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Analysis of anisotropic sandwich structural systems.

I. M. Ibrahim

University of Windsor

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ANALYSIS OF ANISOTROPIC SANDWICH STRUCTURAL SYSTEMS

BY

I. M. IBRAHIM

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES THROUGH THE DEPARTMENT OF CIVIL ENGINEERING IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY AT THE UNIVERSITY OF WINDSOR

THESIS ADVISOR: DR. G. R. MONFORTON

WINDSOR, ONTARIO, CANADA

1973
ANALYSIS OF ANISOTROPIC SANDWICH STRUCTURAL SYSTEMS

ABSTRACT

by

I. M. IBRAHIM

The analysis of anisotropic sandwich structural systems is presented. The structural systems considered are:

1 - Rectangular sandwich plates under transverse loads.
2 - Sandwich cylindrical shell structures.
3 - Sandwich folded plate structures.

The faces of the sandwich systems considered are isotropic, orthotropic or anisotropic and transversely heterogeneous, typical of thin laminated skins (i.e. thin laminated plates and shells). The finite strip method is extended to the analysis of anisotropic sandwich structural systems. Furthermore, a double Fourier-series approach is derived for anisotropic sandwich plates with various boundary conditions. Generally, the formulations are presented in terms of the geometry, the stiffnesses of the faces (bending, membrane and coupling), and the transverse shear stiffnesses of the orthotropic core. Thus, by considering a single face of the sandwich system the methods of analysis are conveniently specialized for predicting the behaviour of thin anisotropic
and transversely heterogeneous structural systems such as those constructed from filamentary composites. In the finite strip approach, the geometric admissibility conditions of the principle of the minimum total potential energy are conveniently satisfied by representing the displacement variables in terms of assumed displacement patterns formed by the sum of products of one-dimensional third and fifth order interpolation polynomials and undetermined displacement coefficients at the two longitudinal edges of the strip, and basic functions which satisfy the force and displacement boundary conditions along the two ends of the strip. The coupling phenomenon between bending and in-plane extension (membrane action) in the unbalanced laminated faces is considered in the analysis. The effect of such a phenomenon on the behaviour of the sandwich systems is studied for various core and face thicknesses, aspect ratios (in case of plates), core and face elastic properties, and lay-ups of the various layers in the faces. Several numerical examples are presented which illustrate the potential of the methods of analysis for predicting the structural behaviour of thin and sandwich anisotropic structural systems.
ACKNOWLEDGEMENTS

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LIST OF SYMBOLS

\( A_{ij}, \ B_{ij}, \ D_{ij} \)  
membrane, coupling and bending stiffnesses of faces, respectively

\( [A]^{(f)}, \ [B]^{(f)}, \ [D]^{(f)} \)  
array containing the membrane, coupling and bending stiffnesses of the faces

\( a_{n}, b_{n}, c_{m}, d_{m}, e_{n}, f_{n} \)  
undetermined coefficients associated with double Fourier series analysis

\( B_{44}, \ B_{55} \)  
core transverse shear stiffness

\( B \)  
width of thin and sandwich plates

\( b \)  
width of sandwich strip

\( \{C\} \)  
vector of undetermined coefficients associated with double Fourier series analysis

\( c \)  
subscript, denoting core

\( \{D\} \)  
vector containing the undetermined edge displacement coefficients of a strip

\( \{D_u\}, \ \{D_v\}, \ \{D_w\} \)  
vector containing the undetermined edge displacement coefficients of a strip for u, v and w displacement components respectively

\( \{\bar{D}\}, \ \{\bar{D}_u\}, \ \{\bar{D}_v\}, \ \{\bar{D}_w\} \)  
vector containing the undetermined edge displacement coefficients of a strip in the reference coordinate system

\( D_{ij}^* \)  
reduced bending stiffnesses

\( d_f \)  
distance from skin reference surface to interface between core and skin
\( E^{(i)}_{11}, E^{(i)}_{22} \) elastic moduli of the \( i^{th} \) lamina

\( [F]^{(i)} \) coefficient matrix represents the elastic constants associated with the \( i^{th} \) lamina

\( f \) subscript or superscript denoting face, \( f = 1, 2 \)

\( G_{xz_c}, G_{yz_c} \) transverse shear moduli of the core

\( [G_c] \) array containing the transverse shear moduli of the core

\( [g] \) array associated with double Fourier series analysis

\( H^{(1)}_{kj}(y_f), H^{(1)}_{kj}(\theta) \) one-dimensional third order interpolation polynomials

\( H^{(2)}_{kj}(y_f) \) one-dimensional fifth order interpolation polynomials

\( [K] \) master stiffness matrix

\( [K^{(n,n)}] \) master stiffness matrix associated with the \( n^{th} \) cycle

\( [K_S] \) strip stiffness matrix

\( [K_S^{(n,n)}] \) partitioned strip stiffness matrix associated with the \( n^{th} \) cycle

\( [K_S^{(n,m)}] \) coupled partitioned strip stiffness matrix

\( \bar{K} \) master stiffness matrix in the reference coordinate system

\( \tilde{K}_S \) strip stiffness matrix in the reference coordinate system

\( L \) length of the structure between end supports

\( M_x, M_y, N, M_{xy}, M_{x\phi} \) moment resultants

\( \{M\} \) vector of moment resultants
applied moments on faces
force resultants
vector of force resultants
applied forces on faces
total number of cycles
work equivalent load vector
vector containing the work equivalent loads of a strip
vector containing the work equivalent loads in the reference coordinate system
intensities of the dead and live loads for shell structures
applied, surface loads
transverse shear stress resultants for core (s=c) and faces (s=f=1,2)
applied transverse forces on faces
applied transverse load
reference surface radius of curvature for core (s=c) and faces (s=f=1,2)
coefficient matrix in the double Fourier series analysis
subscript, denoting core (s=c) and faces (s=f=1,2)
reference surface area of the faces
middle surface area of the core
thickness of faces (s=f=1,2) and core (s=c)
transformation matrix
$U_f$ strain energy of faces

$U_c$ strain energy of core

$U(i)$ total strain energy of the $i^{th}$ strip

$u, v, w$ displacement components in $x, y, z$ directions

$\bar{u}, \bar{v}, \bar{w}$ displacement components in the reference coordinate system

$\bar{u}_c, \bar{v}_c, \bar{w}_c$ displacement components of point at distance $z_c$ from the middle surface of the core

$V_f$ volume of a face of a sandwich strip

$V_c$ volume of the core of a sandwich strip

$W(i)$ external work for the $i^{th}$ strip

$w_c/w_o$ ratio of the center deflection of sandwich plate with cross-plied faces to that of the equivalent sandwich plate with orthotropic faces

$X(x), X(n)$ basic functions

$X_u, X_v, X_w$ basic functions associated with $u, v, w$ displacement components respectively

$\{X\}$ vector of independent degrees of freedom

$\{\bar{X}\}$ vector of independent degrees of freedom in the reference coordinate system

$x_s, y_s$ distance measured on reference surface in cartesian or cylindrical coordinates

$z_s$ distance measured normal to the reference surface

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\[ \gamma_{xz_c}, \gamma_{yz_c} \]

\{\gamma_c\}

\[ \Delta \theta \]

\[ \varepsilon_{x_f}, \varepsilon_{y_f}, \varepsilon_{xy_f} \]

\[ \varepsilon_{x_f}, \varepsilon_{y_f}, \varepsilon_{xy_f} \]

\[ \{\varepsilon_f\}, \{\varepsilon_f^o\} \]

\[ \varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \varepsilon_{12}^{(i)} \]

\[ \varepsilon_x^{(i)}, \varepsilon_y^{(i)}, \varepsilon_{xy}^{(i)} \]

\[ \{\varepsilon_{12}^{(i)}\}, \{\varepsilon_{xy}^{(i)}\} \]

\[ \kappa_{x'}, \kappa_{y'}, \kappa_{xy} \]

\{\kappa\}

\[ \nu \]

\[ \pi_p^{(i)} \]

\[ \pi_p \]

\[ \sigma_x', \sigma_y', \sigma_{xy} \]

\{\sigma\}

\[ \tau_{xz'}, \tau_{yz} \]

\{\tau\}

\[ \phi, \psi \]

core transverse shear strains

vector of core transverse shear strains

contained angle of cylindrical shell strip

general normal and shear strains of faces

reference surface normal and shear strains of faces

vector of general and reference surface strains of faces, respectively

normal and shear strains of the \( i \)th lamina

vector of normal and shear strains of the \( i \)th lamina

curvatures

vector of curvatures

poisson's ratio

potential energy of the \( i \)th strip

total potential energy

normal and shear stresses

vector of stresses

transverse shear stresses

vector of transverse shear stresses

rotations of normal to reference surface

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parameters for basic functions

parameters associated with double Fourier series analysis

cylindrical coordinate

angle between the \( k \)th folded plate and the reference coordinates

angle measured from the longitudinal edges of shell

angle between the longitudinal edges and crown of shell

\( \lambda, \lambda', \alpha_n \)

\( \chi_m, \eta_m, \gamma_n, \beta_n \)

\( \theta \)

\( \theta_k \)

\( \phi \)

\( \delta \)

\( \phi_e \)
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CHAPTER I
INTRODUCTION

The term "sandwich structural system" generally refers to a load-carrying structure composed of two thin high-strength faces which enclose a comparatively thick low-density core. The individual components of the sandwich system have little load-carrying capacity of their own. However, once they are bonded together they produce stiff and light weight structural elements.

Recently, the demand for light-weight high-performance structures has been growing rapidly. An increasing number of structural designs are utilizing sandwich-type constructions in the fabrication of major structural components. This substantial interest in sandwich construction is due to its advantages; it gives high strength-to-weight and stiffness-to-weight ratios, good thermal and acoustical insulation, good surface finish, ease of equipment installation, and makes economical use of materials. Also, sandwich constructions are most ideally suited for mass productions and for use in pre-fabricated structural components. They can, therefore, be used with considerable success in buildings and as light-weight roof structures with a significant reduction in the cost of framework and foundations.
The basic goal of sandwich constructions is best achieved if the low-density core is combined with relatively strong facings. The core of a sandwich structure is used as a spacing device which separates the faces in order to produce a structure with large bending stiffness and strength. The usual core is orthotropic and of such a nature that the effects of transverse shear deformations must be considered in an explicit manner. Honeycomb cores which often consist of thin foils in the form of hexagonal cells perpendicular to the faces have received considerable attention in sandwich constructions. Other types of cores include corrugated sheets, expanded materials such as polyvinyl chloride and synthetic rubber. Honeycomb and corrugated cores are often produced from the same material as the faces. Depending upon the application, aluminum alloy, titanium, reinforced plastic, and other suitable materials have been used for the faces. Composite materials appear to have a great potential in sandwich constructions, owing to its high-strength and low-density properties. The field of filamentary composites has been developed from one dealing primarily with glass-reinforced plastics to a much broader class including boron, graphite and metal fibers.

A particularly interesting phenomenon that arises in unbalanced laminated construction* is coupling between

---

*The term "unbalanced laminated construction" is used to describe a section where the lamina are placed un-symmetrically (elastically and/or geometrically) about the middle-surface.
extensional and flexural actions. This type of behaviour necessitates a greater emphasis on orthotropic, anisotropic and transversely heterogeneous (i.e. heterogeneous through the thickness) structural analysis as a means of predicting the behaviour of these structural types. With the rapidly growing importance of these composite structural materials, used either as solid laminates or as sandwich facings, a major transition in analysis techniques has also been required. However, only a limited number of studies have been done on the behaviour of anisotropic sandwich structures and on sandwich systems used as light-weight roof structures.

In the present work, the finite strip method is extended to the analysis of anisotropic sandwich plate, shell and folded plate structural systems. The finite strip method is a numerical means of structural analysis based on the principle of minimum total potential energy. A double Fourier series approach has also been derived for anisotropic sandwich plates with various boundary conditions, so that the results can be verified by comparing both finite strip and double Fourier series solutions.

1-1 Historical Discussion

The following discussion reviews the contributions of the work which most influenced the present research. The literature devoted to the subject of isotropic sandwich construction is abundant. One of the most important
contributions to sandwich theory was an investigation by Reissner (Ref. 1) in 1948, in which sandwich plates with two identical isotropic faces and isotropic cores were considered. It was assumed that the face-parallel stresses in the core and the variation of the face stresses over the thickness of the face layers were negligible. It was concluded that the effect of the transverse normal stresses in the core is negligibly small compared to the effect of the transverse shear stresses. It was also concluded that the range of deflections for which small deflection theory is valid decreases as the core shear stiffness is made smaller with respect to that of the faces. Also, many studies have been devoted to the solution of sandwich structures for particular loading and boundary conditions. A number of important publications released by the Forest Products Laboratories (Refs. 2 and 3) report on both theoretical and experimental findings dealing with the behaviour of isotropic sandwich structures. Isotropic sandwich shells that undergo small deflections have been considered by a number of investigators including Reissner (Ref. 4), Schmidt (Ref. 5), Monforton and Schmit (Ref. 6), Monforton (Ref. 7), and Schmit and Monforton (Ref. 8). However, the majority of the work presented has been restricted to closed shells of revolution, and to the writer's knowledge no study has been made on the behaviour of sandwich shells used as roof structures. An experimental and theoretical study on the failure modes of isotropic
sandwich folded roofs has been presented by Fazio (Refs. 9 and 10); a direct stiffness method (Ref. 11) extended to take into account the transverse shear deformation of the core was used in the theoretical analysis. It was concluded that folded sandwich plate structures make effective light weight roof structures capable of carrying relatively high loads, and that the ultimate strength depends upon the strength of the bond between facings and core. The broad interest in the present and future possibilities of sandwich structures is attested to by the appearance of several extensive bibliographies (specifically Refs. 12, 13, 14 and 15).

The presence of bending-membrane coupling in thin unbalanced laminated plates was pointed out by Reissner and Stavsky (Ref. 16), and some analytical results were given for a particular type of coupling. The effect of bending-membrane coupling on the behaviour of thin anisotropic rectangular plates with simply supported boundary conditions has been dealt with by Whitney (Ref. 17) and Whitney and Leissa (Refs. 18 and 19) where it was demonstrated that coupling can increase the deflection response by some 300% compared to the response of the corresponding orthotropic plate in which coupling is neglected. It was also concluded that the severity of the coupling effect depends on the degree of anisotropy of the individual layers, the total number of plies in the composite and the aspect ratio of the plate. A double
Fourier series approach was used in the analysis of these plates. The effect of boundary conditions on the response of laminated composites has been presented by Whitney (Ref. 20); a double Fourier series approach presented by Green for the analysis of thin isotropic clamped plates (Réf. 21) extended to take into account the reference surface deformations of the plate was used in the analysis. It was concluded that the effect of bending-extensional coupling is essentially independent of the boundary conditions. An approximate solution for unbalanced laminated plates was presented by Ashton (Ref. 22). This approximation allows an unbalanced laminated plate to be analyzed as a homogeneous plate with unstrained middle plane by replacing the bending stiffnesses of anisotropic plate with reduced bending stiffnesses. The effect of coupling on the behaviour of unbalanced laminated cylindrical shells was presented by Dong, Pister and Taylor (Ref. 23). The subject of anisotropic sandwich shells has received less attention. Recently, a discrete element finite displacement analysis of anisotropic sandwich shells was developed by Monforton (Ref. 7) and Monforton and Schmit (Ref. 6), where the potential energy formulation was presented in terms of the various stiffnesses of the faces (membrane, bending and coupling) and the transverse shear stiffnesses of the orthotropic core. The displacement behaviour was described in terms of the membrane displacements of the individual faces and
the transverse displacement of the sandwich system.
The numerical results reported included isotropic sandwich plates, anisotropic laminated sandwich systems under membrane loading (i.e. buckling cases), axisymmetrical sandwich laminated shells and non-linear problems.

Recently, the finite strip method has been developed and used with considerable success for the analysis of certain types of structures. The work presented by Kantorovich and Krylov (Ref. 24) and by Vlasov (Ref. 25) represent two of the important studies that contributed to the development of the finite strip method by Cheung in 1968. The analysis of thin isotropic and orthotropic plates with various boundary conditions was first presented by Cheung (Ref. 26) followed by the analysis of thin folded plate structures (Ref. 27) in which the membrane displacement components of the structure were expressed as zero-order interpolation polynomials, and the transverse deflection was expressed as third-order interpolation polynomials. More recently (1972) multi-layered isotropic sandwich plates under static and dynamic loadings was presented by Chan and Cheung (Ref. 28). The finite strip method was also used for the analysis of other types of structural systems such as thin skew plates (Ref. 29), the analysis of box girder bridges (Ref. 30), and curved box girder bridges (Ref. 31). It was concluded that the finite strip method is a powerful and convenient approach
for the analysis of various structures and that it is ideally suitable for programming on small computers.

The areas of finite strip analysis of sandwich roof structures and anisotropic sandwich structural systems, the double Fourier series analysis of anisotropic plates with various boundary conditions and the study of the behaviour of sandwich structural systems having unbalanced laminated faces have been virtually untouched.

1-2 Purpose and Scope

The aim of this work is to study the behaviour and to provide a systematic means of analysis capable of predicting displacements and stresses of rectangular plate, cylindrical shell and folded plate anisotropic sandwich structural systems.

The faces of the sandwich systems can be isotropic, orthotropic or anisotropic and transversely heterogeneous, typical of thin laminated skins (i.e. thin plates and shells) and may be composed of an arbitrary number of bonded layers, each of which may have different thickness, linear elastic anisotropic properties, and orientation of elastic axis. Considerable attention is given to the unsymmetric cross-plied and angle-plied laminates (Appendix A). The core considered is orthotropic and typical of that used in honeycomb sandwich constructions.

In this work, the finite strip method is extended to the analysis of the anisotropic thin and sandwich
structural systems. Furthermore, the finite strip analysis of anisotropic sandwich plates with various boundary conditions has been verified by analyzing the structural system using a double-Fourier series approach. In both approaches, the formulations are presented in terms of the structure geometry, the stiffness of the faces (bending, membrane and coupling) and the transverse shear stiffness of the core. Also, the displacement behaviour of the sandwich system is described by five variables: the four reference surface displacement of the two faces, $u_f$ and $v_f$ ($f = 1, 2$), in the $x$ and $y$ directions of the structure respectively, and the transverse displacement, $w$, which is considered to be the same for both faces and core (Fig. 1). Thus, by considering a single face of the sandwich system, the methods of analysis are conveniently specialized for predicting the behaviour of thin anisotropic and transversely heterogeneous structural systems such as those constructed from filamentary composites.

In the finite strip method, the domain of the structure is divided into a number of strips, the length of which reaches from one end of the structure to the other (Fig. 2). The displacement variables of an individual strip are represented by assumed displacement patterns, chosen such that both force and displacement boundary conditions at the two ends of the strip are satisfied. Furthermore, the assumed displacement patterns
conveniently satisfy the geometric admissibility conditions of the principle of minimum total potential energy; these conditions insure continuity of the deformations within the strip and set up linking conditions between the deformation variables (degrees of freedom) of various strips in order to insure that the displacements of the discretized structure are physically compatible with those of the original structure. The displacement functions used are formed in terms of products of one-dimensional third and fifth order interpolation polynomials and undetermined displacement coefficients at the two edges of the strip in the y-direction (Fig. 2), and basic functions which satisfy the force and displacement boundary conditions at the two ends of the strip (x = 0, L). The analysis problem is then transformed into one of determining the deformations of the individual strips. The strips are then pieced together by invoking the geometric admissibility conditions. This can be achieved by a straightforward variable correlation scheme. The finite strip analysis of the anisotropic sandwich structural systems is developed in successive steps starting with a thin structural system (i.e. a single face), and concluding with the required anisotropic sandwich structure. In every step a comparison is made, where possible, with existing references. Thus, some confidence is built into the results of the finite strip analysis for those anisotropic sandwich systems which do not have any verification from alternate approaches.
The finite strip method has been extended to the following types of structural systems.

I. **Rectangular plates with various boundary conditions**
   a) Thin laminated plates
   b) Sandwich plates with orthotropic or anisotropic laminated faces.

II. **Shell structures**
   a) Thin barrel shells
   b) Thin barrel shells with edge beams
   c) Sandwich barrel shells with orthotropic or anisotropic laminated faces.

III. **Folded plate structures**
   a) Thin folded plate structures (using higher order interpolation polynomials)
   b) Sandwich folded plate structures with orthotropic or anisotropic laminated faces.

The effect of bending-membrane coupling in the unbalanced laminated faces on the behaviour of the sandwich systems is studied for various core and face thicknesses, aspect ratios, core and face elastic properties, and orientation of the various layers in the faces.

In the following chapters, the finite strip and the double Fourier series approaches are presented. The general formulations and the assumptions used in the analysis of the structural systems are presented in Chapter II. Chapter III contains the formulations of the finite strip
method in which the discretized potential energies, and the development of the strip stiffness matrices for the various structural systems are presented. Chapter IV is concerned with the "exact" analysis of anisotropic sandwich plates using the double Fourier series approach in which the governing differential equations of the structural systems are presented as well as the solutions for anisotropic sandwich plates having simply supported and clamped boundary conditions. A detailed development of the various finite strip and double Fourier series formulations as well as equations related to the solutions of the various structural systems are presented in the Appendices. The applicability of the finite strip and the double Fourier analysis capabilities to the prediction of the behaviour of anisotropic thin and sandwich structural systems are illustrated in Chapter V through several numerical solutions. Chapter V is also concerned with the study of the effect of bending-membrane coupling on the behaviour of anisotropic sandwich structural systems. A number of conclusions are drawn in Chapter VI with respect to the merits of the analysis capabilities and the behaviour of the anisotropic sandwich structural systems and concludes with recommendations for further research work.
CHAPTER II

GENERAL FORMULATION

In this chapter the general formulations and assumptions used in the analysis of the anisotropic sandwich structural systems are presented. The sandwich system consists of two anisotropic faces separated by a relatively thick orthotropic honeycomb core. A portion of a laminated sandwich cylindrical shell is shown in Figure 1. The subscripts 1 and 2 are used to denote quantities associated with the inner face and the outer face respectively, the subscript c is used to identify quantities that refer to the core, while f is used when attention is limited to the faces (f = 1, 2). The reference surfaces for both the faces and the core are taken as their middle surfaces, that is, $t_f/2$ and $t_c/2$ respectively. The displacement behaviour of the sandwich system is described in terms of the reference surface displacements of the individual faces, $u_f$ and $v_f$, and the transverse deflection, $w$, of the sandwich system (Figure 1).

2-1 Face Considerations

The faces of the sandwich systems are considered to be thin skins (i.e. thin plates or thin shells). Therefore, the usual assumptions of plate and shell theories including
the Kirchhoff-Love hypothesis are retained. As previously mentioned, the faces may be isotropic, orthotropic or anisotropic and transversely heterogeneous, typical of thin filamentary composite constructions in which an individual ply is assumed to be homogeneous and orthotropic with respect to its principle axis 1 and 2 (Figure 3); each ply may also have different thickness, elastic properties and orientation of elastic axis. Therefore, such faces must be considered as transversely heterogeneous since they may only be piecwise homogeneous. The following assumptions are considered for the faces:

1) A normal to the reference surface of a face remains straight and normal to it and undergoes no change in length during deformations. (This is equivalent to neglecting the compressibility and the transverse shear deformation of the faces and is also equivalent to assuming that the layers that make up the faces do not slip over one another.)

2) The transverse normal stress, \( \sigma_{zf} \), is negligible compared with the other reference surface stress components of the facing.

3) Relative rotations about the normal to the reference surface does not occur between bonded layers.

4) In the case of shells the faces are thin so that the ratios of the thickness to the radii of curvature of the reference surface are small compared to unity.
2-1 Strain-Displacement Relations

Based on the foregoing assumptions, the strain-displacement relations of the laminated faces of a sandwich cylinder are represented by \((f = 1,2)\):

\[
\begin{bmatrix}
\varepsilon_x^f \\
\varepsilon_y^f \\
\gamma_{xy}^f \\
\end{bmatrix} = \begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0 \\
\end{bmatrix} + z_f \begin{bmatrix}
\kappa_x^f \\
\kappa_y^f \\
\kappa_{xy}^f \\
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
\varepsilon_f^1 \\
\varepsilon_f^2 \\
\gamma_{xy}^f \\
\end{bmatrix} = \begin{bmatrix}
\varepsilon_f^0 \quad + 
\end{bmatrix} \begin{bmatrix}
\kappa_f^1 \\
\kappa_f^2 \\
\kappa_{xy}^f \\
\end{bmatrix}
\]

where

\[
\begin{align*}
\{\varepsilon_f^0\} & - \text{Contains the normal and shearing strains of the reference surface.} \\
\{\kappa_f\} & - \text{Contains the changes in curvatures and angle of twist of the reference surface during deformation.}
\end{align*}
\]

The components of \(\{\varepsilon_f^0\}\) and \(\{\kappa_f\}\) are given by:

\[
\begin{align*}
\varepsilon_x^0 &= u_x^f \\
\varepsilon_y^0 &= v_y^f + \frac{1}{R^f} \cdot w_f \\
\gamma_{xy}^0 &= v_x^f + u_y^f \\
\kappa_x^f &= -w_{xx}^f \\
\kappa_y^f &= -w_{yy}^f + \frac{1}{R^f} v_y^f \\
\kappa_{xy}^f &= -2(w_{xy}^f - \frac{1}{R^f} v_x^f)
\end{align*}
\]
where

\[ v_{xf} = \frac{\partial v_f(x, y_f)}{\partial x}; \quad v_{yf} = \frac{\partial v_f(x, y_f)}{\partial y} = \frac{1}{R_f} \cdot \frac{\partial v_f(x, \theta)}{\partial \theta} \ldots \text{etc.} \]

The corresponding strain-displacement relations for the faces of a sandwich plate are obtained from Eqs. (2-2a) and (2-2b) by setting \( y_f = y \) and \( \frac{1}{R_f} = 0 \).

### 2.1.2 Force-Deformation Relations

The determination of the force-deformation relations as well as the various elastic stiffness of laminated faces are presented in Appendix A. The resulting force-deformation relations are (Fig. 4):

\[
\begin{bmatrix}
N_{xf} \\
N_{yf} \\
M_{xf} \\
M_{yf} \\
M_{xf} = M_{yf} \\
N_{xf} = N_{yf}
\end{bmatrix}
\begin{bmatrix}
A_{11}^{(f)} & A_{12}^{(f)} & A_{16}^{(f)} & B_{11}^{(f)} & B_{12}^{(f)} & B_{16}^{(f)} \\
A_{22}^{(f)} & A_{26}^{(f)} & B_{22}^{(f)} & B_{26}^{(f)} & B_{26}^{(f)} & B_{26}^{(f)} \\
A_{66}^{(f)} & A_{66}^{(f)} & B_{66}^{(f)} & B_{66}^{(f)} & B_{66}^{(f)} & B_{66}^{(f)} \\
D_{11}^{(f)} & D_{12}^{(f)} & D_{16}^{(f)} \\
D_{22}^{(f)} & D_{26}^{(f)} & D_{26}^{(f)} \\
D_{66}^{(f)}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{xf} \\
\epsilon_{yf} \\
\gamma_{xyf} \\
\kappa_{xf} \\
\kappa_{yf} \\
\kappa_{xyf}
\end{bmatrix}
\]

\[ (2-3a) \]

or

\[
\begin{bmatrix}
N_{xf} \\
N_{yf} \\
M_{xf} \\
M_{yf} \\
M_{xf} = M_{yf} \\
N_{xf} = N_{yf}
\end{bmatrix}
\begin{bmatrix}
\text{SYMMETRIC}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{xf} \\
\epsilon_{yf} \\
\gamma_{xyf} \\
\kappa_{xf} \\
\kappa_{yf} \\
\kappa_{xyf}
\end{bmatrix}
\]

or

\[ \epsilon_{xf}, \epsilon_{yf}, \gamma_{xyf}, \kappa_{xf}, \kappa_{yf}, \kappa_{xyf} \]

or

\[ \epsilon_{xf}^{(f)}, \epsilon_{yf}^{(f)}, \gamma_{xyf}^{(f)}, \kappa_{xf}^{(f)}, \kappa_{yf}^{(f)}, \kappa_{xyf}^{(f)} \]

or

\[ \epsilon_{xf}^{(f)}, \epsilon_{yf}^{(f)}, \gamma_{xyf}^{(f)}, \kappa_{xf}^{(f)}, \kappa_{yf}^{(f)}, \kappa_{xyf}^{(f)} \]
\[
\begin{bmatrix}
\{N_f\} \\
\{M_f\}
\end{bmatrix}
= 
\begin{bmatrix}
[A]^{(f)} & [B]^{(f)} \\
[B]^{(f)} & [D]^{(f)}
\end{bmatrix}
\begin{bmatrix}
\{\epsilon^0_f\} \\
\{\kappa_f\}
\end{bmatrix}
\]  
\hspace{1cm} (2-3b)

where

\{N_f\} - Contains the reference surface force resultant of the face.

\{M_f\} - Contains the moment resultant acting on the faces.

The elements of the matrices \([A]^{(f)}, [B]^{(f)}\) and \([D]^{(f)}\) represent the membrane, coupling and bending stiffnesses of the laminated faces. The coupling stiffnesses vanish when the lamina are placed symmetrically (elastically and geometrically) about the middle surface of the face. Non-zero values of the coupling stiffnesses are characteristic of unbalanced laminated construction in which the chosen reference surface (i.e. middle plane) does not coincide with the centroids of the composite cross-sections in the \(x\) and \(y\) directions of an unbalanced laminated skin. Typical examples of unbalanced laminates are represented by the cross-ply and the angle-ply laminates (Fig. 5 and Appendix A).

2-2 Core Considerations

The core of a sandwich system is used to separate the faces in order to produce a structural member with high bending stiffness and strength. Furthermore, the core
resists the transverse shearing stresses and transmits shear between the faces so that they function effectively as a unit. Therefore, the adhesion which bonds the facings to the core must be capable of transmitting shear stresses between the filler and the faces. In the present work, the core is considered to be an orthotropic honeycomb type (Fig. 6) and of such a nature that the transverse shear deformations must be considered since the transverse shear moduli of the honeycomb cores are low. For the sandwich systems having honeycomb or similar cores, the following assumptions are considered:

(1) The core is incompressible in the transverse direction ($\varepsilon_{zC} = 0$).

(2) The face-parallel shear and normal stiffnesses of the core are negligible compared with that of the faces, as well as to the transverse shear stiffness of the core ($\sigma_{xC} = \sigma_{yC} = \tau_{xyC} = 0$).

(3) The face-parallel displacement $u_{C}$ and $v_{C}$ vary linearly across the thickness of the core.

(4) Bond failure does not occur at the interfaces between the faces and the core.

(5) In the case of shells, the ratio of the core thickness to the radius of curvature of the reference surface, $t_{C}/R_{C}$, is small compared to unity.

The first assumption is equivalent to assuming that the transverse normal stiffness of the core is infinitely
large. It is pointed out that even if the transverse normal stiffness of the core is of the same order of magnitude as the normal stiffness parallel to the faces, the effect of transverse normal strain, $\varepsilon_{Z_C}$, proves to be negligible (Ref. 12). From the first two assumptions it follows that the transverse normal stress in the core, $\sigma_{Z_C}$, is equal to zero. The fifth assumption implies that for sandwich shells, the honeycomb cores will possess the same core cell area throughout the core thickness; that is, the transverse shear moduli of the core, $G_{XZ_C}$ and $G_{YZ_C}$, are constant and equal to those of the flat core (Ref. 7).

The second and fifth assumptions imply that the transverse shear stresses in the core, $\gamma_{XZ_C}$ and $\gamma_{YZ_C}$, are independent of the coordinate, $z$. Therefore, the shear strains will be constant and equal to the shear angles, $\gamma_{XZ_C}$ and $\gamma_{YZ_C}$, which in turn agree with the third assumption. The force resultants acting on an element of honeycomb core are shown in Fig. 7. It is pointed out that the problem of bond failure is critical for sandwich systems with honeycomb cores, since the structure of this core limits the contact area between facings and core to about 5 per cent of the area of the faces.

Based on the foregoing assumptions, the displacement components of the core can be expressed as:

$$\bar{u}_C(x, y_C, z_C) = u_C + z_C \phi_C$$  \hspace{1cm} (2-4a)
\[ \bar{v}_c(x, y_c, z_c) = v_c + z_c \psi_c \]  \hspace{1cm} (2-4b)
\[ \bar{w}_c(x, y_c, z_c) = w_c = w_f = w \]  \hspace{1cm} (2-4c)

From Fig. 8 it can be shown that
\[ u_c = \frac{1}{2} \left[ u_1 + u_2 - (d_1 - d_2) w_x \right] \]  \hspace{1cm} (2-5a)
\[ v_c = \frac{1}{2} \left[ v_1 + v_2 - (h_1 - h_2) w_y \right] \]  \hspace{1cm} (2-5b)

where \( u_c \) and \( v_c \) are the middle surface displacements of the core in the \( x \) and \( y \) direction respectively, and
\[ \phi_c = -\frac{1}{t_c} \left[ u_1 - u_2 - (d_1 + d_2) w_x \right] \]  \hspace{1cm} (2-6a)
\[ \psi_c = -\frac{1}{t_c} \left[ v_1 - v_2 - (h_1 + h_2) w_y \right] \]  \hspace{1cm} (2-6b)

where \( \phi_c \) and \( \psi_c \) are the rotations of the normals to the middle surface of the core (Fig. 8). In the above expressions
\[ d_f = \frac{t_f}{2} \quad (f = 1, 2) \]  \hspace{1cm} (2-7)
\[ h_1 = d_1 \frac{R_c}{R_1} \quad ; \quad h_2 = d_2 \frac{R_c}{R_2} \]
\[ w_{yc} = \frac{\partial w}{\partial y_c} = \frac{1}{R_c} \cdot \frac{\partial w}{\partial y} \]  \hspace{1cm} (2-8)
and the notations

\[ u_f = u_f(x, y_f) \quad v_f = v_f(x, y_f) \quad \text{and} \quad w = w(x, y_f) \quad \ldots \]

e.tc., have been adopted for convenience.

2-2-1 Strain-Displacement Equations

Based on the foregoing assumptions, the transverse shear strain of the core can be expressed in terms of the membrane displacement of the faces, \( u_f \) and \( v_f \), and the transverse deflection, \( w \), as (Fig. 8).

\[
\gamma_{xz_c} = \frac{1}{t_c} \left[ u_2 - u_1 + (d_1 + d_2 + t_c) w_x \right]
\]

\[
(2-9a)
\]

\[
\gamma_{yz_c} = \frac{1}{t_c} \left[ v_2 - v_1 + (h_1 + h_2 + t_c) w_y \right]
\]

\[
(2-9b)
\]

The corresponding strain-displacement equations for the core of a sandwich plate are obtained from Eqs. 2-9 by setting \( y_c = y \) and \( R_c/R_1 = R_c/R_2 = 1 \).

2-2-2 Constitutive Equations

The relationship between stresses and strains for the core under consideration are represented by:

\[
\begin{bmatrix}
\gamma_{xz_c} \\
\gamma_{yz_c}
\end{bmatrix}
= \begin{bmatrix}
G_{xz_c} & 0 \\
0 & G_{yz_c}
\end{bmatrix}
\begin{bmatrix}
\tau_{xz_c} \\
\tau_{yz_c}
\end{bmatrix}
\]

\[
(2-10a)
\]
or
\[ \{ \tau_c \} = [G_c] \cdot \{ \gamma_c \} \]  \hspace{1cm} (2-10b)

where

\[ \{ \tau_c \} \] - Contains the transverse shear stress in the core.

\[ \{ \gamma_c \} \] - Contains the transverse shear strain in the core.

\[ G_{xz_c} \] and \[ G_{yz_c} \] are the transverse shear moduli of the core for the \( x \) and \( y \) directions respectively.

2-2-2 Force-Deformation Relations

The transverse shear force resultants for the core are defined by (Fig. 7).

\[ Q_{xz_c} = \int_{-t_c/2}^{t_c/2} \tau_{xz_c} \, dz_c \]

\[ Q_{yz_c} = \int_{-t_c/2}^{t_c/2} \tau_{yz_c} \, dz_c \]

or

\[
\begin{bmatrix}
Q_{xz_c} \\
Q_{yz_c}
\end{bmatrix} =
\begin{bmatrix}
B_{55} & 0 \\
0 & B_{44}
\end{bmatrix}
\begin{bmatrix}
\gamma_{xz_c} \\
\gamma_{yz_c}
\end{bmatrix}
\]  \hspace{1cm} (2-11b)
where $B_{44}$ and $B_{55}$ are the transverse shear stiffness of the core, defined by:

$$B_{44} = G_{yzC} \cdot t_c \quad ; \quad B_{55} = G_{xzC} \cdot t_c$$  \hspace{0.5cm} (2-11c)

As previously mentioned, the four reference surface displacements, $u_f$ and $v_f$, of the faces and the transverse displacement $w$ are chosen to describe the displacement behaviour of the sandwich system (Fig. 1). This choice of displacement variables admits transverse shear deformation in the core. Also, the analysis of thin anisotropic and transversely heterogeneous structural systems can be considered as a special case obtained by simply considering one face of the sandwich system. It is pointed out that the displacement behaviour of the sandwich system could be described by the middle surface displacement of the core, $u_C$ and $v_C$, the rotations of the normals to the middle surface of the core, $\phi_C$ and $\psi_C$, and the transverse displacement $w$. However, for such a selection, the analysis of thin anisotropic structural systems as well as the choice of assumed displacement patterns (especially for sandwich structural systems having anisotropic faces) will not be as direct as with the set of variables chosen in this work.
CHAPTER III
FINITE STRIP FORMULATION

The discussion in this chapter is focused on the finite strip capability for predicting the behaviour of the anisotropic sandwich structural systems. A substantial number of the problems which arise in structural analysis are not amenable to closed form solutions. Numerical means of structural analysis such as the finite element method has become relatively routine in recent years. However, the finite element method, while versatile and powerful, has some drawbacks mainly in terms of computer storage and time needed to solve the large number of algebraic equations usually generated. Therefore, for certain types of structures it is worthwhile to develop alternative numerical approaches in which a reduced number of simultaneous equations are needed. One of these approaches is the finite strip method. By comparing both finite element and finite strip approaches, a saving in computer storage and time can be realized. The finite element method requires the division of the domain of the structure into elements in both directions, and boundary conditions are imposed only after the overall stiffness matrix has been formed. In the finite strip method the domain is divided only in one direction (y-direction) into a number
of strips the length of which spans the total length of the structure (Fig. 2). The displacement functions are chosen such that both force and displacement boundary conditions at the two ends of the strip (i.e., \( x = 0, L \)) are satisfied. The general two-dimensional problem is then reduced to a one-dimensional problem, with a relatively small number of undetermined edge displacement parameters along the two sides of the strip in the \( y \)-direction. For certain types of structures the size of the strip stiffness matrix is relatively small with a narrow band width; thus, the finite strip method for these types of structures is ideally suitable for programming on small computers. Generally, the finite strip method can be considered to be one of the most powerful tools for the analysis of roof structures, such as sandwich shell and sandwich folded plate structures, as well as the other types of structural systems considered in this work.

The philosophy of the finite strip method is similar to the Kantorovich method (Ref. 24) which reduces a partial differential equation to an ordinary differential equation by adopting continuous functions which satisfy the boundary conditions in one direction. The finite strip method presented herein is based on the principle of minimum total potential energy which can be stated as:

Of all geometrically admissible displacement states, that which minimizes the total potential energy satisfies the equilibrium conditions and is the actual state of displacement.
In order to guarantee that the finite strip method be a monotonically converging potential energy formulation, it is essential that the assumed displacement states satisfy the geometric admissibility conditions of the principle of minimum total potential energy. These conditions can be obtained by examining the potential energy formulations. Since the nature of the energy is undefined until the boundary conditions are specified, satisfaction of the boundary conditions necessarily forms one of the admissibility conditions. As previously mentioned, both force and geometric boundary conditions are satisfied at the two ends of the strip "a priori", while in the y-direction no force boundary conditions are imposed. However, this does not represent a violation of the previously mentioned admissibility condition since at a true minimum of the potential energy the force boundary conditions will be inherently satisfied. In other words, the "closeness" of the finite strip solution can be assessed by the closing error in the natural (force) boundary conditions in the direction in which the force boundary conditions are not satisfied "a priori". The remaining geometric admissibility conditions involve continuity requirements on the assumed displacement patterns, that is, the displacement functions must have a sufficient number of bounded derivatives as required by the form of the total potential energy expression. The total potential energy expression for anisotropic sandwich
structural systems will be presented later in this Chapter. For such an expression the assumed displacement patterns should possess the following geometric admissibility conditions:

(1) The imposed displacement boundary conditions must be satisfied.

(2) The membrane displacements, \( u_f, v_f \) \((f = 1, 2)\), and the transverse deflection \( w \) must be continuous within the strip and along the entire common edge of adjacent strips.

(3) The first derivatives of the transverse deflection must be continuous in the interior of the strip and along the entire common edge of adjacent strips.

In the finite strip analysis of sandwich structural systems, the domain of the structure is imagined to be separated into a number of strips, each strip having a constant thickness (although the thickness can vary from one strip to another). For a particular strip the strain energy expression is formulated in terms of the strip geometry (thickness, width and length), the stiffnesses of the faces (bending, membrane and coupling), the transverse shear stiffness of the core and the displacement variables \((u_f, v_f, \text{ and } w)\). The displacement functions are chosen such that both force and displacement boundary conditions are satisfied at the two ends of the strip. Furthermore, they must satisfy the geometric admissibility conditions of the principle of the minimum total potential energy.
A strip stiffness matrix with preset end conditions can be developed by substituting the assumed displacement functions into the strain energy formulations. It is pointed out that the strip stiffness matrix will always be non-singular because of the preset end conditions. The overall stiffness matrix of the structure can be obtained by imposing the geometric admissibility conditions between adjacent strips. This can be achieved by a variable correlation scheme in which the degrees of freedom of the various strips are linked together in such a way that the displacements of the discretized structure are physically compatible with those of the original structure. The potential of the applied loads is dealt with on a work equivalent basis, in which the work equivalent loads associated with the corresponding degrees of freedom are derived. The total potential energy, $\pi_p$, of a sandwich structure will have the following form:

$$\pi_p = \sum_{s=1,2,c} U_s - \sum_{f \neq 1,2} W^{(f)}$$  \hspace{1cm} (3-1)

where $U_s$ - The total strain energy of the faces ($s=1,2$) and the core ($s=c$).

$W^{(f)}$ - The work done by the externally applied loads.

By minimizing the energy expression, $\pi_p$, the following relations can be obtained.

$$[K] \{x\} = \{p\}$$  \hspace{1cm} (3-2)
where

\[ [K] \] - The overall stiffness matrix of the structure.

\[ \{X\} \] - A vector containing the independent degrees of freedom of the structure.

\[ \{P\} \] - A vector containing the work equivalent loads associated with the corresponding degrees of freedom in \[ \{X\} \].

Solutions for the set of simultaneous equations in expression 3-2 are obtained by standard numerical techniques such as the Gaussian elimination method from which the degrees of freedom are obtained. The force and moment resultants at various locations in the domain of the structure can be obtained by substituting the resulting displacements and their partial derivatives in the force-deformation relations (Eqs. 2-3).

When the various strips are not coplanar or do not form a curvilinear surface (i.e., folded plates and shells with edge beams) it is necessary to work in terms of some reference coordinate system. That is, the strip stiffness matrices and the work equivalent loads of the various strips must be transformed from the local coordinate system of the individual strips to the reference coordinate system. Once the transformation has been carried out, it is possible to assemble the overall stiffness matrix in the conventional manner. The resulting displacements, \[ \{X\} \], will be with respect to the reference coordinate system, and they must be re-transformed to the local
coordinate systems before the force and moment resultants are calculated.

3.1 Strain Energy

The strain energy of an anisotropic laminated face can be written as:

$$U_f = \frac{1}{2} \iiint_{V_f} \{\varepsilon_f\}^T \{\sigma_f\} \, dx \, dy_f \, dz_f \tag{3-3}$$

where

$$\{\sigma_f\} - \text{Contains the normal and shear stress of the reference surface of the face.}$$

$$\{\varepsilon_f\}$$ is given by Eq. 2.1 and $$V_f$$ represents the volume of the face of a sandwich strip. Integrating over the total thickness of the face results in the following expression for the strain energy:

$$U_f = \frac{1}{2} \iint_{S_f} \left( \begin{array}{c} \{\varepsilon_f^0\}^T \\ \{\kappa_f\} \end{array} \right) \left( \begin{array}{c} \{A\}^{(f)} \\ \{B\}^{(f)} \\ \{C\}^{(f)} \end{array} \right) \left( \begin{array}{c} \{\varepsilon_f\} \\ \{\kappa_f\} \end{array} \right) \, dx \, dy_f.$$  \tag{3-4a}

or

\[ \]
\[ U_F = \frac{1}{2} \iint_{S_f} \left[ (\varepsilon_F^o)^T [A] (\varepsilon_F^o) + 2(\varepsilon_F^o)^T [B] (\kappa_F) \right. \\
+ \left. (\kappa_F)^T [D] (\kappa_F) \right] dS_f \]  

(3-4b)

where

\[ (\varepsilon_F^o)^T [A] (\varepsilon_F^o) \] - The strain energy due to membrane action of the faces in their reference planes.

\[ 2(\varepsilon_F^o)^T [B] (\kappa_F) \] - The strain energy due to coupling between membrane and bending action in the faces.

\[ (\kappa_F)^T [D] (\kappa_F) \] - The strain energy due to bending of the faces.

\( S_f \) represents the reference surface area of the face of a sandwich strip \( f = 1, 2 \). The strain energies due to transverse shearing of the faces are considered negligible.

Substitution of the strain-displacement relations (Eq. 2-1) into Eq. 3-4 gives an expression for the strain energy in terms of the reference surface displacements, \( u_F \) and \( v_F \), and the transverse-displacement, \( w \), as well as the various stiffnesses of the face (bending, membrane and coupling).

This expression can be written as:
\[ u_f = \frac{1}{2} \iint \left\{ \left[ A_{11}^{(f)} u_{x_f}^2 + A_{22}^{(f)} v_{y_f}^2 + A_{66}^{(f)} (u_{y_f} + v_{x_f})^2 \right] + 2 A_{12}^{(f)} u_{x_f} v_{y_f} + 2 A_{16}^{(f)} (u_{y_f} + v_{x_f}) u_{x_f} \\
+ 2 A_{26}^{(f)} (u_{y_f} + v_{x_f}) v_{y_f} \right\} - 2 \left[ B_{11}^{(f)} u_{x_f} w_{xx_f} + B_{22}^{(f)} v_{y_f} w_{yy_f} + 2 B_{66}^{(f)} (u_{y_f} + v_{x_f}) w_{xy_f} \right] \\
+ B_{12}^{(f)} (v_{y_f} w_{xx_f} + u_{x_f} w_{yy_f}) + B_{16}^{(f)} (u_{y_f} + v_{x_f}) w_{xx_f} + 2 B_{16}^{(f)} u_{x_f} w_{xy_f} \\
+ B_{26}^{(f)} (u_{y_f} + v_{x_f}) w_{yy_f} + 2 B_{26}^{(f)} v_{y_f} w_{xy_f} \right\} + \left[ D_{11}^{(f)} w_{xx_f}^2 + D_{22}^{(f)} w_{yy_f}^2 + 4 D_{66}^{(f)} w_{xy_f}^2 \right] \\
+ 2 D_{12}^{(f)} w_{xx_f} w_{yy_f} + 4 D_{16}^{(f)} w_{xx_f} w_{xy_f} \right\} \right] \]

...con't
\[ + 4 D_{26}^{(f)} w_{yyf} \overline{w_{xf}} \overline{w_{yf}} + \frac{1}{R_f} \left[ \frac{A_{22}^{(f)}}{R_f} w_{f}^{2} + 2 \left( A_{22}^{(f)} + \frac{B_{22}^{(f)}}{R_f} \right) w_{f} v_{yf} \right. \\
\left. + \left( \frac{D_{22}^{(f)}}{R_f} + 2 B_{22}^{(f)} \right) v_{yf}^{2} - 2 B_{22}^{(f)} w_{f} w_{yyf} - 2 D_{22}^{(f)} v_{yf} w_{yyf} \right. \\
\left. + 4 \left( \frac{D_{66}^{(f)}}{R_f} + B_{66}^{(f)} \right) v_{xf}^{2} + 4 B_{66}^{(f)} v_{xf} u_{yf} - 8 D_{66}^{(f)} v_{xf} w_{yyf} \right. \\
\left. + 2 A_{12}^{(f)} w_{f} u_{xf} - 2 B_{12}^{(f)} w_{f} w_{xxf} + 2 B_{12}^{(f)} u_{xf} v_{yf} \right. \\
\left. - 2 D_{12}^{(f)} v_{yf} w_{xxf} + 4 B_{16}^{(f)} u_{xf} v_{xf} - 4 D_{16}^{(f)} v_{xf} w_{xxf} \right. \\
\left. + 2 A_{26}^{(f)} w_{f} u_{yf} - 4 B_{26}^{(f)} w_{f} w_{xyf} + 2 \left( A_{26}^{(f)} + 2 \frac{B_{26}^{(f)}}{R_f} \right) w_{f} v_{xf} \right. \\
\left. + 2 B_{26}^{(f)} u_{yf} v_{yf} - 4 D_{26}^{(f)} v_{yf} w_{xyf} + 2 \left( \frac{2 D_{26}^{(f)}}{R_f} + 3 B_{26}^{(f)} \right) \right. \\
\left. v_{xf} v_{yf} - 4 D_{26}^{(f)} v_{xf} w_{yyf} \right] \left\{ \right. \\
\left. \right\} dS_{f} \}

(3-5)
where $S_f$ is the reference surface area of the face. Note that the strain energy expression for the faces of a sandwich plate are obtained directly from Equation 3-5 by setting $y_f = y$ and $1/R_f = 0$.

### 3.1.2 Strain Energy of the Core

The strain energy of the core under consideration is limited to the contributions due to transverse shear. Therefore, the strain energy due to the face-parallel deformations in the core are considered negligible. The strain energy expressions can be written in the form:

$$U_C = \frac{1}{2} \iiint_{V_C} \{\tau_C\}^T \{\gamma_C\} \, dx \, dy_C \, dz_C$$  \hspace{1cm} (3-6)

where $V_C$ represents the volume of the core of a sandwich strip. $\{\gamma_C\}$ and $\{\tau_C\}$ are given by Eqs. 2-9 and Eqs. 2-10 respectively. Integrating over the total thickness of the core results in the following expression for the strain energy of the core:

$$U_C = \frac{1}{2} \iiint_{S_C} \begin{bmatrix} \gamma_{xz_C} \\ \gamma_{yz_C} \end{bmatrix}^T \begin{bmatrix} B_{55} & 0 \\ 0 & B_{44} \end{bmatrix} \begin{bmatrix} \gamma_{xz_C} \\ \gamma_{yz_C} \end{bmatrix} \, dx \, dy_C$$  \hspace{1cm} (3-7)

where $S_C$ is the middle surface area of the core of a sandwich strip. Substituting Eqs. 2-9 into Eq. 3-7 gives the strain energy of the core in terms of the reference
surface displacements, \( u_f \) and \( v_f \), the transverse displacement, \( w \), and the transverse shear stiffness of the core as follows:

\[
U_C = \frac{1}{2} \sum_{s_c} \left[ \frac{B_{44}}{t_c^2} \left( v_2 - v_1 + (h_1 + h_2 + t_c) w_{y_c} \right)^2 
+ \frac{B_{55}}{t_c^2} \left( u_2 - u_1 + (d_1 + d_2 + t_c) w_x \right)^2 \right] dS_c
\]

(3-8)

The corresponding strain-energy expression for the core of a sandwich plate is obtained from Eq. 3.8 by setting \( y_c = y \) and \( R_c/R_1 = R_c/R_2 = 1 \).

The total strain energy of the \( i \)th sandwich strip can be expressed by:

\[
U_{(i)} = U_C + \sum_{f=1}^{2} U_f
\]

(3-9)

3.2 Displacement Functions

As previously mentioned, the assumed displacement patterns for the reference surface displacements, \( u_f \) and \( v_f \), and the transverse deflection, \( w \), are chosen such that both force and displacement boundary conditions at the two ends of the strip (Figs. 9) (i.e. \( x = 0, L \)) are satisfied. Also, they must conveniently satisfy the geometric admissibility conditions of the principle of minimum total potential energy. Therefore, convergent solutions are expected. Consider a typical sandwich strip bounded by the
sides 1 and 2; the assumed displacement patterns for the strip can be represented by:

\[ u_f(x, y_f) = \sum_{n=1,2}^N Y_{u_f}^{(n)} \cdot X_{u_f}^{(n)} \]  \hspace{1cm} (3-10a)

\[ v_f(x, y_f) = \sum_{n=1,2}^N Y_{v_f}^{(n)} \cdot X_{v_f}^{(n)} \]  \hspace{1cm} (3-10b)

\[ w(x, y_f) = \sum_{n=1,2}^N Y_{w}^{(n)} \cdot X_{w}^{(n)} \]  \hspace{1cm} (3-10c)

where \( Y_{u_f}^{(n)} \), \( Y_{v_f}^{(n)} \) and \( Y_{w}^{(n)} \) consist of the undetermined displacement coefficients at the two edges of the strip (i.e. edges 1 and 2) for \( u_f \), \( v_f \) and \( w \) respectively, as well as polynomials which describe the variation of the displacement coefficients within the strip in the y-direction (Fig. 9). The polynomials used in this work are one-dimensional third and fifth order interpolation polynomials. It is pointed out that these polynomials permit the satisfaction of the geometric admissibility conditions within the strip in the y-direction. The variations of the displacement coefficients along the length of the strip (i.e. in the x-direction) for \( u_f \), \( v_f \) and \( w \) are described by the basic series functions \( X_{u_f}^{(n)} \), \( X_{v_f}^{(n)} \) and \( X_{w}^{(n)} \) respectively (\( n = 1, 2, \ldots N \), where \( N \) = the total number of cycles in the series). These basic series are functions of \( x \) only, and through them the force and displacement boundary conditions
at $x = 0, L$ are satisfied. Furthermore, they automatically permit the satisfaction of the geometric admissibility conditions since they are continuous functions. From the previous discussion it can be seen that the general two-dimensional problem is reduced to a one-dimensional problem since the displacement coefficients to be determined are in the $y$-direction only.

The $y^{(n)}$ functions can be represented in terms of one-dimensional third order interpolation polynomials by the following expressions for $u_f$, $v_f$ and $w$:

$$u_f(x, y_f) = \sum_{n=1, 2}^{N} \sum_{j=1}^{2} \left[ H_{o_j}^{(1)}(y_f) u_f^{(n)}_{y_f j} + H_{l_j}^{(1)}(y_f) u_f^{(n)} \right]$$

(3-11a)

$$v_f(x, y_f) = \sum_{n=1, 2}^{N} \sum_{j=1}^{2} \left[ H_{o_j}^{(1)}(y_f) v_f^{(n)}_{y_f j} + H_{l_j}^{(1)}(y_f) v_f^{(n)} \right]$$

(3-11b)

$$w(x, y_f) = \sum_{n=1, 2}^{N} \sum_{j=1}^{2} \left[ H_{o_j}^{(1)}(y_f) w_f^{(n)}_{y_f j} + H_{l_j}^{(1)}(y_f) w_f^{(n)} \right]$$

(3-11c)

where $(u_f^{(n)}_{y_f j}, v_f^{(n)}_{y_f j})$, $(v_f^{(n)}_{y_f j}, v_f^{(n)}_{y_f j})$ and $(w_f^{(n)}_{y_f j}, w_f^{(n)}_{y_f j})$ are the
undetermined displacement and slope coefficients at the
two edges of the strip \( j = 1, 2 \) for the \( u_f, v_f \) and \( w \)
displacement functions respectively. \( H_{k,j}^{(1)}(y_f) \) are the one-
dimensional third order interpolation polynomials given by:

\[
H_{01}^{(1)}(y_f) = \left[ 2 \, y_f^3 - 3 \, b_f \, y_f^2 + b_f^3 \right] / b_f^3 ;
\]

\[
H_{02}^{(1)}(y_f) = -\left[ 2 \, y_f^3 - 3 \, b_f \, y_f^2 \right] / b_f^3 ;
\]

\[
H_{11}^{(1)}(y_f) = \left[ y_f^3 - 2 \, b_f \, y_f^2 + b_f^2 \, y_f \right] / b_f^2 ;
\]

\[
H_{12}^{(1)}(y_f) = \left[ y_f^3 - b_f \, y_f^2 \right] / b_f^2
\]  (3-12)

where \( b_f \) is the width of the face of the sandwich strip in
the \( y \)-direction (Fig. 9).

The third order interpolation polynomials (Eq. 3-12)
possess the following properties (Ref. 32):

(a) At \( y_f = 0 \) (i.e. at edge 1 of the strip)

\[
H_{01}^{(1)}(y_f) = 1 \quad ; \quad H_{02}^{(1)}(y_f) = 0
\]  (3-13a)

\[
H_{11}^{(1)}(y_f) = 0 \quad ; \quad H_{12}^{(1)}(y_f) = 0
\]

and
\[
\begin{align*}
\frac{\partial H_{o1}^{(1)}(y_f)}{\partial y_f} &= 0 ; \quad \frac{\partial H_{o2}^{(1)}(y_f)}{\partial y_f} = 0 \\
\frac{\partial H_{l1}^{(1)}(y_f)}{\partial y_f} &= 1 ; \quad \frac{\partial H_{l2}^{(1)}(y_f)}{\partial y_f} = 0 \\
\end{align*}
(3-13b)
\]

(b) At \( y_f = b_f \) (i.e. at edge 2 of the strip)

\[
\begin{align*}
H_{o1}^{(1)}(y_f) &= 0 \quad ; \quad H_{o2}^{(1)}(y_f) = 1 \\
H_{l1}^{(1)}(y_f) &= 0 \quad ; \quad H_{l2}^{(1)}(y_f) = 0 \\
\end{align*}
(3-13c)
\]

and

\[
\begin{align*}
\frac{\partial H_{o1}^{(1)}(y_f)}{\partial y_f} &= 0 ; \quad \frac{\partial H_{o2}^{(1)}(y_f)}{\partial y_f} = 0 \\
\frac{\partial H_{l1}^{(1)}(y_f)}{\partial y_f} &= 0 ; \quad \frac{\partial H_{l2}^{(1)}(y_f)}{\partial y_f} = 1 \\
\end{align*}
(3-13d)
\]

Therefore, the displacement functions (Eqs. 3-11) along the two edges of the strip can be written as:

(a) **Edge 1**

\[
u_f(x,0) = \sum_{n=1,2}^{N} u_{f1}^{(n)} \cdot x_{u_f}^{(n)}
\]  
(3_14a)
\[ v_f(x, o) = \sum_{n=1, 2} v_{f1}^{(n)} \cdot x_{v_f}^{(n)} \]  

\[ w(x, o) = \sum_{n=1, 2} w_{1}^{(n)} \cdot x_{w}^{(n)} \]  

and the first partial derivative of the transverse deflection with respect to \( y \) is given by:

\[ w_y(x, o) = \sum_{n=1, 2} w_{y1}^{(n)} \cdot x_{w}^{(n)} \]

(b) **Edge 2**

\[ u_f(x, b_f) = \sum_{n=1, 2} u_{f2}^{(n)} \cdot x_{u_f}^{(n)} \]  

\[ v_f(x, b_f) = \sum_{n=1, 2} v_{f2}^{(n)} \cdot x_{v_f}^{(n)} \]  

\[ w(x, b_f) = \sum_{n=1, 2} w_{2}^{(n)} \cdot x_{w}^{(n)} \]  

\[ w_y(x, b_f) = \sum_{n=1, 2} w_{y2}^{(n)} \cdot x_{w}^{(n)} \]

Thus, the displacement functions along the two edges of the strip are described by the variation of the undetermined
edge displacement coefficients \( u_{fj}^{(n)} , v_{fj}^{(n)} , w_{j}^{(n)} \), etc.) along the entire length of the strip.

The geometric admissibility conditions between adjacent strips can be satisfied by imposing continuity of the displacement coefficients of the adjacent strips at their common edges. This can be achieved by linking the displacement coefficients of adjacent strips, say I and II as follows (Fig. 10):

\[
\begin{align*}
&u_{f}^{I}(x,b_{f}) = u_{f}^{II}(x,o) \quad \text{or} \quad u_{f2}^{(n)} = u_{f1}^{(n)} II \\
&\quad \text{for } n = 1,2,\ldots,N \tag{3-16a} \\
&v_{f}^{I}(x,b_{f}) = v_{f}^{II}(x,o) \quad \text{or} \quad v_{f2}^{(n)} = v_{f1}^{(n)} II \\
&\quad \text{for } n = 1,2,\ldots,N \tag{3-16b} \\
&w_{x}^{I}(x,b_{f}) = w_{x}^{II}(x,o) \quad \text{or} \quad w_{2}^{(n)} = w_{1}^{(n)} II \\
&\quad \text{for } n = 1,2,\ldots,N \tag{3-16c} \\
&w_{y}^{I}(x,b_{f}) = w_{y}^{II}(x,o) \quad \text{or} \quad w_{y2}^{(n)} = w_{y1}^{(n)} II \\
&\quad \text{for } n = 1,2,\ldots,N \tag{3-16d}
\end{align*}
\]

Note that Eqs. 3-16 represent the variable correlation scheme. In the case of non-coplanar strips (i.e. folded
plate structures), the displacement coefficients \( u_{f}^{(n)} \) and \( v_{f}^{(n)} \) etc.) in Eqs. 3.16 should be with respect to the reference coordinate system of the whole structure; that is, the linking procedure is accomplished only after transforming all the displacement coefficients of the various strip from the local coordinate system of each strip to the reference coordinate system of the structure. It is pointed out that the transverse shear strains of the core will automatically be continuous at the common edge of adjacent strips by satisfying the geometric admissibility conditions.

The third order interpolation polynomials provide the capability of imposing membrane strain continuity of the reference surface of the faces since the displacement expressions for \( u_{f} \) and \( v_{f} \) contain displacement coefficients which represent membrane strains (i.e., \( u_{yf_{j}}^{(n)} \) and \( v_{yf_{j}}^{(n)} \)). The strain continuity between two adjacent strips I and II (Fig. 10) can be achieved as follows:

\[
\frac{\partial u_{f}(x, b_{f})^{I}}{\partial y_{f}} = \frac{\partial u_{f}(x, 0)^{II}}{\partial y_{f}} \quad \text{or} \quad u_{yf_{2}}^{(n)} = u_{yf_{1}}^{(n)} \quad \text{for } n = 1, 2, \ldots N
\]  

(3-17a)

\[
\frac{\partial v_{f}(x, b_{f})^{I}}{\partial y_{f}} = \frac{\partial v_{f}(x, 0)^{II}}{\partial y_{f}} \quad \text{or} \quad v_{yf_{2}}^{(n)} = v_{yf_{1}}^{(n)} \quad \text{for } n = 1, 2, \ldots N
\]  

(3-17b)
Note that the first partial derivative of \( u_f \) and \( v_f \) with respect to \( x \) (i.e. \( u_{xf} \) and \( v_{xf} \)) are both continuous along the common edges of adjacent strips. For example at \( y_f = 0 \)

\[
\frac{\partial u_f(x,0)}{\partial x} = \sum_{n=1,2} u_{f1}^{(n)} \cdot x_{u_f}^{(n)} \quad (3-18a)
\]

\[
\frac{\partial v_f(x,0)}{\partial x} = \sum_{n=1,2} v_{f1}^{(n)} \cdot x_{v_f}^{(n)} \quad (3-18b)
\]

where \( x_{u_f}^{(n)} \) and \( x_{v_f}^{(n)} \) are the first derivatives of the basic functions associated with \( u_f \) and \( v_f \) displacement functions respectively. From the preceding discussions and by considering the force-deformation relations of the faces (Eq. 2-3), it can be seen that for coplanar sandwich elements having zero values of the coupling stiffness matrix \([B]^{(f)}\), the third order polynomials provide continuity of the stress resultants, \( N_{xf} \), \( N_{yf} \) and \( N_{xyf} \), between adjacent strips. However, continuity for the moment resultants, \( M_{xf} \) and \( M_{yf} \), cannot be achieved along the common lines of discretized strips since the transverse deflection (expressed as third order polynomials) does not contain displacement coefficients which represent the second derivative of the transverse deflection (i.e. \( w_{yyf}^{(n)} \)). However, for sandwich systems the problem of continuity of the moment resultants is of secondary importance since the
sandwich system resists the applied loads mainly by membrane actions in the faces. For sandwich structural systems having unbalanced laminated faces, continuity for both the force and moment resultants cannot be achieved along the common lines of adjacent strips because of the existence of the bending-membrane coupling in the faces.

The preceding continuity problems can be overcome using fifth order interpolation polynomials to represent the transverse deflection, as follows:

\[
w(x, y_f) = \sum_{n=1, 2} \sum_{j=1}^{2} \left[ H_{0j}^{(2)}(y_f) \cdot w_j^{(n)} + H_{1j}^{(2)}(y_f) \cdot w_{y_j}^{(n)} + H_{2j}^{(2)}(y_f) \cdot w_{yy_j}^{(n)} \right] + \chi_{n}^{(n)}
\]

where \( w_j^{(n)} \), \( w_{y_j}^{(n)} \), and \( w_{yy_j}^{(n)} \) are the undetermined displacement coefficients at the two edges of the strip (i.e., \( j = 1, 2 \)) for the transverse deflection. \( H_{kj}^{(2)}(y_f) \) are the one-dimensional fifth order interpolation polynomials given by:

\[
H_{01}^{(2)}(y_f) = \left[ b_f^5 - 10 b_f^2 y_f^3 + 15 b_f y_f^4 - 6 y_f^5 \right] / b_f^5
\]

\[
H_{02}^{(2)}(y_f) = \left[ 10 b_f^2 y_f^3 - 15 b_f y_f^4 + 6 y_f^5 \right] / b_f^5
\]
\[ H^{(2)}_{11}(y_f) = \frac{b_f^4 y_f - 6 b_f^2 y_f^3 + 8 b_f y_f^4 - 3 y_f^5}{b_f^4} \]
\[ H^{(2)}_{12}(y_f) = \frac{-4 b_f^2 y_f^3 + 7 b_f y_f^4 - 3 y_f^5}{b_f^4} \]
\[ H^{(2)}_{21}(y_f) = \frac{b_f^3 y_f^2 - 3 b_f^2 y_f^3 + 3 b_f y_f^4 - y_f^5}{2 b_f^3} \]
\[ H^{(2)}_{22}(y_f) = \frac{b_f^2 y_f^3 - 2 b_f y_f^4 + y_f^5}{2 b_f^3} \]  

(3-20)

As with third order interpolation polynomials, the expressions for the transverse deflection and its first and second partial derivatives with respect to \( y \) can, for example at edge 1 \((y_f = 0)\), be written as:

\[ w(x, o) = \sum_{n=1,2}^{N} w^{(n)}_w \cdot x^{(n)}_w \]  

(3-21a)

\[ w_Y(x, o) = \sum_{n=1,2}^{N} w^{(n)}_{y1} \cdot x^{(n)}_w \]  

(3-21b)

\[ w_{YY}(x, o) = \sum_{n=1,2}^{N} w^{(n)}_{yY1} \cdot x^{(n)}_w \]  

(3-21c)

Similar expressions can be written at edge 2 \((y_f = b_f)\) of the strip. Since the displacement expression contains the
terms \( w_{yy_j}^{(n)} (j = 1,2) \), continuity of the stress and moment resultants along the common lines of adjacent strips can be achieved for sandwich systems having orthotropic or unbalanced laminated faces. This can be obtained by imposing the following condition between adjacent strips (I and II), as well as considering the conditions given in Eqs. 3-17.

\[
\frac{\partial^2 w(x,b_2)}{\partial y_f^2} = \frac{\partial^2 w(x_0)}{\partial y_f^2} \quad \text{or} \quad w_{yy_2}^{(n)} = w_{yy_1}^{(n)}
\]

for \( n = 1,2, \ldots N \)  \( (3-22) \)

Note that for the cases where the strips do not have the same thickness, continuity of the moment resultants cannot be achieved at the common edges of discretized strips. It is pointed out that by expressing \( w \) as fifth order interpolation polynomials will result in an increase in the number of degrees of freedom of the structural system.

While it is essential that the linking conditions resulting from the admissibility requirements be carried out, it is often possible to impose linking conditions which are based on the experience and judgement of the analyst. For example, a sandwich system might be constructed and loaded such that it is known that the reference surface membrane displacements and strains of the upper and lower faces are equal in magnitude and opposite in sign. The additional conditions in such a case can be expressed by the following:
\[ u_1(x, y_1) = -u_2(x, y_2) \quad \text{or} \quad u_{1j}^{(n)} = -u_{2j}^{(n)} \quad (3-23a) \]

\[ v_1(x, y_1) = -v_2(x, y_2) \quad \text{or} \quad v_{1j}^{(n)} = -v_{2j}^{(n)} \quad (3-23b) \]

\[ \frac{\partial u_1(x, y_1)}{\partial y_1} = -\frac{\partial u_2(x, y_2)}{\partial y_2} \quad \text{or} \quad u_{y1j}^{(n)} = -u_{y2j}^{(n)} \quad (3-23c) \]

\[ \frac{\partial v_1(x, y_1)}{\partial y_1} = -\frac{\partial v_2(x, y_2)}{\partial y_2} \quad \text{or} \quad v_{y1j}^{(n)} = -v_{y2j}^{(n)} \quad (3-23d) \]

for all values of \( n = 1, 2, \ldots, N \). In such cases, the additional conditions reduce the number of degrees of freedom of the structural system. Other types of addition conditions will be given in Chapter V, associated with the corresponding structural systems.

The boundary conditions in the \( y_f \)-direction of a structure are satisfied by imposing the displacement coefficients to satisfy the prescribed boundary conditions. For example, in the case of a simply supported sandwich plate (at \( x = 0, L \) and \( y_f = 0, B_f \), where \( B_f \) is the width of the plate), the displacement boundary conditions can be described by

\[ u_f(x, y_f) = w(x, y_f) = 0 \quad \text{at} \quad y_f = 0, B_f \quad (3-24a) \]

The satisfaction of these displacement boundary conditions
is achieved through the variable correlation scheme in which the following conditions are imposed on the displacement coefficient

\[
u^{(n)}_f = w(n) = 0 \quad \text{along the boundaries } y_f = 0, B_f.
\]

(3-24b)

The force boundary conditions in the \(y_f\)-direction can be described by:

\[
N_{y_f}(x, y_f) = M_{y_f}(x, y_f) = 0 \quad \text{at } y_f = 0, B_f
\]

(3-25a)

These force boundary conditions will be inherently satisfied at the true minimum of the total potential energy. However, if the force boundary conditions are to be satisfied "a priori" conditions can be imposed on the displacement coefficients through the variable correlation scheme as follows: consider the case where the faces of the sandwich plate are orthotropic. The boundary condition \(N_{y_f}(x, y_f) = 0\) can be satisfied by imposing the following conditions:

\[
u^{(n)}_f = 0 \quad \text{along the boundaries } y_f = 0, B_f
\]

(3-25b)

The condition \(M_{y_f}(x, y_f) = 0\) can be satisfied only if the transverse deflection is expressed as fifth order interpolation polynomials and the following conditions are satisfied:
\[ w^{(n)}_{yy} = 0 \quad \text{along the boundaries} \quad y_f = 0, B_f \quad (3-25c) \]

In case that the faces of the sandwich plate are constructed from unbalanced laminates (for example, cross ply laminates, Appendix A), both conditions (Eqs. 3.25b,c) are required to satisfy each of the force boundary conditions (Eq. 3-25a).

### 3.2.1 Basic Functions

The basic functions for structural systems are derived such that both force and displacement boundary conditions at \( x = 0, L \) are satisfied (Fig. 9). The basic functions for thin isotropic barrel shell structures with various end conditions were derived in Ref. 25. It was mentioned that the basic functions can be obtained by seeking the solution of the differential equation of a vibrating beam spanning between the two ends of the structure. The differential equation is given by:

\[
\frac{\dddot{X}}{4} - \frac{\lambda}{L^4} \cdot X(x) = 0 \quad (3-26)
\]

where

\( L \) - The length of the beam between the two supports (ends).

\( \lambda \) - A certain parameter connected with the frequency of the natural vibration of a beam.

\( X(x) \) - Represent the basic functions.

The general integral of the homogeneous differential
Equation 3-26 can be represented in the following form:

\[ x(x) = c_1 \sin \frac{\lambda x}{L} + c_2 \cos \frac{\lambda x}{L} + c_3 \sinh \frac{\lambda x}{L} + c_4 \cosh \frac{\lambda x}{L} \]  

(3-27)

where \( c_1, c_2, c_3 \) and \( c_4 \) are arbitrary constants, which can be obtained by satisfying the force and displacement boundary conditions at the two ends of the beam (strip). This in turn defines the basic functions for the given boundary conditions. The determination of the basic functions for the isotropic structures with various boundary conditions are given in Appendix B. The same procedure can be used to derive the basic functions for thin laminated structures as well as sandwich structures with various types of facings (i.e. isotropic, anisotropic, ...etc.) and boundary conditions. Since the force-deformation relations (Eqs. 2-3) for a composite system depend upon the various properties of the layers that form the laminated system, the basic functions for thin laminated structures will vary depending on the properties of the layers, for the same boundary conditions. However, for particular types of composites the basic functions for thin isotropic structures can be used as the basic functions for thin laminated structural systems. For such cases the basic functions for the isotropic structure (when used in conjunction with the force-deformation
relations for the laminated system under consideration) permit the satisfaction of the force and displacement boundary conditions at the two ends of the strip. For example, consider the case of a simply supported thin laminated cylindrical shell constructed from cross-plied composites (Appendix A). The boundary conditions of the structure are:

\[ w(x, y) = v(x, y) = M_x(x, y) = N_x(x, y) = 0 \]

at \( x = 0, L \) \hspace{1cm} (3-28)

Considering the force-deformation relations for the cross-plied composites, it can be shown that the basic functions of isotropic simply supported cylindrical shells (Appendix B) will satisfy both the force and displacement boundary conditions for the simply supported thin cross-plied shell. Thus, for this case the basic functions are:

\[ X_u^{(n)} = \cos \left( \frac{n \pi x}{L} \right) \hspace{1cm} (3-29a) \]

\[ X_v^{(n)} = \sin \left( \frac{n \pi x}{L} \right) \hspace{1cm} (3-29b) \]

\[ X_w^{(n)} = \sin \left( \frac{n \pi x}{L} \right) \hspace{1cm} (3-29c) \]

where \( L \) is the length of the shell and \( n = 1, 2, \ldots N \). The basic functions for thin laminated structures for various composites and boundary conditions will be presented in
Chapter V.

Since the displacement behaviour of sandwich systems is expressed in terms of the membrane displacements of the faces \( u_f \) and \( v_f \) and the transverse deflection of the sandwich system, by considering a single face of the sandwich system the problem is converted to one which deals with thin structures. Therefore, in deriving the basic functions of a sandwich system, each face can be treated as a thin structural system; that is, the basic functions for each face are the same as those of the corresponding thin structure, if the resulting basic function for the transverse deflection is the same for both faces. For example, consider an isotropic sandwich plate having two adjacent edges simply supported while the other pair are clamped. The boundary conditions of the structure are:

\[
\begin{align*}
    w(x, y_f) = v_f(x, y_f) = M_{x_f}(x, y_f) = N_{x_f}(x, y_f) &= 0 \\
    \text{at } x = 0 \\
    u_f(x, y_f) = v_f(x, y_f) &= w(x, y_f) = w_x(x, y_f) = 0 \\
    \text{at } x = L
\end{align*}
\]

(3-30a)

(3-30b)

The basic functions for such a system can be represented by the following expressions:
\begin{align}
X_w^{(n)} &= \sin \left( \frac{\lambda_n \cdot x}{L} \right) - a_n \sinh \left( \frac{\lambda_n \cdot x}{L} \right) \quad \ldots \quad (3-31a) \\
X_{u_f}^{(n)} &= \cos \left( \frac{\lambda_n \cdot x}{L} \right) - a_n \cosh \left( \frac{\lambda_n \cdot x}{L} \right) \quad \ldots \quad (3-31b) \\
X_{v_f}^{(n)} &= \sin \left( \frac{\lambda_n \cdot x}{L} \right) - a_n \sinh \left( \frac{\lambda_n \cdot x}{L} \right) \quad \ldots \quad (3-31c) \\
where \\
\lambda_n &= \left( \frac{4n + 1}{4} \right) \pi \quad ; \quad a_n = \frac{\sin(\lambda_n)}{\sinh(\lambda_n)} \quad \ldots \quad (3-31d)
\end{align}

and \( n = 1, 2, \ldots N \)

The previous expressions of basic functions of isotropic sandwich plates (Eqs. 3-31) are the same as those for isotropic thin shells having the same boundary conditions (Appendix B). It is pointed out that the basic functions of shell structures can be used for plates having the same material properties and boundary conditions (at \( x = 0, L \)) as those of the shell. This can be realized by examining the force-deformation relations (Eqs. 2-3).

Some classes of basic functions possess the valuable property of orthogonality:
\[ \int_{x=0}^{x=L} x^{(n)}(x) \cdot x^{(m)}(x) \cdot dx = 0 \quad \text{for } n \neq m \]  
\text{(3-32a)}

\[ \int_{x=0}^{x=L} x^{(n)}(x) \cdot x^{(m)}(x) \cdot dx = 0 \quad \text{for } n \neq m \]  
\text{(3-32b)}

where \( x^{(n)}(x) \) and \( x^{(m)}(x) \) are the second derivatives of the basic function associated with cycles \( n \) and \( m \) respectively.

By examining both sets of basic functions presented in this Chapter (Eqs. 3.29 and 3.31) it can be shown that the basic functions given by Eqs. 3.29 possess the orthogonality property, while the other set of basic functions (Eqs. 3.31) are non-orthogonal. The property of orthogonality has a great influence on the size of the strip and overall matrices of the structure. For orthogonal basic functions the resulting stiffness matrices are smaller and have a narrow band width compared to the non-orthogonal cases.

### 3.3 Development of Strip Stiffness Matrix

As previously mentioned, the total strain energy of a sandwich strip (Eq. 3-9) is expressed in terms of the section stiffnesses, the geometry of the strip and the membrane displacement of the faces and the transverse deflection of the sandwich system. Depending on the type of structural system, as well as the boundary conditions, a suitable set of displacement functions of the strip can be obtained. Substitution of the assumed displacement
patterns into the appropriate strain energy expressions for the structural system under consideration and integrating over the reference surface area of the strip result in the total discretized strain energy of the $i^{th}$ strip, which can be expressed by the following:

$$ U_{(i)} = \frac{1}{2} \{D\}^T \{K_S\} \{D\} \quad (3-33a) $$

where

$$ \{K_S\} - \text{The strip stiffness matrix} $$

$$ \{D\}^T = \begin{bmatrix} \{D(1)\}^T, \{D(2)\}^T, \ldots, \{D(n)\}^T \ldots \end{bmatrix} \quad (3-33b) $$

and $\{D(n)\}^T$ - Contains the undetermined displacement coefficients at the two edges of the strip associated with the $n^{th}$ cycle, and $N$ is the total number of cycles used in the basic functions.

For sandwich systems, $\{D(n)\}$ can be expressed by the following:

$$ \{D(n)\}^T = \begin{bmatrix} \{D(n)\}^T, \{D(n)\}^T, \{D(n)\}^T, \{D(n)\}^T \ldots \end{bmatrix} \quad (3-34) $$

$$ \{D_{u1}^T, \{D_{u2}^T, \{D_{v1}^T \ldots \} \} \} $$

$$ \{D_{v2}^T, \{D_{v2}^T, \{D_{w}^T \ldots \} \} \} $$

$$ \{D_{v2}^T, \{D_{v2}^T, \{D_{w}^T \ldots \} \} \} $$
where

\[
\{D^{(n)}_{u1}\}^T \quad \text{and} \quad \{D^{(n)}_{u2}\}^T
\]

- Contain the undetermined displacement coefficients at the two edges of the strip for the \(u_f(x, y_f)\) displacement functions for the lower and upper face respectively associated with the \(n^{th}\) cycle.

\[
\{D^{(n)}_{v1}\}^T \quad \text{and} \quad \{D^{(n)}_{v2}\}^T
\]

- Contain the undetermined displacement coefficients at the two edges of the strip for the \(v_f(x, y_f)\) displacement functions for the lower and upper face respectively associated with the \(n^{th}\) cycle.

\[
\{D^{(n)}_{w}\}
\]

- Contains the undetermined coefficients at the two edges of the strip for the \(w(x, y_f)\) displacement function associated with the \(n^{th}\) cycle.

In the case where all the displacement functions are expressed as third order interpolation polynomials (Eqs. 3-11), the following relations can be written:

\[
\{D^{(n)}_{u1}\}^T = \begin{bmatrix} u^{(n)}_{11}, u^{(n)}_{12}, u^{(n)}_{y1}, u^{(n)}_{y2} \end{bmatrix} \quad (3-35a)
\]

\[
\{D^{(n)}_{u2}\}^T = \begin{bmatrix} u^{(n)}_{21}, u^{(n)}_{22}, u^{(n)}_{y1}, u^{(n)}_{y2} \end{bmatrix} \quad (3-35b)
\]

\[
\{D^{(n)}_{v1}\}^T = \begin{bmatrix} v^{(n)}_{11}, v^{(n)}_{12}, v^{(n)}_{y1}, v^{(n)}_{y2} \end{bmatrix} \quad (3-35c)
\]

\[
\{D^{(n)}_{v2}\}^T = \begin{bmatrix} v^{(n)}_{21}, v^{(n)}_{22}, v^{(n)}_{y1}, v^{(n)}_{y2} \end{bmatrix} \quad (3-35d)
\]

\[
\{D^{(n)}_{w}\}^T = \begin{bmatrix} w^{(n)}_{1}, w^{(n)}_{2}, w^{(n)}_{y1}, w^{(n)}_{y2} \end{bmatrix} \quad (3-35e)
\]
In the case where the transverse deflection is expressed as fifth order interpolation polynomials (Eq. 3.19), \( \{D_w^{(n)}\} \) can be written as:

\[
\{D_w^{(n)}\} = \left\{ w_1^{(n)}, w_2^{(n)}, w_y^{(n)}, w_{yy}^{(n)} \right\} \tag{3-36}
\]

For thin structural systems, \( \{D^{(n)}\} \) can be written as:

\[
\{D^{(n)}\}^T = \left\{ \{D_u^{(n)}\}^T, \{D_y^{(n)}\}^T, \{D_w^{(n)}\}^T \right\} \tag{3-37}
\]

In the case where all the displacement functions are expressed as third order interpolation polynomials, the following expression can be written:

\[
\{D^{(n)}\}^T = \left\{ u_1^{(n)}, u_2^{(n)}, u_y^{(n)}, u_{yy}^{(n)}, v_1^{(n)}, v_2^{(n)}, v_y^{(n)}, v_{yy}^{(n)}, w_1^{(n)}, w_2^{(n)}, w_y^{(n)}, w_{yy}^{(n)} \right\} \tag{3-38}
\]

The strip stiffness matrix \([K_S]\) (Eq. 3-33a), in its most generalized form, can be partitioned as:
where \([K_S^{(n,n)}]\) is a symmetric partitioned stiffness matrix associated with the undetermined edge displacement coefficients of cycle number \(n\) of the basic functions. For the case of sandwich system where the transverse deflection is expressed as a fifth order interpolation polynomial \([K_S^{(n,n)}]\) can be expressed as follows:
The submatrices on the diagonal of Eq. 3-40 are symmetric, while the off-diagonal arrays are generally non-symmetric. 

\[ [K_S^{(n,m)}] \] in Eq. 3-39 is a partitioned stiffness matrix associated with the undetermined edge displacement coefficients of cycles \( n \) and \( m \) of the basic functions. Therefore, it is a coupled partitioned stiffness matrix which represents the coupling between cycles \( n \) and \( m \) of the basic functions. For the case of a sandwich system where the transverse deflection is expressed as a fifth order interpolation polynomial, \( [K_S^{(n,m)}] \) can be partitioned as:

\[
\begin{bmatrix}
K_S^{(n,n)} & 0 & 0 & 0 \\
0 & K_S^{(n,n)} & 0 & 0 \\
0 & 0 & K_S^{(m,m)} & 0 \\
0 & 0 & 0 & K_S^{(m,m)}
\end{bmatrix}
\]
Generally, the coupled partitioned matrix \([K_S^{(n,m)}]\) is unsymmetrical. In case that all the displacement variables \(u_f, v_f\) and \(w\) are expressed as third order interpolation polynomials, all the submatrices in Eqs. 3-40 and 3-41 will be 4 x 4 dimensional arrays. The various elements of the submatrices are given in Appendix C.
The coupled partitioned matrices appear in the strip stiffness matrix (Eq. 3-39) only if the basic functions used are non-orthogonal. Therefore, for orthogonal basic functions all the elements of the coupled partitional matrices vanish. In such a case, all the cycles are independent (i.e. uncoupled). Therefore, the analysis is converted from one which seeks a solution for all the degrees of freedom of the various cycles as a unit (i.e. coupled cases (Eq. 3-39)) to another in which each cycle is dealt with independently and the final solution is the sum of the solutions of the various cycles. Thus, for orthogonal basic functions the resulting strip stiffness matrix is small compared to the non-orthogonal cases where the number of degrees of freedom of the system is equal to $N$ times the number of degrees of freedom for one cycle.

3.4 Potential of the Applied Loads

The potential of the applied loads is generated using a work equivalent load approach. The external work expression for a face of the $i^{th}$ sandwich strip can be expressed by (Ref. 7):
$$w^{(f)}(i) = \int_{S_f} \left[ p_{x_f} \cdot u_f(x, y_f) + p_{y_f} \cdot v_f(x, y_f) \right] dS_f + \oint_{y_f} \left[ \bar{N}_{x_f} \cdot u_f(x, y_f) \right. + \bar{N}_{y_f} \cdot v_f(x, y_f) - \bar{N}_{x} \cdot w(x, y_f)
abla x_f$$

$$+ \bar{M}_{y_f} \cdot w(x, y_f) + \bar{Q}_{xz_f} \cdot w(x, y_f) - \bar{M}_{x_f} \cdot w(x, y_f)$$

$$- \bar{M}_{y_f} (w(x, y_f) - \frac{1}{R_f} v_f(x, y_f)) \right] \, dy_f$$

$$- \oint_{x} \left[ \bar{N}_{xy_f} \cdot u_f(x, y_f) + \bar{N}_{y_f} \cdot v_f(x, y_f) \right. + \bar{Q}_{yz_f} \cdot w(x, y_f) - \bar{M}_{yx_f} \cdot w(x, y_f)$$

$$- \bar{M}_{y_f} (w(x, y_f) - \frac{1}{R_f} v_f(x, y_f)) \right] \, dx \quad (3.42)$$

where $p_{x_f}$, $p_{y_f}$, and $p_{z_f}$ are applied reference surface tractions in the $x$, $y$, and $z$ directions respectively.

The applied forces $\bar{N}_{x_f}$, $\bar{N}_{y_f}$, etc. and moments $\bar{M}_{x_f}$, $\bar{M}_{y_f}$, etc. are the components acting on the edges of the faces and are positive when they act in the same direction as the force and moment resultants shown in Fig. 4.
The work equivalent loads for the \( i \)th strip are calculated by substituting the assumed displacement patterns into the expression of the potential of the applied loads (Eq. 3-42), and then performing the indicated integrations. The work done by the applied loads can be expressed as:

\[
W_{(i)} = \sum_{f=1}^{2} W_{(i)}^{(f)} = \{P_{S}\}^T \{D\}
\]  

(3-43)

where

\[
\{P_{S}\}^T = \begin{bmatrix}
\{P_{S}^{(1)}\}^T, \{P_{S}^{(2)}\}^T, \ldots, \{P_{S}^{(n)}\}^T
\end{bmatrix}
\]

(3-44)

and

\( \{P_{S}^{(n)}\} \) - Contains the work equivalent loads associated with each of the undetermined edge displacement coefficients in \( \{D^{(n)}\} \) for the \( n \)th cycle.

The derivations of the work equivalent loads for the different structural systems considered in this work are presented in Appendix E.

3.5 Discretized Potential Energy

The discretized potential energy of the \( i \)th sandwich strip is represented by:

\[
\Pi_{p(i)} = U(i) - W(i)
\]

(3-45)
where \( U(i) \) and \( W(i) \) are given by Eq. (3-33a) and Eq. (3-43) respectively.

As previously mentioned, in the case of orthogonal basic functions, the resulting strip stiffness matrix is uncoupled. Therefore, the solution can be obtained by determining the degrees of freedom of each cycle independently, and the final solution is the sum of the solutions of the various cycles. For such systems, the discretized potential energy for the \( n \)th cycle can be written as:

\[
\pi_P^{(n)} = U^{(n)}(i) - W^{(n)}(i) \tag{3-46}
\]

or

\[
\pi_P^{(n)} = \xi \cdot \{D^{(n)}\}^T \{K_S^{(n,n)}\} \{D^{(n)}\} - \{P_S^{(n)}\}^T \{D^{(n)}\} \tag{3-47}
\]

where \( \{D^{(n)}\} \), \( \{K_S^{(n,n)}\} \) and \( \{P_S^{(n)}\} \) are given by Eqs. 3-34, 3-40 and 3-44 respectively. Correspondingly, the total potential energy for an assemblage of coplanar strips is approximated by:

\[
\pi_P^{(n)} = \sum_{i=1}^{I} \pi_P^{(n)}(i) \tag{3-48}
\]

where \( I \) is the number of the strips used to model the original structure. The assemblage of the strips is achieved using the variable correlation scheme, in which
the continuity conditions between adjacent strips are imposed as well as the boundary conditions in the $y_f$-direction of the structure. Therefore, Equation 3-48 can be written as:

$$
\pi_p^{(n)} = \frac{1}{2} \{ X^{(n)} \}^T \{ K^{(n,n)} \} \{ X^{(n)} \} - \{ P^{(n)} \}^T \{ X^{(n)} \}
$$

(3-49)

where

$\{ X^{(n)} \}$ - Contains the independent degrees of freedom of the assembled structure associated with the $n^{th}$ cycle.

$\{ K^{(n,n)} \}$ - The overall stiffness matrix for the structure associated with the degrees of freedom residing in the vector $\{ X^{(n)} \}$.

$\{ P^{(n)} \}$ - Contains the work equivalent loads associated with each degree of freedom residing in the vector $\{ X^{(n)} \}$.

By virtue of the principle of minimum total potential energy, equilibrium positions are associated with the displacement states for which $\pi_p^{(n)}$ is minimum. This condition is achieved when

$$
\frac{\partial \pi_p^{(n)}}{\partial X_j^{(n)}} = 0 \quad j = 1, 2, \ldots, J^{(n)}
$$

(3-50)

where $J^{(n)}$ is the total number of independent degrees of freedom for the $n^{th}$ cycle. The minimization procedure will lead to the following relation.

$$
\{ K^{(n,n)} \} \{ X^{(n)} \} = \{ P^{(n)} \}
$$

(3-51)
The solution for the resulting set of simultaneous equations is obtained by standard numerical techniques such as Gaussian elimination method, from which the independent degrees of freedom \( \{x^{(n)}\} \) for the \( n^{th} \) cycle are obtained. By repeating the same procedure for each cycle, the final results are achieved by substituting the resulting degrees of freedom into the displacement relations (Eqs. 3.10) and summing the results obtained for the various cycles.

In case that the basic functions are non-orthogonal, the resulting strip stiffness matrix is coupled. Therefore, the determination of the degrees of freedom of the various cycles (i.e. \( n = 1,2,\ldots,N \)) is done in one step. For such systems the discretized potential energy for the \( i^{th} \) strip is represented by:

\[
\pi_{P(i)} = \frac{1}{2} \{D\}^T [K_S] \{D\} - \{P_S\}^T \{D\} \tag{3-52}
\]

where \([D],[K_S]\) and \([P_S]\) are given by Eqs. 3-33b, 3-39 and 3-44 respectively. Correspondingly, the total potential energy for an assemblage of strips is given by:

\[
\pi_P = \frac{1}{2} \{X\}^T [K] \{X\} - \{P\}^T \{X\} \tag{3-53}
\]

where

\([X]\) - Contains the independent degrees of freedom for the assembled structure for the total number of cycles, that is,
\( \{x\}^T = \left\{ \{x^{(1)}\}^T, \{x^{(2)}\}^T, \ldots, \{x^{(n)}\}^T, \ldots \right\} \)

\[
[K] = \text{The overall stiffness matrix for the whole structure.}
\]

Similarly, by minimizing the total potential energy expression (Eq. 3-53),

\[
\frac{\partial \pi}{\partial x_j} = 0 \quad j = 1, 2, \ldots, J
\]

where \( J \) is the total number of independent degrees of freedom of the structure (i.e. \( J = N \times J_{(n)} \)). The minimization procedure leads to the following expression:

\[
[K] \{x\} = \{p\}
\]

Note that for orthogonal basic functions the solution of \( N \) sets of \( J_{(n)} \) equations is required. For the non-orthogonal cases the solution is obtained by solving \( J_{(n)} \times N = J \) equations simultaneously.

The stress and moment resultants are obtained by substituting the resulting displacements (Eqs. 3-10) into the force-deformation relations of the faces (Eq. 2.3) and the core (Eq. 2-11).
3.6 Discretized Potential Energy for Strips at Arbitrary Angles

The discussion in the previous section was limited to coplanar strips (i.e. plate problems) and for strips which form a curvilinear surface (i.e. shell problems). However, for the structures where the strips are joined at arbitrary angles (such as folded plates and shells with edge beams) it is necessary to work with respect to a reference coordinate system since the geometric admissibility conditions between adjacent strips can be imposed only if the displacement variables $u_f$, $v_f$, and $w$ of the two adjacent strips at their common edge are related to the same reference coordinate system. That is, the one-to-one linking of the edge displacement coefficients at the common edges between adjacent strips cannot be imposed without transforming all the displacement coefficients at the common edges to a reference coordinate system. The transformation can be done either by transforming the variables of the various strips from their local coordinate system to a single reference coordinate system, or by transforming the variables of a strip to a reference coordinate system which coincides with the local coordinate system of the adjacent strip at their common edge. The first transformation system is most suitable for folded plate structures, while the second transformation system is used for shell structures with edge beams where the transformation is only needed at the common lines between
the shell and the edge beams (Fig. 11). The relation between the edge displacement coefficients of the $i^{th}$ strip before and after transformation is given by:

$$\{D\} = [T] \cdot \{\tilde{D}\} \quad (3-57)$$

where

$[T]$ - The transformation matrix.

$\{\tilde{D}\}$ - The edge displacement coefficients after transformation to the reference coordinate system.

$\{D\}$ is given by Eq. 3-33b.

Substitution of Eq. 3-57 into the discretized potential energy for the $i^{th}$ strips (Eqs. 3-47 or 3-52) results in the transformed energy expression. For example, considering the case of non-orthogonal basic functions (Eq. 3-52), then

$$\pi_p^{(i)} = \frac{1}{2} \{\tilde{D}\}^T [\tilde{K}_S] \{[T] \cdot \{\tilde{D}\}\} - \{P_{S}\}^T [T] \cdot \{\tilde{D}\} \quad (3-58)$$

or

$$\pi_p^{(i)} = \frac{1}{2} \{\tilde{D}\}^T [\tilde{K}_S] \{\tilde{D}\} - \{\tilde{P}_{S}\}^T \{\tilde{D}\} \quad (3-59)$$

where $[\tilde{K}_S]$ and $\{\tilde{P}_{S}\}$ are the strip stiffness matrix and load vector for the $i^{th}$ strip in the reference coordinate system respectively, given by

$$[\tilde{K}_S] = [T]^T[K_S][T] \quad (3-60a)$$
\[ \{ \overline{F}_S \} = [T]^T \{ P_S \} \]  \hspace{1cm} (3-60b)

For the transformation system associated with folded plate structures, once the strip stiffness matrices of the various strips have been transformed into the reference coordinate system, it is possible to assemble the overall stiffness matrix for the whole structure in the conventional manner; therefore

\[ \eta_P = \frac{1}{\lambda} \{ \tilde{X} \} [\tilde{K}] \{ \tilde{X} \} - \{ \tilde{P} \}^T \{ \tilde{X} \} \]  \hspace{1cm} (3-61)

where

\[ \{ \tilde{X} \} \] - The independent degrees of freedom of the structure in the reference coordinate system.

\[ \{ \tilde{P} \} \] - The work equivalent loads in the reference coordinate system associated with the degrees of freedom residing in the vector \( \{ \tilde{X} \} \).

\[ [\tilde{K}] \] - The overall stiffness matrix for the structure in the reference coordinate system.

By minimizing the total potential energy (Eq. 3.61), a similar expression to Eq. 3-56 in the reference coordinate system is obtained.

\[ [\tilde{K}] \{ \tilde{X} \} = \{ \tilde{P} \} \]  \hspace{1cm} (3-62)
Once the vector \( \tilde{X} \) is obtained, the displacement variables in the local coordinate system can be obtained using Eq. 3-57. The displacements as well as the stress and moment resultants at the various locations in the domain of the structure can be obtained as described in the previous section.

For the transformation system associated with a shell with edge beams, once the strip stiffness matrices of the edge beams have been transformed to the new coordinate system, the analysis is done in the same way as in the previous section. The transformation matrices for both folded plates and shells with edge beams will be given in Chapter V.
CHAPTER IV

SERIES SOLUTION OF ANISOTROPIC RECTANGULAR SANDWICH PLATES

The object of this chapter is to present a double Fourier-series approach for predicting the behaviour of anisotropic rectangular sandwich plates under transverse loads having simply supported and clamped boundary conditions. The displacement behaviour of the anisotropic sandwich plate can be obtained by choosing displacement functions which satisfy the governing differential equations as well as the boundary conditions of the sandwich system. The reference surface displacements of the faces, $u_f$ and $v_f$ ($f=1,2$), and the transverse deflection, $w$, are expressed as the sum of products of undetermined parameters and double Fourier-series which describe the distribution of the displacement variables in the $x$ and $y$ directions. The transverse load, $q$, is also expressed in the double Fourier-series form. For the simply supported boundary conditions, the satisfaction of the boundary conditions is assured directly through the choice of the appropriate double series. However, for clamped boundary conditions a direct approach is not possible; the satisfaction of the boundary conditions can be achieved by imposing constraints on the assumed displacement functions.
such that the sum of the terms of the displacement functions at the boundaries satisfy the prescribed clamped boundary conditions. It is pointed out that the analysis of simply supported sandwich plates represents the first step in the analysis of sandwich plates with clamped boundary conditions. Thus, the analysis of sandwich plates with other boundary conditions (for example, two opposite simply supported sides and the other sides clamped) can be dealt with by imposing constraints on the displacement functions in one direction of the sandwich plate.

4-1 Equilibrium Equations

The equilibrium equations for the sandwich plates under transverse load \( q(x,y) \) can be obtained by studying the equilibrium of the force and moment resultants acting on each face separately as well as the force resultants acting on the core. From Figs. 4 and 7 the equilibrium equations are:

\[
\frac{\partial N_{x_1}}{\partial x} + \frac{\partial N_{xy_1}}{\partial y} + \frac{Q_{xz}}{t_c} = 0 \quad (4-1a)
\]

\[
\frac{\partial N_{x_2}}{\partial x} + \frac{\partial N_{xy_2}}{\partial y} - \frac{Q_{xz}}{t_c} = 0 \quad (4-1b)
\]

\[
\frac{\partial N_{xy_1}}{\partial x} + \frac{\partial N_{y_1}}{\partial y} + \frac{Q_{yz}}{t_c} = 0 \quad (4-1c)
\]
\[
\frac{\partial N_{y_2}}{\partial x} + \frac{\partial N_{y_2}}{\partial y} - \frac{Q_{y_2}}{t_c} = 0
\]  \hspace{1cm} (4-1d)

\[
\frac{e}{t_c} \left[ \frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} \right] + \sum_{f=1}^{2} \left[ \frac{\partial^2 M_{xf}}{\partial x^2} + 2 \frac{\partial^2 M_{xyf}}{\partial x \partial y} \right] + q = 0
\]  \hspace{1cm} (4-1e)

where \( e = d_1 + d_2 + t_c \).

It is pointed out the same set of equations can be obtained by taking the first variation of the total potential energy (Ref. 7) and considering the force deformation relations (Eqs. 2-3 and 2-11).

### 4.2 Governing Differential Equations for Sandwich Plates With Unbalanced Cross-Ply Faces

The governing differential equations for the sandwich plates with faces constructed of unbalanced cross-ply laminates can be obtained by substituting the force-deformation relations for the cross-plied faces (Eq. 2-3a and Appendix A) and the core (Eqs. 2-11 and 2-9) into the equilibrium equations. The resulting equations can be written in the following form:
\[
\begin{align*}
A^{(1)}_{11} u^{(1)}_{xx} + A^{(1)}_{66} u^{(1)}_{yy} + [A^{(1)}_{12} + A^{(1)}_{66}] v^{(1)}_{xy} - B^{(1)}_{11} w^{(1)}_{xxxx} & \\
+ \frac{B_{55}}{t^2} [u^{(2)} - u^{(1)} + e w_x] &= 0 \\
\end{align*}
\]

\[
\begin{align*}
A^{(2)}_{11} u^{(2)}_{xx} + A^{(2)}_{66} u^{(2)}_{yy} + [A^{(2)}_{12} + A^{(2)}_{66}] v^{(2)}_{xy} - B^{(2)}_{11} w^{(1)}_{xxxx} & \\
- \frac{B_{55}}{t^2} [u^{(2)} - u^{(1)} + e w_x] &= 0 \\
\end{align*}
\]

\[
\begin{align*}
[A^{(1)}_{12} + A^{(1)}_{66}] u^{(1)}_{xy} + A^{(1)}_{66} v^{(1)}_{xx} + A^{(1)}_{22} v^{(1)}_{yy} - B^{(1)}_{22} w^{(1)}_{yyyy} & \\
+ \frac{B_{44}}{t^2} [v^{(2)} - v^{(1)} + e w_y] &= 0 \\
\end{align*}
\]

\[
\begin{align*}
[A^{(2)}_{12} + A^{(2)}_{66}] u^{(2)}_{xy} + A^{(2)}_{66} v^{(2)}_{xx} + A^{(2)}_{22} v^{(2)}_{yy} - B^{(2)}_{22} w^{(2)}_{yyyy} & \\
- \frac{B_{44}}{t^2} [v^{(2)} - v^{(1)} + e w_y] &= 0 \\
\end{align*}
\]

\[
D^{(1)}_{11} w^{(1)}_{xxxx} + 2[D^{(1)}_{12} + 2 D^{(1)}_{66}] w^{(1)}_{xxyy} + D^{(1)}_{22} w^{(1)}_{yyyy} - B^{(1)}_{11} u^{(1)}_{xxx} - B^{(1)}_{11} u^{(2)}_{xxx} - B^{(1)}_{22} v^{(1)}_{yyyy} - B^{(2)}_{22} v^{(2)}_{yyyy} \\
- \frac{e}{t^2} \left[ B_{55} (u^{(2)} - u^{(1)} + e w_{xx}) \\
+ B_{44} (v^{(2)} - v^{(1)} + e w_{yy}) \right] &= q \\
\end{align*}
\]
In the preceding set of equations, the notations

\[ w_x = \frac{\partial w(x,y)}{\partial x} ; \quad w_{xxxx} = \frac{\partial^4 w(x,y)}{\partial x^4} ; \]

\[ u^{(f)} = u_f(x,y) ; \quad u_{xx}^{(f)} = \frac{\partial^2 u_f(x,y)}{\partial x^2} ; \]

\[ q = q(x,y), \ldots \text{ etc.} \]

have been adopted for convenience.

Also, \( D_{ij} = D_{ij}^{(1)} + D_{ij}^{(2)} \) where \((i,j = 1,2,6)\). For sandwich plates having orthotropic faces, all the coupling stiffnesses, \( B_{ij} \), vanish in Eqs. 4-2. The governing differential equations for sandwich plates having any other type of faces such as the angle-ply laminates can be obtained in a similar way. It is pointed out that the governing differential equations for thin cross-plied plates can be obtained by considering one face of the sandwich system and neglecting the terms associated with the core in Eqs. 4-2.

4-3 Rectangular Sandwich Plates With Four Simply Supported Sides

The simple supports are of the type which allows normal displacements on the boundaries but prevents lateral contraction (i.e. tangential displacements). Thus, the following boundary conditions apply:
\[ v^{(1)} = v^{(2)} = w = N_{x_1} = N_{x_2} = M_{x_1} + M_{x_2} = 0 \]

at \( x = 0, L \) \hspace{1cm} (4-3a)

\[ u^{(1)} = u^{(2)} = w = N_{y_1} = N_{y_2} = M_{y_1} + M_{y_2} = 0 \]

at \( y = 0, B \) \hspace{1cm} (4-3b)

Note that the above boundary conditions imply that the transverse shear strain vanishes along the edges of the core. That is,

\[ \gamma_{xz_c} = \frac{1}{t_c} [u^{(2)} - u^{(1)} + e \, w_x] = 0 \]

at \( y = 0, B \) \hspace{1cm} (4-4a)

\[ \gamma_{yz_c} = \frac{1}{t_c} [v^{(2)} - v^{(1)} + e \, w_y] = 0 \]

at \( x = 0, L \) \hspace{1cm} (4-4b)

The implied boundary conditions (Eqs. 4-4) are necessary for practical sandwich plate applications where it is normal practice to reinforce the border of the sandwich panel in order to prevent damage to the core due to the compressive reactions along the boundaries of the plate.

The boundary conditions (Eqs. 4-3 and 4-4) and the governing differential equations (Eq. 4-2) are identically
satisfied using the following assumed displacement functions:

\[ u^{(f)}(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}^{(f)} \cos \frac{m \pi x}{L} \sin \frac{n \pi y}{B} \]  \hspace{1cm} (4-5a)

\[ v^{(f)}(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{mn}^{(f)} \sin \frac{m \pi x}{L} \cos \frac{n \pi y}{B} \]  \hspace{1cm} (f = 1, 2) \hspace{1cm} (4-5b)

\[ w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m \pi x}{L} \sin \frac{n \pi y}{B} \]  \hspace{1cm} (4-5c)

In the solution that follows, the transverse load \( q \) is expanded into the double-Fourier series:

\[ q = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{m \pi x}{L} \sin \frac{n \pi y}{B} \]  \hspace{1cm} (4-6)

Substitution of Eqs. (4-5) and (4-6) into Eq. (4-2) yields the following simultaneous equations in terms of the undetermined coefficients, \( u_{mn}^{(1)}, u_{mn}^{(2)}, v_{mn}^{(1)}, v_{mn}^{(2)} \), and \( w_{mn} \), for each value of \( m \) and \( n \) (i.e., \( n, m = 1, 2, \ldots, \infty \)).
\[
\begin{bmatrix}
g_{11} & g_{12} & g_{13} & 0 & g_{15} \\
g_{22} & 0 & g_{24} & g_{25} \\
g_{33} & g_{34} & g_{35} & \\
g_{44} & g_{45} & \\
g_{55} & \\
\end{bmatrix}
\begin{bmatrix}
u_{mn}^{(1)} \\
u_{mn}^{(2)} \\
v_{mn}^{(1)} \\
v_{mn}^{(2)} \\
w_{mn} \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
q_{mn} \\
\end{bmatrix}
\]

(4-7)

Generally, the elements \( g_{ij} \) \((i,j = 1, 2, \ldots, 5)\) are functions of \( m \) and \( n \) (Appendix D). The solution of this set of simultaneous equations (Eq. 4-7) is also given in Appendix D. For these boundary conditions, the analysis is uncoupled; that is, for a particular value of \( m \) and \( n \) the solution is obtained and the final solution is the sum of the solutions of the various cycles. In the particular case of a rectangular sandwich plate under a uniformly distributed load of intensity \( q \), the series \( q_{mn} \) will have the following form:

\[
q_{mn} = \frac{16q}{\pi^2 mn} \quad \text{For odd values of } m \text{ and } n
\]

\[
m, n = 1, 3, 5, \ldots
\]

(4-8)

\[
q_{mn} = 0 \quad \text{For even values of } m \text{ and } n
\]

\[
m, n = 2, 4, 6, \ldots
\]

4-4 Rectangular Sandwich Plates with Four Clamped Sides

Considering the following clamped boundary conditions:

\[
u^{(1)} = u^{(2)} = v^{(1)} = v^{(2)} = w = w_x = 0
\]

at \( x = 0 \) and \( x = L \)

(4-9a)
\[ u^{(1)} = u^{(2)} = v^{(1)} = v^{(2)} = w = w_y = 0 \]

at \( y = 0 \) and \( y = B \)

(4-9b)

The governing differential equations (Eqs. 4-2) are satisfied using the following set of displacement functions:

\[ u(f) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} u_{mn}^{(f)} \cos \frac{mnx}{L} \sin \frac{ny}{B} \]

\( (0 < x < L, \ 0 < y < B) \)  

(4-10a)

\[ v(f) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} v_{mn}^{(f)} \sin \frac{mnx}{L} \cos \frac{ny}{B} \]

\( (0 < x < L, \ 0 < y < B) \)  

(4-10b)

and \( f = 1, 2 \)

\[ w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{mnx}{L} \sin \frac{ny}{B} \]

\( (0 < x < L, \ 0 < y < B) \)  

(4-10c)

However, Eqs. 4-10a and 4-10b do not satisfy the boundary conditions given by Eqs. 4-9a and 4-9b, respectively. Furthermore, the first partial derivatives of the transverse deflection with respect to \( x \) and \( y \) do not satisfy both boundary conditions; that is
\[ w_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \left( \frac{m\pi}{L} \right) \cos \frac{m\pi x}{L} \sin \frac{n\pi y}{B} \]

\[ 0 < x < L, \ 0 < y < B \]  

(4-11a)

\[ w_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \left( \frac{n\pi}{B} \right) \sin \frac{m\pi x}{L} \cos \frac{n\pi y}{B} \]

\[ 0 < x < L, \ 0 < y < B \]  

(4-11b)

Therefore, further differentiation for \( u^{(f)} \) and \( w_x \) with respect to \( x \) and for \( v^{(f)} \) and \( w_y \), with respect to \( y \), cannot be accomplished term-by-term. In order to satisfy these boundary conditions the following conditions should be imposed:

1. In order that \( u^{(f)} \) vanish on the edges \( x = 0 \) and \( x = L \)

\[ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} u_{mn}^{(f)} \sin \frac{n\pi y}{B} = 0 \quad \text{at } x = 0 \]  

(4-12a)

\[ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} u_{mn}^{(f)} (-1)^m \sin \frac{n\pi y}{B} = 0 \quad \text{at } x = L \]  

(4-12b)

These two conditions can be satisfied only if

\[ u_{on}^{(f)} + \sum_{m=2}^{\infty} \lambda_m u_{mn}^{(f)} = 0 \quad \text{For all values of } \ n = 1, 2, \ldots, \infty \]

(4-13)
\[
\sum_{m=1}^{\infty} \eta_m u_{mn}^{(f)} = 0 \quad (f=1,2) \quad \text{For all values of } n = 1, 2, \ldots, \infty \quad (4-14)
\]

where

\[
\begin{align*}
\lambda_m &= 1 \quad \text{and} \quad \eta_m = 0 \quad \text{for even values of } m \\
\lambda_m &= 0 \quad \text{and} \quad \eta_m = 1 \quad \text{for odd values of } m
\end{align*}
\]

Eqs. 4-13 and 4-14 imply that the sum of the displacement parameters, \( u_{mn}^{(f)} \), associated with even values of \( m \) and odd values of \( m \) respectively should be equal to zero for each value of \( n \). Therefore, for a particular face each of Eqs. 4-13 and 4-14 contains a number of conditions equal to the number of cycles used for the series \( n \).

(2) In order that \( v_{mn}^{(f)} \) vanish on the edges \( y = 0 \) and \( y = B \)

\[
\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} v_{mn}^{(f)} \sin \frac{m\pi x}{L} = 0 \quad \text{at } y = 0 \quad (4-16a)
\]

\[
\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} v_{mn}^{(f)} (-1)^n \sin \frac{m\pi x}{L} = 0 \quad \text{at } y = B \quad (4-16b)
\]

These two conditions can be satisfied only if

\[
\begin{align*}
v_{m0}^{(f)} + \sum_{n=2}^{\infty} \gamma_n v_{mn}^{(f)} &= 0 \quad \text{For all values of } m = 1, 2, \ldots, \infty \\
\sum_{n=1}^{\infty} \beta_n v_{mn}^{(f)} &= 0 \quad (f=1,2) \quad \text{For all values of } m = 1, 2, \ldots, \infty
\end{align*}
\]

(4-17)
where
\[ \gamma_n = 1 \quad \text{and} \quad \beta_n = 0 \quad \text{for even values of } n \]
\[ \gamma_n = 0 \quad \text{and} \quad \beta_n = 1 \quad \text{for odd values of } n \quad (4-19) \]

For a particular face, the number of conditions included in each of Eqs. 4-17 and 4-18 is equal to the number of cycles used in the series m.

(3) In order that \( w_x \) vanish on the edges \( x = 0 \) and \( x = L \)

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda_m w_{mn} \sin \frac{n\pi y}{B} = 0 \quad \text{at } x = 0 \quad (4-20a)
\]

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mu_m w_{mn} (-1)^m \sin \frac{n\pi y}{B} = 0 \quad \text{at } x = L \quad (4-20b)
\]

These two conditions can be satisfied only if

\[
\sum_{m=2}^{\infty} \lambda_m m w_{mn} = 0 \quad \text{For all values of } n = 1, 2, \ldots, \infty \quad (4-21)
\]

\[
\sum_{m=1}^{\infty} \mu_m m w_{mn} = 0 \quad \text{For all values of } n = 1, 2, \ldots, \infty \quad (4-22)
\]

The number of conditions included in each of Eqs. 4-21 and 4-22 is equal to the number of cycles used in the series m.
(4) In order that \( w_y \) vanish on the edges \( y = 0 \) and \( y = B \)

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n w_{mn} \sin \frac{\pi n x}{L} = 0 \quad \text{at } y = 0 \quad (4-23a)
\]

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n w_{mn} (-1)^n \sin \frac{\pi n x}{L} = 0 \quad \text{at } y = B \quad (4-23b)
\]

These two conditions imply the following

\[
\sum_{n=2}^{\infty} \gamma_n n w_{mn} = 0 \quad \text{For all values of } m = 1, 2, \ldots, \infty \quad (4-24)
\]

\[
\sum_{n=1}^{\infty} \beta_n n w_{mn} = 0 \quad \text{For all values of } m = 1, 2, \ldots, \infty \quad (4-25)
\]

Therefore, the total number of conditions needed to satisfy the boundary conditions for the membrane displacements \( u(f) \) and \( v(f) \) and for the first partial derivatives of the transverse deflection \( w_x \) and \( w_y \) is equal to:

\[6N + 6M\quad (4-26)\]

where \( n = 1, 2, \ldots, N \) and \( m = 1, 2, \ldots, M \) and \( N, M \) are the total number of cycles used.

By imposing the previously mentioned conditions, the assumed displacement functions can now be written in
the following form:

\[ u(f) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} u_{mn}(f) \cos \frac{mn \pi x}{L} \sin \frac{n \pi y}{B} \]

\[ (0 < x < L, \ 0 < y < B) \]  \hspace{1cm} (4-27a)

\[ v(f) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} v_{mn}(f) \sin \frac{mn \pi x}{L} \cos \frac{n \pi y}{B} \]

\[ (0 < x < L, \ 0 < y < B) \]  \hspace{1cm} (4-27b)

and \( f = 1, 2 \)

\[ w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{mn \pi x}{L} \sin \frac{n \pi y}{B} \]

\[ (0 < x < L, \ 0 < y < B) \]  \hspace{1cm} (4-27c)

also

\[ w_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \left( \frac{mn \pi}{L} \right) \cos \frac{mn \pi x}{L} \sin \frac{n \pi y}{B} \]

\[ (0 < x < L, \ 0 < y < B) \]  \hspace{1cm} (4-28a)

\[ w_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \left( \frac{n \pi}{B} \right) \sin \frac{mn \pi x}{L} \cos \frac{n \pi y}{B} \]

\[ (0 < x < L, \ 0 < y < B) \]  \hspace{1cm} (4-28b)
The analysis of this system requires the determination of all the partial derivatives for the assumed displacement functions \( u^{(f)} \), \( v^{(f)} \), and \( w \) which are included in the governing differential equations (Eqs. 4-2). Further differentiation with respect to \( x \) for Eqs. (4-27a and 4-28a) and with respect to \( y \) for Eqs. (4-27b and 4-28b) leads to the following set of equations:

\[
\begin{align*}
    u^{(f)}_x &= - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{mn}{L} \right)^2 u_{mn} \sin \frac{mnx}{L} \sin \frac{n\pi y}{B} \\
    (0 < x < L, \ 0 \leq y \leq B) \\
    (4-29)
\end{align*}
\]

\[
\begin{align*}
    w_{xx} &= - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{mn}{L} \right)^2 w_{mn} \sin \frac{mnx}{L} \sin \frac{n\pi y}{B} \\
    (0 < x < L, \ 0 \leq y \leq B) \\
    (4-30)
\end{align*}
\]

\[
\begin{align*}
    v^{(f)}_y &= - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{nm}{B} \right)^2 v_{mn} \sin \frac{m\pi x}{L} \sin \frac{n\pi y}{B} \\
    (0 \leq x \leq L, \ 0 < y < B) \\
    (4-31)
\end{align*}
\]

\[
\begin{align*}
    w_{yy} &= - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{np}{B} \right)^2 w_{mn} \sin \frac{m\pi x}{L} \sin \frac{n\pi y}{B} \\
    (0 \leq x \leq L, \ 0 < y < B) \\
    (4-32)
\end{align*}
\]
Note that \( u^{(f)}_x \) and \( w_{xx} \) are not valid on the boundaries \( x = 0, L \) and \( v^{(f)}_y \) and \( w_{yy} \) are not valid on the boundaries \( y = 0, B \). This can be realized by considering the force deformation relations (Eqs. 2-3) as follows:

\[
N_x^f = A_{11}^{(f)} u_x^{(f)} + A_{12}^{(f)} v_y^{(f)} - B_{11}^{(f)} w_{xx}
\]

(4-33a)

\[
M_x^f = B_{11}^{(f)} u_x^{(f)} - D_{11}^{(f)} w_{xx} - D_{12}^{(f)} w_{yy}
\]

(4-33b)

For clamped edges at \( x = 0, L \):

\[
N_x^f > 0 ; \quad M_x^f > 0 \quad \text{along edges} \quad x = 0, L
\]

(4-34)

Equation 4-34 cannot be satisfied using Eqs. 4-29 and 4-30. It is pointed out that both \( v_y^{(f)} \) and \( w_{yy} \) do not contribute to \( N_x^f \) and \( M_x^f \) along the boundaries \( x = 0, L \). Similarly, \( N_y^f > 0 \) and \( M_y^f > 0 \) for clamped edges along the boundaries \( y = 0, B \), which cannot be achieved using Eqs. (4-31 and 4-32).

Therefore, further differentiation of Eqs. (4-29 and 4-30) with respect to \( x \) and Eqs. (4-31 and 4-32) with respect to \( y \) cannot be accomplished term-by-term. It can also be shown that for clamped edges the conditions \( N_{xy}^f > 0 \) along the boundaries \( x = 0, L \) and \( y = 0, B \) are fully satisfied using Eqs. 4-27 a and 4-27b to represent the membrane displacements. In turn it results that all the partial derivatives with respect to \( y \) for the displacement function \( u^{(f)}_y \) (i.e. \( u^{(f)}_{xy} \) and \( u^{(f)}_{yy} \)) and with respect to \( x \) for
\( v(f) \) (i.e. \( v_{xy} \) and \( v_{xx} \)) which are included in the governing differential equations can be accomplished term-by-term.

In order to differentiate \( u_{xx}^{(f)} \) and \( w_{xx} \) with respect to \( x \), assume that both \( u_{xx}^{(f)} \) and \( w_{xxx} \) can be represented by a cosine-sine series; partial integration leads to the following results (Appendix D):

\[
u_{xx}^{(f)} = -\frac{1}{\xi} \sum_{n=1}^{\infty} a_n^{(f)} \sin \frac{n\pi y}{B} - \sum_{m=2,4}^{\infty} \sum_{n=1,2}^{\infty} \left[ \frac{a_n^{(f)}}{n^2} + \left( \frac{m}{L} \right)^2 \frac{a_n^{(f)}}{L} \right] \frac{u_{mn}^{(f)}}{\cos \frac{m\pi x}{L} \sin \frac{n\pi y}{B}}
\]

\[
- \sum_{m=1,3}^{\infty} \sum_{n=1,2}^{\infty} \left[ \frac{b_n^{(f)}}{n} + \left( \frac{m}{L} \right)^2 \frac{b_n^{(f)}}{L} \right] \frac{u_{mn}^{(f)}}{\cos \frac{m\pi x}{L}}
\]

\[
\sin \frac{n\pi y}{B} \quad (0 \leq x \leq L, \ 0 \leq y \leq B) \quad (4-35)
\]

\[
a_n^{(f)} = -\frac{4}{LB} \int_{y=0}^{y=B} \left[ u_{x}^{(f)}(L,y) - u_{x}^{(f)}(0,y) \right] \sin \frac{n\pi y}{B} \ dy
\]

\[
b_n^{(f)} = \frac{4}{LB} \int_{y=0}^{y=B} \left[ u_{x}^{(f)}(L,y) + u_{x}^{(f)}(0,y) \right] \sin \frac{n\pi y}{B} \ dy \quad (4-36a)
\]

\[
b_n^{(f)} = \frac{4}{LB} \int_{y=0}^{y=B} \left[ u_{x}^{(f)}(L,y) + u_{x}^{(f)}(0,y) \right] \sin \frac{n\pi y}{B} \ dy \quad (4-36b)
\]
Also,

\[ w_{xxx} = -\frac{1}{4} \sum_{n=1}^{\infty} e_n \sin \frac{n\pi y}{B} - \sum_{m=2,4}^{\infty} \sum_{n=1,2}^{\infty} \left[ e_n + \left( \frac{m\pi}{L} \right)^3 w_{mn} \right] \cos \frac{m\pi x}{L} \sin \frac{n\pi y}{B} \]

\[ + \sum_{m=1,3}^{\infty} \sum_{n=1,2}^{\infty} \left[ f_n + \left( \frac{m\pi}{L} \right)^3 w_{mn} \right] \cos \frac{m\pi x}{L} \sin \frac{n\pi y}{B} \]

\[ 0 < x < L, \ 0 < y < B \quad (4-37) \]

where

\[ e_n = \frac{4}{LB} \int_{y=0}^{y=B} \left[ w_{xx}(L,y) - w_{xx}(0,y) \right] \sin \frac{n\pi y}{B} \ dy \quad (4-38a) \]

\[ f_n = \frac{4}{LB} \int_{y=0}^{y=B} \left[ w_{xx}(L,y) + w_{xx}(0,y) \right] \sin \frac{n\pi y}{B} \ dy \quad (4-38b) \]

Similarly, in order to differentiate \( v_{y}^{(f)} \) and \( w_{yy} \) with respect to \( y \), assume that \( v_{yy}^{(f)} \) and \( w_{yyy} \) can be represented by a cosine-sine series; partial integration leads to the following results:
\[ v_{yy} = - \frac{1}{2} \sum_{m=1}^{\infty} c_m^{(f)} \sin \frac{m \pi x}{L} - \sum_{m=1,2}^{\infty} \sum_{n=2,4}^{\infty} \ \left[ c_m^{(f)} + \left( \frac{n \pi}{B} \right)^2 v_{mn}^{(f)} \right] \sin \frac{m \pi x}{L} \cos \frac{n \pi y}{B} \]

\[ - \sum_{m=1,2}^{\infty} \sum_{n=1,3}^{\infty} \left[ \left( \frac{1}{2} \right)^m \left( \frac{n \pi}{B} \right)^2 \sin \frac{m \pi x}{L} \right] \sin \frac{m \pi x}{L} \cos \frac{n \pi y}{B} \]

\[ (0 \leq x \leq L, \ 0 \leq y \leq B) \quad (4-39) \]

\[ c_m^{(f)} = \frac{4}{LB} \int_{x=0}^{x=L} \left[ v_{yy}^{(f)}(x,B) - v_{yy}^{(f)}(x,0) \right] \sin \frac{m \pi x}{L} \, dx \]

\[ d_m^{(f)} = \frac{4}{LB} \int_{x=0}^{x=L} \left[ v_{yy}^{(f)}(x,B) + v_{yy}^{(f)}(x,0) \right] \sin \frac{m \pi x}{L} \, dx \quad (4-40a) \]

Similarly,

\[ w_{yyy} = - \frac{1}{2} \sum_{m=1,2}^{\infty} \sum_{n=2,4}^{\infty} \ \left[ q_m + \left( \frac{n \pi}{B} \right)^3 w_{mn} \right] \sin \frac{m \pi x}{L} \cos \frac{n \pi y}{B} \]

\[ - \sum_{m=1,2}^{\infty} \sum_{n=1,3}^{\infty} \left[ h_m + \left( \frac{n \pi}{B} \right)^3 w_{mn} \right] \sin \frac{m \pi x}{L} \cos \frac{n \pi y}{B} \]

\[ (0 \leq x \leq L, \ 0 \leq y \leq B) \quad (4-41) \]
where
\[
g_m = -4LB \int_{x=0}^{x=L} \left[ w_{yy}(x,B) - w_{yy}(x,0) \right] \sin \frac{m\pi x}{L} \, dx
\]
\[
(4-42a)
\]
\[
h_m = \frac{4}{LB} \int_{x=0}^{x=L} \left[ w_{yy}(x,B) + w_{yy}(x,0) \right] \sin \frac{m\pi x}{L} \, dx
\]
\[
(4-42b)
\]
All other desired derivatives can be obtained through term-by-term differentiation; for example,
\[
w_{yyyy} = \sum_{m=1,2}^{\infty} \sum_{n=2,4}^{\infty} \left[ g_m + \left( \frac{n\pi}{B} \right)^3 w_{mn} \right] \left( \frac{n\pi}{B} \right)
\sin \frac{m\pi x}{L} \sin \frac{n\pi y}{B} + \sum_{m=1,2}^{\infty} \sum_{n=1,3}^{\infty} \left[ h_m + \left( \frac{n\pi}{B} \right)^3 w_{mn} \right] \left( \frac{n\pi}{B} \right) \sin \frac{m\pi x}{L} \sin \frac{n\pi y}{B}.
\]
\[
(4-43)
\]
Substituting the appropriate partial derivatives of \(u^{(f)}\), \(v^{(f)}\) and \(w\) as well as the load expression (Eq. 4.6) into the governing differential equations (Eqs. 4-2) and equating like Fourier terms leads to the following set of equations:
\[
\sum_{n=1,2}^{\infty} \left[ A_{1n} + A_{2n} u_{on}^{(1)} + A_{3} u_{on}^{(2)} \right] = 0
\]
\[
(4-44a)
\]
\[ \sum_{m=1,2} \sum_{n=1,2} \left[ B_n + g_{11} u_{mn}^{(1)} + g_{12} u_{mn}^{(2)} + g_{13} v_{mn}^{(1)} + g_{15} w_{mn} \right] = 0 \] (4-44b)

\[ \sum_{n=1,2} \left[ c_{1n} + c_{2n} u_{on}^{(1)} + c_{3n} u_{on}^{(2)} \right] = 0 \] (4-44c)

\[ \sum_{m=1,2} \sum_{n=1,2} \left[ D_n + g_{12} u_{mn}^{(1)} + g_{22} u_{mn}^{(2)} + g_{24} v_{mn}^{(2)} + g_{25} w_{mn} \right] = 0 \] (4-44d)

\[ \sum_{m=1,2} \left[ E_{1m} + E_{2m} v_{mo}^{(1)} + E_{3m} v_{mo}^{(2)} \right] = 0 \] (4-44e)

\[ \sum_{m=1,2} \sum_{n=1,2} \left[ F_m + g_{13} u_{mn}^{(1)} + g_{33} v_{mn}^{(1)} + g_{34} v_{mn}^{(2)} + g_{35} w_{mn} \right] = 0 \] (4-44f)

\[ \sum_{m=1,2} \left[ G_{1m} + G_{2m} v_{mo}^{(1)} + G_{3m} v_{mo}^{(2)} \right] = 0 \] (4-44g)

\[ \sum_{m=1,2} \sum_{n=1,2} \left[ H_m + g_{24} u_{mn}^{(2)} + g_{34} v_{mn}^{(1)} + g_{44} v_{mn}^{(2)} + g_{45} w_{mn} \right] = 0 \] (4-44h)
\[
\sum_{m=1,2} \sum_{n=1,2} \left[ I_{mn} + g_{15} u_{mn}^{(1)} + g_{25} u_{mn}^{(2)} + g_{35} v_{mn}^{(1)} + g_{45} v_{mn}^{(2)} + g_{55} w_{mn} \right] = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn}
\]

(4-44i)

In the preceding equations, the series \( A_{n} \), \( B_{n} \), \( C_{n} \) and \( D_{n} \) are functions of the coefficients \( e_{n}, f_{n}, a_{n}(f) \) and \( b_{n}(f) \). The series \( E_{m} \), \( F_{m} \), \( G_{m} \) and \( H_{m} \) are functions of the coefficients \( g_{m}, h_{m}, C_{m}(f) \) and \( d_{m}(f) \). While the series \( I_{mn} \) is a function of both sets of coefficients. These series are given in Appendix D.

The solution of sandwich plate with simply supported sides (Eqs. 4-7) can be obtained from Equations 4-44 by setting the series \( B_{n}, D_{n}, F_{m}, \ldots \) etc. equal to zero. In other words, for the simply supported case:

\[
a_{n} = b_{n}(f) = e_{n} = f_{n}, C_{m} = d_{m}(f) = g_{m} = h_{m} = 0
\]

(4-45)

Thus, for a sandwich plate having two simply supported edges at \( x = 0, L \) and the other sides (\( y = 0, B \)) clamped:

\[
a_{n} = b_{n}(f) = e_{n} = f_{n} = 0 \quad n = 1,2,\ldots \infty
\]

(4-46)

Similarly, if the edges \( y = 0, B \) are simply supported and the other two edges \( (x = 0, L) \) clamped:

\[
c_{m} = d_{m}(f) = g_{m} = h_{m} = 0 \quad m = 1,2,\ldots \infty
\]

(4-47)
From the previous discussion it can be seen that the approach is flexible in dealing with various boundary conditions for the sandwich plate. These cases will be discussed in more details later in this chapter (Section 4-5).

By solving the algebraic equations (Eqs. 4-44) yields

\[ w_{mn} = w_{mn}^{(s)} + B_n BW + B_m BW + F_m FW + H_m HW + I_{mn} IW \]

(4-48a)

\[ u_{mn} = u_{mn}^{(f)} + B_n BU(f) + D_n DU(f) + F_m FU(f) + H_m HU(f) + I_{mn} IU(f) \]

(4-48b)

\[ v_{mn} = v_{mn}^{(f)} + B_n BV(f) + D_n DV(f) + F_m FV(f) + H_m HV(f) + I_{mn} IV(f) \]

(4-48c)

\[ u_{on} = A_l A_U(f) + C_l C_U(f) \]

(4-48d)

\[ v_{mo} = E_l E_V(f) + G_l G_V(f) \]

and

\[ n, m = 1, 2, \ldots, \infty, \ f = 1, 2 \]

(4-48e)
where \( w_{mn}^{(s)}, u_{mn}^{(f)} \) and \( v_{mn}^{(f)} \) correspond to the simply supported solution given by Equations 4-7. \( IWI, BW, IU^{(f)}, IV^{(f)} \), etc. are double series in \( m \) and \( n \), while \( AU^{(f)}, CU^{(f)} \) are series in \( n \) and \( EV^{(f)}, GV^{(f)} \) are series in \( m \) only. The expressions of the various series are given in Appendix D.

The undetermined displacement parameters \( u_{mn}^{(f)}, v_{mn}^{(f)} \) and \( w_{mn} \) are all expressed in terms of the undetermined coefficients \( a_{n}^{(f)}, b_{n}^{(f)}, c_{m}^{(f)} \), etc. These coefficients can be obtained by substituting the displacement parameters (Eqs. 4-48) into Eqs. (4-13, 4-14, 4-17, 4-18, 4-21, 4-22, 4-24, and 4-25). Note that the number of undetermined coefficients is equal to \( 6N + 6M \) which is the same as the number of equations (Eqs. 4-26). The resulting relations can be written in the following matrix form:

\[
\begin{bmatrix}
R(n,n) & R(n,m) \\
6N \times 6N & 6N \times 6M
\end{bmatrix}
\begin{bmatrix}
\{C(n)\} \\
\{C(m)\}
\end{bmatrix}
+ 
\begin{bmatrix}
R(m,n) & R(m,m) \\
6M \times 6N & 6M \times 6M
\end{bmatrix}
\begin{bmatrix}
\{Q(n)\} \\
\{Q(m)\}
\end{bmatrix}
= 0
\]

(4-49a)

or

\[
[R] \{C\} = \{Q\}
\]

(6N+6M) \quad (6N+6M)

(4-49b)

where

\[
[R] - A \text{ coefficient matrix } [(6N + 6M) \times (6N + 6M)]
\]

(4-50a)
\[ \{ C^{(n)} \}^T = \left\{ \{ a^{(1)} \}_n^T, \{ a^{(2)} \}_n^T, \{ b^{(1)} \}_n^T, \{ b^{(2)} \}_n^T \right\}_N^N, \]
\[ \{ e_n \}, \{ f_n \} \quad \text{N N} \]
\[ \{ C^{(m)} \}^T = \left\{ \{ c^{(1)} \}_m^M, \{ c^{(2)} \}_m^M, \{ d^{(1)} \}_m^M, \{ d^{(2)} \}_m^M \right\}_M^M. \]

\[ (4-50b) \]

\[ \{ C^{(m)} \}^T = \left\{ \{ g_m \}_M^M \right\}_M^M. \]

\[ (4-50c) \]

Generally, the terms in the vector \( \{ Q \} \) are functions of the simply supported solutions \( w_{mn(s)}, u_{mn(s)}^{(f)}, \) and \( v_{mn(s)}^{(f)} \).

The solution of Eqs. 4-49 can be obtained by standard numerical techniques such as Gaussian elimination method from which the coefficients \( a_n^{(f)}, b_n^{(f)}, c_m^{(f)}, \ldots \) etc. are obtained. Substitution of the resulting coefficients into Eqs. 4-48 determines the displacement parameters \( (w_{mn}, u_{mn}^{(f)}, v_{mn}^{(f)}, u_{on}^{(f)} \) and \( v_{mo}^{(f)}). \) The displacements \( u^{(f)}, v^{(f)} \) and \( w \) at the various locations in the domain of the sandwich plate can be obtained by substituting the displacement parameters into Eqs. 4-27. Note that for sandwich plates with clamped sides, all the cycles for both \( m \) and \( n \) series are coupled. This can be realized from Eqs. 4-48 and 4-49. On the other hand, the solution of a simply supported sandwich plate does not lead to any coupling between the different cycles; that is, for a particular value of \( m \) and \( n \) the solution is obtained and the final solution is
the sum of the solutions of the various cycles \((n,m = 1, 2, \ldots, \infty)\) (Eq. 4-7).

4-5 Rectangular Sandwich Plates with Two Opposite Sides Simply Supported and the Other Sides Clamped

Consider the following boundary conditions:

\[
N_{x_1} = N_{x_2} = v^{(1)} = v^{(2)} = M_{x_1} = M_{x_2} = w = 0
\]

at \(x = 0, L\) \hspace{1cm} (4-51a)

\[
u^{(1)} = u^{(2)} = v^{(1)} = v^{(2)} = w_y = w = 0
\]

at \(y = 0, B\) \hspace{1cm} (4-51b)

The assumed displacement functions for this case are:

\[
u^{(f)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}^{(f)} \cos \frac{mnx}{L} \sin \frac{nny}{B}
\]

\((0 \leq x \leq L, 0 \leq y \leq B)\) \hspace{1cm} (4-52a)

\[
v^{(f)} = \sum_{m=1}^{\infty} \sum_{q_n=0}^{\infty} v_{mn}^{(f)} \sin \frac{mnx}{L} \cos \frac{nny}{B}
\]

\((0 \leq x \leq L, 0 < y < B)\) \hspace{1cm} (4-52b)
\[ w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m\pi x}{L} \sin \frac{n\pi y}{B} \]

\[ (0 \leq x \leq L, \ 0 \leq y \leq B). \] (4-52c)

In order that \( v^{(f)} \) and \( w_y \) satisfy the boundary conditions (Eq. 4-51b), the conditions given by Eqs. (4-17 and 4-18) and Eqs. (4-24 and 4-25) must be considered for \( v^{(f)} \) and \( w_y \) respectively. Therefore, the number of conditions needed to satisfy the boundary conditions for this case is equal to 6M. Since all the displacements and the stress resultants are satisfied at \( x = 0, L \), the analysis of this system can be achieved by setting

\[ a_n^{(f)} = b_n^{(f)} = c_n = d_n = e_n = f_n = 0 \quad \text{for all values} \quad n = 1, 2, \ldots, \infty \] (4-53)

in the formulations of sandwich plates with four clamped sides. That is, by imposing the conditions given in Eq. 4-53, Equation 4-48 will have the following form.

\[ w_{mn} = w_{mn}^{(s)} + I_{mn}^{(m)} I_W + F_m F_W + H_m H_W \] (4-54a)

\[ u_{mn}^{(f)} = u_{mn}^{(f)} + I_{mn}^{(m)} I_U^{(f)} + F_m F_U^{(f)} + H_m H_U^{(f)} \] (4-54b)

\[ v_{mn}^{(f)} = v_{mn}^{(f)} + I_{mn}^{(m)} I_V^{(f)} + F_m F_V^{(f)} + H_m H_V^{(f)} \] (4-54c)
\[ v_{m0}^{(f)} = E l_m E V^{(f)} + G l_m G V^{(f)} \]  

(4-54d)

where \( I_{mn}^{(m)} \) is obtained by considering Eq. 4-53 in the expression of \( I_{mn}^{'} \). Note that Eq. 4-48d (i.e. \( u_{on}^{(f)} \)) has vanished. This will automatically be obtained by choosing the displacement function for \( u^{(f)} \) in the form given in Eq. 4-52a (i.e. the term \( m = 0 \) is not included in the series). For the case considered, Eq. 4-49 will have the following form

\[ [R^{(m,m)}] \{C^{(m)}\} = \{Q^{(m)}\} \]  

(4-55)

(6Mx6M) 6M

Similarly, for the boundary conditions:

\[ u^{(1)} = u^{(2)} = v^{(1)} = v^{(2)} = w_{x} = w = 0 \]

at \( x = 0, L \)  

(4-56a)

\[ u^{(1)} = u^{(2)} = N_{y_{1}} = N_{y_{2}} = M_{y_{1}} = M_{y_{2}} = w = 0 \]

at \( y = 0, B \)  

(4-56b)

For such boundary conditions, the analysis of the system can be achieved by setting,

\[ c_{m}^{(f)} = d_{m}^{(f)} = g_{m} = h_{m} = 0 \]  

for all values of \( m = 1, 2, \ldots \infty \)  

(4-57)
in the formulations of sandwich plates with four clamped sides. By imposing these conditions, Eq. 4-48 will have the following form:

\[
\begin{align*}
\omega_{mn} &= \omega_{mn}(s) + I_{mn}^{(n)} \omega + B_n B \omega + D_n D \omega \\
\theta_{mn}^{(f)} &= \theta_{mn}^{(s)} + I_{mn}^{(n)} \theta_{U}^{(f)} + B_n B \theta_{U}^{(f)} + D_n D \theta_{U}^{(f)} \\
\gamma_{mn}^{(f)} &= \gamma_{mn}^{(s)} + I_{mn}^{(n)} \gamma_{V}^{(f)} + B_n B \gamma_{V}^{(f)} + D_n D \gamma_{V}^{(f)} \\
\varphi_{on}^{(f)} &= A_{l_n} A \varphi_{U}^{(f)} + C_{l_n} C \varphi_{U}^{(f)}
\end{align*}
\] (4-58a-d)

where \( I_{mn}^{(n)} \) is obtained by considering Eq. 4-57 in the expression of \( I_{mn} \). Note that Eq. 4-48e (i.e. \( \gamma_{mo}^{(f)} \)) has vanished. For this case Eq. 4-53 will have the following form:

\[
[R^{(n,n)}] [C^{(n)}] = [Q^{(n)}]
\] (4-59)

(6N x 6N) 6N 6N

For all the cases considered in this Chapter, the stress and moment resultants are obtained by substituting the resulting displacement functions \( \theta^{(f)} \), \( \gamma^{(f)} \) and \( \omega \) and their partial derivatives into the force-deformation relations (Eq. 2-3).
CHAPTER V

IMPLEMENTATION AND NUMERICAL EVALUATION

The discussion in this chapter is focused on the numerical evaluation for the finite strip and the double Fourier series approaches in analyzing sandwich structural systems. Numerical results are presented for a variety of problems in an attempt to verify and demonstrate the capability of both approaches in predicting the stresses and deflections of the structural systems considered. This chapter is also concerned with the study of the effect of the bending-membrane coupling which exists in unbalanced laminates on the behaviour of the anisotropic sandwich systems. The structural systems considered herein are classified into three major groups as follows:

(1) Rectangular plates.
(2) Cylindrical shell structures.
(3) Folded plate structures.

Generally, the analysis of these systems has been developed in successive steps starting with a thin structural system (i.e. a single face of the sandwich system), and concluding with the required anisotropic sandwich structure. At every step comparison is made, where possible, with existing reference solutions. It is important to note that all the numerical results presented for the finite strip method
adopt the geometric admissibility conditions which are satisfied by linking and dismissing degrees of freedom as outlined in Chapter III and as will be explained in more detail in this chapter. Both the finite strip and the double Fourier series approaches are analysis procedures for predicting structural response by means of digital computers. As such, computer programs were developed for the various structural systems, and the solutions were obtained using the IBM 360/65 digital computer which uses the FORTRAN IV compiler.

5.1. Rectangular Plates

In the following, solutions are obtained for a variety of problems for anisotropic thin laminated plates and sandwich plates with laminated faces under transverse loads and having various boundary conditions. Special attention is given to plates constructed from cross-plied and angle-plied laminates. The finite strip method is used in the analysis of the thin laminated plates while both the finite strip and the double Fourier series approaches are used for the sandwich plate cases. The expressions for the work equivalent loads for the various cases presented herein are given in Appendix E.

5.1.1 Thin Laminated Plates

In this section, numerical evaluation for the finite strip method in analyzing thin cross-plied and angle-plied plates under transverse loads is presented. The cases considered herein are (A) thin cross-plied plates with
four simply supported edges, (B) thin angle-plied plates with four simply supported edges, (C) thin cross-plied plates with four clamped edges, (D) thin angle-plied plates with four clamped edges. It is pointed out that other boundary conditions and types of laminates can be dealt with using the finite strip method. For all the cases, the transverse deflection is expressed as a fifth-order interpolation polynomial. The results obtained are compared with Ref. 19 for the simply supported cases and Ref. 20 for the clamped cases. Both references use the double-Fourier series approach in the analysis of these systems.

(A) Simply Supported Thin Cross-Plied Plates

The simple supports are of the type which allow normal displacements on the boundaries, but prevent lateral contraction (i.e. tangential displacements). This type of simple support is referred to as hinged free normal (HFN).

The boundary conditions for such a case are as follows:

\[ w(x,y) = v(x,y) = 0 \quad \text{at } x = 0, L \quad (5-1a) \]

\[ M_x(x,y) = N_x(x,y) = 0 \quad \text{at } x = 0, L \quad (5-1b) \]

\[ w(x,y) = u(x,y) = 0 \quad \text{at } y = 0, B \quad (5-1c) \]

\[ M_y(x,y) = N_y(x,y) = 0 \quad \text{at } y = 0, B \quad (5-1d) \]

The basic functions for the reference surface displacements
\( u(x, y) \) and \( v(x, y) \) and the deflection \( w(x, y) \) of the laminated plate which satisfy Eqs. (5-1a) and (5-1b) are:

\[
\begin{align*}
X_{u}^{(n)} &= \cos \frac{n\pi x}{L} \quad (5-2a) \\
X_{v}^{(n)} &= \sin \frac{n\pi x}{L} \quad (5-2b) \\
X_{w}^{(n)} &= \sin \frac{n\pi x}{L} \quad n = 1, 2, \ldots, N \quad (5-2c)
\end{align*}
\]

This set of basic functions possesses the orthogonality property and therefore coupling does not occur between cycles.

The satisfaction of the displacement boundary conditions in the \( y \)-direction (Eq. 5-1c) is achieved through the variable correlation scheme in which the following conditions are imposed on the displacement coefficients in the third and fifth order interpolation polynomials along the boundaries \( y = 0, B \) for all values of \( n = 1, 2, \ldots, N \):

\[
u^{(n)} = w^{(n)} = 0 \quad (5-3)
\]

The force boundary conditions in the \( y \)-direction (Eq. 5-1d) will be satisfied in the finite strip analysis at the true minimum of the total potential energy. Note that even if the plate is symmetrically loaded the structural response is not symmetrical about \( y = B/2 \). That is, although
geometric and loading symmetry, exists about the axis \( y = B/2 \) the complete plate must be modeled. This is due to the bending-membrane coupling which exist in this type of plate.

In order to reduce the total number of degrees of freedom of the system, the reduced bending stiffness approximation (R.B.S. of Appendix A) can be used in the finite strip analysis of this system. When the reduced bending stiffnesses are considered, the problem is converted to the analysis of thin orthotropic plates. Therefore, the displacement behaviour of the system is represented by the transverse deflection only. The boundary conditions are:

\[
\begin{align*}
    w(x,y) &= M_x(x,y) = 0 \quad \text{at } x = 0, L \\
    w(x,y) &= M_y(x,y) = 0 \quad \text{at } y = 0, B
\end{align*}
\]  \hspace{1cm} (5-4a, 5-4b)

The basic function for this case is

\[
    x_w(n) = \sin \frac{n\pi x}{L} \quad n = 1, 2, \ldots, N
\]  \hspace{1cm} (5-5)

The numerical results presented herein are for plates under transverse pressure of 1 psi. The plate consists of two orthotropic layers each of 0.05 in. thick and oriented at 0° and 90° with the x axis of the plate. The elastic properties of the individual layer are:
E_{11} = 40 \times 10^6 \text{ psi} \quad \frac{E_{11}}{E_{22}} = 40

G_{12}/E_{22} = 1.0 \quad \nu_{12} = 0.25

The corresponding stiffnesses of the plate are:

A_{11} = A_{22} = 2.0532 \times 10^6 \text{ lb/in.} \quad A_{12} = 2.5039 \times 10^4 \text{ lb/in.}

A_{16} = A_{26} = 0.0 \quad ; \quad A_{66} = 10^5 \text{ lb/in.}

B_{11} = -B_{22} = -4.8826 \times 10^4 \text{ lb} \quad ; \quad B_{16} = B_{26} = B_{66} = 0.0

D_{11} = D_{22} = 1.711 \times 10^3 \text{ lb/in.} \quad ; \quad D_{12} = 20.866 \text{ lb-in.}

D_{16} = D_{26} = 0.0 \quad ; \quad D_{66} = 83.333 \text{ lb-in.}

The reduced bending stiffnesses are:

D_{11}^* = D_{22}^* = 5.4972 \times 10^2 \text{ lb-in.}; \quad D_{12}^* = 6.7039 \text{ lb-in.}

D_{16}^* = D_{26}^* = 0.0 \quad ; \quad D_{66}^* = 83.333 \text{ lb-in.}

The total number of degrees of freedom resulting from modeling the plate with 6 strips are 45 and 19 for the HPN and the RBS cases respectively. The results for the center deflection are compared with the theoretical results.
of Ref. 19 in Table 1 and they are found to be in close agreement. In all the cases, the reduced bending stiffness approximation shows very good agreement with the HFN cases. Therefore, a great reduction in computer time and storage can be achieved by replacing the HFN with the RBS approximation in the finite strip formulations while the accuracy of the displacement results are maintained.

(B) Simply Supported Thin Angle-Plied Plates

The simply supports considered herein are assumed to be smooth pins permitting tangential displacements at the boundaries but normal displacements are prevented. This type of simple support is referred to as hinged free tangential (HFT). The boundary conditions for such cases are:

\[ \begin{align*}
  w(x,y) &= u(x,y) = 0 \quad \text{at } x = 0,L \\
  M_x(x,y) &= N_{xy}(x,y) = 0 \quad \text{at } x = 0,L \\
  w(x,y) &= v(x,y) = 0 \quad \text{at } y = 0,B \\
  M_y(x,y) &= N_{xy}(x,y) = 0 \quad \text{at } y = 0,B
\end{align*} \]  

The basic functions for the reference surface displacements and the transverse deflection which satisfy the boundary conditions, Eqs. (5-6a) and (5-6b), are:
\[ x_u^{(n)} = \sin \frac{n\pi x}{L} \quad (5-7a) \]
\[ x_v^{(n)} = \cos \frac{n\pi x}{L} \quad (5-7b) \]
\[ x_w^{(n)} = \sin \frac{n\pi x}{L} \quad n = 1, 2, \ldots, N \quad (5-7c) \]

Again, this set of basic functions possesses the orthogonality property.

The displacement boundary conditions in the \( y \)-direction (Eq. 5-6c) are satisfied by imposing the following conditions on the displacement coefficients in the third and fifth order interpolation polynomials along the boundaries \( y = 0, B \) for all values of \( n = 1, 2, \ldots, N \):

\[ v(n) = w(n) = 0 \quad (5-8) \]

As in the cross-plied case, symmetry about \( y = B/2 \) cannot be considered and the force boundary conditions (Eq. 5-6d) will be automatically satisfied at the minimum of the potential energy. The basic function associated with the reduced bending stiffness approximation is the same as for the cross-plied case (Eq. 5-5).

The numerical results presented herein are for 20 in. x 20 in. plates under transverse pressure of 1 psi, and the plates consist of two orthotropic layers, each 0.05 in. thick. The elastic properties of a layer are:
\[ E_{11} = 40 \times 10^6 \text{ psi} \quad ; \quad E_{11}/E_{22} = 40 \]

\[ G_{12}/E_{22} = 0.5 \quad ; \quad v_{12} = 0.25 \]

The corresponding membrane, bending, coupling and reduced bending stiffnesses for plates with various lamina orientations (θ = 45°, 30° and 15°) are given in Table 2. The number of degrees of freedom which result from modeling the plated with 6 strips are 45 and 19 for the HFT and the RBS cases respectively. The results are compared with the theoretical results of Ref. 19 in Table 3. Both the HFT and the RBS are in close agreement for the center deflection as reported in the reference. The reduced bending stiffness approximation shows good agreement with the HFT cases except for the case where -15° < θ < 15° (Table 3).

(C) Thin Cross-Plied Plates with Four Clamped Sides

The following clamped boundary conditions are considered:

\[ u(x, y) = v(x, y) = w(x, y) = w_x(x, y) = 0 \]

\[ \text{at } x = 0, L \quad (5-9a) \]

\[ u(x, y) = v(x, y) = w(x, y) = w_y(x, y) = 0 \]

\[ \text{at } y = 0, B \quad (5-9b) \]
The basic functions for the reference surface displacements and the deflection of the plate which satisfy the boundary conditions (Eq. 5-9a) are as follows:

\[ X_u^{(n)} = \cos \left( \frac{\lambda_n x}{L} \right) - \cosh \left( \frac{\lambda_n x}{L} \right) + a_n \left[ \sin \left( \frac{\lambda_n x}{L} \right) \right. \]
\[ + \sinh \left( \frac{\lambda_n x}{L} \right) \]  
\[ (5-10a) \]

\[ X_v^{(n)} = \sin \left( \frac{\lambda_n x}{L} \right) - \sinh \left( \frac{\lambda_n x}{L} \right) - a_n \left[ \cos \left( \frac{\lambda_n x}{L} \right) \right. \]
\[ - \cosh \left( \frac{\lambda_n x}{L} \right) \]  
\[ (5-10b) \]

\[ X_w^{(n)} = \sin \left( \frac{\lambda_n x}{L} \right) - \sinh \left( \frac{\lambda_n x}{L} \right) - a_n \left[ \cos \left( \frac{\lambda_n x}{L} \right) \right. \]
\[ - \cosh \left( \frac{\lambda_n x}{L} \right) \]  
\[ (5-10c) \]

where

\[ \lambda_n = \frac{(2n + 1) \pi}{2} \]
and \( n = 1, 2, \ldots, N \)  
\[ (5-11) \]

The basic functions for this case are non-orthogonal; that is, all the cycles are coupled in the strip stiffness matrix formulations (Eqs. 3-19). Note that this set of basic functions imposes the additional condition that \( N_{xy}(x, y) = 0 \) along the boundaries \( x = 0, L \).
The boundary conditions in the $y$-direction (Eq. 5-9b) are satisfied by imposing the following conditions on the displacement coefficients in the third and fifth order interpolation polynomials along the boundaries $y = 0, B$ for all values of $n = 1, 2, \ldots, N$:

$$u(n) = v(n) = w(n) = w'(n) = 0$$  \hspace{1cm} (5-12)

Again, symmetry about $y = B/2$ cannot be considered in the analysis of this system and therefore the complete plate must be modeled. Since, in the finite strip analysis of laminated plates using the reduced bending stiffness approximations the solution is independent of the in-plane boundary conditions, the basic function for such a case is the same as Eq. (5-10c).

The numerical results presented herein are for a uniform transverse load of 1 psi. Results are presented for plates having aspect ratios of 1 and 2. Each plate is constructed of two orthotropic layers each of 0.12 in. thick and oriented at $0^\circ$ and $90^\circ$ with the $x$ axis of the plate. The elastic properties of an individual layer are:

$$E_{11} = 30 \times 10^6 \text{ psi} \quad ; \quad E_{11}/E_{22} = 40$$

$$G_{12}/E_{22} = 0.5 \quad ; \quad v_{12} = 0.25$$
The corresponding stiffnesses of the plate are:

\[ A_{11} = A_{22} = 3.6958 \times 10^6 \text{ lb/in.}; \ A_{12} = 4.507 \times 10^4 \text{ lb/in.} \]

\[ A_{16} = A_{26} = 0.0 \]

\[ A_{66} = 0.9 \times 10^5 \text{ lb/in.} \]

\[ B_{11} = -B_{22} = -2.1093 \times 10^5 \text{ lb}; \ B_{16} = B_{26} = B_{66} = 0.0 \]

\[ D_{11} = D_{22} = 1.774 \times 10^4 \text{ lb-in.}; \ D_{12} = 2.1634 \times 10^2 \text{ lb-in.} \]

\[ D_{16} = D_{26} = 0.0 \]

\[ D_{66} = 4.32 \times 10^2 \text{ lb-in.} \]

The reduced bending stiffnesses are:

\[ d_{11}^* = d_{22}^* = 5.6995 \times 10^3 \text{ lb-in.}; \ d_{12}^* = 69.506 \text{ lb-in.} \]

\[ d_{16}^* = d_{26}^* = 0.0 \]

\[ d_{66}^* = 4.32 \times 10^2 \text{ lb-in.} \]

Since the basic functions of this system are coupled, the number of degrees of freedom which result from modeling the plate with 6 strips using 3 cycles is \(41 \times 3 = 123\), where 41 represents the number of degrees of freedom per cycle of the basic function. For the reduced bending stiffness approximations, the number of degrees of freedom is \(17 \times 3 = 51\). The center deflection results are compared with the theoretical results of Ref. 20 in Table 4, and are in close agreement. Note that the additional boundary
conditions (i.e., $N_{xy}(x,y) = 0$ at $x = 0, L$) does not have any significant influence on the accuracy of the results obtained. The reduced bending stiffness approximation shows very good agreement with the other finite strip case as well as with the reference solution. Therefore, the reduced bending stiffness approximation in the finite strip formulation is a useful tool to reduce the computer storage and time required in the analysis of this system while maintaining the accuracy of the results. This is a significant saving since all the cycles are coupled and an increase in the number of cycles requires a large increase in the number of the degrees of freedom for this system.

(D) Thin Angle-Plied Plates with Four Clamped Sides

The clamped boundary conditions for this case are the same as those given in the cross-plied clamped plates (Eq. 5-9). However, it was found that the basic functions (Eqs. 5-10) cannot be used for the angle-plied case since this set of basic functions is ideally suitable for the case where $N_{xy}(x,y) = 0$ at $x = 0, L$ and unlike the cross-plied case these conditions were found to have a large influence on the results of the angle-plied plates. However, the finite strip analysis using the reduced bending stiffness approximations (in which the analysis is independent of the in-plane boundary conditions) are suitable for the analysis of this system. The basic function for such a case is the same as for the cross-plied clamped plate (Eq. 5-10c).
The numerical results presented herein are for 20 in. x 20 in. plates under a uniform transverse load of 1 psi. Results are presented for plates constructed from two orthotropic layers each 0.12 in. thick. The cases considered are for angle-plied laminates oriented at ± 45° and ± 35° with the x axis of the plate. The elastic properties of an individual layer are:

\[ E_{11} = 30 \times 10^6 \text{ psi} \quad ; \quad E_{11}/E_{22} = 40 \]

\[ G_{12}/E_{22} = 0.5 \quad ; \quad \nu_{12} = 0.25 \]

The corresponding reduced bending stiffnesses are:

For \( \theta = \pm 45^\circ \)

\[ D_{11}^* = D_{22}^* = 3.3165 \times 10^3 \text{ lb-in.}; \quad D_{12}^* = 2.4525 \times 10^3 \text{ lb-in.} \]

\[ D_{16}^* = D_{26}^* = 0.0 \]

For \( \theta = \pm 35^\circ \)

\[ D_{11}^* = 5.3231 \times 10^3 \text{ lb-in.} \]

\[ D_{12}^* = 2.2079 \times 10^3 \text{ lb-in.} \]

\[ D_{66}^* = 2.5103 \times 10^3 \text{ lb-in.} \]

\[ D_{22}^* = 1.9572 \times 10^3 \text{ lb-in.} \]

\[ D_{16} = D_{26}^* = 0.0 \]
The number of degrees of freedom which result from modeling the plate with 6 strips and using 3 cycles of the basic function is 51. The center deflection results are compared with the theoretical results of Ref. 20 (Table 5) and they are in close agreement. In agreement with the reference solution, it was found that the finite strip analysis using the reduced bending stiffness approximations does not give acceptable agreement with the coupled laminate solutions for certain orientations of the unbalanced angle-plied plates; this is due to the fact that the membrane boundary conditions can significantly influence the plates responses for $-25^\circ \leq \theta \leq 25^\circ$.

5.1.2 Sandwich Plates

In this section the numerical evaluation of the finite strip and the double Fourier series approaches for analyzing various sandwich plates is presented. Sandwich plates with the following boundary conditions are considered: (A) four simply supported sides (B) four clamped sides (C) two opposite simply supported sides and the other two clamped. In all cases, the faces are isotropic or cross-plied and the core is an orthotropic honeycomb type. The behaviour of the sandwich plates with cross-plied faces under transverse loads is studied for the different boundary conditions and properties of the cross-plied faces and core. That is, the effect of the bending-membrane coupling in the unbalanced cross-plied faces on the deflection and stresses
of the sandwich plates will be presented for the cases considered. In the finite strip analysis both the third and the fifth order interpolation polynomials are used. As in the thin laminated plates, symmetry conditions cannot be considered about \( y = B/2 \) for sandwich plates having cross-plied faces because of the unsymmetrical response of the sandwich plate due to the existence of bending-membrane coupling in the faces. For all the cases considered herein, comparison is made between the finite strip and the double Fourier series approaches. In some cases the results are also compared with existing references.

(A) Sandwich Plates with Four Simply Supported Sides

As previously mentioned, the sides of the sandwich plates are usually reinforced in order to prevent any damage to the core due to the reactions of the supports. Therefore, for practical sandwich plate applications the hinged free normal (HFN) type of simple supports are considered. Thus, the boundary conditions are:

\[
v_f(x,y) = w(x,y) = 0 \quad \text{at} \quad x = 0,L \quad (5-13a)
\]

\[
N_{xf}(x,y) = M_{xf}(x,y) = 0 \quad \text{at} \quad x = 0,L \quad (5-13b)
\]

\[
u_f(x,y) = w(x,y) = 0 \quad \text{at} \quad y = 0,B \quad (5-13c)
\]

\[
N_{yf}(x,y) = M_{yf}(x,y) = 0 \quad \text{at} \quad y = 0,B \quad (5-13d)
\]
In the finite strip analysis of this system the basic functions which satisfy the boundary conditions (Eqs. 5-13a and 5-13b) for the isotropic and the cross-plied faces are \( f = 1, 2; \ n = 1, 2, \ldots, N \): 

\[
X_{uf}^{(n)} = \cos \frac{n\pi x}{L} \quad (5-14a)
\]

\[
X_{vf}^{(n)} = \sin \frac{n\pi x}{L} \quad (5-14b)
\]

\[
X_{w}^{(n)} = \sin \frac{n\pi x}{L} \quad (5-14c)
\]

In order to satisfy the displacement boundary conditions in the y-direction (Eq. 5-13c) the following conditions are imposed on the displacement coefficients in the interpolation polynomials along the boundaries \( y = 0, B \) for all values of \( n = 1, 2, \ldots, N \):

\[
u_1^{(n)} = u_2^{(n)} = w^{(n)} = 0 \quad (5-15)
\]

For sandwich plates with isotropic faces, only one half of the plate need be modeled if the plate is symmetrically loaded. For such cases the following conditions are imposed along the axis of symmetry of the plate at \( y = B/2 \) for all values of \( n = 1, 2, \ldots, N \):

\[
v_1^{(n)} = v_2^{(n)} = w_y^{(n)} = 0 \quad (5-16)
\]
Also, if the two faces are identical (geometrically and elastically) the conditions given by Eqs. 3-23 can be imposed since the membrane displacements of both faces are equal in magnitude but opposite in direction. By imposing the symmetry conditions previously mentioned, a large reduction in the number of degrees of freedom is achieved. It is noted that the reduced bending stiffness approximations cannot be used in the analysis of cross-plied sandwich plates since sandwich plates resist the loads mainly by membrane action in the faces.

(A-1) Simply Supported Sandwich Plates with Isotropic Faces

Consider a 20 in. x 20 in. sandwich plate constructed of aluminum alloy faces and an orthotropic aluminum honeycomb core. The properties of the faces and the core are as follows:

(1) Faces

\[ t_f = 0.020 \text{ in.} \quad (f = 1, 2) \]

\[ E = 10 \times 10^6 \text{ psi} \]

\[ v = 0.3 \]

The corresponding stiffnesses of the faces are:

\[ A_{11}^{(f)} = A_{22}^{(f)} = 2.1978 \times 10^5 \text{ lb/in.}; \quad A_{12}^{(f)} = 6.5930 \times 10^4 \text{ lb/in.} \]
\[ A_{16}^{(f)} = A_{26}^{(f)} = 0.0 \quad \text{;} \quad A_{66}^{(f)} = 7.70 \times 10^4 \text{ lb/in.} \]

\[ B_{ij}^{(f)} = 0.0 \quad \text{;} \quad (i,j=1,2,6) \]

\[ D_{11}^{(f)} = D_{22}^{(f)} = 7.3260 \text{ lb-in.} \quad \text{;} \quad D_{12}^{(f)} = 2.1978 \text{ lb-in.} \]

\[ D_{16}^{(f)} = D_{26}^{(f)} = 0.0 \quad \text{;} \quad D_{66}^{(f)} = 2.5667 \text{ lb-in.} \]

\( Core \)

\[ t_c = 1.0 \text{ in.} \]

\[ G_{yz_c} = 75.2 \times 10^3 \text{ psi} \quad \text{;} \quad G_{xz_c} = 32.9 \times 10^3 \text{ psi} \]

The corresponding transverse core shear stiffnesses are:

\[ B_{44} = 75.2 \times 10^3 \text{ lb/in} \quad \text{;} \quad B_{55} = 32.9 \times 10^3 \text{ lb/in.} \]

The plate is subjected to a transverse pressure of 1 psi. For the plate and loading described, the symmetry conditions (Eq. 5-16 and Eq. 3-23) are imposed in the finite strip analysis. However, force boundary conditions are not imposed along the boundaries \( y = 0, B \) nor along the axis of symmetry of the plate. The results obtained from both the finite strip and the double Fourier series approaches are given in Table 6 where they are compared with the finite element method (Ref. 6) as well as the theoretical and experimental results of
Ref. 3. The effect of the number of strips on the accuracy of the results are given in Table 6 where 5 cycles of the basic functions are used. In Table 7 the effect of the number of cycles on the results obtained is presented for a sandwich plate with one half modeled into 4 strips. In Table 8 a comparison of the number of degrees of freedom required is made for the cases where the transverse deflection is expressed as third and fifth order polynomials. Two cases are presented, one for isotropic faces where both symmetry conditions are considered and the other for the cases where symmetry cannot be considered (e.g. cross-plied cases). In Table 9 the results from the various cycles of the double Fourier series are reported.

From the results presented in Table 6, the finite strip and the double Fourier series approaches are in very close agreement for displacements and stresses. Also, both approaches agree with the results given in the two references. In the finite strip analysis, the results obtained for the cases when \( w \) is expressed as a third or fifth order interpolation polynomials show that both have nearly the same accuracy for the stresses and deflections (Table 6 and 7). However, the third order interpolation polynomials have an advantage over the fifth order polynomials in terms of the number of degrees of freedom (Table 8). Therefore, it appears that the third order interpolation polynomials are best suited for sandwich plates with isotropic faces, keeping in mind that the
problem of moment continuity in the faces is of secondary importance in sandwich systems since the loads are resisted mainly by membrane action in the faces. In both the third and the fifth order interpolation polynomials the accuracy of the solution depends on the number of cycles more than the number of strips (Tables 6 and 7). From Table 9 it can be seen that the double Fourier solution does not require many cycles to converge.

(A-2) - Simply Supported, Sandwich Plates with Cross-Plied Faces

In this section, numerical results are presented for simply supported sandwich plates with cross-plied faces under transverse loads in an attempt to study the effect on the structural response due to the coupling between bending and membrane actions in the unbalanced cross-plied faces. The influence of the coupling on the behaviour of the sandwich plates is governed by factors related to the stiffnesses of both facing and core, the aspect ratio of the plate, the core and face thicknesses and the arrangement of the orthotropic layers in the faces. In the following discussion the effect of these factors are presented. For the cases presented herein, the faces of the sandwich plates are constructed from Graphite-Epoxy and Glass-Epoxy cross-plied composites. The elastic properties of these laminates are taken as:
I - Graphite-Epoxy Composites

\[ E_{11} = 30 \times 10^6 \text{ psi} \quad \text{and} \quad E_{11}/E_{22} = 40 \]

\[ G_{12}/E_{22} = 1.0 \quad \text{and} \quad v_{12} = 0.25 \]

II - Glass-Epoxy Composites

\[ E_{11} = 7.5 \times 10^6 \text{ psi} \quad \text{and} \quad E_{11}/E_{22} = 3 \]

\[ G_{12}/E_{22} = 0.4 \quad \text{and} \quad v_{12} = 0.25 \]

Two types of honeycomb cores are considered in the analysis and their properties are as follows:

I - Isotropic Core

\[ G_{xz_c} = 500 \text{ psi} \quad \text{and} \quad G_{yz_c} = 500 \text{ psi} \]

II - Orthotropic Core

(a) \[ G_{xz_c} = 35 \times 10^3 \text{ psi} \quad \text{and} \quad G_{yz_c} = 17 \times 10^3 \text{ psi} \]

(b) \[ G_{xz_c} = 17 \times 10^3 \text{ psi} \quad \text{and} \quad G_{yz_c} = 35 \times 10^3 \text{ psi} \]

For all the cases, the thickness of the core is kept constant \( t_c = 1.0 \text{ in.} \), while the thickness of the faces varies from 0.01 to 0.25 in. The aspect ratio of the
plate varies from 1 (50.0 in. x 50.0 in.) to 10 (50.0 in. x 5.0 in.). For all cases the intensity of the uniform transverse load is 1 psi.

The lay-ups for sandwich plates with two plies in each face are presented in Fig. 12. In Figures 13, 14 and 16, 17, 18 curves are presented for $w_c/w_o$: the ratio of the center deflection of sandwich plates with cross-plied faces to that of the equivalent sandwich plates with orthotropic faces, as function of aspect ratios ($L/B$) of the plate. In all these cases the faces are constructed from two layers of graphite-epoxy composites. Similarly, Figures 19 and 20 are presented for the glass-epoxy cross-plied faces. The curves presented in Fig. 15 show the effect of the ratio $t_c/t_f$ on $w_c/w_o$ for sandwich plates with aspect ratios of 2.5 and 5 and faces constructed from two layers of graphite-epoxy cross-plied composites. Note that in Figs. 13 to 20 graphite-epoxy and glass-epoxy cross-plied composite faces are denoted by GR and GL respectively. Similarly, orthotropic and isotropic cores are denoted by OC and IC, respectively. Some samples of the numerical results are reported in Tables 10 to 13 for the graphite-epoxy cross-plied faces and orthotropic core ($G_{xz_c} = 17 \times 10^3$ psi, $G_{yz_c} = 35 \times 10^3$ psi). In Table 10 the membrane, bending and coupling stiffnesses are presented. The displacement and stress resultants for sandwich plates with aspect ratios of 1, 2.5 and 10 are reported in Tables 11, 12 and 13 respectively. The displacements and stresses
presented are obtained from both the finite strip and double Fourier series approaches. In the finite strip method the transverse deflection is expressed as a fifth order interpolation polynomial in order to be able to achieve continuity in the stress and moment resultants in the faces. It is pointed out that the finite strip and double Fourier series approaches are in very close agreement for all the cases presented in this section.

In Refs. 17, 18 and 19 it was demonstrated that the effect of bending-membrane coupling for thin cross-plied plates can increase the deflection response of such plates by some 300% compared to the response of the corresponding orthotropic plates. It was concluded that the coupling effect for rectangular thin plates is dependent upon the following factors:

1 - The degree of anisotropy \( \frac{E_{11}}{E_{22}} \); the larger the degree of anisotropy the larger the coupling effect.

2 - The aspect ratio \( L/B \); as the aspect ratio increases the plate behaviour rapidly approaches the behaviour for cylindrical bending and the coupling effect is minimum for square plates.

3 - The number of plies; as the number of plies increase the effect of coupling decreases rapidly.

However, by combining such laminates with a core to form anisotropic sandwich plate, it will be shown that the coupling effect can either decrease or increase the
deflection response of the resulting sandwich plates compared to the corresponding sandwich plate with orthotropic faces \( B_{11}^{(f)} = B_{22}^{(f)} = 0 \); the effect depends upon the various properties of the facing and core, as well as the lay-ups of the plies in the two faces. It will be shown that the effect of coupling on the deflection response of sandwich cross-plyed plates is not as severe as in thin laminated plates. For sandwich plates with cross-plyed faces the coupling effect, in addition to the boundary conditions, is dependent upon the following factors:

1 - The ratio between the thickness of the core and the thickness of the faces \( t_c/t_f \).

2 - The lay-ups of the cross-plyed plies in the upper and lower faces.

3 - The aspect ratios of the plate, \( L/B \).

4 - The degree of anisotropy of each ply \( E_{11}/E_{22} \).

5 - The core stiffnesses.

6 - The number of plies in each face.

From the various results presented it can be realized that the ratio \( t_c/t_f \) has a major influence on the coupling effect. It can be concluded that the smaller the ratio \( t_c/t_f \) the larger the coupling effect. In certain circumstances, changing the arrangement of the cross-plyed layers in the two faces (for example lay-ups 1, 2 and 3 in Fig. 13) alters the deflection response from one which has a larger deflection than the corresponding orthotropic case.
to one which has a smaller deflection that the orthotropic case. In other cases (Fig. 17) changing the arrangement of the layers in the faces does not alter the general behaviour of the system but does effect the percentage difference in $w_c/w_o$. In general, then, for certain face and core properties the coupling effect (unlike the thin laminated cases) can decrease the deflection response of the sandwich plate; this can be achieved by choosing a suitable arrangement for the cross-plied layers in the two faces relative to the dimensions of the plate in the $x$ and the $y$ direction of the system. If the coupling has a significant effect on the deflection response, increasing the aspect ratio will decrease the stiffness of the cross-plied sandwich plate when compared with the corresponding orthotropic cases. Examples for such cases are shown in Fig. 13 and Fig. 17 (for lay-ups 1 and 2, $t_c/t_f = 4$).

However, in other cases where coupling is active and adds to the stiffness of the plate (for example Fig. 13 lay-up 3, $t_c/t_f = 4$) increasing the aspect ratio results in an increase in the plate stiffness to a certain limit at which point the effect of coupling diminishes; in some cases (Fig. 16) a further increase in the aspect ratio reverses the original coupling effect and decreases the stiffness of the plate. For cases where coupling has a minor effect on the deflection response of the sandwich plate, for example the cases of thin faces (Fig. 14 and Fig. 13, $t_c/t_f = 10$),
also for the cases where the properties of the facing and the core are balanced (Fig. 19), the aspect ratio does not have any significant influence on the ratio \( w_c/w_o \). For all the cases considered the coupling effect is minimum for square plates. By comparing the cases where the faces are constructed from graphite-epoxy composites with the glass-epoxy cases for the same core stiffnesses (Fig. 13 with Fig. 19 and Fig. 17 with Fig. 20) the effect of the degree of anisotropy of the laminates \( (E_{11}/E_{22}) \) on the coupling effect can be appreciated. It can be concluded that the larger the degree of anisotropy the larger the coupling effect. The effect of the core stiffness on the behaviour of the cross-plied sandwich plates can be studied by comparing Fig. 13 with Fig. 17 and Fig. 19 with Fig. 20, where the properties of the faces are the same while the stiffness of the core is changed; also, the two cases where the stiffnesses of the orthotropic core are reversed (Fig. 13 and Fig. 16) can be compared. It is seen that a reduction in the stiffness of the core tends to reduce the stiffness of the sandwich plates with cross-pled faces compared with the orthotropic cases and vice versa. The influence of the number of plies on the coupling effect is of a secondary importance in sandwich structures, since in such systems the object is to combine a relatively thick light-weight core with thin high-strength faces; thus, a sandwich plate with laminated faces would generally consist of faces with a small number of plies.
It is important to point out that the thickness of the faces has a major influence on the bending-membrane coupling effect. The results indicate that for very thin faces (say for $t_c/t_f$ greater than 50, Fig. 15 and 14), the effect of coupling on the behaviour of sandwich cross-plied plates tends to vanish. That is, the deflection tends to approach those of the corresponding orthotropic cases, and changing the other factors that govern coupling will not have any significant influence on the coupling effect on the behaviour of the plate. On the other hand, for sandwich plates with thick cross-plied faces the coupling effect is active and the other factors influence the coupling effect. The effect of coupling on the stress resultants is presented in Tables 11, 12, 13. The results indicate that the bending-membrane coupling has a significant influence on the stress resultants, $N_{xf}$, $N_{yf}$ and $N_{xyf}$ for the second and third lay-ups. On the other hand, the coupling effect is minimum for the first lay-up. The stress resultants $N_{xf}$, $N_{yf}$ and $N_{xyf}$ in the upper face are slightly different from those in the lower face for the first lay-up. However, for the second and third lay-ups, the stress resultants are the same for both faces. The reason behind the differences for the first lay-up is due to the fact that a cross-plied face in itself represents an unbalanced composite skin where the centroids of the composite cross-sections in the $x$ and $y$ directions are located in different planes parallel to the
middle surface of the skin, and that the arrangement of the plies in the first lay-up is non-symmetric with respect to the middle surface of the core. On the other hand, the plies in the two faces of lay-ups 2 and 3 are symmetric with respect to the middle surface of the core. Similarly, the differences in the behaviour of the cross-plied sandwich plates for the various lay-ups (i.e. lay-ups 1, 2, 3) can be explained.

B - Sandwich Plates With Four Clamped Sides

Since the sides of the sandwich plates are usually reinforced, the following practical clamped boundary conditions are considered:

\[ u_f(x,y) = v_f(x,y) = w(x,y) = w_x(x,y) = 0 \]

at \( x = 0,L \) \hspace{1cm} (5-17a)

\[ u_f(x,y) = v_f(x,y) = w(x,y) = w_y(x,y) = 0 \]

at \( y = 0,B \) \hspace{1cm} (5-17b)

In the finite strip analysis of this system, the basic functions which will satisfy the boundary conditions (Eq. 5-17a) for sandwich plates with isotropic or cross-plied faces are:
\[ X_{u_f}^{(n)} = \cos\left(\frac{\lambda_n x}{L}\right) - \cosh\left(\frac{\lambda_n x}{L}\right) + \alpha_n \left[ \sin\left(\frac{\lambda_n x}{L}\right) \right. \]
\[ \left. + \sinh\left(\frac{\lambda_n x}{L}\right) \right] \quad (5-18a) \]

\[ X_{v_f}^{(n)} = \sin\left(\frac{\lambda_n x}{L}\right) - \sinh\left(\frac{\lambda_n x}{L}\right) - \alpha_n \left[ \cos\left(\frac{\lambda_n x}{L}\right) \right. \]
\[ \left. - \cosh\left(\frac{\lambda_n x}{L}\right) \right] \quad (5-18b) \]

\[ f = 1, 2 \]

\[ X_w^{(n)} = \sin\left(\frac{\lambda_n x}{L}\right) - \sinh\left(\frac{\lambda_n x}{L}\right) - \alpha_n \left[ \csc\left(\frac{\lambda_n x}{L}\right) \right. \]
\[ \left. - \cosh\left(\frac{\lambda_n x}{L}\right) \right] \quad (5-18c) \]

where

\[ \lambda_n = \frac{(2n + 1)\pi}{2}; \quad \alpha_n = \frac{\sin(\lambda_n) - \sinh(\lambda_n)}{\cos(\lambda_n) - \cosh(\lambda_n)} \quad (5-18d) \]

and \( n = 1, 2, ..., N \)

These basic functions are not orthogonal. It is pointed out that the preceding basic functions imply an additional boundary condition that \( N_{xy_f}(x, y) = 0 \) at \( x = 0, L \). The boundary conditions in the \( y \)-directions (Eq. 5-17b) are satisfied by imposing the following conditions on the displacement coefficients in the interpolation polynomials.
along the boundaries $y = 0, b$:

$$u_1^{(n)} = u_2^{(n)} = v_1^{(n)} = v_2^{(n)} = w^{(n)} = w_y^{(n)} = 0$$

for all values of $n = 1, 2, \ldots, N$ \hfill (5-19)

(B-1) Clamped Sandwich Plates With Isotropic Faces

The numerical results presented herein are for a 10 in. x 10 in. clamped sandwich plate with isotropic faces under a uniform transverse load of 1 psi. The plate is analyzed using both finite strip and double Fourier series approaches. In the finite strip approach the displacement functions are expressed as third order interpolation polynomials and the symmetry conditions (Eq. 5-16 and Eq. 3-23) are considered. The properties of the faces and the core are as follows:

(a) Faces

$$t_f = 0.028 \text{ in.}$$

$$E = 10^7 \text{ psi}$$

$$v = 0.3$$

The resulting stiffnesses are:

$$A_{11}^{(f)} = A_{22}^{(f)} = 3.0769 \times 10^5 \text{ lb/in.}; \quad A_{12}^{(f)} = 9.2308 \times 10^4 \text{ lb/in.}$$
\[ A_{16}^{(f)} = A_{26}^{(f)} = 0.0 \quad ; \quad A_{66}^{(f)} = 1.0769 \times 10^5 \text{ lb/in.} \]

\[ B_{ij}^{(f)} = 0.0 \quad (i,j = 1,2,6); \]

\[ D_{11}^{(f)} = D_{22}^{(f)} = 20.103 \text{ lb-in.} \quad ; \quad D_{12}^{(f)} = 6.0308 \text{ lb-in.} \]

\[ D_{16}^{(f)} = D_{26}^{(f)} = 0.0 \quad ; \quad D_{66}^{(f)} = 7.0359 \text{ lb-in.} \]

(b) **Core**

\[ t_c = 0.75 \text{ in.} \]

\[ G_{xz_c} = G_{yz_c} = 3 \times 10^4 \text{ psi} \]

The transverse shear stiffnesses are:

\[ B_{55} = B_{44} = 2.25 \times 10^4 \text{ lb/in.} \]

The results obtained from the finite strip and the double Fourier series approaches are presented in Table 14 where they are compared with the solution given in Ref. 28. The effect of the number of strips on the results obtained are presented in Table 14 where 4 cycles of the basic functions are used. In Table 15 the effect of the number of cycles on the accuracy of the results is presented for sandwich plates with one half modeled into 4 strips. In Table 16 a comparison of the number of degrees of freedom required is made for the third and fifth order polynomial solutions. Two cases are presented, one for isotropic...
faces and the other for the cases where symmetry cannot be considered (e.g. cross-plied). In Table 17 results are presented for deflections and stress resultants for various cycles of the double Fourier series.

The results presented in Table 17 show that the double Fourier series solution requires many cycles to converge for both the deflection and the stress resultants. Note that the number of the resulting simultaneous equations for the case where 20 cycles of the double Fourier series are used is 240. Also, it is pointed out that even cycles do not have any significant effect on the results when the load is uniformly distributed. From the finite strip solutions presented in Tables 14 and 15, it appears that the number of cycles has a greater influence on the results than the number of strips. Table 16 points out the advantage of considering symmetry whenever symmetry exists in order to save in the computer time and storage required (especially for clamped boundary conditions where the basic functions are all coupled and therefore the number of degrees of freedom is dependent on the number of cycles of the basic functions). By comparing the results obtained from the finite strip method with those from the double Fourier series method, it appears that by increasing the number of cycles in the double Fourier series approach the solution will converge to those for the finite strip method and the reference solution. The discrepancies in the results of the stress resultants $N_{xf}$ and $N_{yf}$ (at $x = L/2, y = B/2$) in the finite strip solutions occur
because of the difference in the displacement expressions in the x and y directions of the plate.

(B-2) Clamped Sandwich Plates With Cross-Plied Faces

In this section the effect of coupling on the behaviour of clamped sandwich plates with cross-plied faces is presented. The faces of the sandwich system are constructed from two layers of graphite-epoxy cross-plied composites. The elastic properties for such laminates are the same as those given in section A-2 for simply supported sandwich plates. Numerical results are presented for square plates with \( \frac{t_c}{t_f} = 4 \). The resulting bending, membrane and coupling stiffnesses are presented in Table 10.

The core considered is orthotropic and \( G_{yz_C} = 35 \times 10^3 \) psi and \( G_{xz_C} = 17 \times 10^3 \) psi. The plate is analyzed under uniform transverse load of 1 psi. This system has been analyzed using both finite strip and double-Fourier series approaches. In the finite strip analysis, the transverse deflection is expressed as fifth order interpolation polynomial; and, the plate is modeled by 4 strips using 3 cycles of the basic functions. The resulting number of degrees of freedom is 129. The double Fourier series results are presented after 20 cycles. The results obtained from the finite strip and double Fourier series approaches for the various lay-ups are presented in Table 18. The deflection and stresses obtained from the two approaches are in closest agreement for the third lay-up. However,
for the other lay-ups, the percentage difference between the results of the two approaches is relatively higher than that of the third lay-up. It appears that the finite strip analysis for such cases requires more strips and more cycles. However, for cross-plied sandwich systems (where symmetry cannot be considered), the resulting number of degrees of freedom is usually high (Table 16), especially for clamped boundary conditions where the number of degrees of freedom is dependent upon the number of cycles. This imposes some limitations on the number of cycles and strips to be used. It is pointed out that the double Fourier series results presented in Table 18 have converged; any further increase in the number of cycles does not result in any significant change for the stresses and deflections presented. Also, from Table 18 the effect of coupling can be studied. It can be seen that, as in simply supported sandwich plates, the coupling effect can either decrease the deflection response (lay-up 3) or increase the deflection response (lay-ups 1 and 2) of the resulting clamped cross-plied sandwich plates compared to the corresponding sandwich plates with orthotropic faces. The effect of boundary conditions on the bending-membrane coupling can be studied by comparing the results obtained for \( \frac{w_c}{w_0} \) for the clamped plate with the corresponding simply supported plate, as presented in Table 19. It can be seen that the coupling effect becomes more active by clamping the sides of the plate.
(C) - Sandwich Plates With Two Opposite Sides Simply Supported and the Other Sides Clamped

In this section, two cases are presented for sandwich plates with isotropic and cross-plyed faces and subjected to a uniform transverse load. The finite strip and the double Fourier series approaches are used in the analysis of this system. In the finite strip analysis the basic functions of the simply supported cases (Eqs. 5-14) can be used associated with the boundary conditions in the y-direction for the clamped cases (Eqs. 5-17b), or by using the basic functions of the clamped cases (Eqs. 5-18) with the boundary conditions in the y-direction of the simply supported cases (Eqs. 5-13c and 5-13d).

Practically, the first set is the one to be used in the analysis of this system since the basic functions of the simply supported cases are orthogonal, thus, the computer time and storage for this set is far less than the second set. As in the simply supported case, the symmetry conditions (Eqs. 5-16 and 3-23) can be considered for sandwich plates with isotropic faces.

(C-1) - Sandwich Plates With Isotropic Faces

Consider a 10 in. x 10 in. sandwich plate having the following properties for the faces and the core:

(a) Faces

\[ t_f = 0.028 \text{ in.} \quad (f = 1,2) \]

\[ E = 10^7 \text{ psi} \quad ; \quad \nu = 0.3 \]
The resulting stiffnesses are:

\[ A_{11}^{(f)} = A_{22}^{(f)} = 3.0769 \times 10^5 \text{lb/in.}; \quad A_{12}^{(f)} = 9.2308 \times 10^4 \text{lb/in.} \]

\[ A_{16}^{(f)} = A_{26}^{(f)} = 0.0 \quad ; \quad A_{66}^{(f)} = 1.0769 \times 10^5 \text{lb/in.} \]

\[ B_{ij}^{(f)} = 0.0 \quad (i, j = 1, 2, 6) \]

\[ D_{11}^{(f)} = D_{22}^{(f)} = 20.103 \text{ lb-in.}; \quad D_{12}^{(f)} = 6.0308 \text{ lb-in.} \]

\[ D_{16}^{(f)} = D_{26}^{(f)} = D_{66}^{(f)} = 7.0359 \text{ lb-in.} \]

(b) **Core**

\[ t_c = 0.75 \text{ in.} \]

\[ G_{yz_c} = G_{xz_c} = 3 \times 10^4 \text{ psi} \]

The transverse core shear stiffnesses are:

\[ B_{44} = B_{55} = 2.25 \text{ lb/in.} \]

The plate is analyzed under a transverse uniform load of 1.0 psi. The results obtained from the finite strip and the double Fourier series approaches are presented in Table 20 where they are compared with the deflection and stress resultants given in Ref. 28. The maximum deflection and the stress resultants obtained for the various cycles of the double Fourier series are presented in Table 21. Note that the plate is considered to be clamped along the sides \( x = 0, L \), while the sides \( y = 0, B \) are simply supported. In Table 20, the finite strip case A uses the basic functions of the simply supported plates (Eqs. 5-14), while the finite strip case B uses the basic functions of clamped
plates (Eq. 5-18). For both cases the displacement components are expressed as third order interpolation polynomials. The number of degrees of freedom which result from modeling one half of the plate into 5 strips and using 3 cycles for case A and 4 cycles for case B are 30 for case A and $4 \times 32 = 128$ for case B, where 32 is the number of degrees of freedom for one cycle of the basic functions. Note that the number of simultaneous equations which result from using 20 cycles of the double Fourier series is $6 \times 20 = 120$.

Unlike the simply supported case, the double Fourier series approach requires many cycles to converge for both deflection and stresses (Table 21). It is pointed out that even cycles in the double Fourier series approach do not have any effect on the results when the load is uniformly distributed over the plate (Table 21, cycles 9 and 10).

When comparing the double Fourier series solution with the finite strip case A and the reference solution, the results for the stress resultant $N_{Y_f}$ are in very close agreement in all the cases. Also, it appears that by a slight increase in the number of cycles used in the double Fourier series, the results for the deflection and the stress resultant $N_{X_f}$ will converge to the finite strip and the reference solutions. From the results presented as well as by considering the number of degrees of freedom required for the finite strip solutions cases A and B, it can be realized that the finite strip method case A is preferable.
over case B. Also, the finite strip method case A has the advantages of less computer time and storage over the double Fourier series solution, especially since the basic functions for this case are not coupled (thus any increase in the number of cycles will not affect the computer time and storage when compared with the double Fourier series solution where all the cycles are coupled). Therefore, obviously, it is more practical to use the finite strip method case A in the analysis of sandwich plates with such boundary conditions.

(C-2) - Sandwich Plates With Cross-Plied Faces

In this section the effect of coupling on the behaviour of sandwich plates with cross-plied faces having two sides simply supported and the other two sides clamped is investigated. For rectangular plates the clamped sides are along the two ends of the short span. This system has been analyzed using the finite strip and the double Fourier series approaches. In the finite strip method, the basic functions used are the same as those for the simply supported boundary conditions (Eqs. 5-14). Also, the transverse deflection is expressed as a fifth order interpolation polynomial. For the cases presented herein, the faces of the sandwich plate are constructed from graphite-epoxy cross-plied composites, and the elastic properties for such laminates are the same as those considered in the simply supported sandwich plates, with
cross-plied faces (section A-2). The numerical results are presented for $t_c/t_f = 4$. The resulting bending, membrane and coupling stiffnesses are presented in Table 10. For all the cases, the core is considered to be orthotropic and having stiffnesses $G_{yz} = 35 \times 10^3$ psi and $G_{xz} = 17 \times 10^3$ psi. The plate is analyzed under a uniform transverse load of 1 psi.

In Fig. 21, $w_c/w_o$ is compared for the double Fourier series and the finite strip approaches, as a function of aspect ratio (L/B). Samples of the results obtained from the finite strip and the double Fourier series approaches for various aspect ratios are presented in Tables 22 to 25. In these tables the center deflection and the stress resultants at $x = L/2$ and $y = B/2$ are presented for various lay-ups of the cross-plied laminates in the faces, as well as for the corresponding orthotropic faces. The effect of the boundary conditions on the bending membrane coupling is presented in Fig. 22, where $w_c/w_o$ is compared for sandwich plates having 2 simply supported and two clamped sides with the results of sandwich plates with four simply supported sides and having the same properties of the facing and the core (Fig. 13). The numerical results obtained from the double Fourier series and the finite strip methods are in reasonable agreement for the deflections and the stress resultants (Tables 22 to 25). However, for a few cases it appears that increasing the number of cycles in
the double Fourier series solution is necessary. It is pointed out that the double Fourier series results for the central deflection are always less than the corresponding finite strip results. This implies that by increasing the number of cycles the solution will converge to that of the finite strip method, since the deflection results for all cycles of the double Fourier series solution are additive. As in simply supported sandwich plates with cross-plied faces, the stress resultants in the upper and the lower faces are different when the layers are arranged as in lay-up 1; for the second and third lay-ups the stress resultants are the same in the two faces. From Fig. 22, and as it was concluded in the case of sandwich plates with four clamped sides, it appears that by clamping the sides at y = 0, B the coupling effect tends to have a greater influence on the deflection response than when the four sides of the plate are simply supported. This can be appreciated by comparing the results for the sandwich plate with aspect ratio of one. As previously mentioned in section 5-1-2 (A=2), the larger the aspect ratio of the plate the larger the effect of coupling on increasing $w_c/w_o$. From Fig. 22, it appears that the aspect ratio effect became more severe when the two long sides (y = 0, B) of the plate are clamped.
5.2 Cylindrical Shell Structures

In this section solutions are presented for both thin and sandwich shell roof structures. The shell roof is assumed to be supported at the two ends (i.e. $x = 0, L$) by transverse frames which provide the following support conditions:

(a) Complete rigidity in their planes:

$$v_f(x, y_f) = w(x, y_f) = 0 \quad \text{at} \quad x = 0, L \quad (5-20a)$$

(b) Complete flexibility normal to their planes:

$$M_{x_f}(x, y_f) = N_{x_f}(x, y_f) = 0 \quad \text{at} \quad x = 0, L \quad (5-20b)$$

Therefore, for these simply supported boundary conditions, the basic functions for both thin and sandwich shells having isotropic or cross-plied faces are the same as the corresponding cases for plates. That is, Eqs. 5-2 for thin shells and Eqs. 5-14 for sandwich cylindrical shell. The finite strip method using third order interpolation polynomials is used as means of analysis for all the shell structures considered herein. The expressions for the work equivalent loads for the various cases are given in Appendix E.

5.2.1 Thin Shells

The numerical results presented herein are for thin isotropic cylindrical shell roof and thin isotropic
cylindrical shell roof with edge beams:

(A) Thin Isotropic Cylindrical Shell Roof

Consider a concrete shell roof having the following dimensions and properties (Fig. 25):

\[ L = 62 \text{ ft.} \quad ; \quad R = 31 \text{ ft.} \]
\[ t = 3.75 \text{ in.} \quad ; \quad \phi_e = 40^\circ \]

The elastic properties of the shell are:

\[ E = 3 \times 10^6 \text{ psi} \quad ; \quad \nu = 0.0 \]

This results in the following membrane and bending stiffnesses:

\[ A_{11} = A_{22} = 1.125 \times 10^7 \text{ lb/in.} \quad ; \quad A_{12} = A_{16} = A_{26} = 0.0 \]
\[ A_{66} = 5.625 \times 10^6 \text{ lb/in.} \quad ; \]
\[ D_{11} = D_{22} = 1.3184 \times 10^7 \text{ lb-in.} \quad ; \quad D_{12} = D_{16} = D_{26} = 0.0 \]
\[ D_{66} = 6.5918 \times 10^6 \text{ lb-in.} \]

The shell is analyzed under its own weight (dead load) and a snow load considered as live load. The intensities of the loads are:

Dead load, \( p_d = 47 \text{ lb/ft}^2 \); Live Load, \( p_L = 25 \text{ lb/ft}^2 \)
The shell is symmetrically loaded, therefore only one half of the shell is considered in the analysis which in turn requires that the following conditions be applied at the crown:

\[ v(x, y) = \frac{\partial w(x, y)}{\partial y} = \frac{1}{R} \frac{\partial w(x, \theta)}{\partial \theta} = 0 \quad (5-21a) \]

This in turn requires that the following conditions be imposed on the displacement coefficients in the third order interpolation polynomials at the crown for all values of \( n = 1, 2, \ldots, N \):

\[ v(n) = w_y(n) = 0 \quad (5-21b) \]

The following additional force boundary condition is also imposed in order to ensure that the membrane shear stress \( N_x \phi \) vanishes at the crown for all values of \( n = 1, 2, \ldots, N \):

\[ \frac{\partial u(x, y)}{\partial y} = \frac{1}{R} \frac{\partial u(x, \theta)}{\partial \theta} = 0 \quad (5-22a) \]

or

\[ u_y(n) = 0 \quad (5-22b) \]

The number of degrees of freedom which result from modeling one half of the shell with 8 strips is 51. Only one cycle of the basic function (i.e., \( n = 1 \)) is considered. The results obtained for the reference surface stress resultants and the moment resultants are presented in Table 26 where
comparison is made with Ref. 33. The results are in close agreement for the various stress and moment resultants.

(B) Thin Cylindrical Shell Roofs With Edge Beams

In the analysis of the cylindrical shell roofs with edge beams, the displacement components of the edge beam and of the shell at their line of intersection are measured with respect to different local coordinate systems. Therefore, the satisfaction of the geometric admissibility conditions does not result in a simple one-to-one matching of the displacement coefficients at the line of intersection of the edge beam and shell. Consider two adjacent strips one in the edge beam and the other in the shell (Fig. 26a). The geometric admissibility conditions at the interface between the two strips requires that:

\[ w^{(b)}(x,b) = w^{(s)}(x,0) \cdot \sin \phi_e - v^{(s)}(x,0) \cdot \cos \phi_e \quad (5-23a) \]

\[ v^{(b)}(x,b) = w^{(s)}(x,0) \cdot \cos \phi_e + v^{(s)}(x,0) \cdot \sin \phi_e \quad (5-23b) \]

\[ u^{(b)}(x,b) = u^{(s)}(x,0) \quad (5-23c) \]

\[ \frac{\partial w^{(b)}(x,b)}{\partial y^{(b)}} = \frac{\partial w^{(s)}(x,0)}{\partial y^{(s)}} \quad (5-23d) \]

where \( w^{(b)} \), \( v^{(b)} \) and \( u^{(b)} \) are the displacement components for the beam strip and \( w^{(s)} \), \( v^{(s)} \) and \( u^{(s)} \) are the displacement components for the shell strip, and \( b \) is the width of the
beam strip. From the properties of third order interpolation polynomials (Eqs. 3-13 to 3-15) and Eqs. 5-23, the relationships between the beam and shell displacement coefficients at the interface can be expressed as:

\[ w_2^{(b)} = w_1^{(s)} \cdot \sin \phi_e - v_1^{(s)} \cdot \cos \phi_e \]  \hspace{1cm} (5-24a)

\[ v_2^{(b)} = w_1^{(s)} \cdot \cos \phi_e + v_1^{(s)} \cdot \sin \phi_e \]  \hspace{1cm} (5-24b)

\[ u_2^{(b)} = u_1^{(s)} \]  \hspace{1cm} (5-24c)

\[ w_2^{(b)} = w_1^{(s)} \]  \hspace{1cm} (5-24d)

where \( w_2^{(b)}, v_2^{(b)}, u_2^{(b)}, w_2^{(s)}, v_1^{(s)}, u_1^{(s)} \) and \( w_1^{(s)} \) are the displacement coefficients for the beam strip and the shell strip at the interface between the edge beam and the shell respectively. These conditions must be satisfied for all the cycles of the basic functions independently.

As previously mentioned, the geometric admissibility conditions are imposed between adjacent strips using the variable correlation scheme, in which similar displacements coefficients of two adjacent strips are linked to one another. Therefore, in order to be able to impose the geometric admissibility conditions at the interfaces between the shell and the edge beam (Eqs. 5-24), the displacement components of all strips in the edge beam are transformed from their local coordinate system to a reference coordinate.
system which coincides with the coordinate system of the shell at the line of intersection between the shell and the edge beam. After transformation, the geometric admissibility conditions can be imposed through the variable correlation scheme by a simple one-to-one matching of the displacement coefficients. The transformation from the local to the reference coordinate system of the displacement coefficients at the two edges of a strip in the edge beam can be expressed as follows (Fig. 26b):

(a) At edge 1 (i.e. \( y = 0 \)) of a strip

\[
\begin{align*}
\psi_1^{(b)} &= \psi_1^{(r)} \cdot \sin \phi_e - \psi_1^{(r)} \cdot \cos \phi_e \\
\psi_1^{(b)} &= \psi_1^{(r)} \cdot \cos \phi_e + \psi_1^{(r)} \cdot \sin \phi_e \\
u_1^{(b)} &= \psi_1^{(r)} \\
u_1^{(b)} &= \psi_1^{(r)} \cdot \cos \phi_e + \psi_1^{(r)} \cdot \sin \phi_e \\

(b) At edge 2 (i.e. \( y = b \)) of a strip

\[
\begin{align*}
\psi_2^{(b)} &= \psi_2^{(r)} \cdot \sin \phi_e - \psi_2^{(r)} \cdot \cos \phi_e \\
\psi_2^{(b)} &= \psi_2^{(r)} \cdot \cos \phi_e + \psi_2^{(r)} \cdot \sin \phi_e \\
u_2^{(b)} &= \psi_2^{(r)} \\
u_2^{(b)} &= \psi_2^{(r)} \cdot \cos \phi_e + \psi_2^{(r)} \cdot \sin \phi_e \\

\end{align*}
\]
where \( w_j^{(r)}, v_j^{(r)}, u_j^{(r)} \) and \( w_j^{(r)} \) (\( j = 1, 2 \)) are the displacement coefficients at the two edges of the strip measured with respect to the reference coordinate system. Note that this transformation is required at every cycle of the basic function.

The third order interpolation polynomials for the displacement components \( u(x,y) \) and \( v(x,y) \) contain displacement coefficients which represent membrane strains at each of the two edges of the strip (for example \( u_j^{(n)} \) and \( v_j^{(n)} \)). The transformation of these displacement coefficients from the local to the reference coordinate system can be expressed using the traditional strain transformation relations (Ref. 34). Therefore, the transformation relations for these displacement coefficients are as follows:

(a) At edge 1 of a strip

\[
\begin{align*}
\frac{u^{(b)}}{y_1} &= \frac{u^{(r)}}{y_1} \cdot \sin \phi_e \\
\frac{v^{(b)}}{y_1} &= \frac{v^{(r)}}{y_1} \cdot \sin^2 \phi_e
\end{align*}
\]  
(5-27a)

(b) At edge 2 of a strip

\[
\begin{align*}
\frac{u^{(b)}}{y_2} &= \frac{u^{(r)}}{y_2} \cdot \sin \phi_e \\
\frac{v^{(b)}}{y_2} &= \frac{v^{(r)}}{y_2} \cdot \sin^2 \phi_e
\end{align*}
\]  
(5-28a)

Thus, the resulting transformation matrix is given by:
$$
\begin{align*}
\begin{bmatrix}
u_1^{(b)} \\
u_2^{(b)} \\
u_Y^1 \\
u_Y^2 \\
v_1^{(b)} \\
v_2^{(b)} \\
v_Y^1 \\
v_Y^2 \\
w_1^{(b)} \\
w_2^{(b)} \\
w_Y^1 \\
w_Y^2 \\
\end{bmatrix}
&= 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \sin \phi_e & 0 \\
0 & 0 & 0 & \sin \phi_e \\
\sin \phi_e & 0 & 0 & 0 \\
0 & \sin \phi_e & 0 & 0 \\
0 & 0 & \sin^2 \phi_e & 0 \\
0 & 0 & 0 & \sin^2 \phi_e \\
-cos \phi_e & 0 & 0 & 0 \\
0 & -cos \phi_e & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
u_1^{(r)} \\
u_2^{(r)} \\
u_Y^1 \\
u_Y^2 \\
v_1^{(r)} \\
v_2^{(r)} \\
v_Y^1 \\
v_Y^2 \\
w_1^{(r)} \\
w_2^{(r)} \\
w_Y^1 \\
w_Y^2 \\
\end{bmatrix}
\end{align*}
$$
The geometric admissibility conditions at the interface between the shell and the edge beam can be imposed by a simply one-to-one matching of the displacement coefficients for all the cycles of the basic functions as follows (Fig. 26):

\[ v_2(r) = u_1(s) \quad ; \quad v_2(r) = v_1(s) \quad (5-30a) \]

\[ w_2(r) = w_1(s) \quad ; \quad w_2(r) = w_1(s) \quad (5-30b) \]

The numerical results presented herein are for thin concrete shell roof with edge beams and having the following properties and dimensions (Fig. 27):

\[ L = 83.25 \text{ ft.} \quad ; \quad R = 25.0 \text{ ft.} \]

\[ \phi_e = 35^\circ \]

The elastic properties of the structure are:

\[ E = 3 \times 10^6 \text{ psi} \quad ; \quad \nu = 0.0 \]

(a) Edge Beams

Thickness of the edge beam = 9.0 in.

Depth of the edge beam = 5.0 ft.

The resulting stiffnesses are:
\( A_{11} = A_{22} = 2.7 \times 10^{7} \text{ lb/in.} \); \( A_{12} = A_{16} = A_{26} = 0.0 \)

\( A_{66} = 1.35 \times 10^{7} \text{ lb/in.} \).

\( B_{ij} = 0.0 \) \((i, j = 1, 2, 6)\)

\( D_{11} = D_{22} = 1.8225 \times 10^{8} \text{ lb-in.}; \ D_{12} = D_{16} = D_{26} = 0.0 \)

\( D_{66} = 9.1125 \times 10^{7} \text{ lb-in.} \).

(b) Shell

Thickness of the shell = 3.0 in.

The resulting stiffnesses are:

\( A_{11} = A_{22} = 9.0 \times 10^{6} \text{ lb/in.} \); \( A_{12} = A_{16} = A_{26} = 0.0 \)

\( A_{66} = 4.5 \times 10^{6} \text{ lb/in.} \).

\( B_{ij} = 0.0 \) \((i, j = 1, 2, 6)\)

\( D_{11} = D_{22} = 6.75 \times 10^{6} \text{ lb-in.}; \ D_{12} = D_{16} = D_{26} = 0.0 \)

\( D_{66} = 3.375 \times 10^{6} \text{ lb-in.} \).

The structure is analyzed under its own weight considered as dead load having an intensity of 50 lb/ft\(^2\). The shell
is symmetrically loaded, therefore only one half of the structure is considered in the analysis. This in turn requires that the conditions (Eqs. 5-21, 22) be satisfied. The total number of degrees of freedom which result from modeling one half of the structure by 12 strips (2 strips for the edge beam and 10 strips for the shell) is 78. Only one cycle of the basic functions is considered in the analysis. The results are presented in Figures 28 to 31 where they are compared with the solution of Ref. 35. The Figures show very close agreement between the finite strip and the reference for the various stress and moment resultants. However, at the edge of the shell the results deviate because the reference solution neglects the torsion resistance of the edge beams which is included in the finite strip analysis.

5.2.2 Sandwich Shells

The cases presented herein are for simply supported sandwich shell roof structures. The finite strip analysis of sandwich shells can be considered as a progressive step from the analysis of thin shell structures, in that the thin shell can be treated as a single face of the sandwich shell. Also, the analysis of sandwich shells is not entirely independent from the analysis of sandwich plates. That is, the formulations required in the finite strip analysis of sandwich shell structures incorporate the formulations involved in thin shell and sandwich plate systems. In other words, the strain energy formulations
for the faces of sandwich shells are the sum of the strain energies of each face which in turn represent the strain energy formulations of thin shells. Also, the potential of the applied dead and live loads are the same for thin and sandwich shell roofs. The contribution of the core in the formulations to the total strain energy for sandwich plates are the same as those for sandwich shells except that in sandwich plate cases \( y_c = y_1 = y_2 = y \) and \( R_c / R_f = 1 \). Also, it is pointed out that the basic functions for both sandwich plates and sandwich shell roofs are the same when both structures have the same supporting conditions and type of faces. The numerical results presented herein are for sandwich cylindrical shell roofs having isotropic and cross-plied faces and supported as indicated in Section 5.2.

(A) Sandwich Cylindrical Shell With Isotropic Faces

The following numerical results are for sandwich shells with isotropic aluminum alloy faces and a honeycomb core. The various dimensions and properties of the structures are (Fig. 32):

\[
L = 30.0 \text{ ft} \quad ; \quad R_c = 120.55 \text{ in.}
\]

\[
t_f = 0.1 \text{ in.} \quad (f=1,2) \quad ; \quad t_c = 1.0 \text{ in.}
\]

\( \phi = 30^\circ \)
The elastic properties of the faces are:

\[ E = 10^7 \text{ psi} \quad ; \quad \nu = 0.3 \]

This results in the following stiffnesses:

\[ A_{11}^{(f)} = A_{22}^{(f)} = 1.0989 \times 10^6 \text{ lb/in.}; \quad A_{12}^{(f)} = 3.2965 \times 10^5 \text{ lb/in.} \]

\[ A_{16}^{(f)} = A_{26}^{(f)} = 0.0 \]

\[ A_{66}^{(f)} = 3.8500 \times 10^5 \text{ lb/in.} \]

\[ B_{ij}^{(f)} = 0.0 \quad (i,j = 1,2,6) \]

\[ D_{11}^{(f)} = D_{22}^{(f)} = 9.1575 \times 10^2 \text{ lb-in.}; \quad D_{12}^{(f)} = 2.7472 \times 10^2 \text{ lb-in.} \]

\[ D_{16}^{(f)} = D_{26}^{(f)} = 0.0 \]

\[ D_{66}^{(f)} = 3.2084 \times 10^2 \text{ lb-in.} \]

The core is considered to be isotropic honeycomb type and has the following properties:

\[ G_{yz_c} = 35.0 \times 10^3 \text{ psi} \quad ; \quad G_{xz_c} = 35.0 \times 10^3 \text{ psi} \]

The corresponding transverse shear stiffnesses are:

\[ B_{44} = 35.0 \times 10^3 \text{ lb/in.} \quad ; \quad B_{55} = 35.0 \times 10^3 \text{ lb/in.} \]

In the analysis of this system the weight of the shell is neglected and the shell is analyzed under snow load.
considered as live load, having an intensity of .25 lb/ft\(^2\),
and the load is assumed to be acting on the upper face of
the shell. The shell is symmetrically loaded, therefore
one half of the structure is considered in the analysis.
This requires that the following conditions be imposed at
the crown on the displacement coefficients in the third
order interpolation polynomials for all values of \( n = 1,2,\ldots,N \):

\[
  w_1^{(n)} = v_1^{(n)} = v_2^{(n)} = 0 \tag{5-31}
\]

The following force conditions were also considered at the
crown in order to ensure that the membrane shear stress
\( N_{x\phi_f} \) vanishes at the crown of the shell:

\[
  u_1^{(n)} = u_2^{(n)} = 0 \quad \text{for } n = 1,2,\ldots,N \tag{5-32}
\]

The number of degrees of freedom which result from modeling
one half of the shell by 8 strips is 85. It is noted that
7 cycles are used in the analysis. The stress resultants
are presented in Figs. 33, 34 and 35. The structure
resists the loads mainly by membrane actions in the faces;
the lateral moment of the sandwich shell, \( M_{\phi} \), induces
membrane forces \( N_{x\phi_f} \) in the two faces which result in tensile
stress resultants, \( N_{x\phi_f} \), in the upper face, while \( N_{x\phi_f} \) for the
lower face remains compressive (Fig. 34). For this
structure no solutions are available from other approaches.
However, some confidence is built into the results obtained
since the analysis of this system is not entirely
independent from the thin shell and sandwich plate systems
as previously mentioned.

(B) Sandwich Cylindrical Shell With Cross-Plied Faces

In this section, numerical results are presented for
sandwich cylindrical shell roof having cross-plied faces
which demonstrate the effect on the structural response due
to bending-membrane coupling in the unbalanced cross-plied
faces. The faces of the sandwich shell are constructed from
two layers of graphite-epoxy cross-plied composites each of
0.125" thickness. The resulting membrane, bending and
coupling stiffnesses are given in Table 10. The core
considered is isotropic and \( G_{yzc} = G_{xzc} = 35 \times 10^3 \) psi.
The various dimensions of the structure are (Fig. 36):

\[
L = 30.0 \text{ ft.} \quad ; \quad R_c = 120.55 \text{ in.}
\]

\[
\phi = 30^\circ \quad ; \quad t_c = 1.0 \text{ in.}
\]

In the analysis of this system, the weight of the structure
is neglected and the shell is analyzed under a snow load of
25 lb/ft\(^2\) considered to be acting on the upper face of the
shell. In the finite strip analysis of this system, all the
displacement components are expressed as third order
interpolation polynomials. The number of degrees of freedom
which result from modeling the shell by 10 strips is 110.
The results are presented after 7 cycles of the basic
functions. The stress resultants \( N_{xf} \), \( N_{\phi_f} \), and \( N_{x\phi_f} \) (\( f = 1,2 \)) are presented in Figs. 37, 38 and 39 respectively for the various lay-ups (i.e., 1, 2 and 3) as well as for the corresponding orthotropic faces where the coupling stiffnesses are neglected (i.e., \( B_{1}^{(f)} = B_{22}^{(f)} = 0.0 \)). It is pointed out that despite the fact that the structural response is non-symmetric due to the existence of coupling between membrane and bending actions, the percentage difference between the results obtained for the two halves (i.e., \( \phi = 0 \) to \( \phi = 30^\circ \) at the crown) is negligible. Note that since all the displacement components are expressed as third order polynomials, discontinuity occurs between adjacent strips for the results of the stress resultant \( N_{\phi_f} \).

However, it was found that the magnitude of the discontinuity between the various strips is very small. From the results presented it can be seen that the bending-membrane coupling does not have significant influence on the stress resultants \( N_{xf} \) and \( N_{x\phi_f} \). However, for the stress resultants \( N_{\phi_f} \) the bending-membrane coupling has more influence on the results, and the percentage difference between the results of the cross-plied cases and the corresponding orthotropic faces can reach up to 30% for the second and third lay-ups.
5-3 Folded Plate Structures

In the thin and sandwich folded plate structures considered in the following, it is assumed that the transverse frames which support the folded plate at its two ends \((x = 0, L)\) provide the following supporting conditions:

(a) Complete rigidity in their planes:

\[
v_f(x, y_f) = w(x, y_f) = 0 \quad \text{at } x = 0, L \tag{5-33a}
\]

(b) Complete flexibility normal to their planes:

\[
M_{xf}(x, y_f) = N_{xf}(x, y_f) = 0 \quad \text{at } x = 0, L \tag{5-33b}
\]

It is also assumed that there is no relative rotation between adjacent plates. As in the shell structures, the basic functions which satisfy the above displacement and force boundary conditions (Eqs. 5-33a, b) are the same as for the simply supported plates; that is, Eqs. 5-2, for thin folded plates and Eqs. 5-14 for sandwich folded plate structures. In the finite strip analysis of both thin and sandwich folded plates, the transverse deflection is expressed as fifth order interpolation polynomial while the membrane displacements are expressed as third order polynomials.
5-3-1 Thin Folded Plates

In order to impose the geometric admissibility conditions between adjacent strips in different plates, all the displacement components must be transformed from their local strip coordinate strip system to a reference coordinate system (Fig. 40). After transformation, the overall stiffness matrix can be assembled in the usual manner using the variable correlation scheme. Also, the work equivalent loads associated with the degrees of freedom of the system in the reference coordinate system must be obtained. The stress and the moment resultants for each plate can be obtained after re-transforming the resulting displacement components to the local strip coordinate system and substituting the displacements and their partial derivatives into the force-deformation relations (Eq. 2-3). The folded plate structures are analyzed under distributed loads acting on the plates or by replacing these loads with loads acting on the ridges. The ridge loads are obtained from a one-way slab analysis assuming that each plate is supported along the ridge lines and that the ridges do not deflect. The intensity of the ridge loads is equal to the reaction resulting from the one-way slab analysis. The results obtained from the one-way slab analysis must be added to the results obtained from the finite strip analysis.

Transformation Scheme

Consider a particular plate, \( k \), in the folded plate structure shown in Fig. 40a. Assume that the plate is
modeled into a number of strips and that the local coordinates of a strip in the plate are \( x_k, y_k \) and \( z_k \), and that the reference coordinates of the structure are \( \tilde{x}, \tilde{y} \) and \( \tilde{z} \). The undetermined displacement coefficients in the third and fifth order interpolation polynomials at the two edges of the strip are \( u_e^{(n)}, v_e^{(n)}, w_e^{(n)} \), etc., when measured with respect to the local coordinate system of the plate and \( \tilde{u}_e^{(n)}, \tilde{v}_e^{(n)}, \tilde{w}_e^{(n)} \), etc., when measured with respect to the reference coordinate system of the structure, where \( e \) refers to the edges 1 and 2 and \( n = 1, 2, \ldots, N \). The angle \( \theta_k \) is the angle between the two positive directions of \( y_k \) and \( \tilde{y} \) measured anti-clockwise from the positive direction of the reference coordinate \( \tilde{y} \) (Fig. 40b). The transformation of the displacement coefficients \( u_e^{(n)}, v_e^{(n)} \) and \( w_e^{(n)} \) can be developed by considering the continuity conditions (Fig. 40c). The transformation relations can be written at edge 1 of the strip as follows:

\[
\begin{align*}
\tilde{w}_1^{(n)} &= \tilde{w}_1^{(n)} \cos \theta_k - \tilde{y}_1^{(n)} \sin \theta_k \\
\tilde{v}_1^{(n)} &= \tilde{w}_1^{(n)} \sin \theta_k + \tilde{v}_1^{(n)} \cos \theta_k \\
\tilde{u}_1^{(n)} &= \tilde{u}_1^{(n)}
\end{align*}
\]  

(5-34a)

(5-34b)

(5-34c)

Similarly at edge 2 of the strip:
\[ w_2(n) = \tilde{w}_2 \cos \theta_k - \tilde{v}_2(n) \sin \theta_k \tag{5-35a} \]
\[ v_2(n) = \tilde{w}_2 \sin \theta_k + \tilde{v}_2(n) \cos \theta_k \tag{5-35b} \]
\[ u_2(n) = \tilde{u}_2 \tag{5-35c} \]

In order to impose the continuity of the first partial derivative of the transverse deflection between adjacent strips, the following transformation relations are considered at edges 1 and 2 of the strip:

\[ w(n) \bigg|_{y_1} = \tilde{w}(n) \bigg|_{y_1} \quad \text{and} \quad w(n) \bigg|_{y_2} = \tilde{w}(n) \bigg|_{y_2} \tag{5-36} \]

Similarly, in order to impose continuity of the moment resultants between adjacent strips the following transformation relations are considered:

\[ w(n) \bigg|_{yy_1} = \tilde{w}(n) \bigg|_{yy_1} \quad \text{and} \quad w(n) \bigg|_{yy_2} = \tilde{w}(n) \bigg|_{yy_2} \tag{5-37} \]

The transformation of the displacement coefficients \( u_{y_e}^{(n)} \) and \( v_{y_e}^{(n)} \) \((e = 1, 2)\) can be obtained by using the traditional strain transformation relations (Ref. 34) which results in the following:

\[ u(n) \bigg|_{y_1} = \tilde{u}(n) \bigg|_{y_1} \cos \theta_k \quad \text{and} \quad u(n) \bigg|_{y_2} = \tilde{u}(n) \bigg|_{y_2} \cos \theta_k \tag{5-38} \]
Also,

\[ v_1^{(n)} = \tilde{v}_1^{(n)} \cos^2 \theta_k \quad \text{and} \quad v_2^{(n)} = \tilde{v}_2^{(n)} \cos^2 \theta_k \]  

The transformation relations for other strips in the various plates can be obtained in the same way. Therefore, the transformation matrix for the displacement coefficients associated with cycle number \( n \) of the basic functions, for a particular strip in plate, \( k \), of the folded plate structure can be written as follows (Eq. 5-40):

\[ \ldots \]
<table>
<thead>
<tr>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$w_1$</th>
<th>$w_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x'_1$</td>
<td>$x'_2$</td>
<td>$x''_1$</td>
<td>$x''_2$</td>
</tr>
</tbody>
</table>
\[ [\mathbf{D}] = [T][\tilde{\mathbf{D}}] \]  \hspace{1cm} (5-40b)

Note that the transformation matrix \((\text{Eq. 5-40})\) is valid when the transverse deflection is expressed as a fifth order interpolation polynomial. The transformation matrix when all the displacement components are expressed as third order interpolation polynomials will have the same form as \((\text{Eq. 5-40})\) except that the rows and columns associated with \(w_{YYj} \) \((j = 1, 2)\) are omitted. After transformation, the geometric admissibility conditions can be satisfied through one-to-one matching of the degrees of freedom. For example, consider two adjacent strips I and II; for all values of \(n = 1, 2, \ldots, N:\)

\[ \tilde{u}_2^I = \tilde{u}_1^I \hspace{1cm} \tilde{v}_2 = \tilde{v}_1 \]  \hspace{1cm} (5-41a)

\[ \tilde{w}_2 = \tilde{w}_1 \hspace{1cm} \tilde{w}_2 = \tilde{w}_1 \]  \hspace{1cm} (5-41b)

Also, for continuous moment resultants:

\[ \tilde{w}_{YY2} = \tilde{w}_{YY1} \]  \hspace{1cm} (5-42)

However, when the strips have different thicknesses continuity in the moment cannot be achieved through the last relation since the bending stiffness of the two strips are different.
The numerical results presented herein are for thin concrete folded plate structure having the following properties and dimensions:

\[ E = 3 \times 10^6 \text{ psi} \]

\[ \nu = 0.0 \]

\[ L = 100 \text{ ft.} \]

The other dimensions of the structure are given in Fig. 41. The following are the resulting stiffnesses for the various plates in the structure:

For Plate (1)

\[ A_{11} = A_{22} = 1.8 \times 10^7 \text{ lb/in.}; \quad A_{12} = A_{16} = A_{26} = 0.0 \]

\[ A_{66} = 9.0 \times 10^6 \text{ lb/in.} \]

\[ B_{ij} = 0.0 \quad (i,j = 1,2,6) \]

\[ D_{11} = D_{22} = 5.4 \times 10^7 \text{ lb-in.}; \quad D_{12} = D_{16} = D_{26} = 0.0 \]

\[ D_{66} = 2.7 \times 10^7 \text{ lb-in.} \]
For Plates (2) and (3)

\[ A_{11} = A_{22} = 9 \times 10^7 \text{ lb/in.}; A_{12} = A_{16} = A_{26} = 0.0 \]

\[ A_{66} = 4.5 \times 10^6 \text{ lb/in.} \]

\[ B_{ij} = 0.0 \text{ (i,j = 1,2,6)} \]

\[ D_{11} = D_{22} = 6.75 \times 10^6 \text{ lb-in.}; D_{12} = D_{16} = D_{26} = 0.0 \]

\[ D_{66} = 3.375 \times 10^6 \text{ lb-in.} \]

The structure is analyzed under a system of ridge loads as shown in Fig. 41. Since the structure is symmetrically loaded, one half of the folded plate is considered in the analysis which require that the following conditions be imposed on the displacement coefficients in the interpolation polynomials along the axis of symmetry of the structure:

\[ \tilde{\nu}^{(n)} = \tilde{\omega}^{(n)} = 0 \quad n = 1, 2, \ldots, N \quad (5-43) \]

In the analysis of this system, plates 1, 2 and 3 are modeled into one, three and three strips respectively. The resulting number of degrees of freedom is 66. Note that continuity for the variables \( \tilde{\nu}^{(n)} \) and \( \tilde{\omega}^{(n)} \) is not imposed at the ridges. That is, for two adjacent strips I and II intersecting at a ridge, the variables \( \tilde{\nu}^{(I)}_Y, \tilde{\omega}^{(II)}_Y \) and
\( \bar{\nu}^{(I)} \), \( \bar{\nu}^{(II)} \) are represented by independent degrees of freedom. Also, since plates 1 and 2 have different thicknesses, continuity for \( \bar{w}^{(n)}_{Y} \) is not imposed at the ridge between the first two plates. The number of cycles used in the analysis is seven. The results are presented in Figs. 42 to 45. The results presented for \( \sigma_{x} \) and \( N_{xy} \) are in very close agreement with Ref. 36. However, for \( M_{x} \) and \( N_{y} \) the results are slightly different, but when comparing both with other approaches (Ref. 11) the results are in very close agreement. It is pointed out that this system has also been analyzed by the finite strip method using zero order interpolation polynomials for the membrane displacements \( u \) and \( v \) and a third order interpolation polynomial for the transverse \( \tilde{w} \) deflection (Ref. 27). For this set of displacement functions continuity cannot be achieved for stress and moment resultants between adjacent strips. Also, the zero order interpolation may not produce a sufficiently accurate description of the internal energy when the strain distribution is relatively complicated within a strip (Ref. 7). The results obtained herein using third and fifth order interpolation polynomials for \( u \), \( v \) and \( w \) respectively are closer to the elasticity solution (Ref. 36) than those of Ref. 27.

5-3-2 Sandwich Folded Plates

As in thin folded plate structures, the analysis of sandwich folded plates require the transformation of displacement components of the various plates to a reference
coordinate system. Since thin folded plates represent one face of a sandwich system, the transformation of an individual face will be the same as that for folded plate structures (Eq. 5-40). From the assumptions considered in this work the transverse deflection, \( w_e^{(n)} \) \( (e = 1,2) \), is the same for the two faces and therefore the transformation of the displacement components of the two faces are not independent from one another. Also, it is important to point out that the membrane displacements \( v_{1e}^{(n)} \) and \( v_{2e}^{(n)} \) of the strip (where 1 and 2 refer to the lower and upper faces respectively) are two independent displacement coefficients. Therefore, although \( w_e^{(n)} \) is the same for both faces the displacement coefficients for the upper and lower faces in direction \( \tilde{z} \) of the reference coordinate system will be different. The transformation of a sandwich strip, then, requires additional degrees of freedom to be introduced in the analysis of this system (Fig. 46). Using the transformation equations for thin folded plates, the transformation of the displacement coefficients at the two edges of a sandwich strip in the \( k^{th} \) plate can be written as follows:

At edge (1) of the strip

(a) For the lower face (1)

\[
\begin{align*}
&w_1^{(n)} = \tilde{w}_{11}^{(n)} \cos \theta_k - \tilde{v}_{11}^{(n)} \sin \theta_k \\
&v_{11}^{(n)} = \tilde{w}_{11}^{(n)} \sin \theta_k + \tilde{v}_{11}^{(n)} \cos \theta_k
\end{align*}
\]  

(5-44a) (5-44b)
\[ u_{11}^{(n)} = \tilde{u}_{11}^{(n)} \quad (5-44c) \]
\[ w_{y1}^{(n)} = \tilde{w}_{y11}^{(n)} \quad (5-44d) \]
\[ w_{yy1}^{(n)} = \tilde{w}_{yy11}^{(n)} \quad (5-44e) \]
\[ u_{y11}^{(n)} = \tilde{u}_{y11}^{(n)} \cos \theta_k \quad (5-44f) \]
\[ v_{y11}^{(n)} = \tilde{v}_{y11}^{(n)} \cos^2 \theta_k \quad (5-44g) \]

(b) For the upper face (2)

\[ w_{1}^{(n)} = \tilde{w}_{21}^{(n)} \cos \theta_k - \tilde{v}_{21}^{(n)} \sin \theta_k \quad (5-45a) \]
\[ v_{21}^{(n)} = \tilde{w}_{21}^{(n)} \sin \theta_k + \tilde{v}_{21}^{(n)} \cos \theta_k \quad (5-45b) \]
\[ u_{21}^{(n)} = \tilde{u}_{21}^{(n)} \quad (5-45c) \]
\[ w_{y1}^{(n)} = \tilde{w}_{y21}^{(n)} \quad (5-45d) \]
\[ w_{yy1}^{(n)} = \tilde{w}_{yy21}^{(n)} \quad (5-45e) \]
\[ u_{y21}^{(n)} = \tilde{u}_{y21}^{(n)} \cos \theta_k \quad (5-45f) \]
\[ v_{y21}^{(n)} = \tilde{v}_{y21}^{(n)} \cos^2 \theta_k \quad (5-45g) \]
From the preceding set of transformation equations it can be realized that the number of additional displacement coefficients to be introduced into the system from the transformation of the displacements at edge (1) is one, namely \( \sim w^{(n)}_{11} \) or \( \sim w^{(n)}_{21} \). It is pointed out that from Eqs. (5-44d, 45d) and Eqs. (5-44e, 45e) it can be realized that additional displacement coefficients for the first and second partial derivatives of the deflection are not required.

Since the transformation of the two faces are not independent, Eq. 5-44a and Eq. 5-45a can be replaced by one transformation equation as follows:

\[
\sim w^{(n)}_{1} = \frac{1}{2} \left[ \sim w^{(n)}_{11} + \sim w^{(n)}_{21} \right] \cos \theta_k - \frac{1}{2} \left[ \sim w^{(n)}_{11} + \sim w^{(n)}_{21} \right] \sin \theta_k
\]

(5-46)

It is important to point out that Eq. (5-46) can also be derived through the transformation of the displacement components of the core \((v_c \text{ and } w_c = w_f = w)\) and expressing the resulting transformation equations in terms of the displacement coefficients of the faces using Eq. (2-5b).

Similarly, the transformation of the displacement coefficients at edge 2 of the strip can be obtained. Therefore, the transformation matrix for the displacement coefficients associated with the \(n^{th}\) cycle of the basic functions, for a strip in the \(k^{th}\) plate oriented at an angle \(\theta_k\) with the positive direction of \(\sim y\) can be written as follows:
\[
\begin{bmatrix}
\mathcal{D}^{(n)}_{u_1} \\
\mathcal{D}^{(n)}_{u_2} \\
\mathcal{D}^{(n)}_{v_1} \\
\mathcal{D}^{(n)}_{v_2} \\
\mathcal{D}^{(n)}_{w_1} \\
\mathcal{D}^{(n)}_{w_2}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{T}(u_1, \tilde{u}_1) & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{T}(u_2, \tilde{u}_2) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{T}(v_1, \tilde{v}_1) & 0 & \mathbf{T}(v_1, \tilde{w}_1) & 0 \\
0 & 0 & 0 & \mathbf{T}(v_2, \tilde{v}_2) & 0 & \mathbf{T}(v_2, \tilde{w}_2) \\
0 & 0 & \mathbf{T}(w, \tilde{v}_1) & \mathbf{T}(w, \tilde{v}_2) & \mathbf{T}(w, \tilde{w}_1) & \mathbf{T}(w, \tilde{w}_2)
\end{bmatrix}
\]

(5-47)
where

$$
\begin{bmatrix}
T(u_f, \tilde{u}_f)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \theta_k & 0 \\
0 & 0 & 0 & \cos \theta_k
\end{bmatrix}
$$

$$
\begin{bmatrix}
T(v_f, \tilde{v}_f)
\end{bmatrix} =
\begin{bmatrix}
\cos \theta_k & 0 & 0 & 0 \\
0 & \cos \theta_k & 0 & 0 \\
0 & 0 & \cos^2 \theta_k & 0 \\
0 & 0 & 0 & \cos^2 \theta_k
\end{bmatrix}
$$

$$
\begin{bmatrix}
T(w, \tilde{w}_1)
\end{bmatrix} =
\begin{bmatrix}
\frac{\cos \theta_k}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{\cos \theta_k}{2} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\begin{bmatrix}
T(w, \tilde{w}_2)
\end{bmatrix} =
\begin{bmatrix}
\frac{\cos \theta_k}{2} & 0 \\
0 & \frac{\cos \theta_k}{2} \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
$$
\[
\left[ T(v_1,\tilde{w}_1) \right] = \begin{bmatrix}
sin \theta_k & 0 & 0 & 0 & 0 \\
0 & sin \theta_k & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\left[ T(v_2,\tilde{w}_2) \right] = \begin{bmatrix}
sin \theta_k & 0 \\
0 & sin \theta_k \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

\[
\left[ T(w,\tilde{v}_f) \right] = \begin{bmatrix}
-\sin \theta_k & 0 & 0 & 0 \\
-\frac{\sin \theta_k}{2} & 0 & 0 & 0 \\
0 & -\sin \theta_k & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Also,
\[
\left\{ \tilde{D}^{(n)}_{u_f} \right\}^T = \left\{ \tilde{u}_{f1}^{(n)}, \tilde{u}_{f2}^{(n)}, \tilde{u}_{y1}^{(n)}, \tilde{u}_{y2}^{(n)} \right\}
\]

\[
\left\{ \tilde{D}^{(n)}_{v_f} \right\}^T = \left\{ \tilde{v}_{f1}^{(n)}, \tilde{v}_{f2}^{(n)}, \tilde{v}_{y1}^{(n)}, \tilde{v}_{y2}^{(n)} \right\} \quad (f=1,2)
\]

\[
\left\{ \tilde{D}^{(n)}_{w_1} \right\}^T = \left\{ \tilde{w}_{11}^{(n)}, \tilde{w}_{12}^{(n)}, \tilde{w}_{y1}^{(n)}, \tilde{w}_{y2}^{(n)}, \tilde{w}_{yy1}^{(n)}, \tilde{w}_{yy2}^{(n)} \right\}
\]

\[
\left\{ \tilde{D}^{(n)}_{w_2} \right\}^T = \left\{ \tilde{w}_{21}^{(n)}, \tilde{w}_{22}^{(n)} \right\}
\]
The numerical results presented herein are for a sandwich folded plate roof having two identical aluminum faces and isotropic paper-honeycomb core. The various dimensions of the roof are (Fig. 47):

\[ L = 9.5 \text{ ft.} \]

width of each plate = 2 ft.

\[ t_f = 0.025 \text{ in.} \quad ; \quad t_c = 1.0 \text{ in.} \]

The various elastic properties of the faces and the core are:

(a) Faces

\[ E = 10.25 \times 10^6 \text{ psi} \quad ; \quad \nu = 0.33 \]

The resulting stiffnesses are:

\[ A_{11}^{(f)} = A_{22}^{(f)} = 0.28757 \times 10^6 \text{ lb/in.} \quad ; \quad A_{12}^{(f)} = 0.94897 \times 10^5 \text{ lb/in.} \]

\[ A_{16}^{(f)} = A_{26}^{(f)} = 0.0 \]

\[ B_{ij}^{(f)} = 0.0 \quad (i, j = 1, 2, 6) \]

\[ D_{11}^{(f)} = D_{22}^{(f)} = 14.977 \text{ lb-in.} \quad ; \quad D_{12}^{(f)} = 4.9425 \text{ lb-in.} \]

\[ D_{16}^{(f)} = D_{26}^{(f)} = 0.0 \quad ; \quad D_{66}^{(f)} = 5.0174 \text{ lb-in.} \]

(b) Core

\[ G_{xz_c} = G_{yz_c} = 3 \times 10^3 \text{ psi} \]

The corresponding transverse core shear stiffnesses are:

\[ B_{44} = B_{55} = 3 \times 10^3 \text{ lb/in.} \]
However, in the analysis of this system it is assumed that the transverse shear stiffness $\nu_{55}$ is infinitely large which is the same assumption as that of Refs. 9 and 10. The results are presented for two loading conditions as shown in Fig. 48. For the first loading condition, the ridges 2, 4 and 6 are uniformly loaded with ridge loads of intensity 1600/lb/ft. For the second loading conditions, the ridges 2 to 6 are uniformly loaded with ridge loads of intensity 2260/lb/ft. In the finite strip analysis for the two loading conditions, each plate of the folded roof is modeled into two strips. The resulting number of degrees of freedom is 176. The results are presented after 3 cycles of the basic functions. The resulting stress resultants $N_x$ are presented in Fig. 49 and Fig. 50 for the first and second loading conditions respectively. The resulting displacements $w$ and $v$ at the ridges, in the local coordinate system of each plate are presented in Tables 27 and 28 for the first and second loading conditions respectively. Unfortunately, since the study of sandwich folded plates is relatively recent and only a very limited number of results have been presented in the literature, the results presented herein have not been verified.
CHAPTER VI
SUMMARY AND CONCLUSIONS

6.1 Summary

The finite strip method has been extended to the analysis of sandwich plates, cylindrical shells, and folded plates having anisotropic and transversely heterogeneous faces and orthotropic cores. A double Fourier-series approach has also been applied to anisotropic sandwich plates with various boundary conditions in order that the results can be verified by comparing the finite strip and double Fourier-series solutions. Generally, the formulations were presented in terms of the geometry, the stiffness of the faces (bending, membrane and coupling), and the transverse shear stiffnesses of the orthotropic core. In this way, the analysis is not bound to any one of the several micromechanics theories for laminated structures. Also, the displacement behaviour was described in terms of the membrane displacements of the individual faces and the transverse displacement of the sandwich system. In this way, the transverse shear deformations in the core are taken into account. Also, the analysis of thin anisotropic and transversely heterogeneous laminated systems can be dealt with by considering a single face of the sandwich system. The present work was also concerned with the study of the effect of bending-membrane.
coupling which exists in the laminated faces on the behaviour of anisotropic (specifically cross-plied laminates) sandwich structural systems.

In the finite strip method, the assumed displacement patterns for a particular strip were approximated in terms of the sum of products of one-dimensional third and fifth order interpolation polynomials and undetermined displacement coefficients along the two longitudinal edges of the strip and basic function series which describe the variation of the displacement coefficients along the entire length of the strip and satisfy the force and displacement boundary conditions at the two ends of the strip. Thus, the general two-dimensional problem is converted to a one-dimensional problem with a relatively small number of undetermined displacement coefficients. The resulting strip stiffness matrix is non-singular because of the preset end conditions. The properties of the third and fifth order interpolation polynomials at the two edges of the strip facilitate the satisfaction of the displacement and slope continuity requirements between adjacent strips necessary for geometric admissibility of the displacement state for an assembly of strips, and therefore guarantees conforming solutions. Also, the reference surface strains of the faces and the transverse shear strains of the core can be made continuous between strips so that accurate stress predictions should result for the facings and the core. It is pointed out that
for anisotropic laminated faces continuous stress and moment resultants in the faces between adjacent strips can only be achieved if the transverse deflection is expressed as a fifth order interpolation polynomial. However, for homogeneous orthotropic faces the stress resultants in the faces can be made continuous between adjacent strips by expressing all the displacement components as third order interpolation polynomials. The results indicate that the accuracy of the stress resultants and deflections for the cases where the transverse deflection was expressed as third and fifth order polynomials (e.g. simply supported isotropic sandwich plates), were nearly the same. However, the third order polynomial has an advantage over the fifth order polynomial in terms of the number of degrees of freedom. Thus, the third order polynomial is best suited for sandwich systems with orthotropic homogeneous faces, keeping in mind that the problem of continuous moment resultants in the faces is of a secondary importance since the sandwich system resists the loads mainly by membrane actions in the faces. However, for anisotropic sandwich systems the use of the fifth order polynomial is recommended in order to achieve continuity in the stress and moment resultants in the faces. Also, from the properties of the third and fifth order polynomials a wide variety of alternative imposed displacement boundary conditions can be conveniently handled. These properties also provide the capability of analyzing
sandwich systems having strips connected at arbitrary angles (e.g. folded plates).

The size of the strip stiffness matrix depends upon the properties of the basic functions of the structure. For orthogonal basic functions the size of the strip stiffness matrix is relatively small. Thus, the finite strip method for these types of structures is ideally suitable for programming on small computers. However, for non-orthogonal basic functions the various cycles of the basic functions are coupled in the strip stiffness matrix formulations and the number of degrees of freedom is dependent upon the number of cycles. In some coupled cases (e.g. cross-plied clamped sandwich plates), this results in a relatively large number of degrees of freedom which imposes some limitations on the number of cycles and strips used. Generally, for coupled structural systems it is advantageous to consider all additional conditions which could result in a reduction in the number of degrees of freedom of the system. Obviously, for the structural systems where orthogonal and non-orthogonal basic functions can be used, the orthogonal set is the one to be used in order to save computer time and storage.

In general, the finite strip method can be considered as one of the powerful approaches for the analysis of thin and sandwich structural systems. This can be appreciated by considering the various anisotropic structural systems.
considered in this work. In the majority of the cases considered, the finite strip solutions were in very close agreement with the solutions using other approaches. The finite strip method proved to be a useful tool for the analysis of thin laminated plates. However, for some laminated plates (e.g. clamped angle and cross-plied) it was not possible to choose basic functions which fully represent the structural system; in such cases the basic functions impose additional membrane force boundary conditions. For such cases, the accuracy of the results depends upon the type of laminates; for example, for the clamped cross-plied cases the accuracy of the results was not affected while for the clamped angle-plied cases the accuracy of the results was greatly affected. However, the problem of membrane force boundary conditions in such a case can be circumvented by the use of the reduced bending stiffnesses in the finite strip formulations for which the analysis is independent of the membrane boundary conditions. Generally, the use of the reduced bending stiffness approximation in the finite strip analysis of thin laminated plates is recommended in order to reduce the number of degrees of freedom of the system while the accuracy of the displacement results are maintained. However, the use of the reduced bending stiffnesses in the finite strip analysis is limited to the type of laminates where the membrane boundary conditions do not significantly influence the plate response. It was found that for angle-ply laminates
these orientations are $-15^\circ < \theta < 15^\circ$ for simply supported boundaries and $-25^\circ < \theta < 25^\circ$ for clamped boundaries. It is pointed out that the finite strip method can be used for other types of laminates and boundary conditions than the cases reported in this work. For sandwich plates, the finite strip solution in the majority of cases were in very close agreement with the double Fourier-series solution as well as with other reference solutions. In the cases where the percentage difference between the finite strip and the double Fourier-series solutions were relatively high the differences is attributed to the limitations on either the number of cycles and strips in the finite strip method or on the number of cycles of the double Fourier-series approach. Generally, the finite strip method is ideally suited for the analysis of anisotropic thin and sandwich roof structures, such as cylindrical shells and folded plates where closed form solutions are generally not possible. It is also pointed out that the solutions for these systems reported in this work can be obtained using only a few degrees of freedom.

In the finite strip method presented, the results indicate that the accuracy of the solution depends on the number of cycles more than the number of strips. Also, the results do indicate that relatively few strips and cycles are required in order to accurately predict displacements and stresses for these structural systems. It is pointed out that the numerical examples reported do not provide a
complete evaluation of the potential of the finite strip method presented. In other words, the present finite strip formulations can deal with other types of structural systems than the structures presented in this work, such as anisotropic closed sandwich cylinders. Also, the method of analysis is flexible in dealing with the structural systems considered having other boundary conditions and type of laminates than those reported.

The finite strip capability reported herein has some limitations. The formulation presented cannot be used for the analysis of structural systems with openings nor for structural systems having irregular boundaries. As previously mentioned, the reduction in the undetermined displacement coefficients in the finite strip method is due to the use of basic functions which describe the variation of the displacements in one direction of the system. On the other hand, one of the main limitations on the finite strip analysis for a certain class of problems is due to the incorporation of basic functions. As previously mentioned, in some cases choosing basic functions which fully represent the structure was not possible. Also, in the other cases such as angle-plyed laminates with the hinged free normal (HFN) type of simple supports, the choice of basic functions which satisfy the force boundary conditions was impossible. As previously mentioned, these problems have been circumvented in thin laminated plates by the use of the reduced bending stiffness approximations. However,
these approximations cannot be used in the finite strip analysis of sandwich systems since the load is resisted mainly by membrane actions in the faces. Therefore, it would be worthwhile developing an analysis scheme (similar to the reduced bending stiffness approximations) for sandwich systems in which the bending-membrane coupling is eliminated in the faces. This would overcome the problem of unsatisfied membrane force boundary conditions due to the existence of coupling in the faces. As mentioned in Chapter V, for sandwich cross-plied plates where the faces are very thin (which is the common practice in sandwich construction) the coupling effect on the behaviour of the plate is minimum and can be neglected. However, this might not be the case for other types of laminates such as the angle-plied. However, if a future study proves that the coupling effect is generally small for very thin laminated faces having various lamina orientations, the development of such a scheme previously mentioned would no longer be necessary. Since the accuracy of the finite strip solution is dependent upon the number of cycles and strips used, the limitations on the number of cycles and strips for practical applications of some non-orthogonal cases (e.g. cross-plied clamped sandwich plates), represent a significant limitation on the finite strip analysis of such systems. The problem of the large number of degrees of freedom usually generated in non-orthogonal cases (which require the solution of large numbers of simultaneous equations) may
be overcome by replacing the Gaussian-elimination method considered in this work by a direct search approach which seeks a minimum of the total potential energy using one of the function minimization algorithm such as that presented by Fletcher and Reeves (Ref. 40).

The double Fourier-series approach proved to be a powerful means of analyzing cross-plied simply supported sandwich plates since the various cycles of the double series are independent and the solution converges after a few number of cycles. Also, the analysis requires a simple computer program with little input data. Thus, the computer time and storage required to obtain a numerical solution is very small. The simply supported solution represents the first step in the analysis of clamped boundary conditions, in which constraints are imposed on the displacement functions such that the prescribed clamped boundary conditions are satisfied. Thus, the double Fourier series approach is capable of dealing with plates having various clamped boundary conditions. However, for clamped boundary conditions the cycles are coupled and the solution requires a relatively larger number of cycles than the simply supported case for convergence. Therefore, for practical applications a limit on the number of cycles were imposed. In the majority of cases, the double Fourier-series solutions were in close agreement with the finite strip solution and with other reference solutions. For the cases where the percentage
difference between the double Fourier-series solution and other approaches was significant, the difference is attributed to the limitation on the number of cycles used.

6.2 Conclusions

In addition to assessing the solution methods presented herein, this work also included a thorough study of the effect of bending-membrane coupling on the behaviour of cross-plied sandwich plates. The effect of coupling on the deflection response and stress resultants is governed by factors related to the elastic properties of the faces and core, the aspect ratio (in case of plates), the core and face thicknesses, number of plies in each face, and the lay-ups of the orthotropic plies in the faces.

With regard to the above general statement, the following specific conclusions can be drawn:

1 - The finite strip method is a powerful approach for predicting the structural response of sandwich systems; it is most ideally suited for systems where orthogonal basic functions can be chosen.

2 - The double Fourier Series method presented has limited applicability as compared to the finite strip method; it is most ideally suited for simply supported boundary conditions.
3 - In the finite strip method, the use of third order polynomials for the transverse deflection is sufficient for sandwich systems with isotropic or orthotropic faces; however, fifth order polynomials are recommended for unbalanced laminated faces.

4 - The coupling effect can either increase or decrease the deflection response of cross-plied sandwich plates compared to the corresponding sandwich plates with orthotropic faces where the coupling is neglected.

5 - If a strong core is combined with faces constructed from laminas having either a high or low degree of anisotropy, the deflection response of the resulting cross-plied plate can reverse (in terms of 4 above) according to the lay-ups of the plies in the two faces.

6 - If a very weak core is combined with faces having a relatively high degree of anisotropy, the deflection of the cross-plied plate will generally be larger than the corresponding plate with orthotropic faces, for the various lay-ups.

7 - The larger the degree of anisotropy of each orthotropic layer that make up the faces, the larger the coupling effect.
8 - The weaker the core the larger the deflection of the cross-plied sandwich plate compared to that of the corresponding sandwich plate with orthotropic faces.

9 - The coupling effect decreases as the number of plies increases for a face of constant thicknesses.

10 - By increasing the aspect ratio of the cross-plied sandwich plate, the deflection tends to increase compared to that of the corresponding sandwich plate with orthotropic faces.

11 - The coupling effect is minimum for square plates.

12 - The relative thickness of the faces to that of the core represents the major influence on the coupling effect.

13 - For thin faces \( t_c/t_f > 50 \), the coupling effect tends to vanish and the behaviour of the cross-plied sandwich plate approaches that of the corresponding sandwich plate with orthotropic faces. Also, the effect of the stiffnesses of the faces and core, aspect ratio, and lay-ups of the plies on the coupling effect diminishes.

14 - For practical sandwich constructions where the faces are usually thin \( t_c/t_f > 50 \), the bending-membrane coupling in the laminated faces can be neglected without affecting the
resulting deflection and stresses of cross-ply sandwich systems.

15 - For relatively thick faces \((t_c/t_f < 20)\), the coupling effect is active and can significantly influence both the stress resultants and the deflection response.

16 - Even for thick faces, the coupling effect will not be as severe as in thin laminated plates.

17 - For sandwich systems having relatively thick faces, the magnitude of the coupling effect on the stress resultants is dependent upon the lay-ups of the plies in the two faces. Generally, the coupling effect on the stress resultants and deflection response is minimum for the first lay-up (Fig. 12) where the plies are placed unsymmetrically with respect to the middle surface of the core.

18 - Unlike thin laminated plates, where the coupling is essentially independent of the boundary conditions, it was found that the effect of coupling increases by replacing simple supports with clamped boundaries.

19 - For cylindrical shell roofs, even for very thick faces \((t_c/t_f = 4)\), the effect of coupling on the stress resultants is relatively small.
6.3 Future Research

The success of the finite strip and double Fourier-series approaches presented in this work suggests several natural extensions. A useful extension of the analysis capabilities is to incorporate the geometric non-linearities in the formulations which is of a great importance for the analysis of laminated systems. Although the finite strip method was only applied to anisotropic structural systems which can be adequately modeled using rectangular strips, the method can be extended to parallelogramic strips using skew coordinates for the analysis of anisotropic skew sandwich plates, the solution of which can be verified using the double Fourier-series approach. Also, the incorporation of the finite strip and double Fourier-series approaches within a structural synthesis capability where minimum structural weight represents the basic goal of the synthesis scheme would represent a noteworthy development in the field of light-weight and sandwich structural designs. It is pointed out that the structural synthesis concept requires an analysis approach which can adequately predict the behaviour of the structure at low computer execution time and storage. Thus, the finite strip analysis of structures having orthogonal basic functions and the double Fourier-series analysis of simply supported cross-plied
sandwich plates are ideally suited to the structural synthesis capability. It is noted that the results of the finite strip analysis of sandwich folded plates has not yet been verified. Thus, an experimental study in order to verify the solutions for the cases of isotropic and cross-plied faces would be valuable. In order that the local buckling of the faces can be studied, the analysis approaches should be extended to take into account the possibility of imposing different bending boundary conditions on each face. Therefore, the assumption that the transverse deflection is a constant through the thickness of the sandwich system can be circumvented by assuming a linear strain variation through the core and expressing the transverse deflection of the core in terms of the transverse deflection of each face. Such a formulation would double the number of degrees of freedom associated with the transverse deflection. It is pointed out that extension of the present analysis capabilities to include the effects of the in plane stiffness of the core is straightforward. This would make it possible to deal with a more general class of core materials but would not increase the number of degrees of freedom of the system.
(a) CROSS-SECTION OF LAMINATED FACE

(b) ORIENTATION OF AXES

FIGURE 3. LAMINATED FACE
FIGURE 4. FACE STRESS RESULTANTS
FIGURE 5. CROSS-PLY AND ANGLE-PLY UNBALANCED LAMINATED CONSTRUCTION
FIGURE 6. TYPICAL CONFIGURATION OF HONEYCOMB CORE

FIGURE 7. CORE FORCE RESULTANTS
FIGURE 9. ANISOTROPIC SANDWICH FINITE STRIPS
FIGURE 10. TWO ADJACENT SANDWICH STRIPS
(a) FOLDED PLATE STRUCTURES

(b) CYLINDRICAL SHELL ROOF WITH EDGE BEAMS

FIGURE 11. COORDINATE TRANSFORMATIONS
FIGURE 12. LAY-UPS OF PLIES IN THE FACES
$G_{yz_C} = 35 \times 10^3 \text{ PSI}$, $G_{xz_C} = 17 \times 10^3 \text{ PSI}$.

$\frac{t_c}{t_f} = 4$  $\frac{t_c}{t_f} = 10$  $\frac{t_c}{t_f} = 4, 10$ (Orthotropic)

Graph showing $L/B$ vs. $L$ for GR-OC simply supported sandwich plates.

Figure 13: $\frac{t_c}{t_f}$ vs. $L/B$ for GR-OC simply supported sandwich plates ($t_c/t_f = 4, 10$).
$G_{xz} = 35 \times 10^3$ P.S.I., $G_{yz} = 17 \times 10^3$ P.S.I.

$\frac{t_c}{t_f} = 20$

$\frac{t_c}{t_f} = 50$

$B$

$L = 50''$

Figure 14. $\psi / \psi_0$ vs. $L/B$ for GR-OC simply supported sandwich plates ($t_{oo}/t_{ee} = 20, 50$).
FIGURE 15(a). $w_c/w_o$ VS. $t_c/t_f$ FOR GR-OC SIMPLY SUPPORTED SANDWICH PLATES ($L/B = 2.5$)
$G_{yzc} = 35 \times 10^3$ P.S.I.; $G_{xz c} = 17 \times 10^3$ P.S.I.

(b) $L/B = 5.0$

**FIGURE 15(b).** $w_c/w_o$ vs. $t_c/t_f$ for GR-OC Simply Supported Sandwich Plates ($L/B = 5$).
\[ G_{yz_c} = 17 \times 10^3 \text{ PSI}, \quad G_{xz_c} = 35 \times 10^3 \text{ PSI} \]

**Figure 16.** \( w_c/w_0 \) vs. L/B for GR-OC simply supported sandwich plates \((t_c/t_f = 4, 10)\)
FIGURE 17. $w_c/w_0$ VS $L/B$ FOR GR-IC SIMPLY SUPPORTED SANDWICH PLATES ($t_c/t_\ell = 4, 10$)
FIGURE 18. $w_c/w_0$ VS. $L/B$ FOR GR-IC SIMPLY SUPPORTED SANDWICH PLATES ($t_c/t_f = 20, 50$)
Figure 19: \( \frac{M}{M_0} \text{ vs. } L/B \text{ for } GI-OC \) simply supported sandwich plates \((c/t = 4, 10)\).
FIGURE 20. $w_c/w_0$ vs. $L/B$ for GL-IC simply-supported sandwich plates ($t_c/t_f = 4, 10$)
$G_{yz_c} = 35 \times 10^3$ P.S.I., $G_{xz_c} = 17 \times 10^3$ P.S.I.

FINITE STRIP

DOUBLE FOURIER SERIES

\( t_c/t_f = 4 \)

ORTHOTROPIC

FIGURE 21. $w_c/w_0$ VS. L/B FOR GR-OC SANDWICH PLATES WITH TWO SIDES SIMPLY SUPPORTED AND TWO SIDES CLAMPED ($t_c/t_f = 4$)
FIGURE 22. EFFECT OF BOUNDARY CONDITIONS ON COUPLING IN SANDWICH PLATES WITH GR CROSS-PLIED FACES
FIGURE 23. CYLINDRICAL SHELLS LOADING SYSTEM
(DEAD LOAD)
FIGURE 25. SIMPLY SUPPORTED THIN ISOTROPIC CYLINDRICAL SHELL ROOF
FIGURE 26(a). DISPLACEMENT TRANSFORMATION FOR THIN CYLINDRICAL SHELL ROOF WITH EDGE BEAMS
FIGURE 26(b). DISPLACEMENT TRANSFORMATION FOR THIN CYLINDRICAL SHELL ROOF WITH EDGE BEAMS
FIGURE 27. SIMPLY SUPPORTED THIN ISOTROPIC CYLINDRICAL SHELL ROOF WITH EDGE BEAMS.
FIGURE 29. $N_{\phi}$ $(x=L/2)$ FOR SIMPLY SUPPORTED THIN ISOTROPIC CYLINDRICAL SHELL WITH EDGE BEAMS
FIGURE 30. \( N_{x\phi} \) for simply supported thin isotropic cylindrical shell with edge beams.
FIGURE 32. SIMPLY SUPPORTED SANDWICH CYLINDRICAL SHELL ROOF WITH ISOTROPIC FACES
Figure 34. $N^{\phi_f}(x = L/2)$ for simply supported sandwich cylindrical shell roof with isotropic faces.
FIGURE 35. $N_{x\phi_f}$ ($x = 0, L$) FOR SIMPLY SUPPORTED SANDWICH CYLINDRICAL SHELL ROOF WITH ISOTROPIC FACES.
FIGURE 36. SIMPLY SUPPORTED SANDWICH CYLINDRICAL SHELL ROOF WITH CROSS-PLIED FACES
FIGURE 37. $N_{xf}$ ($x=L/2$) FOR SIMPLY SUPPORTED SANDWICH CYLINDRICAL SHELL ROOF WITH CROSS-PLIED FACES
FIGURE 38. $N_{f}$ (x=L/2) FOR SIMPLY SUPPORTED SANDWICH CYLINDRICAL SHELL ROOF WITH CROSS-PLIED FACES
FIGURE 39. $N_{x\phi_f}$ (x = 0, L) FOR SIMPLY SUPPORTED SANDWICH CYLINDRICAL SHELL ROOF WITH CROSS-PILLED FACES
FIGURE 40(a) (b). DISPLACEMENT TRANSFORMATION FOR THIN FOLDED PLATE STRUCTURES
Figure 40(c). Displacement Transformation for Thin Folded Plate Structures.
FIGURE 41. SIMPLY SUPPORTED THIN FOLDED PLATE ROOF
FIGURE 42. $\sigma_x (x=L/2)$ FOR SIMPLY SUPPORTED THIN FOLDED PLATE ROOF
FIGURE 43. $N_y (x=L/2)$ FOR SIMPLY SUPPORTED THIN FOLDED PLATE ROOF
FIGURE 44. $N_{xy}$ ($x=0,L$) FOR SIMPLY SUPPORTED THIN FOLDED PLATE ROOF
FIGURE 45. $M_y (x=L/2)$ FOR SIMPLY SUPPORTED THIN FOLDED PLATE ROOF
FIGURE 46. DISPLACEMENT TRANSFORMATION FOR SANDWICH FOLDED PLATES.
FIGURE 47. SIMPLY SUPPORTED SANDWICH FOLDED PLATE STRUCTURE
FIGURE 48. LOADING CONDITIONS FOR SANDWICH FOLDED PLATES
TABLE 1. CENTER DEFLECTION OF SIMPLY SUPPORTED THIN CROSS-PLIED PLATES

<table>
<thead>
<tr>
<th>L x B (in. x in.)</th>
<th>Solution Method</th>
<th>Number of Strips</th>
<th>Number of Cycles</th>
<th>( w_{\text{center}} ) at ( x = \frac{L}{2}, y = \frac{B}{2} ) (in.)</th>
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<tbody>
<tr>
<td>20 x 20</td>
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<td>6</td>
<td>3</td>
<td>1.807, 1.800, 1.808, 1.740</td>
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<td>7</td>
<td>3.925, 3.898, 3.926, 3.899</td>
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<td>Finite Strip Ref. 19</td>
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<td>7</td>
<td>3.865, 3.769, 3.866, 3.808</td>
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### TABLE 2. STIFFNESSES FOR THIN ANGLE-PLIED LAMINATES

![Diagram](image)

#### Membrane Stiffnesses \( \times 10^{-3} \) lb/in

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( A_{11} )</th>
<th>( A_{12} )</th>
<th>( A_{16} )</th>
<th>( A_{22} )</th>
<th>( A_{26} )</th>
<th>( A_{66} )</th>
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<tr>
<td>45</td>
<td>1089.1</td>
<td>989.12</td>
<td>0.0</td>
<td>A11</td>
<td>0.0</td>
<td>1014.1</td>
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<td>30</td>
<td>2306.7</td>
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<td>15</td>
<td>3503.6</td>
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<td>0.0</td>
<td>120.79</td>
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<td>291.02</td>
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#### Coupling Stiffnesses \( \times 10^{-3} \) lb

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( B_{11} )</th>
<th>( B_{12} )</th>
<th>( B_{16} )</th>
<th>( B_{22} )</th>
<th>( B_{26} )</th>
<th>( B_{66} )</th>
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<tbody>
<tr>
<td>45</td>
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<td>0.0</td>
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<td>31.579</td>
<td>0.0</td>
<td>10.706</td>
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<td>0.0</td>
<td>22.643</td>
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<td>1.770</td>
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#### Bending Stiffnesses \( \times 10^{-3} \) lb-in

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( D_{11} )</th>
<th>( D_{12} )</th>
<th>( D_{16} )</th>
<th>( D_{22} )</th>
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#### Reduced Bending Stiffnesses \( \times 10^{-3} \) lb-in

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<th>( \theta )</th>
<th>( D_{11}^* )</th>
<th>( D_{12}^* )</th>
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<th>( D_{22}^* )</th>
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<td>0.0</td>
<td>0.1464</td>
<td>0.0</td>
<td>0.2100</td>
</tr>
<tr>
<td>15</td>
<td>1.1579</td>
<td>0.0840</td>
<td>0.0</td>
<td>0.0898</td>
<td>0.0</td>
<td>0.0962</td>
</tr>
<tr>
<td>Number of Strips Cycles</td>
<td>Solution Method</td>
<td>Ref.</td>
<td>RBS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------------------------</td>
<td>----------------</td>
<td>------</td>
<td>------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Finite Strip</td>
<td>19</td>
<td>1.170 1.172</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Ref. 19</td>
<td></td>
<td>1.170 1.172</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Finite Strip</td>
<td>19</td>
<td>1.241 1.292</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Ref. 19</td>
<td></td>
<td>1.241 1.292</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Finite Strip</td>
<td>19</td>
<td>1.142 1.412</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Ref. 19</td>
<td></td>
<td>1.142 1.412</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE 4. CENTER DEFLECTION OF THIN CROSS-PLIED PLATES WITH CLAMPED SIDES.

![Diagram of plate with labeled dimensions x, y, and z axes.]

<table>
<thead>
<tr>
<th>L x B (in.)</th>
<th>Solution Method</th>
<th>Number of Strips</th>
<th>Number of Cycles</th>
<th>( W_{\text{center}} ) x = ( \frac{L}{2} ), y = ( \frac{B}{2} ) (in.) x 10^{-2}</th>
<th>RBS x = ( \frac{L}{2} ), y = ( \frac{B}{2} ) (in.) x 10^{-2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 x 12</td>
<td>Finite Strip Ref. 20</td>
<td>6</td>
<td>3</td>
<td>0.585</td>
<td>0.585</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.581</td>
<td>0.566</td>
</tr>
<tr>
<td>24 x 12</td>
<td>Finite Strip Ref. 20</td>
<td>6</td>
<td>3</td>
<td>0.985</td>
<td>0.985</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.993</td>
<td>0.992</td>
</tr>
</tbody>
</table>
TABLE 5. CENTER DEFLECTION OF THIN ANGLE-PLIED PLATES WITH CLAMPED SIDES.

| ±θ | Solution Method | Number of Strips | Number of Cycles | \( W_{\text{center}} \) \\
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>Finite Strip Ref. 20</td>
<td>6</td>
<td>3</td>
<td>( x = \frac{L}{2} ), ( y = \frac{B}{2} ) (in.) ( \times 10^{-2} )</td>
</tr>
<tr>
<td>35</td>
<td>Finite Strip Ref. 20</td>
<td>6</td>
<td>3</td>
<td>0.584 ( \times 10^{-2} )</td>
</tr>
</tbody>
</table>

\( B = 20" \), \( L = 20" \)
<table>
<thead>
<tr>
<th>Solution Method</th>
<th>Ref. 3</th>
<th>Finite Elements Ref. 5</th>
<th>Double Fourier Series</th>
<th>Experimental Ref. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>ω&lt;sub&gt;max&lt;/sub&gt; (rad/s)</td>
<td>2 Strips</td>
<td>3 Strips</td>
<td>4 Strips</td>
<td>5 Strips</td>
</tr>
<tr>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
</tr>
<tr>
<td>6.3</td>
<td>6.3</td>
<td>6.3</td>
<td>6.3</td>
<td>6.1</td>
</tr>
<tr>
<td>Ny&lt;sub&gt;x, max&lt;/sub&gt; (lb/in)</td>
<td>6.307</td>
<td>6.307</td>
<td>6.307</td>
<td>6.297</td>
</tr>
<tr>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
</tr>
<tr>
<td>18.469</td>
<td>18.469</td>
<td>18.469</td>
<td>18.469</td>
<td>18.469</td>
</tr>
<tr>
<td>Qy&lt;sub&gt;z, max&lt;/sub&gt; (lb/in)</td>
<td>6.304</td>
<td>6.304</td>
<td>6.304</td>
<td>6.304</td>
</tr>
<tr>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>Qx&lt;sub&gt;z, max&lt;/sub&gt; (lb/in)</td>
<td>6.298</td>
<td>6.298</td>
<td>6.298</td>
<td>6.298</td>
</tr>
<tr>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
</tr>
<tr>
<td>0.613</td>
<td>0.613</td>
<td>0.613</td>
<td>0.613</td>
<td>0.613</td>
</tr>
<tr>
<td>N&lt;sub&gt;y, max&lt;/sub&gt; (lb/in)</td>
<td>6.263</td>
<td>6.263</td>
<td>6.263</td>
<td>6.263</td>
</tr>
<tr>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
<td>x = L/2, y = B/2</td>
</tr>
</tbody>
</table>

Results for simply supported sandwich plates with isotropic faces.
<table>
<thead>
<tr>
<th>Solution Method</th>
<th>Finite Strip</th>
<th>4 Strips</th>
<th>Polynomial for w</th>
<th>Polynomial for $\frac{1}{2}$th Order</th>
<th>Polynomial for w</th>
<th>Polynomial for $\frac{1}{2}$th Order</th>
</tr>
</thead>
<tbody>
<tr>
<td># of cycles</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$w_{max} \times 10^{-3}$</td>
<td>$x = L/2$ $y = B/2$</td>
<td>$x = L/2$ $y = B/2$</td>
<td>$x = L/2$ $y = B/2$</td>
<td>$x = L/2$ $y = B/2$</td>
<td>$x = L/2$ $y = B/2$</td>
<td>$x = L/2$ $y = B/2$</td>
</tr>
</tbody>
</table>

TABLE 7. EFFECT OF CYCLES ON THE FINITE STRIP SOLUTION OF SIMPLY SUPPORTED SANDWICH PLATES WITH ISOTROPIC FACES.
TABLE 8. NUMBER OF DEGREES OF FREEDOM FOR SIMPLY SUPPORTED SANDWICH PLATES.

<table>
<thead>
<tr>
<th>Solution Method</th>
<th>Number of Strips</th>
<th>Number of Degrees of Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Isotropic</td>
</tr>
<tr>
<td>Finite Strip</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>Third Order Polynomials</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>26</td>
</tr>
<tr>
<td>Finite Strip</td>
<td>6</td>
<td>24</td>
</tr>
<tr>
<td>Fifth Order Polynomials</td>
<td>8</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>45</td>
</tr>
</tbody>
</table>
TABLE 9. EFFECT OF CYCLES ON THE DOUBLE-FOURIER SERIES SOLUTION OF SIMPLY SUPPORTED SANDWICH PLATES WITH ISOTROPIC FACES.

<table>
<thead>
<tr>
<th>Number of Cycles m, n</th>
<th>$w_{\text{max}}$ (in) $\times 10^{-3}$</th>
<th>$u_{f\text{max}}$ (in) $\times 10^{-3}$</th>
<th>$v_{f\text{max}}$ (in) $\times 10^{-3}$</th>
<th>$N_{x_{f\text{max}}}$ (lb/in)</th>
<th>$N_{y_{f\text{max}}}$ (lb/in)</th>
<th>$N_{xy_{f\text{max}}}$ (lb/in)</th>
<th>$Q_{xz_{f\text{max}}}$ (lb/in)</th>
<th>$Q_{yz_{f\text{max}}}$ (lb/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.505</td>
<td>$\pm$ 0.446</td>
<td>$\pm$ 0.487</td>
<td>$\pm$ 20.42</td>
<td>$\pm$ 21.43</td>
<td>$\pm$ 11.28</td>
<td>$\pm$ 4.98</td>
<td>$\pm$ 5.137</td>
</tr>
<tr>
<td>3</td>
<td>6.275</td>
<td>$\pm$ 0.458</td>
<td>$\pm$ 0.495</td>
<td>$\pm$ 17.95</td>
<td>$\pm$ 18.83</td>
<td>$\pm$ 12.33</td>
<td>$\pm$ 5.511</td>
<td>$\pm$ 5.574</td>
</tr>
<tr>
<td>5</td>
<td>6.304</td>
<td>$\pm$ 0.461</td>
<td>$\pm$ 0.499</td>
<td>$\pm$ 18.47</td>
<td>$\pm$ 19.34</td>
<td>$\pm$ 12.56</td>
<td>$\pm$ 5.943</td>
<td>$\pm$ 6.084</td>
</tr>
<tr>
<td>7</td>
<td>6.295</td>
<td>$\pm$ 0.461</td>
<td>$\pm$ 0.499</td>
<td>$\pm$ 18.27</td>
<td>$\pm$ 19.15</td>
<td>$\pm$ 12.65</td>
<td>$\pm$ 6.051</td>
<td>$\pm$ 6.135</td>
</tr>
<tr>
<td>9</td>
<td>6.299</td>
<td>$\pm$ 0.461</td>
<td>$\pm$ 0.500</td>
<td>$\pm$ 18.36</td>
<td>$\pm$ 19.24</td>
<td>$\pm$ 12.70</td>
<td>$\pm$ 6.180</td>
<td>$\pm$ 6.306</td>
</tr>
<tr>
<td>11</td>
<td>6.297</td>
<td>$\pm$ 0.462</td>
<td>$\pm$ 0.500</td>
<td>$\pm$ 18.31</td>
<td>$\pm$ 19.19</td>
<td>$\pm$ 12.72</td>
<td>$\pm$ 6.226</td>
<td>$\pm$ 6.320</td>
</tr>
</tbody>
</table>
Table 10. Stiffnesses for Graphite-Epoxy Cross-plied Composites.

<table>
<thead>
<tr>
<th>Lay-Up (1)</th>
<th>Lay-Up (2)</th>
<th>Lay-Up (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>0</td>
<td>90</td>
</tr>
<tr>
<td>0</td>
<td>90</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$t_f$ (in)</th>
<th>Membrane Stiffnesses $\times 10^{-3}$ lb/in</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Lambda_{11}^{(f)}$</td>
</tr>
<tr>
<td>0.10</td>
<td>1539.9</td>
</tr>
<tr>
<td>0.25</td>
<td>3849.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$t_f$ (in)</th>
<th>Coupling Stiffnesses $\times 10^{-3}$ lb</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$B_{11}^{(f)}$</td>
</tr>
<tr>
<td></td>
<td>-36.620</td>
</tr>
<tr>
<td>0.25</td>
<td>$B_{11}^{(f)}$</td>
</tr>
<tr>
<td></td>
<td>-228.88</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$t_f$ (in)</th>
<th>Bending Stiffnesses $\times 10^{-3}$ lb-in</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>$D_{11}^{(f)}$</td>
</tr>
<tr>
<td></td>
<td>1.2833</td>
</tr>
<tr>
<td>0.25</td>
<td>20.052</td>
</tr>
<tr>
<td>$\frac{t_C}{t_f}$</td>
<td>Lay-Ups</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td>Orthotropic</td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Orthotropic</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td>t_c/t_f</td>
<td>Lay-Ups</td>
</tr>
<tr>
<td>--------</td>
<td>---------</td>
</tr>
<tr>
<td>0.04</td>
<td>Orthotropic</td>
</tr>
<tr>
<td>0.04</td>
<td>Upper Face</td>
</tr>
<tr>
<td>0.04</td>
<td>Lower Face</td>
</tr>
<tr>
<td>0.40</td>
<td>2</td>
</tr>
<tr>
<td>0.40</td>
<td>3</td>
</tr>
<tr>
<td>0.80</td>
<td>Orthotropic</td>
</tr>
<tr>
<td>0.80</td>
<td>Upper Face</td>
</tr>
<tr>
<td>0.80</td>
<td>Lower Face</td>
</tr>
<tr>
<td>1.00</td>
<td>2</td>
</tr>
<tr>
<td>1.00</td>
<td>3</td>
</tr>
<tr>
<td>Lay-Ups</td>
<td>$w_{\text{center}}$ (in)$\times 10^{-6}$</td>
</tr>
<tr>
<td>---------</td>
<td>---------------------------------</td>
</tr>
<tr>
<td>Orthotropic</td>
<td>45.67 ± 1.100</td>
</tr>
<tr>
<td>1 Upper Face</td>
<td>53.69</td>
</tr>
<tr>
<td>1 Lower Face</td>
<td>50.69</td>
</tr>
<tr>
<td>2</td>
<td>64.67</td>
</tr>
<tr>
<td>3</td>
<td>45.19</td>
</tr>
<tr>
<td>Orthotropic</td>
<td>78.41</td>
</tr>
<tr>
<td>1 Upper Face</td>
<td>79.59</td>
</tr>
<tr>
<td>1 Lower Face</td>
<td>50.59</td>
</tr>
<tr>
<td>2</td>
<td>87.09</td>
</tr>
<tr>
<td>3</td>
<td>73.00</td>
</tr>
</tbody>
</table>
TABLE 14. RESULTS FOR CLAMPED SANDWICH PLATES WITH ISOTROPIC FACES:

<table>
<thead>
<tr>
<th>Solution Method</th>
<th>$w_{\text{max}}$ $(\text{in}) \times 10^{-3}$</th>
<th>$N_{x_f}$ (lb/in)</th>
<th>$N_{y_f}$ (lb/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref. 28</td>
<td>0.424</td>
<td>± 3.033</td>
<td>± 3.043</td>
</tr>
<tr>
<td>Finite Strip Third Order Polynomials</td>
<td>2 Strips</td>
<td>0.416</td>
<td>± 3.215</td>
</tr>
<tr>
<td></td>
<td>3 Strips</td>
<td>0.421</td>
<td>± 2.936</td>
</tr>
<tr>
<td></td>
<td>4 Strips</td>
<td>0.424</td>
<td>± 2.901</td>
</tr>
<tr>
<td>Double Fourier Series</td>
<td></td>
<td>0.405</td>
<td>± 3.128</td>
</tr>
</tbody>
</table>
TABLE 15. EFFECT OF CYCLES ON THE FINITE STRIP SOLUTION OF CLAMPED SANDWICH PLATES WITH ISOTROPIC FACES.

<table>
<thead>
<tr>
<th>Solution Method</th>
<th>Number of Cycles</th>
<th>$w_{\text{max}} \times 10^{-3}$</th>
<th>$N_{x_f}$ (lb/in)</th>
<th>$N_{y_f}$ (lb/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite Strip Third Order Polynomials</td>
<td>1</td>
<td>0.434</td>
<td>± 3.207</td>
<td>± 3.199</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.439</td>
<td>± 3.248</td>
<td>± 3.228</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.420</td>
<td>± 2.909</td>
<td>± 2.923</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.424</td>
<td>± 2.901</td>
<td>± 2.960</td>
</tr>
</tbody>
</table>
TABLE 16. NUMBER OF DEGREES OF FREEDOM FOR CLAMPED SANDWICH PLATES.

<table>
<thead>
<tr>
<th>Solution Method</th>
<th>Number of Cycles</th>
<th>Number of Strips</th>
<th>Number of Degrees of Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Isotropic</td>
</tr>
<tr>
<td>Finite Strip Polynomial</td>
<td>4</td>
<td>4</td>
<td>48</td>
</tr>
<tr>
<td>Finite Strip Polynomial</td>
<td>4</td>
<td>6</td>
<td>72</td>
</tr>
<tr>
<td>Finite Strip Polynomial</td>
<td>4</td>
<td>8</td>
<td>96</td>
</tr>
<tr>
<td>Finite Strip Polynomial for w</td>
<td>4</td>
<td>4</td>
<td>60</td>
</tr>
<tr>
<td>Finite Strip Polynomial for w</td>
<td>4</td>
<td>6</td>
<td>88</td>
</tr>
<tr>
<td>Finite Strip Polynomial for w</td>
<td>4</td>
<td>8</td>
<td>116</td>
</tr>
</tbody>
</table>
TABLE 17. EFFECT OF CYCLES ON THE DOUBLE-FOURIER SERIES SOLUTION OF CLAMPED SANDWICH PLATES WITH ISOTROPIC FACES.

<table>
<thead>
<tr>
<th>Number of Cycles $m, n$</th>
<th>$w_{\text{max}}$ $(\text{in}) \times 10^{-3}$</th>
<th>$N_{x_f}$ $(\text{lb/in})$</th>
<th>$N_{y_f}$ $(\text{lb/in})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x = \frac{L}{2}, y = \frac{B}{2}$</td>
<td>$x = \frac{L}{2}, y = \frac{B}{2}$</td>
<td>$x = \frac{L}{2}, y = \frac{B}{2}$</td>
</tr>
<tr>
<td>5</td>
<td>0.291</td>
<td>± 1.846</td>
<td>± 1.846</td>
</tr>
<tr>
<td>7</td>
<td>0.359</td>
<td>± 3.354</td>
<td>± 3.355</td>
</tr>
<tr>
<td>9</td>
<td>0.356</td>
<td>± 2.310</td>
<td>± 2.310</td>
</tr>
<tr>
<td>10</td>
<td>0.356</td>
<td>± 2.311</td>
<td>± 2.311</td>
</tr>
<tr>
<td>11</td>
<td>0.386</td>
<td>± 3.257</td>
<td>± 3.257</td>
</tr>
<tr>
<td>13</td>
<td>0.386</td>
<td>± 2.537</td>
<td>± 2.537</td>
</tr>
<tr>
<td>15</td>
<td>0.400</td>
<td>± 3.207</td>
<td>± 3.207</td>
</tr>
<tr>
<td>20</td>
<td>0.405</td>
<td>± 3.128</td>
<td>± 3.117</td>
</tr>
</tbody>
</table>
TABLE 18. RESULTS FOR CLAMPED SANDWICH PLATES WITH CROSS-PLIED FACES.

<table>
<thead>
<tr>
<th>Lay-Ups</th>
<th>Solution Method</th>
<th>$w_{center}$ (in) x $10^{-2}$</th>
<th>$N_{xf, center}$ (lb/in)</th>
<th>$N_{yf, center}$ (lb/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orthotropic</td>
<td>Double Fourier</td>
<td>0.689</td>
<td>± 36.197</td>
<td>± 58.175</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.725</td>
<td>± 34.753</td>
<td>± 59.837</td>
</tr>
<tr>
<td>1</td>
<td>Double Fourier</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Upper Face</td>
<td>0.710</td>
<td>+ 37.033</td>
<td>+ 60.213</td>
</tr>
<tr>
<td></td>
<td>Lower Face</td>
<td></td>
<td>- 36.507</td>
<td>- 58.579</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.739</td>
<td>+ 35.399</td>
<td>+ 62.319</td>
</tr>
<tr>
<td></td>
<td>Upper Face</td>
<td></td>
<td>- 34.411</td>
<td>- 61.201</td>
</tr>
<tr>
<td>2</td>
<td>Double Fourier</td>
<td>0.735</td>
<td>± 42.578</td>
<td>± 54.810</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.770</td>
<td>± 40.728</td>
<td>± 56.766</td>
</tr>
<tr>
<td>3</td>
<td>Double Fourier</td>
<td>0.671</td>
<td>± 29.160</td>
<td>± 61.975</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.691</td>
<td>± 28.057</td>
<td>± 61.053</td>
</tr>
</tbody>
</table>
### TABLE 19. EFFECT OF BOUNDARY CONDITIONS ON COUPLING IN SANDWICH PLATES WITH CROSS-PLIED FACES.

<table>
<thead>
<tr>
<th>Lay-Ups</th>
<th>Type of Support</th>
<th>( \frac{w_C}{w_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Simply Supported</td>
<td>1.010</td>
</tr>
<tr>
<td>1</td>
<td>Clamped</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Double Fourier Series</td>
<td>1.031</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>1.020</td>
</tr>
<tr>
<td></td>
<td>Simply Supported</td>
<td>1.020</td>
</tr>
<tr>
<td>2</td>
<td>Clamped</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Double Fourier Series</td>
<td>1.067</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>1.062</td>
</tr>
<tr>
<td></td>
<td>Simply Supported</td>
<td>0.982</td>
</tr>
<tr>
<td>3</td>
<td>Clamped</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Double Fourier Series</td>
<td>0.974</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.953</td>
</tr>
</tbody>
</table>
TABLE 20. RESULTS FOR SANDWICH PLATE WITH ISOTROPIC FACES (TWO SIDES SIMPLY SUPPORTED AND TWO SIDES CLAMPED).

<table>
<thead>
<tr>
<th>Solution Method</th>
<th>( w_{max} \times 10^{-3} )</th>
<th>( N_x ) (lb/in)</th>
<th>( N_y ) (lb/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref. 28</td>
<td>0.545</td>
<td>± 4.094</td>
<td>± 4.418</td>
</tr>
<tr>
<td>Finite Strip Case A</td>
<td>0.545</td>
<td>± 4.016</td>
<td>± 4.352</td>
</tr>
<tr>
<td>Finite Strip Case B</td>
<td>0.546</td>
<td>± 3.907</td>
<td>± 4.334</td>
</tr>
<tr>
<td>Double Fourier Series</td>
<td>0.536</td>
<td>± 4.264</td>
<td>± 4.388</td>
</tr>
</tbody>
</table>
TABLE 21. EFFECT OF CYCLES ON THE DOUBLE-FOURIER SERIES SOLUTION OF SANDWICH PLATES WITH ISOTROPIC FACES (TWO SIDES SIMPLY SUPPORTED AND TWO SIDES CLAMPED)

<table>
<thead>
<tr>
<th>Number of Cycles, m, n</th>
<th>$w_{\text{max}}$ (in) $\times 10^{-3}$</th>
<th>$N_{x_f}$ (lb/in)</th>
<th>$N_{y_f}$ (lb/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x = \frac{L}{2}, y = \frac{B}{2}$</td>
<td>$x = \frac{L}{2}, y = \frac{B}{2}$</td>
<td>$x = \frac{L}{2}, y = \frac{B}{2}$</td>
</tr>
<tr>
<td>3</td>
<td>0.398</td>
<td>$\pm 4.950$</td>
<td>$\pm 3.234$</td>
</tr>
<tr>
<td>5</td>
<td>0.425</td>
<td>$\pm 2.954$</td>
<td>$\pm 3.314$</td>
</tr>
<tr>
<td>7</td>
<td>0.484</td>
<td>$\pm 4.536$</td>
<td>$\pm 3.999$</td>
</tr>
<tr>
<td>9</td>
<td>0.490</td>
<td>$\pm 3.465$</td>
<td>$\pm 3.861$</td>
</tr>
<tr>
<td>10</td>
<td>0.490</td>
<td>$\pm 3.465$</td>
<td>$\pm 3.861$</td>
</tr>
<tr>
<td>11</td>
<td>0.513</td>
<td>$\pm 4.384$</td>
<td>$\pm 4.222$</td>
</tr>
<tr>
<td>13</td>
<td>0.516</td>
<td>$\pm 3.665$</td>
<td>$\pm 4.088$</td>
</tr>
<tr>
<td>15</td>
<td>0.527</td>
<td>$\pm 4.308$</td>
<td>$\pm 4.327$</td>
</tr>
<tr>
<td>20</td>
<td>0.536</td>
<td>$\pm 4.264$</td>
<td>$\pm 4.388$</td>
</tr>
</tbody>
</table>
TABLE 22. SANDWICH PLATE WITH CROSS-PLIED FACES (TWO SIDES SIMPLY SUPPORTED AND TWO SIDES CLAMPED 50 in. x 50 in.)

<table>
<thead>
<tr>
<th>Lay-Ups</th>
<th>Solution Method</th>
<th>$w_{center}$ (in) $\times 10^{-2}$</th>
<th>$N_{xf}$ center (lb/in)</th>
<th>$N_{yf}$ center (lb/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orthotropic</td>
<td>Double Fourier</td>
<td>0.917</td>
<td>$\pm$ 54.148</td>
<td>$\pm$ 76.876</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.959</td>
<td>$\pm$ 56.794</td>
<td>$\pm$ 75.058</td>
</tr>
<tr>
<td>1</td>
<td>Double Fourier</td>
<td>0.946</td>
<td>$+ 55.332$</td>
<td>$+ 77.939$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$- 55.188$</td>
<td>$- 77.875$</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.976</td>
<td>$+ 57.740$</td>
<td>$+ 76.752$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$- 57.678$</td>
<td>$- 75.733$</td>
</tr>
<tr>
<td>2</td>
<td>Double Fourier</td>
<td>1.047</td>
<td>$\pm$ 69.248</td>
<td>$\pm$ 78.194</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>1.087</td>
<td>$\pm$ 71.955</td>
<td>$\pm$ 76.364</td>
</tr>
<tr>
<td>3</td>
<td>Double Fourier</td>
<td>0.838</td>
<td>$\pm$ 42.942</td>
<td>$\pm$ 75.933</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.867</td>
<td>$\pm$ 45.046</td>
<td>$\pm$ 73.153</td>
</tr>
</tbody>
</table>
TABLE 23. SANDWICH PLATE WITH CROSS-PLIED FACES (TWO SIDES SIMPLY SUPPORTED AND TWO SIDES CLAMPED 50 in. x 20 in.).

<table>
<thead>
<tr>
<th>Lay-Ups</th>
<th>Solution Method</th>
<th>w&lt;sub&gt;center&lt;/sub&gt; (in) x 10&lt;sup&gt;-3&lt;/sup&gt;</th>
<th>N&lt;sub&gt;x_f center&lt;/sub&gt; (lb/in)</th>
<th>N&lt;sub&gt;y_f center&lt;/sub&gt; (lb/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orthotropic</td>
<td>Double Fourier</td>
<td>0.848</td>
<td>± 3.371</td>
<td>± 13.021</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.884</td>
<td>± 3.432</td>
<td>± 12.917</td>
</tr>
<tr>
<td>1</td>
<td>Double Fourier</td>
<td>0.924</td>
<td>+ 3.791</td>
<td>+ 13.672</td>
</tr>
<tr>
<td></td>
<td>Upper Face</td>
<td></td>
<td>- 3.681</td>
<td>- 13.656</td>
</tr>
<tr>
<td></td>
<td>Lower Face</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.952</td>
<td>+ 3.753</td>
<td>+ 13.567</td>
</tr>
<tr>
<td></td>
<td>Upper Face</td>
<td></td>
<td>- 3.716</td>
<td>- 13.245</td>
</tr>
<tr>
<td></td>
<td>Lower Face</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Double Fourier</td>
<td>1.126</td>
<td>± 5.353</td>
<td>± 15.112</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>1.164</td>
<td>± 5.405</td>
<td>± 14.984</td>
</tr>
<tr>
<td>3</td>
<td>Double Fourier</td>
<td>0.771</td>
<td>± 2.624</td>
<td>± 12.420</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.790</td>
<td>± 2.590</td>
<td>± 11.923</td>
</tr>
</tbody>
</table>
### Table 24: Sandwich Plate with Cross-Plied Faces (Two Sides Simply Supported and Two Sides Clamped 50 in. x 10 in.)

<table>
<thead>
<tr>
<th>Lay-Ups</th>
<th>Solution Method</th>
<th>$w_\text{center}$ (in) $\times 10^{-3}$</th>
<th>$N_{x_f}\text{center}$ (lb/in)</th>
<th>$N_{y_f}\text{center}$ (lb/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orthotropic</td>
<td>Double Fourier</td>
<td>0.147</td>
<td>$\pm$ 0.345</td>
<td>$\pm$ 2.722</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.150</td>
<td>$\pm$ 0.268</td>
<td>$\pm$ 2.666</td>
</tr>
<tr>
<td>1</td>
<td>Double Fourier</td>
<td>0.181</td>
<td>$+0.492$</td>
<td>$+3.161$</td>
</tr>
<tr>
<td></td>
<td>Upper Face</td>
<td></td>
<td>$-0.381$</td>
<td>$-3.151$</td>
</tr>
<tr>
<td></td>
<td>Lower Face</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.183</td>
<td>$+0.376$</td>
<td>$+3.060$</td>
</tr>
<tr>
<td></td>
<td>Upper Face</td>
<td></td>
<td>$-0.303$</td>
<td>$-3.038$</td>
</tr>
<tr>
<td></td>
<td>Lower Face</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Double Fourier</td>
<td>0.216</td>
<td>$\pm$ 0.551</td>
<td>$\pm$ 3.432</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.220</td>
<td>$\pm$ 0.538</td>
<td>$\pm$ 3.360</td>
</tr>
<tr>
<td>3</td>
<td>Double Fourier</td>
<td>0.154</td>
<td>$\pm$ 0.277</td>
<td>$\pm$ 2.907</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.155</td>
<td>$\pm$ 0.202</td>
<td>$\pm$ 2.777</td>
</tr>
</tbody>
</table>
TABLE 25. SANDWICH PLATE WITH CROSS-PLIED FACES (TWO SIDES SIMPLY SUPPORTED AND TWO SIDES CLAMPED 50 in. x 5 in.).

<table>
<thead>
<tr>
<th>Lay-Ups</th>
<th>Solution Method</th>
<th>( w ) center (in) ( \times 10^{-4} )</th>
<th>( N_{x_f} ) center (lb/in)</th>
<th>( N_{y_f} ) center (lb/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orthotropic</td>
<td>Double Fourier</td>
<td>0.205</td>
<td>( \pm 0.010 )</td>
<td>( \pm 0.415 )</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.206</td>
<td>( \pm 0.008 )</td>
<td>( \pm 0.404 )</td>
</tr>
<tr>
<td>1</td>
<td>Double Fourier</td>
<td>0.331</td>
<td>( \pm 0.075 )</td>
<td>( \pm 0.654 )</td>
</tr>
<tr>
<td></td>
<td>Upper Face</td>
<td></td>
<td>( \pm 0.033 )</td>
<td>( \pm 0.642 )</td>
</tr>
<tr>
<td></td>
<td>Lower Face</td>
<td></td>
<td></td>
<td>( \pm 0.636 )</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.332</td>
<td>( \pm 0.062 )</td>
<td>( \pm 0.597 )</td>
</tr>
<tr>
<td></td>
<td>Upper Face</td>
<td></td>
<td>( \pm 0.035 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Lower Face</td>
<td></td>
<td></td>
<td>( \pm 0.597 )</td>
</tr>
<tr>
<td>2</td>
<td>Double Fourier</td>
<td>0.380</td>
<td>( \pm 0.065 )</td>
<td>( \pm 0.663 )</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.382</td>
<td>( \pm 0.044 )</td>
<td>( \pm 0.669 )</td>
</tr>
<tr>
<td>3</td>
<td>Double Fourier</td>
<td>0.290</td>
<td>( \pm 0.047 )</td>
<td>( \pm 0.610 )</td>
</tr>
<tr>
<td></td>
<td>Finite Strip</td>
<td>0.290</td>
<td>( \pm 0.050 )</td>
<td>( \pm 0.566 )</td>
</tr>
</tbody>
</table>
TABLE 26. THIN ISOTROPIC CYLINDRICAL SHELL ROOF.

<table>
<thead>
<tr>
<th>Stress Resultants</th>
<th>( \phi )</th>
<th>0(^{\circ}) Edge</th>
<th>20(^{\circ})</th>
<th>30(^{\circ})</th>
<th>40(^{\circ}) Crown</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_x ) (lb/in)</td>
<td>Finite Strip</td>
<td>6364.2</td>
<td>-165.0</td>
<td>-1398.5</td>
<td>-660.97</td>
</tr>
<tr>
<td>( x = L/2 )</td>
<td>Ref. 33</td>
<td>6400.0</td>
<td>-167.0</td>
<td>-1410.0</td>
<td>-669.0</td>
</tr>
<tr>
<td>( N_\phi ) (lb/in)</td>
<td>Finite Strip</td>
<td>-0.06</td>
<td>-112.0</td>
<td>-258.9</td>
<td>-314.64</td>
</tr>
<tr>
<td>( x = L/2 )</td>
<td>Ref. 33</td>
<td>0.0</td>
<td>-114.0</td>
<td>-257.0</td>
<td>-312.0</td>
</tr>
<tr>
<td>( N_{x\phi} ) (lb/in)</td>
<td>Finite Strip</td>
<td>0.0</td>
<td>±682.0</td>
<td>±388.75</td>
<td>±91.84</td>
</tr>
<tr>
<td>( x = \pm 0, L )</td>
<td>Ref. 33</td>
<td>0.0</td>
<td>±690.0</td>
<td>±391.00</td>
<td>±92.70</td>
</tr>
<tr>
<td>( M_\phi ) (lb-in/in)</td>
<td>Finite Strip</td>
<td>0.001</td>
<td>159.0</td>
<td>1238.2</td>
<td>2093.5</td>
</tr>
<tr>
<td>( x = L/2 )</td>
<td>Ref. 33</td>
<td>0.0</td>
<td>155.0</td>
<td>1216.0</td>
<td>2077.0</td>
</tr>
</tbody>
</table>
TABLE 27. DISPLACEMENTS FOR SANDWICH FOLDED PLATES (LOADING CONDITION A)

<table>
<thead>
<tr>
<th>Solution Method</th>
<th>Plate</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ridge</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Membrane</td>
<td>Upper</td>
<td>0.562</td>
<td>0.567</td>
<td>-0.565</td>
<td>-0.560</td>
<td>0.560</td>
<td>-0.565</td>
</tr>
<tr>
<td>Displacement</td>
<td>Face</td>
<td>0.562</td>
<td>0.567</td>
<td>-0.565</td>
<td>-0.560</td>
<td>0.560</td>
<td>-0.565</td>
</tr>
<tr>
<td>v (in x 10^-1)</td>
<td>Lower</td>
<td>0.562</td>
<td>0.563</td>
<td>-0.563</td>
<td>-0.562</td>
<td>0.563</td>
<td>-0.562</td>
</tr>
<tr>
<td>Finite Strip</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Membrane</td>
<td>Transverse</td>
<td>-0.935</td>
<td>-0.992</td>
<td>-0.992</td>
<td>-0.984</td>
<td>-0.984</td>
<td>-0.990</td>
</tr>
<tr>
<td>Displacement</td>
<td>Deflection</td>
<td>w (in x 10^-1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solution Method</td>
<td>Plate</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>e</td>
<td>f</td>
</tr>
<tr>
<td>-----------------</td>
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<tr>
<td>Membrane Displacement</td>
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<td>0.114</td>
<td>-0.136</td>
<td>-0.135</td>
<td>0.148</td>
<td>0.149</td>
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<tr>
<td>v (in)</td>
<td>Lower Face</td>
<td>0.113</td>
<td>0.112</td>
<td>-0.137</td>
<td>-0.137</td>
<td>0.148</td>
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<tr>
<td>Finite Strip</td>
<td>Transverse Deflection</td>
<td>w (in)</td>
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<td>-0.226</td>
<td>-0.212</td>
<td>-0.253</td>
<td>-0.246</td>
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LIST OF REFERENCES


APPENDIX A

ELASTIC PROPERTIES OF COMPOSITE FACES

This section is concerned with the determination of the bending, membrane and coupling stiffnesses, as well as the force and moment resultants of laminated faces. Since a laminated face is composed of several layers, the properties of a single layer form the basis on which the properties of the laminated faces are obtained. As mentioned in Chapter II, each lamina is linear elastic, homogeneous and orthotropic with respect to its principal axis 1 and 2 (Fig. 3). Considering the assumptions made for the faces, the stress-strain relations for an orthotropic lamina in the state of plane stress can be characterized by the following:

\[
\begin{bmatrix}
\sigma_1^{(i)} \\
\sigma_2^{(i)} \\
\tau_{12}^{(i)}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{E_{11}^{(i)}}{1 - \nu_{12}^{(i)}\nu_{12}^{(i)}} & \frac{\nu_{12}^{(i)}E_{11}^{(i)}}{1 - \nu_{12}^{(i)}\nu_{21}^{(i)}} & 0 \\
\nu_{12}^{(i)}E_{22}^{(i)} & \frac{E_{22}^{(i)}}{1 - \nu_{12}^{(i)}\nu_{21}^{(i)}} & 0 \\
0 & 0 & G_{12}^{(i)}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1^{(i)} \\
\varepsilon_2^{(i)} \\
\gamma_{12}^{(i)}
\end{bmatrix}
\]  

(A-1a)

or  \[
\{\sigma_{12}^{(i)}\} = [\mathbf{F}^{(i)}] \{\varepsilon_{12}^{(i)}\}
\]  

(A-1b)
where \( v_{12} E_{22} = v_{21} E_{11} \) and the superscript \((i)\) represents the \(i\)th lamina. In case that the material axis (i.e., the principal axis 1 and 2), make an angle \(\theta\) with the reference axis of the structure, \(x\) and \(y\) (Fig. 3b), the stress-strain relations for the \(i\)th lamina referred to the reference axis can be obtained by the following transformation of both stresses and strains:

\[
\begin{bmatrix}
\sigma_{1}^{(i)} \\
\sigma_{2}^{(i)} \\
\tau_{12}^{(i)}
\end{bmatrix} =
\begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & \sin 2\theta \\
\sin^2 \theta & \cos^2 \theta & -\sin 2\theta \\
-\frac{1}{2}\sin 2\theta & \frac{1}{2}\sin 2\theta & \cos^2 \theta - \sin^2 \theta
\end{bmatrix}
\begin{bmatrix}
\sigma_{x}^{(i)} \\
\sigma_{y}^{(i)} \\
\tau_{xy}^{(i)}
\end{bmatrix}
\]

or \(\{\sigma_{12}^{(i)}\} = [T_{\sigma}]{\sigma_{xy}^{(i)}}\) \hspace{1cm} (A-2b)

and

\[
\begin{bmatrix}
\varepsilon_{1}^{(i)} \\
\varepsilon_{2}^{(i)} \\
\gamma_{12}^{(i)}
\end{bmatrix} =
\begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & \frac{1}{2}\sin 2\theta \\
\sin^2 \theta & \cos^2 \theta & -\frac{1}{2}\sin 2\theta \\
-\sin 2\theta & \sin 2\theta & \cos^2 \theta - \sin^2 \theta
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{x}^{(i)} \\
\varepsilon_{y}^{(i)} \\
\gamma_{xy}^{(i)}
\end{bmatrix}
\]

or \(\{\varepsilon_{12}^{(i)}\} = [T_{\varepsilon}]{\varepsilon_{xy}^{(i)}}\) \hspace{1cm} (A-3b)

where \([T_{\sigma}]^{-1} = [T_{\varepsilon}]^T\) \hspace{1cm} (A-4)
Substituting Eqs. (A-2) and (A-3) into (A-1) gives

\[
\begin{pmatrix}
\sigma^{(i)}_x \\
\sigma^{(i)}_y \\
t^{(i)}_{xy}
\end{pmatrix}
= 
\begin{bmatrix}
F^{(i)}_{11} & F^{(i)}_{12} & F^{(i)}_{16} \\
F^{(i)}_{22} & F^{(i)}_{26} & \\
F^{(i)}_{66}
\end{bmatrix}
\begin{pmatrix}
\varepsilon^{(i)}_x \\
\varepsilon^{(i)}_y \\
\gamma^{(i)}_{xy}
\end{pmatrix}
\] (A-5a)

or

\[
\begin{pmatrix}
\sigma^{(i)}_x \\
\sigma^{(i)}_y \\
t^{(i)}_{xy}
\end{pmatrix}
= [F^{(i)}] \begin{pmatrix}
\varepsilon^{(i)}_x \\
\varepsilon^{(i)}_y \\
\gamma^{(i)}_{xy}
\end{pmatrix}
\] (A-5b)

where

\[
[F^{(i)}] = [T^T_{\varepsilon}] \begin{pmatrix}
F^{(i)} \\
T^T_{\varepsilon}
\end{pmatrix}
\] (A-5c)

The coefficient matrix \([F^{(i)}]\) represents the six elastic constants associated with the \(i^{th}\) lamina. Since each layer is assumed homogeneous, the coefficient matrix is constant for a particular lamina. However, the coefficient matrix may vary from lamina to lamina, since each lamina may have different thickness, elastic properties and orientation of elastic principal axis with respect to the reference axis of the structure. Therefore, such faces are only piecewise homogeneous through the thickness and heterogeneous as a whole.

**A-1 Force-Deformation Relations**

The force-deformation relations for the thin laminated faces can be obtained from the following expressions for the force and moment resultants (Figs. 3 and 4):
\[ \{N_f\} = \sum_{i=1}^{I} \int_{z_f=h_{i-1}}^{h_i} \{ \gamma_{xy}^{(i)} \} \ dz_f \]

\[ \{M_f\} = \sum_{i=1}^{I} \int_{z_f=h_{i-1}}^{h_i} \{ \gamma_{xy}^{(i)} \} \cdot z_f \cdot dz_f \]

(A-6)

where \( I \) is the number of layers in the laminated face and \( h_i \) represents the distance between the \( i^{th} \) lamina and the reference surface of the face. Substituting Eqs. (2-1) and (A-5) into Eqs. (A-6) and integrating, result in the following relations:

\[ \{N_f\} = \sum_{i=1}^{I} \left[ \frac{1}{2} F^{(i)} \{ e_{xy}^{(i)} \} (h_i - h_{i-1}) + \frac{1}{3} F^{(i)} \{ \kappa_{xy}^{(i)} \} (h_i^3 - h_{i-1}^3) \right] \]

(A-7a)

\[ \{M_f\} = \sum_{i=1}^{I} \left[ \frac{1}{2} F^{(i)} \{ e_{xy}^{(i)} \} (h_i^2 - h_{i-1}^2) + \frac{1}{3} F^{(i)} \{ \kappa_{xy}^{(i)} \} (h_i^3 - h_{i-1}^3) \right] \]

(A-7b)

Eqs. (A-7) can be rewritten in the following form:
\[
\begin{pmatrix}
N_f \\
M_f
\end{pmatrix} =
\begin{bmatrix}
[A]^{(f)} & [B]^{(f)} \\
[B]^{(f)} & [D]^{(f)}
\end{bmatrix}
\begin{pmatrix}
\varepsilon_f^0 \\
\kappa_f
\end{pmatrix}
\]  

(A-8)

where the elements of the matrices \([A]^{(f)}, [B]^{(f)}\) and \([D]^{(f)}\) represent the membrane, coupling and bending stiffnesses of the laminated faces, defined by (Fig. 3a):

\[
[A]^{(f)} = \sum_{i=1}^{I} \int_{z_f=h_{i-1}}^{h_i} [F^{(i)}] \, dz_f = \sum_{i=1}^{I} [F^{(i)}] \cdot (h_i - h_{i-1})
\]

(A-9a)

\[
[B]^{(f)} = \sum_{i=1}^{I} \int_{z_f=h_{i-1}}^{h_i} [F^{(i)}] \cdot z_f \cdot dz_f
\]

\[
= \frac{1}{2} \sum_{i=1}^{I} [F^{(i)}] \cdot (h_i^2 - h_{i-1}^2)
\]

(A-9b)

\[
[D]^{(f)} = \sum_{i=1}^{I} \int_{z_f=h_{i-1}}^{h_i} [F^{(i)}] \cdot z_f^2 \cdot dz_f
\]

\[
= \frac{1}{3} \sum_{i=1}^{I} [F^{(i)}] \cdot (h_i^3 - h_{i-1}^3)
\]

(A-9c)
Depending upon the various properties of each layer in the laminated face, some elements of the stiffness matrices $A_{ij}^{(f)}$, $B_{ij}^{(f)}$, and $D_{ij}^{(f)}$ (i, j = 1, 2, 6 and f = 1, 2) may vanish (Eqs. A-9). Generally, when the layers are placed symmetrically (elastically and geometrically) about the middle surface of the face all the elements of the coupling matrix will vanish. However, for unsymmetrical laminated constructions such as the unbalanced cross-ply and angle-ply laminates, where the faces consist of an even number of layers all of the same thickness with the orthotropic axis of symmetry (i.e. 1 and 2) in each layer alternately oriented at angles (0° and 90°) and (+θ and -θ) to the plate axis respectively (Fig. 5), the laminated face possesses the following properties.

(a) Cross-ply laminates:

$$A_{16}^{(f)} = A_{26}^{(f)} = D_{16}^{(f)} = D_{26}^{(f)} = 0$$

and

$$A_{22}^{(f)} = A_{11}^{(f)} ; D_{22}^{(f)} = D_{11}^{(f)} \text{ and} \quad (A-10)$$

$$B_{11}^{(f)} = -B_{22}^{(f)}$$

and all the other elements of the coupling matrix $[B]^{(f)}$ vanish.
(b) Angle-ply laminates

\[ A_{16}^{(f)} = A_{26}^{(f)} = D_{16}^{(f)} = D_{26}^{(f)} = 0 \]

and

\[ B_{16}^{(f)} \text{ and } B_{26}^{(f)} \text{ are the only non vanishing elements of the } [B]^{(f)} \text{ matrix.} \]

A-2 Reduced Bending Stiffness Approximation

In the analysis of thin unbalanced laminated plates, the membrane boundary conditions of the plate must be considered in the analysis, since the middle plane of the plate is strained because of the existence of the coupling between the bending and membrane actions. This in turn results in an increase in the size of the formulations required in the analysis. However, the unbalanced laminated plates can be treated as thin orthotropic plates, where the usual assumptions considered for thin homogeneous plates including the assumption that the reference surface of the plate is unstrained, are valid. The analysis, therefore, is independent from the membrane boundary conditions of the plate. This can be achieved by using the reduced bending stiffness-approximation (RBS) (Ref. 22). In this approximation the bending stiffness \( D_{ij} \) \( (i,j = 1,2,6) \) of the heterogeneous anisotropic thin plate is replaced by reduced bending stiffnesses \( D_{ij}^* \) where

\[
[D_{ij}^*] = [D_{ij}] - [B_{ij}][A_{ij}]^{-1}[B_{ij}]
\]

(A-12)
This expression can be obtained by substituting Eq. A-8 into the strain energy formulations of a thin laminated plates (Eq. 3-4).
APPENDIX B
DEVELOPMENT OF BASIC FUNCTIONS

The following discussion outlines the method for determining the basic functions for the structural systems. As mentioned in Chapter III, the basic functions for a strip are obtained by seeking a solution for the general integral of the differential equation of free vibrations of a homogeneous beam, given by:

\[
X(x) = c_1 \sin \left( \frac{\lambda x}{L} \right) + c_2 \cos \left( \frac{\lambda x}{L} \right) + c_3 \sinh \left( \frac{\lambda x}{L} \right) + c_4 \cosh \left( \frac{\lambda x}{L} \right)
\]  

(B-1)

The solution is obtained by determining the values of the constants \( c_j \) \( (j = 1, \ldots, 4) \) such that both the force and displacement boundary conditions are satisfied at the two ends of the strip. It is pointed out that the basic functions are determined in exactly the same manner as the Rayleigh function in the theory of transverse vibration of a homogeneous beam. The basic functions presented herein are limited to the cases of thin isotropic plates and shells with various end conditions.

(1) Both Ends Simply-Supported

The simple supports are of the type which allow normal displacement on the boundary, but prevent lateral contraction (i.e. tangential displacements), therefore:
\[ w(x,y) = v(x,y) = N_x(x,y) = M_x(x,y) = 0 \quad \text{at} \quad x = 0, L \]  
\[ (B-2) \]

These conditions in turn impose the following conditions on the basic function of the transverse deflection, \( x_{x_0} \)

\[ x_{x_0}(0) = x_{x_0}(L) = 0 \quad \text{at} \quad x = 0 \]
\[ (B-3a) \]
\[ x_{x_0}(L) = x_{x_0}(L) = 0 \quad \text{at} \quad x = L \]
\[ (B-3b) \]

Substituting Eq. (B-1) into Eqs. (B-3) lead to:

\[ c_2 + c_4 = 0 \quad \text{ (B-4a)} \]
\[ c_2 + c_4 = 0 \quad \text{ (B-4b)} \]
\[ c_1 \sin (\lambda) + c_2 \cos (\lambda) + c_3 \sinh (\lambda) + c_4 \cosh (\lambda) = 0 \quad \text{ (B-4c)} \]
\[ -c_1 \sin (\lambda) - c_2 \cos (\lambda) + c_3 \sinh (\lambda) + c_4 \cosh (\lambda) = 0 \quad \text{ (B-4d)} \]

From Eqs. (B-4 a,b) \( c_2 = c_4 = 0 \) The remaining two equations take the form:

\[ c_1 \sin (\lambda) + c_3 \sinh (\lambda) = 0 \quad \text{ (B-5a)} \]
\[ -c_1 \sin (\lambda) + c_3 \sinh (\lambda) = 0 \quad \text{ (B-5b)} \]

The determinant of Eqs. (B-5) should be equal to zero, therefore,

\[ \sin(\lambda) = 0 \quad \text{ (B-56)} \]
which yields an infinite set of real roots $\lambda_m$ $(m=1, 2, \ldots, a)$. These roots are:

$$\lambda_m = \pi, 2\pi, \ldots, m\pi$$  \hspace{1cm} (B-7)

In accordance with this, the basic function of the transverse deflection will have the following form:

$$X_w^{(m)} = \sin \frac{m\pi x}{L} \quad (m = 1, 2, \ldots, a)$$  \hspace{1cm} (B-8)

From Eqs. (B-2) and by considering the force-deformation relations associated with this case (Eq. 2.3), the basic functions for the membrane displacements $u$ and $v$ are as follows:

$$X_u^{(m)} = \cos \frac{m\pi x}{L}$$  \hspace{1cm} (B-9)

$$X_v^{(m)} = \sin \frac{m\pi x}{L}$$  \hspace{1cm} (B-10)

Note that this set of basic functions is orthogonal.

(2) **One End Simply Supported, the Other Clamped**

The boundary conditions are as follows:

$$w(x,y) = v(x,y) = N_x(x,y) = M_x(x,y) = 0 \quad \text{at} \ x = 0$$  \hspace{1cm} (B-11a)

$$w(x,y) = v(x,y) = u(x,y) = w_x(x,y) = 0 \quad \text{at} \ x = L$$  \hspace{1cm} (B-11b)

The boundary conditions for the transverse deflection will lead to:

$$X^{(o)}(x) = X^{(o)}(x) = 0 \quad \text{at} \ x = 0$$  \hspace{1cm} (B-12a)

$$X^{(L)}(x) = X^{(L)}(x) = 0 \quad \text{at} \ x = L$$  \hspace{1cm} (B-12b)

Under these conditions the characteristic equation for the parameter, $\lambda$ will be:
\[ \tan \gamma (\lambda) = \tanh (\lambda) \]  
(B-13)

The successive roots of this equation are:
\[ \lambda_m = \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots, \left(\frac{4m + 1}{4}\right)\pi \]  
(B-14)

Therefore, the basic function for the transverse deflection will have the following form:
\[ X_w^{(m)} = \sin \left(\frac{\lambda_m x}{L}\right) - a_m \cdot \sinh \left(\frac{\lambda_m x}{L}\right) \]  
(B-15a)

Where
\[ a_m = \frac{\sin (\lambda_m)}{\sinh (\lambda_m)} \]  
(B-15b)

The basic functions for the membrane displacements, u and v, can be obtained by considering Eqs. (B-11) as well as the force-deformation relations, therefore:
\[ X_u^{(m)} = \cos \left(\frac{\lambda_m x}{L}\right) - a_m \cosh \left(\frac{\lambda_m x}{L}\right) \]  
(B-16)
\[ X_v^{(m)} = \sin \left(\frac{\lambda_m x}{L}\right) - a_m \sinh \left(\frac{\lambda_m x}{L}\right) \]  
(B-17)

Note that the basic functions for this case are non-orthogonal.

(3) Both Ends Clamped

The boundary conditions are as follows:
\[ w(x, y) = v(x, y) = u(x, y) = w_x(x, y) = 0 \quad \text{at } x = 0, L \]  
(B-18)

These conditions impose the following conditions on the basic function of the transverse deflection, \( X(x) \)
\[ X(0) = X(L) = 0 \quad \text{at } x = 0 \]  
(B-19a)
\[ X'(0) = X'(L) = 0 \quad \text{at } x = L \]  
(B-19b)
Similarly, substituting Eq. (B-1) into Eqs. (B-19), the characteristic equation for the parameter $\lambda$ can be obtained:

$$\cos(\lambda) \cdot \cosh(\lambda) = 1$$  \hspace{1cm} (B-20)

The roots of which are:

$$\lambda_m = 0, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \ldots, \frac{(2m+1)\pi}{2}$$  \hspace{1cm} (B-21)

with the condition that $c_1 = 1$, the basic function for the transverse deflection will be as follows:

$$x_w^{(m)} = \sin\left(\frac{\lambda m \cdot x}{L}\right) - \sinh\left(\frac{\lambda m \cdot x}{L}\right)$$

$$- \alpha_m \left[\cos\left(\frac{\lambda m \cdot x}{L}\right) - \cosh\left(\frac{\lambda m \cdot x}{L}\right)\right]$$  \hspace{1cm} (B-22)

Where $m = 0, 1, \ldots$, $\alpha$ and $\alpha_m = \frac{\sin(\lambda_m) - \sinh(\lambda_m)}{\cos(\lambda_m) - \cosh(\lambda_m)}$  \hspace{1cm} (B-23)

Similarly, the basic functions for the membrane displacements $u, v$ can be obtained by considering Eq. (B-18) as well as the force-deformation relations (Eq. 2.3), therefore,

$$x_u^{(m)} = \cos\left(\frac{\lambda m \cdot x}{L}\right) - \cosh\left(\frac{\lambda m \cdot x}{L}\right) + \alpha_m \left[\sin\left(\frac{\lambda m \cdot x}{L}\right) - \sinh\left(\frac{\lambda m \cdot x}{L}\right)\right]$$  \hspace{1cm} (B-24)

$$x_v^{(m)} = \sin\left(\frac{\lambda m \cdot x}{L}\right) - \sinh\left(\frac{\lambda m \cdot x}{L}\right) - \alpha_m \left[\cos\left(\frac{\lambda m \cdot x}{L}\right) - \cosh\left(\frac{\lambda m \cdot x}{L}\right)\right]$$  \hspace{1cm} (B-25)
The basic functions for this case are non-orthogonal. The basic functions for other types of boundary conditions can be derived in the same way as presented herein.
APPENDIX C

STRIP STIFFNESS MATRIX FORMULAS

In this section, the elements of the submatrices which make up the strip stiffness matrix (Eq. 3-39) are presented. The stiffness matrix formulas that follow are valid for orthotropic \( B_{11}^{(f)} = B_{22}^{(f)} = 0 \), or cross-plied, thin and sandwich plate \( 1/R_1 = 1/R_2 = 1/R_C = 0, b_C = b_f = b \), and cylindrical shell strips, having simply supported and clamped ends. For such structural systems, the following relations between the basic functions can be written:

\[
X_{(n)}(x) = X_w^{(n)} = X_{v_f}^{(n)}
\]  

\[
\frac{\partial X_{(n)}(x)}{\partial x} \quad \text{and} \quad X_{u_f}^{(n)} = \frac{X_{(n)}(x)}{\lambda_n} \cdot L
\]

where

\[
\lambda_n = n\pi \quad \text{For simply supported ends}
\]

\[
\lambda_n = \frac{(2n+1)\pi}{2} \quad \text{For clamped ends}
\]

\[
\lambda_n = \frac{(4n+1)\pi}{4} \quad \text{For one end simply supported and the other end clamped}
\]
and \( n = 1, 2, \ldots, N \). Let:

\[
Q_1^{(n,m)} = \int_{x=0}^{x=L} X^{(n)}(x) \cdot X^{(m)}(x) \cdot dx \quad (C-3a)
\]

\[
Q_2^{(n,m)} = \int_{x=0}^{x=L} X^{(n)}(x) \cdot \hat{X}^{(m)}(x) \cdot dx \quad (C-3b)
\]

\[
Q_3^{(n,m)} = \int_{x=0}^{x=L} X^{(n)}(x) \cdot \hat{X}^{(m)}(x) \cdot dx \quad (C-3c)
\]

\[
Q_4^{(n,m)} = \int_{x=0}^{x=L} \hat{X}^{(n)}(x) \cdot X^{(m)}(x) \cdot dx \quad (C-3d)
\]

\[
Q_5^{(n,m)} = \int_{x=0}^{x=L} \hat{X}^{(n)}(x) \cdot \hat{X}^{(m)}(x) \cdot dx \quad (C-3e)
\]

\[
Q_6^{(n,m)} = \int_{x=0}^{x=L} X^{(n)}(x) \cdot \hat{X}^{(m)}(x) \cdot dx \quad (C-3f)
\]

where the superscripts \( n \) and \( m \) refer to the \( n^{th} \) and \( m^{th} \) cycles of the basic functions and \( n, m = 1, 2, \ldots, N \).

Since the strip stiffness matrix is symmetric, only the upper (or the lower) triangle of the matrix need to be formulated (i.e. \( m \geq n \)). The formulas are valid for the
cases where the transverse deflection is expressed as a third or a fifth order interpolation polynomial, and the membrane displacements, $u_f$ and $v_f$, are expressed as third order interpolation polynomials.

**Stiffness Matrix Elements**

$$ K_{S_{ij}}^{u_u} = \frac{b_{ij}}{420 \cdot \lambda_n \cdot \lambda_m} \left[ \frac{\eta_{ij} \cdot L^2}{b_f} \cdot \frac{a_{11}^{(f)} \cdot b_f \cdot q_{ij}^{(1)} \cdot Q_6^{(n,m)}}{420 \cdot \lambda_n \cdot \lambda_m} + \frac{a_{66}^{(f)} \cdot q_{ij}^{(3)} \cdot Q_4^{(n,m)}}{b_f} \right] $$

$$ + \frac{b_{ij} \cdot B_{55} \cdot b_c \cdot q_{ij}^{(1)} \cdot Q_4^{(n,m)} \cdot L^2}{420 \cdot t_c^2 \cdot \lambda_n \cdot \lambda_m} $$

$$ K_{S_{ij}}^{v_v} = \frac{b_f}{420} \left[ \frac{\eta_{ij} \cdot L^2}{b_f} \cdot \frac{a_{22}^{(f)} \cdot q_{ij}^{(3)} \cdot Q_1^{(n,m)}}{b_f} \right] $$

$$ + \frac{a_{66}^{(f)} \cdot q_{ij}^{(1)} \cdot b_f \cdot Q_4^{(n,m)}}{420 \cdot \lambda_n \cdot \lambda_m} $$

$$ + \frac{1}{R_f} \left[ \frac{4 \cdot D_{66}^{(f)} \cdot Q_4^{(n,m)} \cdot q_{ij}^{(1)} \cdot b_f}{R_f} + \left( \frac{D_{22}^{(f)}}{R_f} + 2 \frac{E_{22}^{(f)}}{b_f} \right) \frac{q_{ij}^{(3)} \cdot Q_1^{(n,m)}}{b_f} \right] $$

$$ + \frac{b_{ij} \cdot B_{44} \cdot b_c \cdot q_{ij}^{(1)} \cdot Q_1^{(n,m)} \cdot L^2}{420 \cdot t_c^2} \quad (C-4) $$

$$ (C-5) $$
\[
\begin{align*}
K_{S_i}^{(n), u_2} &= K_{S_i}^{(n), u_1} = \frac{\eta_{ij} \cdot B_{55} \cdot q_{ij}^{(1)} \cdot b_c \cdot Q_4^{(n,m)} \cdot L^2}{420 \cdot t_c^2 \cdot \lambda_n \cdot \lambda_m} \\
(C-6) \\
K_{S_i}^{(n), v_2} &= K_{S_i}^{(n), v_1} = \frac{\eta_{ij} \cdot B_{44} \cdot q_{ij}^{(1)} \cdot b_c \cdot Q_1^{(n,m)}}{420 \cdot t_c^2} \\
(C-7) \\
K_{S_i}^{(n), v_f} &= \frac{\eta_{ij} \cdot L}{420 \cdot \lambda_n} \left[ A_{66}^{(f)} \cdot q_{ij}^{(2)} \cdot Q_4^{(n,m)} + A_{12}^{(f)} \cdot q_{ij}^{(2)} \cdot Q_3^{(m,n)} \right] \\
(C-8) \\
K_{S_i}^{(n), u_f} &= \frac{\eta_{ij} \cdot L}{420 \cdot \lambda_m} \left[ A_{66}^{(f)} \cdot q_{ij}^{(2)}' \cdot Q_4^{(n,m)} + A_{12}^{(f)} \cdot q_{ij}^{(2)} \cdot Q_3^{(n,m)} \right] \\
(C-9) \\
K_{S_i}^{(n), w_2} &= \frac{\alpha_{ij} \cdot b_c \cdot q_{ij}^{(4)} \cdot L}{420 \cdot \lambda_n} \left[ A_{12}^{(f)} \cdot Q_3^{(m,n)} \right] - \frac{B_{11}^{(f)} \cdot Q_6^{(n,m)}}{R_f} \\
+ \frac{(-1)^{f} \cdot B_{55} \cdot (d_1 + d_2 + t_c) \cdot b_c \cdot \alpha_{ij} \cdot q_{ij}^{(4)} \cdot Q_4^{(n,m)} \cdot L}{420 \cdot t_c^2 \cdot \lambda_n} \\
(C-10)
\end{align*}
\]
\[ w^{(n)} u_f^{(m)}_{K_{S_{ij}}} = \frac{a_{ji} \cdot b_f \cdot q_{ji}^{(4)} \cdot L}{420 \cdot \lambda_m} \left[ \frac{A^{(f)}_{12} \cdot Q_3^{(n,m)}}{R_f} - B^{(f)}_{11} \cdot Q_6^{(n,m)} \right] \]

\[ \frac{(-1)^f \cdot B_{55} \cdot (d_1 + d_2 + t_c) \cdot b_c \cdot b_f \cdot q_{ji}^{(4)} \cdot Q_4^{(n,m)} \cdot L}{420 \cdot t_c^2 \cdot \lambda_m} \]

\[ v_f^{(n)} w^{(m)}_{K_{S_{ij}}} = \frac{b_f}{420} \left[ \frac{B_{22}^{(f)} \cdot q_{ij}^{(9)} \cdot Q_1^{(n,m)}}{b_f^2} - \frac{1}{R_f} \right] \]

\[ \left[ \frac{A^{(f)}_{22} + B_{22}^{(f)}}{R_f} \right] \cdot q_{ij}^{(7)} \cdot Q_1^{(n,m)} - \frac{D^{(f)}_{22} \cdot q_{ij}^{(9)} \cdot Q_1^{(n,m)}}{b_f^2} \]

\[ - 4 \cdot D_{66}^{(f)} \cdot q_{ij}^{(5)} \cdot Q_4^{(n,m)} - D_{12}^{(f)} \cdot q_{ij}^{(7)} \cdot Q_3^{(n,m)} \]

\[ \frac{(-1)^f \cdot b_c \cdot B_{44} \cdot q_{ij}^{(5)} \cdot Q_1^{(n,m)} \cdot (h_1 + h_2 + t_c)}{420 \cdot t_c^2} \]

(C-11)

(C-12)
\[
K_{ij}^{w(n), v_f^{(m)}} = \frac{\alpha_{ij}}{420} \left[ \frac{B_{ij}^{(f)} \cdot q_{ji}^{(9)} \cdot Q_{1}^{(n,m)}}{b_f^2} + \frac{1}{R_f} \right]
\]

\[
\left[ \frac{A_{22}^{(f)} + B_{22}^{(f)}}{R_f} \right] \cdot q_{ji}^{(7)} \cdot Q_{1}^{(n,m)} - \frac{D_{ij}^{(f)} \cdot q_{ji}^{(9)} \cdot Q_{1}^{(n,m)}}{b_f^2}
\]

\[
- 4 \cdot D_{66}^{(f)} \cdot q_{ji}^{(5)} \cdot Q_{4}^{(n,m)} - D_{12}^{(f)} \cdot q_{ji}^{(7)} \cdot Q_{3}^{(n,m)}
\]

\[
- (-1)^F \cdot \frac{\alpha_{ij} \cdot B_{ij} \cdot q_{ji}^{(5)} \cdot Q_{1}^{(n,m)} \cdot (h_1 + h_2 + t_c)}{420 \cdot t_c^2}
\]

(C-13)

Where

\[\alpha_{ij} = \eta_{ij}\] when \(w(x, y_f)\) is expressed as a third order interpolation polynomial (Table C1).

\[\alpha_{ij} = \xi_{ij}\] when \(w(x, y_f)\) is expressed as a fifth order interpolation polynomial (Table C2).
\[ K_{S_{ij}} = \frac{\alpha_{ij}}{b_c} \cdot \left[ \frac{B_{44} \cdot q_{ij}^{(13)} \cdot (h_1 + h_2 + t_c)^2 \cdot Q_1^{(n,m)}}{b_c} \right. \]

\[ + B_{55} \cdot b_c \cdot q_{ij}^{(10)} \cdot (d_1 + d_2 + t_c)^2 \cdot Q_4^{(n,m)} \]

\[ + \sum_{f=1}^{2} \frac{\alpha_{ij}}{b_f} \cdot \left[ \frac{D_{11}^{(f)} \cdot q_{ij}^{(10)} \cdot b_f \cdot Q_6^{(n,m)}}{b_f} \right. \]

\[ + \frac{D_{22}^{(f)} \cdot Q_1^{(n,m)} \cdot q_{ij}^{(15)}}{b_f^3} + \frac{4 \cdot D_{66}^{(f)} \cdot q_{ij}^{(13)} \cdot Q_4^{(n,m)}}{b_f} \]

\[ \left. \left\{ 2 \cdot D_{12}^{(f)} \cdot \frac{q_{ij}^{(1)}}{b_f} + \frac{1}{R_f} \left[ \frac{A_{22}^{(f)} \cdot q_{ij}^{(10)} \cdot b_f \cdot Q_1^{(n,m)}}{R_f} \right. \right. \right. \]

\[ + \frac{2 \cdot B_{22}^{(f)} \cdot q_{ij}^{(2)}}{b_f} \right\} \right] \]

Where \( q_{ij}^{(1)} = \left[ q_{ij}^{(12)} \cdot Q_{3}^{(m,n)} + q_{ji}^{(12)} \cdot Q_3^{(n,m)} \right] \left/ 2 \right. ;

\( q_{ij}^{(2)} = \left[ q_{ij}^{(12)} \cdot Q_{1}^{(m,n)} + q_{ji}^{(12)} \cdot Q_1^{(n,m)} \right] \left/ 2 \right. . \]
\[ a_{ij} = \eta_{ij} \quad \text{when } w(x, y_f) \text{ is expressed as a third order interpolation polynomial (Table C1)}. \]

\[ a_{ij} = \xi_{ij} \quad \text{when } w(x, y_f) \text{ is expressed as a fifth order interpolation polynomial (Table C3)}. \]

The constants \( q_{ij}^{(k)}, \ k = 1, 2, \ldots, 15 \) (Eqs. C-4 to C-14) are given in Table C1 if all the displacement components are expressed as third order interpolation polynomials, and in Tables C2 and C3 (\( k = 4, 5, \ldots, 9 \) and \( k = 10, 11, \ldots, 15 \) respectively), for the case where the transverse deflection is expressed as a fifth order interpolation polynomial.
TABLE C1 CONSTANTS AND EXONENTS FOR SUBMATRICES
(FOR THIRD ORDER INTERPOLATION POLYNOMIALS)

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Note: For third order interpolation polynomials

(a) \( q_{ij} = q_{ij} = q_{ij} \)
(b) \( q_{ij} = q_{ij} = q_{ji} \)
(c) \( q_{ij} = q_{ij} = q_{ij} \)
(d) \( q_{ij} = q_{ij} \)
(e) \( q_{ij} = q_{ij} \)

For \( i, j = 1, 2, 3, 4 \)
TABLE C2 CONSTANTS AND EXPONENTS FOR SUBMATRICES
(W IS EXPRESSED AS A FIFTH ORDER INTERPOLATION POLYNOMIALS)

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APPENDIX D

DOUBLE FOURIER - SERIES FORMULAS

The elements of the matrix \([g]\) (Eq. 4-7) are given as follows:

\[
g_{11} = A_{11}^{(1)} \left( \frac{m\pi}{L} \right)^2 + A_{66}^{(1)} \left( \frac{n\pi}{B} \right)^2 + \frac{B_{55}}{t_c}^2 \quad (D-1a)
\]

\[
g_{12} = -\frac{B_{55}}{t_c^2} \quad (D-1b)
\]

\[
g_{13} = \left[ A_{12}^{(1)} + A_{66}^{(1)} \right] \cdot \frac{n\pi}{B} \cdot \frac{m\pi}{L} \quad (D-1c)
\]

\[
g_{15} = -B_{11}^{(1)} \left( \frac{m\pi}{L} \right)^3 - \frac{e \cdot B_{55}}{t_c^2} \cdot \frac{m\pi}{L} \quad (D-1d)
\]

\[
g_{22} = A_{11}^{(2)} \left( \frac{m\pi}{L} \right)^2 + A_{66}^{(2)} \left( \frac{n\pi}{B} \right)^2 + \frac{B_{55}}{t_c^2} \quad (D-1e)
\]

\[
g_{24} = \left[ A_{12}^{(2)} + A_{66}^{(2)} \right] \frac{n\pi}{B} \cdot \frac{m\pi}{L} \quad (D-1f)
\]
\[ g_{25} = -B_{11}(\frac{m_\Pi}{L})^2 + \frac{e \cdot B_{55}}{t_c^2} \frac{m_\Pi}{L} \]  
(D-1g)

\[ g_{33} = A_{66}(\frac{m_\Pi}{L})^2 + A_{22}(\frac{n_\Pi}{B})^2 + \frac{B_{44}}{t_c^2} \]  
(D-1h)

\[ g_{34} = -\frac{B_{44}}{t_c^2} \]  
(D-1i)

\[ g_{35} = -B_{22}(\frac{n_\Pi}{B})^3 - \frac{e \cdot B_{44}}{t_c^2} \frac{n_\Pi}{B} \]  
(D-1j)

\[ g_{44} = A_{66}(\frac{m_\Pi}{L})^2 + A_{22}(\frac{n_\Pi}{B})^2 + \frac{B_{44}}{t_c^2} \]  
(D-1k)

\[ g_{45} = -B_{22}(\frac{n_\Pi}{B})^3 + \frac{e \cdot B_{44}}{t_c^2} \frac{n_\Pi}{B} \]  
(D-1l)

\[ g_{55} = D_{11}(\frac{m_\Pi}{L})^4 + 2[D_{12} + 2 D_{66}](\frac{n_\Pi}{B})^2 \cdot (\frac{m_\Pi}{L})^2 \]

\[ + \frac{D_{22}(n_\Pi/B)^4}{t_c^2} \left[ B_{55}(\frac{m_\Pi}{L})^2 + B_{44}(\frac{n_\Pi}{B})^2 \right] \]  
(D-1m)
where \( m, n = 1, 2, \ldots, \infty \). In Eqs. (D-1) the following relations can be considered for the cross-ply laminates:

\[
A_{11}^{(f)} = A_{22}^{(f)} ; \quad B_{11}^{(f)} = -B_{22}^{(f)} ; \quad D_{11}^{(f)} = D_{22}^{(f)}
\]

(D-1n)

The displacement parameters \( u_{mn}^{(f)}, v_{mn}^{(f)}, \) and \( w_{mn} \) for sandwich plates with four simply supported edges are as follows, (i.e. the solution expressions for Eqs. 4-7).

\[
w_{mn} = g_{12} \cdot g_{34} \cdot L_1 \cdot q_{mn} / \text{DN} \quad \text{(D-2a)}
\]

\[
u_{mn}^{(1)} = g_{12} \cdot g_{34} \cdot L_2 \cdot q_{mn} / \text{DN} \quad \text{(D-2b)}
\]

\[
u_{mn}^{(1)} = g_{12} \cdot g_{34} \cdot L_3 \cdot q_{mn} / \text{DN} \quad \text{(D-2c)}
\]

\[
u_{mn}^{(2)} = -g_{34} \cdot q_{mn} \left[ g_{11} \cdot L_2 + g_{13} \cdot L_3 + g_{15} \cdot L_1 \right] / \text{QN} \quad \text{(D-2d)}
\]

\[
u_{mn}^{(2)} = -g_{12} \cdot q_{mn} \left[ g_{13} \cdot L_2 + g_{33} \cdot L_3 + g_{35} \cdot L_1 \right] / \text{DN} \quad \text{(D-2e)}
\]

where

\[
L_1 = K_1 K_5 - K_2 K_4 \quad \text{(D-3a)}
\]
\[ L_2 = K_2 K_6 - K_3 K_5 \]  
(D-3b)

\[ L_3 = K_3 K_4 - K_1 K_6 \]  
(D-3c)

and

\[ K_1 = -g_{24} \cdot g_{11} \cdot g_{34} - g_{44} \cdot g_{13} \cdot g_{12} \]  
(D-3d)

\[ K_2 = g_{34} \cdot g_{12} \cdot g_{34} - g_{24} \cdot g_{13} \cdot g_{34} \]  
(D-3e)

\[ -g_{44} \cdot g_{33} \cdot g_{12} \]  

\[ K_3 = g_{45} \cdot g_{12} \cdot g_{34} - g_{24} \cdot g_{13} \cdot g_{34} \]  
(D-3f)

\[ -g_{44} \cdot g_{35} \cdot g_{12} \]  

\[ K_4 = g_{12} \cdot g_{12} \cdot g_{34} - g_{24} \cdot g_{11} \cdot g_{34} \]  
(D-3g)

\[ -g_{24} \cdot g_{12} \cdot g_{13} \]  

\[ K_5 = -g_{22} \cdot g_{13} \cdot g_{34} - g_{24} \cdot g_{12} \cdot g_{33} \]  
(D-3h)

\[ K_6 = g_{25} \cdot g_{12} \cdot g_{34} - g_{22} \cdot g_{15} \cdot g_{34} \]  
(D-3i)

\[ -g_{24} \cdot g_{12} \cdot g_{35} \]
\[ DN = L_1 \left[ g_{25} \cdot g_{12} \cdot g_{34} - g_{25} \cdot g_{15} \cdot g_{34} \right. \\
- g_{45} \cdot g_{35} \cdot g_{12} \right] + L_2 \left[ g_{15} \cdot g_{12} \cdot g_{34} \right. \\
- g_{25} \cdot g_{11} \cdot g_{34} - g_{45} \cdot g_{13} \cdot g_{12} \right] \\
+ L_3 \left[ g_{35} \cdot g_{12} \cdot g_{34} - g_{25} \cdot g_{13} \cdot g_{34} \right. \\
- g_{45} \cdot g_{33} \cdot g_{12} \right] \]  \hspace{1cm} (D-3j)

When using a double Fourier series to represent the solution for a structural system, it is inherently assumed that the series is term-by-term differentiable. However, this assumption is only valid if the double series satisfy the prescribed boundary conditions of the structure. That is, in case that a function \( f(x,y) \) is expanded in a double sine series which is valid over the region \( 0 < x < L \) and \( 0 < y < B \) (i.e. the boundary conditions at \( x = 0, L \) are not satisfied), and assuming that the partial derivative \( f_x(x,y) \) can be expanded in a cosine-sine series then the coefficients can be related to the original series, \( f(x,y) \) by partial integration. Note that the first partial derivative of \( f(x,y) \) with respect to \( y \) can be done through term-by-term differentiation. Also, when \( f(x,y) \) is expanded in a sine-cosine series, a cosine-
cosine series, or a cosine-sine series \((0 < x < L, 0 < y < B)\) and correspondingly \(f_x(x,y)\) is expanded in a cosine-cosine series, a sine-cosine series or a sine-sine series. For example, consider the function \(u_x^{(f)}(x,y)\) (Eq. 4-29),

\[
\begin{align*}
  u_x^{(f)}(x,y) &= - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m \pi}{L} \cdot u_{mn} \cdot \sin \frac{m \pi x}{L} \cdot \sin \frac{n \pi y}{B} \\
                  &\quad (0 < x < L, 0 < y < B)
\end{align*}
\]

(D-4)

Assuming that \(u_{xx}^{(f)}(x,y)\) can be represented by a cosine-sine series,

\[
\begin{align*}
  u_{xx}^{(f)}(x,y) &= \sum_{m=0,1}^{\infty} \sum_{n=1,2}^{\infty} A_{mn}^{(f)} \cdot \cos \frac{m \pi x}{L} \cdot \sin \frac{n \pi y}{B} \\
                   &\quad (0 < x < L, 0 < y < B)
\end{align*}
\]

(D-5)

In order to determine the different terms of the series \(A_{mn}^{(f)}\),

For \(m = 0\)

\[
  \begin{align*}
    x &= L, \quad y = B \\
    &\int_0^{L} \int_0^{B} u_{xx}^{(f)}(x,y) \cdot \sin \frac{n \pi y}{B} \cdot dx \cdot dy \\
    &= \int_0^{L} \int_0^{B} \sin \frac{n \pi y}{B} \sum_{n=1,2} A_{0n}^{(f)} \sin \frac{n \pi y}{B} \, dx \, dy \\
    &= \sum_{n=1,2} A_{0n}^{(f)} \int_0^{L} \int_0^{B} \sin \frac{n \pi y}{B} \, dx \, dy
  \end{align*}
\]

\(n = 1,2\) (D-6)
Then
\[
A^{(f)}_{on} = \frac{2}{LB} \int_{0}^{y=B} \left[ u^{(f)}_x(L,y) - u^{(f)}_x(0,y) \right] \sin \frac{n \pi y}{B} \, dy
\]  

(D-7)

For \( m > 0 \)
\[
\sum_{m=1,2} \sum_{n=1,2} \int_{x=0}^{x=L} \int_{y=0}^{y=B} u^{(f)}_{xx}(x,y) \cos \frac{mnx}{L} \cdot \sin \frac{n \pi y}{B} \, dx \, dy
\]

\[
= \sum_{m=1,2} \sum_{n=1,2} \int_{x=0}^{x=L} \int_{y=0}^{y=B} A^{(f)}_m \cdot \cos^2 \frac{mnx}{L} \cdot \sin^2 \frac{n \pi y}{B} \, dx \, dy
\]

(D-8)

Then
\[
A^{(f)}_{mn} = \frac{4}{LB} \int_{0}^{y=B} \left| u^{(f)}_x(x,y) \right| \cos \frac{mnx}{L} \left| x=L \right| \sin \frac{n \pi y}{B} \, dy
\]

\[
+ \frac{4}{LB} \int_{0}^{y=B} \int_{x=0}^{x=L} u^{(f)}_x(x,y) \sin \frac{n \pi y}{B} \, dx \, dy
\]

\[
= \frac{m \pi}{L} \cdot \sin \frac{m \pi x}{L} \cdot \sin \frac{n \pi y}{B} \cdot dx \cdot dy
\]

(D-9)
From Eq. (D-4), 3

\[- \frac{m\pi}{L} \cdot \frac{u_{mn}}{\sin \frac{m\pi x}{L}} = \frac{4}{LB} \int_{x=0}^{x=L} \int_{y=0}^{y=B} u_{x}(x, y) \cdot \sin \frac{m\pi x}{L} \cdot \sin \frac{n\pi y}{B} \cdot dx \cdot dy\]  

(D-10)

Then

\[A_{mn}^{(f)} = \frac{4}{LB} \int_{y=0}^{y=B} \left[ u_{x}(x, y) \cdot \cos \frac{m\pi x}{L} \right]_{x=0}^{x=L} \sin \frac{n\pi y}{B} \cdot dy - \left( \frac{m\pi}{L} \right)^{2} \cdot \frac{u_{mn}}{\sin \frac{m\pi x}{L}}\]  

(D-11)

For even values of m

\[A_{mn}^{(f)} = \frac{4}{LB} \int_{y=0}^{y=B} \left[ u_{x}(L, y) - u_{x}(0, y) \right] \sin \frac{n\pi y}{B} \cdot dy - \left( \frac{m\pi}{L} \right)^{2} \cdot \frac{u_{mn}}{\sin \frac{m\pi x}{L}}\]  

(D-12)

For odd values of m

\[A_{mn}^{(f)} = -\frac{4}{LB} \int_{y=0}^{y=B} \left[ u_{x}(L, y) + u_{x}(0, y) \right] \sin \frac{n\pi y}{B} \cdot dy - \left( \frac{m\pi}{L} \right)^{2} \cdot \frac{u_{mn}}{\sin \frac{m\pi x}{L}}\]  

(D-13)
Setting

\[ a_n^{(f)} = \frac{-4}{LB} \int_{y=0}^{y=B} \left[ u_x^{(f)}(L,y) - u_x^{(f)}(0,y) \right] \sin \frac{n\pi y}{B} \, dy \]  

(D-14)

\[ b_n^{(f)} = \frac{4}{LB} \int_{y=0}^{y=B} \left[ u_x^{(f)}(L,y) + u_x^{(f)}(0,y) \right] \sin \frac{n\pi y}{B} \, dy \]  

(D-15)

Then

\[ A_{0n}^{(f)} = -\frac{a_n^{(f)}}{2} \quad \text{for } m = 0 \]  

(D-16a)

\[ A_{mn}^{(f)} = -a_n^{(f)} - \left( \frac{m\pi}{L} \right)^2 u_m^{(f)} \quad \text{for even values of } m, \ m \neq 0 \]  

(D-16b)

\[ A_{mn}^{(f)} = -b_n^{(f)} - \left( \frac{m\pi}{L} \right)^2 u_m^{(f)} \quad \text{for odd values of } m \]  

(D-16c)

Therefore, equation D-5 will have the following form:
\[ u_{xx}(x,y) = -\sum_{n=1,2}^\infty a_n^{(f)} \cdot \sin \frac{n\pi y}{B} - \sum_{m=2,4}^\infty \sum_{n=1,2}^\infty \left[ a_n^{(f)} + \left(\frac{mn}{L}\right)^2 u_{mn}^{(f)} \right] \cdot \cos \frac{mn L}{L} \cdot \sin \frac{n\pi Y}{B} \]

\[ = \sum_{m=1,3}^\infty \sum_{n=1,2}^\infty \left[ b_n^{(f)} + \left(\frac{mn}{L}\right)^2 u_{mn}^{(f)} \right] \cdot \cos \frac{mn L}{L} \cdot \sin \frac{n\pi Y}{B} \]

\[(0 \leq x \leq L, 0 \leq y \leq B) \quad (D-17)\]

Similarly for \( v_y, w_{xx} \) and \( w_{yy} \)

The series which result from the substitution of the appropriate partial derivatives of the displacement functions of sandwich plates with four clamped sides into the governing differential equations are given as follows (Refer to Eqs. 4-44):

\[ A_{1n} = -\frac{A_{11}^{(1)}}{2} \cdot a_n^{(1)} + \frac{B_{11}^{(1)}}{2} e_n. \quad (D-18a) \]

\[ A_{2n} = -A_{66}^{(1)} \cdot \left(\frac{n\pi}{B}\right)^2 - \frac{B_{55}}{t_c^2} \quad (D-18b) \]

\[ A_3 = \frac{B_{55}}{t_c^2} \quad (D-18c) \]

\[ B_n = a_n^{(1)} - B_{11}^{(1)} \cdot e_n + A_{11}^{(1)} \cdot b_n^{(1)} - B_{11}^{(1)} f_n \]

\[ \delta \quad (D-18c) \]
\[ C_{1n} = - \frac{A_{11}^{(2)}}{2} \cdot a_n^{(2)} + \frac{B_{11}^{(2)}}{2} \cdot e_n \]  

(D-18e)

\[ C_2 = \frac{B_{55}}{t_c^2} \]  

(D-18f)

\[ C_{3n} = - A_{66}^{(2)} \left( \frac{m\pi}{B} \right)^2 - \frac{B_{55}}{t_c^2} \]  

(D-18g)

\[ D_n = A_{11}^{(2)} \cdot a_n^{(2)} - B_{11}^{(2)} \cdot e_n + A_{11}^{(2)} \cdot b_n^{(2)} - B_{11}^{(2)} \cdot f_n \]  

(D-18h)

\[ E_{1m} = - \frac{A_{22}^{(1)}}{2} \cdot c_m^{(1)} + \frac{B_{22}^{(1)}}{2} \cdot g_m \]  

(D-18i)

\[ E_{2m} = - A_{66}^{(1)} \left( \frac{mn}{L} \right)^2 - \frac{B_{44}}{t_c^2} \]  

(D-18j)

\[ E_3 = \frac{B_{44}}{t_c^2} \]  

(D-18k)

\[ F_m = A_{22}^{(1)} \cdot c_m^{(1)} - B_{22}^{(1)} \cdot g_m + A_{22}^{(1)} \cdot d_m^{(1)} - B_{22}^{(1)} \cdot h_m \]  

(D-18l)

\[ G_{1m} = - \frac{A_{22}^{(2)}}{2} \cdot c_m^{(2)} + \frac{B_{22}^{(2)}}{2} \cdot g_m \]  

(D-18m)

\[ G_2 = \frac{B_{44}}{t_c^2} \]  

(D-18n)

\[ G_{3m} = - A_{66}^{(2)} \left( \frac{mn}{L} \right)^2 - \frac{B_{44}}{t_c^2} \]  

(D-18o)
\[ H_m = A_{22}^{(2)} \cdot c_m^{(2)} - B_{22}^{(2)} \cdot g_m + A_{22}^{(2)} \cdot d_m^{(2)} - B_{22}^{(2)} \cdot h_m \]

\[ I_{mn} = D_{11} \cdot \frac{m \pi}{L} \left[ \lambda_m \cdot e_n + \eta_m \cdot f_n \right] - B_{11}^{(1)} \cdot \frac{m \pi}{L} \left[ \lambda_m \cdot a_n^{(1)} + \eta_m \cdot b_n^{(1)} \right] + \lambda_m \cdot b_n^{(1)} \right] - B_{11}^{(2)} \cdot \frac{m \pi}{L} \left[ \lambda_m \cdot a_n^{(2)} + \eta_m \cdot b_n^{(2)} \right] + \frac{n \pi}{B} \left[ \gamma_n \cdot g_m + \beta_n \cdot h_m \right] - B_{22}^{(1)} \cdot \frac{n \pi}{B} \left[ \gamma_n \cdot c_m^{(1)} + \beta_n \cdot d_m^{(1)} \right] + \frac{n \pi}{B} \left[ \gamma_n \cdot c_m^{(2)} + \beta_n \cdot d_m^{(2)} \right] \]

By solving Eqs. 4-44 yields to the displacement parameters, \( u_{on}^{(f)}, u_{mn}^{(f)}, v_{mo}^{(f)}, v_{mn}^{(f)} \) and \( w_{mn} \) in terms of the coefficients \( a_n, b_n^{(f)}, \ldots, h_m \) (Eqs. D-18) and series functions given by the following expressions (Refer to Eqs. 4-48).

\[ IW = -g_{12} \cdot g_{34} \cdot L_1 / DN \]

\[ BW = [g_{25} \cdot g_{34} \cdot L_1 + g_{22} \cdot g_{34} \cdot L_w + g_{24} \cdot g_{34} \cdot K_w] / DN \]

\[ DW = -L_w \cdot g_{12} \cdot g_{34} / DN \]
\[ \text{FW} = [g_{45} \cdot g_{12} \cdot L_{1} + g_{12} \cdot g_{24} \cdot L_{w} \]
\[ + g_{12} \cdot g_{34} \cdot K_{w}] / \text{DN} \quad (D-19d) \]
\[ \text{HW} = - g_{12} \cdot g_{34} \cdot K_{w} / \text{DN} \quad (D-19e) \]

where
\[ L_{w} = g_{15} \cdot g_{12} \cdot g_{34} \cdot K_{2} - g_{25} \cdot g_{11} \cdot g_{34} \cdot K_{2} \]
\[ + g_{25} \cdot g_{13} \cdot g_{34} \cdot K_{1} - g_{35} \cdot g_{34} \cdot g_{12} \cdot K_{1} \]
\[ - g_{45} \cdot g_{13} \cdot g_{12} \cdot K_{2} + g_{45} \cdot g_{33} \cdot g_{12} \cdot K_{1} \]
\[ (D-20a) \]
\[ K_{w} = - g_{15} \cdot g_{12} \cdot g_{34} \cdot K_{5} + g_{25} \cdot g_{11} \cdot g_{34} \cdot K_{5} \]
\[ - g_{25} \cdot g_{13} \cdot g_{34} \cdot K_{4} + g_{35} \cdot g_{34} \cdot g_{12} \cdot K_{4} \]
\[ + g_{45} \cdot g_{13} \cdot g_{12} \cdot K_{5} - g_{45} \cdot g_{33} \cdot g_{12} \cdot K_{4} \]
\[ (D-20b) \]

\[ \text{BU}^{(1)} = [g_{24} \cdot g_{34} \cdot K_{5} - g_{22} \cdot g_{34} \cdot K_{2} \]
\[ + L_{2} \cdot \text{BW}] / L_{1} \quad (D-21a) \]

\[ \text{DU}^{(1)} = [g_{12} \cdot g_{34} \cdot K_{2} + L_{2} \cdot \text{DW}] / L_{1} \quad (D-21b) \]
\[ F_U(1) = [g_{12} \cdot g_{44} \cdot K_5 - g_{12} \cdot g_{24} \cdot K_2 \]
\[ + L_2 \cdot F_W] / L_1 \quad \text{(D-21c)} \]
\[ H_U(1) = [-g_{12} \cdot g_{34} \cdot K_5 + L_2 \cdot H_W] / L_1 \quad \text{(D-21d)} \]
\[ I_U(1) = L_2 \cdot F_{1W} / L_1 \quad \text{(D-21e)} \]
\[ B_V(1) = [g_{22} \cdot g_{34} \cdot K_1 - g_{24} \cdot g_{34} \cdot K_4 \]
\[ + L_3 \cdot B_W] / L_1 \quad \text{(D-22a)} \]
\[ D_V(1) = [-g_{12} \cdot g_{34} \cdot K_1 + L_3 \cdot D_W] / L_1 \quad \text{(D-22b)} \]
\[ F_V(1) = [g_{12} \cdot g_{24} \cdot K_1 - g_{12} \cdot g_{44} \cdot K_4 \]
\[ + L_3 \cdot F_W] / L_1 \quad \text{(D-22c)} \]
\[ H_V(1) = [g_{12} \cdot g_{34} \cdot K_4 + L_3 \cdot H_W] / L_1 \quad \text{(D-22d)} \]
\[ I_V(1) = L_3 \cdot I_W / L_1 \quad \text{(D-22e)} \]
\[ B_U(2) = -[1.0 + g_{11} \cdot B_U(1) + g_{13} \cdot B_V(1) \]
\[ + g_{15} \cdot B_W] / g_{12} \quad \text{(D-23a)} \]
\[ DU(2) = - \left[ g_{11} \cdot DU(1) + g_{13} \cdot DV(1) + g_{15} \cdot DW \right] / g_{12} \]  
(D-23b)

\[ FU(2) = - \left[ g_{11} \cdot FU(1) + g_{13} \cdot FV(1) + g_{15} \cdot FW \right] / g_{12} \]  
(D-23c)

\[ HV(2) = - \left[ g_{11} \cdot HV(1) + g_{13} \cdot HV(1) + g_{15} \cdot HW \right] / g_{12} \]  
(D-23d)

\[ IU(2) = - \left[ g_{11} \cdot IU(1) + g_{13} \cdot IV(1) + g_{15} \cdot IW \right] / g_{12} \]  
(D-23e)

\[ BV(2) = - \left[ g_{13} \cdot BU(1) + g_{33} \cdot BV(1) + g_{35} \cdot BW \right] / g_{34} \]  
(D-24a)

\[ DV(2) = - \left[ g_{13} \cdot DU(1) + g_{33} \cdot DV(1) + g_{35} \cdot DW \right] / g_{34} \]  
(D-24b)

\[ FV(2) = - \left[ 1.0 + g_{13} \cdot FU(1) + g_{33} \cdot FV(1) + g_{35} \cdot FW \right] / g_{34} \]  
(D-24c)

\[ HV(2) = - \left[ g_{13} \cdot HV(1) + g_{33} \cdot HV(1) + g_{35} \cdot HW \right] / g_{34} \]  
(D-24d)

\[ IV(2) = - \left[ g_{13} \cdot IU(1) + g_{33} \cdot IV(1) + g_{35} \cdot IW \right] / g_{34} \]  
(D-24e)
\[ AU^{(1)} = C_{3n} / \{ C_2 \cdot A_3 - C_{3n} \cdot A_{2n} \} \quad (D-25a) \]

\[ CU^{(1)} = -A_3 / \{ C_2 \cdot A_3 - C_{3n} \cdot A_{2n} \} \quad (D-25b) \]

\[ EV^{(1)} = G_{3m} / \{ G_2 \cdot E_3 - E_{2m} \cdot G_{3m} \} \quad (D-26a) \]

\[ GV^{(1)} = -E_3 / \{ G_2 \cdot E_3 - E_{2m} \cdot G_{3m} \} \quad (D-26b) \]

\[ AU^{(2)} = -[1.0 + A_{2n} \cdot AU^{(1)}] / A_3 \quad (D-27a) \]

\[ CU^{(2)} = -A_{2n} \cdot CU^{(1)} / A_3 \quad (D-27b) \]

\[ EV^{(2)} = -[1.0 + E_{2m} \cdot EV^{(1)}] / E_3 \quad (D-28a) \]

\[ GV^{(2)} = -E_{2m} \cdot GV^{(1)} / E_3 \quad (D-28b) \]
APPENDIX E

POTENTIAL OF THE APPLIED LOADS

E-1 Work Equivalent Loads for Rectangular Plates Under Transverse Loads

The work equivalent loads presented herein are for both thin and sandwich rectangular plates under uniformly distributed transverse load of intensity $q$. For such cases, the work done by the applied loads for the $i^{th}$ strip will have the following form:

$$ W(i) = \int_{x=0}^{L} \int_{y=0}^{b} q \cdot w(x,y) \, dx \, dy \quad (E-1) $$

(a) For Third Order Interpolation Polynomials

$$ W(i) = q \int_{x=0}^{L} \int_{y=0}^{b} \sum_{n=1,2} \sum_{j=1}^{N} \left[ H_{0j}^{(1)}(y) \cdot w_{j}^{(n)} ight. $$

$$ + H_{1j}^{(1)}(y) \cdot w_{yj}^{(n)} \right] \cdot x_{w}^{(n)} \cdot dx \cdot dy \quad (E-2) $$

The work expression for the $n^{th}$ cycle, $W(i)$ of the basic function can be written in the following form:

$$ W(i)^{(n)} = \left[ P_{SW}^{(n)} \right]^T \left[ D_{SW}^{(n)} \right] \quad (E-3) $$

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where
\[ \{D_w^{(n)}\}^T = \{w_1^{(n)}, w_2^{(n)}, w_{y1}^{(n)}, w_{y2}^{(n)}\} \]  
(E-4a)

\[ \{P_w^{(n)}\}^T = \{P_{S1}, P_{S2}, P_{Sy1}, P_{Sy2}\} \]  
(E-4b)

where, for example:

\[ P_{S1}^{(n)} = q \int_{x=0}^{x=L} \int_{y=0}^{y=b} H_{01}^{(1)}(y) \cdot x_w^{(n)} \quad dx \quad dy \]
\[ = \frac{q \cdot b}{2} \int_{x=0}^{x=L} x_w^{(n)} \quad dx \]  
(E-5a)

Similarly,

\[ P_{S2}^{(n)} = \frac{qb}{2} \int_{x=0}^{x=L} x_w^{(n)} \quad dx \]  
(E-5b)

\[ P_{Sy1}^{(n)} = \frac{qb^2}{12} \int_{x=0}^{x=L} x_w^{(n)} \quad dx \]  
(E-5c)

\[ P_{Sy2}^{(n)} = -\frac{qb^2}{12} \int_{x=0}^{x=L} x_w^{(n)} \quad dx \]  
(E-5d)
(a-1) Two Ends of The Strip are Simply Supported

\[
\int_{x=0}^{x=L} x_w^{(n)} \, dx = \left( \frac{L}{n\pi} \right) \left[ 1.0 - \cos (n\pi) \right]
\]

(E-6)

(a-2) Two Ends of The Strip are Clamped

\[
\int_{x=0}^{x=L} x_w^{(n)} \, dx = - \left( \frac{L}{\lambda_n} \right) \left[ \cos(\lambda_n) + \cosh(\lambda_n) \right]
\]

\[- 2.0 + \alpha_n \left[ \sin(\lambda_n) - \sinh(\lambda_n) \right]\]

(E-7a)

where

\[
\lambda_n = \frac{(2n + 1)\pi}{2^n} \quad ; \quad \alpha_n = \frac{\sin(\lambda_n) - \sinh(\lambda_n)}{\cos(\lambda_n) - \cosh(\lambda_n)}
\]

(E-7b)

(a-3) One End Clamped and The Other End Simply Supported

\[
\int_{x=0}^{x=L} x_w^{(n)} \, dx = - \left( \frac{L}{\lambda_n} \right) \left[ \cos(\lambda_n) + 1.0 \right]
\]

\[+ \beta_n \left[ \cosh(\lambda_n) - 1.0 \right]\]

(E-8a)

where

\[
\lambda_n = \frac{(4n + 1)\pi}{4^n} \quad ; \quad \beta_n = \frac{\sin(\lambda_n)}{\sinh(\lambda_n)}
\]

(E-8b)
For Fifth Order Interpolation Polynomials

\[ W(i) = q \int_{x=0}^{x=L} \int_{y=0}^{y=b} \sum_{n=1,2} \sum_{j=1}^{b} \left[ H_{0j}^{(2)}(y) \cdot w_j^{(n)} + H_{1j}^{(2)}(y) \cdot w_{Yj}^{(n)} + H_{2j}^{(2)}(y) \cdot w_{YYj}^{(n)} \right] x_w^{(n)} \quad (E-9). \]

\[ w_{w(i)}^{(n)} \mid \begin{bmatrix} P_{S_w}^{(n)} \\ D_w^{(n)} \end{bmatrix} \quad (E-10) \]

where

\[ \{D_w^{(n)}\}^T = \{w_1^{(n)}, w_2^{(n)}, w_{Y1}^{(n)}, w_{Y2}^{(n)}, w_{YY1}^{(n)}, w_{YY2}^{(n)}\} \quad (E-11a) \]

\[ \{P_{S_w}^{(n)}\}^T = \{P_{S1}^{(n)}, P_{S2}^{(n)}, P_{S_{Y1}}^{(n)}, P_{S_{Y2}}^{(n)}, P_{S_{YY1}}^{(n)}, P_{S_{YY2}}^{(n)}\} \quad (E-11b) \]

where, for example,

\[ p_{S1}^{(n)} = q \int_{x=0}^{x=L} \int_{y=0}^{y=b} H_{01}^{(2)}(y) \cdot x_w^{(n)} \cdot dx \, dy \]

\[ = q \cdot \frac{b}{2} \int_{x=0}^{x=L} x_w^{(n)} \, dx \quad (E-12a) \]

Similarly,
\[ p_{S_2}^{(n)} = \frac{q b}{2} \int_{x=0}^{x=L} x_{w}^{(n)} \, dx \]
\[ p_{S_y}^{(n)} = \frac{q b^2}{10} \int_{x=0}^{x=L} x_{w}^{(n)} \, dx \]
\[ p_{S_{y_2}}^{(n)} = -\frac{q b^2}{10} \int_{x=0}^{x=L} x_{w}^{(n)} \, dx \]
\[ p_{S_{y_1}}^{(n)} = \frac{q b^3}{120} \int_{x=0}^{x=L} x_{w}^{(n)} \, dx \]

(E-12b)

(E-12c)

(E-12d)

E-2 Work Equivalent Loads for Shell Roof Structures

The work equivalent load expressions presented herein are for both thin and sandwich simply supported cylindrical shell roof structures. The types of loads considered are the own weight of the shell (dead load) and a snow load (live load). A general work done expression for the \( m \)th shell strip under dead and live loads can be written in the following form:

\[ W_{(m)} = \int_{x=0}^{x=L} \int_{\theta=0}^{\theta=\Delta \theta} \left[ p_z \cdot w(x, \theta) + p_\phi \cdot v(x, \theta) \right] R \cdot d\theta \cdot dx \]

(E-13)

where \( p_z \) and \( p_\phi \) are the components of the applied load in the direction of the \( w(x, \theta) \) and \( v(x, \theta) \) displacements respectively (Fig. 23), and
\[ w(x, \theta) = \sum_{n=1,2}^{N} \sum_{j=1}^{2} \left[ H_{0j}^{(1)}(\theta) \cdot w_{j}^{(n)} + H_{1j}^{(1)}(\theta) \cdot w_{y_j}^{(n)} \right] \sin \frac{n\pi x}{L} \]  

\[ v(x, \theta) = \sum_{n=1,2}^{N} \sum_{j=1}^{2} \left[ H_{0j}^{(1)}(\theta) \cdot v_{j}^{(n)} + H_{1j}^{(1)}(\theta) \cdot v_{y_j}^{(n)} \right] \sin \frac{n\pi x}{L} \]  

(E-14)  

(E-15)  

and the third order interpolation polynomials in terms of \( \theta \) are as follows:

\[ H_{01}^{(1)}(\theta) = \frac{1}{\Delta \theta^3} \left[ 2\theta^3 - 3 \cdot \Delta \theta \cdot \theta^2 + \Delta \theta^3 \right] \]  

(E-16a)  

\[ H_{02}^{(1)}(\theta) = -\frac{1}{\Delta \theta^3} \left[ 2\theta^3 - 3\Delta \theta \cdot \theta^2 \right] \]  

(E-16b)  

\[ H_{11}^{(1)}(\theta) = \frac{R}{\Delta \theta^2} \left[ \theta^3 - 2\Delta \theta \cdot \theta^2 + \Delta \theta^2 \cdot \theta \right] \]  

(E-16c)  

\[ H_{12}^{(1)}(\theta) = -\frac{R}{\Delta \theta^2} \left[ \theta^3 - \Delta \theta \cdot \theta^2 \right] \]  

(E-16d)
Consider a cylindrical shell roof under dead load of intensity \( p_a \) (Fig. 2). The components of the load acting on the \( m \)th strip are as follows:

\[
\begin{align*}
P &= p_a \cos \phi_m \\
P &= p_a \sin \phi_m
\end{align*}
\]

The work done by the loads are:

\[
\begin{align*}
W(x, \theta) &= -p_a \int_0^x \int_0^{\theta_m} \cos(\phi_m - \phi) R \, d\theta \, dx \\
W(x, \theta) &= -p_a \int_0^x \int_0^{\theta_m} \text{cos}(\phi_m - \phi) R \, d\theta \, dx
\end{align*}
\]

\[
\begin{align*}
W &= I_1 + I_2 \\
W &= I_1 + I_2
\end{align*}
\]

\[
\begin{align*}
P &= -p_a \cos \phi_m \\
P &= -p_a \sin \phi_m
\end{align*}
\]

or

\[
\begin{align*}
P &= -p_a \cos \phi_m \\
P &= -p_a \sin \phi_m
\end{align*}
\]
\[ I_2 = \int_{x=0}^{x=L} \int_{\theta=0}^{\theta=\Delta \theta} P_\phi \cdot v(x, \theta) \cdot R \ d\theta \ dx \]

\[ = -P_d \int_{x=0}^{x=L} \int_{\theta=0}^{\theta=\Delta \theta} v(x, \theta) \cdot \sin(\phi_m - \theta) \cdot R \ d\theta \ dx \]

Let \( \theta = \Delta \theta \).

\[ QS_j = \int_{\theta=0}^{\theta=\Delta \theta} \theta^{j-1} \cdot \sin \theta \ d\theta \]

and

\[ QC_j = \int_{\theta=0}^{\theta=\Delta \theta} \theta^{j-1} \cdot \cos \theta \ d\theta \]

Where \( j = 1, 2, 3, 4 \).

Also, let

\[ z(n) = \frac{P_d \cdot R \cdot L \ [1.0 - \cos(n\pi)]}{(n\pi) \cdot \Delta \theta^3} \]

(E-22)

By performing the indicated integrations in Eqs. E-19 and E-20, the work done can be conveniently expressed for the \( n \)th cycle as:

\[ I_1^{(n)} = z(n) \left[ P_w^{(n)} \cdot w_1^{(n)} + P_w^{(n)} \cdot w_2^{(n)} + P_w^{(n)} \cdot w_1^{(n)} + P_w^{(n)} \cdot w_2^{(n)} \right] \]

(E-23)

\[ I_2^{(n)} = z(n) \left[ P_v^{(n)} \cdot v_1^{(n)} + P_v^{(n)} \cdot v_2^{(n)} + P_v^{(n)} \cdot v_1^{(n)} + P_v^{(n)} \cdot v_2^{(n)} \right] \]

(E-24)
where the work equivalent loads for the \( m \)th strip in

Eqs. E-23 and E-24 are as follows:

\[
P_{v1}^{(n)} = z^{(n)} \left[ \sin \phi_m \cdot H_1 - \cos \phi_m \cdot H_2 \right] \tag{E-25a}
\]

\[
P_{v2}^{(n)} = z^{(n)} \left[ -\sin \phi_m \cdot H_3 + \cos \phi_m \cdot H_4 \right] \tag{E-25b}
\]

\[
P_{vY1}^{(n)} = z^{(n)} \left[ \sin \phi_m \cdot H_5 - \cos \phi_m \cdot H_6 \right] \tag{E-25c}
\]

\[
P_{vY2}^{(n)} = z^{(n)} \left[ \sin \phi_m \cdot H_7 - \cos \phi_m \cdot H_8 \right] \tag{E-25d}
\]

\[
P_{w1}^{(n)} = z^{(n)} \left[ \cos \phi_m \cdot H_1 + \sin \phi_m \cdot H_2 \right] \tag{E-25e}
\]

\[
P_{w2}^{(n)} = -z^{(n)} \left[ \cos \phi_m \cdot H_3 + \sin \phi_m \cdot H_4 \right] \tag{E-25f}
\]

\[
P_{wY1}^{(n)} = z^{(n)} \left[ \cos \phi_m \cdot H_5 + \sin \phi_m \cdot H_6 \right] \tag{E-25g}
\]

\[
P_{wY2}^{(n)} = z^{(n)} \left[ \cos \phi_m \cdot H_7 + \sin \phi_m \cdot H_8 \right] \tag{E-25h}
\]

where

\[
H_1 = 2 \cdot QC_4 - 3\Delta \theta \cdot QC_3 + \Delta \theta^3 \cdot QC_1 \tag{E-26a}
\]

\[
H_2 = 2 \cdot QS_4 - 3\Delta \theta \cdot QS_3 + \Delta \theta^3 \cdot QS_1 \tag{E-26b}
\]

\[
H_3 = 2 \cdot QC_4 - 3\Delta \theta \cdot QC_3 \tag{E-26c}
\]

\[
H_4 = 2 \cdot QS_4 - 3\Delta \theta \cdot QS_3 \tag{E-26d}
\]
\[ H_5 = R[\Delta \theta \cdot QC_4 - 2 \Delta \theta^2 \cdot QC_3 + \Delta \theta^3 \cdot QC_2] \]  
(E-26e)

\[ H_6 = R[\Delta \theta \cdot QS_4 - 2 \Delta \theta^2 \cdot QS_3 + \Delta \theta^3 \cdot QS_2] \]  
(E-26f)

\[ H_7 = R[\Delta \theta \cdot QC_4 - \Delta \theta^2 \cdot QC_3] \]  
(E-26g)

\[ H_8 = R[\Delta \theta \cdot QS_4 - \Delta \theta^2 \cdot QS_3] \]  
(E-26h)

(b) **Simply Supported Cylindrical Shell Roof Under Snow Load**

Consider a cylindrical shell roof under snow load of intensity \( p_L \) (Fig. 24), the two components of the load acting on the \( m \)th strip are:

\[ P_z = -p_L \cos \phi \cdot \cos \phi \text{ and } P_\phi = -p_L \cos \phi \cdot \sin \phi \]  
(E-27a)

\[ p_z = -p_L \cos (\phi_m - \theta) \text{ and } P_\phi = -p_L \cos (\phi_m - \theta) \cdot \sin (\phi_m - \theta) \]  
(E-27b)

The work done by the loads are:

\[ W_{(m)} = I_1 + I_2 \]  
(E-28)

where
\[ I_1 = -P_L \int_{x=0}^{x=L} \int_{\theta=0}^{\theta=\Delta \theta} w(x, \theta) \cdot \cos^2(\phi_m - \theta) \cdot R \cdot d\theta \cdot dx \]  
\[ I_2 = -P_L \int_{x=0}^{x=L} \int_{\theta=0}^{\theta=\Delta \theta} v(x, \theta) \cdot \cos (\phi_m - \theta) \cdot \sin (\phi_m - \theta) \cdot R \cdot d\theta \cdot dx \]  
\[ \text{(E-29)} \]

\[ s_j = \int_{\theta=0}^{\theta=\Delta \theta} \theta^{j-1} \cdot \sin^2 \theta \cdot d\theta \]  
\[ \text{(E-31a)} \]

\[ c_j = \int_{\theta=0}^{\theta=\Delta \theta} \theta^{j-1} \cdot \cos^2 \theta \cdot d\theta \]  
\[ \text{(E-31b)} \]

\[ sc_j = \int_{\theta=0}^{\theta=\Delta \theta} \theta^{j-1} \sin \theta \cos \theta \cdot d\theta \quad \text{where} \quad j = 1, 2, 3, 4 \]  
\[ \text{(E-31c)} \]

Also let

\[ z(n) = \frac{P_L \cdot R \cdot L\ [1.0 \cdot \cos(n\pi)]}{(n\pi) \cdot \Delta \theta^3} \]  
\[ \text{(E-32)} \]

By performing the indicated integrations in Eqs. E-29 and E-30, the work done can be conveniently expressed for the \( n \)th cycle of the basic function as given in Eqs. E-23 and
E-24. For this case the work equivalent loads for the $m$th strip are as follows:

$$P_{v_1}^{(n)} = z(n) \left[ \frac{1}{2} \sin(2\phi_m) \left[ E_1 - E_5 \right] + \left[ \sin^2\phi_m - \cos^2\phi_m \right] \cdot E_9 \right]$$

(E-33a)

$$P_{v_2}^{(n)} = z(n) \left[ \frac{1}{2} \sin(2\phi_m) \left[ E_6 - E_2 \right] - \left[ \sin^2\phi_m - \cos^2\phi_m \right] \cdot E_{10} \right]$$

(E-33b)

$$P_{y_1}^{(n)} = z(n) \left[ \frac{1}{2} \sin(2\phi_m) \left[ E_7 - E_3 \right] + \left[ \sin^2\phi_m - \cos^2\phi_m \right] \cdot E_{11} \right]$$

(E-33c)

$$P_{y_2}^{(n)} = z(n) \left[ \frac{1}{2} \sin(2\phi_m) \left[ E_8 - E_4 \right] + \left[ \sin^2\phi_m - \cos^2\phi_m \right] \cdot E_{12} \right]$$

(E-33d)

$$P_{w_1}^{(n)} = z(n) \left[ \cos^2\phi_m \cdot E_1 + \sin^2\phi_m \cdot E_5 + \sin(2\phi_m) \cdot E_9 \right]$$

(E-33e)

$$P_{w_2}^{(n)} = z(n) \left[ \cos^2\phi_m \cdot E_2 + \sin^2\phi_m \cdot E_6 + \sin(2\phi_m) \cdot E_{10} \right]$$

(E-33f)
\[ P_{w_{Y_1}}^{(n)} = \tilde{z}^{(n)} \left[ \cos^2 \phi_m \cdot E_3 + \sin^2 \phi_m \cdot E_7 + \sin(2\phi_m) \cdot E_{11} \right] \]  

\[ P_{w_{Y_2}}^{(n)} = \tilde{z}^{(n)} \left[ \cos^2 \phi_m \cdot E_4 + \sin^2 \phi_m \cdot E_8 + \sin(2\phi_m) \cdot E_{12} \right] \]  

(E-33g)

(E-33h)

where

\[ E_1 = 2c_4 - 3\Delta\theta c_3 + \Delta\theta^2 c_1 \]  

(E-34a)

\[ E_2 = 2c_4 - 3\Delta\theta c_3 \]  

(E-34b)

\[ E_3 = R \Delta\theta \left[ c_4 - 2\Delta\theta c_3 + \Delta\theta^2 c_2 \right] \]  

(E-34c)

\[ E_4 = R \Delta\theta \left[ c_4 - \Delta\theta c_3 \right] \]  

(E-34d)

\[ E_5 = 2s_4 - 3\Delta\theta s_3 + \Delta\theta^2 s_1 \]  

(E-34e)

\[ E_6 = 2s_4 - 3\Delta\theta s_3 \]  

(E-34f)

\[ E_7 = R \Delta\theta \left[ s_4 - 2\Delta\theta s_3 + \Delta\theta^2 s_2 \right] \]  

(E-34g)

\[ E_8 = R \Delta\theta \left[ s_4 - \Delta\theta s_3 \right] \]  

(E-34h)

\[ E_9 = 2sc_4 - 3\Delta\theta sc_3 + \Delta\theta^3 sc_1 \]  

(E-34i)
\[ E_{10} = 2 \, sc_4 - 3 \, \Delta \theta \, sc_3 \]  

\[ E_{11} = R \, \Delta \theta \left[ sc_4 - 2 \, \Delta \theta \, sc_3 + \Delta \theta^2 \, sc_2 \right] \]  

\[ E_{12} = R \, \Delta \theta \left[ sc_4 - \Delta \theta \, sc_3 \right] \]
VITA AUCTORIS

1944 Ibrahim Mahfouz Mohamed Ibrahim was born in Cairo, Egypt, on February 1.

1966 Graduated from Ain Shams University, Cairo, Egypt, with degree of Bachelor of Science in Civil Engineering.

1966 Appointed as an instructor at the Faculty of Engineering, Ain Shams University.

1968 Enrolled at the University of Windsor, Ontario, Canada, in a program leading to the degree of Master of Applied Science in Civil Engineering.

1970 Obtained the degree of Master of Applied Science in Civil Engineering from the University of Windsor, Canada.

1970 Enrolled at the University of Windsor, in a program leading to the degree of Doctor of Philosophy in Civil Engineering.