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Commentary on: Begoῆa Carrascal's "The practice of arguing and the arguments: Examples from mathematics"

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1. INTRODUCTION

For the last decade there has been a growing interest in the interplay between mathematical practice and argumentation. The study of each of these areas promises to shed light on the other, as I and several other authors from a variety of disciplines have argued. I am particularly grateful to Begoña Carrascal for her careful critique of some central assumptions of this programme, as such challenges are vital for its long-term success. In this commentary, I wish to respond to two of her main points in a similar spirit. She writes: "From a review of many of the papers [of the programme] … we can extract two main ideas. First, Johnson's influential definition placed a burden on many of their authors to justify the claim that mathematical products are argumentative. Second, there is a manifest tension in these works between the examples of mathematical products considered as arguments and the process that leads to them" (Carrascal, 2013, p. 6). I will address each of these ideas in turn.

2. THE CLAIM THAT MATHEMATICAL PRODUCTS ARE ARGUMENTATIVE

Carrascal correctly observes that many authors of works on mathematical argumentation have interpreted finished proofs as arguments. She holds that this is a mistake, or at least unnecessary; since the proving process is argumentative, the product need not be: "there is no need to appeal to special cases to defend the assertion that, in mathematical practice, there is a place for argumentation. We only have to distinguish between mathematical products and mathematical practice" (Carrascal, 2013, p. 6). It is on this point that I wish to challenge her. I agree that we do not need to appeal to special cases—because ordinary, rigorous proofs are arguments. The proving process is also argumentative, although we should expect different sorts of argument in these two different contexts.

A preliminary point is that we should absolve Ralph Johnson of any responsibility for the choice supporters of the programme have made to focus on showing that proofs are arguments. Although both Ian Dove and I have published rebuttals of Johnson's view that proofs are not arguments (Dove, 2007; Aberdein, 2011), both of us had already defended the contrary view without reference to Johnson (Dove, 2003; Aberdein, 2005). And other authors Carrascal discusses make

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no reference to Johnson in any of the works she cites (for example Alcolea Banegas, 1998).

Much more importantly, our identification of proofs as arguments is firmly grounded in mathematical practice: mathematicians themselves frequently speak of proofs as arguments, and do not perceive any tension with the rigour of proofs in so doing. For example, “A proof is a rhetorical device for convincing another mathematician that a given statement (the theorem) is true. Thus a proof can take many different forms. The most traditional form of mathematical proof is that it is a tightly knit sequence of statements linked together by strict rules of logic” (Krantz, 2011, pp. 11 f.). Other authors go further, and challenge whether this alleged tradition really is a tradition. For example, Alan Bundy observes that “Prior to the invention of formal logic, a proof was any convincing argument. Indeed, it still is. Presenting proofs in Hilbertian[1] style has never taken off within the mathematical community. Instead, mathematicians write rigorous proofs, i.e. proofs in whose soundness the mathematical community has confidence, but which are not Hilbertian” (Bundy et al., 2005, p. 2377). Bundy observes that Hilbertian proofs may be easily, even mechanically, checked for errors and yet mistakes in rigorous proofs often go unobserved for years. Thus, he concludes, rigorous proofs are not Hilbertian (Bundy et al., 2005, p. 2378).

Many different terminologies have been employed for Bundy’s rigorous/Hilbertian distinction (for a partial list, see Reid & Knipping, 2010, p. 26). For example, Keith Devlin characterizes the distinction as follows: “The right wing (‘right-or-wrong’, ‘rule-of-law’) definition is that a proof is a logically correct argument that establishes the truth of a given statement. The left wing answer (fuzzy, democratic, and human centered) is that a proof is an argument that convinces a typical mathematician of the truth of a given statement” (Devlin, 2003). The first thing to notice here is that Devlin uses ‘argument’ to characterize *both* wings: his is not an opposition between proofs and arguments, but between two sorts of argument that a proof can be. However, where Devlin (and many others) err is in regarding this as an either/or distinction. A faithful and exclusive adherence to the left wing may satisfy Devlin politically, but mathematics needs both wings.

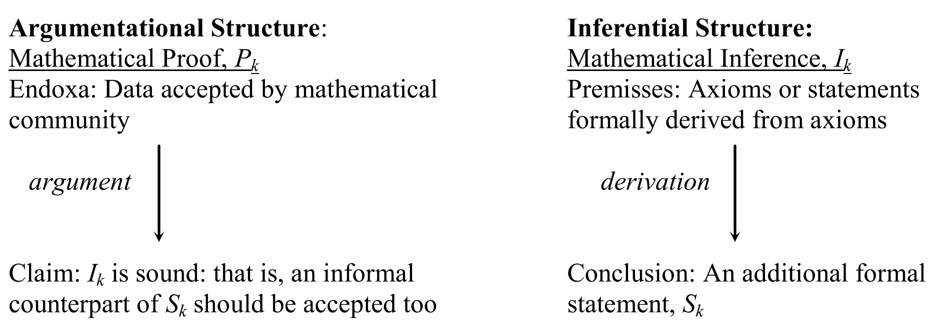

Figure 1: The parallel structure of mathematical reasoning (Aberdein, 2013, p. 363)

[1] Bundy defines a Hilbertian proof as “a sequence of formulae each of which is either an axiom or follows from earlier formulae by a rule of inference” (op. cit.).

In various places (most extensively, Aberdein, 2013) I have defended an account of mathematical justification as parallel in structure: see Fig. 1. The inferential structure is comprised of deductively valid derivations of formal statements from other formal statements: Hilbertian proof, in Bundy's terminology. It is the existence of this structure that underpins the uniquely compelling nature of mathematical results. However, it is not what mathematicians actually do, as most of them freely acknowledge (Krantz, Bundy and Devlin are all mathematicians; for even more examples, see Aberdein, 2013, pp. 361 f.). Instead they reason in a much looser fashion, but in such a way that they are able to become convinced that the derivations of the inferential structure exist. It is this reasoning, which I call the argumentational structure, which comprises the content of ordinary, rigorous mathematics.

If exceptional cases have attracted undue attention it is because many accounts of mathematical reasoning struggle to accommodate them. The parallel structure depicted in Fig. 1 is loose enough to encompass the exceptional cases which Carrascal mentions—unsurveyably long proofs, diagrammatic proofs, contested axioms—and others which she does not—computer-assisted proof, experimental proof, probabilistic methods. However, this should not distract from the fact that it is first and foremost an account of ordinary, rigorous mathematics.

3. THE TENSION BETWEEN MATHEMATICAL PRODUCTS AND THE MATHEMATICAL PROCESS

In this section I wish to alleviate "the tension between the practice and the products of this practice" that Carrascal discerns in works on mathematical argumentation (Carrascal, 2013, p. 7). I will show that the distinction discussed in the last section is a distinction between different types of justification and thereby between different senses of proof-as-object, not proof-as-process. It follows that both aspects of proof are argumentative. Then I will argue that once we appreciate that, we may understand the two aspects as working in concert, not in tension.

The distinction between inferential and argumentational structure is a distinction between propositional and doxastic justification, respectively (on this distinction, see Kvanvig & Menzel, 1990, p. 235; Klein, 2007, p. 6). Propositional justification is a relationship between an individual and a proposition which they may or may not believe: "We can say that a proposition, *h*, is *propositionally justified for S* just in case there is an *epistemically adequate basis* for *h* that is available to S regardless of whether S believes that *h*, or whether S is aware that there is such a basis, or whether if S believes that *h*, then S believes *h* on that basis" (Klein, 2007, p. 6). However, it is doxastic justification that is required for knowledge. This relates the individual to a belief state, not a proposition: "A belief that h is *doxastically justified for S* when and only when S is acting in an epistemically responsible manner in believing that *h*" (ibid.). In mathematics, propositional justification may be Hilbertian in character, but doxastic justification is not. Rather, it requires ordinary rigour, which is inherently argumentative: it is the means by which mathematicians persuade their peers of their results.

To consider a recent mathematical example, Yitang Zhang's proof that there are infinitely many pairs of primes that differ by no more than 70 million proceeds "not via a radically new approach to the problem, but by applying existing methods with great perseverance" (Klarreich, 2013). As one leading number theorist observed, "The big experts in the field had already tried to make this approach work … he succeeded where all the experts had failed" (cited in Klarreich, 2013). That is, an epistemically adequate basis for Zhang's result was available to those "big experts", so it was propositionally justified for them. However, since they could not see how to construct the proof, it was not doxastically justified for them. Another recent mathematical example provides a contrast: Shinichi Mochizuki's claimed proof of the ABC conjecture introduces so many new techniques and concepts that other leading mathematicians in the field describe it as like "reading a paper from the future, or from outer space" and as "very, very weird" (cited in Chen, 2013). If Mochizuki has proved the result, it is propositionally (but presumably not doxastically[2]) justified for him, but it's not yet propositionally justified for other mathematicians. The words "proof" and "proving" are used in lots of different ways and the Hilbertian/rigorous distinction which I have drawn attention to is only one of many possible distinctions. One study of mathematics education research identifies eight different, overlapping senses (Reid & Knipping, 2010, p. 33). Carrascal focuses on a distinction between process and product. Several observations may be made about her use of this distinction. Firstly, this distinction is familiar to argumentation theorists from various sources (for example, Habermas, 1984, p. 18; Pinto, 2001, p. 119). It is also familiar to philosophers of mathematical practice, as the distinction between context of discovery and context of justification: "the thinker's way of finding this theorem and his way of presenting it before a public", as the originator of the distinction illustrated it (Reichenbach, 1938, p. 5). As Carrascal observes, the proving process comprises the discovery of a proof by which the result is justified (Carrascal, 2013, p. 7). However, the identification of the proof as the "product" of the proving process begs the question as to the exact nature of proofs. This point has been made by Geoff Goddu with respect to the corresponding distinction in the sense of 'argument' (Goddu, 2011, p. 87). Rather than distinguishing proof-as-process and proof-as-product we might rather distinguish proof-as-process and proof-as-object.

Each of proof-as-process and proof-as-object could be subdivided further. Indeed, they correspond to respectively five and two of Reid and Knipping's senses of proof (Reid & Knipping, 2010, p. 33). Crucially, the distinction discussed in the last section, between Hilbertian and rigorous proof (or inferential and argumentational structure, or propositional and doxastic justification) is a subdivision of proof-as-object. Conversely, both proof-as-process and proof-as-object may be subsumed under larger headings. In the terminology of the sociologist Erving Goffman, as appropriated by the mathematician Reuben Hersh, the former belongs to the "back" of mathematics: "mathematics as it appears among working

[2] Mochizuki would not be doxastically justified if, as seems plausible, "acting in an epistemically responsible manner" includes successfully explaining the proof to others and he has not yet done this.

mathematicians, in informal settings, told to one another in an office behind closed doors"; the latter to the "front" of mathematics: "mathematics in 'finished' form, as it is presented to the public in classrooms, textbooks, and journals" (Hersh, 1991, p. 128). Philosophy of mathematical practice needs to pay attention to the front and the back, and thereby to both proof-as-process and proof-as-object, and especially to their interaction. Mechanisms devised in argumentation theory may be crucial to the last of these. For example, Chris Reed and Douglas Walton have argued that argumentation schemes represent a point of contact between the two views (Reed and Walton, 2003).

4. CONCLUSION

In conclusion, we have seen that argumentation theory can make a critical contribution to three projects in the philosophy of mathematical practice: analysis of proof-as-process, analysis of proof-as-object, and analysis of the relationship between the two. Carrascal is correct to insist on the importance of distinguishing the projects, and she offers many valuable insights into how the first project should proceed, but she is wrong to suppose that argumentation theory has nothing to say about the other two.

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