\end{bmatrix}
= 
[ R ] [ C_T ] [ N ] [ V_{mU} ]
\]

Similarly

\[
\begin{bmatrix}
I_{mT}
\end{bmatrix}
= 
[ R ] [ C_T ] [ N ] [ I_{mU} ]
\]

(65)

where \( C_T \) is the row cycling matrix

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

From the above, we can see that premultiplication of equation (62) by the matrix product \( NC_T \) would give an expression for the \( V_{mT} \) and \( I_{mT} \) matrices in terms of the receiving end voltage matrix \( V_{mR} \).

The \( A, B, C, D \) constants for the section of line between points \( P \) and \( T \) are identical in nature to those used between points.
U and R. Both the \( v_{xT} \) and \( i_{xT} \) matrices are known in terms of the \( v_{xR} \) matrix. An expression of the form shown in equation (58) will give the \( v_{xF} \) and \( i_{xF} \) matrices in terms of the \( v_{xR} \) matrix.

Equation (65) derived by Barthold is not of practical use because of their general nature that would require solution by trial and error. No numerical application to a practical line is demonstrated in his analysis. Barthold gave a general solution for the radio frequency propagation in closed form including transposition and faults.

The analysis of 230 mile long Pinard Hammer line indicates that the line reflects less than 1% of the incident power. Transposition induces a reduction of the received signal by about -6db due to the reason that transposition acts as a natural mode converter and the most predominant mode 3 incident power is converted into 23.6% for mode 3 and 74.5% for mode 2 of the refracted power respectively.

The value of -6db is not applicable to lines with multiple transpositions because three closely spaced transpositions restore the balance of modes and the average loss per transposition should be much lower than that for a single transposition in a long line.

2.4.4 Receiver Couplings:

The attenuated mode components incident at the receiving end are recombined into phase conductor quantities \( v_k \) and \( i_k \). The mode values are equal to the sum of the incident and reflected values. Equations (49) to (52) hold good for this case.
For Fig. 15(b) the network matrix

\[
\begin{bmatrix}
\gamma_r & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \gamma_r
\end{bmatrix}
\]  \hspace{1cm} (63)

Substituting the matrix \gamma into equation (63) into equation (66), \(v_1, v_2\) and \(v_3\) can be found and reflected values are calculated from \(V_k = v_k - V_k\).

A similar procedure applies for figure (c).

For known values of load impedances the received currents and powers are calculated.

The theoretical analysis of the line propagation characteristics showed that the lowest attenuation would be obtained with centre to outer phase coupling and that centre phase to ground would provide the optimum "phase to ground" coupling.

These findings were verified by experiments by Jones and Sogoki of the Ontario Hydro.\textsuperscript{24, 25} Since the reliability of the carrier PLC channel depends to a large extent in the method of coupling, experiments were made on the 230 kV, 500 kV Pigeon River line to compare the effects of various coupling methods on the line propagation characteristics.

The attenuation of a 40 mile untransposed section was measured first with different combinations of transmitter and receiver coupling. The results show that at 50 kHz the overall attenuation is lowest for phase to ground coupling.
For termination (a), the code voltage and currents \( v_k \) and \( i_k \) are expressed in terms of the incident phase values \( v_k \) and \( i_k \) from equation (51)

\[
\begin{align*}
\begin{bmatrix}
 y_{11} & y_{12} & y_{13} \\
 y_{12} & y_{22} & y_{12} \\
 y_{13} & y_{12} & y_{11}
\end{bmatrix}
&= 
\begin{bmatrix}
 y_r & 0 & 0 \\
 0 & y_r & 0 \\
 0 & 0 & y_r
\end{bmatrix}
\end{align*}
\]

\[ v_1 = 2 \begin{bmatrix} y_k \end{bmatrix} \begin{bmatrix} v_k \\ i_k \end{bmatrix} = 2 \begin{bmatrix} y_k \\ i_k \end{bmatrix} \]

Phase 2 is short circuited to ground, so \( v_2 = 0 \)

From these equations

\[
\begin{align*}
 v_1 &= \frac{2((y_{11} + y_r)I_1 - y_{13}I_3)}{(y_{11} + y_r)^2 - y_{13}^2} \\
 v_3 &= \frac{2((y_{11} + y_r)I_3 - y_{13}I_1)}{(y_{11} + y_r)^2 - y_{13}^2}
\end{align*}
\]
At 225kHz however the same coupling method results in 28dB more attenuation than with centre phase to ground coupling. In the second experiment, similar effects were observed in measurement of the attenuation of the full 230 mile line with phase to phase coupling.

**Fig. 16. Centre Phase to Outer Phase Coupling**
At a low carrier frequency the cross-coupled arrangement gave a much lower attenuation. It was found to be the most advantageous. The tests have shown the importance of careful selection of the phase conductors for carrier coupling with few transpositions. The measured input impedances and transmitter coupling efficiencies were found to be in reasonable agreement with calculated values.
2.5. Propagation Analysis of Lossy Lines

There is a renewed interest in the application of power line carrier on long 33kV lines which are somewhat lossy at PLC frequencies. For a finite line with lossy conductors above a poorly conducting earth, each conductor is charged to well-determined potential relative to the earth's surface. In general the conductors will behave like a system of charged capacitances with earth at zero potential. Furthermore, each conductor surface is equipotential and the electric field in space is the same as for a lossless system, so that the principle of ideal images still applies to the electric field of a lossy line.

Alternating currents injected into this system of conductors produce a magnetic field in the surrounding space. However, return currents now penetrate into the earth, the depth of penetration depending on frequency and earth resistivity. Similarly, conductor currents penetrate into the imperfectly conducting metal.

The self and mutual inductances increase since the return circuit does not consist of an infinitely thin ground plane but is formed by a thick layer of earth. This is equivalent to an increase of conductor height varying with frequency and earth resistivity. Hence, no conductor images can be established to calculate the magnetic field or inductances by classical means. Furthermore, there are ohmic losses associated with the
penetrating currents. To account for the penetration of currents, complicated correction factors are introduced for the increase in self and mutual inductances and resistances.

Thus lossless line parameters are uniquely defined by the geometry of the line whereas only the capacitance can be so determined. For a lossy system, inductances depend on geometry, frequency and resistivities and hence, have no fixed geometrical equivalent that could represent the real line.

Starting from the fundamental differential equations (7) of a system,

\[
\frac{d[V]}{dx} = [Z][I]
\]

\[
\frac{d[I]}{dx} = [Y][V]
\]

We have already derived Eq. (14) the fundamental solution for conductor voltages and currents of a lossy or lossless line which is:

\[
[V] = [Z][I]
\]

Where \([Z]\), characteristic impedance matrix, is given by:

\[
[Z] = [Y]^{-1} [Y] [Z]^{-1} = [Z] [Y]^{-1} [Z]
\]

D. E. Hidran\textsuperscript{28} investigated the technique of modal analysis applied to propagation of signals on overhead transmission lines. The propagation problem was first analyzed in phase quantities at a constant frequency and then with nodal
transformation in the original differential equations.

From the modal analysis, properties of modal transformations are deduced. This analysis shows that when earth correction terms are used in the description of the transmission line, the propagation matrix can be used to determine the modal transformations. It is also demonstrated that to evaluate the phase quantity results numerically it is necessary to perform a modal transformation.

The initial work with modal analysis in radio noise design calculations of transmission lines was performed by Adams using transmission line constants calculated for a perfect earth meaning an earth with infinite conductivity. From equation (13) and (14) the solution for voltage and current can be written as

\[
\begin{bmatrix}
  \mathbf{V} \\
  \mathbf{I}
\end{bmatrix} = \begin{bmatrix}
  \mathbf{F}_1 \\
  \mathbf{F}_2
\end{bmatrix} \begin{bmatrix}
  \mathbf{Z}^{-1} & \mathbf{V}_0 \\
  \mathbf{Z}^{-1} & \mathbf{V}_0
\end{bmatrix}
\]

where \( \mathbf{Z} = \mathbf{[y]}^{-1} \mathbf{[y]} \mathbf{[z]} \)

Hedman applied Sylvester's formula in the expansion of \( \mathbf{C} \) to a matrix exponent in numerical terms. Sylvester's theorem states that if \( \mathbf{F}(\mathbf{A}) \) is any well behaved function of a matrix \( \mathbf{A} \) then \( \mathbf{F}(\mathbf{A}) \) can be expanded in terms of matrix \( \mathbf{A} \) and its latent roots, \( \lambda_1, \lambda_2, \ldots, \lambda_n \). Equation (3) of the Hedman paper can be written as

\[
\begin{bmatrix}
  \mathbf{V} \\
  \mathbf{I}
\end{bmatrix} = \frac{-\lambda_1^x}{\epsilon} \mathbf{[e]}_1 + \frac{-\lambda_2^x}{\epsilon} \mathbf{[e]}_2 + \frac{-\lambda_3^x}{\epsilon} \mathbf{[e]}_3 \\
\]

\( \mathbf{e}_1 \) is an eigenvector of \( \mathbf{[y]} \) matrix and it is assumed equal to \( \mathbf{V}_0 \).
where \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are the eigenvalues of the \( Y \) matrix.

This equation shows that any arbitrary signal applied to the system will propagate as three distinct natural signals or in three natural modes of propagation. Modal quantities are related to phase quantities by

\[
\begin{bmatrix}
L_0 \\
L_1
\end{bmatrix}
\begin{bmatrix}
V' \\
I'
\end{bmatrix} =
\begin{bmatrix}
V \\
I
\end{bmatrix}
\]

where \( [L_0] \) is a square matrix transforming the modal voltages to phase voltages. \( [L_1] \) is a square matrix transforming the modal currents to phase currents.

\[
\begin{bmatrix}
V' \\
I'
\end{bmatrix} =
\begin{bmatrix}
Z_c' \\
I'
\end{bmatrix}, \quad \text{where} \quad [Z_c'] = [L_0]^{-1} [Z_c] [L_1]
\]

Thus the modal characteristic impedance can be evaluated from the known transmission line parameters. The modal characteristic impedance is not uniquely defined and the value of \( Z_c' \) depends on the method of normalizing the eigenvectors.

Wadepohl\(^{26,27}\) developed a completely generalised method of solving travelling wave phenomena on polyphase lines by the use of methods of matrix algebra. He summarized the solution to the single conductor wave equation and developed the solution for the two conductor case by classical methods. Matrix four-port equations for transmission line analysis are set out as an alternative set of boundary equations.

In the case of electrically long lines, a method of calculating the modal parameters of the line is described.
cases, the solutions have made use of the fact that the voltages and currents in the system may be resolved into natural modes by evaluating the eigenvalues and eigenvectors of a matrix basic to the transmission line. Wedepohl developed the theory of natural modes of propagation, taking into account the admittance and impedance matrices in order to calculate the characteristic impedance and propagation matrices implicitly. In his work, the relation between modes and eigenvalue analysis is quite confusing and is not clear.

Perz 31, 32 analyzed the propagation of high frequency currents and voltage on lossy power lines. Matrix analysis is described of voltage and current propagation along a resistive conductor system above an imperfectly conducting ground. The approach in the analyses presented is based on the concept of vector spaces with voltage and current modal components representing the co-ordinates.

**Modal Transformation:**

The n entries corresponding to n conductor voltages in vector \([V]\) can be represented as components, \(V_1, V_2, V_3, \ldots\), \(V_n\) of the vector from the origin to the point. The same representation applies to vector \([I]\). The derivation of the ratios of modal components of lossy lines, the phase-shift between corresponding voltage and current components, and modal propagation constants, is based on matrix eigenvalue analysis.
The following properties which are quoted from reference 31, will be referred to in the analysis of a system of parallel conductors:

1. The square matrix \([P] = [z] [y]\) of order \(n\) has distinct linearly independent eigenvector axes \([z^{(1)}], [z^{(2)}], \ldots, [z^{(n)}]\) each associated with the eigenvalue \(\lambda_1, \lambda_2, \ldots, \lambda_n\) of the matrix \([P]\). Therefore, the modal matrix \([M]\) whose column vectors are the eigenvectors of nonzero length diagonalizes \([P], [M]^{-1} [P] [M] = [\Lambda]_d\)

where the elements of the diagonal matrix \([\Lambda]_d\) are the \(n\) eigenvalues of \([P]\).

2. Similarly, the modal matrix \([M]'\) whose column vectors are the non-zero eigenvectors of the transpose of \([P]\), diagonalizes \([P]' = [y] [z]'\)

\([M]' [P]' [M]' = [\Lambda]_d\)

3. Functions of matrix \([P]\) or \([P]'\) of order \(n\) reduce to square matrices of order \(n\) and their eigenvectors are the eigenvectors of \([P]\) or \([P]'\), namely \([z^{(j)}], [z^{(j)}]'\), respectively, \(j = 1, 2, \ldots, n\)

\([M]' [P] [M] = [F(r)]_d\)

\([M]' [P]' [M]' = [F(r)]_d\)

\([F(r)]_d\) is the diagonal matrix of the eigenvalues \(F(r_j)\) of \([F([P])]\), \(r_j\) eigenvalues of \([P]\) or \([P]'\).
4. Modal matrices \([\tilde{M}]\) and \([\tilde{N}]\) with eigenvector columns of any length satisfy the equation:
\[
[\tilde{M}]_t [\tilde{M}] = [\tilde{N}]_t [\tilde{N}] = [D_1]
\]
where \([D_1]\) is a diagonal matrix obtained from the product
\([\tilde{M}]_t [\tilde{N}]\),
\([\tilde{M}]_t [\tilde{N}]\) and \([\tilde{N}]\) are mutually orthogonal, if
\[
[\tilde{N}]_t [\tilde{M}] = [D_2] \quad \text{and} \quad [\tilde{M}]_t [\tilde{N}] = [D_2]
\]
\([D_2]\) is diagonal matrix, (asterisk means conjugate.)

5. By definition \([H]_t [M] = [U]\), the identity matrix \([H]\)
when multiplied by \(j^{th}\) column vector of \(M\) gives a column vector, having \(j^{th}\) entry equal to one and remaining elements are all zero.

\[
[M]_t [\delta(j)] = [1]_j
\]

**Modal Voltages and Currents**

For a voltage vector \(V\), which does not coincide with any of \(M(j)\) eigenvector axes, the \(n\) co-ordinates of vector \(V\) are \(V(1), V(2), \ldots, V(n)\).

where
\[
\begin{bmatrix}
V_1(j) \\
V_2(j) \\
\vdots \\
V_n(j)
\end{bmatrix} = [V(j)] = [1]_j \begin{bmatrix}
\delta_1(j) \\
\delta_2(j) \\
\vdots \\
\delta_n(j)
\end{bmatrix}
\]
for \(j = 1, 2, 3, \ldots, n\) (69)

where
\[
\delta_n(j) = \frac{V_n(j)}{V_1(j)}
\]
The modal voltages add up to the actual conductor voltages,

\[
\begin{bmatrix}
    V_1 \\
    V_2 \\
    \vdots \\
    V_n
\end{bmatrix}
= \begin{bmatrix}
    V_1^{(1)} \\
    V_2^{(1)} \\
    \vdots \\
    V_n^{(1)}
\end{bmatrix} + \begin{bmatrix}
    V_1^{(2)} \\
    V_2^{(2)} \\
    \vdots \\
    V_n^{(2)}
\end{bmatrix} + \cdots \begin{bmatrix}
    V_1^{(n)} \\
    V_2^{(n)} \\
    \vdots \\
    V_n^{(n)}
\end{bmatrix}
\]  \hspace{1cm} (70)

\[
\begin{bmatrix}
    V_1 \\
    V_2 \\
    \vdots \\
    V_n
\end{bmatrix}
= \begin{bmatrix}
    V_1^{(1)} + V_2^{(1)} + \cdots + V_n^{(1)} \\
    V_1^{(2)} + V_2^{(2)} + \cdots + V_n^{(2)} \\
    \vdots \\
    V_1^{(n)} + V_2^{(n)} + \cdots + V_n^{(n)}
\end{bmatrix}
\]  \hspace{1cm} (71)

\[
\begin{bmatrix}
    V_1 \\
    V_2 \\
    \vdots \\
    V_n
\end{bmatrix}
= \begin{bmatrix}
    1 & 1 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
    V_1^{(1)} \\
    V_2^{(2)} \\
    \vdots \\
    V_n^{(n)}
\end{bmatrix}
= \begin{bmatrix}
    m_1 \\
    m_2 \\
    \vdots \\
    m_n
\end{bmatrix}
\begin{bmatrix}
    V_1^{(1)} \\
    V_1^{(2)} \\
    \vdots \\
    V_1^{(n)}
\end{bmatrix}
\]  \hspace{1cm} (72)

\([m] \) is normalized modal matrix.

By use of the above equation, the actual conductor voltages can be calculated, if conductor 1 modal voltages are known. Conversely, modal voltages on conductor 1 can be calculated for known phase voltages.
\[
\begin{bmatrix}
V_1^{(1)} \\
V_1^{(2)} \\
\vdots \\
V_1^{(n)} \\
V_2 \\
\vdots \\
V_n
\end{bmatrix} = \left[ n \right]^{-1} \begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_n
\end{bmatrix} = \left[ m \right]^{-1} \begin{bmatrix}
V
\end{bmatrix}
\]

(73)

A square matrix with columns equal to the modal voltage components is derived with the help of Eq. (69) and (73).

\[
\begin{bmatrix}
V_1^{(1)} & V_1^{(2)} & \cdots & V_1^{(n)} \\
V_2^{(1)} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
V_n^{(1)} & \cdots & \cdots & V_n^{(n)}
\end{bmatrix} \left[ B \right] = \left[ A \right] = \begin{bmatrix}
0 & V_1^{(1)} & 0 & \cdots & 0 \\
0 & 0 & V_1^{(2)} & \cdots & 0 \\
0 & 0 & 0 & \cdots & V_1^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & V_1^{(n)}
\end{bmatrix}
\]

(74)

The resolution of conductor currents into modal components at one point of the line and their synthesis at another point of the line into actual conductor currents follows exactly the same procedure as that for voltages. Hence, it is sufficient to replace the voltage symbol \( V \) with \( I \) and \( A \) with \( B \) in the above equations to obtain relations between actual conductor currents and modal values.

Summary of properties of lossy lines:

1. There is a phase shift between conductor currents and voltages on a system of \( k \) conductors.
2. Voltages are linearly related to currents through a symmetric matrix \( [Z] \) whose elements are the complex self and mutual characteristic impedances. Contrary to a lossless line, elements of \( [Z] \) cannot be derived from only the geometry of a lossy system.

3. There are two different sets of \( k \) nodes with complex components, one for voltages and another for currents. There is no coupling between voltage components of one node with corresponding current components of the remaining nodes. The two modal sets are not mutually orthogonal.

4. Modal resolution on lossy lines is not power invariant.

5. The system behaviour depends on frequency, and resistivities of conductors and earth. For a given line geometry complete analysis has to be repeated for new values of frequency and resistivities.

2.6. Effects of Earth Resistivity on Modal Parameters

The complex elements of this matrix and therefore the modal quantities are influenced by the earth resistivity. The effect of earth resistivity on modal components general and modal reflection free terminations for field measurements was described.\(^{(29)}\) For a three phase, 500 kV Ontario Hydro Pinard hanger line, the changes in modal attenuation and velocity of propagation with changing earth resistivity, from \( 20 \) \( \Omega \cdot \text{m} \).
At three frequencies 50kHz, 100kHz and 200kHz are described in reference 29 and been shown in figure (17) and (18).

Fig. 17 - Effect of Earth Resistivity on Modal Attenuations
Fig. (17) illustrates the effect of earth resistivity on modal attenuation. Modes are numbered in order of decreasing modal impedances. Examination of Fig. 17 indicates that mode (3) and (2) attenuations decrease with increasing earth resistivity. Attenuation of mode 3 changes little with $\rho_e$.

Fig. (18) illustrates the effect of earth resistivity on modal velocities of propagation. The modal voltages and currents propagate with different velocities always less than the velocity of light.

![Graph showing effect of earth resistivity on modal velocities](image)

**Fig. 18 - Effect of Earth Resistivity on Modal Velocities**

Mode (1) "ground" mode has the highest attenuation and the lowest velocity of propagation and hence it is of little importance in PIG problems.

### 2:6:1 Earth Conduction Effects

A general method used for studying the effects of the finitely-conducting earth on overhead transmission line is to express the wave equation in integral form, both within and above the earth, and then to match these two equations at the earth's surface through the use of boundary conditions. Resultant solutions are usually complex integrals which may be quite difficult to evaluate numerically.
One technique used by Carson and Wise to evaluate these integrals in the high frequency region is to use an asymptotic expansion of the integrand and to integrate term by term. Resultant solutions valid for high frequency are detailed in Hadam's paper. Series expansion solutions are developed for different frequency regions. Results were plotted on a logarithmic scale as shown in Figs. (19) and (20) for a single conductor above earth.

**Fig. 19 - Earth Correction Resistance**

From Fig. 19, it can be interpreted that in the low frequency region of the curve when resistance increases 1 decade for each decade increase in frequency, the resistance is proportional to frequency. In the high frequency region of the curve with the increase of 1 decade in resistance for each 2 decades increase in frequency, resistance is increasing proportional to the square root of frequency. For small values of \( \frac{d}{\sqrt{\varepsilon}} \), that is when the height of the conductor is low and earth resistivity is large, and in the extreme case when \( \frac{d}{\sqrt{\varepsilon}} \) equals to 1 in the region of 100 c/s.
to 1Mc/s the earth resistance is proportional to frequency. When \( \frac{d}{dF} \) is large in the range of 20, the effective resistance is proportional to square root of frequency in the range of 10Mc/s to 1Mc/s.

Similar, interpretation can be made from figure 20. In the low frequency region, with the increase of frequency the inductance term decreases less than proportionally to the square root of frequency. In the high frequency region, the inductance correction terms decrease proportionately with the square root of frequency.

In the lossy line analysis the elements of the series impedance matrix depend on conductor resistivity and line geometry and must be corrected for the effects of finite earth resistivity. These are Carson's correction factors.

2.6.2. Carson Correction Factors

The complete solution of the actual problem of wave propagation along a transmission system composed of an overhead wire parallel to the surface of the earth has not been completely solved for account of the nonhomogeneity of the earth's surface.

Carson solved the problem of calculating the electric field parallel to an alternating current flowing in a straight, infinitely long wire placed above and parallel to a plane homogeneous earth with the following assumptions:

1. The ground permeability (\( \mu_g \)) is unity.
2. The wave is propagated with the velocity of light and without attenuation.

3. The frequency is so low that polarization currents may be neglected.

Carson and Pollaczek\textsuperscript{15} in 1926 developed independently theoretical solutions of propagation along a system of parallel conductors above lossy ground. The solution of propagation problems and the derivation of correction factors are quite complex even if some simplifying assumptions are adopted. Magnetic field intensity at any point in space is a combination of two field components, field due to the current in the wire and the field of the ground current. Series self and mutual impedances consists of two parts, self and mutual impedances on the assumption of a perfectly conducting ground and the second part is impedance due to ground currents.

Series impedance correction factors were developed which have been expressed by infinite series with coefficients also in the form of infinite convergent series. Carson's earth-correcting resistance terms are proportional to frequency and to the square root of frequency respectively, in the low and high frequency regions. The correction
factors of Carson's and Pollack's apply to lossy ground for earth dielectric constant and permeability equal to those of free space ($\varepsilon = \varepsilon_0$ and $\mu = \mu_0$).

Recently Ritz and Houghton proposed a direct method for a numerical computation of series impedance line correction factors. A general solution for any $\varepsilon$ and $\mu$ in the form of a generalized plane wave integral is formed.

**Line Above Perfectly Conducting Ground Plane**

The analysis of the guided electromagnetic field in a lossless system helps in the understanding of the generalized plane wave solution in integral form and of the application of boundary conditions. The conductor current has a uniform density around the cylindrical conductor. The density of the earth return current is the largest directly under the conductor and diminishes with the lateral distance. The magnetic and electric fields produced by each of these sinusoidally time varying currents are related through classical Maxwell's equations.

For a single wire line at a constant height above and parallel to a perfectly conducting ground plane, the line oscillating current $I_z$ flowing in the positive $z$ direction with the ground plane forming the return circuit is given as

$$I_z = I_0 e^{j\omega t - \gamma}$$
electric field $E$ and magnetic field $H$ are associated with this oscillating current, these fields being at right angles to each other and in a plane perpendicular to the line and ground. This implies that there are no field components perpendicular to this plane or $E_z = 0$, $H_z = 0$.

![Diagram of magnetic field $H^0$ produced at distance $r$ from a wire carrying current $I$](image)

**Fig. 21. Magnetic field $H^0$ produced at distance $r$ from a wire carrying current $I$.**

The value of components of a plane wave being dependent on the distance from the line conductor are given by relations:

$$E = E_0 e^{j\omega t - Y}$$
$$H = H_0 e^{j\omega t - Y}$$

The total magnetic field vector $H$ at any point in space is a combination of two fields, the field $H^0$ produced by the line current and the field $H'$ due to current penetrating in the ground.

$$H = H^0 + H'$$
The generalized wave equations for the magnetic field components due to line current has been derived using Amperes Law and definite integrals

\[ B_x^0 = \frac{-(y-h)}{(h-y)^2+x^2} \frac{I_z}{2\pi} \int_0^\infty e^{-(y-h)k_1} \cos xk_1 dk_1 \quad y \geq h \]  

\[ H_y^0 = \frac{x}{(h-y)^2+x^2} \frac{I_z}{2\pi} \int_0^\infty e^{-(y-h)k_1} \sin xk_1 dk_1 \quad y \geq h \]  

Similarly, the magnetic field at a point \( P(x, y) \) due to the ground current is given by the relations

\[ B_x^I = \frac{I_z}{2\pi} \int_0^\infty e^{-(y+h)k_1} \cos xk_1 dk_1 = \frac{-(y+h)}{(y+h)^2+x^2} \frac{I_z}{2\pi} \]  

\[ H_y^I = \frac{-I_z}{2\pi} \int_0^\infty e^{-(y+h)k_1} \sin xk_1 dk_1 = \frac{x}{(y+h)^2+x^2} \frac{-I_z}{2\pi} \]  

By comparison of equations (36) and (37), fields due to ground current have been found to be the same as those for a single line conductor in the z-direction located at \( y = -h \) which carries current \(-I\). Hence the ground line can be removed and this part of the system replaced by the electrical imaginary conductor as shown (7).

The conductor series impedance per unit length for a conductor above lossless earth is given by the relation

\[ Z_c = R_c + \omega L_0 + j\omega L_1 = j\omega L_0 + Z_1 \]

where \( R_c \) is the resistance of the conductor at \( \omega \), \( L_1 \) is the internal inductance of the conductor and \( L_0 \) is the external
self-inductance per unit length and \( L \) internal impedance which is quite small as compared with \( Z_1 \).

\( Z_1 \) is the correction factor, practically equal to \( R \), the conductor resistance at lower and higher frequencies.

The entries of the series impedance matrix, for a system of parallel conductors are

\[
Z_{pq} = j\omega L_{pq} = j\omega 2.10^{-7} \frac{D_{pq}}{x_{pq}} \text{ for } p = q
\]

\[
Z_p = Z^*_{1p} + j\omega L_p = R_{0p} + j 2.10^{-7} \ln \frac{2h_p}{r_p} \text{ for } p \neq q
\]

**Line Above Lossy Ground**

In actual practice, for a system of parallel conductors, the ground is far from being a superconductor, therefore, the return current is not confined to the surface; instead it flows with decreasing density, at depths much greater than the height of the conductors. The ground return circuit cannot be replaced by image conductors. The presence of conductor losses is associated with an electric field component in the direction of transmission. This leads to negligibly small deformation of the plane waves and the attenuation of transmitted energy.

With the application of electromagnetic theory, self and mutual inductances and resistances of the actual line conductors above a reference or zero potential plane have been derived. These series impedances obtained can be seen to consist of two parts, first part represents the parameters of the line.
and the second the correction factors. Simplifying assumptions are made in the model:

1. The electromagnetic system is in the steady state, sinusoidally time-varying condition.
2. The electromagnetic field propagates in the form of essentially plane non-uniform waves.
3. The conductors have a circular cross-section and their radii are very small as compared with other dimensions. The conductors are ideally parallel to each other and the ground.
4. The principle of superposition is applied to the field waves along a lossy transmission line.

The magnetic fields components due to the ground current in the z direction has been derived in reference 30, as

\[ h_x' = \frac{I_z}{2\pi} \int_{0}^{\infty} k_1 F_a(k_1) e^{-\gamma h} \cos \pi v k_1 dz \]

\[ h_y' = -\frac{I_z}{2\pi} \int_{0}^{\infty} k_1 F_a(k_1) e^{-\gamma h} \sin \pi v k_1 dz \]

where \[ F_a(k_1) = \frac{ik_1}{2\pi} \frac{e^{-\gamma h} \sqrt{1 - \chi^2}}{k_1(\sqrt{1 - \chi^2} + \mu k_1)} \]

To compute magnetic field at any point in space, for a number of conductors with any currents, principle of superposition is applied to equation (35) together with equation (38).
Series Self-Inductance:

For the conductor with internal impedance \( Z_1 \) and current \( I \), the coaxial electric field intensity is equal to the potential drop per unit length on the surface of the conductor. Equation (34) of the paper gives:

\[
\frac{\partial V(x, y)}{\partial z} - \frac{\partial V(x, 0)}{\partial z} = \frac{\partial V}{\partial z} = -Z_0 I
\]

\[
= -\left[ Z_1 \frac{i \omega \mu}{2\pi} \int_{-2\pi k_1}^{2\pi k_1} \frac{-2\pi k_1 \, dk_1}{\pi} \right] I
\]

\[
\int_{\gamma}^{\infty} \frac{-2\pi k_1 \, dk_1}{\pi} \left[ \frac{\sqrt{\gamma^2 - \pi^2}}{\sqrt{\gamma^2 - \pi^2}} \right]
\]

(89)

It can be observed that total series impedance consists of two parts. The first two terms are the same as that of a lossy conductor above lossless ground whereas the remaining integral constitutes the series resistive and inductive correction factors.

Mutual Impedances:

From equation (19) and (33) of reference 30, the mutual series impedance is

\[
\frac{\partial V_1}{\partial z} = -Z_2 = -\frac{i \omega \mu_0}{2\pi} \int_{\gamma}^{\infty} \frac{-2\pi k_1 \, dk_1}{\pi} \left[ \frac{\sqrt{\gamma^2 - \pi^2}}{\sqrt{\gamma^2 - \pi^2}} \right]
\]

(90)

The first part represents the mutual impedance between conductor \( p \) and \( q \) above a lossless ground.

The remaining term in the equation is the mutual resistive and inductive correction factor due to the lossy ground.
This single solution for the earth series-impedance correction factors, which is not in the rather complex form of infinite series, provides the same value of correction terms as those developed in different forms and by different analytical methods by Carson, Poliazeck, and Wise. The derivation of the correction factors can be directly included in the digital computer programming of lossy line modal analysis.
2.7 Application of Lossless Modal Analysis to IBS System

Modal analysis is applied to the solution of the problem of high frequency transmission on bundles with insulated subconductors. The subconductors in a conventional bundle conductor transmission line are electrically connected through spacers and suspension clamps and hence, each bundle may be considered as a single conductor.

Since the IBS propagation is balanced within each bundle, the electromagnetic field produced at the earth's surface is of much lower value than for conventional PLC.

The losses due to the imperfectly conducting earth can be neglected and the lossless line assumptions should lead to fairly accurate results.

2.7.1 Modal Analysis

Two Sub-conductor Bundle:

For a two sub-conductor bundle, the analysis becomes quite simple as given in Appendix 'B'. The impedance and modal transformation matrices are also given. For a two pole d.c. line with two conductor bundles there are four nodes present for such a system. Nodes 1 and 2 are push pull nodes within each bundle, node 3 is a push pull node, for the two conductor groups whereas node 4 is the ground node.

Three Sub-conductor Bundle:

Modal impedances and transformation matrices have been derived as shown in Appendix 'B'. There are two modes of equal
nodal impedance nodal analysis indicates that there are two push-pull excitations for the three subconductor configuration. In fact, two subconductors are interconnected and the return path is provided by the third conductor. This excitation is not balanced since the voltage on the return subconductor is twice the voltage on the remaining subconductors. A balanced excitation can be obtained when the signal is applied to one pair of subconductors with the third subconductor floating.

Four Conductor Bundle:

The analysis is shown in the Appendix B. It is shown that four modes exist in each bundle.

Node 1 is the ground mode with currents flowing in the same direction in all subconductors and returning through the ground.

Node 2, 3 and 4 are push-pull nodes utilizing two subconductors.

Node 4 is a diagonal push-pull node with currents in all four subconductors.
CHAPTER 3

UNCONVENTIONAL TERMINAL COUPLING ARRANGEMENTS

3.1. Introduction

Communication is achieved by a power-line carrier signal that is coupled from a grounded transmitter to the line through a high voltage capacitor, whose reactive impedance is very high at the power frequency but very low at carrier frequencies. In addition to the capacitor, a tuned wavetrap is provided, to prevent a low station impedance from short circuiting the carrier signal.

The present practice of coupling PLC signals onto a power transmission line in a directional manner requires the use of coupling capacitors, line traps, and filters. Sophistication of this type of coupler has permitted low loss, broad-band coupling on either a phase to phase or phase to ground basis, and has made possible the widespread use of carrier for vital services.

Two factors, however, now limit the capacitor coupling method. The first is the high cost of line traps that are suitable for the large short-circuit currents now experienced on power systems, and of high-voltage capacitors and line trap structures for EHV systems. The second is the need for greater channel isolation, that is, for confinement of the carrier signal to the desired line section. This is of increasing importance because of crowding of the 30-300 kHz frequency
spectra by more and more carrier channels. As of today, no further expansion is possible without some improvement in channel isolation.

Recent trends in system development have encouraged research into alternate means of coupling. One approach uses directional coupling which can avoid any need for a wave trap and at the same time, provide an increased communication capacity.

Oliver pointed out that the natural coupling between transmission lines was investigated about 50 years ago with the object of eliminating the effect which caused crosstalk in communication systems. Oliver and other researchers suggested that the natural coupling could be used to make directional couplers with the remarkable properties of perfect match and isolation over the entire frequency band for the dominant TE11 modes.

W.W. Hunsford in 1947 published a paper in which he considered two adjacent transmission lines coupled together weakly at two places by means of simple link circuits and shows how a large number of coupling elements can be used to improve bandwidth.

H.J. Riblett in the same year gave a theoretical discussion approaching the directional coupler problem from the point of view of a pair of uniform transmission lines coupled together at discrete points. For several years directional couplers had been used only at microwave frequencies, until 1954
when Firestone\textsuperscript{36,37} extended the theory of microwave couplers to suit PLC conditions. He was the first person to investigate the possibility of transferring the carrier energy in a directional manner by means of open wire transmission lines. This was followed by further independent analytical and experimental studies one of Ontario Hydro Research Division\textsuperscript{38,40} and one at the Central Electricity Research Laboratories in England,\textsuperscript{11,12,39,41} which point to very practical parallel-wire couplers.
3.2 Transverse Line Directional Coupler

3.2.1 Basic Concepts

Directional Coupler:

An ideal directional coupler is a passive, linear network having the property that power fed in at terminal (1) divides in some ratio between terminals 2 and 4 without appearing at terminal (3) while power fed in at terminal 2 divides between terminal 1 and 3 without appearing at 4. An alternative description is that of a four port formed by physically locating two transmission lines close enough to each other so that they are coupled via the electric and magnetic fields associated with the voltages and currents on the lines. The performance of a directional coupler is measured by two parameters, its Coupling and Directivity.

If $P_1$ is the incident power in Port 1 and $P_4$ the power coupled in the forward direction in arm 4, then,

$$\text{Coupling} = 10 \log \frac{P_1}{P_4} \text{ dB}$$

(98)
Ideally power $P_3$ coupled in the backward direction in arm 3 should be zero. (The extent to which this is achieved is measured by Directivity,

$$D = 10 \log \frac{P_4}{P_3} \text{ db}$$

(99)

3.2.2. Assumptions in the Analysis of Transmission Line Couplers:

The following assumptions will be made to analyze the general treatment of directional coupling action of transmission lines:

1. The transmission lines are lossless. This implies that there is no penetration of currents in the conductors and earth. No electromagnetic energy can penetrate and be lost or trapped inside the conducting media. Internal inductance can be ignored and when the depth of penetration is very much smaller than the radius of curvature, we can imagine the surface to be perfectly flat over a small region.

2. The conductor sizes and spacings are very very small, compared with wavelength. Since in transmission line analysis we assume that only T.E.k. waves exist, when the wavelength is comparable with the distance between the conductors we must recognize that other types of waves (T.E or T.M) may exist. These higher order modes are undesirable. Another reason for a small spacing is that when the distance between the wires of an unshielded line approaches a quarter wavelength, the line acts as an antenna and
radiates a considerable portion of the energy that it carries.

3. The dielectric medium is homogeneous and isotropic.

4. Lines are balanced and coupling is symmetrical. That is to say, whenever a given meter reading is taken, it is a necessity that the meter leads be reversed and exactly the same reading should be obtained.

3.2.3. Characteristic Equation of Two Coupled Lines

![Diagram of two coupled lines with labels D1, D2, D3, D4, D12, D13, D14, D24.]

*Fig. 26. Perspective View of Two Transmission Lines.*

Firestone has shown that a matched four wire transmission system will display a directional coupling characteristic, if the following relationship is maintained.

\[
\frac{L_m}{C_m} = Z_{01} \cdot Z_{03}
\]  

\[(100)\]
Where $L_M$ is the mutual inductance per unit length between the two transmission lines,

$C_M$ is the mutual capacity per unit length between two transmission lines,

$Z_{01}$ is the characteristic impedance of line $(1,2)$, and $Z_{03}$ is the characteristic impedance of line $(3,4)$.

Equation (100) must be satisfied to obtain directional coupling with infinite directivity.

To prove that this condition actually exists for an open wire transmission line, the relations for $L_M$ and $C_M$ which have been derived in reference 32 are:

$$L_M = 4 \cdot 10^{-7} \ln \left( \frac{D_{23} \cdot D_{14}}{D_{13} \cdot D_{24}} \right) H/m$$

and

$$C_M = \frac{4 \cdot \varepsilon_r \cdot 10^{-7}}{Z_{01} \cdot Z_{03}} \ln \left( \frac{D_{23} \cdot D_{14}}{D_{13} \cdot D_{24}} \right) F/m$$

(101)

For free space $\varepsilon_r = 1$, forming the desired quotient,

$$\frac{L_M}{C_M} = \frac{4 \cdot 10^{-7} \ln \left( \frac{D_{23} \cdot D_{14}}{D_{13} \cdot D_{24}} \right)}{Z_{01} \cdot Z_{03} \ln \left( \frac{D_{23} \cdot D_{14}}{D_{13} \cdot D_{24}} \right)} = \frac{Z_{01} \cdot Z_{03}}{Z_{01} \cdot Z_{03}}$$

Hence, Eq. (100) is satisfied and directional coupling should exist.
3.2.4. Coupling equation without reflections

Coupling in decibels is defined as:

\[ D_{bc} = 10 \log_{10} \frac{P_{L3}}{P_{11}} \]  \hspace{1cm} (102)

Where \( P_{11} \) is the input power to line (1,2) at terminal pair (1),

\( P_{L3} \) is the power delivered to the matched load \( Z_{03} \) at

terminal pair (3).

The coupling equation for such a system has also been derived:

\[ D_{bc} = 20 \log_{10} \left\{ \frac{4\lambda L \cdot 10^{-7} \ln \sqrt{\frac{D_{14} \cdot D_{23}}{D_{13} \cdot D_{24}}}}{\sqrt{Z_{01} \cdot Z_{03}}} \right\} \] \hspace{1cm} (103)

The factor \( \lambda \) in meters has been added to take into account an

arbitrary coupling length. It can be seen from equation (103),

that when

\[ \ln \sqrt{\frac{D_{14} \cdot D_{23}}{D_{13} \cdot D_{24}}} = 0, \] no coupling occurred. This means,

\[ \frac{D_{14} \cdot D_{23}}{D_{13} \cdot D_{24}} = 1 \]

or,

\[ \frac{D_{14}}{D_{24}} = \frac{D_{13}}{D_{23}} \]
3.2.5. Coupling and Directivity with Reflections:

Fig. 27. Configuration for Reflections on Both Sides

For the arrangement of the above configuration in which $Z_{L3}$ and $Z_{L4}$ are connected to terminals which are at some distance from the ends of the coupled section if $Z'_{L3}$ represents the impedance at A when looking toward the $Z_{L3}$ load, then

$$Z_{L3} = Z_0 \left( \frac{\left( Z_{L3}\cos \beta l_1 + jZ_{L3}\sin \beta l_3 \right)}{Z_0 \cos \beta l_3 + jZ_{L3}\sin \beta l_3} \right)$$  \hspace{1cm} (104)

Similarly, the impedance $Z_{L4}$ can be referred to point B:

$$V_3 = -L \frac{di_1}{dt} = -\frac{dq_3}{dt} = -C \frac{dv_{12}}{dt}$$  \hspace{1cm} (105)

Kirchoff's voltage law will give:

$$-L \frac{di_1}{dt} - (Z_{L3} + Z'_{L4}) i_3 = 0$$  \hspace{1cm} (106)
Final expressions for Directivity and Coupling have been derived:\footnote{37}

\[
D_{\text{bd}} = 20 \log_{10} \left| \sqrt{\frac{R_{33}'}{R_{44}'}} \left( \frac{Z_{22} Z_{44}'}{Z_{22} Z_{33}' - 1} \right) \right| \tag{107}
\]

where the prime indicates reflected quantities.

\[
R_{33}' = \frac{R_{L3}}{Z_{03}}, \quad R_{44}' = \frac{R_{L4}}{Z_{03}},
\]

\[
Z_{22}' = \frac{Z_{L2}}{Z_{01}}, \quad Z_{33}' = \frac{Z_{L3}}{Z_{03}}.
\]

That is, all \( Z \)'s and \( R \)'s are normalised with respect to the proper characteristic impedances.

When \( \beta L_3 = \beta L_4 = 0 \)

\( Z_{L3}, Z_{L4} \) and \( Z_{L2} \) are purely resistive, the expression for Directivity becomes

\[
D_{\text{bd}} = 20 \log_{10} \left| \sqrt{\frac{Z_{22}}{Z_{44}'}} \left( \frac{Z_{22} Z_{44}'}{Z_{22} Z_{33}' - 1} \right) \right| \tag{108}
\]

The final expression for coupling 37 is

\[
D_{\text{bc}} = 20 \log_{10} \left( \frac{\mu_c L_1}{Z_{01} Z_{03}} \right) \left( \sqrt{\frac{D_{14} D_{23}}{D_{13} D_{24}}} \left( \frac{1}{Z_{22}} \right) \right) \left( \frac{1 + Z_{44}'}{Z_{22}} \right) \left( \frac{Z_{22} Z_{33}'}{Z_{22} Z_{33}' - 1} \right) \right| \tag{109}
\]

It can be observed from equation (108) that if \( Z_{22} \times Z_{33} = 1 \)
\[ \log_{10} \left( \frac{Z_{44} \left( Z_{44} + 1 \right)}{Z_{44} (0)} \right) = \infty \]

We then have infinite directivity regardless of the termination on terminals (4), provided the other terminals are matched. Equation (107) will show that, if \( Z_{22} Z'_{33} = 1 \), an infinite directional coupler will result regardless of the other terminations. This means that any degree of mismatch at terminals (2) or (3) is tolerable, provided \( Z_{22} \) and \( Z_{33} \) are reciprocals.

Observation can be made from Eq. (109). If \( Z_{22} = Z'_{33} = Z_{44} = 1 \), this means there are no reflections; the second bracketed term within the log expression reduces to unity, and the coupling equation becomes equation (103). It can be seen that effect of reflections on the two lines is to add a factor

\[ \left| \frac{Z_{22} Z'_{33}}{Z'_{33} + Z_{44}} \left( \frac{1}{Z_{22}} + Z_{44} \right) \right| \]

inside the log term, which is called the Loss Factor. It is desirable to see what happens to the coupling when the condition for infinite directivity

\[ Z_{22} Z'_{33} = 1 \]

is satisfied.

The loss factor reduces to a constant. This implies that coupling remains constant when \( Z'_{33} \) is varied, provided the reciprocal relationship \( Z_{22} \) and \( Z_{33} \) is maintained.
3.3. Carrier Coupling to Transmission Line by Parallel Conductor

3.3.1. Principle of Operation

The principle of operation of the two arrangements of a parallel wire coupler, the Directional Coupler\(^{40,42}\) conceived by Keyser and Hicks of Ontario Hydro and the Inductive Coupler devised and developed by Hooper and Pullen\(^{39}\) of Central Electricity Research Laboratories, England, shown in figure (28) can be described as follows.

**Directional Coupler**

Considering the coupler in fig. 28(a) namely a wire about one span length strung parallel to a phase conductor as close to it as insulation restrictions permit. It is terminated at one end in its characteristic impedance.

On application of a signal to this coupling wire, currents are induced in the phase conductor as a result of capacitive and magnetic coupling. In one direction, toward the right in fig. 28(a), these currents add, and in the other direction they cancel. Thus a signal is launched on the power line in one direction only. The magnitude of this current is unaffected by changes in station impedance. The only theoretical restrictions are that the coupling be "loose" which is satisfied by the insulation requirements and that the coupling wire be terminated in its characteristic impedance to avoid reflections which would induce reverse direction currents on the power line.
Fig. 28. Two arrangements of parallel wire coupler with coupling wire adjacent to an outer phase conductor.
This directional coupler overcomes the problems mentioned earlier. Since a signal is transmitted in one direction only and conversely can be received from one direction only, good station isolation is possible without the use of line traps or capacitors. The coupler is however quite inefficient. Because of the loose coupling, most of the transmitter output power is dissipated in the terminating resistor, and only a small fraction reaches the power line.

Inductive Coupler:

The Inductive coupler which Hooper and Pullen call simply a "Parallel Transmission Line Coupler", gains improved efficiency at a cost of much smaller bandwidth. As shown in fig. 28(b), it is a short circuited coupling wire, a quarter wavelength long. In this case, however, the power line must be short circuited 'behind' the coupler by means of a inductance-capacitance shunt trap, possibly aided by low station impedance. If the transmitter output impedance is properly selected with reference to the power line characteristic impedance, multiple reflections from the two short circuits and the transmitter terminals results in addition and cancellation of currents on the power line, such that all the power is launched in the desired direction.

This coupler is called the inductive coupler, since it is not inherently directional, is theoretically lossless since it contains no intentionally dissipative components. It appears
to satisfy the need for acceptable station isolation without line traps. It will, however, usually require a capacitor on the power line immediately adjacent to the station end of the coupling wire. It can be concluded that the parallel wire coupler in either its high loss, broadband form or its low loss, narrow-band form appears to be feasible. Its application would depend on the relative economics of line-trap and capacitor costs compared with coupling wire and transmitter-power costs, and on the relative need for increased bandwidth or improved channel isolation. Keyser and Hicks verified Firestone's equations by laboratory experiments on a scale model at 50 MHz. In consequence they applied the Firestone analysis to a three-phase, 230 kV line terminated for a symmetrical coupler and found that in this application the coupling loss is of the order of -12 db. This loss is due to the relatively large spacing required for safety reasons between the phase and the coupler conductors. The signal injected into the horizontal line is propagated in the push-pull mode (2). This should have a relatively high attenuation. It may be concluded that for these reasons the terminated symmetrical couplers are inferior for horizontal power lines as compared with conventional PLC coupling through a high voltage capacitor.
FIG. 29. DIRECTIONAL COUPLER ON A SINGLE CIRCUIT LINE
Ground was simulated by a high conductivity plane large enough to approximate an infinite plane. The coupler as shown in figure (30) was constructed to operate at a frequency of 50 Hz/s (scale frequency 250 kHz).

Fig. 30. MODEL TRANSMISSION LINE

The three centre wires represent the three phase transmission line and the two outside wires represent the line coupler.
3.3.2. **Theory of Transmission Line Directional Coupler**

By applying the theory of coupled transmission lines, J. Cooper and F. D. Pullen showed that it is possible to couple carrier frequency power onto and off power transmission lines with high efficiency using a parallel conductor as the coupling element.

![Diagram of coupled two wire transmission lines](image)

**Fig. 31. COUPLED TWO WIRE TRANSMISSION LINES**

Two lines, A and B are electromagnetically coupled together over a length. There are four pairs of terminals 1 to 4. As we know, for the coupled transmission lines to act as a perfect directional coupler, the lines should be symmetrically coupled and lossless and impedance relations as shown in the figure should be maintained.
We can say that if the coupled sections of the transmission have characteristic impedances

\[
\begin{align*}
Z_a &= \frac{Z_A}{\sqrt{1-k^2}}, \quad Z_b = \frac{Z_3}{\sqrt{1-k^2}}
\end{align*}
\]

and are terminated in impedances \(Z_A\) and \(Z_3\) respectively, a unit wave incident on a pair of terminals such as 1, in fig. 31 will be transmitted to terminal 2 with complex amplitude \(\mathfrak{G}\) and will be coupled to terminals 4 with complex amplitude \(\mathfrak{F}\). At terminal 3 there will be no output. We call this form of directional coupler a contra-directional coupler, because the output wave on the coupled line travels in the opposite direction to the incident wave. For a co-directional coupler there will be output at terminal 3 and none at terminal 4.

**Factor \(k\)**

The coupling factor \(k\rightarrow 1\) for tight coupling and from equation (110), it can be seen that the characteristic impedances of the coupled lines become large.

In case of loose coupling \(k\rightarrow 0\) and the characteristic impedances approach those of the connecting transmission lines.

The effective characteristic impedance could be reduced by the factor \(\sqrt{1-k^2}\) over the coupled length. By increasing \(Z_0\) of the coupled lengths by \(\frac{1}{\sqrt{1-k^2}}\), this effect will be compensated.
Propagation of waves on two parallel lossless lines are described by the following set of differential equations.

\[
\frac{\partial v_1}{\partial z} + L_1 \frac{\partial i_1}{\partial t} + \mu_m \frac{\partial i_2}{\partial t} = 0
\]
\[
\frac{\partial v_2}{\partial z} + L_2 \frac{\partial i_2}{\partial t} + \mu_m \frac{\partial i_1}{\partial t} = 0
\]
\[
\frac{\partial i_1}{\partial z} + C_{11} \frac{\partial v_1}{\partial t} - C_m \frac{\partial v_2}{\partial t} = 0
\]
\[
\frac{\partial i_2}{\partial z} + C_{22} \frac{\partial v_2}{\partial t} - C_m \frac{\partial v_1}{\partial t} = 0
\]

or,

\[
\frac{\partial}{\partial z} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{\partial}{\partial t} \begin{bmatrix} L_1 & L_m \\ L_m & L_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}
\]

\[
\frac{\partial}{\partial z} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \frac{\partial}{\partial t} \begin{bmatrix} C_{11} & -C_m \\ -C_m & C_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
\]
\( L_{11}, L_{22}, C_{11}, C_{22} \) are the self inductance and capacitance per unit length of lines 1 and 2 in the presence of lines 2 and 1 taking \( v_2, v_1, \) and \( i_2, i_1 \) zero respectively.

\( L_m \) is the mutual inductance per unit length.

\( C_m \) is the mutual capacity per unit length.

The fundamental solution for conductor voltages and currents assuming sinusoidal of a lossy or lossless line is

\[
[V] = [Z] [I]
\]

\([Z]\) is the characteristic Impedance Matrix.

For a two conductor line

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} =
\begin{bmatrix}
Z_{11} & Z_m \\
Z_m & Z_{22}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
\]

(112)

The entries of matrix \([Z]\) for a lossless line are purely resistive and independent of frequency.

For a lossless system,

\[
[V] = 60 [G] [I]
\]

\([G]\) = Geometry Matrix

\[
V_1 = Z_{11} I_1 + Z_m I_2
\]

\[
V_2 = Z_m I_1 + Z_{22} I_2
\]

\[
V_1 I_1 = Z_{11} I_1^2 + Z_m I_1 I_2
\]

\[
V_2 I_2 = Z_m I_1 I_2 + Z_{22} I_2^2
\]
\[ v_1 i_1 - v_2 i_2 = z_{11} i_1^2 - z_{22} i_2^2 = 0 \]

\[ \frac{v_2}{v_1} = \frac{i_1}{i_2} = \sqrt{\frac{z_{22}}{z_{11}}} = \sqrt{\frac{L_{22}}{L_{11}}} = \sqrt{\frac{C_{11}}{C_{22}}} \]

Equal power is associated with each line.

From eq. (15) and fig. (?)

\[ [V] = [P] [Q] \]

\[ [C] = [P]^{-1} \Sigma \epsilon_0 [Q]^{-1} \]

For two conductor configuration as shown in fig. (33)

\[ [C] = \frac{2 \pi \epsilon_0}{G_{11} G_{22} - G_{12}^2} \begin{bmatrix} G_{22} & -G_{21} \\ -G_{12} & G_{11} \end{bmatrix} \]

**Fig. 33. Two Conductors and Their Image Configuration**
From the geometrical configuration

\[ L_1 = 2 \times 10^{-7} \frac{\ln \frac{d_1}{a_1}}{H/m} \]

\[ L_{22} = 2 \times 10^{-7} \ln \frac{d_2}{a_2} \frac{H}{m} \]

\[ C_{11} = \frac{1}{D} \ln \frac{d_2}{a_2} \]

\[ C_{12} = C_{21} = -\frac{1}{D} \ln \frac{c}{b} \]

\[ C_{22} = \frac{1}{D} \ln \frac{d_1}{a_1} \]

Where

\[ D = 18.10^{11} \left\{ \ln \frac{d_1}{a_1} \ln \frac{d_2}{a_2} - \left( \ln \frac{c}{b} \right)^2 \right\} \]

The co-efficient of capacitive and inductive coupling is defined by\(^{31}\)

\[ k_L = \frac{L_1}{\sqrt{L_{11} \cdot L_{22}}} = \frac{2 \times 10^{-7} \ln \frac{c}{b}}{2 \times 10^{-7} \sqrt{\ln \frac{d_1}{a_1} \ln \frac{d_2}{a_2}}} \]

\[ k_C = \frac{C_{11}}{\sqrt{C_{11} \cdot C_{22}}} = \frac{\frac{1}{D} \ln \frac{c}{b}}{\sqrt{\frac{1}{D} \ln \frac{d_2}{a_2} \cdot \frac{1}{D} \ln \frac{d_1}{a_1}}} \]

\[ \kappa = k_C = k \]

\[ C_1 = \frac{2 \pi \varepsilon \ln \frac{d_2}{a_2}}{\ln \frac{d_1}{a_1} \ln \frac{d_2}{a_2} - \left( \ln \frac{c}{b} \right)^2} = \frac{c_1}{1-k^2} \]

\[ L_{..} = 2 \times 10^{-7} \ln \frac{d_1}{a_1} \frac{H}{m} = L_1 \]
Similarly, \[ C_{22} = \frac{C_2}{1-k^2}, \quad L_{22} = L_2 \]

and \( L_j \) \((j=1,2)\) are the self capacity and inductance of either line in the absence of the other.

The differential equations (iii) become

\[ \frac{\partial v_1}{\partial z} + L_{11}(1+k)\frac{\partial i_1}{\partial t} = 0 \]

\[ \frac{\partial i_1}{\partial z} + C_{11}(1-k)\frac{\partial v_1}{\partial t} = 0 \]

The velocity of propagation:

\[ \frac{1}{\sqrt{L_{11}C_{11}(1-k^2)}} \]

Assuming sinusoidal excitation and letting

\[ Z_{11} = j\omega L_{11}, \quad Y_{11} = j\omega C_{11}, \quad Z_{22} = j\omega L_2, \quad Z_m = j\omega L_m, \]

\[ Y_m = j\omega C_m, \quad Y_{22} = j\omega C_{22}, \]

These equations become

\[ \frac{d}{dx} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = - \begin{bmatrix} Z_{11} & Z_m \\ Z_m & Z_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \]

\[ \frac{d}{dx} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = - \begin{bmatrix} Y_{11} & -Y_m \\ -Y_m & Y_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \]

Differentiating

\[ \frac{d^2}{dx^2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Z_{11}Y_{11} - \frac{Z_mY_m}{Z_{11}Y_{11}} & Z_mY_m - Z_{11}Y_m \\ Z_mY_{11} - \frac{Z_mY_m}{Z_{22}Y_{22}} & Z_mY_m + Z_{22}Y_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \]
Similarly, equation for current can be written

\[
\frac{d^2}{dx^2} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Y_{11} \\ Z_{22} & Y_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

\[
\frac{d^2}{dx^2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \frac{d^2}{dx^2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} A_3 & B_3 \\ A_4 & B_4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

\[
A_1 = Z_{11} Y_{11} - Z_{m} Y_{m}, \quad A_2 = Z_{22} Y_{22} - Z_{m} Y_{m},
\]

\[
A_3 = Z_{11} Y_{11} - Z_{m} Y_{m}, \quad A_4 = Z_{22} Y_{22} - Z_{m} Y_{m},
\]

\[
B_1 = Z_{m} Y_{22} - Z_{22} Y_{m}, \quad B_2 = Z_{m} Y_{11} - Z_{22} Y_{m},
\]

\[
B_3 = Z_{m} Y_{11} - Z_{11} Y_{m}, \quad B_4 = Z_{m} Y_{22} - Z_{11} Y_{m},
\]

It can be seen that

\[
A_3 = A_1, \quad B_1 = B_2 \]

\[
A_2 = A_4, \quad B_2 = B_3
\]

Voltage and current equations becomes

\[
\frac{d^2}{dx^2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
\]

\[
\frac{d^2}{dx^2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} A_1 & B_2 \\ A_2 & B_1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

(118)
We have already seen

\[ L_{11} = 2.10^{-7} \ln \frac{d}{a_1} \quad C_{11} = \frac{2\pi e}{\ln \frac{d}{a_1}(1-k^2)} \]

\[ k = \text{co-efficient of coupling} \]

\[ L_{11} = 2.10^{-7} \ln \frac{d_2}{a_2} \quad C_{22} = \frac{2\pi e}{\ln \frac{d_2}{a_2}(1-k^2)} \]

\[ L_{11}C_{11} = L_{22}C_{22} \quad \text{therefore} \quad Z_{11}Y_{11} = Z_{22}Y_{22} \]

\[ A_1 = Z_{11}Y_{11} - Z_m Y_m \quad A_2 = Z_{22}Y_{22} - Z_m Y_m \]

Gives \[ A_1 = A_2 = A \]

\[ B_1 = Z_m Y_{22} - Z_{11}Y_m \quad k = \frac{Z_m}{\sqrt{Z_{11}Z_{22}}} \]

\[ B_1 = k\sqrt{Z_{11}Z_{22}}Y_{22} - Z_{11}k\sqrt{Y_{11}Y_{22}} = 0 \quad \text{since} \quad (Z_{11}Y_{11} = Z_{22}Y_{22}) \]

Similarly

\[ B_2 = Z_m Y_{11} - Z_{22}Y_m = k\sqrt{Z_{11}Z_{22}}Y_{11} - Z_{22}k\sqrt{Y_{11}Y_{22}} \]

\[ B_2 = k\sqrt{Z_{22}Y_{11}}(\sqrt{Z_{11}Y_{11}} - \sqrt{Z_{22}Y_{22}})^2 = 0 \]

The differential equations for voltages and currents become

\[ \frac{d^2}{dx^2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \]

\[ \frac{d^2}{dx^2} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \quad \text{where} \quad A = Z_{11}Y_{11} - Z_m Y_m \]
\[ A = Z_{11} \gamma_{11} \left( 1 - \sqrt{\frac{1 - k^2}{k Z_{11} Y_{11}^2}} \right) \]

\[ A = Z_{11} \gamma_{11} \left( 1 - \frac{k \sqrt{Z_{11} Y_{11}^2 - Z_{21} Y_{22}^2}}{k \sqrt{Z_{11} Y_{11}^2}} \right) \]

\[ A = Z_{11} \gamma_{11} \left( 1 - \frac{k^2 Z_{11} Y_{11}^2 + Z_{22} Y_{22}^2}{Z_{11} Y_{11}} \right) \quad \text{since} \quad (Z_{11} Y_{11} = Z_{22} Y_{22}) \]

\[ A = Z_{11} \gamma_{11} (1 - k^2) = Z_{22} \gamma_{22} (1 - k^2) \]

(120)

### 3.3 Derivation of Transmission and Coupling Factors

The values of transmission factor \( G \) and coupling factor \( F \) depend upon the length of the coupled lines, the phase change per unit length along the line and the coupling factor \( k \).

Oliver derived the expressions for \( G \) and \( F \) by considering a sequence of reflected and transmitted waves generated by a unit impulse input solution for the transmitted and coupled waves were quite complex and are

\[ G = \frac{\sqrt{1 - k^2}}{\left(\sqrt{1 - k^2 \cos \beta l + j \sin \beta l}\right)} \]

\[ F = \frac{j k \sqrt{Z_3}}{Z_1} \frac{\sin \beta l}{\left(\sqrt{1 - k^2 \cos \beta l + j \sin \beta l}\right)} \]

(121)

Another approach to find the expression for \( G \) and \( F \) is by solving transmission line equations with the help of boundary conditions. Results gave the same expressions as were derived by Oliver.
Consider the case of a long transmission line excited by a sinusoidal voltage and coupled to a very short line segment as shown in Fig. (34).

![Diagram of two lines coupled over a short distance]

**Fig. 34. Two lines coupled over a short distance**

A wave is travelling from left to right in line (1) and voltages and currents are induced in an elemental length $dx$ of line 2 terminated in impedances $Z_a$ and $Z_b$.

A wave travelling from source to load will induce a current and voltage given by

\[ i_c = C_m \frac{dv_1}{dt}, \quad v_L = L_m \frac{di_1}{dt} \]
Total voltage across $BB'$ will be

$$V_b = Z_b \left( \frac{Z_a Z_c}{Z_a + Z_b} - \frac{Z_b V_L}{Z_a + Z_b} \right)$$

$$V_b = \frac{Z_b}{Z_a + Z_b} \frac{\partial}{\partial t} \left( Z_a C_{n1} v_{11} - L_{n1} i_{11} \right) \quad (122)$$

This voltage will be zero

$$V_b = \frac{Z_a}{C_{n1}} \frac{i_{11}}{v_{11}} \quad (123)$$

Since we are taking an elemental length of line $(2)$ neglecting $L_{n1}$ and $C_{n1}$, the transmission line equation becomes

$$\frac{\partial v_{11}}{\partial z} = -L_{11} \frac{\partial i_{11}}{\partial t} = -j \omega L_{11} i_{11}$$

$$\frac{\partial i_{11}}{\partial z} = -C_{11} \frac{\partial v_{11}}{\partial t} = -j \omega C_{11} v_{11}$$

These equations are of the form

$$\frac{\partial v_{11}}{\partial z} = z i_{11}$$

$$\frac{\partial i_{11}}{\partial z} = y v_{11}$$

The solution from reference 31 for a two conductor line gives

$$\frac{v_{11}}{i_{11}} = \sqrt{\frac{L_{11}}{C_{11}}} \quad (124)$$

$$Z_a = \frac{L_a}{C_{n1}} \sqrt{\frac{C_{11}}{L_{11}}}$$
From Eq. (114)

\[
\begin{align*}
\kappa_L &= \kappa_C = k \quad \text{and} \quad \frac{L_{11}}{C_{11}} = \sqrt{\frac{L_{11}L_{22}}{C_{11}C_{22}}} \\
Z_a &= \sqrt{\frac{L_{11}L_{22}}{C_{11}C_{22}}} \sqrt{\frac{C_{11}}{L_{11}}} = \sqrt{\frac{L_{22}}{C_{22}}} 
\end{align*}
\] (125)

This is the characteristic impedance of line (2) in the presence of line (1).

The voltage and current at \( BB' \) will still be zero if the conductors \( AB \) and \( A'B' \) are extended an arbitrary distance to the left and then terminated in \( Z_a \). This follows since the induction in each added element section produces no voltages and current at its right hand terminations.

**Directional coupling for any length of coupled section on open wires:**

The directional coupling properties of a section of line which was small compared to the wavelength of the applied frequency so that coupling could be considered a lumped circuit phenomenon was previously examined.

Now consider two lossless transmission line systems which are coupled \(^{39}\) over any given length as shown in figure (35).

The coupling is constant over a length \( \ell \), if the connecting lines are of impedance

\[
Z_{01} = \frac{L_{11}}{C_{11}} = Z_A = Z_B = Z \quad \text{and are properly}
\]
Fig. 35. Two transmission lines lines coupled over a appreciable distance terminated. From previous results we should expect a wave incident on terminals a-a' to produce an output at b-b' and at c-c', but none at a-a'.

In order to analyze the behavior of this coupler for sinusoidal excitation, we start with the coupled transmission line equations (120) in the form

\[
\frac{d^2}{dx^2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Z_{11} \gamma_{11} (1-k^2) \\ Z_{12} \gamma_{12} (1-k^2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
\]

(126)

\[
\frac{d^2}{dx^2} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} Z_{11} \gamma_{11} (1-k^2) \\ Z_{12} \gamma_{12} (1-k^2) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}
\]

The final solution of these equations by applying boundary conditions has been solved in Appendix 'C'.

The voltages as a function of distance along the coupler are equations (141) and (142) as shown in Appendix 'C'.

\[
\frac{v_1}{v_s} = \frac{\sqrt{1-k^2} \cos \beta (l-x) + j \sin \beta (l-x)}{\sqrt{1-k^2} \cos \beta l + j \sin \beta l} \quad \text{(143)}
\]

for line (1)

\[
\frac{v_2}{v_s} = \frac{j k \sin \beta (l-x)}{\sqrt{1-k^2} \cos \beta l + j \sin \beta l} \quad \text{(144)}
\]

for line (2)
For Unit Amplitude Source:

The transmitted output at \( x = l \) is:

\[
G = \frac{\sqrt{1-k^2}}{\sqrt{1-k^2 \cos \beta l + j \sin \beta l}}
\]  

(145)

The coupled output

\[
F = \frac{ik \sin \beta l}{\sqrt{1-k^2 \cos \beta l + j \sin \beta l}}
\]  

(146)

At \( x=0 \), the equations for transmission factor \( G \) and the coupling factor \( F \) are same as derived by Oliver in equation (121).

Since the coupler is lossless

\[
G^2 + F^2 = 1
\]  

(147)

\[
\frac{F}{G} = \frac{ik \sin \beta l}{\sqrt{1-k^2}}
\]

Moduli and phases of \( G \) and \( F \) are given by

\[
|G| = \sqrt{\frac{1-k^2}{1-k^2 \cos^2 \beta l}}
\]

\[
|F| = \frac{k \sin \beta l}{\sqrt{1-k^2 \cos^2 \beta l}}
\]

(143)

Phase \( G = g = \tan^{-1} \left( \frac{-\sin \beta l}{\sqrt{1-k^2 \cos \beta l}} \right) \)

Phase \( F = f = \tan^{-1} \left( \frac{\sqrt{1-k^2 \cos \beta l}}{\sin \beta l} \right) \)
\[ g = f(\frac{\pi}{2}) \]

The function \( g \) lags \( f \) by 90°.

Coupling efficiency \( \frac{\text{Coupled Power}}{100} \cdot \frac{\text{Input Power}}{F^2} \cdot 100 \)  

From \( |\hat{r}| = \frac{k |\sin \beta l|}{\sqrt{1 - k^2 \cos^2 \beta l}} \)

Coupling efficiency increases with \( k \) and is a maximum for \( \sin \beta l = 1 \)

\[ \beta l = \frac{\pi}{2} \left( \lambda = \frac{2\pi}{\beta} \right) \]

\[ l = \frac{\pi \lambda}{2.2\pi} = \frac{\lambda}{4} \]  

It implies that when the length of the coupler is a quarter wave length or any odd multiple, the coupling efficiency is maximum.
3.4. Insulated Sky Wire Coupling

With the trend to progressively higher transmission line voltages, however, the complexity and cost of the PLC coupling have also become greater. Also saturation in the conventional PLC frequency spectrum namely 30 kHz to 300 kHz has been approached on some of the large power systems.

Consideration of other media of transmission for reliable and economical communication channels has therefore become necessary. Microwave has been only a partial answer because of the high cost per circuit where only a relatively small number of circuits are required.

The use of insulated overhead sky wires offers another method of overcoming some of the limitations associated with PLC. In this case the term, "sky wires" or "ground wires", is used to designate the small size conductors that are strung parallel to and above the power-carrying conductors of the transmission line. They are normally grounded at each tower and are for the purpose of providing protection from lightning. The method of insulating the wire consists of placing an insulator at the point of contact with the supporting structure. An insulation level sufficiently low to avoid affecting the lightning protection afforded by the sky wire is essential of course.

The use of sky wires is concerned with the economy of providing a medium for transmission of the communication
Fig. 36. Sky wire coupling

Fig. 37. Distributed capacitance and inductance between sky wire and phase conductors
circuit with a high degree of reliability. This should include
circuits for telephone, teletype, remote control, relaying,
telemetering, and data transmission systems.

There are two fundamental ways of coupling energy into
the phase wires. One is by magnetic coupling and the other is
by capacitive coupling as shown in figures 36 and 37.

From a study of Figs. 36 and 37 several things become
apparent:

1. Carrier current flow in the insulated section of sky wire
creates a magnetic field around it which induces a voltage
in the parallel phase wires.

2. Fig. 37(b) shows the distributed capacitance between the
static wire and the parallel phase conductors. The radio
frequency voltage on the static wire is coupled through the
capacitance to the phase wires.

The authors consider the sky wire as a lumped element
coupled capacitively or inductively with the power line.
This approach is correct provided the length of the coupling
element is a very small fraction of a wavelength and therefore,
the capacitive and inductive coupling is proportional to the
length of the element. The self and mutual values of inductances
and capacitances per unit length are then derived from classical formulas for a lossless or a lossy system. The sky wire
coupler is too long to be considered as a lumped parameter
circuit.
J. Reichard at Ontario Hydro, in 1964 began to study the practicability of using insulated overhead ground wires for carrier-transfer trip relaying. He concluded that a carrier channel using a single insulated sky wire is practicable and is as reliable as conventional PLC in the presence of line faults, so that it could be used for transfer trip relaying. Such an installation might be desirable to reduce the cost of EHV coupling equipment and to provide a simple means to utilize new frequencies on an overcrowded 230kV system without interfering with the existing coupling tuning and terminating devices need for channels already in service. During the past few years some utilities in U.S.A. and Europe have already begun or are planning to use thousands of miles of ground wires for communication telemetering and supervisory control.

A carrier channel using short insulated skywire sections at both ends also appears to be practicable. It has the same features as mentioned above except that it may require an increase in transmitter power. However, the installation costs are smaller than for a continuous-wire channel because drainage coils are eliminated and skywire insulation costs are considerably reduced.

In addition, this type of coupling installation may provide a practical means to bypass an intermediate station without the need for expensive coupling equipment. The performance of the experimental transfer trip relay channel using a single
ground wire on the test line was generally satisfactory and encourages further investigations.

There are some distinct advantages to a coupler of this type:

1. Simplified line trap installations and possible elimination of coupling capacitors in future EHV systems allow considerable saving.

2. The ground wire circuit is not subjected to the frequency selectivity that is imposed on power line carrier circuits by conventional line tuning and trapping equipment.

3. The insulated ground wire circuit will not be adversely affected by power switching which results in the simplification of maintenance problems.
Chapter 4

Power Line Carrier on Insulated Bundle Subconductors

In primary voltage transmission networks for 220kV and higher power levels, single conductors are being steadily superseded by bundle conductors consisting of metallo-
ically interconnected individual conductors. Efforts were made in England, Ireland, Australia, Japan and various other countries to equip primary high voltage transmission networks with multichannel telephone systems. Conventional PLC systems have become highly developed and at least 25,000 channels are in service throughout the world. However the advent of EHV transmission systems requiring the use of bundle conductors leads to the concept of intrabundle communication.

In intrabundle communication the carrier signals are injected into suitably insulated subconductors within a phase bundle. It was found\textsuperscript{9} that the insulated bundle subconductor (IBS) communication system presents some useful applications, not to be expected in conventional PLC installations. It is possible to satisfy almost any demand for large groups of communication channels operating over the bundle conductors of a 220kV or any other high voltage transmission network. When a carrier system is operated over such a bundled line consisting of
isolated individual conductors it will, on account of the relatively close spacing of the individual conductors, radiate very little signal power and pick up very little noise. This presents appreciable advantages in power line carrier communication with regard to sectionalization and the renewed assignments of the same frequencies.

The shortage of spectrum space is primarily due to three factors: the increasing demand for communication channels, the narrowness of the available frequency range and the severe restrictions as to the renewed assignments of the same carrier channels in the high voltage transmission network of interconnected power stations. Economic considerations usually restrict the assignment of radio frequency wave traps to only one or two conductors used for communications. No change in the operating state of the high-voltage transmission network will greatly effect attenuation on the communications sector. Owing to the capacitive coupling of the conductors, some of the radio frequency power of a carrier transmitter will flow over those phase conductors that are without wave traps into other branches of the network, so disallowing any renewed assignment of the same carrier channel. In recent years a further factor has been added. The faster the telemetering and data transmission systems are required to operate, the broader their transmission channel bandwidth must be. It is thus seen that very solid reasons exist for seeking new approaches to the problem of shortage
of carrier channels in high voltage transmission networks.

The requirements for additional speech channels and for increased speed in protection signaling, together with the trend towards digital data transmission, call for wider band communication channels. Experiments have shown that communication signals can be superimposed between suitably insulated subconductors of the same phase bundle using conventional line traps and coupling equipment operating at similar frequencies and bandwidth to PLC. A theoretical study of intra communication (I.C.) systems using a novel quarter wave-length coupling method and a different band, shows their potentialities to be much greater than for PLC by several orders of magnitude for some parameters.

The principle of PLC operation on IBS is shown in Figure (38). The subconductors are energized from the station bus through a centre tapped line trap and are separated from one another along the line by insulated spacers and suspension clamps.

![Diagram](image)

**Fig. 38. I.B.S. TRANSMITTER COUPLING TO TWO SUBCONDUCTOR BUNDLES**
The PLC signal is injected into the subconductors through a planned step up autotransformer and two coupling capacitors from a push pull matching filter network at the transmitter output. The coupling arrangement utilized at the receiving end is essentially of the same form.

Nakamura\textsuperscript{10} proposed that PLC coupling should be applied at a quarter-wavelength from the bus end of the bundle. This eliminates the need for a costly line trap, since the bus may be connected directly to the bundle. F.D. Pullen\textsuperscript{11} extended the work of Nakamura. He developed a coupling method for use with intrabundle communication which avoids using line traps and is cheaper and more efficient than the conventional method.
4.1. Simplified Analysis

Consider a twin bundle system for which the conductors of the bundle have been insulated from one another throughout the length.

![Diagram showing a twin bundle system with a quarter wavelength (λ/4) and impedance symbols including Z_{in}, Z_{ob}, Z_{L}, and PLC EQUIP.]

Fig. 20. TWIN BUNDLE CONDUCTOR WITH DIRECT INJECTION OF PLC SIGNALS

A carrier signal is injected at a distance \( l = \lambda/4 \) from the short circuited end.

The input impedance of a line of length \( l \) terminated in load impedance \( Z_L \) is given by (38)

\[
Z_1 = Z_{ob} \frac{Z_{ob} \tan \beta l + Z_L \tan^2 \beta l}{Z_{ob} + jZ_L \tan \beta l} \quad (151)
\]

Where \( Z_{ob} \) is the characteristic impedance of the line.

For a short circuit at load end, \( Z_L = 0 \)

\[
Z_1 = jZ_{ob} \tan \beta l \quad , \quad \beta = 2 \pi / \lambda \quad (152)
\]
A transmission line of length $\lambda/4$ short circuited at one end will appear as an open circuit at the other. The input impedance of such a coupler formed is a short circuited transmission line in parallel with the characteristic impedance $Z_{ob}$ of the line.

\[
Z_{in} = \frac{Z_{ob} \tan \phi l}{Z_{ob}^* \tan \phi l}
\]

\[
Z_{in} = \frac{jZ_{ob} \tan \phi l}{1 + j \tan \phi l}, \quad \frac{1 - j \tan \phi l}{1 + j \tan \phi l}
\]

\[
Z_{in} = \frac{Z_{ob} \tan \phi l (\tan \phi l + j)}{1 + \tan \phi l}
\]  \hspace{1cm} (153)

To find the power coupled:

\[
I = \frac{|V|}{|Z_{ob} + Z_{in}|}
\]

Power coupled:

\[
P = \left( \frac{|V|}{|Z_{ob} + Z_{in}|} \right)^2 \Re \{Z_{in} \} \quad (\Re \text{ stands for "real part of"})
\]

For maximum power: $Z_{ob} = Z_{in}^*$. (* indicates conjugate.)

The current $I$ for maximum power:

\[
I_{max} = \frac{|V|}{2\Re(Z_{ob})}
\]

\[
P_{max} = \frac{|V|^2 \Re(Z_{ob})}{(2\Re(Z_{ob}))^2} = \frac{|V|^2}{4\Re(Z_{ob})}
\]
\[ \frac{P}{P_{\text{max}}} = \frac{\frac{4 \text{Re}(Z_{ob} + Z_{in})}{|Z_{ob} + Z_{in}|^2}}{1 + \tan^2 \phi} \]  

(154)

\[ = \frac{(Z_{ob} + Z_{ob}^*) (Z_{in} + Z_{in}^*)}{|Z_{ob} + Z_{in}|^2} \]

(asterisk means conjugate)

Substituting the value of \( Z_{in} \) from equation (153)

\[ \frac{P}{P_{\text{max}}} = \frac{(Z_{ob} + Z_{ob}^*) (Z_{ob} \tan \phi + 2 \tan \phi)}{Z_{ob} (1 + \frac{\tan \phi}{\tan \phi + 4})^2} \]

\[ = \frac{2(Z_{ob} + Z_{ob}^*) (1 + \tan^2 \phi)}{Z_{ob} Z_{ob}^* (1 + 5 \tan^2 \phi + 4 \tan^2 \phi)} \]

\[ \frac{P}{P_{\text{max}}} = \frac{2(Z_{ob} + Z_{ob}^*) \tan^2 \phi}{Z_{ob}^*} \quad \frac{\tan^2 \phi}{(1 + 4 \tan^2 \phi)} \]

If \( Z_{ob} \) is real, \( \frac{Z_{ob} + Z_{ob}^*}{Z_{ob}^*} = 2 \)

\[ \frac{P}{P_{\text{max}}} = \frac{4 \tan^2 \phi}{1 + 4 \tan^2 \phi} = \frac{4}{1 + 4 \cot^2 \phi} \]  

(155)

\( P_{\text{max}} \) is the power coupled at \( f_0 \), the resonance frequency, when the source resistance of the carrier frequency generator is made equal to \( Z_{ob} \).

To obtain the bandwidth of such a coupled circuit, the 3dB points or half power points will be obtained from Eq.(155) when \( \cot \phi \) is

\[ \cot \phi = \pm 2, \phi = 26.5^\circ \]

Where \( \phi = \frac{\pi}{2} \times \frac{f}{f_0} \) or \( \frac{f}{f_0} = \frac{2 \times 26.5 \times \pi}{180} \text{ radians} \)

\[ \frac{f_{dB} - f}{f_0} = \frac{12.7}{18} \text{ gives} \]

Bandwidth = \( 2(f_0 - f) = 1.4f_0 \)
4.2. The Effect of Distributed Coupling

![Diagram of simplified circuit of intrabundle distributed coupling]

**Fig. 41 SIMPLIFIED CIRCUIT OF INTRABUNDLE DISTRIBUTED COUPLING**

Treating the signal conductors as a parallel wire coupler, the transmission line equations of this assumed lossless system can be written as:

\[
\begin{align*}
\begin{bmatrix}
V_3 \\
V_4
\end{bmatrix} &= \cos \beta l \begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} - j \sin \beta l \begin{bmatrix}
Z_c
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} \quad (156) \\
\begin{bmatrix}
I_3 \\
I_4
\end{bmatrix} &= -j \sin \beta l \begin{bmatrix}
Z_c
\end{bmatrix}^{-1} \begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} + \cos \beta l \begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} \quad (157)
\end{align*}
\]
\[
\begin{bmatrix}
 Z_02 \\
 Z_m \\
 Z_01
\end{bmatrix}
\]
\[
Z_0 = \begin{bmatrix}
 Z_02 & k \sqrt{Z_02 \cdot Z_01} \\
 k \sqrt{Z_02 \cdot Z_01} & Z_01
\end{bmatrix} = Z_{02} \begin{bmatrix}
 1 & ka \\
 ka & a^2
\end{bmatrix}
\]

where \( a = \frac{Z_{01}}{Z_{02}} \)

\[
Z_{in} = \frac{V_i}{I_1}
\]

From Fig. (41)
\[
V_3 = V_4 = (I_3 + I_4)Z_{02}
\]

Expanding equations (156) and (157) and noting that \( V_2 = 0 \)

and \( V_3 = V_4 \)

\[
V_3 = V_1 \cos \theta - jZ_{02} a^2 I_1 \sin \theta - jZ_{02} k a I_2 \sin \theta
\]

\[
V_3 = -jZ_{02} k a \sin \theta I_1 - j \sin \theta Z_{02} I_2
\]

\[
I_3 = -\frac{j \sin \theta^2 V_1}{Z_{02} a^2 (1 - x^2)} + I_1 \cos \theta
\]
\[
I_4 = \frac{-j \sin \theta l (1-k \sin \phi)}{Z_{02} a^2 (1-k^2)} + I_2 \cos \phi l
\]

Using (158),

\[
V_3 = \frac{-j V_l (1-k \sin \phi l)}{a^2 (1-k^2)} + I_1 Z_{02} \cos \phi l + I_2 Z_{02} \cos \phi l
\]

Equating (159) and (160),

\[
I_2 = \frac{-V_l \cos \phi l + I_1 Z_{02} \sin \phi l I_1 (ka-a^2)}{j Z_{02} \sin \phi l (ka-1)}
\]

From 160 and 161,

\[
-j Z_{02} k a \sin \phi l \quad I_1 - j \sin \phi l \quad Z_{02} I_2 = \frac{-V_l (1-k \sin \phi l)}{a^2 (1-k^2)} + \quad + I_1 Z_{02} \cos \phi l + I_2 Z_{02} \cos \phi l
\]

Substituting \( I_2 \) and evaluating \( \frac{V_l}{I_1} \),

\[
Z_{in} = \frac{V_l}{I_1} = Z_{02} a^2 (1-k^2) \sin \phi l
\]

\[
\left( \frac{(ka-1) (\cos \phi l + j \sin \phi l)}{(ka-a^2) (\cos \phi l + j \sin \phi l)} \right) \left( \frac{-j(1-k) \sin \phi l}{\cos \phi l a^2 (1-k^2) (\cos \phi l + j \sin \phi l)} + \right)
\]

After simplification,

\[
Z_{in} = Z_{02} a^2 (1-k^2) \sin \phi l
\]

\[
\left( \frac{m (\cos \phi l (1+1) + j \sin \phi l) e^{j(\Pi/2-\phi l)}}{(j \sin \phi l e^{j(\Pi/2-\phi l)} + m \cos \phi l)} \right) \left( \frac{a^2 (1-k) a^2 (1-k^2)}{(1-ak)^2} \right)
\]
\[ Z_{in} = Z_0 a^2 (1 - k^2) \sin \theta \left\{ e^{i \left( \frac{\pi}{2} - \beta \right)} \cos \beta \frac{1}{2} + j \sin \frac{1}{2} \right\} \]

The input impedance for a quarter wavelength coupler \( l = \frac{\lambda}{4}, \beta l = \frac{\pi}{2} \) at centre frequency \( f_0 \) is

\[ R_e = \frac{Z_0 \cdot a^2 (1 - k^2) \sin \theta}{j} = \frac{mZ_0 a^2 (1 - k^2)}{j} = -mZ_0 (1 - k^2) \]

To find the power coupled, consider the following network:

\[ V \quad \rightarrow \quad I \quad \rightarrow \quad Z_{in} \]

The current flowing is

\[ I = \frac{|V|}{|R_e + Z_{in}|} \]

The power coupled is

\[ P = \frac{|V|^2 Re[Z_{in}]}{|R_e + Z_{in}|^2} \]

For maximum power \( Z_{in} = \frac{R_e}{g} \), so

\[ P_{\text{max}} = \frac{|V|^2}{4R_e} \]
The relative power coupled is

\[
\frac{P}{P_{\text{max}}} = \frac{\text{Re} \left\{ Z_{1n} \right\}}{\text{Re} + Z_{1n}}
\]

\[
\frac{\text{Re}}{P_{\text{max}}} = \frac{2\pi \text{Re} \left\{ Z_{1n} \right\}}{\text{Re} + Z_{1n}^2}
\]

\[
\frac{P'}{P_{\text{max}}'} = \frac{2\pi (Z_{1n} + Z_{1n}^*)}{R_c + Z_{1n}^2} \quad (* \text{ indicates conjugate})
\]

4.3 Coupling Characteristics

4.3.1 Effect of Coupling Factor:

The analysis is presented here for a bundle of two subconductors having outside diameter \(2r=0.9\) inch \((0.075\) feet) average height \(h = 54\) feet, distance between two subconductors \(d = 18\) inches \((1.5\) feet)

![Diagram](image)

**Figure 2 Configuration of Single and Coupled Conductors**
For the simplified analysis where the effect of distributed coupling is neglected, Equation (155) gives

\[
\frac{\rho}{F_{\text{max}}} = \frac{4}{4 + \cot^2 \beta l}
\]

where \( \beta l \) at frequency \( f \) is \( \frac{\pi}{2} \cdot \frac{f}{f_0} \); \( f_0 \) is the principal resonance frequency at which the length of a coupler is a quarter wavelength. A plot of \( \rho/F_{\text{max}} \) and normalized frequency is shown in Figure (43) which gives a bandwidth of \( 1.4f_0 \).

When the effect of distributed coupling is taken into account, Equation (164) is used which can be written as

\[
\frac{\rho}{F_{\text{max}}} = \frac{2 \cdot \text{Re}\left\{ \frac{Z_{\text{in}}R_C}{R_C + Z_{\text{in}}} \right\}^2}{R_C + Z_{\text{in}}}
\]

where \( Z_{\text{in}} \) is given by Equation (162).

The analysis is conducted with coupling factor, diameter of coupling wire and spacing between subconductor and coupling wire as variables. Calculations are made with the help of a digital computer and curves are plotted for bandwidth versus coupling factor \( k \), diameter of signal conductor and spacing between subconductor and coupling wire as shown in Figures (44), (45) and (46).

With an increase in coupling factor, the bandwidth decreases appreciably. An increase in diameter of coupling wire results in a decrease of bandwidth, and as the spacing between the subconductor and coupling wire is increased, there is an increase in bandwidth.
Fig. 43 GRAPH BETWEEN RELATIVE POWER COUPLED AND NORMALIZED FREQUENCY

BANDWIDTH = 1.4 \xi_0
Fig. 44 EFFECT OF COUPLING FACTOR ON BANDWIDTH
CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

The natural node analysis allows one to estimate the propagation performance of PLC signals on power lines. The resolution of phase currents and voltages into natural nodes allows one to solve the propagation problem between two points of a uniform three phase line.

Modal theory of analysis does have the disadvantage of involving some rather sophisticated and difficult mathematics but in spite of this difficulty the results obtained are of substantial value in evaluating PLC on long distance HV transmission lines prior to construction. Modal analysis assists in the better understanding of how carrier signals propagate on a three phase transmission line. Application of matrices leads to a direct extension of classical theory of wave propagation on single conductor lines to a system of parallel conductors at constant height above ground.

Application of lossless modal analysis to propagation problems on single circuit horizontal three phase lines leads to meaningful results that have been found consistent with field measurements. In a lossless system with air as the insulating medium, there are two parameters defining the propagation of plane waves: surge impedance $Z_0$, and propagation velocity $v_0$ (equal to velocity of light). In a lossless horizontal line node current components are in the same ratio as the node voltage
components and the components, proportional to the columns of the transformation matrix M are not only linearly independent but also orthogonal. Hence, there is no exchange of energy among the natural modes of a loss-less line.

Practical lines are not loss-less but if the effects of imperfectly conducting ground and conductors are small the lossless line method is very helpful in analyzing line terminations, faults and transpositions. It has been seen that for a long PLC link, it is important that most of the input power appears in node 3 which can be achieved with the centre phase to outer phase coupling of the push pull type (figure 3(d)). Transposition introduces a reduction of 6db, since it acts as a natural code converter.

Phase to phase should be used for final installation because of the following advantages:

1. The expected noise level will be lower for phase to phase coupling than phase to ground.
2. The better possibility of carrier signals being received
even with one phase wire in trouble, when phase to phase coupling is used.

At a low carrier frequency, the cross-coupled arrangement gave a much lower attenuation. It was found to be the most advantageous.

The lossy system analysis is based on the diagonalizing of the functions of a complex matrix \([P]\) by means of the eigenvector matrix of \([P]\). The entries of matrix \([P]\) are complex and their values vary with frequency \(F\) and earth resistivity \(\rho\), hence the modal attenuation \(\alpha_m\) and phase propagation constants \(\beta_m\) also depend on \(F\) and \(\rho\). The modal transformations of voltages and currents on lossy lines are not power invariant because the voltages of one mode are not orthogonal to the currents of the remaining modes. There is no coupling between the current components of one mode and voltage components of other modes, hence, the modes are independent of each other.

Examination of effects of earth resistivity revealed that:

1. It increases substantially the self and mutual inductances and resistances per unit length of the line.
2. The attenuation of mode 3, the most useful mode for propagation of PLC signals is little affected by earth resistivity.
3. The attenuation of mode 2 decreases with increasing earth resistivity.
For the complete solution of the actual problem of wave propagation along a transmission system composed of an overhead wire parallel to the surface of the earth, the elements of the series impedance matrix must be corrected using Carson's correction factors, which produce satisfactory results.

It is hoped that the concept of natural modes will be more generally accepted as an efficient tool by power and communication engineers. This will advance the understanding of FLC problems and could lead to more efficient use of existing installations and to possible savings in the design of future installations.

Results of the theoretical investigations show that there is a good possibility of coupling carrier frequencies efficiently to power lines by means of an adjacent transmission line. The parallel wire coupler in either its high-loss broad-band form or its low-loss narrow-band form appears to be technically feasible. Its application would depend on the relative economics of line tap and capacitor costs compared with coupling wire and transmitter power costs and on the relative need for increased bandwidth or improved channel isolation.

The theoretical coupling for two coupled transmission lines is very wideband and has maximum efficiency for a transmission line which is a quarter of a wavelength at the carrier frequency. The directional coupler has been examined in theory as a means of coupling carrier signals onto transmission.
lines. It would appear to be an alternative to coupling methods requiring line traps which are costly, expensive and would at the same time provide more communication capacity.

The skywires properly terminated and insulated can likely be used for carrier current communication by insulating them from the towers and providing suitable transpositions. Because of their location, however, the coupling loss in most of the cases will be higher. There also be some problems encountered in feeding and properly terminating such a skywire coupler so as to maintain its properties over a wide bandwidth, while allowing it to carry out its primary function.

It has been shown how carrier channels can be operated over bundle conductors with isolated individual conductors. A special advantage lies in the fact that considerably more frequencies can be allocated and therefore a considerably larger number of communication channels can be established. Other advantages are:

1. In bundles of three and four subconductors the current is reduced to one-third and one-fourth and the centre-tapped line trap carrier only half the current carried by a line trap in conventional PLC coupling, thereby reducing the current rating and cost of line traps.

2. Since all subconductors in a bundle are at the same 60Hz potential, a step-up transformer inserted between them and the coupling capacitor results in less capacitance and hence
the cost for the same bandwidth as in conventional line to line coupling.

3. The performance of the line trap is not affected by the impedance of the station. Hence, the line trap can be designed as a complementary part of the coupling network resulting in lower attenuation and wider useful bandwidth.

4. The two coupling capacitors could be housed in a common porcelain shell since there is no large potential difference between the two capacitors stacks.

For the complete elimination of line traps, PLC coupling should be applied at a quarter wavelength from the bus end of the bundle. This method of coupling is cheaper and more efficient than the conventional method. Input impedance and power coupled has been calculated which gives a bandwidth of 1.4 $f_0$ where $f_0$ is the centre frequency. Examination of curves between coupling factor, spacing and diameter of coupling wire on bandwidth of such a coupler shows that bandwidth reduces with an increase in coupling and tends to a maximum when coupling is very weak and tends to zero for tight coupling. With the increase in diameter of coupling wire, bandwidth decreases and an increase of spacing results in an increase in bandwidth.

From the above study, it is recommended that further theoretical and experimental investigations on the possibility of the conventional method of coupling PLC signals should be carried out. Since for EHV transmission multiple bundle
conductors used, investigation is needed of coupling RLC signals on bundle of four and multiple conductors. Modal theory can be extended for a complete analysis of RLC signal propagation on lossy lines above lossy ground which is of great importance for further study.
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i) Theory of Modal Analysis


APPENDIX A

NATURAL MODE ANALYSIS

For a horizontal single circuit line with two symmetrical sky wires as shown in Fig. 8. The equation of phase voltages and currents are as given Eq. (20)

\[ Z_{11} I_1 + Z_{12} I_2 + Z_{13} I_3 + Z_{14} I_4 + Z_{15} I_5 = V_1 \]
\[ Z_{21} I_1 + Z_{22} I_2 + Z_{23} I_3 + Z_{24} I_4 + Z_{25} I_5 = V_2 \]
\[ Z_{31} I_1 + Z_{32} I_2 + Z_{33} I_3 + Z_{34} I_4 + Z_{35} I_5 = V_3 \]
\[ Z_{41} I_1 + Z_{42} I_2 + Z_{43} I_3 + Z_{44} I_4 + Z_{45} I_5 = 0 \]
\[ Z_{51} I_1 + Z_{52} I_2 + Z_{53} I_3 + Z_{54} I_4 + Z_{55} I_5 = 0 \]

Solving the last two equations for \( I_4 \) and \( I_5 \) in terms of \( I_1, I_2, I_3 \), and substituting in the first three equations

\[ I_4 = -\frac{1}{Z_{44}} (Z_{41} I_1 + Z_{42} I_2 + Z_{43} I_3 + Z_{45} I_5) \]
\[ I_5 = -\frac{1}{Z_{55}} (Z_{51} I_1 + Z_{52} I_2 + Z_{53} I_3 + Z_{54} I_4) \]
\[ I_4 = -\frac{1}{Z_{44}} \left( Z_{41} I_1 + Z_{42} I_2 + Z_{43} I_3 + Z_{45} I_5 \right) \]
\[ I_5 = -\frac{1}{Z_{55}} \left( Z_{51} I_1 + Z_{52} I_2 + Z_{53} I_3 + Z_{54} I_4 \right) \]
\[ I_4 = -\frac{1}{Z_{44} Z_{55} - Z_{44} Z_{55}} \left( I_1 Z_{43} Z_{54} - I_2 Z_{42} Z_{54} + I_3 Z_{41} Z_{54} \right) \]
\[
I_5 = \frac{Z_{14}}{Z_{54}Z_{45} - Z_{44}Z_{55}} \left( I_1 (Z_{51} - \frac{Z_{54}Z_{44}}{Z_{54}}) + I_2 (Z_{52} - \frac{Z_{52}Z_{45}}{Z_{52}}) + \\
+ I_3 (Z_{53} - \frac{Z_{55}Z_{44}}{Z_{55}}) \right) \\
+ \left( \frac{Z_{14}Z_{55}}{Z_{45}Z_{54} - Z_{44}Z_{55}} (\frac{Z_{41}Z_{55} - Z_{42}Z_{44}}{Z_{55}}) + \\
+ \frac{Z_{14}Z_{15}}{Z_{45}Z_{54} - Z_{44}Z_{55}} (\frac{Z_{55}Z_{44} - Z_{54}Z_{41}}{Z_{44}}) \right) \\
+ \left( \frac{Z_{14}Z_{55}}{Z_{45}Z_{54} - Z_{44}Z_{55}} (\frac{Z_{42}Z_{55} - Z_{45}Z_{42}}{Z_{55}}) + \\
+ \frac{Z_{14}Z_{44}}{Z_{45}Z_{54} - Z_{44}Z_{55}} (\frac{Z_{55}Z_{44} - Z_{54}Z_{41}}{Z_{44}}) \right) \\
\right) \\
\text{or: } A_1 + 3I_2 + CI_3 = V_1 \ \\
\text{Similarly: } 3I_1 + 2I_2 + BI_3 = V_2 \\
\text{and: } CI_1 + 3I_2 + AI_3 = V_3 \\
\text{where: } A = Z_{14} \frac{Z_{41}Z_{55} - Z_{42}Z_{55}}{Z_{45}Z_{54} - Z_{44}Z_{55}} + \\
+ \frac{Z_{15}}{Z_{45}Z_{54} - Z_{44}Z_{55}} (\frac{Z_{55}Z_{44} - Z_{54}Z_{41}}{Z_{44}}) \\
\]

\[ Z = z_{12} + \frac{z_{14}}{z_{45}z_{54} - z_{44}z_{55}} (z_{42}z_{55} - z_{45}z_{52}) \]

\[ C = z_{13} + \frac{z_{14}}{z_{45}z_{54} - z_{44}z_{55}} (z_{43}z_{55} - z_{45}z_{43}) + \]

\[ + \frac{z_{15}}{z_{45}z_{54} - z_{44}z_{55}} (z_{55}z_{44} - z_{54}z_{42}) \]

\[ D = z_{22} + \frac{z_{24}}{z_{45}z_{54} - z_{44}z_{55}} (z_{42}z_{55} - z_{45}z_{52}) + \]

\[ + \frac{z_{25}}{z_{45}z_{54} - z_{44}z_{55}} (z_{52}z_{44} - z_{54}z_{42}) \]

For simplicity in showing the relations between voltage and current, letters A, B, C and D are introduced. Diagonal terms A and D are self surge impedances, off diagonal terms B and C are the mutual impedances of the phase conductors. For a long reflection free single conductor line, relation between current \( I(0) \) flowing and voltage \( V(0) \) impressed by a high frequency generator is

\[ V(0) = Z_0 I(0) \]

Voltage and current follow exponential law with distance \( x \) from the generator:

\[ V(x) = V(0) e^{\alpha x}, \quad I(x) = I(0) e^{-\alpha x} \quad (23) \]
\( \alpha \) = attenuation in nepers /unit length and \( V(x) / I(x) = Z_0 = \text{constant} \)

for three conductor long reflection free line, relations between phase voltages and currents can be written as

\[
\begin{align*}
V_k^{(n)}(x) &= V_k^{(n)}(0) e^{-\alpha^{(n)} x} \\
I_k^{(n)}(x) &= I_k^{(n)}(0) e^{-\alpha^{(n)} x} \\
V_k^{(n)}(x) &= Z^{(n)} I_k^{(n)}(x)
\end{align*}
\] (24)

\( k = 1, 2, 3 \) designates the phase and \( n = 1, 2, 3 \) denotes the particular set corresponding to natural mode of propagation.

The phase currents and voltages are related by expression (24) and at the same time must satisfy the line equation (22):

\[
\begin{align*}
A I_1^{(n)} + B I_2^{(n)} + C I_3^{(n)} &= V_1^{(n)} = Z^{(n)} I_1^{(n)} \\
B I_1^{(n)} + D I_2^{(n)} + E I_3^{(n)} &= V_2^{(n)} = Z^{(n)} I_2^{(n)} \\
C I_1^{(n)} + D I_2^{(n)} + E I_3^{(n)} &= V_3^{(n)} = Z^{(n)} I_3^{(n)}
\end{align*}
\] (25)

In matrix form

\[
\begin{bmatrix}
A & B & C \\
B & D & E \\
C & D & A
\end{bmatrix}
\begin{bmatrix}
I_1^{(n)} \\
I_2^{(n)} \\
I_3^{(n)}
\end{bmatrix} =
\begin{bmatrix}
V_1^{(n)} \\
V_2^{(n)} \\
V_3^{(n)}
\end{bmatrix} =
\begin{bmatrix}
I_1^{(n)} \\
I_2^{(n)} \\
I_3^{(n)}
\end{bmatrix}
\] (26)
From this homogeneous set of equations can be obtained
\[(A - Z^{(n)})I_1^{(n)} + BI_2^{(n)} + CI_3^{(n)} = 0 \]
\[BI_1^{(n)} + (D - Z^{(n)})I_2^{(n)} + CI_3^{(n)} = 0 \tag{27} \]
\[CI_1^{(n)} + BI_2^{(n)} + (A - Z^{(n)})I_3^{(n)} = 0. \]

Matrix equivalent of equation (27) is
\[
\begin{bmatrix}
A - Z^{(n)} & B & C \\
B & D - Z^{(n)} & B \\
C & 0 & A - Z^{(n)}
\end{bmatrix}
\begin{bmatrix}
I_1^{(n)} \\
I_2^{(n)} \\
I_3^{(n)}
\end{bmatrix} = 0 \tag{28}
\]

Equation (28) has a solution defining the relative values of \(I_1^{(n)}, I_2^{(n)}, I_3^{(n)}\) only if the characteristic determinant is zero.

\[
\begin{vmatrix}
A - Z^{(n)} & B & C \\
B & D - Z^{(n)} & B \\
C & 0 & A - Z^{(n)}
\end{vmatrix} = 0
\]

Expansion gives
\[
\left((A - Z^{(n)}) - C\right)\left((D - Z^{(n)}) (A - Z^{(n)}) + B^2\right) = 0 \tag{29}
\]

Equation is cubic in \(Z^{(n)}\), roots of this equation are the eigenvalues.

For \((A - Z^{(n)}) - C\) for \(n = 2\), \(Z^{(2)} = A - C \tag{30}
\]

For \((D - Z^{(n)}) (A - Z^{(n)}) + B^2 = 0\)
\(Z^{(2)} = Z(n(A + C + D) + (D(A + C) - 2B^2) = 0 \)
\[ Z^n = \frac{(A+C+D) + \sqrt{(A+C+D)^2 - 4(2(A+C) - 2B^2)}}{2} \]

mode 3 \[ Z^{(3)} = \frac{A+C+D}{2} - \frac{\sqrt{\varepsilon^2}}{2} > 0 \]

mode 1 \[ Z^{(1)} = \frac{A+C+D}{2} + \frac{\sqrt{\varepsilon^2}}{2} > 0 \]

where \[ \varepsilon^2 = A^2 + 3B^2 + C^2 + D^2 + 2(AC - AD - CD) > 0 \]

For each value of \[ Z^{(n)} \], equation (32) the important ratios of mode (n) currents, and therefore voltages of various phase conductors are found by solving equation (27)

\[ (A-Z^{(n)})I_1^n + BI_2^n + CI_3^n = 0 \] (33)

\[ BI_1^n + (D-Z^{(n)})I_2^n + CI_3^n = 0 \] (34)

\[ CI_1^n + BI_2^n + (A-Z^{(n)})I_3^n = 0 \] (35)

Equation (33) \times (A-Z^{(n)}) - ((35) \times C) gives

\[ (A-Z^{(n)})^2 I_1^{(n)} + B(A-Z^{(n)})I_2^{(n)} - C^2 I_1^{(n)} - BCI_2^{(n)} = 0 \]

\[ I_1^{(n)} \left( (A-Z^{(n)} - C) (A-Z^{(n)} + C) \right) = I_2^{(n)}(B(C-A+Z^{(n)}) \right) \]

\[ \frac{I_1^{(n)}}{I_2^{(n)}} = \frac{B}{Z^{(n)} - A - C} \] (36)

From Eq. (36) \[ A+Z^{(n)} = \frac{-2B^2}{Z^{(n)} - D} \] (37)

\[ \frac{I_1^{(n)}}{I_2^{(n)}} = \frac{B(Z^{(n)} - D)}{2B^2} = \frac{Z^{(n)} - D}{2B} \]
Also $(34 \times \mathbf{C}) - (35 \times \mathbf{B})$ gives

\[
\frac{I_2(n)}{I_3(n)} = \frac{B(n-Z(n)-C)}{B(n-Z(n)+C)} \frac{(A-Z(n)+C)}{B(n-Z(n)-D)}
\]

\[
= \frac{Z(n-A-C)}{3} = \frac{I_2(n)}{I_1(n)} = \frac{2B}{Z(n-D)}
\]

From Eq. (24)

\[
V_k(n) = Z(n)I_k(n) \quad , \quad V_2(1) = Z(1)I_2(1)
\]

\[
V_1(1) = Z(1)I_1(1) \quad , \quad \frac{I_2(1)}{I_1(1)} = \frac{V_2(1)}{V_1(1)} \quad \text{and so on.}
\]

\[
\frac{I_2(1)}{I_1(1)} = \frac{I_2(1)}{I_3(1)} = \frac{V_2(1)}{V_1(1)} = \frac{V_2(1)}{V_3(1)} = \frac{Z(1)-A-C}{B} = \frac{2B}{Z(1)-D} = p
\]

(38)

\[
\frac{I_2(2)}{I_1(2)} = \frac{V_2(2)}{V_1(2)} = 0 \quad , \quad \frac{I_2(3)}{I_1(3)} = \frac{I_2(3)}{I_3(3)} = \frac{V_2(3)}{V_1(3)} = \frac{V_2(3)}{V_3(3)} = \frac{Z(3)-A-C}{B}
\]

(39)

Putting $Z(2) = A-C$ in Eq. (27)

\[
(A-A+C)I_1(2) + 3I_2(2) + CI_3 = 0
\]

\[
BI_1(2) + (D-A+C)I_2(2) + BI_3 = 0
\]

\[
CI_1(2) + 3I_2(2) + (A-A+C)I_3(2) = 0
\]

Solving

\[
\frac{I_3(2)}{I_1(2)} = \frac{V_3(2)}{V_1(2)} = -1 \quad , \quad \frac{I_2(2)}{I_1(2)} = \frac{V_2(2)}{V_1(2)} = 0 \quad , \quad I_2(2) = V_2(2) = 0
\]

(40)
For any problem the values of mode phase currents or voltages can be calculated from the actual currents.

\[ I_1 = I_1^{(1)} + I_1^{(2)} + I_1^{(3)} \]
\[ I_2 = I_2^{(1)} + I_2^{(2)} + I_2^{(3)} = pI_1^{(1)} + qI_1^{(3)} \] \hspace{1cm} \text{(4.1)}
\[ I_3 = I_3^{(1)} + I_3^{(2)} + I_3^{(3)} = I_1^{(1)} - I_2^{(2)} + I_1^{(3)} \]

In matrix form,

\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 \\
p & 0 & q \\
1 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
I_1^{(1)} \\
I_1^{(2)} \\
I_1^{(3)}
\end{bmatrix}
\hspace{1cm} \text{(4.2)}
\]

where

\[ p = \frac{I_2^{(1)}}{I_1^{(1)}} = \frac{I_2^{(1)}}{I_3^{(1)}} = \frac{V_2^{(1)}}{V_1^{(1)}} = \frac{V_2^{(1)}}{V_3^{(1)}} = \frac{Z^{(1)}}{B} - A - C \]

\[ q = \frac{I_2^{(3)}}{I_3^{(3)}} = \frac{I_3^{(3)}}{I_3^{(3)}} = \frac{V_2^{(3)}}{V_1^{(3)}} = \frac{V_2^{(3)}}{V_3^{(3)}} = \frac{Z^{(3)}}{B} - A + C \]

and

\[ \frac{I_2^{(1)}}{I_1^{(2)}} = \frac{V_2^{(1)}}{V_1^{(2)}} = -1 \hspace{1cm} \frac{I_2^{(2)}}{I_1^{(2)}} = \frac{V_2^{(2)}}{V_1^{(2)}} = 0 \]

Columns of equation (4.2) or eigenvectors can be normalized and the normalized modal transformation matrix \([ \hat{M} \]) with columns equal to eigenvectors of unity length is
\[
\hat{m} = \begin{bmatrix}
\frac{1}{\sqrt{2+p^2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2+q^2}} \\
\frac{3}{\sqrt{2+p^2}} & 0 & \frac{a}{\sqrt{2+q^2}} \\
\frac{1}{\sqrt{2+p^2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2+q^2}}
\end{bmatrix}
\]
APPENDIX B

MODELLING ANALYSIS OF TWO, THREE AND FOUR SUBCONDUCTORS BUNDLES

Two Subconductors Bundles:

\[
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{12} & Z_{22}
\end{bmatrix}
\]

\[
Z_{11} = Z_{22} = 60 \ln \frac{2H}{r}, \quad Z_{12} = 60 \ln \frac{\sqrt{(2H^2 + d^2)}}{d} = 60 \ln \frac{2H}{d}
\]

Fig. 22 BUNDLE OF TWO SUBCONDUCTOR
\[
[Z] = \begin{bmatrix}
A & \frac{1}{2}
\end{bmatrix}
\]

The modal impedance are,

\[Z(1) = A + B = 60 \ln \left( \frac{2H}{Rd} \right) \text{ ohms}\]

\[Z(2) = A - B = 60 \ln \frac{d}{r} \text{ ohms}\]

The eigen vector matrix is

\[
\begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\]

The normalized eigenvector matrix or modal transformation matrix is

\[
\mathbf{M} = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{bmatrix} = \begin{bmatrix}
0.5 & 0.5 \\
0.5 & -0.5
\end{bmatrix}
\]

(91)

**Bundle of Three Superconductors**

Surge impedance matrix of cyclically transposed bundle is of the form

\[
Z = \begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} \\
Z_{21} & Z_{22} & Z_{23} \\
Z_{31} & Z_{32} & Z_{33}
\end{bmatrix}
\]
\[ Z_{11} = Z_{33} = Z_{22} = A \approx 60 \ln \left( \frac{2h}{r} \right) \text{ ohms} \]
\[ Z_{12} = Z_{32} = Z_{13} = B \approx 60 \ln \left( \frac{2H}{d} \right) \text{ ohms} \]

\[ Z = \begin{bmatrix}
  A & B & B \\
  B & A & B \\
  B & B & A \\
\end{bmatrix} \]

Modal impedances are

\[ z^{(1)} = A + 2B \approx 60 \ln \left( \frac{2h}{rd^2} \right) \text{ ohms} \]

\[ z^{(3)} = z^{(3)} = A - B \approx 60 \ln \left( \frac{d}{r} \right) \text{ ohms} \]

**Fig. 23 Bundle of Three Subcon Ductors**

For each value of \( Z^{(n)} \), the ratios of node \((n)\), \(n = 1, 2, 3\) currents and voltages of various phase conductors can be found as shown in equation (43) of Appendix A.

Normalized modal transformation matrix for a three conductor bundle is

\[ \begin{bmatrix}
  \frac{1}{\sqrt{6}} & 0.5\sqrt{2} & \frac{1}{3}\sqrt{3} \\
  -\frac{1}{3}\sqrt{3} & 0 & \frac{1}{3}\sqrt{3} \\
  \frac{1}{\sqrt{6}} & -0.5\sqrt{2} & \frac{1}{3}\sqrt{3} \\
\end{bmatrix} \quad (92) \]
**Fig. 24. Bundle of Four Subconductors**

Surge impedance matrix of a bundle is

\[
\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} & Z_{14} \\
Z_{21} & Z_{22} & Z_{23} & Z_{12} \\
Z_{31} & Z_{23} & Z_{22} & Z_{12} \\
Z_{41} & Z_{13} & Z_{12} & Z_{11}
\end{bmatrix}
= \begin{bmatrix}
A & B & C & D \\
E & F & C & \text{ } \\
F & E & B & \text{ } \\
D & C & B & A
\end{bmatrix}
\]
From Eq. (21)

\[ Z_{kk} = \frac{60 \ln \frac{2h_k}{r_k}}{r_k} \]

\[ Z_{ij} = \frac{60 \ln \frac{d_{ij}}{d_{ij}}}{d_{ij}} \]

- \( h_k \) = height above ground
- \( r_k \) = geometric mean radius of conductor \( k \)
- \( d_{ij} \) = distance between conductor \( i \) and image of conductor \( j \)
- \( d_{ij} \) = distance between conductor \( i \) and \( j \)

\[ Z_{11} = A = 50 \ln \frac{2H+d}{r} \]

\[ Z_{12} = Z_{34} = B = 60 \ln \frac{2H}{d} \]

\[ Z_{13} = Z_{24} = C = 60 \ln \frac{4H^2+d^2}{2d} = 30 \ln \frac{4H^2+d^2}{2d^2} \]

\[ Z_{14} = D = 30 \ln \frac{2H^2+2d^2+2Hd}{d^2} \]

\[ Z_{22} = E = 50 \ln \frac{2H-d}{r} \]

\[ Z_{23} = F = 30 \ln \frac{2(2H^2+d^2-2Hd)}{d^2} \]

Eigenvalues of the matrix \( Z \) are the roots of the characteristic equation of the determinant.

\[
\begin{vmatrix}
A - \lambda & B & C & D \\
B & E - \lambda & F & C \\
C & F & E - \lambda & B \\
D & C & B & A - \lambda \\
\end{vmatrix} = 0
\]
Replacing $A - \lambda = A'$, $B - \lambda = B'$:

$$
\begin{vmatrix}
A' & B & C & D \\
B' & E & F & C \\
C & F & E' & B \\
D & C & B & A'
\end{vmatrix}
\quad = 0
$$

Adding elements of 4th row to 1st row and 3rd row to 2nd row:

$$
\begin{vmatrix}
A' + D & B + C & C + D & D + A' \\
B' + D & E + F & E' + F & F + B' \\
C & F & E' & B \\
D & C & B & A'
\end{vmatrix}
\quad = 0
$$

Subtract elements of column 4th from 4th and elements of column 2nd from 3rd column:

$$
\begin{vmatrix}
A' + C & B + C & 0 & 0 \\
B + C & E' + F & E' - F & B - C \\
C & E' & E' - F & B - C \\
D & C & C - B & A' - D
\end{vmatrix}
\quad = 0
$$

Expanding the determinant:

$$
(A' + D)(E' + F - (B + C)^2)((A' - D)(E' - F - (B - C)^2)) = 0
$$

$A' = \lambda - \lambda$, $E' = E - \lambda$

$$(A - \lambda)(E - \lambda)(E' - F - (B + C)^2) = 0$$

$$(A - \lambda)(E - \lambda)(E - F - (B - C)^2) = 0$$
\[ \lambda_{2} = \frac{1}{2} (A+D)+(E+F) \pm \sqrt{\left(\frac{(A+D)+(E+F)}{2}\right)^2 + 4(\delta+C)^2} \]
\[ \lambda_{3} = \frac{1}{2} (A-D)+(E-F) \pm \sqrt{\left(\frac{(A-D)+(E-F)}{2}\right)^2 + 4(\delta-C)^2} \]

Modal impedances \( Z^{(n)} \) are arranged in order of decreasing value.

\[ Z^{(1)} = \frac{1}{2} (A+D)+(E+F) + \frac{i}{2} \sqrt{\left(\frac{(A+D)+(E+F)}{2}\right)^2 + 4(\delta+C)^2} \]
\[ Z^{(2)} = \frac{1}{2} (A-D)+(E-F) + \frac{i}{2} \sqrt{\left(\frac{(A-D)+(E-F)}{2}\right)^2 + 4(\delta-C)^2} \]
\[ Z^{(3)} = \frac{1}{2} (A+D)+(E-F) - \frac{i}{2} \sqrt{\left(\frac{(A+D)+(E-F)}{2}\right)^2 + 4(\delta-C)^2} \]
\[ Z^{(4)} = \frac{1}{2} (A-D)+(E+F) - \frac{i}{2} \sqrt{\left(\frac{(A-D)+(E+F)}{2}\right)^2 + 4(\delta+C)^2} \]

Normalized modal components form the columns of the modal transformation matrix necessary for modal resolution and is of the form.

\[
\begin{bmatrix}
  m_{1}^{(1)} & m_{2}^{(1)} & m_{3}^{(1)} & m_{4}^{(1)} \\
  m_{1}^{(2)} & m_{2}^{(2)} & m_{3}^{(2)} & m_{4}^{(2)} \\
  m_{1}^{(3)} & m_{2}^{(3)} & m_{3}^{(3)} & m_{4}^{(3)} \\
  m_{1}^{(4)} & m_{2}^{(4)} & m_{3}^{(4)} & m_{4}^{(4)} \\
\end{bmatrix}
\]

(94)

Where

\[
\frac{m_{k}(n)}{\sqrt{(I_{1}^{(n)})^2 + (I_{2}^{(n)})^2 + (I_{3}^{(n)})^2 + (I_{4}^{(n)})^2}}
\]

For a practical bundle of four subconductors, the value of modal impedance \( Z^{(2)} \) is very close to that of \( Z^{(3)} \).
Modal analysis is simplified for a cyclically rotated bundle. Elements of self and mutual impedances in (94) are now
\[ A = E = 60 \ln \frac{2\pi r}{d} \]
\[ B = D = F = 60 \ln \frac{2\pi r + d}{d} \]
\[ C = 60 \ln \frac{2\pi + d}{d \sqrt{2}} \approx B - 20.8 \]

Modal impedances are calculated from
\[ z(1) = A + 2B + C \approx A + 3B - 20.8 \]
\[ z(2) = z(3) = A - C \approx A - B + 20.8 \]
\[ z(4) = A - 2B + C \approx A - B - 20.8 \]

Matrix \( \mathbf{\mu} \) of normalized modal impedance for this system is obtained in a similar way as for three conductor bundle and is
\[
\mathbf{\mu} = \begin{bmatrix}
0.5 & 0 & 0.5\sqrt{2} & 0.5 \\
-0.5 & 0.5\sqrt{2} & 0 & -0.5 \\
0.5 & 0 & -0.5\sqrt{2} & 0.5 \\
0.5 & -0.5\sqrt{2} & 0 & -0.5
\end{bmatrix}
\]
(97)
Coupled transmission line equations have been derived as

\[
\frac{d^2}{dx^2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = Z_{11} \gamma_{11} (1-x^2) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
\]

The solution to these equations can be obtained by taking into considerations positive and negative travelling waves existing on each line and directional coupling is obtained only by these two oppositely travelling waves adding in one direction to give a net energy flow and subtracting in the other direction to give no energy flow.

Assuming the solution of the form

\[
v_1 = V_1 e^{-\gamma x} + V_1' e^{\gamma x} \\
v_2 = V_2 e^{-\gamma x} + V_2' e^{\gamma x} \\
I_1 = I_1 e^{-\gamma x} + I_1' e^{\gamma x} \\
I_2 = I_2 e^{-\gamma x} + I_2' e^{\gamma x}
\]

(127)

The primed quantities \((V_1', V_2', I_1', I_2')\) represent the amplitudes of the negatively travelling waves while unprimed quantities \((V_1, V_2, I_1, I_2)\) represent the amplitudes of the positively travelling waves.

Differentiating Eq. (127) twice

\[
\frac{d^2 v_1}{dx^2} = \gamma^2 v_1
\]
\[
\frac{d^2v_1}{dx^2} = Z_1 Y_1(1-k^2)v_1
\]

\[
Y^2 = Z_1 Y_1(1-k^2) = Z_{22} Y_{22}(1-k^2)
\]

\[
Y = \sqrt{Z_1 Y_1(1-k^2)} = \sqrt{Z_{22} Y_{22}(1-k^2)} \quad (128)
\]

Since current and voltage of a given travelling wave are always related by the line parameters it is possible to get the current expressions in terms of the voltages.

From Eq. (111)
\[
\frac{dI_1}{dx} = -Y_{11}v_1 + Y_{12}v_2
\]

From Eq. (127)
\[
I_1 = I_1 e^{-\gamma x} + I_1' e^{\gamma x}
\]

Gives
\[
-I_1 e^{-\gamma x} + I_1' e^{\gamma x} = -Y_{11}(V_1 e^{-\gamma x} + V_1' e^{\gamma x}) + Y_{12}(V_2 e^{-\gamma x} + V_2' e^{\gamma x})
\]

Comparing co-efficients of \(e^{-\gamma x}\) and \(e^{\gamma x}\) gives
\[
I_1 = \frac{Y_{11}V_1}{\gamma} - \frac{Y_{12}V_2}{\gamma}
\]

\[
I_1' = -\frac{Y_{11}V_1'}{\gamma} + \frac{Y_{12}V_2'}{\gamma}
\]

Similarly
\[
I_2 = \frac{Y_{22}V_2}{\gamma} - \frac{Y_{21}V_1}{\gamma}
\]

\[
I_2' = \frac{Y_{22}V_2'}{\gamma} + \frac{Y_{21}V_1'}{\gamma}
\]

(129)
The set of equations becomes
\[\begin{align*}
V_1 &= V_1 e^{-\gamma x} + V_1 e^{\gamma x} \\
V_2 &= V_2 e^{-\gamma x} + V_2 e^{\gamma x} \\
l_1 &= \frac{Y_{11} V_1}{\gamma} - \frac{Y_{12} V_2}{\gamma} e^{-\gamma x} - \left(\frac{Y_{11} V_1}{\gamma} - \frac{Y_{12} V_2}{\gamma}\right) e^{\gamma x} \\
l_2 &= \frac{Y_{22} V_2}{\gamma} - \frac{Y_{12} V_1}{\gamma} e^{-\gamma x} - \left(\frac{Y_{22} V_2}{\gamma} - \frac{Y_{12} V_1}{\gamma}\right) e^{\gamma x}
\end{align*}\]

Substituting
\[P_{11} = \frac{Y_{11}}{\gamma}, \quad P_{22} = \frac{Y_{22}}{\gamma}, \quad \{P_m = \frac{Y_m}{\gamma}\}
\]
The equations become
\[\begin{align*}
V_1 &= V_1 e^{-\gamma x} + V_1 e^{\gamma x} \\
V_2 &= V_2 e^{-\gamma x} + V_2 e^{\gamma x} \\
l_1 &= (P_{11} V_1 - P_m V_2) e^{-\gamma x} - (P_{11} V_1 - P_m V_2) e^{\gamma x} \\
l_2 &= (P_{22} V_2 - P_m V_1) e^{-\gamma x} - (P_{22} V_2 - P_m V_1) e^{\gamma x}
\end{align*}\]

To find out the unknown constants, we apply the boundary conditions. Referring to Figure (24) at the end of line (1) at \(x = l\), the total voltage must equal the total current times the load impedance. On line (2) at \(x = l\) and \(x = r\), the total voltage must be total current times the load impedance.
At \( x=0 \), the voltage across line (1) is the source voltage minus the IR drop in the source impedance as

\[
v_1 \bigg|_{x=\ell} = Z_1 I_1 \bigg|_{x=\ell} \\
v_2 \bigg|_{x=\ell} = Z_{B12} \bigg|_{x=\ell} \\
v_2 \bigg|_{x=0} = -Z_{A12} \bigg|_{x=0} \tag{132} \\
v_1 \bigg|_{x=0} = V_s - R_s I_{11} \bigg|_{x=0}
\]

With the help of boundary conditions, equations (131) can be written as:

\[
e^{-\gamma \ell} (1-Z_{P11})V_1 + e^{+\gamma \ell} (1+Z_{P11})V_1' = e^{-\gamma \ell} Z_{Pm} V_2 - e^{+\gamma \ell} Z_{m} V_2 = 0 \\
e^{-\gamma \ell} Z_{Bm} V_1 - e^{+\gamma \ell} Z_{Bm} V_1' + e^{-\gamma \ell} (1-Z_{3P22})V_2 + e^{+\gamma \ell} (1+Z_{3P22})V_2' = 0 \\
-Z_{Ap} V_1 + Z_{Ap} V_1' + (1+Z_{A22})V_2 + (1-Z_{A22})V_2' = 0 \\
(1+i\delta P_{11}) V_1 - (1-i\delta P_{11}) V_1' = R_{m} V_2 + R_{m} V_2' = V_s \tag{133}
\]

For infinite directivity \( V_2 = 0 \) at the end of the line. It is necessary to have

\[
v_2 e^{-\gamma \ell} + V_2' e^{+\gamma \ell} = 0 \\
\frac{V_2'}{V_2} = e^{-2\gamma \ell}
\]

If the lines are assumed to be symmetrical \( Z_{01} = Z_{01} \), and the source impedance is equal to zero \( (R_s = 0) \) and

\[
Z_L = Z_{01} = Z_A = Z_3
\]
\[ P_{11} = \frac{Y_{11}}{Y} = \sqrt{Z_{11} Y_{11}(1-k^2)} \]

\[ P_{11} = \frac{Y_{11}}{\sqrt{Z_{11}^2(1-k^2)}} = \frac{Y_{11}}{Z_{11} \sqrt{1-k^2}} = \frac{1}{\sqrt{1-k^2} Z_{01}} \]

\[ P_{22} = \frac{Y_{22}}{Y} = \frac{Y_{22}}{\sqrt{Z_{22}^2(1-k^2)}} = \frac{Y_{22}}{Z_{22} \sqrt{1-k^2}} = \frac{1}{\sqrt{1-k^2} Z_{02}} \]

\[ P_m = \frac{Y_m}{Y} = \frac{k Y_{11} Y_{22}}{\sqrt{Z_{11}^2 Y_{11}(1-k^2)}} = \frac{k}{\sqrt{1-k^2} Z_{01}} \]

\[ Z_{01} = Z_{02} \text{ gives } \frac{Z_{11}}{Y_{11}} = \frac{Z_{22}}{Y_{22}} \]

and \[ Z_{11} Y_{11} = Z_{22} Y_{22} \text{ gives } \frac{Y_{11}}{Y_{22}} = \frac{Z_{11}}{Z_{22}} \]

After substituting these values, of \( Z_{11}, P_{11}, P_{22}, P_m, Z_0, Z_3, \)

\( \epsilon_3 \) and \( V_2 \) equation (133) can be simplified as follows:

\[ e^{-\gamma L} \left( \frac{1}{\sqrt{1-k^2}} \right) V_1 + e^{\gamma L} \left( 1 + \frac{1}{\sqrt{1-k^2}} \right) V_1^+ \left( e^{-\gamma L} \frac{\omega - \epsilon}{\sqrt{1-k^2}} + e^{-\gamma L} \frac{\epsilon}{\sqrt{1-k^2}} \right) V_2 = 0 \]

\[ e^{-\gamma L} \frac{\kappa}{\sqrt{1-k^2}} V_1 - e^{\gamma L} \frac{\kappa}{\sqrt{1-k^2}} V_1^+ \left( e^{-\gamma L} \frac{1 - \frac{1}{\sqrt{1-k^2}}} {\sqrt{1-k^2}} + e^{-\gamma L} \frac{1}{\sqrt{1-k^2}} \right) V_2 = 0 \]
\[ V_1 = V_2 = 0 \]

Replacing
\[ \frac{1}{\sqrt{1-a^2}} = a \]

Gives
\[ e^{-\tau l} (1-N) V_1 + e^{\tau l} (1+N) V_1' + 2e^{-\tau l} k\nu V_2 = 0 \]
\[ e^{-\tau l} k\nu V_1 - e^{\tau l} k\nu V_1' - 2e^{-\tau l} NV_2 = 0 \]
\[ -k\nu V_1 + k\nu V_1' + (1+N)(1-N) e^{-2} V_2 = 0 \]

We need three equations to solve for \( V_1, V_1', \) and \( V_2 \)

\[ a_1 V_1 + b_1 V_1' + c_1 V_2 = 0 \]
\[ a_2 V_1' + b_2 V_2' + c_2 V_2 = 0 \]
\[ a_3 V_1' + b_3 V_2' + c_3 V_2 = V_s \]

Where
\[ a_1 = e^{-\tau l} (1-N) \]
\[ b_1 = e^{\tau l} (1-N) \]
\[ c_1 = 2e^{-\tau l} k\nu \]

and so on.
\[
\begin{bmatrix}
  c_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 
\end{bmatrix}
\begin{bmatrix}
  v_1 \\
  v_2 \\
  v_3 
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  v_s 
\end{bmatrix}
\]  
(138)

After simplification
\[
\frac{v_1}{v_s} = \frac{1 + N(1-k^2)}{1 + N(1-k^2) - e^{-2\gamma \ell} (1+N(1-k^2))}  
\]  
(139)

We are interested in finding
\[
\frac{v_1}{v_s} \quad \text{and} \quad \frac{v_2}{v_s}
\]

Where \( v_1 \) and \( v_2 \) have already been defined in Eq. (131).

\[
v_1 = V_1 e^{-\gamma \ell} + V_1' e
\]

\[
v_2 = V_2 e^{-\gamma \ell} + V_2' e
\]

Assuming no losses \( \gamma = j\beta \)

\[
v_2 = V e^{-j\beta \ell} + V_2 e^{j\beta \ell}
\]

\[
\frac{v_2}{v_s} = \frac{k e^{-j\beta \ell}}{e^{-2\gamma \ell} (N(1-k^2)-1)+1+N(1-k^2)}
\]

\[
+ \frac{n_\ell e^{-j\beta \ell} e^{-2j\beta \ell}}{e^{-2\gamma \ell} (N(1-k^2)-1)+1+N(1-k^2)}
\]

\[
\frac{v_2}{v_s} = \frac{k(e^{-j\beta \ell} e^{-2j\beta \ell})}{N(1-k^2)(1+e^{-2\gamma \ell})+(1-e^{-2\gamma \ell})}
\]
After simplification

\[
\frac{v_2}{v_s} = \frac{2jk \sin \beta (L-x) (\cos \beta l - j \sin \beta l)}{2\left( \frac{1}{(1-k^2)^2} \cos \beta l + j \sin \beta l \right) (\cos \beta l - j \sin \beta l)}
\]

Since \( \frac{v}{v_s} = \frac{1}{\sqrt{1-k^2}} \)

\[
\frac{v}{v_s} = \frac{jk \sin \beta (L-x)}{\sqrt{1-k^2} \cos \beta l + j \sin \beta l}
\]  \hspace{1cm} (141)

From Eq. (127)

\[
v_1 = V_1 e^{-j \beta x} + v_1' e^{j \beta x}
\]

\[
v_1 = v_s \left( \frac{\left(1+N(1-k^2)\right) e^{-j \beta x}}{1+N(1-k^2) - e^{-2j k l} (1-N(1-k^2))} + \frac{e^{j \beta x} (1-N(1-k^2))}{1-N(1-k^2) - e^{2j k l} (1+N(1-k^2))} \right)
\]

After simplification

\[
\frac{v_1}{v_s} = \frac{\cos \beta (L-x) \sqrt{1-k^2} + j \sin \beta (L-x)}{\sqrt{1-k^2} \cos \beta l + j \sin \beta l}
\]  \hspace{1cm} (142)
VITA AUCTORIS

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