CAUCHY-RIEMANN (CR)- SUBMANIFOLDS OF SEMI-RIEMANNIAN MANIFOLDS WITH APPLICATIONS TO RELATIVITY AND HYDRODYNAMICS.

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CAUCHY-RIEMANN, (CR) - SUBMANIFOLDS
OF SEMI-RIEMANNIAN MANIFOLDS
WITH APPLICATIONS TO
RELATIVITY AND HYDRODYNAMICS

by

Ramesh Sharma

A Dissertation
Submitted to the
Faculty of Graduate Studies and Research
through the department of
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of Doctor of Philosophy at
the University of Windsor

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ABSTRACT

Recently, there has been a keen interest of showing an interplay between definite and indefinite (in particular, Lorentzian) Riemannian geometries [Flaherty (1976), Duggal (1978, 86), Beem and Ehrlich (1981), O'Neil (1983) etc].

The objective of my dissertation is to present a few fresh ideas on this fruitful relationship, in reference to some applications in Relativity and Hydrodynamics. Our working spaces are the Cauchy-Riemann (CR)-submanifolds of a Hermitian manifold, introduced by Bejancu (1978). This choice is motivated by the fact that a CR-submanifold can be Lorentzian as opposed to a Hermitian manifold which, according to Flaherty (1976), cannot have Lorentzian signature. First part of the thesis is devoted to the characterization of Ricci-flat, Einstein and conformally flat space-times; followed by a few solutions of the Einstein Field equations. We have also shown the existence of an orthogonally transitive abelian isometry group which leads to the study on Killing Horizon [Carter (1969)].

In the second part, we present a few fresh ideas on the mutual interplay between the CR-structure and physical space-time with respect to the conformal geometry and its applications to
hydrodynamics. The conformal geometry is further related with a symmetry property called Conformal Collineation, which has the prospect of potential physical applications. In this respect, we have studied locally symmetric manifolds and classified the shape operator of pseudo-Einstein hypersurfaces in conformally flat space. Our work provides some more information on a recent result on singularity theorems (Beem and Ehrlich (1985)).

Finally, we improve a recent result of Herrera et al (1985) who showed that stiff equation of state is singled out if a special conformal motion is orthogonal to the 4-velocity of an isotropic fluid. We have avoided the stiff state by using special conformal collineation and generated some new solutions of isotropic/anisotropic fluids.
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असतो मा सत्यमपि,
tam slo ma jyotirmay,
mukhyam apust gajam
au shaanti, shaanti, shaanti: ||

(From unreality, take me into reality;
from darkness, take me into light;
from death, take me into immortality.)

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DEDICATED

To

THE LOTUS FEET

of

"SHREE SATYA SAI BABA"

And

My Parents

मंगलं गृह देवायं, मंगलं शान्त दायिने ।
मंगलं पवित्रायं, मंगलं सत्यसाइने ॥
॥ श्री जय साई राम ॥

(May the Divine Guru be auspicious;
may the Bestower of Wisdom be auspicious;
may the Lord who manifested in Parthi be auspicious;
may Bhagvan Sathya Sai Baba be auspicious to us.)
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CHAPTER 0

INTRODUCTION

Semi-Riemannian geometry is the study of a smooth manifold equipped with a non-degenerate metric of arbitrary signature. Its special cases are: Riemannian geometry with positive definite metrics, and Lorentzian geometry. For many years these two geometries have developed almost independently. But recently there has been a trend of showing an interplay between them (for example, Flaherty (1976) [37], Duggal (1978) [27], (1986) [26], Beem and Ehrlich (1981) [2], O'Neill (1983) [82]).

The objective of this dissertation is to present a few fresh ideas on this fruitful relationship, in reference to some applications in Relativity and Hydrodynamics. The mathematical tool used is the concept of Cauchy-Riemann (CR) submanifolds of a Hermitian manifold introduced by Bejancu (1978) [4].

The reason for studying CR-submanifolds is motivated by a theorem of Flaherty [37]: "A Hermitian metric on a complex manifold cannot have Lorentzian signature." Contrary to this, the metric induced on the CR-submanifold of a Hermitian manifold can have a Lorentzian signature as explained below:

Let $J$ be a complex structure tensor ($J^2 = -I$) on a
Hermitian manifold \( \mathbb{M} \). Consider a submanifold \( M \) isometrically embedded in \( \mathbb{M} \). \( M \) is called a CR-submanifold of \( \mathbb{M} \) if it has an invariant distribution \( D \) (\( JD = D \)); and an anti-invariant complementary orthogonal distribution \( D^\perp \). In this thesis, the restrictions of \( g \) to \( D \) and \( D^\perp \) are positive definite and indefinite respectively. Thus the metric on the entire submanifold is indefinite (in particular, Lorentzian).

The purpose of 1st chapter is to present the requisite notation and terminology of tensor analysis on manifolds, followed by brief aspects of semi-Riemannian manifolds and submanifolds. The concept of a CR-submanifold of a Hermitian manifold has also been reviewed.

In chapter II we develop the theory of CR-submanifolds of Kaehler and locally conformal Kaehler manifolds, with the view of its applicability to indefinite (in particular, Lorentzian) spaces, such as the physical space-time of relativity. Our study has been confined to totally umbilical, mixed foliate and normal mixed totally geodesic CR-submanifolds (Yano and Kon [111]).

Chapter III deals with the applications of CR-submanifolds considered in chapter II, within the framework of general relativity and Kaluza-Klein unified field theory [34,68]. We have characterized Ricci-flat, Einstein and conformally flat spaces. Two mathematical models of general relativistic space-time (one
having a scalar field and other having a non-singular electromagnetic field) have been constructed. A few solutions of the Einstein's field equations have been generated. Next, we have shown how a totally umbilical (dim $D^\perp = 1$) CR-submanifold $M$ of a Kaehler manifold can become the 5-dimensional Kaluza-Klein space-time and the Sasakian structure induced on $M$, can be associated with electromagnetic field and its potential vector field. The normal $f$-structure induced on normal mixed totally geodesic CR-submanifold (with flat normal connection) has been employed to generate a $q$-parameter orthogonally transitive Abelian isometry group. We have also shown the existence of a non-singular electromagnetic field in the above-mentioned CR-submanifolds. This result leads to the study on Killing Horizon, a concept of vital importance in general relativity (see Carter [18]).

As a recent excellent example of mutual interplay between the CR-structure and space-time geometry [86], we have presented in chapter IV a few fresh ideas on this fruitful relationship with respect to the conformal symmetries of semi-Riemannian (in particular, Lorentzian) manifolds. This opens the door for the study of a relatively new symmetry property called "Conformal Collineations".
Chapter V has been devoted to the study of conformal collineations (which includes motions, Homothetic motions, Affine collineations and Conformal motions as special cases). A non-trivial example of conformal collineation has been carved out by embedding technique. An interesting result has been established for a locally symmetric space admitting special conformal collineation (S.Conf C). Magid [72] classified the shape operators of Einstein hypersurfaces in a space-form. Beem and Ehrlich [3] deduced a singularity theorem by establishing strong energy and generic conditions of the space-time embedded in Minkowski space, assuming the shape operator to be diagonalizable. We have further studied this problem by classifying the shape operators of Einstein and Pseudo-Einstein hypersurfaces of a conformally flat space. It has been shown that the physical energy condition \( (\nu + p \geq 0, \nu = \text{energy density and } p = \text{pressure} \) of embedded perfect fluid is related to the diagonalizability of the shape operator. Thus our work throws some more light on a recent result on singularity theorems [3].

Chapter VI deals with some applications in Relativistic hydrodynamics. Recently, Herrera et al [55] studied the consequences of the existence of a one-parameter group of conformal motions in isotropic and anisotropic fluids. They showed that the stiff equation of state (pressure = density) is
singled out provided the special conformal motion is orthogonal to the fluid 4-velocity. We have shown that the stiff equation of state can be avoided by considering a special conformal collineation and also have generated new solutions for isotropic-anisotropic fluids.

In the appendices we have suggested a few interesting problems for further study.
CHAPTER 1

SEMI-RIEMANNIAN MANIFOLDS AND SUBMANIFOLDS

1. Notations and Terminology: Although we shall use mainly the coordinate-free notations in this thesis, nevertheless coordinates will be employed whenever they are convenient to work with. The reason for doing so is that the coordinate-free approach enjoys the invariant description of geometric objects. In general we shall use the following notations:

- $M$: A smooth manifold
- $X, Y, Z, W$: Arbitrary vector fields on $M$
- $\nabla$: Linear, (Affine) connection in $M$
- $d$: Exterior derivative operator
- $L$: Lie-derivative operator
- $[,]$: Lie-bracket
- $\wedge$: Exterior (Wedge) product
- $A$: Shape Operator

An $n$-dimensional smooth manifold is a Hausdorff topological space $M$ covered by a family of subsets $(U_\alpha)$ of $M$ such that $U_\alpha$ are
mapped homeomorphically onto open subsets of $\mathbb{R}^n$ by the family of maps $(s_{\alpha})$ and for $U_{\alpha} \cap U_{\beta}$ non-empty, $s_{\alpha} s_{\beta}^{-1} : s_{\alpha}(U_{\alpha} \cap U_{\beta}) \rightarrow s_{\alpha}(U_{\alpha} \cap U_{\beta})$ is a smooth map of an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$. Every $U_{\alpha}$ is a coordinate neighborhood with local coordinates $x^i$ ($i = 1$ to $n$) of a point $p$ in $U_{\alpha}$, given by $s_{\alpha}(p) = x = (x^i)$ in $\mathbb{R}^n$. Let $s_{\beta}(p) = y = (y^i)$ and $p$ be in $U_{\alpha} \cap U_{\beta}$. Then the transformation $x \mapsto y$ is smooth. If $(U_{\alpha})$ is the maximal covering then it is the set of all possible coordinate neighborhoods covering $M$. The lower half space $\mathbb{R}^n/2$ is that region for which $x \leq 0$. The boundary $\partial M$ of $M$ is the set of all points of $M$ whose image under a map $s_{\alpha}$ lies on the boundary of $\mathbb{R}^n/2$. $\partial M$ is an $(n-1)$-dimensional smooth manifold without boundary. For example, a unit 2-sphere $S^2$ in $\mathbb{R}^3$ is the boundary of the three-dimensional ball:

$$\{(x^1, x^2, x^3) : (x^1)^2 + (x^2)^2 + (x^3)^2 \leq 1\}.$$

A function $f$ on a smooth manifold $M$ is a map from $M$ to $\mathbb{R}$. $f$ is said to be smooth at a point $p$, if for $s_{\alpha}$ is a smooth function of the local coordinates at $p$ for any coordinate neighborhood $U_{\alpha}$ containing $p$. $f$ is said to be smooth on $M$ if it is so everywhere in $M$. Under the usual algebraic addition and multiplication, the collection of all smooth functions on $M$ forms a commutative ring denoted by $\mathcal{F}(M)$. A tangent vector to $M$ at $p$ is a real-valued function $X_p : \mathcal{F}(M) \rightarrow \mathbb{R}$, which obeys
(1) $\mathbb{R}$-linearity: $X_p(af + bg) = aX_p f + bX_p g$, 
(2) Leibnitz rule: $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$, 
for any real numbers $a$, $b$ and $f$, $g$ in $\mathcal{F}(M)$. The set of all tangent vectors at $p$ becomes a real vector space called the tangent space $T_p(M)$. If $(x^i)$ is a coordinate system in $M$ at $p$, then the vectors $(\partial_i)$ at $p$ form a basis for $T_p(M)$.

By a vector field $X$ on $M$ we mean a map that assigns a tangent vector $X_p$ to every point of $M$. For a vector field $X$ and a function $f$ we define $Xf$ in $\mathcal{F}(M)$ as: $(Xf)(p) = X_p(f)$ for all $p$ in $M$. Thus $X$ is called a smooth vector field if $Xf$ is smooth for every $f \in \mathcal{F}(M)$. Under the addition and scalar multiplication:

$(X + Y)_p = X_p + Y_p$, $(fX)_p = f(p)X_p$ for every $p$ in $M$; the set of all smooth vector fields constitute a module $\mathcal{F}(M)$ over the ring $\mathcal{F}(M)$. Obviously a basis of $\mathcal{F}(M)$ in a coordinate system $(x^i)$ is $(\partial_i)$.

The Lie-bracket of $X$ and $Y$ is a vector field $[X, Y]$ such that $[X, Y]f = X(Yf) - Y(Xf)$ and satisfies the following properties

(a) $[X, Y]$ is skew-symmetric and $\mathbb{R}$-linear.
(b) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$. (Jacobi identity)
(c) $[\partial_i, \partial_j] = 0$.

The union of all the tangent spaces $T_p(M)$ over all the points of $M$ is called tangent bundle $T(M)$ and can be rendered a smooth manifold structure by choosing coordinate systems of the type $(x^0, \ldots, x^m, x^1, \ldots, x^n)$, where $\pi : T(M) \to M$, $(x^i, \ldots, x^n)$
is a coordinate system in $M$ and $x^1, \ldots, x^n$ are the coordinates of the tangent vector field at a point $p$. Alternatively, $X$ in $\mathfrak{X}(M)$ is a smooth section of $T(M)$, i.e., a smooth map $X: M \to T(M)$ such that $\pi \circ X = \text{identity map}$.

An integral curve of a vector field $X$ is a curve $c: I \to M$, such that $dc/\,dt = X_{c(t)}$ for all $t$ in $I$.

The elements of dual module $\mathfrak{X}^*(M)$ of $\mathfrak{X}(M)$ are called 1-forms. Any 1-form can be written as a linear combination of the basis vectors $dx^i$ of $\mathfrak{X}^*(M)$. In fact $(dx^i)(\partial_j) = \delta^i_j$, makes $(dx^i)$ a basis of $\mathfrak{X}^*(M)$. A tensor field $T$ of type $(r,s)$ is an $\mathfrak{X}(M)$-multilinear function $T$ on the cartesian product of $\mathfrak{X}^*(M)$'s and their duals.

Before introducing the concept of Lie-derivative we shall introduce two well-known maps on manifolds. Let $\sigma: M_1 \to M_2$ be a smooth map. Then we have two induced maps:

$\sigma_*: T_p(M_1) \to T_{\sigma(p)}(M_2)$

$\sigma^*: \mathfrak{X}_{\sigma(p)}(M_2) \to \mathfrak{X}_p(M_1)$

such that $(\sigma^*f)(p) = f(\sigma(p))$ and $X(\sigma^*f)\big|_p = \sigma_*X(f)\big|_{\sigma(p)}$. The first map is the well-known Jacobian differential of $\sigma$. The second map is called the pull-back and can be extended straightaway to differential forms of any degree. One can easily verify that $\sigma^*$ commutes with $d$.

Consider a smooth vector field $X$ on $M$. There exists a unique
maximal curve \( c(t) \) through each point \( p \) of \( M \) such that \( c(0) = p \) and whose tangent vector at \( c(t) \) is \( X_c(t) \). This curve is called the integral curve of \( X \) through \( p \). In local coordinate system \((x^i)\), the integral curve \( x^i = x^i(t) \) is the unique solution of the system of differential equations:

\[
\frac{dx^i}{dt} = X^i
\]

where \( X^i \) are the components of the vector field \( X \). Thus the point \( p \) can be moved a parameter distance \( t \) along the integral curve of \( X \) through \( p \). This generates a one-parameter local group of diffeomorphisms \( \sigma_t : U \rightarrow M \), where \( U \) is an open neighborhood of \( p \).

In fact, \( \sigma_t \) maps a tensor field \( T \) of type \((r, s)\) at \( p \) into \( \sigma_{t*} T \) at \( \sigma_t(p) \). The Lie-derivative \( L_X T \) of \( T \) is defined as

\[
(L_X T)_p = \lim_{t \rightarrow 0} \frac{1}{t} \{ T_{\sigma_t(p)} - (\sigma_{t*} T)_p \}.
\]

Following Lie-derivatives will be frequently used (for the Lie-derivative of a general tensor field \( T \) and its properties we refer to [56]):

\[
L_X f = Xf, \quad L_X Y = [X, Y], \quad (L_X \alpha)(Y) = X(\alpha(Y)) - \alpha([X, Y])
\]

\[
(L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y)
\]

\[
(L_X [\nabla])(Y, Z) = L_X(\nabla_Y Z) - \nabla([X, Y], Z) - \nabla(Y, [X, Z])
\]

We shall also use the relation

\[
L_X = \text{div} X + \text{i}_X \omega, \text{ where } \text{i}_X \text{ is the interior product.}
\]

A distribution \( D \) on \( M \) is a mapping \( D : p \rightarrow D_p \) where \( p \) is a point of \( M \) and \( D_p \) is a \( q \)-dimensional subspace of \( T_p(M) \). \( D \) could be
defined in general, over a coordinate neighborhood \( U \) of \( M \),
instead of over the entire \( M \). In case, the subspaces \( D_p \)
determine a \( q \)-dimensional submanifold of \( M \) such that \( D_p \) may be
the tangent space of \( M \) at \( p \), we say that \( D \) is integrable or
involutive. A necessary and sufficient condition for the
integrability of \( D \) is that \( [X, Y] \) lies in \( D \) whenever \( X, Y \) lie in \( D \).
An integrable distribution is called a foliation and the induced
submanifold of \( M \) is called a leaf. A distribution \( D \) is said to be
parallel with respect to a connection \( \nabla \) on \( M \) if \( \nabla_Y X \) lies in \( D \) for
any \( X \) lying in \( D \) and any vector field \( Y \).

2. Semi-Riemannian Manifolds. The notion of semi-Riemannian
manifold is based on a special \((0, 2)\)-tensor on each \( T_p(M) \). A
bilinear form on \( T_p(M) \) is a map \( g_p : T_p(M) \times T_p(M) \to \mathbb{R} \).
Let \( g \) denote the tensor field that assigns \( g_p \) at \( p \). \( g \) is symmetric if \( g \)
\( (X, Y) = g(Y, X) \). \( g \) is said to be

1. positive definite if \( X \neq 0 \) \( g(X, X) > 0 \),
2. positive semi-definite if \( g(X, X) \geq 0 \) for all \( X \) in \( T(M) \)
3. non-degenerate if \( g(X, Y) = 0 \) for all \( Y \), \( X \neq 0 \).

If \( D_p \) is a subspace of \( T_p(M) \) then the restriction \( g_p \) \( (D_p \times D_p) \) is
denoted by \( g_p D_p \). The index of a symmetric bilinear form \( g_p \) on
\( T_p(M) \) is the dimension of the maximal subspace \( D_p \) of \( T_p(M) \) on
which \( g \) \( D_p \) is negative definite. A scalar product \( g_p \) on the
tangent space $T_p(M)$ is a non-degenerate symmetric bilinear form on $T_p(M)$. A metric tensor $g$ on a smooth manifold $M$ is a symmetric non-degenerate $(0,2)$ tensor field on $M$ of constant index. A smooth manifold $M$ furnished with a metric tensor $g$ is called a semi-Riemannian manifold. The common value $\nu$ of index $g_p$ on $M$ is called its index. In particular, for index $1$ ($n \geq 2$) $M$ is called a Lorentzian manifold.

Let $(x^i)$ be a coordinate system on an open coordinate neighborhood $U$ of $M$. The components of $g$ can be expressed by $g_{ij} = g(\partial_i, \partial_j)$. Hence we can write $g(X,Y) = g_{ij}x^i x^j$, for $X = x^i \partial_i$ and $Y = y^j \partial_j$. Therefore, $g = g_{ij}dx^i dx^j$.

The geometric significance of the index of a semi-Riemannian manifold is manifested by the following causal characters of a vector:

A tangent vector $X$ to $M$ is called

(1) Spacelike if $g(X,X) > 0$ or $X = 0$

(2) Null if $g(X,X) = 0$ and $X \neq 0$

(3) Timelike if $g(X,X) < 0$.

The set of all null vectors at a point $p$ of $M$ forms the nullcone at $p$. The causal character of a tangent vector is decided by the three alternatives mentioned above. The term causal has been adapted from relativity theory and therefore the null vectors are also cited as lightlike vectors.
The metric tensor $g$ defines a quadratic form $ds^2$ whose coordinate expression is $g_{ij}dx^i dx^j$. Now we state a result that is the backbone of semi-Riemannian geometry:

Given a semi-Riemannian manifold $M$, we can single out a unique symmetric connection $\nabla$ such that $\nabla g = 0$. More clearly, there is a unique connection $\nabla$ such that $\nabla_X Y - \nabla_Y X = [X, Y]$ and $X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$. $\nabla$ is called Levi-Civita connection of $M$ and is determined by the Koszul formula:

$$g(\nabla_X Y, Z) = \{X, g(Y, Z)\} + Y g(Z, X) - Z g(X, Y)$$
$$- g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) / 2$$

In the coordinate system $(x^i)$ the vector field $\nabla_{\partial_i} \partial_j$ can be spanned as $\Gamma^k_{ij} \partial_k$, where $\Gamma^k_{ij}$ are the Christoffel symbols defined by:

$$\Gamma^k_{ij} = g^{km} \partial_i \partial_j g_{jm} + \partial_j g_{im} - \partial_m g_{ij} / 2$$

A tensor field $T$ of type $(r, s)$ is said to be parallel if it is covariant constant.

The generalization of a straight line in a semi-Riemannian manifold $M$ is a geodesic defined by a curve $c : I \to M$ (I being any open or closed interval of $\mathbb{R}$) whose tangent vector field is parallel. In other words, a geodesic is an auto-parallel curve or a curve of zero acceleration. If $x^i = x^i(t)$ denotes a parametrization of the curve in a coordinate system $(x^i)$ then the curve is geodesic iff.
\[
d\omega^i = \frac{\partial}{\partial x^j}(dx^j)(dx^k)(dx^l) = 0.
\]

The gradient of a smooth function \( f \) on \( M \) is a vector field metrically equivalent to the differential 1-form \( df \). It is denoted by \( \text{grad} f \) such that \( g(\text{grad} f, X) = (df)(X) = Xf \). The divergence of \( X \) is denoted by \( \text{div} X = \sum_i g(\nabla_{e_i} X, e_i) \), where \( (e_i) \) is an orthonormal frame: \( C_1 = g(e_1, e_1) = 1 \) or \(-1\). The \( \text{div}(\text{grad} f) = g^{ij}\nabla_i \nabla_j f \) is called the Laplacian for Riemannian \( M \) and d'Alembertian for Lorentzian \( M \).

The curvature tensor of a semi-Riemannian manifold (known as Riemann-Christoffel curvature tensor) satisfies:
\[
\begin{align*}
g(R(X, Y)Z, W) &= -g(R(X, Y)W, Z) \\ g(R(X, Y)Z, W) &= g(R(Z, W)X, Y) \\ C_1^i R &= -C_2^i R, \ C_3^i R = 0.
\end{align*}
\]

The trace \( C_1^i R \) of \( R \) is the so-called Ricci tensor of \( M \) (denoted by \( \text{Ric} \)) is a \((0, 2)\)-symmetric tensor, i.e., \( \text{Ric}(X, Y) = \text{Ric}(Y, X) \).

The \((1, 1)\)-tensor \( Q \) defined by \( g(QX, Y) = \text{Ric}(X, Y) \), is called the Ricci tensor of type \((1, 1)\) and is self-adjoint. The scalar curvature denoted by \( r \) is the trace of Ricci map, viz. \( r = C_1^i Q \). The Einstein tensor \( G \) of \( M \) is defined as \( \text{Ric} = \frac{1}{2} \text{rg} \), which is divergence-free, as can be seen from the contracted Bianchi second identity:
\[ \text{div} Q = dr/2. \]

This plays a key role in the establishment of Einstein's Field Equations of General Relativity. The Weyl conformal curvature tensor \( C \) of \( M \) is the trace-free part of \( R \) and
defined by
\[ C(X,Y)Z = R(X,Y)Z - (1/n-2)Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX \]
\[ - g(X,Z)QY + \{r/(n-1)(n-2)\}[g(Y,Z)X - g(X,Z)Y], \]
where \( C \) satisfies all the symmetry and skew-symmetry properties of curvature tensor and it is trace-free. An important feature of \( C \) is that it is conformally invariant.

A semi-Riemannian manifold is Einstein iff \( Ric = (r/n)g \), where \( r \) is constant for \( n > 2 \). \( M \) is conformally flat iff \( C = 0 \) (for \( n > 3 \)). \( M \) is recurrent iff \( \nabla R = \alpha \otimes R \) for some 1-form \( \alpha \) on \( M \).

In particular, when \( \alpha = 0 \), \( M \) is called locally symmetric. A locally symmetric \( M \) is symmetric if it is connected and complete. \( M \) would be called Ricci-symmetric if \( \nabla Ric = 0 \). A locally symmetric \( M \) is obviously Ricci-symmetric. \( M \) is called conformally recurrent iff \( \nabla C = \alpha \otimes C \). \( M \) would be called conformally symmetric iff \( \alpha = 0 \). A recurrent (locally symmetric) \( M \) is conformally recurrent (conformally symmetric) but the converse may not hold. The Riemannian curvature tensor \( R \) is rather complicated. In order to determine \( R \) completely we have a simpler notion, viz. sectional curvature. A 2-dimensional subspace \( P \) of the tangent space \( T_p(M) \) is called a tangent plane to \( M \) at \( p \). The plane \( P \) is said to be non-degenerate if \( g(X,X)g(Y,Y) - g(X,Y)^2 \neq 0 \) for \( X,Y \) in \( P \), otherwise is degenerate. The quotient \( g(R(X,Y)X,Y)/[g(X,X)g(Y,Y) - g(X,Y)^2] \)
for \(X, Y\) in a non-degenerate plane \(P\), is independent of the choice of \(X, Y\) spanning \(P\) and is called the sectional curvature \(K(P)\) of \(P\). Hence the sectional curvature \(K\) of \(M\) is a real valued function on the set of all non-degenerate tangent planes to \(M\). It can be shown that \(K\) determines \(R\) completely. A semi-Riemannian manifold \(M\) is said to be of constant curvature if its sectional curvature function is constant. In this case we have

\[
R(X, Y)Z = K[g(X, Z)Y - g(Y, Z)X]
\]

where \(K\) = constant = sectional curvature. Harris[53] has defined the sectional curvature of \(M\) with respect to a null (degenerate) tangent plane as follows: Let \(P\) be a degenerate tangent plane to \(M\), containing a null vector \(N\). The null sectional curvature with respect to \(N\) of the plane \(P\) is:

\[
K_N(P) = g(R(X, N)N, X)/g(X, X)
\]

where \(X\) is any non-null vector in \(P\). Here \(K_N(P)\) is independent of the choice of \(X\) in \(P\), but it does depend quadratically on \(N\).

Therefore, it is best to restrict attention to a set of 'normalized' null vectors which contains exactly one representative for each null direction. Let \(U\) be a timelike unit vector field on \(M\). The null congruence associated with \(U\) is the set of null vectors. On a semi-Riemannian product \(M \times N\) of semi-Riemannian manifolds \(M\) and \(N\) the metric tensor is \(\tau_M^*(g_M) + \tau_N^*(g_N)\), where \(\tau_M\) and \(\tau_N\) are the projections of \(M \times N\) onto \(M\) and
Let $f$ be a smooth function on $M$. The warped product $M \times _f N$ of $M$ and $N$, with $f$, is the product manifold $M \times N$ furnished with the metric:

$$g = TM^* (g_M) + (f_{TM})^2 TN^* (g_N)$$

Locally, if $(x^i)$ is a coordinate system on $M$ and $(y^a)$ is a coordinate system on $N$, then the line-element of $M \times _f N$ is:

$$ds^2 = g_{ij} dx^i dx^j + f^2 g_{ab} dy^a dy^b$$

$M$ is called the base of $M \times _f N$ and $N$ the fibre. The general relativity theory has many exact solutions as the warped products of two semi-Riemannian manifolds. For a detailed account of the geometry of $M \times _f N$ in terms of the warping function $f$ and individual geometries of $M$ and $N$ we refer to O'Neil [82].

Let us now see how the subspaces of the tangent spaces of semi-Riemannian and Riemannian manifolds differ from each other.

Consider the Schwarzschild half-plane defined as the region:

$r > 2M$ in the $t-r$ plane, with metric $ds^2 = -h dt^2 + h^{-1} dr^2$, where $h = 1-(2M/r)$ and $M$ is a constant. This defines a Lorentzian scalar product $g = -h dt \otimes dt + h dr \otimes dr$. There are two linearly independent null vectors $\partial_t + h \partial_r$ and $\partial_t - h \partial_r$. These are two straight lines in the $t-r$ plane, inclined at an angle of $45^\circ$ with either axis.

When we say $X$ and $Y$ are orthogonal, we mean that $g(X,Y) = 0$.

Two distributions of $M$ are said to be orthogonal if each vector
of one is orthogonal to each vector of the other. In a Riemannian manifold we picture two orthogonal vectors as two lines perpendicular to each other. For a semi-Riemannian (non-Riemannian) manifolds this picture does not fit at all. The reason could be illustrated through the fact that a null vector is orthogonal to itself. Let $D$ be a distribution of $M$ and $D^\perp$ be orthogonal to $D$. This is justified by the fact that a null ray in Schwarzschild half-plane is the orthogonal subspace of itself.

The following results can be verified easily:

1. $\dim D_p + \dim D^\perp_p = \dim T_p(M) = n$.
2. $(D^\perp_p)\perp = D_p$. The fact that $g_p$ is non-degenerate on the whole tangent space $T_p(M)$ is equivalent to $T_p(M)^\perp = 0$ (the zero subspace).

A distribution $D$ of $M$ is called non-degenerate if $g|_D$ is non-degenerate. For a Riemannian manifold, any distribution $D$ is non-degenerate. But for an indefinite $g$ there do exist degenerate distributions, e.g. a null vector field spans a light ray which is degenerate distribution. A subspace of a scalar product space may be degenerate. A characterization of non-degenerate distribution is given as: A distribution $D$ of $T(M)$ is non-degenerate iff. $T(M) = D \oplus D^\perp$. A scalar product space always has an orthonormal basis $(e_1)$ such that $g(e_1, e_1) = c_1 = 1$ or $-1$. For any orthonormal basis $(e_1)$ of $T(M)$ the number of negative signs in $(c_1, \ldots, c_n)$ is the index of $M$. 
3. Isometric Embedding Of Semi-Riemannian Manifolds:
A smooth manifold $M$ is said to be a submanifold of a smooth manifold $\bar{M}$ if (i) $M$ is a topological subspace of $\bar{M}$ and (ii) the inclusion map $j : M \subset \bar{M}$ is smooth and at each point $p$ of $M$ its differential map $j_\ast : T_p(M) \to T_j(p)(\bar{M})$ is one-to-one. In particular, when $j$ is one-to-one such that the induced map $M \to j(M)$ is a homeomorphism onto the subspace $j(M)$ of $\bar{M}$, we say that $j$ is an embedding, and $M$ is an embedded submanifold of $\bar{M}$. By submanifold we shall always mean an embedded submanifold. A vector field $\bar{x}$ on $\bar{M}$ is said to be tangent to $M$ if $\bar{x}_p$ lies in $T_p(M)$ for all $p$ of $M$. In fact $T_p(M)$ is regarded as a subspace of $T_p(\bar{M})$. If a vector field $\bar{x}$ on $\bar{M}$ is tangent to $M$, then its restriction $x$ to $M$ is a vector field on $M$. Furthermore, if a vector field $\bar{y}$ on $\bar{M}$ is also tangent to $M$, then $[\bar{x}, \bar{y}]$ is tangent to $M$ and $[\bar{x}, \bar{y}]/M = [\bar{x}/M, \bar{y}/M]$. The co-dimension of $M$ in $\bar{M}$ is the difference between the dimensions of $\bar{M}$ and $M$. In particular, if the co-dimension of $M$ in $\bar{M}$ is one, we say that $M$ is a hypersurface of $\bar{M}$. Let $c$ be the value of the function $f$ on $\bar{M}$. If at each point of $f^{-1}(c) = \{p \in \bar{M} : f(p) = c\}$ the differential $df_p$ is non-zero, then $f^{-1}(c)$ is called a level hypersurface of $f$.

Our study will be mainly focussed on isometrically embedded submanifolds, i.e. $j^*g$ equals the restriction of $g$ to $M$. Although
they agree on this measurement, nevertheless $M$ appears different to observers within and outside $M$. This will be accounted for by the introduction of the second fundamental form or the shape tensor.

Before we proceed ahead we would like to define three categories of submanifolds: A submanifold $M$ of $\bar{M}$ is said to be spacelike, timelike or null (degenerate) according as the induced metric $J^*g$ on $M$ is definite, indefinite or degenerate. In fact, the semi-Riemannian submanifolds of $M$ are either space-like or time-like (i.e. non-degenerate). Null submanifolds are not semi-Riemannian. The causal characters of submanifolds are preserved by isometries and conformal transformations.

Henceforth we will consider only semi-Riemannian submanifolds, unless otherwise specified. Thus, $T_p(M)$ at each point $p$ of $M$ is a non-degenerate subspace of $T_p(\bar{M})$, and the following decomposition holds:

$$T_p(\bar{M}) = T_p(M) \oplus T_p(M)^\perp$$

The subspace $T_p(M)^\perp$ is also non-degenerate. Vectors of $T_p(M)$ and $T_p(M)^\perp$ are respectively referred to as tangent and normal vectors to $M$.

The Levi-Civita connection $\nabla$ of $\bar{M}$ gives rise to an induced Levi-Civita connection $\nabla$ of $M$. Arbitrary tangent vector fields to $M$ will be denoted by $X, Y, Z$ and normal vector fields to $M$ by $V, N$.
given by

\[ \nabla_X N = (\nabla_X N)^\perp \]

The basic equations for the submanifold are:

\[ \nabla_X Y = \nabla_X Y + B(X, Y) \quad \text{(GAUSS FORMULA)} \]

\[ \nabla_X N = - \Delta N + \nabla_X N \quad \text{(WEINGARTEN FORMULA)} \]

where \( B: T(M) \times T(M) \to T(M)^\perp \) and \( \Delta N: T(M) \to T(M) \) are the second fundamental form and shape operator of \( M \). They are related by

\[ g(B(X, Y), N) = g(\Delta N X, Y) \]

A normal vector field \( N \) on \( M \) is said to be parallel in the normal bundle, if \( \nabla_X N = 0 \). The submanifold \( M \) is called totally geodesic if \( B = 0 \) (equivalently \( \Delta N = 0 \)). \( M \) is called umbilical with respect to \( N \) a normal vector field \( N \) if \( \Delta N = \alpha I \), for some function \( \alpha \) on \( M \). If \( M \) is umbilical with respect to any normal vector field then it is said to be totally umbilical. For an orthonormal frame \( (e_1) \) of \( M \), the mean curvature vector \( \mu \) of \( M \)

\[ \mu = (\text{Tr}.B)/n = (1/n) \sum C_1 B(e_1, e_1) \]

where \( g(e_1, e_1) = C_1 \) and \( n \) is the dimension of \( M \). Actually speaking, \( \mu \) is independent of the choice of the frame \( (e_1) \). In particular, when \( \mu = 0 \) we say \( M \) is extremal \([82]\). It is quiet plain that a submanifold \( M \) which is extremal and totally umbilical is totally geodesic. In fact, \( \Delta N = \alpha I \) implies \( \alpha = g(\mu, N) \); which proves the assertion.

The characteristic equations of Gauss, Codazzi and Ricci are

\[ g(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + g(B(X, Z), B(Y, W)) - g(B(Y, Z), B(X, W)) \]
\[ [\mathcal{R}(X, Y)Z]^\perp = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \]
\[ g(\mathcal{R}(X, Y)V, N) = g(\mathcal{R}^+(X, Y)V, N) + g([\mathcal{A}_N, \mathcal{A}_V]X, Y) \]
respectively, where the covariant derivative \( \nabla_X B \) is given by

\[(3.1) \quad (\nabla_X B)(Y, Z) = D_X B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z) \]

and the curvature tensor \( \mathcal{R}^+ \) of the normal connection \( D \) is defined as

\[ \mathcal{R}^+(X, Y)N = D_X D_Y N - D_Y D_X N - D_{[X, Y]} N \]

The second fundamental form is said to be parallel if \( \nabla_X B = 0 \).

The preceding formalism is not valid for degenerate submanifolds.

The theory of degenerate submanifolds has drawn considerable attention recently \([91, 92]\). Particularly important is the case of null hypersurfaces of space-time \([13, 100]\).

By the embedding of space-time, physicists understand the local or global isometric embedding of the space-time into a flat space. But, as there is no reason for choosing a flat embedding space, we can consider ambient spaces such as (i) Ricci-Flat spaces, (ii) Conformally flat spaces and (iii) Spaces of constant curvature. The following two theorems specify the minimal semi-Riemannian manifolds wherein their submanifolds can be locally (Physics is usually a local affair) embedded:

1. (FRIEDMAN) A semi-Riemannian manifold \( \mathcal{M}^n(p, q) \) (\( p \) and \( q \) denote the numbers of timelike and spacelike eigenvectors of the metric tensor, respectively) with analytic metric can be locally,
isometrically and analytically embedded in $\mathbb{M}(r,s)$ where $m \geq \frac{n(n+1)}{2}$, $r \geq p$, $s \geq q$. [42]

2. (GREENE) A semi-Riemannian manifold $\mathbb{M}(p,q)$ with a smooth metric can be locally, isometrically and smoothly embedded in $\mathbb{M}(r,s)$ where $m \geq \frac{n(n+3)}{2}$ and $r \geq p$, $s \geq q$. [42]

The first theorem demands the embedding space for the 4-dimensional space-time to be of minimum dimension 10 and the second one (which assumes weaker conditions) demands it to be of minimum dimension 14. In many interesting cases, the dimension of the flat embedding space is much lower than the values demanded by the above two theorems.

It is also well known [17] that a semi-Riemannian manifold M with analytic metric can be locally, isometrically and analytically embedded into a certain unique Ricci-flat $\mathbb{M}^{n+1}$. The embedding class of a semi-Riemannian manifold is its codimension determined by its embedding into a specified embedding space.

The global isometric embedding of semi-Riemannian manifolds deserves more attention when we view the singularity as a boundary of the space-time manifold. Singularities are unavoidable in cosmological models and models of gravitational collapse. Clarke [21] has proved: A $C^\infty$ semi-Riemannian manifold $\mathbb{M}^n$ with $C^k$ Riemannian metric ($k \geq 3$) of rank $r$ and signature $s$
can be globally and isometrically embedded in $\Sigma^m(p, q)$, where $m = p+q$, $p \geq n - (r + s)/2 + 1$ and $q \geq (n/2)(3n + 11)$ for a compact $M^n$; $q \geq (n/6)(2n^2 + 37) + (5 n^2/2) + 1$ for non-compact $M^n$. For non-degenerate metrics with Lorentz signature of $M$ we conclude from the above theorem that $p \geq 2$, $m = q + 2$. Thus two timelike directions in $\mathcal{E}^q + 2$ are adequate to accommodate any space-time globally. For the 4-dimensional space-time, in particular we need the embedding space $\mathcal{E}^9$.

4. Complex, Contact and f-structures

An almost complex structure on a smooth manifold $M$ is a $(1,1)$-tensor field $J$ such that $J^2 = -I$ ($I$ denotes the identity operator on the tangent space at each point). $M$ furnished with almost complex structure is called an almost complex manifold. Such a manifold is even-dimensional and orientable. The Nijenhuis' tensor $[J, J]$ of $J$ is a $(1,2)$-tensor field given by


for arbitrary vector fields $X$ and $Y$ on $M$. An almost complex structure is integrable (i.e. induced from the complex structure of a complex manifold) iff $[J, J]$ vanishes identically. An almost complex manifold is called an almost Hermitian manifold if there exists a semi-Riemannian metric $g$ on $M$ such that $g(JX, JY) = g(X, Y)$. A Hermitian manifold is an almost Hermitian manifold
whose underlying almost complex structure is integrable. A Kaehler manifold is a Hermitian manifold \( M \) whose complex structure \( J \) is parallel with respect to the Levi-Civita connection of \( g \). The holomorphic section of a Kaehler manifold \( M \) is a section obtained by a plane element spanned by a non-null tangent vector \( X \) at a point and \( JX \). The sectional curvature of \( M \) with respect to a holomorphic section \( (X, JX) \) is called holomorphic sectional curvature. A Kaehler manifold is said to be a complex space-form if its holomorphic sectional curvature is independent of the direction of \( X \) and is same at every point of \( M \). A complex space-form with constant holomorphic sectional curvature \( c \) is denoted \( M(c) \) whose curvature tensor is given by

\[
\mathbf{R}(X,Y)Z = c\left[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\right]
\]

A smooth manifold \( M \) is called an almost contact manifold if there exists on \( M \) a tensor field \( \varphi \), a vector field \( \xi \) and a 1-form \( \eta \) such that \( \varphi^2 = -I + \eta \otimes \xi \), \( \varphi \xi = 0 \), \( \eta \varphi = 0 \), \( \eta(\xi) = 1 \). Then \( M \) is odd-dimensional. The almost contact structure is said to be normal if \( \{\varphi, \varphi\} + d\eta \otimes \xi = 0 \).

An almost contact manifold is said to be an almost contact metric manifold if it has a semi-Riemannian metric \( g \) such that

\[
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)
\]

In particular, if \( d\eta(X, Y) = g(\varphi X, Y) \) then an almost contact metric manifold is called a contact metric manifold. More
specifically, if the structure of a contact metric manifold is normal, then it is called a normal contact metric (Sasakian) manifold. In a Sasakian manifold the following properties hold:

$$\nabla_X \xi = s_X, \ (\nabla_X s)Y = -g(X, Y)\xi + \eta(Y)X = R(X, \xi)Y$$

$$R(X, Y)Z = -s(R(X, Y)sZ) - g(Y, Z)X - g(X, Z)Y - g(sY, Z)sX + g(sX, Z)sY.$$

Yano [109] introduced a generalisation of an almost complex and an almost contact structure called the $f$-structure of rank $r$. This structure is defined by a non-null $(1,1)$ tensor field $f$ such that $f^3 + f = 0$. M has a pair of complementary distributions $D$ and $D^\perp$ defined by their projection operators $-f^2$ and $f^2 + I$ respectively. Obviously, $r = \dim D$. For $r = n$ it reduces to an almost complex structure and for $r = n-1$ it is an almost contact structure.

5. Cauchy-Riemann (CR) Submanifolds

Let $M$ be an almost Hermitian manifold with almost complex structure $J$ and Hermitian metric $g$. By a CR-submanifold [4] of $\tilde{M}$ we mean a non-degenerate submanifold $M$ which admits a smooth distribution $D$: $x \to D_x \subset T_x(M)$ such that:

1. $D$ is invariant, $JD_x = D_x$ for each $x$ in $M$.
2. The orthogonal complementary distribution $D^\perp$ is anti-invariant, i.e., $JD_x^\perp \subset T_x(M)^\perp$ for each $x$ in $M$.

Remark: The above definition would not be possible in case
of degenerate $D$, due to the fact that $D$ and $D^\perp$ are not necessarily complementary for degenerate $D$.

A CR-structure on a smooth manifold $M$ is a complex sub-bundle $H$ of the complexified tangent bundle $CT(M)$ of $M$ such that $(H \cap \overline{H})_x = \{0\}$ and $H$ is involutive, i.e. for $X, Y$ in $H$, $[X, Y]$ is in $H$. It is known that on a CR-manifold there exist a real distribution $D$ and a field of endomorphisms $p: D \to D$ such that $p^2 = -I$, $D = \text{Re}(H \cdot \overline{H})$ and $H_x = \{X - iFX : X \in D_x\}$. The following theorem justifies the name of CR-submanifolds:

Theorem (Blair and Chen [11]). Let $M$ be a CR-submanifold of a Hermitian manifold $\bar{M}$. If $M$ is a proper submanifold then $M$ is a CR-manifold.

In particular, if the holomorphic distribution $D$ is same as $T(M)$ then $M$ reduces to an invariant submanifold of $\bar{M}$, and if the totally real distribution $D^\perp$ is same as $T(M)$ then $M$ reduces to a totally real submanifold of $\bar{M}$. For $\dim D^\perp = \dim T(M)^\perp$, $M$ reduces to a generic submanifold of $\bar{M}$.

The transvects of a tangential vector field $X$ and a normal vector field $V$ by the almost complex structure tensor $J$ decompose into their tangential and normal parts as

\((5.1)\) \hspace{1cm} (a) \hspace{0.5cm} JX = PX + FX, \hspace{0.5cm} (b) \hspace{0.5cm} JV = tV + fV\)

It therefore follows that $g(FX, V) + g(X, tV) = 0$.

Operating $J$ to both sides of (5.1)a and (5.1)b, we get
(5.2) \[ p^2 = -I - tf, \quad FP + ff = 0 \]

(5.3) \[ Pt + tf = 0, \quad f^2 = -I - Ft \]

Then it follows that \( FP = 0 \). Thus (5.2) implies \( ff = 0 \), and hence \( tf = 0 \). Therefore from (5.3) we get \( Pt = 0 \). Operating \( P \) on (5.2) and using the last relation we find that \( P \) defines an \( f \)-structure.

(5.4) \[ p^3 + P = 0 \]

Likewise equation (5.3) and \( ff = 0 \) implies an \( f \)-structure:

(5.5) \[ f^3 + f = 0, \]

in the normal bundle. It is known that a submanifold of an almost Hermitian manifold is a CR-submanifold iff \( FP = 0 \).
CHAPTER II

CR-SUBMANIFOLDS

1. Motivation and Basic Results

According to the theorem of Flaherty [37], we know that the Hermitian metric on an almost complex manifold cannot have the Lorentzian signature i.e., index 1 which is essential for a physical space-time manifold of relativity. Moreover, for a four dimensional space-time we can choose a coordinate system comprising two real coordinates x, y and complex conjugate nullcoordinates z + it and z - it. These two facts suggest that a complex structure can be defined only on its two-dimensional submanifold: x = const., y = const. With this motivation and the purpose of applying our results in relativity theory, we consider a class of submanifolds of a Hermitian manifold (in particular, a Kaehler manifold and locally conformal Kaehler manifold) such that there may be complementary complex and real distributions. One of the setting for such a distribution can be provided by singling out Holomorphic distributions of the CR-submanifolds (see for example, Penrose [86]).

As pointed out in chapter I (Sec. 5) we have an f-structure on the CR-submanifold and accordingly the Lorentzian signature can
be assigned to the metric induced on the CR-submanifold. In fact our study is not only applicable within the framework of general relativity, but in the theory of semi-Riemannian manifolds whose metrics have signatures compatible with the f-structure. Such a study is consistent with the current trend of interplay between definite and indefinite Riemannian geometries [2, 37, 29, 82].

The covariant derivatives of the operators P, F, t and f are defined along the CR-submanifold as

\begin{align}
(1.1) \quad (\nabla_X P)Y &= \nabla_X (PY) - P(\nabla_X Y) \\
(1.2) \quad (\nabla_X F)Y &= D_X (FY) - F(\nabla_X Y) \\
(1.3) \quad (\nabla_X t)V &= \nabla_X (tV) - t(D_X V) \\
(1.4) \quad (\nabla_X f)V &= D_X (fV) - f(D_X V)
\end{align}

The Kaehlerian condition $\nabla J = 0$, the Gauss and Weingarten formulae provide the following expressions:

\begin{align}
(1.5) \quad (\nabla_X P)Y &= A_{FYX} + \gamma B(X, Y) \\
(1.6) \quad (\nabla_X F)Y &= B(X, PY) + fB(X, Y) \\
(1.7) \quad (\nabla_X t)V &= A_{FYX} - PA_Y X \\
(1.8) \quad (\nabla_X f)V &= -FA_Y X - B(X, tV)
\end{align}

The following lemma (Yano and Kon [iii], page 90) will be used:

Lemma 1.1. Let $M$ be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then, for any vector fields $X$ and $Y$ in $D^\perp$ we have

\begin{equation}
(1.9) \quad A_{FXY} = A_{FYX}.
\end{equation}

We also mention two known theorems for our further use:
Theorem 1.1 (Yano and Kon [111]). Let M be a CR-submanifold of a Kaehler manifold M. Then the totally real distribution D^± is integrable and its leaves are totally real submanifolds of M.

Definition 1.1. The f-structure induced on the CR-submanifold of a Kaehler manifold is said to be partially integrable if D is integrable and the almost complex structure induced on each leaf of D is integrable.

Theorem 1.2 (Yano and Kon [111]). Let M be a CR-submanifold of a Kaehler manifold M. Then the f-structure induced on M is partially integrable iff:

\[(1.10) \quad B(\mathbf{F}X, \mathbf{Y}) = B(X, \mathbf{PY})\]

for all X and Y in D.

Definition 1.2. A CR-submanifold of a Kaehler manifold is said to be mixed totally geodesic if B(X, Y) = 0 for any X in D and Y in D^±.


This section deals with a subclass of mixed totally geodesic CR-submanifolds characterized by the partial integrability of f-structure induced on them.

Definition 2.1. A CR-submanifold of a Kaehler manifold is called mixed foliate if it is mixed totally geodesic and the f-structure induced on it is partially integrable.
Lemma 2.1. Let $M$ be a mixed foliate CR-submanifold of a Kaehler manifold $\bar{M}$. Then, for all $\nu$ in $T(M) \perp$ we have:

$$A\nu P + P\nu = 0.$$  

(2.1)

The following known theorem [7] holds for a positive definite Kaehler metric.

Theorem 2.1 If $M$ is a mixed foliate proper CR-submanifold of a complex space-form $\bar{M}(c)$, then $c \leq 0$.

We would like to see what sort of constraint is imposed on the possible values of $c$ when the metric of $\bar{M}(c)$ is indefinite. For this purpose, we consider a particular case of this general situation in the following:

Theorem 2.2 If $M$ is a mixed foliate proper CR-submanifold of a complex space-form $\bar{M}(c)$ such that the metric $g|D$ is definite and $g|D^\perp$ is indefinite, then $c = 0$.

Proof: The curvature tensor of $\bar{M}(c)$ is given by eqn. (4.1) of chapter I. Restricting the vector fields $X, Y$ to $D$ and $Z$ to $D^\perp$ we observe that:

$$\{R(X, Y)Z\} \perp = (c/2) g(\nu Y, X)JZ.$$  

(2.2)

Now equation (3.1) [chapter I] gives

$$(\nabla_X B)(Y, Z) = -B(\nabla_Y X, Z) - B(Y, \nabla_X Z)$$

since $M$ is mixed totally geodesic. Interchanging $X$ and $Y$ in the last equation and subtracting we get

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = -B([X, Y], Z) - B(Y, \nabla_X Z) + B(X, \nabla_Y Z)$$
By our hypothesis, \( M \) is mixed foliate and hence, by the integrability of \( D \), we obtain:

\[
\]

As \( Z \) is any vector field in \( D^\perp \), there is a normal vector field \( V \) such that \( Z = JV \). Therefore \( Z = tv \) and \( tV = 0 \). Thus we have:

\[
\nabla V = (\nabla t)V + tDvV = - PAvY + tDvV,
\]

where use has been made of eqn. (1.7). Now lemma 2.1 shows that:

\[
(2.3) \quad (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = B(X, AvPY) + B(PY, AvX)
\]

Using eqns. (2.2) and (2.3) in the Codazzi equation provides:

\[
(\frac{c}{2}) g(PY, X)JZ = B(X, AvPY) + B(PY, AvX)
\]

Substituting \( X = PY \) and replacing \( Z \) by \( JV \) we obtain:

\[
(2.4) \quad g(AvPY, AvPY) = - (\frac{c}{4}) g(PY, PY) g(V, V)
\]

By hypothesis, \( g|D \) is definite. Moreover, from lemma 2.1 we have \( AvPY = - PAvY \). Employing these facts in eqn. (2.4) we obtain the inequality:

\[
 cg(V, V) \leq 0, \quad \text{that is,} \quad cg(Z, Z) \leq 0 \quad \text{for any} \quad Z \in D^\perp.
\]

Again by hypothesis \( g|D^\perp \) is indefinite and therefore \( D^\perp \) does contain at least one spacelike vector field \( Z_1 \) and a timelike vector \( Z_2 \). Consequently \( c \leq 0 \) and \( c \geq 0 \). Hence, \( c = 0 \). This completes the proof.

**Corollary:** Under the hypothesis of theorem 2.2, we have:

\[
(2.5) \quad AvP = 0, \quad \text{for every} \quad V \in JD^\perp.
\]

**Proof:** It follows from eqn. (2.4) and the conclusion \( c = 0 \) of
Theorem 2.2 that \( g(AvPY, AvPY) = 0 \). Since \( AvP = -PAv \), the vector field \( AvPY \) lies in \( D \). The hypothesis, \( g \mid D \) is definite implies that (2.5) holds.

Remark. Chen[20] has proved that "A CR-submanifold of \( C^1 \) is mixed foliate iff it is a CR-product, i.e. the product of the leaves of \( D \) and \( D^\perp \)." I have verified that this result is valid for both definite and indefinite metrics. Employing it for the mixed foliate CR-submanifold \( M \) under the hypothesis of Theorem 2.2, it follows straightaway that \( M \) is a CR-product. We will independently (i.e., without using Chen's theorem) prove this important result in another way to gain more insight into the structure of \( M \). First, we establish the following lemma:

Lemma 2.2 A necessary and sufficient condition for the integrability of the \( f \)-structure of a mixed foliate proper CR-submanifold \( M \) of a Kaehler manifold \( \tilde{M} \) is:

\[ AvPY = 0, \text{ for any vector field } Y \text{ tangent to } M. \]

Proof: We know that a proper mixed foliate CR-submanifold \( M \) of a Kaehler manifold \( \tilde{M} \) has a partially integrable \( f \)-structure and integrable \( D^\perp \). The \( f \)-structure would be completely integrable if its Nijenhuis' tensor \( [P, P] \) vanishes identically (Sec. 4, Chapter I), i.e.

\[ [P_X, PY] + P^2(X, Y) - P[PY, X] - P[X, PY] = 0. \]

For \( X, Y \) in \( D \) the integrability of \( D \) implies \( [P, P](X, Y) = 0 \). For
X, Y in D̅ implies [P, P](X, Y) = 0.

Finally, if X is in D and Y in D̅ then we observe that:

\[ [P, P](X, Y) = (\nabla_X P) Y - (\nabla_Y P) X - P((\nabla_X P) Y - (\nabla_Y P) X) \]

\[ = A_{XY} X - P A_{XY} X = 2 A_{XY} X, \]

where we have used (1.5) and (2.1). So, for integrability of the f-structure on M, it is necessary and sufficient that A_{XY} = 0, or equivalently, A_{XY} = 0 for any Y in JD̅.

**Definition 2.2** The f-structure induced on the CR-submanifold of a Kaehler manifold is said to be normal if the following (1, 2)-tensor field S vanishes:

\[ S(X, Y) = [P, P](X, Y) - t[:(\nabla_X P) Y - (\nabla_Y P) X]. \]

It has been shown in [11] that the normality of the f-structure induced in the CR-submanifold of a Kaehler manifold is equivalent to A_{XY} = P A_{XY}, for any vector field X tangent to M. This holds for a definite and an indefinite metric. The following result characterizes the intrinsic structure of M hypothesized as in theorem 2.2.

**Theorem 2.3** Under the hypothesis of theorem 2.2 the f-structure induced on M is integrable and normal. Moreover, if D̅ is parallelizable then M is locally a CR-product M_T X M⊥, where M_T is flat and M⊥ is a totally geodesic real submanifold of M.

**Proof:** From eqn. (2.4) and the consequence \( c = 0 \) of thm. 2.2
we conclude that \( A_Y P = 0 \). Hence lemma 2.2 asserts that the f-structure on \( M \) is integrable. Now the necessary and sufficient condition for the normality of the f-structure on \( M \) is \( P A Y = A_Y P \) for any \( Y \) in \( JD^\perp \). This is automatically satisfied since we have \( A_Y P = 0 \) [cor. to thm. (2.2) and lemma 2.1]. Hence the structure is also normal. It can be shown with the aid of eqn. (1.5) that the fundamental 2-form \( \Omega \) of the f-structure is closed. It therefore follows from the work of Goldberg [44] that \( (\nabla_X P) Y = 0 \) for \( X \) in \( D \). The expression (1.5) for \( (\nabla_X P) Y \) ensures that it lies in \( D^\perp \) (clear from the result \( P A_Y X = -A_Y P X = 0 \) so that \( A_Y X \) lies in \( D^\perp \)).

Next, from equation (1.5) and lemma 1.1 we have \((\nabla_X P) Y = (\nabla_Y P) X\) for all \( X, Y \) in \( D^\perp \). Therefore, \( g((\nabla_X P) Y, Z) = g((\nabla_Y P) X, Z) \) for all \( X, Y \) in \( D^\perp \) and \( Z \) in \( T(M) \). This means, \( (\nabla_X \Omega) (Y, Z) = -(\nabla_Y \Omega)(Z, X) \) whence we get \( (\nabla_Z \Omega)(X, Y) = 0 \). Hence \( (\nabla_Z P) X \) lies in \( D \), but as shown earlier, \( (\nabla_Z P) X \) lies in \( D^\perp \) for any \( Z \) and \( X \) tangent to \( M \). We had also proved that \( (\nabla_Z P) X = 0 \) whenever \( X \) is in \( D \) and \( Z \) tangent to \( M \). Eventually, we obtain \( (\nabla_Z P) X = 0 \) for any \( Z \) and \( X \) tangent to \( M \), i.e. \( \nabla P = 0 \). Thus according to Chen's theorem [20] "A CR-submanifold of a Kaehler manifold is a CR-product iff. \( \nabla P = 0 \)" (Proved also by Bejancu), \( M \) turns out to be \( M_T \times M^\perp \); where \( M_T \) is a leaf of \( D \) totally geodesic in \( M \) and \( M^\perp \) is a leaf of \( D \) totally geodesic in \( M \). It follows that \( M_T \) is flat. This completes the proof.
Proposition 2.1. Under the hypothesis of theorem 2.2, if $D^\perp$ is parallelizable and the normal connection is flat then $M$ is locally flat.

Proof: Since $D^\perp$ is parallelizable, we can choose an orthonormal base $(\xi_a)$ of $D^\perp$. If $(\eta^a)$ denotes its dual then one can show that $FX = \eta^a(X)J\xi_a$ and $tJ\xi_a = -\xi_a$. Hence

$$S(X,Y) = Np(X,Y) - t((-\nabla_X Y) - (\nabla_Y X))$$

Therefore we get

$$d\eta^a(X,Y)\xi_a - \eta^a(Y)tD_X J\xi_a + \eta^a(X)tD_Y J\xi_a = 0$$

since the $f$-structure is integrable and normal. But, as the normal connection is flat, we get $d\eta^a = 0$. For such a structure we know from Blair [10] that $L_{\xi_a}g = 0$. Consequently, $\nabla\xi_a = 0$ and $R(X,Y)\xi_a = 0$, i.e. $M^\perp$ is locally flat. Hence $M$ is locally flat.

Note: I am very grateful to Professor David E. Blair for his extremely significant help in understanding the normality condition of the $f$-structure induced on the CR-submanifold of a Kaehler manifold, in terms of the framed $f$-structure.

Remark: If $M$ of proposition 2.1 were complete and the $f$-structure globally framed then $M$ would be flat and hence could represent the Minkowski space-time of special relativity [54], when the dimension of $M$ is 4.

Here we consider a class of CR-submanifolds of a Kaehler manifold, which are mixed totally geodesic with distribution \( D \) not necessarily integrable (unlike that of a mixed foliate CR-submanifold) and the \( f \)-structure induced on \( M \) is normal.

Definition 3.1 A CR-submanifold \( M \) of a Kaehler manifold \( \tilde{M} \) is said to be normal mixed totally geodesic if it is mixed totally geodesic and the \( f \)-structure induced on \( M \) is normal.

Theorem 3.1. Let \( M \) be a normal mixed totally geodesic CR-submanifold of a complex space-form \( \tilde{M}(c) \). Then:

1. if \( g \) and \( W = Av^2 + AFAv_Z \) (\( V \) any vector field of \( JD^L \) and \( Z = JV \)) are positive definite on \( D \), then \( c \geq 0 \) and
   
2. if \( g \) is positive definite on \( D \) and indefinite on \( D^L \), then \( W \) cannot be definite on \( D \). Also, \( c = 0 \) iff \( W = 0 \) on \( D \).

Proof: Supposing \( X, Y \) to be in \( D \), using Codazzi equation and the expression for the curvature tensor of \( \tilde{M}(c) \) we can show

\[
B(Y, PAvX) - B(X, PAvY) - B([X,Y], Z) = (c/2)g(PY, X)JZ
\]

where \( Z = JV \) is in \( D^L \). Taking its scalar product with \( V \) we get

\[
(3.1) \quad g(AvY, PAvX) - g(AvX, PAvY) - g(AvZ, [X,Y]) = -cg(PY,X)g(V,V)/2
\]

It can be shown that \( B(PX, Y) - B(X, PY) = F(Y, X) \Rightarrow g(AvPX - PAvX, Y) = g([X,Y], Z) \). Putting \( X = PY \) in eqn. (3.1) gives:

\[
(3.2) \quad g(WY, Y) = (c/2)g(V,V)g(Y,Y)
\]

where we have used the normality condition \( AvP = PAv \). If \( g \) and \( W \)
(as defined in theorem 3.1) are positive definite on \( D \) then (3.2) implies that \( c \geq 0 \), which proves part (1). Let \( W \) be definite on \( D \). If \( g \) is positive definite on \( D \) and indefinite on \( D^\perp \) then (3.2) implies \( cg(V, V) = cg(Z, Z) \geq 0 \). Now \( Z \) being in \( D^\perp \) could be space-like or time-like. Hence \( c = 0 \) and therefore the operator \( W = 0 \) on \( D \), which contradicts our hypothesis that \( W \) is definite. The last part of (2) follows from (3.2).

Remark 1: In view of eqn. (3.2) and Chen's theorem [20], "A CR-submanifold of a Kaehler manifold is a CR-product iff \( AvP = 0 \), for any \( V \) in \( JD^+ \)" we observe that if \( M \) of theorem 3.1 is CR-product then \( c = 0 \). Thus for \( M \) to be a non CR-product, \( c \) must not be zero.

Remark 2: Part (1) of the theorem can be compared with theorem 2.1, where we concluded \( c = 0 \). This contrast arises because the CR-submanifold in theorem 2.1 was mixed foliate whereas in the present theorem it is normal mixed totally geodesic. Certainly the mixed totally geodesic feature is common to both the theorems. However, the main distinction lies in the integrability of the distribution \( D \). In theorem 2.1, \( D \) was integrable, whereas in theorem 3.1 it is not. Note that if we assume further in theorem 3.1 that \( D \) is integrable then, in virtue of theorem 2.1 we would have ended up with \( c = 0 \). Thus, eqn. (2.4) would provide \( AvP = 0 \) which satisfies eqn. (3.2). But
AvP = 0 would then (for a framed \( f \)-structure on \( M \)) imply that \( M \) is a CR-product.

Remark 3: For the case where \( g \) is definite on \( D \) and indefinite on \( D^c \) we compare theorem 2.2 and part (2) of theorem 3.1. As a consequence of theorem 2.2, \( M \) reduces to a CR-product provided the \( f \)-structure on it is framed. On the other hand theorem 3.1 involves the operator \( W \) on \( D \). The condition that \( c \) may vanish, is that \( W \) must vanish identically on \( D \). This is quite compatible with the consequence of theorem 2.2 in that if we assume that \( M \) of theorem 3.1 (part(2)) is a CR-product then we must have \( Av \) vanishes on \( D \) and hence the operator \( W \) vanishes on \( D \), thus reducing \( c \) to 0. Hence we claim to have got a wider class of CR-submanifolds, as hypothesized in part(2) of theorem 3.1, which can be embedded in \( C^n \). In fact, this class does accommodate non CR-product submanifolds of \( C^n \).

Remark 4: If we relax the condition that the normal connection of \( M \) is flat in the hypothesis of the preceding theorem, we could show using Goldberg's theorem [44], that \( VP = 0 \), locally. Hence \( M \) would be locally decomposable as \( M^T \times M^L \); where \( M^T \) is still locally flat but \( M^L \) is not necessarily so.

In this section we shall characterize totally umbilical CR-submanifolds of a Kaehler manifold. The dimension of the totally real distribution \( D^\perp \) will be denoted by \( q \). We mention the following basic theorem first given by Bejancu [6] for a positive definite metric:

Theorem 4.1 Let \( M \) be a totally umbilical proper CR-submanifold of a Kaehler manifold \( M \). For \( q > 1 \), \( M \) reduces to a totally geodesic submanifold and is locally a Riemannian product of an invariant and an anti-invariant submanifold of \( M \).

We shall prove below that this theorem is still valid for semi-Riemannian metrics. The proof is slightly longer than that for the Riemannian metrics:

Proof: By lemma 1.1, we have \( A\mu X = A\mu Y \) for all \( X, Y \) in \( D^\perp \). As \( t\mu \) belongs to \( D^\perp \), for any \( X \) in \( D^\perp \) we have \( A\mu X t\mu = A F t\mu X \). As \( M \) is totally umbilical, for any \( X, Y \) tangent to \( M \) we have \( \nabla(X, Y) = g(X, Y)\mu \). Therefore, \( A\mu X = g(\mu, V)X \). Hence \( g(\mu, FX) t\mu = g(\mu, Ft\mu) X \)

which implies

\[
(4.1) \quad g(t\mu, X) t\mu = g(t\mu, t\mu) X, \text{ for all } X \in D^\perp.
\]

Since \( q > 1 \), it follows on contraction of (4.1) at \( X \) with respect to an orthonormal base of \( D^\perp \) that \( g(t\mu, t\mu) = 0 \). Putting this into (4.1) yields \( g(t\mu, X) t\mu = 0 \). Thus \( t\mu = 0 \).
Now let $X$ be any vector field tangent to $M$. Then

$$(\nabla_X t)_\mu = \nabla_X t_\mu - t D_X t_\mu = -t D_X t_\mu.$$  

Using (1.7) in the above equation we obtain

$$-t D_X t_\mu = A_{\mu}^\nu X_\nu - P\mu_\nu X_\nu = -g(\mu, \nu)PX_\mu t_\nu$$

Using $P t = 0$, gives $g(\mu, \nu)P^2 X_\mu = 0$. $M$ is proper CR-submanifold and hence $D = \{0\}$. Thus, $g(\mu, \nu) = 0$. Further, $(\nabla_X P)Y = A_{\mu}^\nu X_\nu + t_B(X, Y) = -g(Y, t_\nu) = 0$. Thus we get $\nabla_X P = 0$.

This implies, through Chen's result [20], that $M$ is locally a product of an invariant submanifold $M^T$ and an anti-invariant submanifold $M^\perp$ of $M$. What remains to be proved is that $\nu = 0$.

Suppose $Y$ is in $D$ so that $FY = 0$. As $D$ is parallel, $\nabla X Y$ is in $D$ and therefore $F(\nabla X Y) = 0$. Consequently, $(\nabla_X F)Y = 0$ and using (1.6) we obtain $g(X, FY)_\mu = g(X, Y)F_\mu$, for every $Y$ in $D$. Substituting $X = FY$ and noting $g(FX, Y) = -g(FY, X)$ we find that $\nu = 0$. As a result $B = 0$. Hence $M$ is totally geodesic and is locally a CR-product of the leaves of $D$ and $D^\perp$. This proves the theorem.

The case $q = 1$ was not covered in the preceding theorem. Chen [19] proved the following theorem for $q = 1$:

**Theorem:** Let $M$ be a totally umbilical CR-submanifold of a Kaehler manifold $\tilde{M}$. Then

(i) $M$ is totally geodesic, or

(ii) $q = 1$, or

(iii) $M$ is totally real.
Note that if M were a proper CR-submanifold in the above theorem, then the possibility (iii) would be ruled out.

Theorem 4.2 Let M be a proper totally umbilical CR-submanifold of a Kähler manifold M with q = 1. Suppose the mean curvature vector \( \nu \) is non-vanishing over M. Then \( \nu \) does have a non-zero component in \( JD^- \) and it lies entirely in \( JD^- \). Moreover, the following statements are equivalent:

1. M is a normal contact metric (Sasakian) manifold.
2. \( \nu \) has a non-zero constant norm.
3. \( \nu \) is parallel in the normal bundle.
4. second fundamental form of M is parallel.

Proof: Suppose \( \nu \) has no non-zero component in \( JD^- \). Then we have \( t_N = 0 \). Proceeding exactly as in the proof of theorem 4.1 we can show that M is totally geodesic and hence by Chen's theorem stated above theorem 4.2, we arrive at a contradiction. Therefore \( \nu \) does have a non-zero component in \( JD^- \). Thus \( t_N \neq 0 \) and \( \nu \) lies entirely in \( JD^- \). Total umbilicity of M implies that it is mixed geodesic. Using the Bejancu's [5] result: "If M is a mixed totally geodesic CR-submanifold of a Kähler manifold then \( A_{J\nu}X = J\nu X \) for any \( X \) in D and \( V \) in the complement of \( JD^- \) in \( T(M)^- \).", we find \( g(\nu, JV) = 0 \). As the complement is a holomorphic subbundle of \( T(M)^- \) and \( V \) is any vector field in it we conclude that \( g(\nu, V) = 0 \). Thus \( \nu \) lies entirely in \( JD^- \) and \( f_N = 0 \). Now since \( q = 1 \), all
other vectors of \( D^\perp \) are scalar multiples of \( t_\mu \). For any \( X \) tangent to \( M \) we can show that

\[
g(\nu, \nu)P^2X = g(t_\nu, X)t_\mu - g(t_\nu, t_\mu)X.
\]

Operating \( P \) on this gives

\[
(4.2) \quad g(\nu, \nu) = g(t_\nu, t_\mu)
\]

Hence we obtain

\[
(4.3) \quad g(\nu, \nu)(P^2X + X) = g(t_\nu, X)t_\mu
\]

In this case too, equation (4.1) holds, which shows (as \( q = 1 \)) that \( g(\nu, \nu) \neq 0 \) and hence \( g(t_\nu, t_\mu) \neq 0 \). Equation (4.3) becomes

\[
(4.4) \quad P^2X = -X + [g(t_\nu, t_\mu)]^{-1}g(t_\nu, X)t_\mu
\]

Under the setting \( \xi = -[g(t_\nu, t_\mu)]^{-1/2}t_\mu \) and \( \eta \) as the dual 1-form of \( \xi \), equation (4.4) takes the form

\[
(4.5) \quad P^2X = -X + \eta(X)\xi
\]

It can be deduced that \( P\xi = 0 \), \( \eta_0P = 0 \), \( \text{rank}(P) = n-1 \) and

\[
(4.6) \quad g(PX, PY) = g(X, Y) - \eta(X)\eta(Y)
\]

With the help of equations (1.5), (4.4) and

\[
\nabla_X(t_\mu) - tD_\mu = -g(\nu, \mu)PX
\]

we derive the following equations:

\[
(4.7) \quad (\nabla_XP)Y = [g(\nu, \nu)]^{1/2} [g(\xi, Y)X - g(X, Y)\xi]
\]

\[
(4.8) \quad \nabla_X\xi = [g(\nu, \nu)]^{1/2} PX
\]

Eqns. (4.5)-(4.8) show that \( M \) is a Sasakian manifold [28] iff. \( g(\nu, \nu) \) is a non-zero constant. This proves the equivalence of (1) to (2). In virtue of the equality (the proof is easy)
\[ \text{t} D_{\mu} = (X \ln |g(t, t)|) t_{\mu}, \]

the statement \( (2) \) is equivalent to \( \text{t} D_{\mu} = 0 \). Differentiating the result \( f_{\mu} = 0 \) (obtained earlier), and operating \( f^2 \) on the derived equation provides \( f(D_{\mu}) = 0 \). Hence \( (2) \) is equivalent to \( D_{\mu} = 0 \), i.e., the statement \( (3) \). The statement \( (4) \) means

\[ D_{\mu}(B(Y, Z)) = B(\nabla_{\mu} Y, Z) + B(Y, \nabla_{\mu} Z). \]

Substituting \( B(X, Y) = g(X, Y)_{\mu} \) in the preceding equation shows that \( (4) \) is equivalent to \( (3) \). This completes the proof.

Remark: In particular, if \( M \) were a real hypersurface of \( \tilde{M} \) (as hypothesized in theorem 4.2, second case), then the statement \( (3) \) would have been automatically true (as apparent from the fact that \( D_{\mu} \) does not belong to \( JD^{\perp} \)).

5. CR-submanifolds of locally Conformal Kaehler Manifolds

Now we are going to study CR-submanifolds of a class of Hermitian manifolds slightly larger than that of Kaehler manifolds for which \( d\tilde{\Omega} = \Omega \wedge w \), \( w \) being a 1-form called the Lee-form \([67]\) and \( \Omega \) being the 2-form defined by \( \Omega(X, Y) = g(JX, Y) \), where \( g \) is the Hermitian metric, \( J \) is the complex structure and \( X, Y \) are arbitrary vector fields. When \( w \) is closed these manifolds are known \([105]\) as locally conformal Kaehler (l.c.K) manifolds. First I review the necessary background on these manifolds for the purpose of getting some new results and their applications.
Let us denote the vector field metrically equivalent to the Lee-form \( w \) by \( W \). It is well-known that many particular classes of Hermitian manifolds have been intensively studied. Among them, Hermitian manifolds whose metric is globally conformal to a Kaehler metric have been encountered (see, A. Gray [47] and L. Vanhecke [48]). But these manifolds have the same topological properties as the Kaehler manifolds. Therefore, it seems interesting to study Hermitian manifolds which are only locally conformal to a Kaehler manifold. For a detailed account on l.c.K manifolds, we refer to [105]. Before we proceed further, let us derive an important identity for a Hermitian manifold:

For a Hermitian manifold, \([J, J] = 0\) which implies

\[
(\bar{\nabla}_{J Y} J Z) + (\bar{\nabla}_{Z J} Y) - J\{(\bar{\nabla}_{Y} J Z) - (\bar{\nabla}_{J Z} Y)\} = 0.
\]

Taking the scalar product with \( X \) and using the identities:

\[
(\bar{\nabla}_{X \Omega})(J Y, J Z) = - (\bar{\nabla}_{X \Omega})(Y, Z),
\]

\[
(\bar{\nabla}_{X \Omega})(J Y, Z) = (\bar{\nabla}_{X \Omega})(Y, J Z)
\]

we obtain

\[
(\bar{\nabla}_{J Y \Omega})(J Z, X) - (\bar{\nabla}_{Z \Omega})(X, Y) - (\bar{\nabla}_{Y \Omega})(Z, X) + (\bar{\nabla}_{J Z \Omega})(X, J Y) = 0
\]

Moreover,

\[
3 (d \Omega)(X, J Y, J Z) - (d \Omega)(X, Y, Z) = - 2 (\bar{\nabla}_{X \Omega})(Y, Z)
\]

From the last two equations it follows that

\[
(\bar{\nabla}_{X \Omega})(Y, Z) = (3/2)(d \Omega)(X, Y, Z) - (d \Omega)(X, J Y, J Z)
\]

Use of the above identity in an l.c.K manifold provides
\[(\nabla_X Y)(x, z) = \left(\frac{1}{2}\right) \left[ \omega(X, Y)w(z) + \omega(Z, X)w(y) + g(Z, X)w(JY) - g(X, Y)w(JZ) \right] \]

Therefore we obtain the identity:

\[(5.1) \quad (\nabla_X J)Y = \left(\frac{1}{2}\right) \left[ \omega(X, Y)w - w(Y)jX + w(JY)X + g(X, Y)jW \right], \]

which holds also for Hermitian manifolds with \(d\omega = \omega \wedge w\) but not necessarily \(dw = 0\), i.e., not necessarily 1-c.K.

Now, we study CR-Submanifolds of an 1-c.K Manifold. Let \((M, g)\) be a non-degenerate submanifold of an 1-c.K manifold \((M, g)\). We set the orthogonal decompositions of the transforms of any tangential vector field \(X\) and any normal vector field \(V\), by \(J\) into their tangential and normal parts as

\[(5.2) \quad (a) \quad JX = PX + FX, \quad (b) \quad JV = TV + fV \]

Differentiating (5.2)(a) along the submanifold we have

\[ (\nabla_X J)Y = J(\nabla_X Y) = \nabla_X PY + B(X, PY) - AfYX + Dx FY, \]

Using the identity (5.1) in the last equation and projecting the resulting equation onto tangent and normal bundles we obtain:

\[(5.3) \quad (\nabla_X P)Y = AfYX + tB(X, Y) + [\omega(X, Y)w_T - w(Y)PX + g(X, Y)(JW)_T + w(JY)X]/2 \]

\[(5.4) \quad (\nabla_X F)Y = -B(X, PY) + fB(X, Y) + [\omega(X, Y)w_N - w(Y)FX + g(X, Y)(JW)_N]/2 \]

where the subscripts \(T\) and \(N\) denote the tangential and normal components. Likewise, the differentiation of (5.1)(b) provides

\[(5.5) \quad (\nabla_X T)V = AfYX - PAwX + [\omega(X, V)w_T - w(V)PX + w(JV)X]/2 \]
\[(5.6) \quad (\nabla_X f)V = -B(X, \tau V) - FAuX + [\varphi(X, V)wN - w(V)FX]/2\]

Blair and Chen [11] proved the following important result:

**Theorem 6.1** In order that a submanifold \( M \) of a Hermitian manifold \( \tilde{M} \) with \( d\varphi = \varphi \wedge w \), be a CR-submanifold it is necessary that \( D^* \) be integrable.

This theorem is valid, in particular for \( M \) an l.c.K manifold. Now we shall establish the following lemma:

**Lemma 6.1** Let \( M \) be a CR-submanifold of an l.c.K manifold.

Then we have

\[
AF_X Y - AF_Y X = \{w(JY)X - w(JX)Y\}/2
\]

for any \( X, Y \) in \( D^* \).

**Proof:** Let \( X \) and \( Y \) be in \( D^* \). Then for any vector field tangent to \( M \), we have

\[
g(\nabla Z X, Y) = g(\nabla Z X, Y) - g(\nabla Z X, Y) = 0
\]

Use of eqn. (5.3) in the above equation gives

\[
0 = g(AF_X Y, Z) - g(AF_Y X, Z) + [\varphi(Z, X)g(\omega T, Y) - g(Z, X)g((JW) T, Y) + w(JX)g(Y, Z)]
\]

But \( \varphi(Z, X) = -\varphi(X, Z) = -g(JX, Z) = 0 \), as \( JX \) is normal to \( M \).

Therefore we obtain

\[
0 = AF_X Y - AF_Y X + [g((JW) T, Y)X + w(JX)Y]/2
\]

which proves the lemma.

**Theorem 6.2** Let \( M \) be a semi-Riemannian totally umbilical proper CR-submanifold of an l.c.K manifold. If \( q > 1 \), then
(1) $J(\nu + \frac{1}{2} \omega_N)$ is normal to $M$,

(2) $\nu + \frac{1}{2} \omega_N$ is orthogonal to $\nu$ and

(3) $J\nu$ is orthogonal to $\omega_N$.

(4) $M$ is locally a product of a complex and a totally real submanifold of the $1.c.K$ manifold if the tangential part of the Lee-vector $W$ is zero. In this case the mean curvature vector $\nu$ turns out to be $-\frac{1}{2} \omega_N$.

(5) A necessary condition for $M$ to reduce to a totally geodesic CR submanifold of the $1.c.K$ manifold is that $\omega_N$ vanishes.

Proof: In virtue of the total umbilicity of $M$, the lemma 6.1 implies, for $X, Y$ in $D^{-}$, that

$$g(\nu, FX)Y - g(\nu, FY)X = \frac{1}{2} g(t\omega_N, Y)X - g(t\omega_N, X)Y$$

Employing eqns. (5.2)(a) and (b), and the skew-symmetry property: $g(JX, Y) = -g(JY, X)$ we transform the last equation into

$$g(t\nu + \frac{1}{2} t\omega_N, Y)X - g(t\nu + \frac{1}{2} t\omega_N, X)Y = 0$$

Contracting the above-obtained equation with respect to an orthonormal base of $D^{-}$ provides

$$(q - 1)g(t\nu + \frac{1}{2} t\omega_N, X) = 0$$

As $q$ exceeds 1, we conclude that

$$t\nu + \frac{1}{2} t\omega_N = 0$$

For any $X$ tangent to $M$ we have $(\nabla_X t)\nu = \nabla_X t\nu - tD_X \nu$. Thus,

$$F((\nabla_X t)\nu) = F\nabla_X t\nu = -(\nabla_X F)t\nu$$

Using (5.3) and (5.5) in the above equation gives
\[ \text{PA}_T X = P^2 \alpha X + \{ \Omega(X, \mu) \text{FW}_T - \text{W}(\mu)P^2 X + \text{W}(J \mu)PX \}/2 \]

\[ = -[\text{PA}_T X + \text{FB}(X, t \mu) + \{ \Omega(X, t \mu) \text{FT} - \text{W}(t \mu)PX + \text{W}(Jt \mu)X \}/2] \]

The umbilicity condition and \( \Omega(X, t \mu) = 0 \) reduce it to

\[ g(\mu + \frac{1}{2} \text{W}, \mu)P^2 X - \frac{1}{2} \Omega(X, \mu) \text{FW}_T - \frac{1}{2} \text{W}(J \mu)PX \]

\[ = \frac{\Omega(X, t \mu) \text{FT} - \text{W}(t \mu)PX + g(X, t \mu) \text{FW}_T}{2} \]

But \( \Omega(X, t \mu) = g(JX, t \mu) = g(PX + FX, t \mu) = 0 \) and therefore we get

\[ g(\mu + \frac{1}{2} \text{W}, \mu)P^2 X = \frac{1}{2} \text{W}(f \mu)PX \]

Hence we obtain

\[ g(\mu + \frac{1}{2} \text{W}, \mu) = 0 \]

Therefore eqn. (5.7) also provides \( \text{W}(f \mu) = 0 \) which is equivalent to saying that \( \mu \) is orthogonal to \( \text{FW} \). Thus we have proved (1), (2) and (3).

Now we have

\[ (\nabla X \cdot Y) = g(FY, X) + g(X, Y)t \mu + \frac{1}{2} \{ g(X, Y) \text{FW}_T - \text{W}(Y)PX + g(X, Y)(JW)_T + \text{W}(Y)X \} \]

\[ = g(JY, \mu + \frac{1}{2} \text{W})X - \frac{1}{2} g(Y) \text{FW}_T + \frac{1}{2} \Omega(X, Y) \text{FW}_T - \frac{1}{2} g(X, Y) \text{FW}_T \]

\[ = -g(Y, t \mu)X - [g(Y, \text{W} + \text{FW}_T)X + \text{W}(Y)PX - \Omega(X, Y) \text{FW}_T - g(X, Y) \text{FW}_T]/2 \]

\[ = [g(FY, \text{W}_T)X - g(W_T, Y)PX + g(FY, \text{W}_T)X + g(X, Y) \text{FW}_T]/2 \]

Obviously, if \( \text{W}_T = 0 \) then \( \nabla X \cdot P = 0 \). Conversely, if \( \nabla X \cdot P = 0 \), then applying contraction to both sides of the above equation we get \( \text{W}_T = 0 \). So \( \nabla P = 0 \) is equivalent to \( \text{W}_T = 0 \). Let \( \text{W}_T = 0 \) so that \( \nabla P = 0 \) which implies that \( \mathbb{D} \) and \( \mathbb{D}^- \) are parallel. This shows that \( \mathbb{M} \)
is locally a product of a complex and a totally real submanifold of the l.c.K manifold. Let us pick up \( Y \) arbitrarily from \( D \). As \( D \) is parallel we must have \( (\nabla_X F)Y = 0 \). This shows that

\[
0 = g(PX, Y)(\mu + \frac{1}{2} \lambda N) + g(X, Y)f(\mu + \frac{1}{2} \lambda N)
\]

for any \( X \) tangent to \( M \) and \( Y \) in \( D \). Substituting \( X = PY \), gives

\[
g(Y, Y)(\mu + \frac{1}{2} \lambda N) = 0.
\]

Hence we obtain \( \mu + \frac{1}{2} \lambda N = 0 \). This proves part (4). To prove the last part, we start with the totally geodesic \( M \), viz. \( B = 0 \).

Differentiating the identity: \( FPY = 0 \) for CR-submanifolds and using (5.3) and (5.4) entails

\[
g(X, PY)N - w(PY)FX + g(X, PY)(FW - fN) + g(X, Y)FW + g(X, Y)(FW + fN) + w(Y)FX = 0
\]

whence, using (5.2)(a) and (b), we get

\[
g(PX, PY)N - g(tN, Y)FX + g(X, PY)FN + g(X, Y)FtN = 0.
\]

But we found earlier that \( tN = -2t \mu \). Here \( \mu = 0 \) and so \( tN = 0 \). Incorporating this into the above equation, we get

\[
g(PX, PY)N + g(X, PY)FN = 0.
\]

Substituting \( Y = X \) and noting \( g(X, PX) = 0 \) we, therefore obtain

\[
g(PX, PX)N = 0 \text{ implying } N = 0.
\]

This completes the proof.

Remark: Under the hypothesis of the theorem 6.2, the identical vanishing of the tangential part of the Lee-vector is equivalent to the \( f \)-structure of \( M \) being parallel.

Corollary 1. Under the hypothesis of theorem 6.2, if \( M \) were
generic then the normal part of the Lee-vector $= (-2)$ times the mean curvature vector.

Corollary 2. Under the hypothesis of theorem 6.2, if the Lee-vector of the l.c.K manifold is tangential to $M$ then $\nu$ is a null vector of the complement of $JD^\perp$ in the normal bundle.

Definition 6.1 (O'Neil [82]) Let $M$ be a semi-Riemannian submanifold of $\overline{M}$ with mean curvature vector field $\nu$. The convergence of $M$ is the real-valued function $k$ on the normal bundle of $M$ such that

$$k(V) = g(\nu, V) \text{ for } V \in T(M)^\perp.$$

Based on this definition, we can state the following:

Corollary 3 Under the hypothesis of theorem 6.2, if the Lee-vector is tangential to $M$ then $\nu$ is a null vector in the holomorphic normal sub-bundle $T(M)^\perp - JD^\perp$ and hence the convergence of $M$ with respect to the mean curvature vector is zero.

Remark: The idea of convergence has been employed by O'Neil in stating the Hawking and Penrose singularity theorems in general relativity [82].
CHAPTER III

APPLICATIONS OF CR-SUBMANIFOLDS TO PHYSICAL SPACE-TIMES.

1. Decomposable Space-Times

First we would like to discuss locally decomposable CR-submanifolds obtained by virtue of theorems 2.3 and 4.1 of chapter II. Later on we shall specialize to the 4-dimensional $M$ required for general relativity theory.

The CR-submanifolds under the hypothesis of theorem 2.3 or theorem 4.1 are locally decomposable as $M^T \times M^\perp$. Also, in case of theorem 2.3, $M^T$ is locally flat; whereas in case of theorem 4.1, it is not necessarily flat. Thus, it is sufficient to discuss the CR-submanifolds, under the hypothesis of theorem 4.1. These submanifolds carry a parallel f-structure:

\[(1.1) \quad P^3 + P = 0, \quad \text{rank}(P) = 2p, \quad \nabla P = 0,\]

where $2p$ is the dimension of D. $M$ has a pair of complementary orthogonal distributions $D^\perp$ (of dimension $q$) and $D$ defined respectively by the projection operators $-P^2$ and $P^2 + I$ acting on the tangent space of $M$ at every point. Let $D^\perp$ be parallelizable so that there exist $q$ vector fields $\xi_a(1 \leq a \leq q)$ spanning $D^\perp$ and their duals $\eta^a$ such that
(1.2) \[ P^2X = -X \cdot \eta(X)\xi_a, \; P\xi_a = 0, \; \eta(PX) = 0 \]

As \( g(PX, Y) = -g(X, PY) \) we can set

(1.3) \[ g(X, Y) = \sum \sigma \eta^a(X) \eta^a(Y) + g(PX, PY) \]

where \( \sigma \) is a real smooth (analytic) positive function, \( \sigma = 1 \) or \(-1\), and \( X, Y \) are arbitrary vector fields tangent to \( M \). Obviously, \( g(\xi_a, X) = e_a \sigma \eta^a(X) \) (a not summed). As \( \nabla\eta = 0 \), both \( D^+ \) and \( D \) are parallel. Let us follow the notations: \( q + 1 \leq a, b \leq q + p, \; q + p + 1 \leq a^k, b^k \leq q + 2p, \; q + 1 \leq A, B \leq q + 2p \) and \( i \leq j, j \leq 2p + q \).

Let \( \{\xi_i\} \) be an adapted orthogonal frame with its dual \( \{\eta^i\} \) such that \( P\xi_a = \xi_a \), \( P\xi_a = -\xi_a \). Thus \( g \) has the form:

(1.4) \[ g(X, Y) = \sigma e_a \eta^a(X)\eta^a(Y) + \tau e_A \eta^A(X)\eta^A(Y) \]

where \( \tau \) is a real smooth (analytic) positive function, \( \sigma = \sigma_k \) and \( e_A = 1 \) or \(-1\). Let us call such a manifold a semi-Riemannian framed (SRF)-manifold.

Theorem 1.1 The SRF-manifold \( M \) has \( (q) \) and \( (2p) \) smooth (analytic) \( i \)-forms \( w_a^B \) and \( w_A^B \) respectively such that the Ricci tensor and the scalar curvature are:

(1.5) \[ \text{Ric}(X, Y) = \sum \sigma \eta^a(X)\eta^b(Y) + \sum \eta^A(X)\eta^B(Y), \]

(1.6) \[ H_a^B = \sum e_c e_d \Gamma^c_d \]

(1.7) \[ H_A^B = \sum e_c e_d \Gamma^c_d \]

(1.8) \[ \Gamma_{j1}^{1j} = (d\omega_j^1 - 2 \omega_j^m \omega_m^1)(\xi_k, \xi_1) \]

(1.9) \[ \chi = (H/\sigma) + (W/\tau) \]

(1.10) \[ H = \sum e_a H_a^a \text{ and } W = \sum e_A H_A^A \]
Proof: Using $\nabla g = 0$ and the fact that $D^\perp$ and $D$ are both parallel we have the set-up

\begin{align}
(1.11) \quad \nabla_X \xi_A &= \frac{1}{2} X(1n\gamma) \xi_A + \sum_{b \neq A} w^A_B(X) \xi_B \\
(1.12) \quad \nabla_X \xi_A &= \frac{1}{2} X(1n\gamma) \xi_A + \sum_{b \neq A} w^A_B(X) \xi_B
\end{align}

such that $w^A_B = e_a e_b w^A_a$, $w^B_B = e_a e_b w^B_a$; $[w^A_B]$ and $[w^A_B]$ being $q \times q$ and $2p \times 2p$ matrices of 1-forms, respectively. Using (1.11) and (1.12) we obtain the following components of the curvature tensor:

\begin{align}
(1.13) \quad R(X,Y) \xi_J &= \sum_{i \neq j} \sum_k \Gamma^k_{j1} \eta_i^k(X) \eta^j(Y)
\end{align}

Bianchi identities and the symmetry of the Ricci tensor imply that only the components $\Gamma_{bcd}^a$ and $\Gamma_{BCD}^A$ of $\Gamma_{JK11}^I$ are non-zero (i.e. no cross components survive) and $\Gamma_{JK11}^I = \Gamma_{KJ11}^I$ (1, J, K being all different). $\Gamma_{JK11}^I = -\epsilon_{IJK} \Gamma_{1111}$. Using these we get (1.5) and then (1.9) can be recovered by contracting (1.5).

**Corollary 1.** For flat SRF-manifolds $M$, each $H_{KJ} = 0$.

**Corollary 2.** The SRF-manifolds are Einstein iff.

\[ 2p \gamma H = q \phi W \text{ and } H_{KJ} = 0 \text{ for } J \neq K. \]

Using the non-zero components of the Weyl conformal tensor $C$ we can prove the following:

**Theorem 1.2** For SRF-manifolds we have

\begin{align}
(1.14) \quad H_{1J} &= 0 \text{ (1 \neq J), for } q = 2p \\
(1.15) \quad H &= q(q - 1)\sigma \tau / (2p + q - 1)(q - 2p) \text{ and } \nonumber \\
W &= 2p(2p - 1)\tau \phi / (2p + q - 1)(2p - q) \text{ for } q \neq 2p.
\end{align}
Corollary 1. SRF-manifolds with \( q = 2, \ p = 1 \) are conformally flat iff. \( r = 0 \).

Corollary 2. For conformally flat SRF-manifolds with \( q \) different from \( 2p \), \( r = 0 \) iff. they are flat.

Corollary 3. SRF-manifolds of constant curvature are necessarily flat.

At this point, we study the local geometry with a view of obtaining some interesting physical applications. Consider the frame \( \{ \xi_1 \} \) as a coordinate frame \( \{ \partial_1 \} \) for a local coordinate system \( (x^1) \). Computing all the Lie-brackets \( [\xi_1, \xi_j] = 0 \), with the help of (1.11) and (1.12), we get the following metric:

\[
\begin{align*}
\text{ds}^2 &= \sigma \sum_a e_a (dx^a)^2 + \tau \sum_A e_A (dx^A)^2 \\
\text{grad} \sigma &= \sum_a e_a (\partial_a \ln \sigma) \partial_a, \\
\text{grad} \tau &= \sum_A e_A (\partial_A \ln \tau) \partial_A.
\end{align*}
\]

(1.16) \( w_a^b \) lies in \( (D^-)^* \), \( w_A^B \) lies in \( D^* \),

\[
\begin{align*}
2w_a^b(\xi_B) &= \xi_a(\ln \sigma), & 2w_A^B(\xi_B) &= \xi_A(\ln \tau) \\
w_a^b(\xi_B) &= w_B^c(\xi_a), & (a \neq b \neq c) \\
w_A^C(\xi_B) &= w_B^C(\xi_A), & (A \neq B \neq C)
\end{align*}
\]

Thus, 4-dimensional SRF-manifolds \( M (q = 2, \ p = 1, \ e_1 = -1, \ e_2 = e_3 = e_4 = 1) \) are models of a class of decomposable space-times. For the general study of decomposable space-times and its significance in relativity, please see [67]. In particular, we construct a model by the transformation:
\[ x^1 = t, \ x^2 = z, \ x^3 = 2 \cot(\theta/2) \cos \sigma, \]
\[ x^4 = 2 \cot(\theta/2) \sin \sigma, \ \tau = 16[4-(x^3)^2-(x^4)^2]^{-2}. \]

Thus, the metric is:

\[ (1.17) \quad ds^2 = \sigma(-dt^2 + dz^2) + (d\sigma^2 + \sin^2 \sigma \ d\theta^2) \]

Therefore, we have a class of space-times, endowed with an SRF-structure, whose underlying topology is \( \mathbb{R}^2 \times S^2 \). This means that, according to Geroch et al [40], 4-dimensional SRF-manifolds can be physical models of the maximally extended positive mass Schwarzschild, the negative mass Schwarzschild and the Reissner-Nordstrom solutions. As a special case, \( \mathbb{R}^2 \) can be seen as a Kruskal plane of mass \( M \) [82] by setting \( \sigma \) equal to \( F(r) \) where \( r \) is defined by \( 2(r - 2M)\exp[(r/2M) - 1] = z^2 - t^2 \), and by warping [82] the metric \( (1.17) \) homothetically on the fibre \( p \times S^2 \) for any \( p \) in \( \mathbb{R}^2 \times \frac{1}{2} S^2 \) having the line element \( ds_K^2 = F(r)(-dt^2 + dr^2) + r^2(d\sigma^2 + \sin^2 \sigma \ d\theta^2) \) and \( F(r) = (8M^2/r)\exp[1 - (r/2M)] \).

We would like to discuss a couple of physically interesting examples by specifying the values of the first Beltrami operator \( B: \alpha \mapsto g(\text{grad} \alpha, \text{grad} \alpha) \) as follows:

**Example 1.** Let \( \alpha = \tau, \ e_A^2 = 1, \ \text{Ker}(B) = \{ \tau: B(\tau) = 0 \}. \)

Under this setting we have \( \sum_A \tau_A \otimes \tau = 0. \) Thus \( \tau \) is a constant and each \( \xi_A \) is Killing. As \( \{ \xi_A \} \) is linearly independent and the abelian property comes from \( [\xi_A, \xi_B] = 0 \), we infer that, under the above conditions, on an SRF-manifold, there exists a 2p-parameter
abelian isometry group generated by \( \{ \xi_A \} \). Physically, on \( M \) \((q = 2, p = 1)\) considered as a space-time, this property is useful, in particular, for derivation of type D solutions of the Einstein's field equations \([64]\).

Example 2. Let \( \sigma = \sigma, q = 2, \xi_1 = \xi_2 = 1, B : B(\sigma) = \sigma \) Hence \((\partial_1 \ln \sigma)^2 + (\partial_2 \ln \sigma)^2 \geq 1\). Such equations arise in geometric optics \([61]\). Their solutions describe the 2-dimensional phenomena of geometric optics whose level curves represent wave-fronts, while the characteristics represent light rays.

2. A Class Of 4-dimensional Decomposable Space-Times

In this section we shall discuss the SRF-manifolds of dimension 4 and of Lorentzian signature. We shall call them Lorentzian Framed (LF)-manifolds. Here we have an orthonormal frame:

\( \{ \xi_1, \xi_2, \xi_3, \xi_4 \} \) in \( M \) such that the first two vector fields span \( D^- \) and the last two span \( D \). In fact:

\[
P \xi_1 = P \xi_2 = 0, \quad P \xi_3 = \xi_4, \quad P \xi_4 = -\xi_3.
\]

\[
P^2(X) = -X + \eta^1(X)\xi_1 + \eta^2(X)\xi_2,
\]

where \((\eta^1, \eta^2)\) is the dual of \((\xi_1, \xi_2)\). Thus \( g \) is expressed canonically as:

\[
g = -\eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2 + \eta^3 \otimes \eta^3 + \eta^4 \otimes \eta^4.
\]

As \( \nabla P = 0 \), we can show that

\[
(2.1) \quad \nabla \xi_1 = h(X)\xi_2, \quad \nabla \xi_2 = h(X)\xi_1
\]
(2.2) \[ \nabla \xi_3 = w(X)\xi_4, \quad \nabla \xi_4 = -w(X)\xi_3 \]

where \( h \) and \( w \) are smooth (analytic) 1-forms. From (2.2) we obtain the following components of the curvature tensor:

\[
R(X, Y)\xi_1 = (dh)(X, Y)\xi_2, \\
R(X, Y)\xi_2 = (dh)(X, Y)\xi_1, \\
R(X, Y)\xi_3 = (dw)(X, Y)\xi_4, \\
R(X, Y)\xi_4 = -(dw)(X, Y)\xi_3
\]

By virtue of Bianchi identities we observe, upon a lengthy computation, that the only non-zero components of \( dh \) and \( dw \) are \((dh)(\xi_2, \xi_1) = H\) and \((dw)(\xi_4, \xi_3) = W\) respectively. Thus the Riemann curvature tensor has the form:

(2.3) \[
R(X, Y)Z = 2H(\eta^2 \wedge \eta^1)(X, Y)(\eta^4(Z)\xi_2 + \eta^2(Z)\xi_1) \\
+ 2W(\eta^4 \wedge \eta^3)(X, Y)(\eta^3(Z)\xi_4 - \eta^4(Z)\xi_3)
\]

Successive contractions of (2.3) yield:

(2.4) \[ \text{Ric} = H(-\eta^2 \eta^3 + \eta^2 \eta^2) + W(\eta^4 \eta^3 - \eta^4 \eta^4) \]

(2.5) \[ r = 2(H + W) \]

Remarks: (1) Ricci-flat (Empty) LF-spaces are characterised by closed 1-forms \( h \) and \( w \).

(2) Einstein LF-spaces are characterised by \( 4H = 4W = r \).

The Weyl conformal tensor of LF-spaces is found to have the following non-vanishing components:

(2.6) \[ C(1, 2, 1, 2) = -C(3, 4, 3, 4) = r/6, \]

\[ C(1, 3, 1, 3) = C(1, 4, 1, 4) = -C(2, 3, 2, 3) = -C(2, 4, 2, 4) = r/12 \]
where \( C(\xi_1, \xi_2, \xi_3, \xi_4) \) has been denoted \( C(1, 1, 1, 1) \). Hence we obtain the result:

"LF-spaces are conformally flat iff. \( r = 0 \)."

This is in agreement with corollary 1 of theorem 1.2. Let us now consider a coordinate frame \((\xi_1) (1 \leq 1, j \leq 4)\), compatible with the LF-structure, for a local coordinate system \((t, x, y, z) = (x^1)\) such that:

\[
(2.7) \quad \xi_1 = \sigma \xi_1, \quad \xi_2 = \sigma \xi_2, \quad \xi_3 = \tau \xi_3, \quad \xi_4 = \tau \xi_4
\]

where \( \sigma \) and \( \tau \) are non-zero smooth (analytic) functions. Under such a coordinate system the metric takes the form:

\[
(2.8) \quad ds^2 = \sigma^2 (-dt^2 + dx^2) + \tau^2 (dy^2 + dz^2),
\]

where \( \sigma = \sigma(t, x) \) and \( \tau = \tau(y, z) \) and

\[
(2.9) \quad \left( \ln \sigma \right), tt - \left( \ln \sigma \right), xx = H\sigma^2,
\]

\[
\left( \ln \tau \right), yy + \left( \ln \tau \right), zz = -W\tau^2
\]

At this point we would like to compare the above metric with the metrics obtained by Debever, McLenaghan and Tariq [25] in their theorem. If a space-time admits a Riemannian-Maxwellian invertible structure, then it possesses locally an invertible 2-parameter abelian isometry group. There exists a system of coordinates \((u, v, w, x)\) such that the metric has one of the forms:

\[
ds^2 = (L du + M dv)^2 - R^2 dw^2 - (Ndu + Pdv)^2 - S^2 dx^2
\]

\[
ds^2 = R^2 dw^2 - (Ldu + Mdv)^2 - (Ndu + Pdv)^2 - S^2 dx^2
\]

where \( L, M, N, P, R \) and \( S \) are arbitrary functions of \( w \) and \( x \) only.
For the first metric the orbits of the group are time-like while for the second metric they are space-like. Under the strong condition, viz. $\sigma = \text{constant}$, or $\gamma = \text{constant}$, our metric (2.8) for LF-spaces is compatible with the hypothesis of the theorem of Debever et al, as stated above.

In terms of the coordinates $(t, x, y, z)$ the Ricci tensor is:

\[(2.10) \quad \text{Ric} = H(-dt \otimes dt + dx \otimes dx) + W(dy \otimes dy + dz \otimes dz)\]

Here we shall discuss the fundamental problem of determining Lorentzian metrics with a prescribed Ricci curvature. In this connection we prove:

Theorem 2.1 If the Ricci curvature of an LF-space is prescribed to be a smooth (analytic) function, then the metric tensor can be obtained locally, as smooth (analytic) functions.

Proof: If there is a prescribed smooth (analytic) Ricci curvature then, in view of eqn. (2.10), the functions $H$ and $W$ are known. Hence let $H = H_0$ and $W = W_0$ in the local coordinate system $(t, x, y, z)$, for which the frame $\{\xi_i\}$ is related to $\{\partial_t, \partial_x, \partial_y, \partial_z\}$ by (2.7). Thus we obtain a pair of partial differential equations:

\[
(1n\sigma^2), tt - (1n\sigma^2), xx = 2\sigma^2H_0
\]

\[
(1n\tau^2), yy + (1n\tau^2), zz = -2\tau^2W_0
\]

which are essentially Liouville's equations [65] and can be cast into the compact form:
(\ln U^2), \varphi = k U^2 F^2

where \varphi and \overline{\varphi} are complex conjugate coordinates, k a constant and F an analytic real function of \varphi and \overline{\varphi}. Its solution is [64]

U^2 = k F^2 (1 + f \overline{f})^2 / (f, \varphi \overline{f}, \overline{\varphi})

f being an arbitrary smooth function of \varphi. Thus we can obtain a class of solutions of \sigma and \tau and hence the metrics for the prescribed Ricci curvature.

In [26] De Turck has shown the local existence of a smooth (analytic) metric whose Ricci curvature is prescribed to be smooth (analytic) and non-singular. Theorem 2.1 provides the solution locally for the metric of an LF-space in terms of smooth /analytic functions, if its Ricci curvature is prescribed to be smooth /analytic, but may be singular.

Next we would like to determine the Petrov types [61] of LF-spaces, by the standard procedure of finding the roots of the quartic equation:

(2.11) \psi_0 - 4E \psi_1 + 6E^2 \psi_2 - 4E^3 \psi_3 + E^4 \psi_4 = 0

where \psi's are the (Newman-Penrose)-curvature components of the Weyl conformal tensor C, defined as

\psi_0 = C(e_4, e_1, e_4, e_1), \psi_1 = C(e_4, e_3, e_4, e_1),

\psi_2 = C(e_4, e_3, e_4, e_3) - C(e_4, e_3, e_1, e_2),

\psi_3 = C(e_4, e_3, e_2, e_2), \psi_4 = C(e_3, e_2, e_3, e_4)

where \(e_1, e_2, e_3, e_4\) is the complex null tetrad of the
Newman-Penrose formalism [79]. \( e_1, e_2 \) are conjugate complex null vectors and \( e_3, e_4 \) are real null vectors and they can be derived from the orthonormal tetrad \((\xi_1, \xi_2, \xi_3, \xi_4)\) as:

\[
\begin{align*}
\xi_1 &= (\xi_3 - i\xi_4)/\sqrt{2}, & e_1 &= (\xi_3 - i\xi_4)/\sqrt{2} \\
\xi_2 &= (\xi_3 + i\xi_4)/\sqrt{2}, & e_2 &= (\xi_3 + i\xi_4)/\sqrt{2} \\
\xi_3 &= (\xi_1 - \xi_2)/\sqrt{2}, & e_3 &= (\xi_1 - \xi_2)/\sqrt{2} \\
\xi_4 &= (\xi_1 + \xi_2)/\sqrt{2}, & e_4 &= (\xi_1 + \xi_2)/\sqrt{2}
\end{align*}
\]

A straightforward computation using the equations (2.6), (2.12) and (2.13) shows that \( \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \) and \( \Psi_2 = r/12 \).

Hence equation (2.11) reduces to \( E^2 = 0 \). Thus we conclude that the LF-spaces are of Petrov type D or 0 according as \( r \) is not equal to or equal to 0.

Having determined the Petrov types of LF-spaces we proceed to find certain exact solutions as follows:

We consider the Einstein field equations, with a suitable choice of units, to be of the form:

\[
(2.14) \quad \mathbf{Ric}(X,Y) + (\Lambda - (r/2))g(X,Y) = T(X,Y)
\]

where \( \Lambda \) is the cosmological constant and \( T \) is the energy-momentum tensor of all the matter fields present [54]. We shall discuss both geometric and physical solutions.

(A) Flat LF-spaces: Such spaces have \( H = W = 0 \) which give the wave and Laplace equations:

\[
(1n\sigma)_{tt} - (1n\sigma)_{xx} = 0, \quad (1n\tau)_{yy} + (1n\tau)_{zz} = 0
\]

Their local solutions are well known. These are trivially of type 0. Besides the trivial example of Minkowski space, the plane wave
solution:
\[ ds^2 = \exp[e(t + x)](-dt^2 + dx^2) + dy^2 + dz^2, \]
\( e=0 \) or \( i \), (we refer to [64]) corresponds to the flat LF-spaces for \( \sigma = \exp[e(t + x)/2] \) and \( r = 1 \).

(B) Einstein LF-spaces: These spaces are of Petrov type D as \( 4H = 4W = r \) (a non-zero constant). Locally, these are Liouville's equations:
\[ 2[(\ln \sigma^2), \xi \xi = (\ln \sigma^2), \xi \xi] = r \sigma^2 \]
\[ 2[(\ln \tau^2), \eta \eta = (\ln \tau^2), \eta \eta] = -r \tau^2 \]
whose solutions are \( \sigma^2 = (16/\pi)U'(u)V'(v)[U(u) + V(v)]^{-2} \), where \( U \) and \( V \) are arbitrary functions of \( u = t+x \) and \( v = t-x \) respectively; and \( \tau^2 = (16/\pi)(\eta, \eta)\xi(1 + \eta \eta)^{-2} \) where \( \eta \) is an arbitrary analytic function of \( \xi = y + iz \) [64]. One can verify that the Einstein LF-spaces are locally symmetric. If, in addition, an Einstein LF-space is complete and simply connected, then it follows [62] that it would be symmetric and hence homogeneous. In this case, the Einstein LF-spaces would be a sub-class of the only homogeneous Einstein spaces of type D, given by
\[ ds^2 = [2d\xi d\bar{\xi} + (1 + \bar{c}c \xi)^2] - [2dudv + d(1 - cuv)^2] \]
where \( \xi = y + iz \), \( u = t + x \), \( v = t-x \) and \( r = 8c \); as shown by Cahen [15]. The other possible cases of type D solutions with a \( G_6 \) are those with an isotropy group composed of boosts and
rotations. The metric of such spaces is of the form:

\[ ds^2 = A^2(dy^2 + \Sigma^2(y, k)dz^2) + B^2(dx^2 - \Sigma^2(x, k')dt^2) \]

where A and B are constants [64].

Remark: In particular, for Einstein LF-spaces which deviate from the Minkowski space-time up to the first order (weak gravitational forces), i.e. such that the powers of \( \ln \sigma \) and \( \ln r \) higher than 1 may be neglected, the above Liouville's equations would reduce to the Telegraph and Helmholtz equations:

\[
\begin{align*}
(\ln \sigma^2 + 1),tt - (ln \sigma^2 + 1),xx &= (\tau/2)(\ln \sigma^2 + 1) \\
(\ln r^2 + 1),yy + (\ln r^2 + 1),zz &= (-\tau/2)(\ln r^2 + 1)
\end{align*}
\]

respectively. Their solutions are well-known [39] in terms of Bessel functions of order 0.

(C) Conformally flat LF-spaces: Such spaces are of Petrov type 0 as \( r = 0 \) and \( H = -W \). These conditions are locally expressed as

\[
\sigma^{-2}[(\ln \sigma^2),tt - (\ln \sigma^2),xx] = r^{-2}[(\ln r^2),yy + (\ln r^2),zz].
\]

The left hand side of this equation is a function of \( t \) and \( x \) only; while the right hand side is a function of \( y \) and \( z \) only. Hence each side must be a constant = \( C \) (say). Eventually, we obtain a pair of Liouville's equations whose solutions have already been described in (B). It can be shown that the LF-spaces of type 0 are locally symmetric. As an example, the Bertotti-Robinson-like metric
\[ ds^2 = \frac{2dqd\phi}{(1 + cq^2)^2} - \frac{2dudv}{(1 + cuv)^2} \]
corresponds to conformally flat LF-spaces such that: \( C = -4c \),
\( q = y + iz, u = t + x, v = t - x \) and \( \sigma^2 = 2(1 + cuv)^2 \).
\( \tau^2 = 2(1 + cq^2)^{-2} \). Observe that there exist spaces of type 0 other than LF-spaces. For instance, all spaces of non-zero constant curvature are not LF-spaces. Furthermore, the recurrent spaces of type 0 [64] satisfying \( \nabla R = k \otimes R \) with the non-zero recurrence form \( k \), representing the plane-wave solutions:
\[ ds^2 = dy^2 + dz^2 - 2dudv - \phi^2(u)(y^2 + z^2)du^2/2, \]
\[ 2k = d\ln\phi, \nabla e_4 = 0, \text{Ric}(X,Y) = \phi^2 e_4(X)e_4(Y) \]
can not be the LF-spaces of type 0 as they are not locally symmetric.

(D) A Scalar Field Of Rest Mass \( m \): For the scalar field (like \( \pi_0 \)-meson) the energy-momentum tensor is given by [54]
\[ T(X,Y) = (X\psi)(Y\psi) - \frac{1}{15} [g(\text{grad}\psi, \text{grad}\psi) + (m\psi/h)^2] g(X,Y) \]
where \( h \) stands for the Planck's constant. The field equations can be written as
\[ \{H - (r/2)\} [\eta^4(X)\eta^4(Y) + \eta^2(X)\eta^2(Y)] + \]
\[ \{W - (r/2)\} [\eta^3(X)\eta^3(Y) + \eta^4(X)\eta^4(Y)] + \Lambda g(X,Y) \]
\[ = (X\psi)(Y\psi) - \frac{1}{15} [g(\text{grad}\psi, \text{grad}\psi) + (m\psi/h)^2] g(X,Y) \]
Contraction of the above equation provides
\[ r = g(\text{grad}\psi, \text{grad}\psi) + 2(m\psi/h)^2 + 4\Lambda. \]
Plugging \( \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \) for \( X=Y \) in the last but one equation obtains
\[ (2.15) \begin{cases} 
\xi_1 \psi = \xi_2 \psi = 0, 
(\xi_3 \psi)^2 = (\xi_4 \psi)^2 \\
H = \frac{1}{2} (m \psi / h)^2 + \Lambda, 
W = (\xi_3 \psi)^2 + \frac{1}{2} (m \psi / h)^2 + \Lambda.
\end{cases} \]

Thus it turns out that
\[ (2.15)' \quad r = 2(\xi_3 \psi)^2 + 2(m \psi / h)^2 + 4 \Lambda. \]

Moreover, it follows from the conservation law: \( \text{div } \Gamma = 0 \) that \( \text{div } \text{grad } \psi = (m \psi / h)^2 \psi \). Thus we obtain a mathematical model of LF-spaces carrying a scalar field of mass \( m \).

Remarks: (1) LF-spaces with a scalar field \( \psi \) and \( \Lambda > 0 \), can not be conformally flat as it is evident from (2.15) that \( r > 0 \).

(2) LF-spaces with zero rest mass scalar field and \( \Lambda = 0 \), have singular Ricci-tensor \( (H = 0) \). Also, for this case, \( \psi \) must be non-constant as otherwise \( W = 0 \) and hence the space is empty.

(3) The above remark implies that LF-spaces with a non-zero scalar field and \( m = \Lambda = 0 \), can not be non-empty Einstein spaces.

Now I present a new exact solution for a scalar field of zero-rest mass. Expressing eqns. (2.15) locally we get
\[
(\ln r^2), yy + (\ln r^2), zz = -2(\psi, y)^2 - (m \psi, z)^2 / (2r^2 + 2r^2 + 2 - 2 \psi^2) \\
(\ln \sigma^2), xx - (\ln \sigma^2), tt = -(m \psi, h)^2 - 2(\psi, z)^2 - 2 \sigma^2 + 2.
\]

\[ \psi^i \nabla_i \psi = (m / h)^2 \psi, \quad \psi, y = \psi, z, \quad \psi, x = \psi, t = 0. \]

Therefore, \( \psi = \psi(s) \), where \( s = y z \). The third equation of the above system becomes \( \psi, y y + \psi, z z = (m \psi / h)^2 \psi \). Suppose \( m \) is non-zero. Then the second equation demands \( \psi = \text{constant} \) and the third equation then implies \( m = 0 \). Thus the scalar field turns
out to be of zero rest-mass. Consequently, the second equation becomes Liouville's equation. The third equation implies
\[ \psi, yy + \psi, zz = 0 \text{ and as } \psi = \psi(s), \text{ we get } \psi = As + B, \text{ A and B being arbitrary constants. Accordingly, } \psi, y = A \text{ and the first equation assumes the form:} \]
\[ (\ln r^2), yy + (\ln r^2), zz = -2 \Lambda r^2 - 2A^2 \]
This can be transformed into Liouville's equation as discussed by Bieberback [9]. In particular, for \( \Lambda = 0 \), the last equation reduces to Poisson equation with a solution:
\[ r = \exp[-\Lambda^2(y^2 + z^2)/4]. \]
The second equation reduces to the wave-equation \( (\ln \sigma), xx - (\ln \sigma), tt = 0 \) whose solution is \( \ln \sigma = F(x + t) + G(x - t) \), \( F \) and \( G \) being arbitrary functions of \( x + t \) and \( x - t \) respectively.

(E) Non-Singular Simple Electromagnetic Field: Let this field be represented by a simple bivector \( F \). Then we have
\[ T(X, Y) = g(F \times X, F \times Y) + (\text{tr.} F \times 2) g(X, Y) / 4 \]
where \( F(X, Y) = g(F \times X, Y) \). The Maxwell equations demand \( dF = 0 \) and \( \text{div} F(X) = g(J, X) \) (in suitable units), where \( J \) denotes the current 4-vector. For the non-singular simple electromagnetic field we have \( \det(F) = 0 \). Also, with the aid of an orthochronous Lorentz transformation, it is possible to show that there exists a non-zero Maxwell scalar \( \sigma = F(e_2, e_1) + F(e_4, e_3) \), where \( \sigma \) is either real or pure imaginary. For the sake of economy, we set \( \sigma \)
to be imaginary (the results are identical for $\sigma$ real). In this case, it is easy to see that $F$ is compatible with the LF-structure if

\[ F(X,Y) = \sigma g(PX, Y), \quad i = J(-1) \]

Then it follows that

\[ PX = i[\theta^3(X)e_3 - \theta^4(X)e_4] \]

where $(\theta^3, \theta^4)$ is the dual of $(e_3, e_4)$. One can verify that $P$ satisfies the minimum recurrent relation: $P^3 + P = 0$. This shows that the space-time carrying a simple non-singular electromagnetic tensor field $F$ is an example of LF-space. With the help of relation (2.16) and the facts that the fundamental 2-form $\omega$ associated with $P$, and $F$ are both closed, we can prove

Proposition 2.1 For a simple non-singular electromagnetic field in an LF-space defined by the relation (2.16) the gradient of $\omega$ belongs to $D$. Using the second Maxwell equation $(\text{div} F)(X) = g(J, X)$ we obtain $J = i P(\text{grad} \omega)$ and hence we state

Proposition 2.2 An LF-space carrying a simple non-singular electromagnetic field $F$ is homogeneous iff. $J = 0$.

The NP-operators [64], Maxwell equations and $J$ are given by:

\[ \delta(i\sigma) = e_1(i\sigma), \quad \delta(i\sigma) = e_2(i\sigma), \quad \Delta(i\sigma) = e_3(i\sigma), \]

\[ D(i\sigma) = e_4(i\sigma), \quad J(i\sigma) = 2 (i\sigma) - I_1, \quad \Delta(i\sigma) = -2\nu(i\sigma) - I_2 \]

\[ \delta(i\sigma) = 2\tau(i\sigma) + I_0, \quad \delta(i\sigma) = -2\pi(i\sigma) - I_0 \]
\[ J = J_0 e_1 + 10e_2 + 11e_3 + 12e_4 \]

where \( \rho, \mu, \tau \) and \( \pi \) are NP-spin coefficients. By relating the above-mentioned material with the LF-structure replacing \( X, Y \) in (2.14) by \( \xi_1 \)'s we obtain

\[ \Delta \xi = D_\xi = 0, \quad \langle 2 \rangle \xi \xi = (\xi_3 + \xi_4) \xi, \quad \langle 2 \rangle \xi \xi = (\xi_3 + \xi_4) \xi \]

(2.17) \[ J = (1/2)[(e_1 + e_2)(1)\xi_4 - (e_1 - e_2)(\rho)\xi_3], \]

\[ H = (s^2/2) \Lambda, \quad W = -(s^2/2) \Lambda. \]

Thus we have a mathematical model of LF-spaces satisfying (2.17).

Now we discuss a special case when \( F = dV^* \), where \( V^* \) is the dual of a homothetic vector field \( V \) (potential field) such that \( L_V g = 2\kappa g \) (\( \kappa \) - constant) and \( J \) is non-zero. For the LF-spaces, this means that \( s \) is non-constant and hence we see, from the last equation (2.17), that \( H \) is non-zero and \( W \) is non-zero everywhere. Following McIntosh [74] we have \( QV = J \) (upto a constant factor) where \( g(QX, Y) = Ric(X, Y) \). With the foregoing hypothesis and on using the expressions of \( J \) and \( Q \) from the structure equations we conclude that \( V = (1/\omega)J \). Hence we have established the following:

Theorem 2.2 In an LF-space-time, a homothetic potential vector field is parallel to the current 4-vector.

Finally we will discuss the local exact solutions of a non-singular simple electromagnetic field. The field equations can be expressed locally as
\begin{align}
(2.18) & \quad (1n s^2), tt - (1n s^2), xx = \sigma^2 s^2 + 2 \sigma^2 \Lambda \\
(2.19) & \quad (1n r^2), yy + (1n r^2), zz = r^2 s^2 - 2 r^2 \Lambda
\end{align}

where $s = s(y, z)$. As $r = 4 \Lambda$, the non-singular electromagnetic field equations of LF-spaces are either of Petrov type O or D, according as $\Lambda$ is zero or non-zero. For $\Lambda = 0$ we conclude from the above equations that $s$ must be constant and hence these equations become Liouville's equations whose solutions have already been discussed. In particular, for a source-free ($J = 0$, $s =$constant) non-singular electromagnetic field, if we substitute $x = r$, $y = \cosh^{-1} (\cosec \theta)$, $z = \phi$, $\sigma = r^{-1}$, $\tau = \sin \theta$, then we obtain the well known Bertotti-Robinson metric [64]:

$$ds^2 = (2s^{-2})(dr^2 + r^2 d\phi^2 + r^2 \sin^2 \theta d\phi^2 - dt^2)$$

which is the only conformally flat solution of the source-free non-singular electromagnetic field. For non-null electromagnetic field with $J$ non-vanishing we observe that the two null eigen vectors $e_3$ and $e_4$ of the electromagnetic field are also the double eigen vectors of the Weyl tensor. This shows that it is an aligned case [64]. As $s = s(y, z)$ we can transform (2.18) and (2.19) into a pair of Liouville's equations whose solutions have been already discussed.

(F) Kruskal Space-Time: We construct a model of Schwarzschild geometry in Kruskal-Szekeres coordinates [76] by using the warped product technique. The LF-space is a direct product of the leaves
of $D$ and $D^\perp$ whose line-elements, in terms of local coordinates are $\sigma^2(t, x)(-dt^2 + dx^2)$ and $\tau^2(y, z)(dy^2 + dz^2)$. Under the transformation:

$$y = 2\cot(\theta/2)\cos \varphi, \quad z = 2\cot(\theta/2)\sin \varphi, \quad \tau = 4/(4 + y^2 + z^2)$$

the metric of the LF-space becomes

$$(2.20) \quad ds^2 = \sigma^2(-dt^2 + dx^2) + (d\varphi^2 + \sin^2 \varphi ds^2)$$

i.e. the leaf $Q$ of $D^\perp$ is a 2-plane and that $S$ of $D$ is a unit 2-sphere respectively. Now $Q$ can be seen as a Kruskal plane of mass $M$ [82] by setting $\sigma^2 = F(r)$ where $r$ is defined as:

$$(r/2M) - 1\exp(r/2M) = x^2 - t^2.$$  Warping (2.20) homothetically on the fibre $S$ at each point we get the Kruskal space-time $K = Q \times S$ having metric:

$$ds_k^2 = F(r)(-dt^2 + dr^2) + r^2(d\varphi^2 + \sin^2 \varphi ds^2)$$

where $F(r) = (32M^3/r)\exp(-r/2M)$. Using the corollary [82] and the fact that $K$ is empty we get

$$Q\text{Ric}(X, Y) = 2g(\nabla_X\text{grad} r, Y) \text{ for all } X, Y \text{ tangent to } Q$$

$$S\text{Ric}(U, V) = r^2g(U, V)r^\# \text{ for all } U, V \text{ tangent to } S$$

where $r^\# = \Delta r/r + g(\text{grad } r, \text{ grad } r)/r^2$, $\Delta$ denoting the Laplacian on $Q$. Since $\dim Q = 2$ and $S$ is a 2-sphere we have

$$Q\text{Ric}(X, Y) = Wg(X, Y), \quad r^\# = 1/r^2 = H/r^2.$$  Hence $H = 1$ and $2\nabla_X\text{grad } r = rW$ which yields

$$W = [r, xx - (F, tt)/r\sigma^2]$$

Thus we have got
\[(\ln r^2) \cdot y_y + (\ln r^2) \cdot z_z + 2r^2 = 0, \quad (\ln \sigma^2) \cdot t_t - (\ln \sigma^2) \cdot \sigma = (12\sigma M^4/r^4) \exp(-r/2M).\]

The first equation is the Liouville's equation, whose solution has been discussed already. The second equation is an inhomogeneous wave equation (\(r\) being known in terms of \(x\) and \(t\)) whose solutions can be obtained for suitable initial and boundary conditions.

3. Application To Kaluza-Klein Theory:

We shall show how theorem 4.2 can be utilized for designing the framework of the 5-dimensional Kaluza-Klein (K-K) theory of relativity [68]. In fact, K-K theory unifies the gravitational and electromagnetic fields by considering the space-time as a 5-dimensional Lorentzian manifold \((M, g)\) of signature \((-++++)\). The real 4-dimensional space-time \(V_4\) is actually embedded as a hypersurface in \(M\). \(V_4\) carries the gravitational field of general relativity. In order to bring electromagnetic field into the space-time K-K theory increases the dimension by 1 (although the modern versions of the theory increase the dimension by an arbitrarily chosen number [14]), nevertheless the extra dimension(s) does(do) not carry any physical significance. In order to bring out a physical significance of the extra dimension Einstein and Bergmann [34] had generalized this theory to some
extent. The basic formalism of the K-K theory consists of a 5-dimensional connected orientable smooth manifold M with a Lorentzian metric g such that there exists a space-like Killing vector field ξ which satisfies: \( L_\xi g = 0 \), and which is also a geodesic vector field, i.e. \( \nabla_\xi \xi = 0 \), where \( \nabla \) is the Levi-Civita connection of g. The Killing equation can be written in the form \( g(\nabla_\xi X, Y) + g(\nabla_\xi Y, X) = 0 \), where X, Y are arbitrary vector fields tangent to M. Substituting \( \xi \) for Y in the last equation entails \( g(\nabla_\xi \xi, \xi) = 0 \) (because \( \nabla_\xi \xi = 0 \)). Hence \( X(g(\xi, \xi)) = 0 \) showing that the norm of \( \xi \) is constant throughout. The skew-symmetric part of the covariant derivative of the dual form \( \eta \) of \( \xi \) plays the role of electromagnetic field tensor. In order to apply theorem 4.2 we let the dimensions of the distributions \( D \) and \( D^\perp \) be 4 and 1 respectively. Under the hypothesis of theorem 4.2, M has a unit space-like Killing vector field \( \xi \) which is also geodesic, as can be seen from equation (4.8, II). Thus all the axioms of K-K theory are fulfilled under the hypothesis of this theorem. The skew-symmetric part of \( (\nabla_\xi \eta) \) is \( (d\eta)(X,Y) \) whose value as computed from equation (4.8, II) is \( |g(\mu, \nu)|^{1/2} \Omega(X,Y) \), where \( \Omega \) is the 2-form associated with P. Thus the 2-form \( (|g(\mu, \nu)|^{1/2})\Omega \) represents the electromagnetic field tensor if any one of the statements (1-4) of the theorem is satisfied. Thus the Sasakian structure of M characterizes the Kaluza-Klein theory. The
electromagnetic tensor field is represented by the fundamental 2-form \( \Omega \) and the potential 1-form by \( \eta \), as \( \Omega = \varphi \eta \). That the potential vector field \( \xi \) defines the axis of symmetry of the electromagnetic field, follows from the consequence:

\[
L_{\xi} \Omega = (d\xi \circ d\eta + i_{\xi} d\eta)\eta = 0,
\]

where \( i_{\xi} \) denotes the interior product operator.

Remark: We observe that in the above application, the signature of \( g \) restricted to \( D \) is \((-+++\)) and \( F \) restricted to \( D \) is an almost Hermitian structure. But we know \([37]\) that the almost Hermitian structure cannot admit a Lorentz signature, unless the underlying almost complex structure is defined by a complex-valued (not a real) \((1,1)\)-tensor field. Thus we must assume \( P \) to be complex-valued. For such a complex-valued almost Hermitian structure we refer to Flaherty \([37]\). In this case the electromagnetic field would be provided by the real part of the complex-valued 2-form \( d\eta \).
4. Application To Space-Times With Orthogonal Transitive Isometry Groups

In this section we discuss the application of CR-submanifolds of Kaehler manifolds such that the $f$-structure induced on them is normal and the normal connection is flat.

For a general $f$-structure one does not have global complementary frames. Hence the notion of normality is normality with respect to a connection in the vector bundle over the manifold $M$ with fibre determined by the complementary (real) part [57]. Let $\{\xi^a\}$ (a, b range from 1 to q) be a local orthogonal basis of $D^\bot$ (it will be global if we take $D^\bot$ parallelizable) and $\{\eta^a\}$ its dual. Then,

\[ P^2 X = -X + \eta^a(X)\xi_a, \quad P\xi_a = 0, \quad \eta^a(PX) = 0. \]

Now let us compute the torsion tensor of this $f$-structure.

First we note: $FX = \Sigma \eta^a(X)J\xi_a$ and $tJ\xi_a = -\xi_a$. Thus,

\[ S(X, Y) = Np(X, Y) - t((\nabla_X F)Y - (\nabla_Y F)X) \]
\[ = Np(X, Y) - t(D_X FY - F_{\nabla_Y} X - D_Y FX - F_{\nabla_X} Y) \]
\[ = Np(X, Y) + d\eta^a(X, Y)\xi_a - \eta^a(Y)tD_X J\xi_a - \eta^a(X)tD_Y J\xi_a. \]

So in general, the last two terms in the above expression are not zero. Of course they would be zero in certain cases: e.g. when $q = 1$; when $M$ has flat normal connection; $M$ has complementary frames and $M = M^* \times R^q$ with $M^*$ Kaehler. Hence in the
aforementioned cases the normality condition \( S = 0 \) reduces to
\[
(4.2) \quad N\eta(X, Y) + d\eta^a(X, Y)\xi_a = 0,
\]
which is the usual normality condition for globally framed \( f \)-manifolds [57]. As the 2-form \( \Omega \) of the CR-submanifold of a Kaehler manifold is closed and under the above mentioned cases the \( f \)-structure is normal in the sense of equation (4.2), we shall call such a CR-submanifold a \( K \)-manifold in Blair's terminology [10].

**Lemma 4.1** On a \( K \)-manifold the following identities hold:
\[
(4.3) \quad (1) \, L_{\xi_a} P = 0, \quad (2) \, [\xi_a, \xi_b] = 0, \quad (3) \, L_{\xi_a} \Omega = 0
\]

Proof: The identities (1) and (2) follow from Goldberg and Yano [46] and (3) follows from Duggal [27].

As the frame \( \{\xi_a\} \) is orthogonal we can choose a positive scalar function \( \sigma \) on \( M \) such that
\[
(4.4) \quad g(X, Y) = e_a\sigma \eta^a(X)\eta^a(Y) + g(\xi X, \xi Y)
\]
where \( g(\xi_a, X) = e_a\sigma \eta^a(X), \quad (e_a = 1 \text{ or } -1) \).

**Theorem 4.1** On a \( K \)-manifold there exists a \( q \)-parameter abelian isometry group generated by the vector fields \( \xi_a \) iff. \( \grad \sigma \) belongs to \( D \).

Proof: First we have
\[
(L_{\xi_a} \Omega)(X, Y) = \xi_a(\Omega(X, Y)) - \Omega([\xi_a, X], Y) - \Omega(X, [\xi_a, Y]) - (L_{\xi_a} g)(\xi X, \xi Y) = 0
\]
where use has been made of (1) and (2) of lemma 4.1. Replacing \( X \) by \( \xi X \) in the preceding equation and the use of structure
equations (4.1) yield
\[ L_{\xi_a} g = \xi_a(\sigma) \sum_b \gamma_b \gamma_0 \gamma^D \eta^D \]
Thus each \( \xi_a \) is Killing iff. \( \text{grad } \sigma \) lies in \( D \). As \( \{\xi_a\} \) is linearly independent set and the abelian property comes from (2) of Lemma 4.1, we infer that there exists a \( q \)-parameter abelian isometry group generated by \( \{\xi_a\} \) iff. \( \text{grad } \sigma \) belongs to \( D \).
For physical applications of this theorem, we first recall the necessary basic theory introduced by Carter [16]. An isometry group in \( M_{2p,q} \) is said to be orthogonally transitive if the \( q \)-dimensional orbits of the group are orthogonal to a family of \( 2p \)-dimensional surfaces. The group is invertible if the isotropy subgroup is an involution which inverts the sense of \( q \) independent directions in the surface of transitivity at a point but leaves unaltered the sense of the directions orthogonal to the surface of transitivity at \( x \). Orthogonal transitivity together with non-null transitive surfaces is a necessary but not sufficient condition for a group to be invertible in a neighborhood of \( x \). A null hypersurface in \( M_{2p,q} \) is said to be an LIH (Local isometry horizon) with respect to a group of isometries if (1) it is invariant under the group, and (2) each null geodesic generator is a trajectory of the group. In particular, an LIH for a 1-parameter group is said to be a Killing horizon. Carter [16] has shown that whenever the surfaces of transitivity
of an orthogonal transitive group do become null, an LIH occurs and that if the group is Abelian, an LIH is a Killing horizon. Now it is clear from theorem 4.1, structure equations (4.1), (4.4) and the above remarks that the abelian isometry group generated by \( \xi_a \) acts orthogonally transitively on \( M_{2p+q} \) with non-null orbits. Thus, there exists a local coordinate system \((x^a, x^A)(A = q+1, \ldots, q+2p)\) on \( M_{2p+q} \) such that the metric \( g \) has its line-element

\[
ds^2 = g_{ab} dx^a dx^b + g_{AB} dx^A dx^B
\]

with \( g_{ab}, g_{AB} \) as functions of \( x^{q+1}, \ldots, x^{q+2p} \) only [18]. Therefore, under the hypothesis of the theorem 4.1, it follows from sections 3 and 4 of [18] that there exists a Killing horizon in a K-manifold whenever an orthogonally transitive abelian group has null surfaces of transitivity.

To regard \( M \) as the space-time of general relativity we let \( p = 1 \) and \( q = 2 \). Since \( M \) is a proper CR-submanifold we cannot take either of \( p \) and \( q \) equal to 0. By theorem 4.1, \( M \) has a 2-parameter abelian isometry group generated by the vector fields \( \xi_1 \) and \( \xi_2 \) iff. \( \text{grad} \sigma \) belongs to \( D \). The 2-parameter abelian isometry groups have been immensely useful for idealized problems in general relativity, of which cylindrical symmetry is perhaps the most popular. However, the most important case is that of stationary axial symmetry applicable to large classes of finite
In the homogeneous and-homogeneous flat Minkowski space-time, the normal vector to the current vector field is $\mathbf{n} = \frac{\partial}{\partial t}$, where $\partial$ is the partial derivative with respect to $t$. In the totally real diagonal part $P$ of the current vector field $J$, we observe that $J = 0$ for $t = 0$. Thus, the vector field $J$ is the current vector field $J$.

The second Maxwell equation (c) yields $\mathbf{D} \cdot \mathbf{E} = 0$, therefore, $\mathbf{D} \cdot \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$. Thus, the differential equation $\mathbf{D} \cdot \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$ provides the current vector field $J$.

The integral of the Maxwell equation (c) yields $\mathbf{D} \cdot \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$.

Therefore, $\mathbf{D} \cdot \mathbf{E} = -\mathbf{A}$.

Furthermore, using Lemma 1.2 of Chapter II, we have:

$$\mathbf{D} \cdot \mathbf{E} = -\mathbf{A}$$

where $\mathbf{D}$ is the mean curvature vector of the homomorphic mapping.

Consider the integral of the Maxwell equation (c) to obtain:

$$\mathbf{D} \cdot \mathbf{E} = -\mathbf{A}$$

Therefore, $\mathbf{D} \cdot \mathbf{E} = -\mathbf{A}$.

Given $\mathbf{G} = \mathbf{G}(\mathbf{E}, \mathbf{A}) = \mathbf{G} \cdot \mathbf{E} + \mathbf{A}$ and $\mathbf{G} = \mathbf{G}(\mathbf{E}, \mathbf{A}) = \mathbf{G} \cdot \mathbf{E} + \mathbf{A}$, we observe that $\mathbf{E} = \mathbf{E}(\mathbf{A})$, $\mathbf{E} = \mathbf{E}(\mathbf{A})$, and $\mathbf{E} = \mathbf{E}(\mathbf{A})$, where $\mathbf{E}$ and $\mathbf{A}$ are the electric and magnetic fields, respectively. Since $\mathbf{A}$ is closed, the Maxwell equation is satisfied.

Consider the fundamental 2-form $\Omega$ of a realistic approximation.
bundle. It is plain from the above analysis that if \( M \) were a
generic submanifold, in particular, then \( J \) would have
automatically vanished identically. Thus the generic \( M \) would
carry a source-free Einstein-Maxwell field. Moreover, we observe
\[ \Omega(\xi_1, \xi_2) = g(P\xi_1, \xi_2) = 0, \]
which is the condition that the tensor
\( \Omega \) be skew-invertible, i.e. that it be affected only by an overall
change of sign when the senses of Killing vectors \( \xi \) are
simultaneously inverted. As the energy-momentum tensor of the
electromagnetic field is homogeneous quadratic in the
electromagnetic tensor, viz.

\[ (\mathcal{Q}) T(X, Y) = g(P^2 X, Y) - g(X, Y) \operatorname{tr} P^2, \]

therefore the relation \( \Omega(\xi_1, \xi_2) = 0 \) implies that the
energy-momentum tensor is invertible, and consequently when no
matter other than the electromagnetic field is present \( (J = 0) \),
that the Einstein tensor is invertible so that the conditions of
the following theorem of Carter [16] are satisfied:

Theorem 4.2(Carter) Let \( D \) be a connected open subdomain of
an \( n \)-dimensional \( C^3 \) manifold with a \( C^2 \) semi-Riemannian metric and
an abelian \( (n-2) \)-parameter isometry group, whose surfaces of
transitivity, which in general are \( (n-2) \)-dimensional, become
degenerate on a subset \( F \) where the group has fixed points.

Then the group will be orthogonally transitive everywhere in \( D \),
and consequently invertible in \( D \), except where the surfaces of
transitivity are null, provided that:— (I) Ricci tensor is invertible in the group everywhere in \( D \); and (II) one of the following holds: (a) \( F \) is non-empty, (b) there is a discrete isometry in some neighborhood in \( D \) consisting of an inversion in a direction orthogonal to the surfaces of transitivity (in other words, an inversion about a hypersurface to which the surfaces of transitivity are tangent). (c) it is known, for any other reason, that the group is orthogonally transitive on at least one point in \( D \).

In our case the condition (I) is true, seen by the following argument: Let \( \xi_3, \xi_4 \) form an orthogonal base of \( D \). We form the scalars:

\[
R_{1J}\xi_1^1\xi_2^J, R_{1J}\xi_3^1\xi_1^1, R_{1J}\xi_2^J, \xi_2^1, R_{1J}\xi_4^1\xi_4^1, R_{1J}\xi_4^1\xi_2^1, R_{1J}\xi_3^1\xi_4^1.
\]

These are invariant of the transformation:

\[
\xi_1 \rightarrow -\xi_1, \xi_2 \rightarrow -\xi_2, \xi_3 \rightarrow \xi_3, \xi_4 \rightarrow \xi_4.
\]

The reason is that the first and last scalars are obviously invariant. The intermediate scalars involve \( R_{1J}\xi_1^1 \) and \( R_{1J}\xi_2^1 \) which render those scalars zero as follows:

The field equations for an Einstein-Maxwell field are:

\[
R_{1J} - \frac{1}{2}r \delta_{1J} = P_{1K}p_{KJ} - (\text{tr} \cdot P^2/4) \delta_{1J}
\]

which implies that

\[
R_{1J}\xi_a^1 = \left[(r/2) - (\text{tr} \cdot P^2/4)\right] \xi_a^J
\]

The above expression coupled with any of \( \xi_3^J \) and \( \xi_4^J \) gives 0. The
condition (II)(c) is true because the distributions $D$ and $D^\perp$ are orthogonal complements of each other.

Thus Carter's theorem which implies that the group is orthogonally transitive, is compatible with our case, because the abelian isometry group generated by $\xi_1, \xi_2$ is already orthogonally transitive everywhere in a connected region of $M$. Moreover, the source current vector $J = (1/2\pi)\nu \cdot D$ lies in the distribution $D^\perp$ and hence is parallel to the 2-surfaces of transitivity (leaves of the foliation $D^\perp$).
CHAPTER IV

CR-STRUCTURES AND
CONFORMAL SYMMETRIES

As a recent excellent example of mutual interplay between the Cauchy-Riemann structure and physical space-time geometry [66], I present, in this chapter, a few fresh ideas on this fruitful relationship with respect to the conformal geometry and the groups of collineations of Lorentzian (more generally, semi-Riemannian) manifolds.

1. Introduction
In the early 1930's, the Riemannian geometry and the theory of complex variables were synthesized by Kaehler [55] and Schouten et al [96] which further developed (during 1950's) into what is now known as the complex manifold theory [31, 33, 38, 77]. For these spaces, there is an atlas of complex coordinate charts such that in the intersection region, the coordinate transformations are holomorphic. Simple examples are: A Riemann surface, $\mathbb{C}R$ and its projective space $\mathbb{C}P^{n-1}$. In general, any complex manifold of dimension $n$ can be considered as a real (analytic) manifold of
dimension 2n with \( z_T = x_T + iy_T \) as complex coordinates for \( x_T, y_T \) as real coordinates. There is a holomorphic tangent bundle associated with any complex manifold whose tangent vector has the form \( V = v^z \partial z_T \), as opposed to the form of a complex-valued vector, \( V = v^z \partial z_T + V^z \partial z_T \). There is a direct relation between the holomorphic tangent bundles and the Cauchy-Riemann (CR)-structures in the following way. Let us recall the definition: A differentiable manifold \( M \) has a CR-structure if there exists a holomorphic sub-bundle \( H \) of the complexified tangent bundle \( CTM \) of \( M \), such that \( H \cap \bar{H} = \{0\} \) and \( H \) is involutive, i.e. \([X,Y] \) belongs to \( H \) for every \( X, Y \) in \( H \). The complex manifold theory has also been used in many areas of Mathematical Physics (see, for example, [69]). In particular, Penrose [86] has recently shown that a CR-structure on a 5-real-dimensional manifold, as a hypersurface of a complex manifold, has direct connection with the physical space-time geometry (see also Lebrun [66]).

2. CR-structure

Let \((M,g)\) be a semi-Riemannian manifold of dimension \( n \), isometrically embedded in a complex manifold. \( M \) has a CR-structure if in the tangent space \( T_x(M) \) at each point \( x \) of \( M \), a 2p-real-dimensional subspace \( H_x \) of \( T_x(M) \) is singled out, called
the holomorphic tangent space. $H_x$ regarded as $p$-dimensional complex space and spanned by the vectors $Z_r = X_r + iY_r$ for every $r$ $(1 \leq r \leq p)$ provides a linear operator $J$, satisfying $J^2 = -I$. Explicitly, $JX_r = -Y_r$, $JY_r = X_r$ so that $JZ_r = iZ_r$. To complete a basis for the entire $T_x(M)$ one needs a complementary set of $n-2p$ vectors. In the sequel, we assume that such a basis exists in some neighborhood $U_x$ of $x$ in $M$. It follows from a result given by Newlander and Nirenberg (76) that a real-analytic CR-structure can be realized in the above way if the following integrability relations hold:

\[(2.1) \quad [Z_r, Z_s] = \text{complex linear combinations of } Z_r's\]

where $1 \leq r, s \leq p$. However, for a $C^\infty$ CR-structure it has been shown (80) by counter examples that the relations (2.1) are not sufficient. Therefore, in the latter case a non-realizable CR-structure may arise (for example, see Penrose [86]). We shall confine our study only to those CR-structures which can be realized in the above way. The mathematical theory for the existence of a CR-structure on $M$ is primarily based on the theorems of Clarke (21) and Greene (49), on the local (global) isometric embeddings of semi-Riemannian manifolds.

To answer, "why study CR-structures?" for physical interpretations we quote the following (besides the reference of Penrose [86]):
Clarke [21] "isometric embeddings of space-times might shed some light on a number of 'global questions' concerning singularities and causality properties."

Kramer et al [64] "Global embedding may give a deeper insight into the geometric properties of space-time. In fact, the maximal analytic extension of the Schwarzschild solution was found by the method of embedding. In particular reference to the exact solutions, they said, "No systematic analysis of global embedding of exact solutions has yet been done."

Beem and Ehrlich [3] "The knowledge that space-times may be regarded as embedded submanifolds of certain semi-Euclidean spaces allows one to use semi-Euclidean knowledge to obtain space-time knowledge. Hence a detailed study relating extrinsic geometric structures to physically meaningful properties is needed."

3. CR-structure and Conformal Geometry

It was first pointed out by Hermann Weyl [107] that the existence of a family of null cones leads to the conformal geometry of the space-time. Since then, the conformally invariant property has been an essential geometric prescription for a good part of physics. For example: all equations of massless particles [84], including the Yang-Mills equations [106], are conformally
invariant. The Maxwell equations governing the electromagnetic field are also known to be conformally invariant. The conformal ideas have played a key role in finding the exact solutions of Einstein's field equations (see, e.g., Goldberg and Sachs [43], Robinson and Trautman [91], Robinson and Schild [90], and Kerr [662]). Also, the conformal invariance is the root of the twistor program [85]. As there are evidences that a CR-structure is physically meaningful [80, 86], it is reasonable to seek for its possible interplay with the conformal geometry of the physical space-time. We proceed as follows:

A vector field \( k \) on \( M \) generates a local one-parameter group of transformations called a flow \( (f_t) \) on \( M \). For non-vanishing \( k \) there exists a distribution \( K \) of \( 1 \)-dimensional subspaces of \( T_x(M) \) to \( M \) which is invariant under the action of the flow \( (f_t) \). Let there exist a conformal structure on \( M \). This means that there is an equivalence class of semi-Riemannian metrics \( \langle g \rangle \) such that two metrics \( g \) and \( g^* \) belong to the same class iff. \( g^* = \Omega g \), where \( \Omega \) is a positive function on \( M \). A conformal structure \( \langle g \rangle \), along with \( K \), defines another distribution \( K^1 \) of \( (n-1) \)-dimensional subspaces. At any point \( x \) of \( M \), the sign \( e(\hat{x}) \) of \( g(k(x), k(x)) \) is a conformal invariant which we assume to be constant over \( M \). The distributions \( K \) and \( K^1 \) define vector bundles over \( M \). If \( k \) is non-null, then
(3.1) \[ TM = K \otimes K^\perp \quad \text{and} \quad K \cap K^\perp = \{0\} \]

In case \( K \) is null, i.e., \( e = 0 \), \( K \) is a sub-distribution of \( K^\perp \), and we have the exact sequence

(3.2) \[ 0 \rightarrow K \rightarrow K^\perp \rightarrow K^\perp / K \rightarrow 0 \]

The fibres of the quotient bundle \( S = K^\perp / K \) are \((n-2)\)-dimensional, and are well-known as screen spaces.

For a physical interpretation, let \( M \) be the real 4-dimensional space-time and \( K \) a null vector field (in the next section we will consider the case when \( K \) is time-like). The conformal structure of \( M \) induces a complex structure on the screen space associated with \( K \) in the following way. \( S \) can be seen as an oriented plane with a conformal structure. Let \( J \) be a rotation in \( S \) through 90°. Clearly, \( J^2 = -I \). Thus, \( J \) defines a complex structure on \( S \). The complexified space \( CS \) is represented as a direct sum \( S_+ \oplus S_- \), where

(3.3) \[ S_\pm = \{ Z \in CS : JZ = \pm \text{sign} Z \} \]

Let \( K_+ \) be the subspace of \( CK^\perp \) projecting onto \( S_+ \) by the canonical map \( CK^\perp \rightarrow CS \). Clearly, \( K_+ \) is maximal as it is 2-dimensional, and also is a totally null subspace of \( CTM \). The fact that a CR-structure can be locally realized on \( M \) with respect to \( K_+ \) comes from the Riemann mapping theorem, which states that any smooth bounded simply connected region in the Argand plane \( C^1 \) is holomorphically identical to the unit disk. Thus, a CR-structure
of a physical space-time is directly related to its conformal geometry.

Remarks: (a) It was pointed out by Poincare [86] back in 1907, that the Riemann mapping theorem for $C^1$ has no direct analogue in higher complex dimension. This is why, in general, the CR-structures can be locally distinct from one another. For detailed study on this matter and non-realizable CR-structures we refer to Penrose [86].

(b) Lebrun [86] has shown a relation between the conformal geometry of a 3-dimensional manifold and CR-5-manifolds.


In this section, we assume that the vector field $k$ introduced in the previous section, is time-like. Thus, if $M$ has a conformal structure [36] then the equation (3.1) will hold. It is easy to see that the vector bundle $K^2 \rightarrow M$ will have a conformal structure, in a natural way, induced by [9]. Indeed, for any $x$ in $M$, the vector space $K(x)$ will be endowed with an equivalence class of scalar products, two scalar products being equivalent iff. one is a positive multiple of the other.

To see an interplay between a CR-structure of $M$ and a family of time-like curves generated by $k$, we let $M$ be a 4-dimensional space-time manifold isometrically embedded as a hypersurface of a
5-dimensional manifold $M$. Thus, following Lebrun [66], one can show that the 3-dimensional manifold, of the distribution $K$, with its conformal structure induced by $[g]$, has an associated CR-5-manifold $M$.

Remarks (c) The time-like curves are of vital importance to the study of fluids in general relativity. In fact, the integral curves of the fluid 4-velocity vector form a family of time-like curves. The general study on such curves can be traced back to the work of Eisenhart [36, 1924], followed by Synge [101, 1937], Gödel [41, 1952], Lichnerowicz [71, 1955] and several others. The most important result was the Raychaudhuri's equation [89] showing directly the influence of the fluid matter on the convergence of a family of time-like geodesics, together with the related property for time-like geodesics. This convergence property for time-like geodesics, together with the related property for null geodesics (emerging from the subject matter of section 3 above), is now the main root of the singularity theorems [54].

(d) Recently, Crampin and Prince [23] have studied geodesics not as curves and local vector fields but in the form of a global vector field on the tangent bundle, called "geodesic spray". In view of the importance of this topic, we reproduce some of their remarks: "It is in principle more straightforward to study a
single globally defined vector field than an infinite collection of vector fields each of which is, in general, only locally defined. The concept of geodesic completeness of the space-time manifold is replaced by the concept of completeness, as a vector field, of the geodesic spray on the tangent bundle. Certain important geometrical construction; such as Jacobi fields, have a more straightforward definition and interpretation in terms of the spray than on the space-time manifold.

5. CR-structure and Conformal Collineation

It has been known since the work of Noether [61], in 1918, that the existence of certain symmetry properties described by continuous groups of motions, has a direct relation with the conservation laws in the form of first integrals (constants of motion) of a dynamical system. For example, see Trautmann [104], Sachs [94] and Komar [63]. In particular, for physical space-time, the study on various symmetry properties, such as groups of motions, has been of fundamental importance [22, 27, 59]. In this section, I will show that there exists a relationship between a particular symmetry called "Conformal Collineation" (Conf C) and a CR-structure. In the sequel, we assume that (M,g) is a space-time, i.e., a time oriented Lorentzian manifold.

A vector field ξ on M generates a conformal motion (Conf M)
if it satisfies:

\[(5.1) \quad L_\xi g = 2\sigma g,\]

where \(\sigma\) is a real scalar function on \(M\). A (conf \(M\)) must satisfy

\[(5.2) \quad (L_\xi \nabla)(X, Y) = (X\sigma)Y - (Y\sigma)X - g(X, Y)\text{grad} \sigma,\]

\(\nabla\) being the Levi-Civita connection of \((M, g)\), but the converse is not necessarily true [110]. However, it has been shown [98] that the equation (5.2) is equivalent to:

\[(5.3) \quad L_\xi g = 2\sigma g + h\]

where \(h\) is a symmetric parallel \((0, 2)\)-tensor, i.e. \(\nabla h = 0\). Thus the equation (5.3) does not pull back to \((\text{Conf} M)\) unless \(h\) is proportional to \(g\). Therefore, it is logical to know the geometric/physical meaning (if any) of the equation (5.3). For this purpose, we are obliged to introduce the following concept:

Definition. \((M, g)\) admits a Conformal Collineation (Conf C) if there exists a vector field \(\xi\) for which (5.3) holds. \(\xi\) will be called an Affine Conformal Vector field.

It will be shown in the following chapters that the notion of (conf C) has the prospect of potential geometric and physical applications. Let us now examine how the above subject matter may be related with a CR-structure:

Let \(M\) be a submanifold of a semi-Riemannian manifold \(\bar{M}\), defined by an isometric embedding. Consider an affine conformal vector field \(\xi\) in \(\bar{M}\). \(\xi\) can be decomposed into its tangential part
and the normal part \( n \). Using equation (5.3) we have

\[(5.4) \quad (L_k + n g)(X, Y) = 2g(X, Y) + h(X, Y),\]

Thus, we find

\[(L_k g)(X, Y) = 2g(X, Y) - g(\overline{\nabla}_X n, Y) - g(X, \overline{\nabla}_Y n) + h(X, Y),\]

where \( \overline{\nabla} \) is the Levi-Civita connection of \( \overline{M} \). Using the Weingarten equation provides that

\[(L_k g)(X, Y) = 2g(X, Y) + 2g(AX, Y) + h(X, Y),\]

where \( A \) stands for the shape operator of \( M \) with respect to \( n \).

Suppose \( 2g(AX, Y) = - h(X, Y) \). Thus, with the above hypothesis we get \( L_k g = 2g \). Hence we have shown that a (Conf \( C \)) in \( M \) can generate a (Conf \( M \)) in \( M \).

To see an interplay of a CR-structure with the (Conf \( C \)), we let \( (M, g) \) be a space-time of dimension 4. If the conformal vector field \( k \) on \( M \) is null, then a CR-structure can be realized through the screen space \( S = \mathbb{R}^4 / \mathbb{R} \) as explained in section 3. However, if \( k \) is time-like, then we let \( M \) be a hypersurface isometrically embedded in a 5-dimensional manifold \( \overline{M} \). Therefore, as explained in section 4, the 3-dimensional manifold, of the distribution \( \mathbb{R}^4 \), with its conformal structure induced from \( (M, g) \) has an associated CR-structure on \( \overline{M} \) with a (Conf \( C \)). Thus, we have established a relationship between a CR-structure and a conformal collineation in a 5-dimensional manifold \( \overline{M} \).
CHAPTER V

CONFORMAL COLLINEATIONS IN A SEMI-RIEMANNIAN MANIFOLD

1. Conformal collineations.
We would like to recall the concept of a conformal collineation introduced in the preceding chapter and then characterize it in certain geometric and physical spaces.

A vector field \( \xi \) is said to generate a one-parameter group of conformal motions (Conf \( M \)) in \((M,g)\) if

\[
(1.1) \quad L_\xi g = 2\sigma g,
\]

where \( \sigma \) is a smooth scalar function on \( M \). The generating vector field \( \xi \) is called conformal vector field. In particular, when \( \sigma \) is zero (non-zero constant), we call \( \xi \) a Killing (Homothetic) vector field and the corresponding infinitesimal transformations generated by them as motion (homothetic motion). A conformal vector field \( \xi \) satisfies

\[
(1.2) \quad (L_\xi \nabla)(X,Y) = (X\sigma)Y + (Y\sigma)X - g(X,Y)\text{grad} \sigma
\]

Conversely, if a vector field \( \xi \) satisfies (1.2) it is not necessarily conformal. Yano [110] has mentioned indirectly that, for a compact orientable positive definite Riemannian manifold
without boundary, equation (1.2) implies: \( L_\xi g = 2\sigma g + cg \) (\( c \) being a constant). This implication may not be valid for a non-compact or an indefinite Riemannian manifold (e.g. the space-time manifold of general relativity). Relaxing the non-compactness but still retaining the positive definite metric, Tashiro [102] proved: if a Riemannian (positive definite) manifold is irreducible, then (1.2) implies \( L_\xi g = 2\sigma g + cg \) (\( c \) constant).

Definition: 1.1 (Tashiro [102]) A vector field \( \xi \) generates a 1-parameter group of conformal collineations (Conf C) if it satisfies (1.2). \( \xi \) is called an affine conformal vector field (cf. the definition of an affine Killing vector field [110]).

In particular, when \( \sigma = \text{constant} \), a (Conf C) is called an affine collineation (AC) and the generating vector field is called an affine Killing vector field.

The foregoing results led me to study the (Conf C) in a semi-Riemannian manifold in general (not necessarily positive definite or compact or irreducible). A basic characterization of (Conf C) is given by

Theorem 1.1. A vector field \( \xi \) generates a (Conf C) iff

\[
L_\xi g = 2\sigma g + h
\]

where \( h \) is a second order covariant constant tensor field (also known as a special quadratic first integral).

Proof: Following the setting \( \nabla(X,Y) = \nabla_X Y \) we have
(1.4) \[ L_\xi \nabla_X Y - \nabla_X L_\xi Y = (L_\xi \nabla)(X, Y) + \nabla_{[\xi, X]} Y. \]

Using this it is straightforward to show that

(1.5) \[ (\nabla_X L_\xi g)(Y, Z) = g(Y, L_\xi \nabla_X Z - \nabla_X L_\xi Z) - \]

\[ [\xi, X] g(Y, Z) + g(Z, [L_\xi \nabla_X, \nabla_X L_\xi Y]. \]

The above two equations give

(1.6) \[ (\nabla_X L_\xi g)(Y, Z) = g(Y, (L_\xi \nabla)(X, Z)) + g(Z, (L_\xi \nabla)(X, Y)). \]

Now let \( \xi \) generate a (Conf C). Then use of (1.2) in (1.6) yields

(\nabla_X L_\xi g)(Y, Z) = 2(\sigma)g(Y, Z).

Thus, \( \nabla_X (L_\xi g - 2\sigma g) = 0. \) Hence \( L_\xi g = 2\sigma g + h, \) where \( \nabla h = 0. \)

Conversely, if \( L_\xi g = 2\sigma g + h, \) such that \( \nabla h = 0; \) then (1.6) implies

\[ (g(Y, (L_\xi \nabla)(X, Z)) + g(Z, (L_\xi \nabla)(X, Y)) = 2(\sigma)g(Y, Z). \]

Hence one readily obtains.

\[ g(Z, (L_\xi \nabla)(X, Y)) = (\sigma)g(Y, Z) + (Y\sigma)g(X, Z) - g(X, Y)(Z\sigma) \]

Hence (1.2) follows. This completes the proof.

Now we discuss some geometric properties of the (Conf C).

Let \( \{\sigma_t : t = \text{an infinitesimal real parameter}\} \) be the \( 1 \) parameter group of local transformations (Conf C's) generated by the affine conformal vector field \( \xi. \) Then the equation (1.3) and its equivalent equation (1.2) can be expressed as follows:

(1.7) \[ g^* = (1 + 2t\sigma)g + t h \]

(1.8) \[ \nabla^* X Y = \nabla_X Y + t \{(X\sigma)Y + (Y\sigma)X - g(X, Y)(\text{grad} \sigma)\} \]

where \( g^*(X, Y) = g((d\sigma_t)X, (d\sigma_t)Y), \) and
\[ \nabla^t_X Y = \nabla_t (d\omega_t)_X (d\omega_t)_Y, \]

\(d\omega_t\) denotes the Jacobian differential of \(\omega_t\). It follows that:

(i) A null vector field \(U\) will be transformed by \(\omega_t\) into a null vector field iff. \(h(U, U) = 0\) (i.e. the causal character is, in general, not preserved by (Conf C)).

(ii) A non-null vector field retains its causal character. In general, Conf C is not angle preserving. In particular, two orthogonal vector fields \(U\) and \(V\) will be transformed into orthogonal vector fields \((d\omega_t)U\) and \((d\omega_t)V\) iff. \(h(U, V) = 0\).

It is well-known [106] that, under a conformal motion a geodesic does not, in general, remain a geodesic unless it is null and in this case the transformed geodesic is again null. Let us consider the analogous situation in the case of (Conf C). Let \(U\) be the tangent vector field to an affinely parametrized geodesic so that \(\nabla^t_t U = 0\). Thus (1.8) implies:

\[ \nabla^t_t U = t(2(U U_U) - g(U, U)\text{grad } \omega)\]

We observe that the image of the geodesic is not in general, a geodesic unless \(g(U, U) = 0\) in which case:

\[ \nabla^t_t U = 2t(U U_U)\]

It follows, therefore, that the transformed geodesic is not affinely parametrized. This is also true of a Conf M. But, as discussed earlier, a null geodesic is transformed by a (Conf C) into a geodesic which may not be again null.
2. A Non-trivial Example of Conformal Collineations

Consider a non-degenerate submanifold \((M, g)\) of a semi-Riemannian manifold \((\tilde{M}, \tilde{g})\), defined by an isometric embedding. Let us consider a conformal vector field \(V\) on \(\tilde{M}\). Then \(V\) is uniquely decomposed into its tangential part \(\xi\) and normal part \(v\). As \(V\) is conformal in \(M\), we have

\[(L_{\xi} + v)g(X, Y) = 2\sigma g(X, Y),\]

An alternative form of the above equation is:

\[(L_{\xi} g)(X, Y) = 2\sigma g(X, Y) - \tilde{g}(\nabla_{X} V, Y) - \tilde{g}(X, \nabla_{Y} V)\]

Application of Weingarten formula to the foregoing eqn. gives

\[(L_{\xi} g)(X, Y) = 2\sigma g(X, Y) + 2g(AX, Y),\]

where \(A\) stands for the shape operator of \(M\) corresponding to the normal vector field \(v\). Let \(A\) be parallel with respect to the Levi-Civita connection of \((M, g)\). Then we define a second order tensor field \(h\) on \(M\) such that \(2g(AX, Y) = h(X, Y)\). Consequently we obtain:

\[L_{\xi} g = 2\sigma g + h,\]

where \(h\) is parallel. Thus \(\xi\) is an affine conformal vector field and hence generates a \((\text{Conf } C)\) in \(M\).

Remark 1. In the above example, for a totally umbilical \((M, g)\), \(\xi\) would reduce to a conformal vector field.

Remark 2. In the above example, if \((M, g)\) were taken as a compact orientable Riemannian submanifold without boundary, of \((M, g)\); then it would reduce to a totally geodesic submanifold,
such that the tangential part of $V$ would reduce to a conformal vector field in $(M,g)$.

3. Geometric Description of Conformal Collineations

Tashiro [102] has proved that if a Riemannian manifold $M$ is irreducible, then a $(\text{Conf} \ C)$ on $M$ is a $(\text{Conf} \ M)$. Also, he has characterized $(\text{Conf} \ C)$ in various reducible Riemannian manifolds and has concluded that in those cases the $(\text{Conf} \ C)$ is a sum of an affine collineation and a Conformal motion.

A semi-Riemannian manifold is conformally recurrent [75] if $\nabla_X C = k(X)C$, where $k$ is the recurrence 1-form and $C$ is Weyl conformal curvature tensor (A more general class of manifolds has been discussed, as 4-dimensional space-times, by McLenaghan and Leroy [75] as follows: A complex recurrent space-time is a space for which the self dual Weyl tensor satisfies:

$$C^*_{abcd;e} = C^*_{abcd} \cdot k_e,$$

where $k_e$ is the complex recurrence vector, semi-colon denotes covariant differentiation and $C^*_{abcd} = C_{abcd} + iC\tilde{a}bcd$ and $C\tilde{a}bcd$ is the dual of $C_{abcd}$ defined as $C\tilde{a}bcd = (1/2)C_{cdef}C_{ab}^{cdef}$, $C_{cdef}$ being the components of the Levi Civita 4-form

$$C = \sqrt{-g} e^1 \wedge e^2 \wedge e^3 \wedge e^4$$

where $g$ is the determinant of the metric tensor components $g_{ab}$ with respect to a positively oriented basis $\{e_a\}$ such that its
chial is \( \mathbb{R}^3 \). If \( k_0 \) is real, then the space is conformally recurrent as defined above. When \( k_0 = 0 \), it is conformally symmetric. A recurrent (symmetric) manifold is conformally recurrent (symmetric). It is called essentially conformally recurrent if it is conformally recurrent but is neither conformally flat nor recurrent. Goryachev [51] proved that if an essentially conformally recurrent manifold with a locally exact recurrence form, admits a (Conf C), then

\[
(3.1) \quad h = \sigma g + \tau \text{Ric}
\]

The above result also holds for a conformally flat space [70]. Thus it would be interesting to consider (Conf C) in a locally symmetric manifold. We will accomplish it later (Theorem 4.1).

In fact, the study of a (Conf C) involves the study of covariant, constant \((0,2)\)-symmetric tensor fields, or the special quadratic first integrals (SQFI's). Eisenhart [35] proved that a positive definite Riemannian manifold admitting a SQFI linearly independent of the metric tensor, is reducible. Patterson [63] generalized this result for the indefinite metrics as follows:

"If an indefinite Riemannian manifold admits a SQFI (linearly independent of the metric tensor) which has at least two distinct latent roots at any point, then it is reducible." Thus it is more interesting to study (Conf C) in indefinite spaces, than in definite spaces. We now present the effect of a (Conf C) on the
Various curvature quantities as follows:

\[(3.2) \quad (L_\xi R)(X,Y,Z) = (((\nabla_\xi \sigma)Z)Y - ((\nabla_\xi \sigma)Z)X\]
\[+ g(X,Z)\nabla_\gamma \text{grad } \sigma - g(Y,Z)\nabla_\chi \text{grad } \sigma\]

\[(3.3) \quad (L_\xi \text{Ric})(Y,Z) = -(n-2) (\nabla_\xi \sigma)Z + (\Delta \sigma)g(Y,Z)\]

\[(3.4) \quad (L_\xi Q)X = -(n-2) \nabla_\sigma \text{grad } \sigma + (\Delta \sigma)X - 2\text{Q}_{X} - HQX\]

\[(3.5) \quad L_\xi F = 2(n-1)\Delta \sigma - 2\sigma_{\text{tr.}} \text{tr.} HQ\]

where \(\Delta \sigma = - \text{div grad } \sigma\) and \(g(HX,Y) = h(X,Y)\).

It is very important to bear in mind that \(\delta \xi = - n\sigma - \frac{1}{2}(\text{tr.} H)\) and \(\text{tr.} H\) is constant. The variation in the Weyl tensor is:

\[(3.6) \quad (L_\xi C)(X,Y,Z) = (1-n-2)g(Y,Z)HQX - g(X,Z)HQY - h(Y,Z)QX\]
\[+ h(X,Z)QY + \text{tr.}(h(Y,Z)X - (h(X,Z)Y)/(n-1)(n-2))\]
\[= (\text{tr.} HQ)(g(Y,Z)X - g(X,Z)Y)/(n-1)(n-2).\]

4. Special Conformal Collineation

In this section we shall study the (Conf C's) that can define other important symmetry properties.

Definition 4.1. (Katzin, Levine and Davis [59]) A vector field \(\xi\) is said to generate a Curvature Collineation \((C C)\) if it satisfies: \(L_\xi R = 0\).

Definition 4.2. A vector field \(\xi\) is said to generate a Ricci Collineation \((R C)\) if it satisfies: \(L_\xi \text{Ric} = 0\).

Definition 4.3. A vector field \(\xi\) is said to generate a Weyl Collineation \((W C)\) if it satisfies: \(L_\xi C = 0\).
(C C) is a fundamental symmetry property of the physical space-time. Its investigation is strongly motivated by the all-important role of the Riemann curvature tensor $R$ in the general theory of relativity. Katzin, Levine and Davis [59] have shown that the existence of a certain type of (C C) leads directly to the existence of a cubic first integral of a mass particle with geodesic trajectory. Besides, they have found that if a 4-dimensional space-time with non-vanishing Ric and with vanishing scalar $r$ admits a (C C), then a field conservation law results. This conservation law is directly related to a conservation law obtained by Sachs [94] for null electromagnetic radiation fields. Also, they have shown that their result can be extended to pure null gravitation radiation fields which were also treated by Sachs. Furthermore, the identity of Komar [63], which serves as a covariant generalization of field conservation laws in the general relativity theory, appears in a natural way as an essentially trivial necessary condition for the existence of a (C C) in an n-dimensional semi-Riemannian manifold. Katzin et al [59] have shown that for space-times with zero Ricci Ricci tensor, more familiar symmetries such as projective and conformal collineations (including affine collinations, motions, conformal and homothetic motions) are subcases of a (C C). Thus (C C) appears to be the fundamental symmetry property of space-time to
consider in the study of conservation laws pertaining to gravitational radiation. We give below a necessary and sufficient condition for a (Conf C) to be a (C C):

Proposition 4.1. A (Conf C) is a (C C) iff. $\nabla \nabla \sigma = 0$.

Proof. In virtue of the identity (110):

$$(\nabla_x (L_x^C))(Y, Z) - (\nabla_y (L_y^C))(X, Z) = (L_x^R)(X, Y, Z)$$

we see that $\xi$ generates (C C) iff.

$$(\nabla_x (L_x^C))(Y, Z) = (\nabla_y (L_y^C))(X, Z).$$

As $\xi$ generates a (Conf C), the above condition reduces to:

$$g(\nabla_x \text{grad} \sigma, Z)Y - g(\nabla_y \text{grad} \sigma, Z)X = g(Y, Z)\nabla_x \text{grad} \sigma$$

$$= g(X, Z)\nabla_y \text{grad} \sigma$$

By contradiction, we can easily show that the above condition is equivalent to $\nabla \nabla \sigma = 0$. This proves the proposition.

Remark: The expression (3.6) for $L_x^C$ does not involve $\nabla \nabla \sigma$ at all and therefore remains (indirectly) the same even when we restrict the (Conf C) to (C C).

Let us consider (Conf C) in a Ricci-flat (Empty, in general relativity) manifold of dimension $\geq 3$. Then (3.3) gives $\nabla \nabla \sigma = 0$. Thus, we obtain $L_x^R = 0$, in view of proposition 4.1. Eventually $L_x^C = L_x^R = 0$. Hence, we state:

Proposition 4.2. A (Conf C), in a Ricci-flat space is (C C) and hence ($\xi^C$).

More generally, suppose $\xi$ generates (Conf C) in an Einstein
space. Then $\text{Ric} = \frac{1}{n} g$ provides $L_g \text{Ric} = \frac{1}{n} L_g g$ (r being constant) which becomes, in view of (3.3) and (1.3):

$$-(n-2)(\nabla^2 \phi) + (\Delta \phi) g(Y, Z) = \frac{1}{n} (2\sigma + h)(Y, Z).$$

Also, from (3.5) we observe

$$\sigma = 2(n-1) \Delta \phi - 2\sigma - \frac{1}{n} \text{tr} H.$$

The last two equations together obtain

$$(4.1) \quad \nabla \sigma = -\left[\frac{1}{n} (n-1) \sigma + \frac{1}{n} 2n(n-1)(n-2) \right] \left[ \text{tr} H \right] g - 2(n-1) h.$$  

Operating by $\nabla$ yields

$$(4.2) \quad \nabla \nabla \sigma = -(\frac{1}{n} (n-1)) (\nabla \sigma) \otimes g.$$  

Thus $\sigma$ satisfies a system of third order partial differential equations. For a $(\text{Conf} M)$, we would have had the following system of second order partial differential equations

$$\nabla \sigma = -(\frac{1}{n} (n-1)) \sigma.$$

A $(\text{Conf} C)$ in an Einstein space of dimension $> 2$ is a $(C \ C)$ iff it is merely a motion (isometry). For, let $\xi$ generate $(\text{Conf} \ C)$ in an Einstein space. Then we have equation (4.1). It follows, in the light of proposition 4.1 that $\xi$ generates $(C \ C)$ iff

$$2(n-1) h = (\text{tr} H) g - 2n(n-2) g.$$  

This can be seen, upon a little computation, to be equivalent to:

$L_\xi g = 0$. Trivially, $\xi$ generates $(\text{WC})$. In general, it remains an open problem to find a proper $(\text{Conf} C)$ which is also $(\text{WC})$.

Definition 4.4. A $(\text{Conf} \ C)$ is called a special conformal collineation $(S \text{Conf} C)$ if it is also a $(C \ C)$. 
For $(S \text{ Conf } C)$, the eqns. (3.3)–(3.5) assume the simpler forms

\begin{align*}
(4.3) \quad & L_\xi \text{Ric} = 0 \\
(4.4) \quad & L_\xi \text{Q} = -2\sigma \text{Q} - H\text{Q} \\
(4.5) \quad & L_\xi F = -2\sigma r - \text{tr.} H\text{Q}
\end{align*}

As pointed out in Sec. 3, we now investigate $(S \text{ Conf } C)$ in a locally symmetric (non-Einstein) manifold. In this connection we prove the following theorem:

**Theorem 4.1.** If a non-Einstein locally symmetric manifold admits an $(S \text{ Conf } C)$, then either

1. the $(S \text{ Conf } C)$ reduces to an $(A \text{ C})$, or
2. grad $\sigma$ is a Killing vector with light velocity and the scalar curvature of the manifold vanishes identically.

**Proof:** Suppose a non-Einstein locally symmetric manifold $M$ admits an $(S \text{ Conf } C)$ generated by a vector field $\xi$. Then we already know that $\nabla \nabla \sigma = 0$. Using the well-known identity:

\[ L_\xi \nabla X - \nabla X L_\xi - \nabla_{[\xi, X]} Y = (L_\xi \nabla)(X, Y) \]

we can straightforwardly derive:

\begin{align*}
(4.6) \quad & (L_\xi \nabla R - \nabla X L_\xi R - \nabla_{[\xi, X]} R)(Y, Z, W) = (L_\xi \nabla)(X, R(Y, Z, W)) - R((L_\xi \nabla)(X, Y), Z)W - R(Y, (L_\xi \nabla)(X, Z))W - R(Y, Z)(L_\xi \nabla)(X, W).
\end{align*}

As the manifold is locally symmetric, $\nabla R = 0$, and hence $\nabla \text{Ric} = 0$ and $\nabla r = 0$. Hence $r$ turns out to be constant. Now using the fact that $\xi$ generates $(C \text{ C})$, we have $L_\xi R = 0$. Hence from the equation (4.6) we find.
\[ g(R(Y, Z)W, \text{grad} \sigma)X - g(R(Y, Z)W, X)\text{grad} \sigma - (Z\sigma)R(Y, Z)W \\
- (Yo)R(X, Z)W - (Z\sigma)R(Y, X)W - (Yo)R(Y, Z)X + g(X, Y)R(\text{grad} \sigma, Z)W \\
+ g(X, Z)R(Y, \text{grad} \sigma)W + g(X, W)R(Y, Z)\text{grad} \sigma = 0. \]

Substituting \( \text{grad} \sigma \) for \( X \) in the above equation yields:
\[ g(\text{grad} \sigma, \text{grad} \sigma)R(Y, Z)W = 0. \]

This implies that \( g(\text{grad} \sigma, \text{grad} \sigma) = 0 \). Because the space is not flat. Moreover, since \( \forall \sigma = 0 \), \( \text{grad} \sigma \) is a Killing vector field. Thus we proved that \( \text{grad} \sigma \) is a Killing vector with light velocity. Contraction of the last but one equation entails:
\[ (Yo)QZ - (Z\sigma)QY = 0. \]

Contracting further gives
\[ (4.7) \quad (Yo)r = (QY)\sigma = \text{Ric}(\text{grad} \sigma, Y) \]

Now, the fact that \( \text{grad} \sigma \) is parallel, leads to \( Q(\text{grad} \sigma) = 0 \).

Employing this consequence in (4.7) we get \( (Yo)r = 0 \). As a result either (1) \( \sigma \) is constant, or (2) \( r = 0 \). We, therefore, conclude that either (1) \( (S, \text{Conf}(C)) \) reduces to \( \text{an} \quad (A, C) \), or (2) \( \text{grad} \sigma \) is a Killing vector with light velocity and the scalar curvature vanishes identically. This completes the proof.

Remark 1: We shed some light on the hypothesis of the foregoing theorem with the background of the example cited in section 2. Let us take \( M \) as a space-form and \( M \) as its real hypersurface. Set \( n = \sigma N \), where \( N \) is the unit normal vector field to \( M \). As the shape operator \( A \) was taken parallel in \( M \),
therefore $\Lambda$ is recurrent and hence by the theorem of Matsuyama (Thm. B, [73]): A recurrent hypersurface $H$ in a real space-form is locally symmetric. We observe that $M$ is locally symmetric. The tangential part $\xi$ of $\xi$ in $\bar{M}$, defines a $(\text{Conf C})$. We also note that $\xi$ can define $(\text{C C})$ in $M$ without reducing to a Killing vector field, whereas $\xi$ could not have done so in $M$.

Remark 2. As mentioned in the beginning of Sec. 4, Katzin et al. [59] showed that if a 4-dimensional space-time with non-vanishing Ricci tensor and with vanishing scalar curvature admits a $(\text{C C})$ then a field conservation law results. The theorem 4.1 concludes that $R = 0$, but $\text{Ric} \neq 0$ (otherwise $M$ would become Einstein, violating the hypothesis of the theorem). Moreover $M$ has a $(\text{C C})$. Therefore all the requirements of the above mentioned result of Katzin et al. are met by the hypothesis of theorem 4.1. Hence a field conservation law occurs.

Remark 3. The conclusion (2) of the theorem 4.1 gives rise to a Killing horizon (a concept of vital importance in general relativity [18]) formed by the null hypersurface $\sigma = \text{constant}$ of the locally symmetric manifold.

The above three remarks justify the hypothesis and the usefulness of theorem 4.1.
5. Shape Operators of Einstein Hypersurfaces

In A Conformally Flat Space

Magid [72] has classified algebraically the shape operators for the isometric embedding of an Einstein hypersurface into a space form. As indicated in section 3 of Chapter 1, the embedding space could be taken as a space-form (including the flat space). Einstein spaces (including the Ricci-flat spaces) and conformally flat spaces. We choose the embedding space to be a conformally flat space $\tilde{M}$ and embed an Einstein space $M$ into $\tilde{M}$ with the assumption that $\tilde{M}$ admits an affine conformal vector field $\tilde{\xi}$ whose tangential part $\xi$ is a conformal vector field in $M$. This assumption is consistent with the fact that a conformal vector field preserves the causal character of a semi-Riemannian manifold (actually $M$ would be, in particular, considered as the space-time of general relativity, which is known to have the conformal geometry). The ambient space need not be physical space-time, i.e. need not have the conformal geometry only and this is why we have assumed the existence of an affine conformal vector field (not necessarily conformal) in $M$. We will show that the shape operator is either diagonalizable at each point or can be cast into certain special forms. As the shape operator for the definite spaces is always diagonalizable at each point we shall discuss the embedding of indefinite spaces. A hypersurface whose
shape operator is diagonalizable at each point is called proper, otherwise it is improper. We prove the following by using the Petrov classification scheme [87].

Theorem 5.1. If an indefinite Einstein space $\mathcal{M}(n \geq 3)$ is embedded isometrically as a hypersurface, into a conformally flat space $\mathcal{M}^{n+1}$ admitting a (Conf.C) generated by a vector field whose tangential part generates a (Conf. M) in $\mathcal{M}$, then the shape operator $A_x$ at each point $x$ of $\mathcal{M}$ is either diagonalizable or can be put into one of the following forms:

$$A_x = \begin{bmatrix}
  a & 0 & 0 \\
  0 & a & 0 \\
  0 & 0 & a
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
  a & b \\
  -b & a
\end{bmatrix}$$

where $a$ and $b$ are arbitrary functions on $\mathcal{M}$, with respect to some specially chosen basis. In the last case, $n$ is even and the signature of $\mathcal{M}$ is $(n/2, n/2)$.

Proof: Since $\mathcal{M}$ is conformally flat and has a vector field $\xi$ generating a (Conf C), by the theorem of Levine and Katzin [70] we have

$$(5.1) \quad L_\xi g = 2\sigma g + \tau \overline{\text{Ric}}$$

where $\sigma$ and $\tau$ are smooth functions and $\overline{\text{Ric}}$ is the Ricci tensor of
the ambient space $\mathbb{M}$. The vector field $\vec{\xi}$ can be decomposed uniquely into its tangential part $\xi$ and normal part $\ell N$ as:

$$(5.2) \quad \vec{\xi} = \xi + \ell N$$

$N$ being the unit (space-like or time-like) vector field normal to $\mathbb{M}$. For vector fields $X, Y$ tangent to $\mathbb{M}$, equation (5.1) gives

$$g(\nabla_X X, X) + g(\nabla_Y X, X) = 2\sigma g(X, Y) - \tau \text{Ric}(X, Y)$$

Employing (5.2), the Gauss and Weingarten formulae we get

$$(5.3) \quad (L_{\xi} g)(X, Y) = 2\sigma g(X, Y) + 2\tau g(AX, Y) + \tau \text{Ric}(X, Y)$$

As $\mathbb{M}$ is conformally flat, we have

$$(5.4) \quad g(R(X, Y)Z, W) = [\text{Ric}(Y, Z)g(X, W) - \text{Ric}(X, Z)g(Y, W) -
\begin{align*}
&g(Y, Z)\text{Ric}(X, W) + g(X, Z)\text{Ric}(Y, W) + (n-1) \\
&-\left\{\tau n(n-1)\right\} g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \end{align*}$$

where $\nabla$ denotes the curvature tensor of $\mathbb{M}$ and $X, Y, Z, W$ are arbitrary vector fields in $\mathbb{M}$. The Gauss equation for $\mathbb{M}$ is:

$$(5.5) \quad g(R(X, Y)Z, W) = g(R(X, Y)Z, W) - \partial g(AX, Z)g(AX, W) - g(AX, Z)g(AY, W))$$

for $X, Y, Z, W$ tangent to $\mathbb{M}$. Using this in (5.4) gives

$$(5.6) \quad g(R(X, Y)Z, W) = [\text{Ric}(Y, Z)g(X, W) - \text{Ric}(X, Z)g(Y, W) -
\begin{align*}
&g(Y, Z)\text{Ric}(X, W) + g(X, Z)\text{Ric}(Y, W) + (n-1) \\
&-\left\{\tau n(n-1)\right\} g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \end{align*}$$

Now, if we substitute $\vec{X} = \vec{W} = \vec{N}$ and $\vec{Y} = \vec{Z} = \vec{Z}$ in (5.4) then

$$(5.7) \quad g(R(N, Y)Z, N = [\text{Ric}(N, N) - (\ell \nabla n) + (n-1) g(Y, Z) + (\ell \nabla n) \text{Ric}(Y, Z).$$

By the substitution $X = W = e_1$, where $\{e_1\}$ is an orthonormal
frame in $\mathcal{M}$ in equation (5.5), multiplying by $C_1 = g(e_1, e_1)$ and finally summing over $i$, we find

$$\tilde{\text{Ric}}(Y, Z) - C_1 g(R(N, Y) Z, N) = \text{Ric}(Y, Z) - C_1 (\text{tr}.A) g(AY, Z) - g(AY, AZ).$$

Keeping in mind that $\{e_1, N\}$ constitutes an orthonormal frame in $\mathcal{M}$. Eliminating $g(R(N, Y) Z, N)$ from (5.7) and (5.8) provides

$$\tilde{\text{Ric}}(Y, Z) = \{(n-1)/(n-2)\} \text{Ric}(Y, Z)$$

$$+ \{(Cn\tilde{\text{Ric}}(N, N) - F) / n(n-1)\} g(Y, Z)$$

$$- \{(\text{tr}.A) g(AY, Z) - g(AY, AZ)\}.$$ 

Again, substituting $Y = Z = e_1$, multiplying by $C_1$ and summing over $i$ in the above equation yields

$$\tilde{F} - r = 2C \tilde{\text{Ric}}(N, N) - C_1 (\text{tr}.A)^2 - \text{tr}.A^2.$$

Putting (5.10) into (5.9) we get

$$\tilde{\text{Ric}}(Y, Z) = \{(n-1)/(n-2)\} \{\text{Ric}(Y, Z) - C_1 (\text{tr}.A)^2 - \text{tr}.A^2\}$$

$$+ g(Y, Z) \{(n-2)(F / n) - r + C((\text{tr}.A)^2 - \text{tr}.A^2)\} / 2(n-1)$$

Through equation (5.3) and the hypothesis that $\xi$ generates a (Conf $\mathcal{M}$) in $\mathcal{M}$ we find that

$$\tilde{\text{Ric}}(Y, Z) = - (2f / r) g(AY, Z) + \pi g(Y, Z),$$

where $\pi$ is a scalar function. The eqns. (5.11) and (5.12) can be used, through a cumbersome computation, to derive

$$\xi Q = \xi I + \theta A - A^2$$

where we have taken

$$\xi \kappa = r + (2f(n-2) / r(n-1)) \text{tr}.A - C((\text{tr}.A)^2 - \text{tr}.A^2),$$

$$\theta = \text{tr}.A - 2\xi f(n-2) / r(n-1)$$
Now, using the Petrov classification scheme for symmetric tensors, \( A \) can be cast into the form:

\[
A = \begin{bmatrix}
B_1 & B_2 & \cdots & B_k \\
& C_1 \\
& & C_2 \\
& & & C_m
\end{bmatrix}
\]

where

\[
B_1 = \begin{bmatrix}
d_1 \lambda_1 & d_1 & \cdots & d_1 \\
& d_1 \lambda_1 & d_1 & \cdots & d_1 \\
& & \ddots & \ddots & \ddots \\
& & & d_1 \lambda_1 & d_1 \\
& & & & d_1 \lambda_1
\end{bmatrix}
\]

is an \((s_1 \times s_1)\)-matrix and

\[
C_j = \begin{bmatrix}
\alpha_j \beta_j & 1 & 0 \\
-\beta_j \alpha_j & 0 & 0 \\
\alpha_j \beta_j & 1 & 0 \\
-\beta_j \alpha_j & 0 & 0 \\
1 & 0 \\
0 & 1 \\
\alpha_j \beta_j & 0 & 0 \\
-\beta_j \alpha_j & 0 & 0 \\
\end{bmatrix}
\]

is a \((2t_j \times 2t_j)\)-matrix, with their squares.
As per our hypothesis, $M$ is Einstein, i.e. $Q = (r/n)I$. Thus, (5.13) implies that the orders of block matrices $B_i$ and $C_j$ are less than or equal to 2. Consequently, $A$ has blocks of type:

$$[\nu_i] \quad \text{or} \quad \begin{bmatrix} d_j \lambda_j & d_j \\ 0 & d_j \lambda_j \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{bmatrix}$$

with their respective squares:

$$[\nu_i^2] \quad \text{or} \quad \begin{bmatrix} \lambda_j^2 & 2 \lambda_j \\ 0 & \lambda_j^2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha_k^2 - \beta_k^2 & 2 \alpha_k \beta_k \\ -2 \alpha_k \beta_k & \alpha_k^2 - \beta_k^2 \end{bmatrix}$$
The block \( \begin{bmatrix} d_j & \lambda_j & d_j \\ 0 & d_j & \lambda_j \end{bmatrix} \) can be transformed into \( \begin{bmatrix} \lambda_j & 1 \\ 0 & \lambda_j \end{bmatrix} \) by a change of the basis: \( \{1, \hat{1}\} \rightarrow \{-1, \hat{1}\} \). Eventually, (5.13) can be put as

\[
\ell \frac{\pi}{n} I = \phi I + \hat{\theta} \begin{bmatrix} \nu_1 \\ \begin{bmatrix} \lambda_j & 1 \\ 0 & \lambda_j \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \nu_1^2 \\ \begin{bmatrix} \lambda_j^2 & 2\lambda_j \\ 0 & \lambda_j^2 \end{bmatrix} \end{bmatrix}
\]

Matching corresponding entries we obtain

(5.16) \quad \theta = 2 \lambda_j, \quad (\theta - 2\alpha_k) \beta_k = 0

\[
\theta \nu_1 - \nu_1^2 = \theta \lambda_j - \lambda_j^2 = \theta \alpha_k - \alpha_k^2 + \beta_k^2 = \phi
\]

where \( \phi = (c \Gamma / n) - s \).

If there are any blocks with \( \alpha \)'s and \( \beta \)'s (\( \beta \)'s being non-zero) then \( \alpha_k = \lambda_j = \theta / 2 \), for every \( j \) and \( k \). Hence \( \alpha \)'s and \( \beta \)'s are all equal to \( \theta / 2 \). From the last relation we also note that the \( \beta \)'s are all equal. Thus, the relations (5.16) is equivalent to:

\[
\theta = 2\lambda, \quad \theta = 2\alpha
\]

\[
\nu_1 = (\theta \pm \sqrt{(\theta^2 - 4\phi)})/2, \quad \lambda^2 = \phi, \quad \alpha^2 + \beta^2 = \phi
\]

Clearly, A cannot have both \( \alpha \) and \( \lambda \) blocks, otherwise \( \lambda = \alpha \) and \( \beta = 0 \). So, if A has only \( \lambda \)-blocks then \( \nu_1 = \lambda \) and
\[
A = \begin{bmatrix}
\lambda & 1 \\
0 & \lambda & 1 \\
0 & 0 & \lambda
\end{bmatrix}, \quad \text{tr} \cdot A = n\lambda.
\]

From \(\lambda = \theta/2\), we get \(\text{tr} \cdot A = n\theta/2\). Thus, in virtue of eqn. (5.15) we find \(\theta = 2 \cdot \text{tr} \cdot A = 2\lambda = 2\epsilon f/2(n-1)\). Consequently,

\[
A = \begin{bmatrix}
\alpha & \alpha \\
0 & \alpha & 1 \\
0 & 0 & \alpha
\end{bmatrix}
\]

where \(\alpha(n - 1) = 2\epsilon f\). On the other hand, if \(A\) has only \(\alpha\)-blocks then \(\lambda_1 = \alpha + \sqrt{-\beta^2}\) which cannot be real, because \(\beta \neq 0\); and hence there cannot occur any \(\mu\)-term in \(A\). Eventually, we find that all \(\alpha\)'s = \(\theta/2\) and

\[
A = \begin{bmatrix}
\alpha & \beta \\
-\beta & \alpha \\
\alpha & \beta \\
-\beta & \alpha
\end{bmatrix}, \quad \text{tr} \cdot A = n\alpha = n\theta/2.
\]

Thus, \(\theta = 2\alpha = a\) and
This proves the theorem.

Corollary. Under the hypothesis of theorem 5.1, if $M$ is improper then, either $(A - aI)^2 = 0$, or $(A - aI)^2 = -b^2I$ where $a$ and $b$ are arbitrary functions on $M$.

6. Shape Operators Of Pseudo-Einstein Hypersurfaces

In A Conformally Flat Space

Here I shall study Pseudo-Einstein hypersurface in the same context as the preceding section and correlate it with the physically realistic perfect fluid of general relativity.

Definition: 6.1. A semi-Riemannian manifold is said to be pseudo-Einstein if there exists a 1-form $u$ such that:

$$\text{Ric} = s g + t u \otimes u$$

and $g(U, U) = e^{-e^2}$, where $g(U, X) = u(X)$ and $s, t$ are scalar functions. For $t = 0$, it reduces to Einstein manifold and for $s = t = 0$, it reduces further to Ricci-flat (empty in general relativity) space.

The above definition is motivated by two issues: (1) Yano's definition (111) of a pseudo-Einstein hypersurface of a Kaehlerian manifold, given by $g_{ij}$ with $s$ and $t$ as constants.
The dimension of such a hypersurface is odd (see also [97]).

(2) Einstein's field equations in the framework of general relativity (for instance, the perfect fluid equations:

$$\text{Ric} + (\Lambda - (r/2))g = \rho g + (\nu + \lambda) u \otimes u,$$

where \(\rho\) and \(\nu\) are the pressure and energy densities of the fluid and \(u\) is the 1-form metrically equivalent to 4-velocity vector field \(U\). Contraction of the above eqn. gives: \(t = e(\nu - \lambda \nu))\).

Remark: In the above definition one could also have taken \(U\) to be a null vector in order to deal with the field equations governing plane waves [64]:

$$\text{Ric} = \phi^2 K \otimes K (K \equiv \text{a real null vector}).$$

But we shall restrict the definition of pseudo-Einstein space, to only non-null \(U\).

We shall prove the following theorem for 4-dimensional hypersurfaces only because it is cumbersome to consider the blocks of the shape operator in higher dimensions.

Theorem 6.1. If a 4-dimensional pseudo-Einstein hypersurface \(M^4\) is isometrically embedded into a conformally flat manifold \(M^5\) admitting a (conf C) generated by a vector field whose tangential part generates a (conf M) in \(M^4\), then either the shape operator \(A\) of \(M^4\) is diagonalizable at each point or it can be put into the following forms:
A = \begin{bmatrix} a & \ast \\ \ast & a \end{bmatrix} \text{ or } \begin{bmatrix} a & b \\ b & a \end{bmatrix} \text{ or } \begin{bmatrix} u \pm \sqrt{-c}e^t \\ e^{-t} \\ 0 \\ 0 \end{bmatrix}

where \( a, b, u \) and \( e^t \) are scalar functions and \( c^2 = e^2 = 1 \), with respect to some specially chosen basis.

Proof: With respect to an orthonormal basis formed by \( U \) and three orthonormal vectors orthogonal to \( U \), we observe that the Ricci tensor can be represented by

\[
Q = \begin{bmatrix}
\ast & e^t \\
\ast & \ast \\
\ast & \ast \\
\ast & \ast
\end{bmatrix}
\]

Proceeding exactly as in the proof of theorem 5.1, up to equation (5.13), we get

\[
\begin{bmatrix}
\ast & e^t \\
\ast & \ast \\
\ast & \ast \\
\ast & \ast
\end{bmatrix} = \ast \begin{bmatrix} 1 \\ \ast \\ \ast \\ \ast \end{bmatrix} + \ast \begin{bmatrix} A \\ \ast \\ \ast \\ \ast \end{bmatrix} - \ast A^2.
\]

As we argued in the case of theorem 5.1, \( A \) has blocks of type...
Therefore, either \( A \) will be diagonalizable at each point, or can be put into one of the following forms:

\[
\begin{bmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{bmatrix}
\]

\[
\begin{bmatrix}
\alpha_1 & \beta_1 \\
-\beta_1 & \alpha_1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{bmatrix}
\]

\[
\begin{bmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{bmatrix}
\]
Plugging the above listed forms of $A$ into equation (5.13) shows that the form (2) is not possible whereas the forms (3)-(7) hold only if $\mathcal{M}$ were Einstein. The remaining form (1) leads to:

$$\nu_2 = \lambda = (\theta/2), \quad \nu_1 = \nu_2 = \pm (\theta e^{t})$$

Thus, in this case:

$$A = \begin{bmatrix} \nu & \nu \pm \sqrt{-\varepsilon e^t} \\ \nu & 0 \end{bmatrix}$$

where $\nu = (\theta/2)$ and $\tr A = (8\varepsilon f/3\gamma) + \mathfrak{g}(\theta e^t)$. This completes the proof of the theorem.

**Corollary.** Under the hypothesis of theorem 6.1, if $\mathcal{M}$ is not Einstein then $A$ is either diagonalizable at each point or has the form:

$$A^\nu = \begin{bmatrix} \nu & \nu \pm \sqrt{-\varepsilon e^t} \\ 0 & \nu \end{bmatrix}$$
where $\nu$ and $t$ are scalar functions and $e^2 = e^2 = 1$, with respect to some specially chosen basis.

As a physical example supporting the above corollary, we consider perfect fluid space-time $M^4$ embedded in $\mathbb{M}^5$. The field equations are:

\begin{equation}
(6.2) \quad \text{Ric} - (\Lambda - \frac{r}{2})g = \rho g + (\nu + p') u \otimes u,
\end{equation}

where $\rho$ and $\nu$ are pressure and energy densities, $\Lambda$ is the cosmological constant and $u$ is the fluid velocity vector such that $g(u, u) = -1$ and $g(u, x) = u(x)$. This clearly shows that $M^4$ is a pseudo-Einstein space, but not an Einstein space (which is possible only if $\nu + p = 0$, a non-physical case). Comparing $(6.2)$ with $(6.1)$ yields

\begin{equation}
\begin{aligned}
 s &= \rho + \frac{r}{2} - \Lambda, \\
 t &= \nu + p
\end{aligned}
\end{equation}

To be more specific, we consider $M^4$ as a time-like hypersurface of $\mathbb{M}^5$. Furthermore, the signature of $\mathbb{M}^5$ could be either $(---++)$ or $(---++)$ only. If $\mathbb{M}^5$ has signature $(---++)$ (i.e. the Lorentz signature) then the unit normal $N$ is space-like and $e = 1$.

Moreover, since $u$ is time-like, $e = -1$. Hence we get: $\sqrt{-(\mathbf{e} \cdot \mathbf{p})} = \sqrt{(\nu + p)}$ which is always non-zero real (in view of the physical condition $\nu + p > 0$). Thus, in this case, either $A$ is diagonalizable at each point or

\begin{equation}
A = \begin{bmatrix}
\nu \pm \sqrt{\nu + p} & 0 \\
0 & 1
\end{bmatrix}
\end{equation}
On the other hand, if the signature of $\mathcal{M}^5$ were $(-+++)$ then $N$ would be time-like and $\mathcal{E} = e = -1$. $\int(-\mathcal{E} e^t) = \int \nu + p$ which is always imaginary for physically realistic fluid. Consequently, $A$ is diagonal in this case.

Conclusion: (A) Under the hypothesis of theorem 6.1, if $\mathcal{M}^4$ is a perfect fluid of general relativity, and

1. $\mathcal{M}^5$ is Lorentzian then $\mathcal{M}^4$ is not necessarily proper
2. $\mathcal{M}^5$ has signature $(-+++)$ then $M$ is proper.

Beem and Ehrlich [3] have studied time-like hypersurfaces of a Minkowski space, with the property that the shape operator is diagonalizable at each point. They have established a singularity theorem by showing the validity of generic and strong energy condition, under the assumption that the shape operator is diagonalizable at each point. We have studied pseudo-Einstein spaces embedded in conformally flat spaces with a certain symmetry property and have shown the possible non-diagonal forms of the shape operator.

(B) In case (1) of the conclusion (A) we observe that the physical energy condition: $\nu + p > 0^*$ is equivalent to saying that $M$ is not necessarily diagonalizable at each point.
The following abbreviations have been and will be used:

(M) Motion
(H M) Homothetic Motion
(A C) Affine Collineation
(C C) Curvature Collineation
(R C) Ricci Collineation
(Conf M) Conformal Motion
(S Conf M) Special Conformal Motion
(Conf C) Conformal Collineation
(S Conf C) Special Conformal Collineation
(W C) Weyl Conformal Collineation

Relations between the above symmetries have been shown (An arrow indicates "implies"). In the following flow chart: (Note that the relation between (Conf C) and (W O) was shown incorrect in [59]. It was corrected in [24].)
Pictorial Illustration of a (Conf C) With Reference to
Example Given in Sec. 2

Totally Umbilical Submanifold

Non-Totally Umbilical Submanifold
CHAPTER VI

CONFORMAL COLLINEATIONS AND RELATIVISTIC HYDRODYNAMICS

The idea of self-similarity plays an important role in Newtonian fluid mechanics in deriving dynamical solutions which represent blast waves or explosive fluid motions [1]. In general relativity, by imposing self-similarity on the space-time as well as fluid motion, more interesting and important solutions can be derived, which represent the blast waves or inhomogeneous cosmological models. General relativistic self-similar solutions were derived and discussed by Cahill and Taub [16], Bicknell and Henriksen [8], Taub [103], Eardley [31], McIntosh [74] and others. Recently, Herrera et al [55] studied the consequences of the existence of a one-parameter group of conformal motions for isotropic and anisotropic fluid. They concluded that for special conformal motions (Conf M which are CC too), the stiff equation of state ($\mu = p$, for which the speed of sound equals the velocity of light [64]) is singled out in a unique way, provided the generating conformal vector field is orthogonal to the 4-velocity. In this chapter we shall study the same problem by using conformal collineations (which include conformal motions

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and self-similar motions). It is shown that the stiff equation of state is not singled out when the fluid velocity vector is orthogonal to the (S Conf C).

1. Conformal Collineations and Hydrodynamical Variables

Let \((x^a) (a = 0, 1, 2, 3)\) be a coordinate system on the 4-dimensional space-time \(M\) carrying an isotropic/anisotropic fluid. In suitable units and with zero cosmological constant, we can write Einstein's field equations as:

\[(1.1) \quad R_{ab} - (R/2)g_{ab} = T_{ab}\]

where \(R_{ab}\) is the Ricci tensor, \(R\) is scalar curvature, \(g_{ab}\) is the metric tensor of signature \((-+++)\) and \(T_{ab}\) is energy-momentum tensor:

\[(1.2) \quad T_{ab} = (\mu + \bar{p})U_aU_b + \bar{p}g_{ab} + (\bar{p} - \bar{p})S_aS_b,\]

where \(U^a\) is the four-velocity, \(S^a\) is a unit spacelike vector orthogonal to \(U^a\), \(\mu\) is the energy density, \(p\) is the pressure in the direction of \(S^a\) and \(\bar{p}\) is the pressure on the two-space orthogonal to \(U^a\) and \(S^a\). Contraction of \((1.1)\) and \((1.2)\) yields

\[(1.3) \quad R = - T = - T_{ab}^a = \mu - p - 2\bar{p}\]

Note that the fluid is isotropic (perfect) when \(p = \bar{p}\), otherwise is anisotropic.

Let a vector field \(\xi\) generate (Conf C) in \(M\). Then

\[(1.4) \quad L_\xi g_{ab} = 2g_{ab}, \quad \Sigma_{ab} = 0\]

\[(1.5) \quad L_\xi R_{ab} = -2 \nabla_a \nabla_b \sigma - (\Box \sigma)g_{ab}\]
(1.6) \[ L_c R = -6 \Box \sigma - 2 \varphi - h a b R^a_b \]

where \( \Box \) represents the d'Alembert operator. Using eqns. (1.1) and (1.2) we obtain

(1.7) \[ h a b R^a_b = (\mu - \varphi)(h^a_a/2) + (\mu - \bar{\varphi})h a b u a b + (\mu - \bar{\varphi})h a b s a s b \]

The four-velocity \( U^a \) can be represented by \( dx^a/ds \), where \( x^a \) are the coordinates of a point on the affinely parametrized world line: \( x^a = x^a(s) \), of the fluid. It is well-known that \( L_c dx^a = 0 \).

Using \( ds^2 = -g a b d x^a d x^b \), we find that

\[ (L_c ds)/ds = \Box - (h a b U^a U^b)/2. \]

Thus, we find

(1.8) \[ L_c U^a = (2\sigma g a b + h a b)U^b + g a b L_c U^b \]

\[ = 2\sigma U^a + h a b U^b - g a b (L_c ds)/ds \]

\[ = (\sigma + (h b e U^b c)/2)U^a + h a b U^b. \]

Likewise, we can show that

(1.9) \[ L_c S^a = \{\sigma - (h b e S^b c)/2\} S^a + h a b S^b \]

The Lie-derivation of (1.1) via \( \xi \), using (1.2)-(1.9) and taking projections yield

(1.10) \[ -2(\nabla a \nabla c)U^a U^b - 2\Box \sigma = L_c \nu + 2\varphi + (\mu - \bar{\varphi})(h/4) \]

\[ - [(\mu + \bar{\varphi})h a b U^a U^b - (\mu - \bar{\varphi})h a b s a s b]/2 \]

(1.11) \[ -2(\nabla a \nabla c)S^a S^b + 2\Box \sigma = L_c \tilde{p} + 2\varphi - (\mu - \bar{\varphi})(h/4) \]

\[ - [(\mu + \bar{\varphi})h a b U^a U^b - (\mu - \bar{\varphi})h a b s a s b]/2 \]

(1.12) \[ -2(\nabla a \nabla c)W^a W^b + 2\Box \sigma = L_c \tilde{p} + 2\varphi - (\mu - \bar{\varphi})(h/4) \]

\[ - [(\mu + \bar{\varphi})h a b U^a U^b + (\mu - \bar{\varphi})h a b s a s b - (\mu - \bar{\varphi})h a b w a w b]/2 \]
(1.13) \[ 4(\nabla_a \nabla_b \sigma) U^{ab} = (\mu + p + 4\bar{p})(h_{ab} U^{ab}) \]

(1.14) \[ 4(\nabla_a \nabla_b \sigma) U^{ab} = (\mu + p + 2\bar{p})(h_{ab} U^{ab}) \]

(1.15) \[ 4(\nabla_a \nabla_b \sigma) S^{ab} = (2\bar{p} - \mu - p)(h_{ab} S^{ab}) \]

where \( w^a \) is a unit space-like vector orthogonal to \( S^a \) and \( U^a \). Furthermore, by contraction of eqn. (1.10) we have

(1.16) \[ 6 \sigma = L e \tau + 2\sigma - (\mu - p) (\h / 2) \]

\[ -[(\mu + \bar{p})h_{ab} U^{ab} + (p - \bar{p})h_{ab} S^{ab}] \]

where we have denoted \( h_a \) by \( h \). Eqns. (1.10)-(1.12) express the infinitesimal variations in the hydrodynamical variables \( \mu \), \( p \) and \( \bar{p} \) of the fluid. Eqns. (1.13)-(1.15) correlate the corresponding cross-components of \( \nabla_a \nabla_b \sigma \) and \( h_{ab} \) through \( \mu \), \( p \) and \( \bar{p} \).

Remark: It appears from the Eqns. (1.13)-(1.15) that \( h_{ab} \) might be proportional to \( \nabla_a \nabla_b \sigma \). Unfortunately, this is ruled out by the observation that the desired proportionality demands \( \mu - p + 4\bar{p} = \mu + p + 2\bar{p} = 2\bar{p} - \mu - p = p - \mu \), which are equivalent to \( p = \bar{p} \) and \( \mu + p = 0 \). This is a non-physical solution. Thus, we infer that \( h_{ab} \) cannot be proportional to \( \nabla_a \nabla_b \sigma \). However, we could have the following two possibilities: (1) \( h_{ab} = 0 \), reducing (Conf C) to (Conf M) and (2) \( \nabla_a \nabla_b \sigma = 0 \), giving (S Conf C).

In the rest of this chapter, unless otherwise explicitly stated...
(in the last section), we focus our attention on the possibility (2), viz. the (S Conf C). We shall also point out certain advantages of the use of (S Conf C) over (S Conf M), in particular reference to the study of Herrera et al. [55] on special conformal motions. Let us consider (S Conf C) generated by \( \xi \), in the fluid in question. Then eqns. (1.10)-(1.16) become

\[
\begin{align*}
(1.17) & \quad L_\xi \nu + 2\nu = \frac{((\nu - \nu)\nu_{ab}u_{ab} - (\nu - \nu)\nu_{ab}a_{ab}}{2} - (\mu - p)h/4, \\
(1.18) & \quad L_\xi \nu + 2\nu = \frac{((\nu - \nu)\nu_{ab}u_{ab} - (\nu - \nu)\nu_{ab}a_{ab}}{2} + (\mu - p)h/4, \\
(1.19) & \quad L_\xi \nu + 2\nu = \frac{((\nu - \nu)\nu_{ab}u_{ab} + (\nu - \nu)\nu_{ab}a_{ab}}{2} - (\mu - p)h/4, \\
(1.20) & \quad (\nu - p + 4p)\nu_{ab}a_{ab} = 0, \\
(1.21) & \quad (\nu + p + 2p)\nu_{ab}a_{ab} = 0, \\
(1.22) & \quad (\mu - p + 2p)\nu_{ab}a_{ab} = 0, \\
(1.23) & \quad L_\xi T + 2\nu T = \frac{((\nu - \nu)\nu_{ab}u_{ab} + (\nu - \nu)\nu_{ab}a_{ab}}{2} + (\mu - p)h/2.
\end{align*}
\]

Remark 1. We observe, from eqns. (1.20)-(1.22): (1) If \( \nu_{ab} \) is not diagonal with respect to an orthonormal base formed by \( u_a, s_a, w_a \), and a fourth vector, then either \( \mu = p - 4p \) or \( \mu + p + 2p = 0 \) (whose isotropic counterpart is \( \mu + 3p = 0 \) called curvature fluid [74]).

Remark 2. If \( \nu_{ab}a_{ab} \) and either of \( \nu_{ab}u_{ab} \) and \( \nu_{ab}a_{ab} \) survives, then the anisotropy disappears \( (p = \nu) \), and again we have \( \mu + 3p = 0 \).

A general geometric-physical interpretation of equations (1.17)-
(1.19) is difficult to obtain. Therefore, we concentrate on a few special cases in the following sections. But, before that, we would like to set up a conservation law generator [24] with the help of (S.Conf C). We have
\[
\nu_\alpha(R^\beta_{\alpha\beta}) = (\nu_\alpha R^\beta_{\alpha\beta})_\xi^\beta + \bar{R}^\beta_\alpha(\nu_\alpha \xi^\beta)
\]
\[
= (L_\xi R/2) + R^\beta_\alpha(\nu_\alpha \xi^\beta)
\]
\[
= [L_\xi R + \bar{R}^\alpha_\beta L^\beta_\xi \xi^\alpha]/2
\]
\[
= [L_\xi R + 2\sigma R + \bar{R}^\alpha_\beta \xi^\alpha]/2
\]
\[
= 0,
\]
where we have used eqn. (1.6) and \(\nu_\alpha \nabla_\beta \sigma = 0\) (and hence we also have \(\Box \sigma = 0\)). Thus we state:

**Lemma 1.1.** An (S.Conf C) generated by \(\xi\) gives rise to a conservation law generator:
\[
(1.24) \quad \nu_\alpha(R^\beta_{\alpha\beta}) = 0
\]
This generalizes the conservation law generator evolved by a special conformal motion, as mentioned by Herrera et al [55].

2. Space-like Special Conformal Collineation

We consider two specific directions, namely (A) \(\xi\) is collinear with \(S\); (B) \(\xi\) is orthogonal to \(U\) and \(S\). These cases are discussed to compare our results with Herrera et al [55].

(A) \(\xi = \alpha S\), where \(\alpha\) is a scalar function. Then (1.2) shows
\[
\bar{R}^\beta_\alpha \xi^\beta = -\mu \xi^\alpha.
\]
Taking its divergence and using lemma 1.1, eqns.
(1.17)-(1.19) and the relation \( \nabla_a \xi^a = 4\sigma (h/2) \), we get

\[(2.1) \quad (\nu + \rho - 2\overline{\rho})(4\sigma + h - 2h_{ab}S^{ab}) = 0.\]

This leads to the following two non-trivial possibilities:

\[(2.2) \quad (1) \nu + \rho - 2\overline{\rho} = 0, \; \text{or} \; (11) 4\sigma + h - 2h_{ab}S^{ab} = 0.\]

For perfect fluids, eqn. (2.2)(1) reduces to the stiff equation of state \((\nu = \rho)\). To prove that the stiff equation of state is not singled out, we must show that other solutions are possible out of the (2.2)(11). For this purpose, we consider the following case: As mentioned in chapter V (sec. 3), for a non-Einstein conformally flat space we have \( h_{ab} = \tau R_{ab} \), but the converse need not be true. We study the converse by prescribing

\[(2.3) \quad h_{ab} = \tau R_{ab} (\tau \neq 0)\]

For a (S-Conf C), \( h_{ab} \) has to be covariant constant. Thus, our space-time must be Ricci-recurrent [52]. Using the field eqns. (1.1) and (1.2) and eqn. (2.3), we get

\[(2.4) \quad (i) 2h_{ab}U^aU^b = \tau (2\nu - R), \; (ii) 2h_{ab}S^{ab} = \tau (2\rho + R), \; (iii) \tau R = h, \; \text{where} \; R = \nu - \rho - 2\overline{\rho}.\]

For \( R \neq 0 \), using eqns. (2.4)(i) and (ii) in (2.2)(11), yields

\[(2.5) \quad 2\sigma (\nu - \rho - 2\overline{\rho}) = ph, \; \sigma \neq 0.\]

As \( h_{ab} \) is parallel, \( h \) is constant. Thus, eqn. (2.5) can provide different classes of solutions for different suitable choices of the function \( \sigma \) (satisfying, of course, \( \nabla_a \nabla_b \sigma = 0 \)).

Therefore, the stiff equation of state is not singled out if \( \xi \) is
orthogonal to \( U \).

(B): \( \xi \) is orthogonal to \( U \) and \( S \). Proceeding exactly as in (A), we obtain the following

\[
(\omega - p)(4\sigma - h - h_{ab}U^aU^b + h_{ab}S^aS^b) = 0
\]

which gives two possibilities:

\[
(2.6) \quad \begin{align*}
(1) \quad & \omega = p, \text{ or } (II) \quad 4\sigma = h + h_{ab}U^aU^b - h_{ab}S^aS^b.
\end{align*}
\]

For Ricci-recurrent spaces satisfying equations (2.3) and (2.4), we obtain

\[
(2.7) \quad 2\sigma(\omega - p - 2R) = h\sigma, \sigma \neq 0.
\]

Thus, for this case, either \( \omega = p \) with no constraints on \( p \) or there are other possible solutions generated from eqn. (2.7) for suitable value of \( \sigma \).

The dominant energy condition \([64]\) for (A) and (B) is satisfied if \( 4\sigma + h \geq 0 \).

**Conclusion:**

1. The stiff equation of state is not singled out when \( \xi \) is orthogonal to \( U \).
2. The equations (2.5) and (2.7) provide perfect fluid solutions iff.

\[
(2.8) \quad \sigma = h\sigma/(2\mu - 6p), \quad R \neq 0, \quad \sigma \neq 0.
\]

3. For choices of non-zero \( \sigma \) other than (2.8) (satisfying the energy condition) different anisotropic solutions are possible.

**Remark:** Herrera et al \([55]\) have recently discussed this topic with respect to special conformal-motion. Some of their
Conclusions are:

(a) Under their assumptions, the existence of a one-parameter group of special conformal motions introduces specific restrictions on the hydrodynamical variables.

(b) Furthermore, for a (S Conf M), the stiff equation of state is singled out in a unique way, provided $\xi$ is orthogonal to $U$.

Comparing their conclusion (b) with our analysis, it is clear from (2.2)-(2.8) that the use of special conformal collineations has certain advantages. For example, the stiff equation of state is not singled out when $\xi$ is orthogonal to $U$.

3. Time-like, Special Conformal Collineations

In this section, we assume $\xi = \alpha U$, $\alpha$ being a scalar. Proceeding exactly as in sec. 2(A), we can obtain the following non-trivial possibilities:

\[(3.1) \quad (1) \; \mu + p - 2\rho = 0, \quad \text{or} \quad (11) \; 4\sigma + h - 2h_{ab}U^aU^b = 0\]

For perfect fluids, eqn. (3.1)(1) reduces to $\mu - 3\rho = 0$. This equation of state (representing a curvature fluid) has been used earlier in several exact solutions (see, i.e., conformally flat solutions, § 32.5.3 p. 370-71 of [64]); cf. also McIntosh [74].

To investigate the case (3.1)(11) we take the Lie-derivative of $\xi = \alpha U$. This provides

\[L_U U_a = (2\sigma - LuL)U_a + h_{ab}U^b\]
Making use of eqn. (1.3) in the above equation, we get

\[ (3.2) \quad L\sigma = \sigma - \frac{\partial (\partial U)}{2}. \]

Now, taking the divergence of \( \nabla \sigma = \sigma \cdot U \) we get

\[ (3.3) \quad L\sigma = 4\sigma - \alpha \theta + \left( \frac{\mathbf{h}}{2} \right), \]

where \( \theta \equiv \nabla \sigma U \) is the volume expansion of the fluid. Now, eliminating \( L\sigma \) from eqns. (3.2)-(3.3), and using eqn. (3.1) (11) we obtain

\[ (3.4) \quad \delta \sigma = 4\alpha \theta - \mathbf{h}. \]

Thus, \( \sigma \) is constant iff. \( \alpha \theta \) = constant, i.e., generates an (A C) iff. \( \alpha \theta \) = constant. To show that solutions other than curvature fluid: \( \mu - 3\rho = 0 \), are possible from the eqns. (3.1)(11), we use the Ricci-recurrent spaces satisfying (2.3) and (2.4). Hence:

\[ (3.5) \quad 2\sigma (\mu - p - 2\rho) + h\sigma = 0, \quad R = 0 \quad \text{and} \quad \sigma \neq 0. \]

Thus, it is possible to generate solutions other than the curvature fluid for the case (3.1) (11) by assigning in (3.5) suitable values of the function \( \sigma \) (satisfying \( \nabla \sigma = 0 \)). The dominant energy condition is satisfied for \( 4\sigma - \mathbf{h} > 0 \).

Conclusions: (1) The curvature fluid solution is not singled out when \( \xi \) is collinear with \( U \).

(2) The eqn. (3.5) provides perfect fluid solutions iff.

\[ (3.6) \quad \sigma = \frac{h\sigma}{(6\rho - 2\mu)}, \quad R \neq 0, \quad \sigma \neq 0. \]

(3) For any choice of nonzero \( \sigma \) other than (3.6) and satisfying energy condition, different anisotropic solutions are possible.
(4) For $\sigma$ constant, (S Conf C), in a non-Einstein conformally flat space reduces to (A C) [60]. In particular, the volume expansion $\theta = 0$ implies $\delta \sigma = -\dot{h}$ and hence, from (3.6) we obtain $\mu + 3p = 0$. An example of such a space is the Einstein cosmological model [60]. Hence the value $-\dot{h}/8$ of $\sigma$, in this case, renders the fluid unphysical and non-expanding. So we should take $\sigma = -\dot{h}/8$.

4. Spherically Symmetric Solutions

In this section we first investigate (Conf C) in static and spherically symmetric space-times. Thus, we consider a comoving coordinate system $(t, r, \theta, \phi)$ with the line-element:

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $\nu$, $\lambda$ and $r$ are functions of $r$ alone. Suppose the components of a (Conf C) vector field are $\xi^a = (0, \xi^1, 0, 0)$. Obviously, $\xi^1 = (0, e^{2\lambda}1, 0, 0)$ are the covariant components of $\xi$. In view of the static and spherical symmetry of (4.1) and the eqn:

$$L_{\sigma}g_{ab} = 2\sigma g_{ab} + h_{ab} (\nabla g_{ab} = 0)$$

it appears that $\xi^1$ is a function of $r$ alone. Computing all the Christoffel symbols $\Gamma_{bac}^a$ and using (4.2) we see that $(h_{ab})$ is a diagonal matrix and $h_{ab} = c g_{ab}$ ($c$ = a constant). Thus, we state

Theorem 4.1 In a spherically symmetric static space-time, a (Conf C) along a radial direction reduces to (Conf M).
Thus retaining spherical symmetry, in order to explore the possibility of a (Conf C) which is not (Conf M), the line-element can be written as (4.1) with $\dot{v}$, $\lambda$, and $Y$ as functions of $t$ and $r$ only. The non-zero Christoffel symbols, after a lengthy computation, can be found to be:

\[
\Gamma_{00} = \dot{v}, \quad \Gamma_{01} = \dot{v}', \quad \Gamma_{11} = e^{2(\lambda - \dot{v})}, \quad \Gamma_{22} = e^{-2\dot{v}}Y, \quad \Gamma_{33} = e^{-2\dot{v}}YY \sin^2 \theta,
\]

\[\text{(4.3)} \quad \Gamma_{01} = e^{2(\lambda - \dot{v})}Y, \quad \Gamma_{01} = \dot{v}, \quad \Gamma_{11} = \lambda, \quad \Gamma_{22} = e^{-2\dot{v}}Y, \quad \Gamma_{33} = -e^{-2\dot{v}}YY \sin^2 \theta,
\]

\[
\Gamma_{02} = \dot{v} \dot{\lambda}, \quad \Gamma_{12} = \dot{v}' \dot{\lambda}, \quad \Gamma_{33} = -\sin \theta \cos \theta,
\]

\[
\Gamma_{03} = \dot{v} \dot{\lambda}, \quad \Gamma_{13} = \dot{v}' \dot{\lambda}, \quad \Gamma_{23} = \cot \theta,
\]

where the indices 0, 1, 2, 3 correspond to the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$. The prime and dot denote partial differentiations with respect to $r$ and $t$ respectively. The fluid 4-velocity $U^a = (U^0, 0, 0, 0)$ and the unit space-like vector field $S^a = (0, S^1, 0, 0)$, so that $U^0 = e^{-\dot{v}}$ and $S^1 = e^{\dot{v}}$. Computing the surviving components of the Einstein tensor $R_{ab} - (1/2)Rg_{ab}$ for the non-static line-element (4.1) we write the field equations for isotropic/anisotropic fluid as:

\[\text{(a)} \quad \nu = Y - 2Y - 1 e^{-2\lambda} (Y'' + Y'Y' + Y''/2Y) + 2Y - 1 e^{-2\dot{v}} (Y' + Y2/2Y)
\]

\[\text{(b)} \quad \rho = -Y - 2Y - 1 e^{-2\dot{v}} (Y + Y') + 2Y - 1 e^{-2\dot{v}} (Y' + Y2/2Y)
\]

\[\text{(4.4)} \quad \gamma = e^{-2\lambda} \left[(\nu'' + \nu'Y' - Y')Y + Y'Y'' - YY''ight] + e^{-2\dot{v}} \left[(\lambda' - 2\dot{v})Y + Y' + Y'' + YVight]
\]
\[ (d) \quad 0 = Y - \dot{Y} J - Y' \]

We first consider the case when \( (n_0) \) is a diagonal matrix. Therefore it follows from eqns. (1.13)-(1.15) that the matrix \( (\nabla a \nabla b) \) is also diagonal. The equations \( \nabla c h_{4b} = 0 \) provide the following relations:

\[
\begin{align*}
h_{00} &= c_0 e^{2\lambda}, \\
h_{11} &= c_1 e^{2\lambda}, \\
h_{22} &= c_2 e^{2\lambda}, \\
h_{33} &= h_{22} \sin^2 \theta.
\end{align*}
\]

(4.5) \[
\begin{align*}
(h_{11} + e^{2\lambda} - 2u h_{00}) \lambda &= 0, \\
(h_{11} + e^{2\lambda} - 2u h_{00}) \dot{u} &= 0, \\
Y(e^{2\lambda} h_{22} + Y^2 h_{00}) &= 0, \\
Y'(e^{2\lambda} h_{22} - Y^2 h_{11}) &= 0.
\end{align*}
\]

We shall discuss various possibilities, viz. (1) \( Y = \) constant, which is, physically, not of much significance, in that the corresponding perfect fluid solution is physically irrelevant (\( Y = \) constant, would imply, in view of eqn. (4.4)(v) that \( \nu - \nu' = 0 \)).

(11) \( Y = Y(r, t) \). In this case, \( \dot{Y} \neq 0 \) and \( \dot{Y}' \neq 0 \) and the last pair of eqns. in (4.5) show that \( h_{00}, h_{11} \) and \( h_{22} \) are proportional to \( -e^{2\lambda}, e^{2\lambda} \) and \( Y^2 \) respectively. Thus, in this case (Conf. C) reduces to (Conf. M). (iii) \( Y = Y(r) \). For this case we observe from eqn. (4.5) that \( \xi \) generates (Conf. C), that is not (Conf. M), only if \( \lambda = \lambda(r) \) and \( \mu = \mu(r) \). Here \( C_1 = C_2 \). (iv) \( Y = Y(t) \). Here we find by observing eqn. (4.5) again, that \( \xi \) generates a (Conf C) other than a (Conf. M) only if \( \lambda = \lambda(r) \) and \( \mu = \mu(t) \). In this
case, $C_0 = -C_2$. Based on the above mentioned consequences we may state the following:

Theorem 4.2. The only spherically symmetric solutions admitting a (Conf C) which is not (Conf M), and has diagonal hab.
are of the forms:

\[ (a) \quad ds^2 = -e^{2\nu(t)}dt^2 + e^{2\lambda(r)}dr^2 + r^2d\Omega^2 \]
\[ (b) \quad ds^2 = -e^{2\nu(t)}dt^2 + e^{2\lambda(r)}dr^2 + y^2(t)d\Omega^2 \]

where $d\Omega^2 = d\sigma^2 + \sin^2\sigma d\omega^2$. Let us investigate further into the above solutions. As $\partial_\sigma$ and $\partial_\omega$ are Killing we shall set $e^\lambda = \lambda_1 U$ $+ \lambda_2 S = (\lambda_1 e^{-\nu}, \lambda_2 e^{-\lambda}, 0, 0)$. In virtue of the eqn. $L_gab = 2\phi_gab$

$\nabla$, we get

\[ h_{00} = 2[\sigma e^{\nu} - \lambda_1 - \lambda_2 e^{\nu} - \lambda] e^{\lambda} \]
\[ h_{11} = 2[-\sigma e^{\lambda} + \lambda_2 + e^{\lambda} - \lambda] \lambda_1 \lambda \ e^{\lambda} \]
\[ h_{22} = 2[\sigma y^2 + \lambda_2 y^2 e^{-\lambda} + \lambda_1 y^2 e^{-\nu}] \]
\[ \lambda_1 = \lambda_1(r, t), \quad \lambda_2 = \lambda_2(r, t) \]

Let us take up the metric (4.6)(a). In this case we already know that $c_1 = c_2$ and hence $h_{00} = c_0 e^{2\nu}$, $h_{11} = c_1 e^{2\lambda}$, $h_{22} = c_1 y^2$, $h_{33} = h_{22} \sin^2\sigma$. Eqns. (4.7) reduce to

\[ (a) \quad c_0 = 2(\sigma - e^{-\nu} \lambda_1) \]
\[ (b) \quad c_1 = 2(\nu e^{-\lambda} \lambda_2 - \sigma) \]
\[ (c) \quad c_1 = 2(\lambda_2 e^{-\lambda} \frac{\nu}{\nu} - \sigma) \]

Thus, $\sigma = \sigma(r, t)$. Further, (Conf C) reduces to (A C) if either $\lambda_1 = \lambda_1(r)$ or $\lambda_2 = \lambda_2(t)$. Especially, if $\lambda_2 = \lambda_2(t)$, we get $\sigma = -$
\( c_{1/2} \) and \( \lambda 2Y' e^{-\lambda} = 0 \) which gives \( \lambda_2 = 0 \). Thus, \( \lambda_2 \) cannot be a non-zero function of \( t \) alone. In general, from eqns. (4.8)(b) and (c) we conclude that \( \lambda_2 = T(t)Y \). The field equations become

\[
\begin{align*}
\nu &= Y^{-2} - Y^{-2} e^{-2\lambda} (2YY'' - 2Y'Y'') + \frac{Y'}{2} \\
p &= -Y^{-2} + Y^{-2} e^{-2\lambda} (Y')^2 \\
\bar{\rho} &= e^{-2\lambda} (Y'' - Y' Y')
\end{align*}
\]

(4.9)

It follows that \( \nu + p + 2\bar{\rho} = 0 \) whose isotropic counterpart represents a curvature fluid. Further, eqn. (4.8) provides \( 2\sigma + c_1 = 2e^{-\lambda} T_1(t)Y' \), where \( T_1(t) \) is a function of \( t \) alone. Thus, we get \( e^{\lambda} (2\sigma + c_1') = 2T_1Y' \). Consequently, the last two eqns. in (4.9) transform into:

\[
\begin{align*}
p &= \frac{1}{2} - \frac{(2\sigma + c_1)^2}{4 T_1^2} \\
\bar{\rho} &= \sigma' (2\sigma + c_1) / (2YY'T_1^2)
\end{align*}
\]

For a perfect fluid, we therefore get the relation

\[
(1nY) = \frac{2}{\sigma'} \{(2\sigma + c_1) / ((2\sigma + c_1)^2 - 4 T_1^2) \}
\]

Solving it we obtain

\[
Y^2 = T_2(t) [(2\sigma + c_1)^2 - 4 T_1^2]
\]

A subcase of special interest is \( (S \text{ Conf} C) \) which restricts \( \sigma \) to \( \nabla_a \nabla_b \sigma = 0 \). Use of the Christoffel symbols (4.3) shows that

\[
(4.10) \quad \sigma = \sigma(t), \quad \dot{\sigma} = \dot{\omega} \sigma
\]

which integrate to

\[
(4.11) \quad C \sigma = \epsilon
\]

Through eqns. (4.8) and (4.11) we obtain
\[ \lambda_2 = k \left( Y(2\sigma + c_1) / 2 \right), \quad \lambda_1 = k \cdot Y, \]
\[ \lambda_2 = C \left( \sigma^2 - c_0 \sigma + R(r) \right) / 2, \]

where \( C \) and \( k \) are arbitrary constants. The field eqns. reduce to
\[ \mu \triangleq \rho = \bar{\rho} = 0 \] (Fluid, does not exist).

Hence we draw the following inference:

The only possible spherically symmetric anisotropic fluid admitting a proper \((\text{Conf} \ C)\) with \((h_{ab})\) diagonal and described by the metric \((4.6)(a)\) has the equation of state: \( \mu + \rho + 2\bar{\rho} = 0. \)

Its isotropic counterpart is \( \mu + 3\rho = 0 \) (curvature fluid). If we restrict the \((\text{Conf} \ C)\) to a \((S \text{ Conf} \ C)\) then we have \( \bar{\rho} = 0 \) and \( \mu + \rho = 0 \) (Non-physical case).

Next we consider the metric \((4.6)(b)\). Here \( c_0 = -c_1 \) and \( h_{00} = c_0 e^{2\mu}, \ h_{11} = c_1 e^{2\lambda_1}, \ h_{22} = -c_0 X^2, \ h_{33} = h_{22} \sin^2 \theta. \) With these values in eqns. \((4.7)\) we get
\[ c_0 = 2(\sigma - \lambda_1 e^{-\lambda_2}), \]
\[ c_1 = 2(-\sigma + e^{-\lambda_2}), \]
\[ -c_0 = 2(-\sigma - \lambda_1 e^{-\lambda_2} Y/Y), \]
\[ \sigma = \sigma(r, t) \]

The above system of equations solves for \( \lambda_1 \) as
\[ (4.13) \quad \lambda_1 = R_1(r) Y \]

The field eqns. become
\[ \rho = Y^{-2}(1 + R_1^{-2}(2\sigma - c_0)^2 / 4), \]
\[ p = -Y^{-2} - 2Y^{-1}(Y/2Y) + 2\sigma/(2\sigma - c_0)(2\sigma - c_0)^2 / 4R_1^2 Y^2. \]
\[ \overline{p} = -\overline{\sigma}(2\sigma - c_0)/2R_1^2 \]  

Hence we have the equation of state \( \nu + p - 2 \overline{p} = 0 \) (whose isotropic counterpart represents stiff matter). For the isotropic (perfect) fluid, we observe that

\[ \dot{Y}[(2\sigma - c_0)^2 + 4R_1^2] + 2\dot{\sigma}(2\sigma - c_0)Y = 0 \]

Solving this linear equation one finds

\[ Y^2 = 4R_2(r)/[(2\sigma - c_0)^2 + 4R_1^2] \]

In the subcase of a \( S \) Conf C we find:

\[ 2\dot{K}Y = C\dot{\sigma}^2 - Cc_0\dot{\sigma} + K_1 \]

\[ 2\dot{\lambda}_1 = C\dot{\sigma}^2 - Cc_0\dot{\sigma} + K_1 \]

\[ \dot{\lambda}_2 = e^\lambda(2\sigma + c_1)/2 \]

\( K_1 \) and \( K_2 \) being constants. The field eqns. reduce to:

\[ \nu = Y^{-2}[4 + K^{-2}(2\sigma - c_0)^2]/4 \]

\[ p = -Y^{-2} - [4 + C(2\sigma - c_0)^2/(C\dot{\sigma}^2 - Cc_0\dot{\sigma} + K_1)]/2KYC \]

\[ \overline{p} = -(1/KYC) \]

For a perfect fluid, \( \sigma \) reduces to a constant and hence \( Y \) constant. The solution is therefore non-physical.

Finally, we analyze the case when \( (\lambda_{ab}) \) is non-diagonal.

Here eqn. (4.2) brings out the following:

(a) \[ h_{00} = 2(\sigma e^\nu - \lambda_1 e^\nu \lambda_2 e^\nu - \lambda) e^\nu \]

(b) \[ h_{11} = 2(-\sigma e^\lambda + \lambda e^\lambda - \nu \lambda_1 e^\lambda \]

(c) \[ h_{22} = K\dot{Y}^2 = 2(-\sigma e^\lambda + \lambda_2 e^\lambda e^{-\lambda} + \lambda_1 Y e^{-\nu}) \]

(d) \[ h_{33} = h_{22} \sin^2 \theta \]
(e) \( h_{01} = (\lambda_2 - \lambda_2^2)e^\lambda + (\lambda_1 - \lambda_1^2)e^\lambda \)

(f) \( h_{00} = 2(\lambda h_{01} + \dot{\lambda} h_{00}) \)

(4.14) (g) \( \dot{h}_{00} = 2(\dot{\lambda} e^{2\lambda} - 2\dot{\lambda} h_{01} + \dot{\lambda} h_{00}) \)

(h) \( \dot{h}_{11} = 2(\dot{\lambda} h_{11} + e^{2\lambda} - 2\dot{\lambda} h_{01}) \)

(i) \( \dot{h}_{11} = 2(\dot{\lambda} h_{11} + \dot{\lambda} h_{01}) \)

(j) \( h_{01} = (\lambda + \dot{\lambda}) h_{01} + \dot{\lambda} \left( h_{11} + e^{2\lambda} - 2\dot{\lambda} h_{00} \right) \)

(k) \( h_{01} = (\lambda + \dot{\lambda}) h_{01} + \dot{\lambda} \left( h_{00} + e^{2\lambda} - 2\dot{\lambda} h_{11} \right) \)

(l) \( Y h_{22} + e^{-2\lambda} Y h_{00} = Y e^{-2\lambda} h_{01} \)

(m) \( Y h_{22} + e^{-2\lambda} Y h_{01} = Y e^{-2\lambda} h_{11} \)

where \( k \) stands for a constant. Clearly, from (4.14)(1) and (m) we see that either \( Y \) is constant or is a function of both \( r \) and \( t \).

In case, \( Y = \text{constant}, \) (4.14)(c) implies that \( 2\sigma = -k \) and hence (Conf C) reduces to (AC). But this gives a non-physical solution \( \mu = p = 0 \). When \( Y \) is a function of both \( r \) and \( t \), neither \( \dot{Y} \) nor \( Y' \) can be zero and hence equations (4.14)(f) — (4.14)(m) can be transformed into

\[
\begin{align*}
\dot{h}_{00} &= 2(\dot{\lambda} + e^{2\lambda} - 2\dot{\lambda} Y' - Y h_{00}) = 2k e^{2\lambda} \dot{Y} Y' \\
\dot{h}_{11} &= 2(\lambda + e^{2\lambda} - 2\dot{\lambda} h_{11}) = -2ke^{2\lambda} \dot{Y}' Y' \\
\dot{h}_{11} &= 2(\dot{\lambda} + 2\dot{\lambda} Y' - 2\dot{\lambda} h_{11}) = -2ke^{2\lambda} \dot{Y}' Y' \\
\dot{h}_{01} &= (\lambda + e^{2\lambda} - 2\dot{\lambda} Y' - Y' h_{01}) = 2k e^{2\lambda} \dot{Y} Y' \\
\dot{h}_{01} &= (\lambda + e^{2\lambda} - 2\dot{\lambda} Y' - Y' h_{01}) + (\dot{Y} + Y' h_{01}) \\
\dot{h}_{00} &= -k e^{2\lambda} + e^{2\lambda} - 2\dot{\lambda} h_{01} \dot{Y}.
\end{align*}
\]
In general, we can not integrate the above system of equations.

As \( n_{0x} \neq 0 \), so if \( (\text{Conf} \ C) \) is a \( (S \text{ Conf} \ C) \), then eqn. \((1.20)\) immediately gives \( \mu - p + 4p = 0 \) whose isotropic counterpart is the curvature fluid.
APPENDIX A

Special Conformal Collineation as Matter Collineation

Consider the (S Conf C) generated by \( \xi \), in a space-time of general relativity. The Einstein's Field Equations are:

\[ \text{Ric}(X,Y) - \frac{(r/2)}{g(X,Y)} = T(X,Y) \]

Lie-derivation and using (3.3) and (3.5) of chapter V, yield

\[ (\text{tr.HQ})g(X,Y) - rh(X,Y) = 2(L_{T})T(X,Y) \]

In general, the solutions of the field eqns. can be unique only upto a diffeomorphism [54]. In particular, we consider here the one-parameter group of special conformal collineation generated by \( \xi \). If the matter tensor \( T \) is invariant of \( \xi \) (\( L_{T}T = 0 \)), then

\[ (\text{tr.HQ})g = rh \]

which yields, \( \text{tr.HQ} = r(\text{tr.H})/4 \). Hence eqn. (2) implies

(a) \( h = (\text{tr.H}/4)g \) if \( r \) is nowhere zero.

(b) no restriction on \( h \) if \( r = 0 \) everywhere.

Case (a) reduces (S,Conf C) to (S,Conf M). Case (b) leaves \( h \) as such but demands \( r = 0 \). In this case, therefore, pure radiation, and electromagnetic fields are included.

Conclusion: An (S,Conf C) can be further investigated in pure radiation and electromagnetic fields.
APPENDIX B

A new symmetry property

Here we first like to refer to sec. 2 of chapter V. If we relax the condition that \( A \) is parallel with respect to the Levi-Civita connection, we shall obtain:

\[
(L_\epsilon g)(X,Y) = 2\epsilon g(X,Y) + 2g(AX,Y)
\]

where \( A \) is the shape operator with respect to the normal part of a conformal vector field in \( M \). Equation \((1)\), therefore, defines an infinitesimal transformation which deviates from a conformal motion by an embedding term. We call such an infinitesimal transformation an Almost Conformal Motion (A-Conf M).

Example 1: Collinson [21] has shown that a (CC) vector \( \xi \) in empty space of Petrov type \( N \) generates an (A-Conf M) such that:

\[
(L_\epsilon g)(X,Y) = 2\epsilon g(X,Y) + \alpha 1(X)1(Y)
\]

where \( \alpha \), \( \alpha \) are scalar functions and \( 1 \) is the 1-form metrically equivalent to a unique principal vector of the Weyl tensor.

Example 2: Katzin, Levine and Davis [60] found that the (CC) vector in a non-Einstein conformally flat space satisfies:

\[
(L_\epsilon g)(X,Y) = 2\epsilon g(X,Y) + \tau \text{Ric}(X,Y)
\]

Clearly, \((2)\) and \((3)\) are special cases of \((1)\). Based on the above information, I believe that further study on this new symmetry (A-Conf M) has the prospect of physical applications.
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