D numbers and the D distribution.

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D NUMBERS AND THE D DISTRIBUTION

by

Mei Ling Huang

A Dissertation
Submitted to the Faculty of Graduate Studies and Research
Through the Department of Mathematics and Statistics
in Partial Fulfillment of the Requirements for the
Degree of Doctor of Philosophy at
The University of Windsor

Windsor, Ontario, Canada
1990
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ISBN 0-315-61891-4
Out of Great Complexity

Comes Great Simplicity
Ph.D. Dissertation
D NUMBERS AND THE D DISTRIBUTION

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ABSTRACT

Three new distributions are derived for sums of independent truncated Poisson variates, namely, the generalized Stirling, the R, and the D distributions. They depend respectively on the generalized Stirling, R, and D numbers which are defined, studied, and tabulated. Recursion, decomposition, and recurrence relations, limiting and modal properties of these new distributions and numbers are investigated. The moments are obtained. MVU estimators of the probability functions (p.f.'s) of these distributions and computational methods for these numbers and p.f.'s are also given.

In addition, the D distribution is extended to the D compound distribution when the number of truncated Poisson variables to be summed is considered as a random variable.

Applications are given to a variety of problems: e.g. occupancy, queueing, estimation, medical and systems design problems.
ACKNOWLEDGMENTS

The author wishes to express her sincerest gratitude and appreciation to Dr. Karen Yuen Fung for her constant support, helpful supervision and great patience during the preparation of this dissertation. The author is deeply indebted to her for her scholarly advice at every stage of development of this work.

The author is also grateful to the external examiner Dr. N. Balakrishnan and other members of the doctoral committee: Dr. M. Hlynka, Dr. S. R. Paul and Dr. J. Morrissey, for their valuable time and suggestions which greatly improved this dissertation.

I would like to thank the Department of Mathematics and Statistics and the University of Windsor for financial support in terms of scholarships and teaching assistantship. Also, the author wishes to thank Delta Gamma Kappa for the award of the Delta Gamma Kappa World Fellowship during my Ph.D. studies.

Special thanks are due to all faculty members, staff, fellow students and friends for their interests in my research, and for their inspiration.

Finally, thanks are extended to my family, for their deep love, support and encouragement.
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CHAPTER 1

INTRODUCTION

1.1 THE TRUNCATED POISSON DISTRIBUTION

A random variable $X$ is said to have a Poisson distribution with parameter $\lambda$ if

$$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad \lambda > 0, \ k = 0, 1, 2, \ldots \quad (1.1)$$

It is possible to modify the Poisson probabilities by omitting from the domain of $X$ some of the non-negative integers. In general, the values removed are either theoretically meaningless under the model, or they are practically unobservable. The most common form of truncation is the omission of the zero class because the observational apparatus becomes active only when at least one event occurs.

(1) The zero truncated Poisson distribution (ZTPD) is given by

$$P(X=k) = e^{-\lambda} (1-e^{-\lambda})^{-1} \frac{\lambda^k}{k!}$$

$$= (e^{-\lambda} - 1)^{-1} \frac{\lambda^k}{k!}$$
\[ \{ e_2(1, \lambda) \}^{-1} \frac{\lambda^k}{k!}, \quad \lambda > 0, \quad k = 1, 2, \ldots \quad (1.2) \]

where \[ e_2(1, \lambda) = \sum_{j=1}^{\infty} \frac{\lambda^j}{j!} = e^\lambda - 1. \]

This distribution has been studied by David and Johnson (1952), Plackett (1953), Irwin (1959), Cohen (1954) and others in terms of estimation. Many applications of this distribution have been given, e.g. McKendrick (1926), Finney and Varley (1955), Gross (1973).

(2) The left truncated Poisson distribution (LTPD)

If the first \( N \) categories are truncated, the resulting distribution is called the left truncated Poisson distribution,

\[
P(X=k) = e^{-\lambda}(1 - e^{-\lambda} \sum_{j=N}^{\infty} \frac{\lambda^j}{j!}) \frac{\lambda^k}{k!}
\]

\[
= \left( \sum_{j=N}^{\infty} \frac{\lambda^j}{j!} \right) \frac{\lambda^k}{k!}
\]

\[
= \{ e_2(N, \lambda) \}^{-1} \frac{\lambda^k}{k!}, \quad \lambda > 0, \quad k = N, N+1, \ldots \quad (1.3)
\]

where \[ e_2(N, \lambda) = \sum_{j=N}^{\infty} \frac{\lambda^j}{j!} . \]

Problems involving this distribution were investigated by Rider (1953), Moore (1956), Patil (1962), Rao and Rubin (1964), Subrahmaniam (1965), Haight and Fisher (1966).

(3) The right truncated Poisson distribution (RTPD)

Similarly, the Poisson distribution can be truncated on
the right. This occurs if the counting mechanism is unable to deal with large numbers. The right truncated Poisson distribution is given by

$$P(X=k) = e^{-\lambda}(1 - e^{-\lambda} \sum_{j=M+1}^{\infty} \frac{\lambda^j}{j!} \frac{\lambda^k}{k!})$$

$$= (\sum_{j=0}^{M} \frac{\lambda^j}{j!} \frac{\lambda^k}{k!})$$

$$= [e(M,\lambda)]^{-1} \frac{\lambda^k}{k!}, \quad \lambda > 0, \ k = 0, \ldots, M. \quad (1.4)$$

where $e(M,\lambda) = \sum_{j=0}^{M} \frac{\lambda^j}{j!}.$

Tippett (1932) discussed this distribution for $M = 2, 3, 4$; Cohen (1961) provided tables to obtain the maximum likelihood estimate of $\lambda$ in this case. Also, Moore (1952), Patil (1962), Haight and Fisher (1966) gave some results on the estimate of parameter $\lambda.$

(4) **The doubly truncated Poisson distribution (DTPD)**

If the Poisson distribution is truncated below $N$ and above $M,$ we have the doubly truncated Poisson distribution given by

$$P(X=k) = e^{-\lambda}(1 - e^{-\lambda}(\sum_{j=0}^{N-1} \frac{\lambda^j}{j!} + \sum_{j=M+1}^{\infty} \frac{\lambda^j}{j!})^{-1} \frac{\lambda^k}{k!})$$

$$= (\sum_{j=N}^{M} \frac{\lambda^j}{j!} \frac{\lambda^k}{k!})$$

$$= [e(N, M, \lambda)]^{-1} \frac{\lambda^k}{k!}, \quad \lambda > 0, \ k = N, N+1, \ldots, M. \quad (1.5)$$
where $e(N, M, \lambda) = \sum_{j=N}^{M} \frac{\lambda^j}{j!}$.

Moore (1954) suggested a simple form of an estimate needing no tables but having a larger standard error than the maximum likelihood estimate of $\lambda$. Doss (1963) did some comparative studies on the efficiency of the maximum likelihood estimate. Cohen (1954), Patil (1962) also provided some results on the estimate of parameter $\lambda$.

Various other types of truncated Poisson distributions have been studied by Moore (1954), Hartley (1958), Cohen (1960a,b,c), Swamy (1962), and Patil (1962), etc.

1.2 MOTIVATION FOR STUDYING THE D NUMBERS AND D DISTRIBUTION

1.2.1 INTUITIVE BACKGROUND —
"DISTRIBUTION OF THE SUM OF TRUNCATED POISSON VARIATES"

The problem of finding the distribution of the sum of random variables is important both in the theory of probability and in the practice of statistical inference. From the theoretical point of view it is a problem in convolutions of distributions, and from the practical point of view the mean value $\bar{x}$ of a sample $x_1, x_2, ..., x_n$ is, apart from the constant factor $(1/n)$, just the sum of the sample values. Sometimes, one is also interested in studying the total value of a number of entities, each of which is either impossible, unnecessary, or simply not recorded.
It is well known that the distribution of the sum of two independent Poisson variables is still a Poisson distribution. But for truncated Poisson variables the distribution of the sum of these variables is not truncated Poisson any more but some very complicated totally new kind of distribution. It is interesting and important to investigate these distributions as many authors have studied them in the past.

Let us look at the following two facts:

**FACT I** :

If random variables $Y_i \sim \text{Poisson}(\lambda)$ are independent, 

\[ \lambda > 0; \ y_i = 0, 1, \ldots; \ i = 1, \ldots, n; \]

then

\[ X = \sum_{i=1}^{n} Y_i \sim \text{Poisson}(n\lambda), \quad x = 0, 1, \ldots. \]

The probability function of $X$ is

\[ P(X = x) = e^{-n\lambda(n\lambda)^x} \quad (1.6) \]

**FACT II** :

If random variables $Y_i \sim \text{ZTPD}(\lambda)$ are independent, 

\[ \lambda > 0; \ y_i = 1, 2, \ldots; \ i = 1, \ldots, n; \]

then

\[ X = \sum_{i=1}^{n} Y_i \sim \text{Stirling-DSK}(n, \lambda), \quad x = n, n+1, \ldots. \]
Stirling-DSK\( (n, \lambda) \) is the Stirling distribution of the second kind with parameters \( \lambda \) and \( n \). It was named by Ahuja (1971) and Singh (1975).

The probability function of \( X \) is

\[
P(X = x) = (e^\lambda - 1)^{-n}(n! S(x, n)) \frac{\lambda^x}{x!}, \quad x = n, n+1, \ldots \tag{1.7}
\]

where

\[
S(x, n) \text{ is the Stirling number of the second kind}
\]

\[
S(x, n) = \left\{ \begin{array}{ll}
\frac{(-1)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (k)^x, & x = n, n+1, \\
0, & \text{otherwise}
\end{array} \right. \tag{1.8}
\]

(see Section 1.5).

Tate and Goen (1958) used characteristic functions to study the distribution of \( n \) i.i.d. left truncated Poisson random variables (r.v.s) and find the minimum variance unbiased estimator (MVUE) of the parameter \( \lambda \) of the left truncated Poisson distribution (LTPD). Later, various authors did more studies, for example, Gross (1970) used mathematical induction to get the exact distribution of \( n \) i.i.d. ZTPD r.v.s; Cacoullos (1972) gave simpler proofs of the above results by using a combinatorial derivation approach (Cacoullos, 1961). Ahuja (1971) and Singh (1975) defined the Stirling distribution of the second kind (Stirling-DSK\( (n, \lambda) \)); they started investigating the relation between the Stirling numbers and the probability function of the Stirling distribution. Charalambides (1974), and
Cacoullos and Papageorgiou (1984) studied generalized cases of Stirling distribution of the second kind.

In real life, many restricted probability models are truncated not only from the left side. For example, the waiting room of a queue has a maximum capacity, a delivery from inventory has at least one item, a bulk arrival at a queue may consist of a to b customers. Depending on the circumstances, the left, right or doubly truncated Poisson distribution is used. Sometimes, the sum of these random variables is of interest.

However, much less work is found in the literature corresponding to the right and doubly truncated Poisson distribution. One main reason is that the MVUE of the parameter $\lambda$ of RTPD$(\lambda)$ and DTPD$(\lambda)$ does not exist (see proof by Patil, 1963), and most researchers want to find a best estimate for the parameter $\lambda$. Another reason is that the distribution of the sum of the doubly (or right) truncated Poisson variables is much more complicated.

Saleh (1970) obtained the distribution of the sum of $n$ i.i.d. random variables from the LTPD as well as from the RTPD. Saleh and Rahim (1972) provided a general formula for the distribution of the sum of independent variables from a discrete population, truncated by any set of $s$ distinct values. Using these results, the exact distribution of the sum of $n$ i.i.d. random variables from a Poisson population, truncated arbitrarily, can be obtained.

Cacoullos (1977) discussed the best estimate for the multiply truncated power series distribution. His research work still focused on best estimates of the truncated distribution.
The author of this dissertation investigated the distribution of the sum of \( n \) independent doubly truncated Poisson random variables. Some new results are obtained:

**RESULT:**

If random variables \( Y_i \sim \text{DTPD}(\lambda_i) \) are independent,

\[
\lambda_i > 0; \ y_i = N_i, N_i + 1, \ldots, M_i; \ 0 \leq N_i < M_i, \ i = 1, \ldots, n; \ n, N_i, M_i \text{ are integers},
\]

then

\[
X = \sum_{i=1}^{n} Y_i \sim \text{D distribution} \ (n; \mathcal{L}, \Lambda),
\]

\[
x = \sum_{i=1}^{n} N_i, \sum_{i=1}^{n} N_i + 1, \ldots, \sum_{i=1}^{n} M_i
\]

\[
\mathcal{L} = \langle (N_i, M_i) | 0 \leq N_i < M_i, i = 1, \ldots, n; n, N_i, M_i \text{ are integers} \rangle
\]

\[
\Lambda = \langle \lambda_i | \lambda_i \text{ is a positive real number}, i = 1, 2, \ldots, n \rangle
\]

The probability function of \( X \) is

\[
P(X = x) = \prod_{i=1}^{n} e(N_i, M_i; \lambda_i) \frac{1}{\Gamma(x_n)} G(x, n; \mathcal{L}, \Lambda) / x!.
\]  \hspace{1cm} (1.9)

There are two major parts in (1.9),

(i) \( e(N_i, M_i; \lambda_i) \) is an incomplete exponential function,

\[
e(N_i, M_i; \lambda_i) = \begin{cases} 
\sum_{j=N_i}^{M_i} \frac{\lambda_i^j}{j!} , & \lambda_i > 0; \ 0 \leq N_i < M_i, \\
0 , & \text{otherwise.}
\end{cases}
\]  \hspace{1cm} (1.10)

More details of \( e(N_i, M_i; \lambda_i) \) will be discussed in Section 1.3.
(ii) $G(X, n, \lambda)$ which will be defined in Chapter 2 (see formula (2.4)) are called D numbers in this dissertation. More results of research on $G(X, n, \lambda)$ are also given in Chapter 2. Chapter 5 will illustrate the D distribution and its properties and applications.

Most of the research work in the literature involves n i.i.d. random variables. One reason is that to find the MVUE of the parameter $\lambda$ from a random sample we must have n i.i.d. random variables. Another reason is that the non i.i.d. case is much more complicated. However, in many situations, the random variables are not identical. For example, Shanmugam (1985) derived the distribution of the sum of a zero truncated Poisson variable and an ordinary Poisson variable. He obtained its statistical properties and gave an example to illustrate a medical application.

**EXAMPLE (Epidemic of cholera in a village in India)**

Let $Y$ be the number of cholera cases in a household. Event $Y = 0$ is not observable since the observational apparatus (i.e., diagnosis) is activated only when $Y > 0$. Consider $Y$ to be a $ZT$-Poisson $(\lambda)$ r.v., where $\lambda$ is the cholera incidence parameter. Health service agencies resort to various preventive treatments after they discover cholera in a household. These have the effect of changing $\lambda$ to $\rho \lambda$. That is $E(Z) = \rho \lambda$, where $0 \leq \rho < \infty$ is an intervention parameter, and $Z$ is the number of cholera cases in a household that occurred after preventive treatments. $Z$ is a Poisson r.v., and $Y$ and $Z$ are stochastically independent.

Assume that the observational apparatus has kept a
record of only the r.v. $X = Y + Z$, $X$ being the total number of cholera cases in a household that occurred altogether. Shanmugam called $X$ an intervened Poisson distribution r.v..

**FACT III:**

If random variables $Y \sim ZTPD(\lambda)$, $\lambda > 0$, $y = 1, 2, \ldots$

$Z \sim \text{Poisson}(\rho \lambda)$, $0 \leq \rho \leq 1$, $z = 0, 1, \ldots$

then

$X = Y + Z \sim \text{Intervened Poisson}(\rho, \lambda), x = 1, 2, \ldots$

The probability function of $X$ is

$$P(X = x) = \left( e^{\rho \lambda} (e^{\lambda} - 1) \right)^{-1} \frac{\lambda^x}{x!} \left[ (1 + \rho)^x - \rho^x \right], x = 1, 2, \ldots (1.11)$$

In this dissertation, we research cases which do not have identical parameters. When the parameters $\lambda_i$, $N_i$, $M_i$, may be different for $i = 1, 2, \ldots, n$ in (1.9), we get the general D numbers $GD(x, n; \lambda, \Lambda)$ and the D distribution $Dn(x; \lambda, \Lambda)$.

1.2.2 LOGICAL REASONING

The previous section illustrated the intuitive background and initial results. Another reason which motivated the author to study the D numbers and the D distribution is the finding and investigating of the relationship between the numbers and the distributions. This dissertation is a logical reasoning process which has four steps:
Step I: INTUITIVE BACKGROUND

- n-fold convolution of independent truncated Poisson random variables
  (See Section 1.2.1)

Step II: PROPERTIES

- Using relationship between D numbers and D distribution

Step III: NEW CALCULATION METHOD FOR THE P.F.

- Without D numbers

Step IV: APPLICATIONS and BASIC THEORY

(See Section 1.3)

The logic tree is given below.

Two major relations: (—→)
(1) Relations between D numbers and D distribution.
(2) Relations between incomplete exponential function and the D distribution.

Two other relations: (…………→)
(3) n-fold convolution of independent but not necessarily identical truncated Poisson random variables.
(4) Special and generalized cases.

Now, we illustrate each Step.

Step I: INTUITIVE BACKGROUND

The previous section (1.2.1) described the background and initial results of the sum of n independent truncated Poisson random variables.
LOGIC TREE

Queueing Theory  Occupancy Problem  Statistical Estimation  Medical Problem  Systems Design

Step IV  ---  Applications  ---  ---  ---  ---

With D Numbers  Without D Numbers

Step III  ---  Calculations  ---  ---  ---  ---

CHAPTER 8  IDTPD  CHAPTER 7  MGSDK  CHAPTER 5  D Distribution  CHAPTER 6  R Distribution

CHAPTER 9  D-PC Distribution

Step II  ---  ---  ---  ---  Properties  ---  ---  ---  ---

CHAPTER 1  Incomplete Exponential Function

Step I  ---  ---  ---  Sum of n independent r.v.'s  ---  ---  ---  ---

Poisson Dist.  Left Truncated Poisson Dist.  Doubly Truncated Poisson Dist.  Right Truncated Poisson Dist.

---  ---  ---  ---  Basic Theory  ---  ---  ---  ---

Distribution Theory  Number Theory  MVU Estimation Theory  Combinatorial Theory
Step II: PROPERTIES

D numbers and IEF are defined and properties are studied and presented in PART I (Chapters 2, 3, 4) of this dissertation. These properties were discovered either by observations from tables of these numbers or directly from proofs using basic combinatorial theory.

Using the relationship between D numbers and the D distribution, properties of the D distribution were then derived. The results are presented in PART II (Chapter 5, 6, 7, 8) of this dissertation.

In PART III we define the D compound distribution which treats n, the number of truncated Poisson variables to be summed, as a random variable. Properties of this distribution are given in Chapter 9.

Step III: NEW CALCULATION METHOD FOR P.F.

As D numbers get very large (beyond the word length of the computer) when x and n are large, the computation becomes more difficult. We recommend using the recurrence relations of the p.f. of the D distribution to calculate the p.f., thereby without using the D numbers.

Step IV: BASIC THEORY and APPLICATIONS (see Section 1.3)

We use the following basic theories in this dissertation:

- Probability distribution theory
- MVU estimation theory
- Number theory
- Combinatorial theory
1.3 NEW RESULTS OF THIS DISSERTATION

In this dissertation, the main results obtained from research are:

(1) A NEW KIND OF NUMBERS AND DISTRIBUTION HAS BEEN DERIVED, NAMELY, D NUMBERS AND THE D DISTRIBUTION. (Chapters 2 and 5)

Special and generalized cases of D numbers and the D distribution have also been investigated:

SPECIAL CASES:
Generalized Stirling numbers and distribution (Chapters 4 and 7);
R numbers and the R distribution (Chapters 3 and 6);
Intervened truncated Poisson distribution (Chapter 8).

GENERALIZED CASE:
D compound distribution (Chapter 9).

Some of these numbers and the p.f. of related distributions have been

\[
\begin{align*}
\text{Defined} & \\
\text{Tabulated} & \\
\text{Investigated} & 
\end{align*}
\]

(2) GOOD PROPERTIES OF THESE NUMBERS AND THEIR RELATED DISTRIBUTIONS HAVE BEEN DISCOVERED.
These are: Recurrence properties,
Modal properties,
Limiting properties,
Recursion and decomposition relations,
Moments.
(3) **MVU ESTIMATES OF P.F. OF THESE DISTRIBUTIONS HAVE BEEN GIVEN.**

Also, we have given the MVU estimate of the variance of the above MVU estimate of p.f.

(4) **COMPUTATIONAL METHODS.**

We have investigated new computational methods which are:

\[
\begin{align*}
\text{Efficient} \\
\text{Convenient} \\
\text{Accurate}
\end{align*}
\]

for

\[
\begin{align*}
\text{D numbers} \\
P.f. \text{ of the D distribution} \\
\text{MVUE of p.f. of the D distribution}
\end{align*}
\]

(5) **WIDE APPLICATIONS.**

Many interesting applications of the D numbers and the D distribution are found in this dissertation:

(1) **RESTRICTED SUMMATION IN REAL LIFE.**

**EXAMPLE 1.** (n-channel of M/M/c/c Queue in Telephone Systems) 
(see Chapter 6, Section 6.8, Example 1).

**EXAMPLE 2.** (Medical Problem) 
(see Chapter 8, Section 8.5, Example 2).

**EXAMPLE 3.** (Applications in the NBA Basketball World Championship Series) 
(see Chapter 5, Section 5.8, Example 2).

**EXAMPLE 4.** (University Registration Problem) 
(see Chapter 8, Section 8.5, Example 1).

**EXAMPLE 5.** (Ambassador Bridge Problem) 
(see Chapter 9, Section 9.6, Example 1).
(ii) **Statistical Applications.**

The sum of \( n \) i.i.d. random variables can be used to find the complete statistic, and MVU estimates for parameters or p.f. of distribution. (see Chapters 5, 6, 7, Section 5.7, 6.7, 7.7). Interestingly, the MVUE of the p.f. of the \( D \) distribution depends only on three \( D \) numbers.

(iii) **Restricted Occupancy Problems.**

In general, these are problems of finding the number of ways that \( x \) objects can be randomly assigned to \( n \) urns, each of which contains several cells, and each cell may have restricted capacity. There are many models of this problem that can be formulated depending on whether the objects, urns and cells are distinguishable or not, and whether there is a maximum or minimum capacity restriction put on each urn and each cell. The \( D \) numbers, \( R \) numbers, and MG-Stirling numbers are solutions to three particular models of this problem.

1.4 **Incomplete Exponential Function**

**Definition 1.** An incomplete exponential function (IEF) is defined by

\[
e(N,M; \lambda) = \begin{cases} 
\sum_{i=N}^{M} \frac{\lambda^i}{i!}, & \lambda > 0; \ 0 \leq N < M, \ N, M \text{ are integers} \\
\sum_{i=0}^{M} \frac{\lambda^i}{i!}, & \lambda > 0; \ N = -1, -2, \ldots \\
0, & \text{otherwise.}
\end{cases}
\]  

(1.12)

Special cases include
\[ e_1(M, \lambda) = e(0, M; \lambda), \quad M = 1, 2, 3, \ldots \]  
(1.13)

\[ e_2(N, \lambda) = e(N, \omega; \lambda), \quad N = 0, 1, 2, \ldots \]  
(1.14)

We can easily get the following properties of IEF:

**PROPERTY 2.1**

\[ e(N, M, \lambda) = e_1(M; \lambda) - e_1(N-1, \lambda) \]  
(1.15)

\[ e_2(N, \lambda) = e^\lambda - e_1(N-1, \lambda) \]  
(1.16)

**PROPERTY 2.2 (Recurrence Formula)**

\[ e(N, M, \lambda) = e(N+1, M; \lambda) + \frac{\lambda^N}{N!} = e(N, M-1; \lambda) + \frac{\lambda^M}{M!}. \]  
(1.17)

**PROPERTY 2.3 (Derivative Properties)**

\[ \frac{d^r e(N, M; c\lambda)}{d\lambda^r} = c^r e(N-r, M-r; c\lambda), \]  
where \( c \) is a constant, \( r \) is a positive integer.

**PROPERTY 2.4**

\[ \lim_{N \to 0, M \to \infty} e(N, M; \lambda) = e^\lambda. \]  
(1.19)

IEF is related to many important functions (see Abramowitz and Stegun, 1965):

(i) \[ e_1(M-1, \lambda) = \frac{\Gamma(M, \lambda)}{\Gamma(M)} e^\lambda, \quad \lambda > 0, \ M = 1, 2, \ldots \]  
(1.20)

where
\[ \Gamma(M, x) = \int_x^\infty t^{M-1} e^{-t} \, dt \text{ is an incomplete gamma function,} \quad (1.21) \]

\[ \Gamma(M) = \int_0^\infty t^{M-1} e^{-t} \, dt \text{ is the gamma function.} \quad (1.22) \]

(ii) \[ e_i(M-1, \lambda) = [1 - \Gamma(\lambda / \sqrt{M}, M-1)] e^\lambda, \]

\[ \lambda > 0, \quad M = 1, 2, \ldots \quad (1.23) \]

where

\[ I(u, p) = \frac{1}{\Gamma(p+1)} \int_0^{\sqrt{p+1}} t^p e^{-t} \, dt \text{ is Pearson's incomplete gamma function.} \]

For convenience, we have tabulated the values of \( e_i(M, \lambda) \) in Table VII of Appendix A, which is very helpful in applications of the D distribution. Otherwise, note that values of \( I(u, p) \) were tabulated by Pearson (1922).

1.5 STIRLING NUMBERS

Stirling numbers of the first and second kind were first computed by Stirling (1730), and have been studied by many authors, e.g. Riordan (1958), Jordan (1965) and Comtet (1974), etc.

**DEFINITION 1** (Stirling Numbers of the First Kind)

Stirling numbers of the first kind are defined as (Jordan, 1965)

\[ s(x, n) = \left\{ \frac{1}{n!} D^n(t) \right\}_{x=t=0} \quad (1.24) \]

or
\[ (t)_x = \sum_{n=1}^{x} s(x,n)t^n \]  \hspace{1cm} (1.25)

where

\[ D^n f(t) \text{ is the n-th derivative of } f(t), \]

\[ D^n f(t) = \lim_{h \to 0} \frac{\Delta^n f(t)}{h^n} \]  \hspace{1cm} (1.26)

\[ \Delta^n \text{ is the n-th difference operator} \]

\[ \frac{\Delta f(t)}{h} = f(t+h) - f(t) \]

\[ \frac{\Delta^n f(t)}{h} = \frac{\Delta^{n-1} f(t+h)}{h} - \frac{\Delta^{n-1} f(t)}{h} \]  \hspace{1cm} (1.27)

and \((t)_x = t(t-1)(t-2)\ldots(t-x+1)\) is called the \(t\) factorial of degree \(x\).

**PROPERTIES.** (of Stirling numbers of the first kind, Jordan, 1965)

1. \( s(x,0) = 0; \)
   \[ s(1,1) = 1; \]  \hspace{1cm} (1.28)
   \[ s(x,n) = 0; \text{ if } n > x. \]

2. Recurrence relation:
   \[ s(x+1,n) = s(x,n-1) - x s(x,n) \]  \hspace{1cm} (1.29)

3. \( s(x,1) + s(x,2) + s(x,3) + \ldots + s(x,x) = 0 \)  \hspace{1cm} (1.30)

4. \((-1)^x x! = \sum_{n=1}^{x+1} (-1)^n s(x,n)\)  \hspace{1cm} (1.31)

or

\[ x! = \sum_{n=1}^{x+1} |s(x,n)| \]  \hspace{1cm} (1.32)

5. Approximation formula:
   \[ s(x+1,2) \sim (-1)^{x+1} x! [\log(x+1) + c] \]  \hspace{1cm} (1.33)

where \(c\) is Euler's constant.
\[
\lim_{x \to \infty} \frac{s(x+1, n+1)}{(x+1)!} = 0 \tag{1.34}
\]

\[
|s(x, n)| \sim \frac{(x-1)!}{(n-1)!} [\log x + c]^{n-1} \tag{1.35}
\]

where \(c\) is Euler's constant.

(6) Table (see Appendix A.8 Table XI).

**DEFINITION 2. (Stirling Numbers of the Second Kind)**

The Stirling numbers of the second kind are defined as (Jordan, 1965)

\[
s(x, n) = \frac{(-1)^n}{n!} \sum_{k=1}^{n} (-1)^k \binom{n}{k} k^x \tag{1.36}
\]

\[
= \left[ \frac{1}{n!} \Delta^x t^x \right]_{t=0} \tag{1.37}
\]

or

\[
t^x = \sum_{n=1}^{x+1} s(x, n) n t^n . \tag{1.38}
\]

**REMARK.**

(i) Formula (1.8) is identical to (1.36) and (1.37).

(ii) Generating function of \(s(x, n)\) is

\[
(e^t - 1)^n = \sum_{x=n}^{\infty} s(x, n) t^x , \quad t > 0 \tag{1.39}
\]

**PROPERTIES. (of Stirling numbers of the second kind, Jordan, 1965)**

(1) \(s(x, 0) = 0\); \(s(x, 1) = 1\); \(s(0, 0) = 0\);

\(s(x, n) = 0\), if \(n > x\);
\[ S(x+1,1) = S(x,1) = S(1,1) = 1 ; \] (1.40)
\[ S(x+1,n+1) = S(x,x) = S(1,1) = 1 . \]

(2) Recurrence relation:
\[ S(x+1,n) = S(x,n-1) + n S(x,n) \] (1.41)

(3) \[ S(x,n) = \frac{x!}{n!} \sum \frac{1}{r_1! r_2! \ldots r_m!} , \] (1.42)

where the above sum is extended to every value of \( r_i > 0 \)
(with repetitions and permutations), satisfying the condition
\[ r_1 + r_2 + \ldots + r_m = x . \]

(4) Approximation formula
\[ \lim_{x \to \infty} \frac{S(x,n)}{n^x} = \frac{1}{n!} \] (1.43)

or
\[ S(x,n) \sim \frac{n^x}{n!} , \quad x \to \infty . \] (1.44)

\[ \lim_{x \to \infty} \frac{S(x+1,n)}{S(x,n)} = n . \] (1.45)

(5) Table (see Appendix A.8 Table XII).
PART I

THE D NUMBERS
Chapter 2

THE D NUMBERS

2.1 INTRODUCTION

The $D(x,n;L,A_0)$ numbers introduced here are an extension of the Stirling numbers of the second kind $S(x,n)$ (see section 1.3). $S(x,n)$ have been extensively studied, generalized and widely applied in combinatorial analysis and statistical estimation problems. See for example, Tate and Goen (1958), Ahuja (1971), Charalambides (1974), Singh (1975), Cacoullos and Papageorgiou (1984), Broder (1984), Charalambides and Singh (1988), Huang and Fung (1988, 1989a, b, c, d, e), and Huang (1990).

One of the applications of $S(x,n)$ is the restricted occupancy problem (see Riordan, 1958). $S(x,n)n!$ is the number of ways of putting $x$ different objects into $n$ distinguishable urns with no urns empty.

In this chapter, we consider a more general model of the above restricted occupancy problem. We present $D(x,n;L,A_0)$ as the number of ways of putting $x$ different objects into $n$ distinguishable urns with $\lambda_1, \lambda_2, \ldots, \lambda_n$ distinguishable cells and each urn containing at least
\(N_1, N_2, \ldots, N_n\) and at most \(M_1, M_2, \ldots, M_n\) objects, respectively. The Stirling numbers of the second kind (1.8), generalized Stirling number of the second kind (4.3), multiparameter Stirling number of the second kind (4.5), and \(R\) numbers (3.1) are all special cases of the \(D\) numbers.

2.2 D NUMBERS

**Definition 1.** The \(D\) numbers are defined as

\[
D(x, n; \mathcal{L}, \lambda) = \text{the number of ways of putting } x \text{ different objects into } n \text{ distinguishable urns having } \lambda_1, \ldots, \lambda_n \text{ distinguishable cells with each urn containing at least } N_i, N_2, \ldots, N_n \text{ and at most } M_1, M_2, \ldots, M_n \text{ objects, respectively}
\]

\[
= \sum_{\mathcal{Y}} \left( \left[ \begin{array}{c} x \\ y_1, y_2, \ldots, y_n \end{array} \right] \right) \prod_{i=1}^{n} \lambda_i^{y_i}
\]

(2.1)

where \(\mathcal{Y} = \{ \mathcal{Y} | \mathcal{Y} = (y_1, y_2, \ldots, y_n), y_i \text{ is an integer}, \sum y_i = x, \max(N_i, x - \sum M_j) \leq y_i \leq \min(x - \sum N_i, M_j), i = 1, \ldots, n \}, \)

\[
x = \sum_{i=1}^{n} N_i, \sum_{i=1}^{n} N_i + 1, \ldots, \sum_{i=1}^{n} M_i; n \text{ is a positive integer},
\]

\[
\mathcal{L} = \{ (N_i, M_i) | N_i, M_i \text{ are nonnegative integers}, N_i < M_i, i = 1, 2, \ldots, n \},
\]

\[\lambda = \{ \lambda | \lambda_i \text{ is a positive integer, } i = 1, 2, \ldots, n \}.
\]

**Remark.**

The summation is to be carried out over all sets of values of \((y_1, y_2, \ldots, y_n)\) for which \(y_i\) is between \(\max(N_i, x - \sum M_j)\) and \(\min(x - \sum N_i, M_j)\)
and \( \min(x - \sum_{j \neq i} N_j, M_i) \) for \( i = 1, 2, \ldots, n \), and which satisfy
\[
y_1 + y_2 + \ldots + y_n = x.
\]

If all \((N_i, M_i) \equiv (N, M)\) in \( \mathcal{L} \), we write \( \mathcal{L} \equiv (N, M) \).

If all \( \lambda_i \equiv \lambda \), we write \( \Lambda_0 \equiv \lambda \).

**Definition 2.** The d-numbers are defined by
\[
d(x, n; \mathcal{L}) = \text{the number of ways of putting } x \text{ different objects into } n \text{ distinguishable urns with each urn containing at least } N_1, N_2, \ldots, N_n \text{ and at most } M_1, M_2, \ldots, M_n \text{ objects, respectively}
\]
\[
= D(x, n; \mathcal{L}, \lambda) \land \lambda^x
\]
\[
= \sum_{\chi \in \mathcal{G}} \binom{x}{y_1, y_2, \ldots, y_n} \tag{2.2}
\]

where \( \chi \) and \( \mathcal{G} \) are defined in (2.1);
\[
x = \sum_{i=1}^{n} N_i, \sum_{i=1}^{n} N_{i+1}, \ldots, \sum_{i=1}^{n} M_i; \ n \text{ is a positive integer.}
\]

**Definition 3.** The D\(_i\)-numbers are defined as
\[
D_i(x, n; (N, M)) = \text{the number of ways of putting } x \text{ different objects into } n \text{ distinguishable urns with each urn containing at least } N \text{ and at most } M \text{ objects, respectively}
\]
\[
= d(x, n; (N, M))
\]
\[
= \sum_{\chi \in \mathcal{G}_i} \binom{x}{y_1, y_2, \ldots, y_n} \tag{2.3}
\]
where \( \mathcal{Y}_i = \{ \chi \mid \chi = (y_1, y_2, \ldots, y_n), y_i \text{ is an integer}, \sum_{i=1}^{n} y_i = x, \)
\[
\max(N, x-(n-1)M) \leq y_i \leq \min(x-(n-1)N, M), i=1, \ldots, n\}.
\]
\[x = nN, nN +1, \ldots, nM; \text{ } n \text{ is a positive integer;}
\]
\[0 \leq N < M; \text{ } N, M \text{ are integers.}
\]

Note that
\[
D(x, n; (N, M), \lambda) / \lambda^x = d(x, n; (N, M)) = D_i(x, n; (N, M)).
\]

**Remark.**
For the case when \( \lambda \) is not necessarily an integer in (2.1), we define the GD-numbers.

**Definition 4.** The GD-numbers are defined by
\[
GD(x, n; \mathcal{L}, \Lambda) = \sum_{\chi \in \mathcal{Y}} \left( \prod_{i=1}^{n} \lambda_i \right)^{\chi} \left( \prod_{i=1}^{n} y_i \right) \ldots (2.4)
\]
where \( \chi, \mathcal{Y}, \mathcal{L} \) are defined in (2.1)
\[
\Lambda = \{ \lambda_i \mid \lambda_i \text{ is a real positive number, } i = 1, 2, \ldots, n\}.
\]
\[x = \sum_{i=1}^{n} N_i, \sum_{i=1}^{n} N_i +1, \ldots, \sum_{i=1}^{n} M_i; \text{ } n \text{ is a positive integer.}
\]

The following relationships of D numbers with other numbers are obtained.
(i) When \( N, M > 0, M_i = \omega, GD(x, n; \mathcal{L}, \Lambda) \) is the more generalized Stirling number of the second kind (See Huang and Fung, 1988, and see Chapter 4). Also in this case
\(d(x,n;\mathcal{L})\) is \(g(x,n;\mathcal{M})\) (Huang and Fung, 1988), where \(g(x,n;\mathcal{M})/n!\) have been named by Cacoullos and Papageorgiou (1984) as the multiparameter Stirling numbers of the second kind.

(ii) When \(N>0, M = \infty\), \(D_I(x,n;(N,\infty))/n!\) is the generalized Stirling number of the second kind (Tate and Goen, 1958).

(iii) \(D_I(x,n;(1,\infty))/n!\) is the Stirling number of the second kind (Jordan, 1958).

(iv) GD\((x,n;\mathcal{L}_0,\Lambda)\) is the R-number \(R(x,n;\mathcal{M},\Lambda)\) (Huang and Fung, 1989a, also see Chapter 3), where \(\mathcal{L}_0 = \{(0,M_i) | M_i\) is a positive integer, \(i = 1, 2, \ldots, n\}\).

**Lemma 1.** The exponential generating function of \(d(x,n;\mathcal{L})\) is

\[
\prod_{i=1}^{n} e(N_i,M_i;\lambda) = \sum_{x=L_1}^{L_2} d(x,n;\mathcal{L}) \frac{\lambda^x}{x!},
\]

(2.5)

where \(L_1 = \sum_{i=1}^{n} N_i, L_2 = \sum_{i=1}^{n} M_i\),

\[e(N_i,M_i,\lambda) = \sum_{k=0}^{M_i} \frac{\lambda^k}{k!}\]

is an incomplete exponential function (Chapter 1, Section 1.4), tabulated in Appendix A.4, Table VII.

**Proof.**

\[
\prod_{i=1}^{n} e(N_i,M_i;\lambda) = \prod_{i=1}^{n} \left( \frac{\lambda^{N_i}}{(N_i)!} + \frac{\lambda^{N_i+1}}{(N_i+1)!} + \cdots + \frac{\lambda^{M_i}}{(M_i)!} \right)
\]
\[
= \sum_{x=L_1}^{L_2} \sum_{y \in \mathcal{Y}} \frac{1}{y_1! y_2! \cdots y_n!} \lambda^x
\]

where \( \mathcal{Y} = \{ y \mid y = (y_1, y_2, \ldots, y_n), y_i \text{ is an integer, } \sum_{i=1}^{n} y_i = x, \max(N_i, x - \Sigma M) \leq y_i \leq \min(x - \Sigma N, M), i=1, \ldots, n \} \),

\[
\sum_{y \in \mathcal{Y}} \frac{1}{y_1! y_2! \cdots y_n!}
\]

is the coefficient of \( \lambda^x \).

Therefore

\[
\prod_{i=1}^{n} e(N_i, M_i; \lambda) = \sum_{x=L_1}^{L_2} \sum_{y \in \mathcal{Y}} \binom{x}{y_1, y_2, \ldots, y_n} \frac{\lambda^x}{x!}
\]

\[
= \sum_{x=L_1}^{L_2} d(x, n; L, \mathcal{C}) \frac{\lambda^x}{x!} . \quad \square
\]

**COROLLARY 1.** The exponential generating function of \( D_1(x, n; (N, M)) \) is

\[
\{ e(N, M; \lambda) \}^n = \left[ \frac{\lambda^N}{N!} + \frac{\lambda^{N+1}}{(N+1)!} + \cdots + \frac{\lambda^M}{M!} \right]^n
\]

\[
= \sum_{x=nN}^{nM} D_1(x, n; (N, M)) \frac{\lambda^x}{x!}
\]

(2.7)

where \( x = nN, nN+1, \ldots, nM \).
REMARK.

\[ D_i (x,n; (N, M)) \] can also be written in the following algebraic form

\[
D_i (x,n; (N, M)) = x! \sum_{x = \Sigma; k_j}^{n} \left( \frac{1}{\Pi_{j=N}^{M} (j!)} \right) k_j^{n} k_{N+1} \ldots k_{M} \]

by using multinomial expansion formula, and the summation is to be carried out over all sets of values \( (k_N, k_{N+1}, \ldots, k_M) \) for

\[
Nk_N + (N+1)k_{N+1} + \ldots + Mk_M = x, \quad k_j \text{ is positive integer;}
\]

\[
k_N + k_{N+1} + \ldots + k_M = n.
\]

where \( x = nN, nN+1, \ldots, nM; \quad j = N, N+1, \ldots, M. \)

2.3 RECURRENCE RELATIONS

THEOREM 1. (RECURRENCE RELATIONS FOR d NUMBERS)

The d numbers have the following recurrence relations:

\[
d(x+1, n; \xi) = nd(x, n; \xi)
\]

\[
+ \sum_{i=1}^{n} \left( \left( \begin{array}{c} x \\ N - 1 \\ i \\ \end{array} \right) d(x-N-1, n-1; \xi_{\text{ex}}) - \left( \begin{array}{c} x \\ M - 1 \\ i \\ \end{array} \right) d(x-M, n-1; \xi_{\text{ex}}) \right),
\]

where

\[
\xi_{\text{ex}} = \xi \mid j \neq i);
\]

\[
x = \sum_{i=1}^{n} N, \sum_{i=1}^{n} N+1, \ldots, \sum_{i=1}^{n} M - 1.
\]
PROOF.

Let \( L_1 = \sum_{i=1}^{n} N_i, \quad L_2 = \sum_{i=1}^{n} M_i. \)

Differentiating both sides of (2.5) with respect to \( \lambda, \)

\[
\sum_{i=1}^{n} \prod_{j \neq i} e(N_j, M_j; \lambda) \frac{d}{d\lambda} e(N_i, M_i; \lambda) = \sum_{x=L_1}^{L_2} \frac{\lambda^{x-1}}{x!} d(x, n; \xi) \frac{\lambda^{x-1}}{(x-1)!},
\]

\[
\sum_{i=1}^{n} \prod_{j \neq i} e(N_j, M_j; \lambda) e(N_i - 1, M_i - 1; \lambda) = \sum_{x=L_1}^{L_2} \frac{\lambda^{x-1}}{(x-1)!} d(x, n; \xi) \frac{\lambda^{x-1}}{x!},
\]

\[
\sum_{i=1}^{n} \prod_{j \neq i} e(N_j, M_j; \lambda) \left( e(N_i, M_i; \lambda) + \frac{\lambda^{N_i-1}}{(N_i-1)!} \frac{M_i}{M_i!} \right)
\]

\[
= \sum_{x=L_1+1}^{L_2-1} \frac{\lambda^{x}}{x!} d(x+1, n; \xi),
\]

\[
\sum_{i=1}^{n} \left( \prod_{i=1}^{n} e(N_i, M_i; \lambda) + \prod_{j \neq i} e(N_j, M_j; \lambda) \left( \frac{\lambda^{N-i-1}}{(N_i-1)!} - \frac{M_i}{M_i!} \right) \right)
\]

\[
= \sum_{x=L_1+1}^{L_2-1} \frac{\lambda^{x}}{x!} d(x+1, n; \xi),
\]

\[
\prod_{i=1}^{n} e(N_i, M_i; \lambda) + \sum_{i=1}^{n} \prod_{j \neq i} e(N_j, M_j; \lambda) \left( \frac{\lambda^{N_i-1}}{(N_i-1)!} - \frac{M_i}{M_i!} \right)
\]

\[
= \sum_{x=L_1+1}^{L_2-1} \frac{\lambda^{x}}{x!} d(x+1, n; \xi) \frac{\lambda^{x}}{x!}.
\]

Using (2.5) again, we have
\[ r \left( \sum_{x=L_2}^{L_1} d(x, n; \xi) \frac{\lambda^x}{x!} \right) + \sum_{i=1}^{n} \left( \sum_{j=L_1}^{L_2-M_i} d(j, n-1; \xi_{(i)}) \frac{\lambda^j}{j!} \frac{\lambda^{N-1-i}}{(N-1)!} - \frac{\lambda^i}{M_i!} \right) \]

\[ = \sum_{x=L_1}^{L_2-M_i} d(x+1, n; \xi) \frac{\lambda^x}{x!}. \]

Equating the coefficients of \( \lambda^x \) on both sides of the equation, we get (2.9).

Similarly we can obtain recurrence relations for \( D_i \) numbers as follows.

**Theorem 2. (Recurrence Relations for \( D_i \) Numbers)**

\[ D_i(x+1, n; (N, M)) = nD_i(x, n; (N, M)) - n \binom{x}{M} D_i(x-M, n-1; (N, M)) \]

\[ + n \binom{x}{N-1} D_i(x-N+1, n-1; (N, M)), \quad (2.10) \]

where \( x = nN, nN+1, \ldots, nM-1 \).

**Proof.**

Differentiating both sides of (2.7) with respect to \( \lambda \), we have

\[ n! e(N, M; \lambda)_{(N)} \frac{d}{d\lambda} e(N, M; \lambda) = \sum_{x=nN}^{nM} D_i(x, n; (N, M)) x \frac{\lambda^{x-1}}{x!} \]

\[ n! e(N, M; \lambda)_{(1)} \left( \sum_{j=N}^{M} \frac{\lambda^{j-1}}{(j-1)!} \right) = \sum_{x=nN}^{nM} D_i(x, n; (N, M)) \frac{\lambda^{x-1}}{(x-1)!} \]
\[ \text{n}\{e(N, M; \lambda)\}^{n-1}\{e(N, M; \lambda)\} + \frac{\lambda^{N-1}}{(N-1)!} - \frac{\lambda^M}{M!} \]

\[ = \sum_{x=nN-1}^{nM-1} D_I(x+1, n; (N, M)) \frac{\lambda^x}{x!} \]

\[ \text{n}\{e(N, M; \lambda)\}^{n} + \text{n}\{e(N, M; \lambda)\}^{n-1}\left[\frac{\lambda^{N-1}}{(N-1)!} - \frac{\lambda^M}{M!}\right] \]

\[ = \sum_{x=nN-1}^{nM-1} D_I(x+1, n; (N, M)) \frac{\lambda^x}{x!} . \]

**Using (2.7) again,** we have

\[ \text{n}\sum_{x=nN}^{nM} D_I(x, n; (N, M)) \frac{\lambda^x}{x!} \]

\[ + \text{n} \sum_{j=(n-1)N}^{(n-1)M} D_I(j, n-1; (N, M)) \frac{\lambda^j}{j!} \left( -\frac{\lambda^M}{M!} + \frac{\lambda^{N-1}}{(N-1)!} \right) \]

\[ = \sum_{x=nN-1}^{nM-1} D_I(x+1, n; (N, M)) \frac{\lambda^x}{x!} . \]

Equating the coefficients of \( \lambda^x \) on both sides of the equation, we get (2.10). 

**Theorem 3.** For a given \( n, \ 0 \leq M_i < \infty \) for \( i = 1, 2, \ldots, n \), the last two d numbers in terms of \( x \) are always equal, i.e.,

\[ d(\sum_{i=1}^{n} M_i, n; \lambda) = d(\sum_{i=1}^{n} M_i - 1, n; \lambda) . \]  

(2.11)
PROOF.

\[ d_1(\sum_{i=1}^{n} M_i, n; \mathfrak{L}) = n! \left( \sum_{i=1}^{n} \frac{M_i}{M_1, M_2, \ldots, M_n} \right) \] = \frac{n! \left( \sum_{i=1}^{n} M_i \right)!}{\prod_{i=1}^{n} (M_i !)}

and

\[ d_1(\sum_{i=1}^{n} M_i - 1, n; \mathfrak{L}) = \sum_{j=1}^{n} n! \left( \sum_{i=1}^{n} \frac{M_i - 1}{M_1, \ldots, M_j - 1, \ldots, M_n} \right) \]

\[ = n! \sum_{j=1}^{n} \frac{M_j \left( \sum_{i=1}^{n} M_i - 1 \right)!}{M_j (M_j - 1)! \prod_{k \neq j} (M_k !)} \]

\[ = n! \sum_{j=1}^{n} \frac{n (\sum_{i=1}^{n} M_i - 1)!}{\prod_{i=1}^{n} (M_i !)} \]

\[ = n! \frac{C \left( \sum_{i=1}^{n} M_i \right)!}{\prod_{i=1}^{n} (M_i !)}. \]

From Theorem 3, we can easily get

**COROLLARY 2.**

\[ D_1(nM, n; (N, MD)) = D_1(nM-1, n; (N, MD)). \]  \hspace{1cm} (2.12)
COROLLARY 3.

\[ D_i(nM-1, n; (N, MD)) = \frac{n}{n-1} \binom{nM-1}{M} D_i((n-1)M-1, n-1; (N, MD)) \]  

(2.13)

PROOF.

Using Theorem 2, we have

\[ D_i(nM, n; (N, MD)) = n D_i(nM-1, n; (N, MD)) \]

\[- n \binom{nM-1}{M} D_i((nM-1)-M, n-1; (N, MD)) \]

\[ + n \binom{nM-1}{M} D_i((nM-1)-N+1, n-1; (N, MD)). \]

Since \((nM-1)-N+1 > (n-1)M\), hence

\[ D_i((nM-1)-N+1, n-1; (N, MD)) = 0, \]

and using Corollary 2, we have

\[ D_i(nM-1, n; (N, MD)) \]

\[ = n D_i(nM-1, n; (N, MD)) - n \binom{nM-1}{M} D_i((nM-1)-M, n-1; (N, MD)). \]

From this we get (2.13).

THEOREM 4.

(i) For fixed \(N \geq 0\), and \(n > 1\), the first \(M-N+1\) numbers in terms of \(x\) are always equal, i.e.,

\[ D_i(nN+j, n; (N, MD)) = D_i(nN+j, n; (N, M+i)), \]

(2.14)

\[ j = 1, 2, \ldots, (M-ND); \quad i = 1, 2, \ldots. \]
(ii) For fixed \( M, 0 < N < M < \infty \), and \( n > 1 \), the last \( M-N+1 \)
numbers in terms of \( x \) are always equal, i.e.,

\[
D_i(nM-j, n; (N, M)) = D_i(nM-j, n; (N-i, M)) \quad (2.15)
\]

\( j = 1, 2, \ldots, (M-N); \quad i = 1, 2, \ldots \)

**Proof.**

For simplification, we use the following notation

\[
\sum_{y_i = A}^{B} \left( y_1^x, y_2^x, \ldots, y_n^x \right) = \sum_{y \in \mathcal{Y}} \left( y_1^x, y_2^x, \ldots, y_n^x \right),
\]

where \( \mathcal{Y} = \{ y \mid y = (y_1, y_2, \ldots, y_n), y_i \text{ is an integer}, \sum_{i=1}^{n} y_i = x, A \leq y_i \leq B, i=1, \ldots, n \} \).

From (2.3), we have

\[
(i) \quad D_i(nN+j, n; (N, M)) = \sum_{y_i = \max(N, nN+j-(n-1)M)}^{\min(nN+j-(n-1)N, M)} \left( y_1^x, y_2^x, \ldots, y_n^x \right).
\]

For \( n > 1 \) and \( j \leq M-N \), we have

\[
\min(N+j, M) = N+j, \quad \text{and} \quad \max(nN+j-(n-1)M-N) = nN+(M-N)-(n-1)M-N = (n-2)(N-M) \leq 0,
\]

therefore \( \max(N, nN+j-(n-1)M) = N \).

Therefore,
\[ D_i(nN+j, n; (N, M)) = \sum_{y_i = n}^{N+j} \left( y_1, y_2, \ldots, y_n \right) \]

for \( j = 1, 2, \ldots, (M-N) \).

On the other hand,

\[ D_i(nN+j, n; (N, M+i)) = \sum_{y_i = \max(N, nN+j-(n-1)iM+i)}^{nN+j} \left( y_1, y_2, \ldots, y_n \right) \]

\[ = \sum_{y_i = N}^{N-j} \left( y_1, y_2, \ldots, y_n \right). \]

For \( n > 1, j \leq M-N, \) and \( i = 1, 2, \ldots, \) we have

\[ \min(N+j, M+i) = N+j, \]

and

\[ \max(N, nN+j-(n-1)(M+i)) = \max(N, nN+j-(n-1)M) = N. \]

Hence we have (2.14).

(ii) Similarly, we can show that both \( D_i(nM-j, n; (N, M)) \) and \( D_i(nM-j, n; (N-i, M)) \) in (2.15) are equal to

\[ \sum_{y_i = M-j}^{M} \left( y_1, y_2, \ldots, y_n \right). \]

THEOREM 5.

Let \( (x)_a = x(x-1) \ldots (x-a+1) \). \( D_i(x, n; (N, M)) \) satisfies the following recurrence relations:

(i) \( D_i(x, n; (N+1, M)) = \sum_{s=0}^{A} (-1)^s \frac{(N)_s}{(N!)^s} D_i(x-Ns, n-s; (N, M)) \)

(2.16)
where \( A = \left[ \frac{nM-x}{M-N} \right] \leq n; \quad x = n(N+1), \ldots, nM, \)

\([x] = \) largest integer less than or equal to \(x\).

\[
D_I(x, n; (N, M)) = \sum_{s=0}^{A} \frac{\binom{n}{s}(x)}{(N!)^s} D_I(x-Ns, n-s; (N+1, M)) \tag{2.17}
\]

for \(x = nN, \ldots, nM\).

(iii) \(D_I(x, n; (N, M-1)) = \sum_{s=0}^{B} \frac{(-1)^s \binom{n}{s}(x)}{(M!)^s} D_I(x-Ms, n-s; (N, M)) \tag{2.18}\)

where \( B = \left[ \frac{x-nN}{M-N} \right] \leq n; \quad x = nN, nN+1, \ldots, n(M-1). \)

\[
D_I(x, n; (N, M)) = \sum_{s=0}^{B} \frac{\binom{n}{s}(x)M^s}{(M!)^s} D_I(x-Ms, n-s; (N, M-1)) \tag{2.19}
\]

for \(x = nN, \ldots, nM\).

**PROOF.**

If we write the exponential generating function (2.7) of \(D_I(x+1, n; (N, M))\) as

\[
[e(N+1, M; \lambda)]^n = \left[ e(N, M; \lambda) = \frac{\lambda^N}{N!} \right]^n,
\]

and expand the right hand side with the binomial formula, we obtain

\[
[e(N+1, M; \lambda)] = \sum_{s=0}^{n} (-1)^s \binom{n}{s}[e(N, M; \lambda)] \frac{\lambda^s}{(N!)^s}.
\]

Using (2.7) on both sides of the equation above,
\[
\sum_{x=n(N+1)}^{nM} D_i(x, n; (N+1, M)) \frac{\lambda^x}{x!} = \sum_{s=0}^{n} (-1)^s \binom{n}{s} \sum_{x=(n-s)N}^{nN} D_i(x, n-s; (N, M)) \frac{\lambda^x}{(N!)^s x!}.
\]

Hence
\[
\sum_{x=n(N+1)}^{nM} D_i(x, n; (N+1, M)) \frac{\lambda^x}{x!} = \sum_{s=0}^{nM-s(M-N)} \sum_{x=nN}^{ns} (-1)^s \binom{n}{s} D_i(x-Ns, n-s; (N, M)) \frac{\lambda^x}{(N!)^s x!}.
\]

Equating the coefficients of \(\lambda^x\) on both sides of above equation we get (2.16).

Similarly we get (2.17), (2.18), and (2.19).

**REMARK.**

When \(M = \infty\), \(D_i(x, n; (N, \infty))/n!\) is the generalized Stirling number of the second kind. Equations (2.17) and (2.18) are in agreement with results of Charalambides (1974) and Cacoullos and Papageorgiou (1984) (see Chapter 4).

### 2.4. LIMITING PROPERTIES

**THEOREM 6.** (Limiting properties)

(i) \(\lim_{N \to 0} \lim_{M \to \infty} D_i(x, n; (N, M)) = n^x.\) \hspace{1cm} (2.20)
\[(iii) \lim_{n \to \infty} \left( e(N, M; 1) \right)^n \sum_{k=nN}^{\mu} \frac{D(k, n; (N, M))}{k!} = 1/2 ,\]  

where

\[\mu = n \frac{e(N-1, M-1; 1)}{e(N, M; 1)} ;\]

\[\lfloor \mu \rfloor = \text{largest integer less than or equal to } \mu .\]

(iii) When \(n\) is large,

\[D_1(x, n; (N, M)) \approx \frac{x! \left( e(N, M; 1) \right)^n}{\sqrt{2\pi \sigma}} \{\bar{\Phi}(y_1)-\bar{\Phi}(y_2)\} \]  

\[\approx \frac{x}{\sigma} \left( e(N, M; 1) \right)^n \{\bar{\Phi}(y_1)-\bar{\Phi}(y_2)\} \]  

where

\[y_1 = \frac{(x+1/2)-\mu}{\sigma} , \quad y_2 = \frac{(x-1/2)-\mu}{\sigma} ;\]

\(\bar{\Phi}(\cdot)\) is the cumulative standard normal;

\[\sigma^2 = n \left( \frac{e(N-2, M-2; 1)}{e(N, M; 1)} + \frac{e(N-1, M-1; 1)}{e(N, M; 1)} - \left( \frac{e(N-1, M-1; 1)}{e(N, M; 1)} \right)^2 \right) .\]

(iv) When \(n\) is large

\[D_1(\lfloor \mu \rfloor, n; (N, M)) \approx \frac{\lfloor \mu \rfloor!(e(N, M; 1))^n}{\sqrt{2\pi \sigma}} \]  

\[\approx \frac{\lfloor \mu \rfloor^{\lfloor \mu \rfloor - 1/2} e(e(N, M; 1))^n}{\sigma} .\]  

**Proof.**

(i) By (2.3),
\[ \lim_{N \to 0, M \to \infty} D_{I}(x, n; (N, M)) = \sum_{y_1 = 0}^{x-1} \cdots \sum_{y_n = 0}^{x-1} = n^x. \]

(ii) Let \( X \) be the sum of \( n \) i.i.d. doubly truncated Poisson random variables with parameter \( \lambda = 1 \) and left and right truncation points at \((N, M)\) respectively. The distribution of \( X \) is called the D-distribution with p.f. (see Chapter 5)

\[
p(x, n; (N, M), 1) = \left( \frac{e(N, M; 1)}{e(N, M; 1)} \right)^x D_{I}(x, n; (N, M)) / x!, \quad (2.26)
\]

where \( x = nN, nN+1, \ldots, NM. \)

The mean and variance are given by

\[
\mu = E[X] = n \frac{e(N-1, M-1; 1)}{e(N, M; 1)}
\]

\[
\sigma^2 = Var[X] = n \left( \frac{e(N-2, M-2; 1)}{e(N, M; 1)} + \frac{e(N-1, M-1; 1)}{e(N, M; 1)} - \left( \frac{e(N-1, M-1; 1)}{e(N, M; 1)} \right)^2 \right).
\]

When \( n \) is large, we can use the Lindeberg-Lévy Central Limit Theorem (Fisz, 1963, p. 197) to obtain the following.

\[ Z = \frac{X - E(X)}{\sqrt{Var(X)}} \sim N(0, 1), \text{ when } n \to \infty, \]

\[ \lim_{n \to \infty} P(X \leq \mu) = \lim_{n \to \infty} P(Z \leq 0) = \Phi(0) = 1/2. \]

\[ \left\{ \frac{\mu}{\sqrt{Var(X)}} \right\} \]

In other words, \( \sum_{x=nN}^{[\mu]} p(x, n; (N, M), 1) = 1/2. \)

Hence we get (2.21).

(iii) Using the fact that \( X \) is approximately normally distributed when \( n \) is large, we can find
\[
P(X=x) \approx \frac{1}{\sqrt{2\pi \text{Var}(X)}} \int_{x^{-1/2}}^{x^{-1/2}} e^{-\frac{(t-E(X))^2}{2\text{Var}(X)}} \, dt. \quad (2.27)
\]

Equating this to the p.f. of \( X \), we get (2.22).

By applying Stirling formula \( n! \approx \sqrt{2\pi n} \frac{n^{n+1}}{e^n} \), we get (2.23).

(iv) When \( x = [\mu] \), it is easy to prove that
\[
\frac{1}{\sqrt{2\pi \sigma^2}} \int_{[\mu]-1/2}^{[\mu]+1/2} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \, dt = 1, \quad \text{when } n \text{ is large}.
\]

Since \([\mu]\) and \( \sigma^2 \) are functions of \( n \) respectively, they can be written as \( \mu_n \) and \( \sigma_n^2 \).

Let
\[
B_n(x) = \frac{(x-\mu_n)^2}{2\sigma_n^2},
\]

\( f_n(x) = e^{-B_n(x)} \),

and
\[
A_n = \int_{\mu_n-1/2}^{\mu_n+1/2} f_n(x) \, dx.
\]

Clearly
\[
\lim_{n \to \infty} B_n(\mu) = \lim_{n \to \infty} B_n(\mu + 1/2) = \lim_{n \to \infty} B_n(\mu - 1/2) = 0.
\]
Hence
\[
\lim_{n \to \infty} f(n, \mu) = \lim_{n \to \infty} f(n, \mu + 1/2) = \lim_{n \to \infty} f(n, \mu - 1/2) = 1.
\]

Also \( f_n(x) \) is a decreasing function of \( n \) for all \( x \in [\mu_n, \mu_{n+1/2}] \), such that
\[
\lim_{n \to \infty} f_n(x) = 1 \quad \text{for all} \quad x \in [\mu_n, \mu_{n+1/2}].
\]

Similarly \( f_n(x) \) is an increasing function of \( n \) for all \( x \in [\mu_n - 1/2, \mu_n] \), such that
\[
\lim_{n \to \infty} f_n(x) = 1 \quad \text{for all} \quad x \in [\mu_n - 1/2, \mu_n].
\]

Hence \( \lim_{n \to \infty} \Lambda_n = 1 \).

Then (2.27) can be written as
\[
p(X=\mu) \approx \frac{1}{\sqrt{2\pi} \sigma}.
\]

Using (2.26), we obtain
\[
[e(N, M; 1)]^{-n} D_1(\mu, n; (N, M))/[\mu]! \approx \frac{1}{\sqrt{2\pi} \sigma}.
\]

Therefore, we get (2.24) and (2.25).

**Remark.**

For convenience, one can use the following log-form of (2.23),
\[
\log D_1(x, n; (N, M)) = (x + \frac{1}{2}) \log x - x + n \log e(N, M; 1) - \log \sigma
\]
\[
+ \log (\tilde{y}_1 - \tilde{y}_2).
\]
2.5 MONOTONICITY PROPERTIES

PROPERTY 1. For fixed $x$, 

(i) when $N > 0$, $D_I(x, n; (N, MD))$ has a single maximum. For any $x > nN + 4$, there exists a unique $k_x$ such that

$$D_I(x, n-1; (N, MD)) < D_I(x, n; (N, MD)), \quad n = 1, 2, \ldots, k_x;$$

$$D_I(x, n; (N, MD)) \geq D_I(x, n+1; (N, MD)), \quad n = k_x, \ldots, x; \quad (2.29)$$

Also $k_x = k_{x-1}$ or $k_{x-1} + 1$.

(ii) When $N = 0$

$$D_I(x, n; (0, MD)) \geq D_I(x, n-1; (0, MD)), \quad \text{for any } n. \quad (2.30)$$

(iii) When $x > nN$,

$$\frac{D_I(x, n-1; (N, MD))}{D_I(x, n; (N, MD))} > \frac{D_I(x, n-2; (N, MD))}{D_I(x, n-1; (N, MD))}. \quad (2.31)$$

When $x > nN + 4$,

$$\frac{(n-1)^{n-1}}{n^{n-1}} \frac{D_I(x, n-1; (N, MD))}{D_I(x, n; (N, MD))} > \frac{D_I(x, n-2; (N, MD))}{D_I(x, n-1; (N, MD))}. \quad (2.32)$$

For proof of this property, refer to Dobson (1968).

PROPERTY 2. For fixed $n \geq 1$,

$$h(x, n; (N, MD)) = \frac{x D_I(x-1, n; (N, MD))}{D_I(x, n; (N, MD))}, \quad x = nN, nN + 1, \ldots, nM, \quad (2.33)$$
(i) \( h(x,n; (N,M)) \) is an increasing function of \( x \).

(ii) If \( N>0 \), \( h(x,n; (N,M)) \geq \frac{N+1}{n} \) for \( nN<x<nM \), with equality for \( x = nN+1 \).

If \( N=0 \), \( h(x,n; (N,M)) \geq \frac{1}{n} \) for \( 0<x\leq nM \), with equality for \( x=2 \), and \( M \geq 2 \). For \( M=1 \) there is no equality.

For proof of this property, refer to Ahuja (1971).

**COROLLARY 9.** For fixed \( n \geq 1 \),

\[
\frac{D_{1}(x,n)}{D_{1}(x+1,n)} \geq \frac{D_{1}(x-1,n)}{D_{1}(x,n)} .
\]

(2.34)

Property 1 and 2 can also be verified by checking the tables of D-numbers in Appendix A.1.

**2.6 APPLICATIONS**

(1) **PROPERTIES OF THE D DISTRIBUTION DEPEND ON THE D NUMBERS.**

As we can see in (1.9), the D numbers \( G(\xi_x, n; \xi, \lambda)/x! \) are the main factors in the expression of the p.f. of \( D(\xi_x, n; \xi, \lambda) \).

In Chapter 5, we will investigate the properties of the D distribution. Most of them depend on the properties of the D numbers. For example, D numbers are used in the proofs of recurrence, recursion, and modal properties of the D distribution.

(2) **STATISTICAL ESTIMATION.**

The D number is also an important factor in the
expression of the MVU estimate of the p.f. of the D
distribution (see Chapter 5, section 5.7).

When all $\lambda_i \equiv \lambda$, the MVU estimate of (5.29) is

$$
\hat{p}(Z, m) = \frac{\left(\sum_{x} \mathcal{D}(x, n; \mathcal{L}) \mathcal{D}(z-x, (m-1)n; \mathcal{L}^{(m-1)})\right)}{\mathcal{D}(z, mn; \mathcal{L}^{(m)})}, \quad (2.35)
$$

where $z$ is the observed value of $Z = \sum_{i=1}^{m} Z_i$, $Z_i \sim \text{DD}(n; \mathcal{L}, \Lambda)$
are i.i.d., $i = 1, 2, \ldots, m$.

Here $(m-1)L + x \leq z \leq (m-1)Q + x$;
$L = \sum_{i=1}^{m} N_i$,
$Q = \sum_{i=1}^{m} M_i$;
$x = L, L+1, \ldots, Q$,
$\mathcal{L}^{(m)} = m$ identical sets of $\mathcal{L}$.

When $\mathcal{L} = (N, M)$, (2.35) becomes

$$
\hat{p}_x(Z, m) = \frac{\left(\sum_{x} \mathcal{D}_I(x, n; (N, M) \mathcal{D}_I(z-x, (m-1)n; (N, M))\right)}{\mathcal{D}_I(z, n; (N, M))}, \quad (2.36)
$$

where $(m-1)nN + x \leq z \leq (m-1)nM + x$;
$x = nN, nN+1, \ldots, nM$.

With $\mathcal{D}_I(x, n; (N, M))$ and $\mathcal{D}(x, n; \mathcal{L})$ numbers available,
one can easily calculate the values of (2.35) and (2.36).
Selected cases of the D numbers and d numbers are given in
Tables I and II in Appendix A.1.

(3) **RESTRICTED OCCUPANCY PROBLEM**.

As given in Definition 1, D numbers are the number of
ways of putting \( x \) different objects into \( n \) distinguishable urns having \( \lambda_1, \lambda_2, \ldots, \lambda_n \) distinguishable cells with each urn containing at least \( N_1, N_2, \ldots, N_n \) and at most \( M_1, M_2, \ldots, M_n \) objects, respectively.

2.7. COMPUTATIONAL METHODS FOR D NUMBERS

We can use Definition 1, 2, 3, 4 or Theorem 1 and 2 (recurrence relations of D numbers) to calculate the values of D numbers with a computer subroutine. The numerical results in Appendix A.1, Tables I and II were computed on an IBM4381 machine with double precision.
CHAPTER 3

R NUMBERS

3.1 INTRODUCTION

In Chapter 2, we introduced the \( D(x, n; \mathcal{L}, \Lambda_0) \) numbers, where \( \mathcal{L} \) is a set of truncation points \( (N_i, M_i) \), \( i = 1, 2, ..., n \). If there are no left truncation points \( N_i \), i.e. when all \( N_i = 0 \), for \( i = 1, 2, ..., n \), then the \( D(x, n; \mathcal{L}, \Lambda_0) \) numbers become a special kind of numbers which we call \( R \) numbers \( R(x, n; \mathcal{M}, \Lambda) \). Since \( R \) numbers are defined and treated as a special case of the \( D \) numbers, all the properties of \( D \) numbers also apply to \( R \) numbers. Some proofs will be omitted in this chapter.

We will obtain recurrence, limiting and monotonicity properties of \( R \) numbers. Finally, we will describe a method of calculation of \( R \) numbers.

3.2. R NUMBERS

**DEFINITION 1.** The \( R \) Numbers are defined as

\[
R(x, n; \mathcal{M}, \Lambda_0) = \text{the number of ways of putting } x \text{ different objects into } n \text{ distinguishable urns having } \lambda_i
\]
,...,\lambda_n\text{ distinguishable cells with each urn containing at most } M_1, M_2, \ldots, M_n \text{ objects, respectively}

\begin{equation}
= \sum_{\gamma \in \mathcal{Y}} \left( \prod_{i=1}^{n} \lambda_i \right)^{y_i} \prod_{i=1}^{n} y_i
\end{equation}

where \( \mathcal{Y} = \{ \gamma \mid \gamma = (y_1, y_2, \ldots, y_n), y_i \text{ is an integer, } \sum_{i=1}^{n} y_i = x, \max(0, x - \sum_{j \neq i} M_j) \leq y_i \leq \min(x, M_i), i = 1, \ldots, n \} \),

\( x = 0, 1, \ldots, \sum_{i=1}^{n} M_i; \ n \text{ is a positive integer, } \)

\( \Pi = \{ M_i \mid M_i \text{ is a positive integer, } i = 1, 2, \ldots, n \} \),

\( \Lambda_0 = \{ \lambda_i \mid \lambda_i \text{ is a positive integer, } i = 1, 2, \ldots, n \} \).

\textbf{REMARK.}

The summation is to be carried out over all sets of values of \((y_1, y_2, \ldots, y_n)\) for which \( y_i \) is between \( \max(0, x - \sum_{j \neq i} M_j) \) and \( \min(x, M_i) \) for \( i = 1, \ldots, n \), and which satisfy \( y_1 + y_2 + \ldots + y_n = x \).

If all \( M_i \equiv M \) in \( \Pi \), we write \( \Pi \equiv M \).

If all \( \lambda_i \equiv \lambda \), we write \( \Lambda_0 \equiv \lambda \).

\textbf{DEFINITION 2.} The \( r \)-numbers are defined by

\( r(x, n; \Pi, \lambda) = \text{the number of ways of putting } x \text{ different objects into } n \text{ distinguishable urns with each urn containing at most } M_1, M_2, \ldots, M_n \text{ objects, respectively} \)

\( = R(x, n; \Pi, \lambda) \lambda^x \)
\[ = \sum_{\gamma \in \mathcal{J}} \left( \frac{X}{\gamma_1, \gamma_2, \ldots, \gamma_n} \right) \]

where \( \gamma \) and \( \mathcal{J} \) are defined in (3.1);

\[ x = 0, 1, \ldots \sum_{i=1}^{n} M_i; \quad n \text{ is a positive integer.} \]

**DEFINITION 3.** The \( R_i \) numbers are defined as

\[ R_i(x, n; MD) = \text{the number of ways of putting } x \text{ different objects into } n \text{ distinguishable urns with each urn containing at most } M \text{ objects} \]

\[ = r(x, n; MD) \]

\[ = \sum_{\gamma \in \mathcal{J}_i} \left( \frac{X}{\gamma_1, \gamma_2, \ldots, \gamma_n} \right) \]

where \( \mathcal{J}_i = \{ \gamma \mid \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n), \gamma_i \text{ is an integer, } \sum_{i=1}^{n} \gamma_i = x, \]

\[ \max(0, x-(n-1)M) \leq \gamma_i \leq \min(x, MD), i=1, \ldots, n \}, \]

\[ x = 0, 1, \ldots, nM; \quad n, M \text{ are positive integers.} \]

Note that

\[ R(x, n; M, \lambda^x) = R(x, n; MD) = R_i(x, n; MD). \]

For the case when \( \lambda_i \) is not necessarily an integer in (3.1), we define the GR-numbers.

**DEFINITION 4.** THE GR-numbers are defined by

\[ \text{GR}(x, n; M\lambda) = \sum_{\gamma \in \mathcal{J}} \left( \frac{X}{\gamma_1, \gamma_2, \ldots, \gamma_n} \right) \prod_{i=1}^{n} \lambda_i \]

\[ = \sum_{\gamma \in \mathcal{J}} \left( \frac{X}{\gamma_1, \gamma_2, \ldots, \gamma_n} \right) \prod_{i=1}^{n} \lambda_i \]
where \( \chi, \gamma, \Pi \) are defined in (3.1),
\[
\Lambda = \{ \lambda_i | \lambda_i \text{ is a positive real number, } i = 1, 2, \ldots, n \},
\]
\[
x = 0, 1, \ldots, \sum_{i=1}^{n} M_i, \text{ and } n \text{ is a positive integer.}
\]

**Lemma 1.** The exponential generating function of \( r(x, n; \mathcal{L}) \) is
\[
\prod_{i=1}^{n} e^x_{i} (M_i, \lambda) = \sum_{x=0}^{L} r(x, n; \mathcal{M}) \frac{\lambda^x}{x!}, \tag{3.5}
\]
where \( L = \sum_{i=1}^{n} M_i \),
\[
e^x_{i} (M_i, \lambda) = \sum_{k=c}^{M_i} \frac{\lambda^k}{k!} \text{ is an incomplete exponential function (Chapter 1, Section 1.4), tabulated in Appendix A.4, Table VII.}
\]

The proof is similar to the one given in Chapter 2, Section 2.2, Lemma 1.

**Corollary 1.** The exponential generating function of \( R_I(x, n; \mathcal{M}) \) is
\[
[e^x_{i} (M_i, \lambda)]^n = (1 + \lambda^1 + \ldots + \lambda^{M_i} / M_i!)^n = \sum_{x=0}^{n^M} R_I(x, n; \mathcal{M}) \frac{\lambda^x}{x!}. \tag{3.6}
\]

It is easy to show that \( r(x, n; \mathcal{M}) \) in (3.5) and \( R_I(x, n; \mathcal{M}) \) in (3.6) are the same as given in (3.2) and (3.3).

**Remarks.**

(i) \( GR(x, n; \mathcal{M}, \Lambda) \) is \( GD(x, n; \mathcal{L}, \Lambda) \) with \( N_i = 0, M_i > 0 \). Also, in this case, \( r(x, n; \mathcal{M}) \) is \( d(x, n; \mathcal{L}) \), \( R_I(x, n; \mathcal{M}) \) is
$D(x, n; 0, M)$ (see Chapter 2, Section 2.2; Huang and Fung, 1989b).

(ii) R numbers $R(x, n; M, \Lambda)$ can be used in restricted occupancy problems. Fang (1982) discussed the problem of finding the number of ways that $x$ objects can be randomly assigned to $n$ urns, each of which contains $M$ cells, in such a way that each cell contains at most one object. He presented many models arising from this restricted occupancy problem (Table 1), depending on whether the urns, cells and objects are distinguishable (DT) or indistinguishable (IDT), whether empty urns are permitted or not. Many authors have worked on this kind of restricted occupancy model also by using different conditions of restrictions. For example, see Barton and David (1959), Freund and Pozner (1956), Johnson and Kotz (1977), Charalambides (1983).

**TABLE 1**

Restricted Occupancy Models

<table>
<thead>
<tr>
<th>Models with empty urns permitted</th>
<th>Models with no empty urns</th>
<th>Urns</th>
<th>Balls</th>
<th>Cells</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>DT</td>
<td>IDT</td>
<td>IDT</td>
</tr>
<tr>
<td>(3)</td>
<td>(4)</td>
<td>DT</td>
<td>IDT</td>
<td>DT</td>
</tr>
<tr>
<td>(5)</td>
<td>(6)</td>
<td>DT</td>
<td>DT</td>
<td>DT</td>
</tr>
<tr>
<td>(7)</td>
<td>(8)</td>
<td>DT</td>
<td>DT</td>
<td>IDT</td>
</tr>
<tr>
<td>(9)</td>
<td>(10)</td>
<td>IDT</td>
<td>DT</td>
<td>IDT</td>
</tr>
<tr>
<td>(11)</td>
<td>(12)</td>
<td>IDT</td>
<td>DT</td>
<td>DT</td>
</tr>
<tr>
<td>(13)</td>
<td>(14)</td>
<td>IDT</td>
<td>IDT</td>
<td>DT</td>
</tr>
<tr>
<td>(15)</td>
<td>(16)</td>
<td>IDT</td>
<td>IDT</td>
<td>IDT</td>
</tr>
</tbody>
</table>
\( R_I(x, n; M) \) in (2.3) corresponds to model (7) in Fang (1982).

\( r(x, n; M) \) in (2.2) corresponds to an extension of model (7) in Table 1 with each urn having \( M_1, M_2, \ldots, M_n \) indistinguishable cells, respectively.

Freund and Pozner (1956) introduced the \( NC(x, n; M) \) numbers which correspond to model (1) of Table 1. The only difference from model (7) is that the objects are indistinguishable here. \( R_I(x, n; M) \) is naturally different from \( NC(x, n; M) \).

By using the inclusion-exclusion principle (Riordan, 1958), we have

\[
NC(x, n; M) = \sum_{j=1}^{n} (-1)^j \binom{n}{j} \left( \frac{n+x-j(M+1)-1}{n-1} \right)
\]

(3.7)

where \( x = 0, 1, \ldots; n = 1, 2, \ldots; \) and \( M \) is a positive integer.

\( NC(x, n; M) \) has the generating function

\[
f(y; n, M) = (1+y+y^2+\ldots+y^M)^n = \sum_{x=0}^{nM} NC(x, n, M)y^x.
\]

(3.8)

Also, Freund and Pozner (1956) gave the following recurrence relations of \( NC(x, n, M) \):

\[
NC(x, n, M) = \sum_{i=0}^{M} NC(x-i, n-1, M)
\]

(3.9)

\[
NC(x, n-1, M) = \sum_{i=1}^{M} \binom{n-x}{i} NC(x-i, n-1, M)
\]

(3.10)

\[
NC(x, n, M) = NC(x-1, n, M) + NC(x, n-1, M) - NC(x-M-1, n-1, M).
\]

(3.11)

We will compare the recurrence relations of \( R_I(x, n; M) \)
in the next section with (3.9) - (3.11).

Table 2 shows the $R_i(x,n;M)$ numbers as well as the $N(x,n;M)$ numbers in brackets for $M = 2$.

**TABLE 2 $R_i(x,n;M)$ and $N(x,n;M)$ Numbers**

$M = 2$

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>1(1)</td>
<td>2(2)</td>
<td>3(3)</td>
<td>4(4)</td>
<td>5(5)</td>
<td>6(6)</td>
</tr>
<tr>
<td>2</td>
<td>1(1)</td>
<td>4(3)</td>
<td>9(6)</td>
<td>16(10)</td>
<td>25(15)</td>
<td>36(21)</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>6(2)</td>
<td>24(7)</td>
<td>60(16)</td>
<td>120(30)</td>
<td>210(50)</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>6(1)</td>
<td>54(6)</td>
<td>204(19)</td>
<td>540(45)</td>
<td>1170(90)</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>-</td>
<td>90(3)</td>
<td>600(16)</td>
<td>2220(51)</td>
<td>6120(126)</td>
</tr>
<tr>
<td>6</td>
<td>-</td>
<td>-</td>
<td>90(1)</td>
<td>1440(10)</td>
<td>8100(45)</td>
<td>29520(141)</td>
</tr>
<tr>
<td>7</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2520(4)</td>
<td>25200(30)</td>
<td>128520(126)</td>
</tr>
<tr>
<td>8</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2520(1)</td>
<td>63000(15)</td>
<td>491400(90)</td>
</tr>
<tr>
<td>9</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>113400(5)</td>
<td>1587600(50)</td>
</tr>
<tr>
<td>10</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>113400(1)</td>
<td>4082400(21)</td>
</tr>
<tr>
<td>11</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>7484400(6)</td>
</tr>
<tr>
<td>12</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>7484400(1)</td>
</tr>
</tbody>
</table>

We can see that

(a) $R_i(x,n;M)$ is much larger than $N(x,n;M)$ for the same parameters $x$, $n$, and $M$.

(b) $R_i(x,n;M)$ increases (in general) with argument $x$ but $N(x,n;M)$ increases first with argument $x$ then decreases to 1.

(iv) By using the exponential generating function (3.6), $R_i(x,n;M)$ can also be written in the following algebraic form

$$R_i(x,n;M) = x! \sum_{\Sigma jk} \prod_{j=0}^{M} (k!)^{x} (j!)^{k}.$$  

(3.12)
The summation is to be carried out over all sets of values \( (k_0, k_1, \ldots, k_M) \) for
\[
\begin{align*}
k_1 + 2k_2 + \ldots + Mk_M &= x; \\
k_0 + k_1 + \ldots + k_M &= n;
\end{align*}
\]
where \( x = 0, 1, \ldots, nM; \ j = 0, 1, \ldots, M; \ k_j = 0, 1, \ldots, n. \)

3.3 Recurrence Relations

**Theorem 1. (Recurrence Relations for \( r \) Numbers)**

The following recurrence relations hold for \( r \) numbers,
\[
r(x+1,n; M) = nr(x,n; M) - \sum_{i=1}^{n} \binom{x}{M_i} r(x-M_i, n-1; M_{(i)}),
\]

where
\[
M_{(i)} = \{ M_j | j \neq i \};
\]
\[
x = 0, 1, \ldots, \sum_{i=1}^{n} M_i - 1.
\]

**Proof.**

Let \( L = \sum_{i=1}^{n} M_i \).

Differentiating both sides of (3.5) with respect to \( \lambda \),
\[
\sum_{i=1}^{n} \prod_{j \neq i} e_{i}(M_j, \lambda) \frac{d}{d\lambda} e_{i}(M_i, \lambda) = \sum_{x=0}^{L} r(x, n; M) \frac{\lambda^{x-1}}{x!},
\]
\[
\sum_{i=1}^{n} \prod_{j \neq i} e_{i}(M_j, \lambda)e_{i}(M_i - 1, \lambda) = \sum_{x=0}^{L} r(x, n; M) \frac{\lambda^{x-1}}{(x-1)!},
\]
\[
\sum_{i=1}^{n} \prod_{j \neq i} \left( e_{i}(M_j, \lambda) - \frac{\lambda^{M_i}}{M_i!} \right) = \sum_{x=0}^{L-1} r(x+1, n; \mathcal{M}) \frac{\lambda^{x}}{x!},
\]

\[
\sum_{i=1}^{n} \left( \prod_{j=i+1}^{n} e_{i}(M_j, \lambda) - \lambda^{M_i}/M_i! \right) = \sum_{x=0}^{L-1} r(x+1, n; \mathcal{M}) \frac{\lambda^{x}}{x!},
\]

\[
\sum_{i=1}^{n} \prod_{j \neq i} e_{i}(M_j, \lambda) = \sum_{x=0}^{L-1} r(x+1, n; \mathcal{M}) \frac{\lambda^{x}}{x!}.
\]

Using (3.5) again, we have

\[
n \left( \sum_{x=0}^{L} r(x, n; \mathcal{M}) \frac{\lambda^{x}}{x!} \right) + \sum_{i=1}^{n} \left( \sum_{j=0}^{L-M_i} r(j, n-1; \mathcal{M}) \frac{\lambda^{j}}{j!} \left( -\frac{\lambda^{M_i}}{M_i!} \right) \right)
\]

\[
= \sum_{x=0}^{L-1} r(x+1, n; \mathcal{M}) \frac{\lambda^{x}}{x!}.
\]

Equating the coefficients of \( \lambda^{x} \) on both sides of the equation, we get (3.14).

Similarly we can obtain recurrence relations for \( R_i \) numbers.

**THEOREM 2. (RECURSIVE RELATIONS FOR \( R_i \) NUMBERS)**

\[ R_i(x+1, n; \mathcal{M}) = n R_{i}(x, n; \mathcal{M}) - n^{x} \binom{x}{M_i} R_{i}(x-M, n-1; \mathcal{M}) \]  \( (3.15) \)

where \( x = 0, 1, \ldots, nM-1. \)

**PROOF.**

Differentiating both sides of (3.6) with respect to \( \lambda \), we have
\[ \frac{d^n e_i(M, \lambda)}{d\lambda^n} = \sum_{x=1}^{\infty} R_i(x, n; M) \frac{\lambda^x}{x!}, \]

\[ \frac{n-1}{M!} \left( \sum_{j=1}^{\infty} \frac{\lambda^j}{(j-1)!} \right) = \sum_{x=1}^{\infty} R_i(x, n; M) \frac{\lambda^x}{(x-1)!}, \]

\[ \frac{n-1}{M!} \left( e_i(M, \lambda) - \frac{\lambda^M}{M!} \right) = \sum_{x=0}^{n-1} R_i(x+1, n; M) \frac{\lambda^x}{x!}, \]

\[ \frac{n}{M!} \left( n-1 \right) \frac{n-1}{M!} \left( -\frac{\lambda^M}{M!} \right) = \sum_{x=0}^{n-1} R_i(x+1, n; M) \frac{\lambda^x}{x!}. \]

Using (3.6) again, we have

\[ \sum_{x=0}^{nM} R_i(x, n; M) \frac{\lambda^x}{x!} + \sum_{j=0}^{(n-1)M} R_i(j, n-1; M) \frac{\lambda^j}{j!} \frac{\lambda^M}{M!} = \sum_{x=0}^{nM-1} R_i(x+1, n; M) \frac{\lambda^x}{x!}. \]

Equating the coefficients of \( \lambda^x \) on both sides of the equation, we get (3.15).

**Remark.**

By comparing (3.15) with (3.11), we find that \( R_i(x, n; M) \) numbers have totally different recurrence relations from those of \( N(x, n; M) \) numbers.

**Theorem 3.** For a given \( n \), \( 0 \leq M < \omega \) for \( i = 1, 2, \ldots, n \), the last two \( r \) numbers in terms of \( x \) are always equal, i.e.,

\[ r(\sum_{i=1}^{n} M_i, n, M) = r(\sum_{i=1}^{n-1} M_i, n, M) \]

(3.16)
We omit the proof here because it is similar to the one for Theorem 3 in Section 2.3 of Chapter 2.

From Theorems 1 to 3, we can also obtain the following Corollaries.

**COROLLARY 2.**

\[ R_i(nM, n; M) = R_i(nM-1, n; M). \]  
(3.17)

**COROLLARY 3.**

\[ R_i(nM-1, n; M) = \frac{n}{n-1} \binom{nM-1}{M} R_i((n-1)M-1, n-1; M). \]  
(3.18)

**PROOF.**

Using Theorem 2, we have

\[ R_i(nM, n; M) = nR_i(nM-1, n; M) - n \binom{nM-1}{M} R_i((nM-1) - M, n-1; M) \]

and using Corollary 2, we have

\[ R_i(nM-1, n; M) = nR_i(nM-1, n; M) - n \binom{nM-1}{M} R_i((nM-1) - M, n-1; M). \]

From this we get (3.18). \[\square\]

**THEOREM 4.**

For \( n > 1 \), the first \( M \) \( R_i \) numbers in terms of \( x \) are always equal, i.e.,

\[ R_i(x, n; M) = R_i(x, n; M+i) = n^x \]  
(3.19)

\( x = 1, 2, \ldots, M; \quad i = 1, 2, \ldots \)
PROOF.

Since for $n > 1$ and $x \leq M$, we have

$$\min(x, M) = x,$$

$$\max(x-(n-1)M) = M(2-n) \leq 0.$$  Hence

$$\max(0, x-(n-1)M) = 0.$$

Then

$$R_I(x, n; M) = \sum_{y_i = \max(0, x-(n-1)M)}^{\min(x, M)} \left(y_1^x, y_2^x, \ldots, y_n^x\right) = \sum_{y_i = 0}^{\min(x, M)} \left(y_1^x, y_2^x, \ldots, y_n^x\right) = n^x.$$

On the other hand, for $n > 1$ and $x \leq M$, we have

$$\min(x, M+1) = x,$$

$$\max(0, x-(n-1)(M+1)) = \max(0, x-(n-1)M) = 0.$$  Hence

Then

$$R_I(x, n; M+1) = \sum_{y_i = \max(0, x-(n-1)(M+1))}^{\min(x, M+1)} \left(y_1^x, y_2^x, \ldots, y_n^x\right) = \sum_{y_i = 0}^{\min(x, M+1)} \left(y_1^x, y_2^x, \ldots, y_n^x\right) = n^x.$$

Hence we have (3.19).

THEOREM 5. Let $(x)_a = x(x-1)\ldots(x-a+1)$. $R_I(x, n; M)$
satisfies the following recurrence relations:

\[(i) \quad R_I(x, n; M-1) = \sum_{s=0}^{B} (-1)^{s, s} \frac{\binom{n}{s}(x)}{(M!)^s} R_I(x-Ms, n-s; M) \quad (3.20)\]

where \( B = \left[ \frac{n}{M} \right] \leq n ; \quad x = 0, 1, \ldots, n(M-1). \)

\[(ii) \quad R_I(x, n; M) = \sum_{s=0}^{B} \frac{\binom{n}{s}(x)}{(M!)^s} R_I(x-Ms, n-s; M-1) \quad (3.21)\]

for \( x = 0, 1, \ldots, nM. \)

**PROOF.**

If we write the exponential generating function (3.6) of \( R_I(x, n; M-1) \) as

\[\left[ e_1(M-1, \lambda) \right]^n = \left[ e_1(M, \lambda) - \frac{\lambda^M}{M!} \right]^n,\]

and expand the right hand side by the binomial formula, we obtain

\[\left[ e_1(M-1, \lambda) \right]^n = \sum_{s=0}^{n} (-1)^{s, s} \binom{n}{s} \left[ e_1(M, \lambda) \right]^{n-s} \frac{\lambda^{Ms}}{(M!)^s}.\]

Using (3.6) again, we get

\[\sum_{x=0}^{n(M-1)} R_I(x, n; M-1) \frac{\lambda^x}{x!} = \sum_{s=0}^{n} (-1)^{s, s} \binom{n}{s} \sum_{x=0}^{(n-s)M} R_I(x, n-s; M) \frac{\lambda^x}{(M!)^s x!}\]

\[= \sum_{s=0}^{n} \sum_{x=Ms}^{nM} (-1)^{s, s} \binom{n}{s}(x) \frac{\lambda^x}{(M!)^s x!} R_I(x-Ms, n-s; M).\]
Equating the coefficients of $\lambda^x$ on both sides of the last relation we get (3.20).

Similarly we get (3.21).

3.4. LIMITING PROPERTIES

THEOREM 6. (Limiting properties)

(i) $\lim_{M \to \infty} R_1(x, n; M) = n^x$. \hspace{1cm} (3.22)

(ii) $\lim_{n \to \infty} [e_i(M,1)]^n \sum_{k=0}^{\mu} \frac{R_1(k, n; M)}{k!} = 1/2$ \hspace{1cm} (3.23)

where

$\mu = \max \left\{ \left[ \frac{1}{\mu} \right] \right\}$

$\mu = \text{largest integer less than or equal to } \mu.$

(iii) When $n$ is large,

$R_1(x, n; M) \approx \frac{x! [e_i(M,1)]^n}{\sqrt{2\pi} \sigma} \frac{1}{\int \frac{1}{\sigma} \left\{ \Phi(y_1) - \Phi(y_2) \right\}}$ \hspace{1cm} (3.24)

$\approx \frac{x^{x+1/2} - x}{x^2 e^{[e_i(M,1)]^n} \int \frac{1}{\sigma} \left\{ \Phi(y_1) - \Phi(y_2) \right\}}$ \hspace{1cm} (3.25)

where

$y_1 = \frac{(x+\frac{1}{2})-\mu}{\sigma}, \quad y_2 = \frac{(x-\frac{1}{2})-\mu}{\sigma}$;

$\Phi(\cdot)$ is the cumulative standard normal;
\( \sigma^2 = n \left( \frac{e^{(M-2,1)}}{e^{(M,1)}} + \frac{1}{e^{(M-1,1)}} - \left( \frac{e^{(M-1,1)}}{e^{(M,1)}} \right)^2 \right). \)

(iv) When \( n \) is large

\[
R_i([\mu], n; M) \cong \frac{[\mu]! (e^{(M,1)})^n}{\sqrt{2\pi n}}
\]

(3.26)

\[
\cong \frac{[\mu]^{-\frac{1}{2}} - (\mu)}{\sqrt{\frac{1}{2} e^{\left( -e^{(M,1)} \right)}}}
\]

(3.27)

**PROOF.**

(i) By (3.1),

\[
\lim_{M \to \infty} R_i(x, n; M) = \sum_{y_i = 0}^{x} \left( y_1, y_2, \ldots, y_n \right) = n^x.
\]

(ii) Let \( X \) be the sum of \( n \) i.i.d. right truncated Poisson random variables with parameter \( \lambda = 1 \) and right truncation points at \( M \). The distribution of \( X \) is called the \( R \) distribution with p.f. given in Chapter 6 (see also Huang and Fung, 1989a).

\[
p(x, n; M, 1) = [e^{(M,1)}]^{-n} R_i(x, n; M) / x!,
\]

where \( x = 0, 1, \ldots, nM. \)

The mean and variance are given by

\[
\mu = \mathbb{E}(X) = n \frac{e^{(M-1,1)}}{e^{(M,1)}}.
\]

\[
\sigma^2 = \text{Var}(X)
\]
\[
= n \left( \frac{e^{(M-2,1)}}{e^{(M,1)}} + \frac{e^{(M-1,1)}}{e^{(M,1)}} - \left( \frac{1}{e^{(M,1)}} \right)^2 \right).
\]

When \( n \) is large, we can use the Lindeberg-Levy Central Limit Theorem (Fisz, 1963, p.197) to obtain the following

\[
Z = \frac{X - E(X)}{\sqrt{\text{Var}(X)}} \sim N(0,1), \quad \text{when } n \to \infty,
\]

\[
\lim_{n \to \infty} P(X \leq \mu) = \lim_{n \to \infty} P(Z \leq \mu) = \Phi(0) = 1/2.
\]

In other words, \( \lim_{n \to \infty} \sum_{x=0}^{\{\mu\}} p(x; n; M, 1) = 1/2. \)

Hence we get (3.23).

(iii) Using the fact that \( X \) is approximately normally distributed when \( n \) is large, we find

\[
P(X = x) \approx \frac{1}{\sqrt{2\pi \text{Var}(X)}} \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} e^{-\frac{(t - E(X))^2}{2\text{Var}(X)}} dt
\]

Equating this with the p.f. of \( X \), we get (3.24).

Applying the Stirling formula \( n! \approx \sqrt{2\pi n} \frac{n^{1.5}}{e^n} \), we get (3.25).

(iv) When \( x = \{\mu\} \), we proved the following formula in Chapter 2 (see section 2.4, proof of Theorem 6)

\[
\int_{\{\mu\} - \frac{1}{2}}^{\{\mu\} + \frac{1}{2}} e^{-\frac{(t - \mu)^2}{2\sigma^2}} dt = 1, \quad \text{when } n \text{ is large.}
\]

Hence we get (3.26) and (3.27). \( \square \)
REMARK.

For convenience, one can use the following log-form of (3.25),

\[ \log R_i(x, n; M) = (x + \frac{1}{2}) \log x - x + n \log e_1(M, 1) - \log \sigma \]

\[ + \log \left( e(y_1) - e(y_2) \right). \]

(3.28)

3.5 MONOTONICITY PROPERTIES

PROPERTY 1.

(i) For each \( x = 0, 1, 2, \ldots, nM \)

\[ R_i(x, n; M) \geq R_i(x, n-1; M), \quad n = 1, 2, \ldots \]  

(3.29)

(ii) For each \( x = 1, 2, \ldots, nM, \)

\[ \frac{R_i(x, n-1; M)}{R_i(x, n; M)} > \frac{R_i(x, n-2; M)}{R_i(x, n-1; M)}, \quad n = 2, 3, \ldots. \]  

(3.30)

(iii) For each \( x = 5, 6, \ldots, nM, \)

\[ \left( \frac{n-1}{n} \right) \frac{R_i(x, n-1; M)}{R_i(x, n; M)} > \frac{R_i(x, n-2; M)}{R_i(x, n-1; M)}, \quad n = 2, 3, \ldots. \]  

(3.31)

For the proof of this property, refer to Dobson (1968).

PROPERTY 2. For each \( n=1, 2, \ldots, \)

\[ h(x, n; M) = \frac{x R_i(x-1, n; M)}{R_i(x, n; M)}, \quad x = 1, 2, \ldots, nM. \]  

(3.32)

Then \( h(x, n; M) \) is an increasing function of \( x. \)

\[ h(x, n; M) \geq \frac{1}{n} \]  

for \( 0 < x \leq nM, \) with equality for \( x=2, \) and \( M \geq 2. \) For \( M=1, \) there is no equality.
The proof of this property is similar to the one given by Ahuja (1971) for Stirling numbers.

**COROLLARY 9.** For each \( n=1,2,..., \)

\[
\frac{R_i(x,n;M)}{R_i(x+1,n;M)} \geq \frac{R_i(x-1,n;M)}{R_i(x,n;M)}, \quad x = 1,2,...,nM. \tag{3.33}
\]

Properties 1 and 2 can also be verified by checking the tables of \( R \) numbers in Appendix A.2., Tables III and IV.

### 3.6 COMPUTATIONAL METHODS FOR \( R \) NUMBERS

We can use Definition 1,2,3,4 or Theorems 1 and 2 (recurrence relations of \( R \) numbers) to calculate the values of \( R \) numbers with a computer subroutine. The numerical results in Appendix A.2, Tables III and IV were computed on an IBM4381 machine with double precision.
CHAPTER 4

THE GENERALIZED STIRLING NUMBERS

4.1 INTRODUCTION

In Chapter 2, we introduced the \( D(x, n; \mathcal{L}, \Lambda_o) \) numbers, where \( \mathcal{L} \) is a set of truncated points \( (N_i, M_i) \), \( i = 1, 2, \ldots, n \). If there is no truncation on the right, that means all \( M_i = \infty \), for \( i = 1, 2, \ldots, n \). Then the \( D(x, n; \mathcal{L}, \Lambda_o) \) numbers become the more generalized Stirling numbers of the second kind \( GSC(x, n; \mathcal{L}, \Lambda_o) \). Since the more generalized Stirling numbers of the second kind are defined and treated as a special case of \( D \) numbers, all of the properties of \( D \) numbers are also true for the more generalized Stirling numbers of the second kind. Some proofs will be omitted in this chapter.

The more generalized Stirling numbers of the second kind \( GSC(x, n; \mathcal{M}) \) are also extensions of the Stirling numbers of the second kind \( S(x, n) \) (see Section 1.2., (1.8)) and the generalized Stirling numbers of the second kind (4.3) which were studied by Tate and Goen (1958), Ahuja (1971), Charalambides (1974), and Cacoullos and Papageorgiou (1984).
4.2. MORE GENERALIZED STIRLING NUMBERS OF THE SECOND KIND

**DEFINITION 1.** The more generalized Stirling numbers of the second kind are defined as

\[ GS(x, n; \mathcal{N}, \Lambda_0) = \text{the number of ways of putting } x \text{ different objects into } n \text{ distinguishable urns having } \lambda_1, \ldots, \lambda_n \text{ distinguishable cells with each urn containing at least } N_1, N_2, \ldots, N_n \text{ objects, respectively} \]

\[ = \sum_{\gamma \in \mathcal{Y}} \left( \gamma_1, \gamma_2, \ldots, \gamma_n \right)^n \prod_{i=1}^{n} \lambda_i^{y_i} \]  

(4.1)

where \( \mathcal{Y} = \{ \gamma | \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n), \ y_i \text{ is an integer, } \sum_{i=1}^{n} y_i = x, \ i=1, \ldots, n \}, \]

\[ N_i \leq y_i \leq x - \sum_{i=1}^{n} N_j, \ j \neq i \}\)

\[ x = \sum_{i=1}^{n} N_i, \sum_{i=1}^{n} N_i + 1, \ldots; \ n \text{ is a positive integer,} \]

\( \mathcal{N} = \{ N_i | N_i \text{ is a nonnegative integer, } i = 1, 2, \ldots, n \}, \)

\( \Lambda_0 = \{ \lambda_i | \lambda_i \text{ is a positive integer, } i = 1, 2, \ldots, n \} \).

**REMARK.**

The summation is to be carried out over all sets of values of \( (\gamma_1, \gamma_2, \ldots, \gamma_n) \) for which \( \gamma_i \) is between \( N_i \) and \( x - \sum_{j \neq i} N_j \) for \( i = 1, 2, \ldots, n \), and which satisfy \( y_1^+ + y_2^+ + \cdots + y_n^+ = x \).

If all \( N_i \equiv N \) in \( \mathcal{N} \), we write \( \mathcal{N} \equiv N \).

If all \( \lambda_i \equiv \lambda \), we write \( \Lambda_0 \equiv \lambda \).
DEFINITION 2. The $g$-numbers are defined by

$$g(x,n;\mathcal{N}) = \text{the number of ways of putting } x \text{ different objects into } n \text{ distinguishable urns with each urn containing at least } N_1, N_2, \ldots, N_n \text{ objects, respectively}$$

$$= GS(x,n;\mathcal{N},\lambda) \lambda^x$$

$$= \sum_{y \in \mathcal{J}} \binom{x}{y_1, y_2, \ldots, y_n} (4.2)$$

where $\lambda$ and $\mathcal{J}$ are defined in (4.1);

$$x = \sum_{i=1}^{n} N_i, \sum_{i=1}^{n} N_i+1, \ldots; n \text{ is a positive integer.}$$

DEFINITION 3. The generalized Stirling numbers of the second kind are defined as

$$S(x,n;N) = \text{the number of ways of putting } x \text{ different objects into } n \text{ distinguishable urns with each urn containing at least } N \text{ objects}$$

$$= \sum_{y \in \mathcal{J}} \binom{x}{y_1, y_2, \ldots, y_n} (4.3)$$

where $\mathcal{J} = \{ y \mid y = (y_1, y_2, \ldots, y_n), \sum_{i=1}^{n} y_i = x, N \leq y_i \leq x-(n-1)N, i=1, \ldots, n \}$,

$$x = nN, nN+1, \ldots; n \text{ is a positive integer; } \quad N \text{ is a nonnegative integer.}$$

DEFINITION 4. (Tate and Goen, 1958) A generalized Stirling number of the second kind is given by
\[ S^*(x, n; N) = \frac{(-1)^n x!}{n!} \sum_{k_0! k_1! \cdots k_n!} \frac{(-1)^{N-1} k_0^{N-1} (x - \sum_{j=0}^{N-1} j k_{j+1})}{(x-\sum_{j=0}^{N-1} j k_{j+1})!} \prod_{j=0}^{N-1} (j!)^{k-j}, \]

where

\[ k = 0, 1, \ldots, n; \quad i = 0, 1, 2, \ldots, N; \quad x = nN, nN+1, \ldots; \]

the summation is taken over all \((k_0, k_1, \ldots, k_N)\) such that

\[ k_0 + k_1 + \cdots + k_N = n. \]

\begin{remark}
We can prove that

\[ S(x, n; N! n!) = S^*(x, n; N), \]

for \(x = nN, nN+1, \ldots\); \(n\) is a positive integer; and \(N\) is an integer.

Note that

\[ G(x, n; N, \lambda) / \lambda^x = g(x, n; N) = S(x, n; N) = n! S^*(x, n; N). \]

\begin{remark}
For the case when \(\lambda_i\) is not necessarily an integer in (4.1), we define the \(G\)-numbers.

\begin{definition}
THE \(G\)-numbers are defined by

\[ G(x, n; \lambda) = \sum_{\chi \in \mathcal{Y}} \chi \left( y_1^x y_2^y \cdots y_n^{y_n} \right) \prod_{i=1}^{n} \lambda_i \]  \quad (4.5)

where \(\chi, \mathcal{Y}, \Pi\) are defined in (4.1)

\[ \Lambda = \{ \lambda_i \mid \lambda_i \text{ is a positive real number, } i = 1, 2, \ldots, n \}. \]
\end{definition}
\[ x = \sum_{i=1}^{n} \sum_{j=1}^{N_i} N_i+j, \ldots; \text{n is a positive integer.} \]

**Remark.**

The following relationships of G numbers with other numbers are obtained.

(i) \( G(x, n; \mathcal{M}, \Lambda) \) is \( GD(x, n; \mathcal{M}, \Lambda) \) when \( N > 0 \), and \( M = \infty \) (see Chapter 2, Section 2.2). Also, in this case \( g(x, n; \mathcal{M}) \) is \( d(x, n; \mathcal{M}) \) (see Chapter 2, Section 2.2), and \( g(x, n; \mathcal{M}/n! \) have been named by Cacoullos and Papageorgiou (1984) as the multiparameter Stirling numbers of the second kind.

(ii) \( S(x, n; \mathcal{M}) \) is \( D_I(x, n; (N, \omega)) \) when \( N \neq 0 \), \( M = \infty \).

(iii) \( g(x, n; 1) \) is the Stirling number of the second kind (Jordan, 1965).

**Lemma 1.** The exponential generating function of \( g(x, n; \mathcal{M}) \) is

\[
\prod_{i=1}^{n} e_{2}(N_i, \lambda) = \sum_{x=L}^{\infty} g(x, n; \mathcal{M}) \frac{\lambda^x}{x!}, \quad (4.6)
\]

where \( L = \sum_{i=1}^{n} N_i \),

\[
e_{2}(N_i, \lambda) = \sum_{k=N_i}^{\infty} \frac{\lambda^k}{k!}
\]

is an incomplete exponential function (See Chapter 1, Section 1.4 and Appendix A.4, Table VII).

The proof is similar to one given in Chapter 2, Section 2.2, Lemma 1.

**Corollary i.** The exponential generating function of \( S(x, n; \mathcal{N}) \) is
\[(e_2\langle N, \lambda \rangle)^n = \sum_{x=nN}^{\infty} S(x, n; \mathcal{M}) \frac{\lambda^x}{x!} \]  

(4.7)

It is easy to show that \(g(x, n; \mathcal{M})\) in (4.6) and \(S(x, n; \mathcal{M})\) in (4.7) are the same as given in (4.2) and (4.4), respectively.

4.3 Recurrence Relations

**Theorem 1.** (Recurrence Relations for \(g\) Numbers)

\[g(x+1, n; \mathcal{M}) = ng(x, n; \mathcal{M}) + \sum_{i=1}^{n} \binom{x}{n-1} g(x-N_i+1, n-1; \mathcal{M}_i)\]  

(4.8)

where

\[\mathcal{M}_i = \{N_j | j \neq i\};\]

\[x = \sum_{i=1}^{n} N_i, \quad \sum_{i=1}^{n} N_i + 1, \ldots.\]

**Proof.**

Differentiating both sides of (4.6) with respect to \(\lambda\),

\[\sum_{i=1, j \neq 1}^{n} \prod_{j \neq 1}^{n} e_2\langle N_j, \lambda \rangle \frac{d}{d\lambda} e_2\langle N_i, \lambda \rangle = \sum_{x=L}^{\infty} g(x, n; \mathcal{M}) \frac{\lambda^{x-1}}{x!},\]

\[\sum_{i=1, j \neq 1}^{n} \prod_{j \neq 1}^{n} e_2\langle N_j, \lambda \rangle e_2\langle N_i-1, \lambda \rangle = \sum_{x=L}^{\infty} g(x, n; \mathcal{M}) \frac{\lambda^{x-1}}{(x-1)!},\]

\[\sum_{i=1, j \neq 1}^{n} \frac{\prod_{j \neq 1}^{n} e_2\langle N_j, \lambda \rangle + \frac{\lambda^x}{(N_{i-1})!}}{(N_{i-1})!} = \sum_{x=L-1}^{\infty} g(x+1, n; \mathcal{M}) \frac{\lambda^x}{x!},\]

\[\sum_{i=1}^{n} \left( \prod_{j=1}^{n} e_2\langle N_j, \lambda \rangle + \prod_{j \neq i}^{n} e_2\langle N_j, \lambda \rangle \frac{\lambda^x}{(N_{i-1})!} \right) = \sum_{x=L-1}^{\infty} g(x+1, n; \mathcal{M}) \frac{\lambda^x}{x!},\]
\[ n \prod_{j=1}^{n} e_{2}(N, \lambda_{j}) \sum_{k=1}^{n} e_{2}(N, \lambda_{k}) \frac{\lambda}{(N-1)!} = \sum_{x=L-1}^{\infty} g(x+1, n, ND \frac{\lambda}{x!}}.

Using (4.7) again, we get

\[ n \left( \sum_{x=L}^{\infty} g(x, n; ND \frac{\lambda}{x!}) + \sum_{i=1}^{n} \left( \sum_{j=L-N}^{\infty} g(j, n-1; ND \frac{\lambda}{j!}) \frac{\lambda}{(N-1)!} \right) \right) \]

\[ = \sum_{x=L-1}^{\infty} g(x+1, n; ND \frac{\lambda}{x!}) \]

Equating the coefficients of \( \lambda^x \) on both sides of the equation, we get (4.8). □

When \( ND = N \), the recurrence relation for \( S(x, n; N) \) can be obtained from Theorem 1.

**COROLLARY 2.** (RECURRENCe RELATIONS FOR \( S(x, n; N) \))

\[ S(x+1, n; N) = nS(x, n; ND + n_{N-1}) S(x-N+1, n-1; ND), \quad (4.9) \]

where \( x = nN, nN+1, \ldots \).

The result in Corollary 2 is in agreement with the result of Jordan (1965) and Charalambides (1974).

The generalized Stirling numbers of the second kind \( S(x, n; ND \) also have the following recurrence relations which follow directly from the properties of \( D \) numbers (2.16), (2.17).

**THEOREM 2.** Let \( (x)^a = x(x-1) \ldots (x-a+1) \). \( S(x, n; N) \) satisfies the following recurrence relations:
\[(i) \quad S(x, n; N+1) = \sum_{s=0}^{n} (-1)^s \binom{n}{s}^{N_s} \frac{(x)^{N_s}}{(N!)^s} S(x-Ns, n-s; N) \quad (4.10)\]

where \( x = n(N+1), \ldots, nM. \)

\[(ii) \quad S(x, n; N) = \sum_{s=0}^{n} \frac{(\binom{n}{s})^{N_s}}{(N!)^s} S(x-Ns, n-s; N+1) \quad (4.11)\]

for \( x = nN, nN+1, \ldots. \)

**Proof.**

If we write the exponential generating function (4.7) of \( S(x, n; N) \) as

\[
[e_2(N+1, \lambda)]^n = \sum_{s=0}^{n} (-1)^s \binom{n}{s} (e_2(N, \lambda))^n \frac{\lambda^{N_s}}{(N!)^s},
\]

and use (4.7) again, we get

\[
\sum_{x=n(N+1)}^{\infty} S(x, n; N+1) \frac{\lambda^x}{x!}
\]

\[
= \sum_{s=0}^{n} (-1)^s \binom{n}{s} \sum_{x=n-N}^{\infty} S(x, n-s; N) \frac{\lambda^{N_s+x}}{(N!)^s x!}
\]

\[
= \sum_{s=0}^{n} \sum_{x=nN}^{\infty} (-1)^s \binom{n}{s} (x)^{N_s} S(x-Ns, n-s; N) \frac{\lambda^x}{(N!)^s x!}
\]

Equating the coefficients of \( \lambda^x \) on both sides of the last relation we get (4.10).

Similarly we can get (4.11).

**Remark.** Equations (4.10) and (4.11) are in agreement with results of Charalambides (1974) and Cacoullos and Papageorgiou (1984).
4.4 LIMITING PROPERTIES

THEOREM 3. (Limiting properties)

(i) \( \lim_{n \to 0} S(x, n; N) = n^x. \) \hspace{1cm} (4.12)

(ii) \( \lim_{n \to \infty} \left[ \frac{e_2(N, 1)}{n} \right]^{-n} \sum_{k=nN}^{\mu} \frac{S(k, n; N)}{k!} = 1/2. \) \hspace{1cm} (4.13)

where

\[ \mu = \left\lfloor \frac{n}{e_2(N, 1)} \right\rfloor, \]

\[ [\mu] = \text{largest integer less than or equal to } \mu. \]

(iii) When \( n \) is large,

\[ S(x, n; N) \approx \frac{x! \left[ e_{2}(N, 1) \right]^{n}}{\sqrt{2\pi \sigma}} \left\{ \bar{\Phi}(y_1) - \bar{\Phi}(y_2) \right\} \]

\[ \approx \frac{n^{-x/2} e^{-n} \left[ e_{2}(N, 1) \right]^{n}}{\sigma} \left\{ \bar{\Phi}(y_1) - \bar{\Phi}(y_2) \right\} \]

where

\[ y_1 = \frac{(x + \frac{1}{2}) - \mu}{\sigma}, \quad y_2 = \frac{(x - \frac{1}{2}) - \mu}{\sigma}; \]

\( \bar{\Phi}(\cdot) \) is the standard cumulative normal;

\[ \sigma^2 = n \left\{ \frac{e_{2}(N-2, 1)}{e_{2}(N, 1)} + \frac{e_{2}(N-1, 1)}{e_{2}(N, 1)} - \left( \frac{e_{2}(N-1, 1)}{e_{2}(N, 1)} \right)^2 \right\}. \]

(iv) When \( n \) is large.
\[
S(\mu, n; N) \equiv \frac{[\mu]! \left( e_2(N, 1) \right)^n}{\sqrt{2\pi\sigma}}.
\]

\[
= \frac{[\mu]^{\frac{1}{2}} e^{-\frac{\mu}{2}} \left( e_2(N, 1) \right)^n}{\sigma}.
\]

**Proof.**

(i) By (4.1),

\[
\lim_{N \to \infty} S(x, n; N) = \sum_{y_1=0}^{x} \left( \begin{array}{c} x \\ y_1 \end{array} \right) y_1^{x} = n^x.
\]

(ii) Let \( X \) be the sum of \( n \) i.i.d. left truncated Poisson random variables with parameter \( \lambda = 1 \) and left truncation points at \( N \). The distribution of \( X \) is called the generalized Stirling distribution of the second kind with p.f. (see Chapter 7, also Huang and Fung, 1988).

\[ p(x, n; N, 1) = \left( e_2(n, 1) \right)^{-n} S(x, n; N) / x! \]

where \( x = nN, nN+1, \ldots \)

The mean and variance are given by

\[
\mu = E[X] = n \frac{e_2(N-1, 1)}{e_2(N, 1)}
\]

\[
\sigma^2 = \text{Var}[X]
\]

\[
= n \left( \frac{e_2(N-2, 1)}{e_2(N, 1)} + \frac{e_2(N-1, 1)}{e_2(N, 1)} - \left( \frac{e_2(N-1, 1)}{e_2(N, 1)} \right)^2 \right).
\]

When \( n \) is large, we can use the Lindeberg-Levy Central Limit Theorem (Fisz, 1963, p.197) to obtain the following
\[ Z = \frac{X - E(X)}{\sqrt{\text{Var}(X)}} \sim N(0,1) \text{, when } n \to \infty, \]

\[ \lim_{n \to \infty} P(X \leq \mu) = \lim_{n \to \infty} P(Z \leq 0) = \Phi(0) = 1/2. \]

In other words, \( \lim_{n \to \infty} \sum_{x=nN}^{\lfloor \mu \rfloor} p(x,n;N,1) = 1/2. \)

Hence we get (4.13).

(iii) Using the fact that \( X \) is approximately normally distributed when \( n \) is large, we find

\[ P(X = x) \approx \frac{1}{\sqrt{2\pi \text{Var}(X)}} \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} e^{-\frac{(t - E(X))^2}{2\text{Var}(X)}} \, dt \quad (4.18) \]

Equating this with the p.f. of \( X \), we get (4.14).

By applying the Stirling formula \( n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \), we get (4.15).

(iv) When \( x = \lfloor \mu \rfloor \), we proved in Chapter 2, Section 2.4 that

\[ \int_{(\mu)^{1/2}}^{(\mu)^{-1/2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \, dt = 1, \text{ when } n \text{ is large.} \]

Hence, we get (4.16) and (4.17).

**Remark.** For convenience, one can use the following log-form of (4.15),

\[ \log S(x,n;N) = (x+\frac{1}{2}) \log x - x + n \log e \frac{1}{2} (N,1) - \]

\[ - \log \sigma + \log(\Phi(y_1) - \Phi(y_2)). \quad (4.19) \]
4.5 MONOTONICITY PROPERTIES

PROPERTY 1. For fixed \( x \),

(i) when \( N > 0 \), \( S(x, n; N) \) has a single maximum. For any \( x > nN + 4 \), there exists a unique \( k_x \) such that

\[ S(x, n-1; N) < S(x, n; N), \quad n = 1, 2, \ldots k_x; \]

\[ S(x, n; N) \geq S(x, n+1; N), \quad n = k_x, \ldots, x. \]

Also \( k_x = k_{x-1} + 1 \).

(ii) When \( N = 0 \)

\[ S(x, n; 0) \geq S(x, n-1; 0), \text{ for any } n. \]

(iii) When \( x > nN, \)

\[
\frac{S(x, n-1; N)}{S(x, n; N)} \geq \frac{S(x, n-2; N)}{S(x, n-1; N)}. \quad (4.22)
\]

When \( x > nN + 4 \)

\[
\left(\frac{n}{n-1}\right) \frac{S(x, n-1; N)}{S(x, n; N)} \geq \frac{S(x, n-2; N)}{S(x, n-1; N)}. \quad (4.23)
\]

For proof of this property, refer to Dobson (1968).

PROPERTY 2. For \( n \geq 1 \),

\[ h(x, n; N) = \frac{x S(x-1, n; N)}{S(x, n; N)} \quad (4.24) \]

(i) \( h(x, n; N) \) is an increasing function of \( x \).

(ii) If \( N > 0 \), \( h(x, n; N) \geq \frac{n+1}{n} \) for \( x > nN \), with equality for \( x = nN + 1 \).

If \( N = 0 \), \( h(x, n; N) \geq \frac{2}{n} \) for \( x \geq 0 \), with equality for \( x = 2 \).
For proof of this property, refer to Ahuja (1971).

**COROLLARY 3.** For \( n \geq 1 \),

\[
\frac{S(x,n)}{S(x+1,n)} \geq \frac{S(x-1,n)}{S(x,n)}.
\] (4.25)

Properties 1 and 2 can also be verified by checking the tables of \( G \)-numbers in Appendix A.3, Tables V and VI.

**4.6 COMPUTATIONAL METHODS FOR MG-STIRLING NUMBERS OF THE SECOND KIND**

We can use Definitions 1, 2, 3, 4 or Theorems 2 and 3 (recurrence relations of \( g \) numbers and \( S \) numbers) to calculate the values of the more generalized Stirling numbers of the second kind on the computer.

Tables given in Appendix A.3 (Tables V and VI).
PART II

THE D DISTRIBUTION
CHAPTER 5

THE D DISTRIBUTION

5.1 INTRODUCTION

In Chapter 1, we discussed the motivation for studying the D distribution. In this chapter, we introduce the D distribution. Its p.f. can be expressed in terms of a D number and an incomplete exponential function which are defined in Chapter 2 and Chapter 1. The Poisson distribution, the truncated (left, right, doubly) Poisson distribution, the Stirling distribution of the second kind and its generalized case (GSDK), ITPD, MGSDK, and the R distribution are all special cases of this D distribution. Also, we will prove that the D distribution is the distribution of the sum of n independent but not identical doubly truncated Poisson variables.

We also derive some properties of the D distribution, such as recursion and decomposition relations, modal properties, etc. They are based on the properties of the D numbers. They cannot be derived from the general properties of the truncated power series distribution.

In this chapter a MVU estimate of the p.f. of the D distribution is given. It can easily be calculated from tables of the incomplete exponential function (Appendix A.4,
Table VII) and the D numbers (Appendix A.1, Table I, II). Also, we provide the computational method of the p.f. of the D distribution without using D numbers. Tables of the p.f. of the D distribution are included.

Finally, some examples are included at the end to illustrate the use of the D distribution.

5.2 D DISTRIBUTION

A random variable \( X \) is said to have a doubly truncated Poisson distribution (DTPD) with parameters \( \lambda, N, M \), if

\[
P(X=k) = \left( \sum_{j=N}^{M} \frac{\lambda^j}{j!} \right) \frac{\lambda^k}{k!}, \quad \lambda > 0, \quad 0 \leq N < M, \tag{5.1}
\]

where \( k = N, N+1, \ldots, M \).

**Definition**: A random variable \( X \) is said to have a D distribution \( (DDC(n; \mathcal{E}, \Lambda)) \), if its p.f. is

\[
p(x, n; \mathcal{E}, \Lambda) = P(X=x) = \prod_{i=1}^{n} e(N_i, M_i; \lambda_i)^{-1} \cdot \Gamma(x, n; \mathcal{E}, \Lambda) / x! \tag{5.2}
\]

where \( x = \sum_{i=1}^{n} N_i, \sum_{i=1}^{n} N_{i+1}, \ldots, \sum_{i=1}^{n} M_i \).

\( \Gamma(x, n; \mathcal{E}, \Lambda) \) is the D number (see Chapter 2, Section 2.2) and

\( e(N_i, M_i; \lambda_i) \) is the incomplete exponential function (see Chapter 1, Section 1.4).

Next, we verify that (5.2) is a probability function of the discrete-type random variable \( X \) (Fisz, 1963, p. 34).
PROOF.

Let \( L_1 = \sum_{i=1}^{n} N_i \), \( L_2 = \sum_{i=1}^{n} M_i \).

Let \( x_j, \quad j = 1, 2, \ldots, (L_2 - L_1 + 1) \), be an arbitrary jump point of the random variable \( X \), \( x_j \) is an integer, \( L_1 \leq x_j \leq L_2 \).

(i) Since \( e(N, M; \lambda_i) \geq 0 \) and \( GD(k, n; \mathcal{L}, \Lambda) \geq 0 \) in (5.2), then
\[
P(X = x_j) \geq 0
\]
for each \( x_j, \quad j = 1, 2, \ldots, (L_2 - L_1 + 1) \).

\[
\sum_{j=1}^{L_2 - L_1 + 1} P(X = x_j) = \sum_{k=L_1}^{L_2} P(X = k)
\]

\[
= \sum_{k=L_1}^{L_2} \prod_{i=1}^{n} e(N_i, M_i; \lambda_i)^{-1} GD(k, n; \mathcal{L}, \Lambda) \cdot k!
\]

\[
= \prod_{i=1}^{n} e(N_i, M_i; \lambda_i)^{-1} \sum_{k=L_1}^{L_2} GD(k, n; \mathcal{L}, \Lambda) \cdot k!
\]

Note that

\[
\prod_{i=1}^{n} e(N_i, M_i; \lambda_i) = \prod_{i=1}^{n} \left( \frac{\lambda_i}{N_i!} + \frac{\lambda_i}{(N_i+1)!} + \cdots + \frac{\lambda_i}{M_i!} \right)
\]

\[
= \sum_{k=L_1}^{L_2} \sum_{y \in Y} \prod_{i=1}^{n} \frac{\lambda_i}{(y_i)!}
\]
where $\mathcal{Y} = \{ y \mid y = (y_1, y_2, \ldots, y_n), y_i \text{ is an integer}, \sum_{i=1}^n y_i = k, \max(N, k - \sum_{i \neq j} M_j) \leq y \leq \min(M_i, k - \sum_{i \neq j} N_j) \}$.  

\[
L^2 = \sum_{k=L_1}^{L_2} \sum_{y \in \mathcal{Y}} \frac{1}{y_1!y_2!\ldots y_n!} \prod_{i=1}^n \lambda_i^{y_i}
\]

\[
L^2 = \sum_{k=L_1}^{L_2} \sum_{y \in \mathcal{Y}} \left( \prod_{i=1}^k y_i \right) ^n \prod_{i=1}^n \lambda_i ^{y_i} \cdot (k!)
\]

\[
L^2 = \sum_{k=L_1}^{L_2} \text{GD}(k, n; \mathcal{L}, \Lambda) / (k!)
\]

Hence \[
\sum_{j=1}^{L_2-L_1+1} P(X = x) = 1.
\]

Therefore (5.2) is a probability function of discrete-type random variable $X$. 

**Lemma 1.** If $Y_i \sim \text{DTPDC}(N_i, M_i; \lambda_i)$ are independent, $i = 1, \ldots, n$, then the probability generating function of $X = \sum_{i=1}^n Y_i$ is

\[
\varphi_{X}(s) = \prod_{i=1}^n \frac{e(N_i, M_i; s\lambda_i)}{e(N_i, M_i; \lambda_i)}, \quad 0 < s < 1.
\]

**Proof.**

For $0 < s < 1$, 

\[
\varphi_{X}(s) = \prod_{i=1}^n \frac{e(N_i, M_i; s\lambda_i)}{e(N_i, M_i; \lambda_i)}, \quad 0 < s < 1.
\]
\[ \psi_x(s) = \sum_{x} P(X=x) s^x = E(s^X) = \prod_{i=1}^{n} E(s^{Y_i}) \]
\[ = \prod_{i=1}^{n} \sum_{y_i = N_i}^{M_i} P(Y_i = y_i) s^{y_i} \]
\[ = \prod_{i=1}^{n} e(N_i, M_i; \lambda_i)^{-1} \left( \sum_{y_i = N_i}^{M_i} \frac{\lambda_i^{y_i}}{y_i!} s^{y_i} \right) \]
\[ = \prod_{i=1}^{n} e(N_i, M_i; \lambda_i)^{-1} \left( \sum_{y_i = N_i}^{M_i} \frac{(s\lambda_i)^{y_i}}{y_i!} \right) \]
\[ = \prod_{i=1}^{n} \frac{e(N_i, M_i; s\lambda_i)}{e(N_i, M_i; \lambda_i)}. \]

**THEOREM 1.** If \( Y_i \sim \text{DTPD}(N_i, M_i; \lambda_i) \) are independent, \( i = 1, \ldots, n \), then \( X = \sum_{i=1}^{n} Y_i \sim \text{DD}(N, M; \Lambda) \) with p.f. (5.2).

**PROOF.**

From the probability generating function of \( X \) given in Lemma 1, we can get \( P(X=x) \) by collecting the coefficients of \( s^x \) such that \( \sum_{i=1}^{n} y_i = x \).

**COROLLARY 1.** If \( (N_i, M_i) \equiv (N, M), \lambda_i \equiv \lambda \) in Theorem 1 (i.e., \( Y_i \)'s are i.i.d.), then the p.f. of \( X \) reduces to

\[ p(x, n; (N, M; \lambda)) = e(N, M; \lambda)^{-n} D_1(x, n; (N, M)\lambda^x/x!) \]

where \( x = nN, nN+1, \ldots, nM \).
COROLLARY 2. Special cases:

(i) $\text{DD}(1; (0, \omega), \lambda)$ is the ordinary Poisson distribution.
    $\text{DD}(1; (N, M), \lambda)$ is the doubly (or left, right) truncated Poisson distribution.

(ii) $\text{DD}(n; (1, \omega), \lambda)$ is the Stirling distribution of the second kind.

(iii) $\text{DD}(n; (N, \omega), \lambda)$ is GSDSK($n; N, \lambda$). When all $N_i > 0$, $M_i \equiv \omega$,
    $\text{DD}(n; \mathbb{R}, \lambda)$ is MGSDSK($n; M, \lambda$).

(iv) $\text{DD}(n; (0, M), \lambda)$ is R distribution.

(v) $\text{DD}(2; (0, \omega), (N, M), \lambda)$ is ITPD.

5.3 RECURRENCE RELATIONS OF P.F.

In this section, we derive the recurrence relations of the p.f. of the D distribution by using the recurrence properties of the D numbers. The properties of the D-numbers are all given in Chapter 2 and the readers may refer to them.

But interestingly, the results below for the D distribution do not depend on the D numbers. Since the D numbers become very large with increasing arguments, the results of this section provide a more convenient alternative in the calculations of the p.f. of the D distribution.

Let $d(x, n; \mathcal{L}) = G\text{DD}(x, n; \mathcal{L}, \lambda) / \lambda^x$. Using the recurrence relation for d-numbers, we obtain

THEOREM 2. (RECURRENCE RELATION FOR THE P.F. OF $\text{DD}(n; \mathcal{L}, \lambda)$)

If a random variable $X \sim \text{DD}(n; \mathcal{L}, \lambda)$, then the p.f.
p(x, n; \Theta, \lambda) in (5.2) satisfies the recurrence relation

\[ p(x+1, n; \Theta, \lambda) = \frac{n\lambda}{(x+1)} p(x, n; \Theta, \lambda) + \sum_{i=1}^{n} \frac{\lambda}{(x+1)(N_i-1)! e(n_i, M_i; \lambda)} p(x-N_i+1, n-1; \Theta \setminus \{i\}, \lambda) \]

\[ - \frac{\lambda}{(x+1)(M_i-1)! e(n_i, M_i; \lambda)} p(x-M_i, n-1; \Theta \setminus \{i\}, \lambda) \] \]

(5.5)

where

\[ \Theta \setminus \{i\} = \{ (N_j, M_j) \mid j \neq i \}; \]

\[ x = \sum_{i=1}^{n} N_i, \sum_{i=1}^{n} N_i + 1, \ldots, \sum_{i=1}^{n} M_i - 1. \]

**PROOF.**

Using (2.9),

\[ d(x+1, n; \Theta) = nd(x, n; \Theta) \]

\[ + \sum_{i=1}^{n} \left[ \binom{x}{N_i-1} d(x-N_i+1, n-1; \Theta \setminus \{i\}) - \binom{x}{M_i} d(x-M_i, n-1; \Theta \setminus \{i\}) \right], \]

where

\[ \Theta \setminus \{i\} = \{ (N_j, M_j) \mid j \neq i \}; \]

\[ x = \sum_{i=1}^{n} N_i, \sum_{i=1}^{n} N_i + 1, \ldots, \sum_{i=1}^{n} M_i - 1. \]

So

\[ p(x-M_i, n-1; \Theta \setminus \{i\}, \lambda) = \prod_{j \neq i} e(n_j, M_j; \lambda)^{-1} d(x-M_i, n-1; \Theta \setminus \{i\}) \frac{\lambda^{x-M_i}}{(x-M_i)!}, \]
\[ p(x-N_i+1, n-1; \mathcal{G}_{(i)}, \lambda) = \prod_{j \neq i}^{n} e(N_j, M_j; \lambda) \frac{\lambda^x}{(x-N_i-1)!}, \]

and

\[ p(x, n; \mathcal{G}, \lambda) = \prod_{i=1}^{n} e(N_i, M_i; \lambda) \frac{\lambda^x}{x!}. \]

Hence,

\[ p(x+1, n; \mathcal{G}, \lambda) = \prod_{i=1}^{n} e(N_i, M_i; \lambda) \frac{\lambda^x}{(x+1)!} \]

\[ + \sum_{i=1}^{n} \left[ \binom{x}{N_i-1} d(x-N_i+1, n-1; \mathcal{G}_{(i)}) - \binom{x}{M_i} d(x-M_i, n-1; \mathcal{G}_{(i)}) \right] \frac{\lambda^{x+1}}{(x+1)!} \]

\[ = \prod_{i=1}^{n} e(N_i, M_i; \lambda) \frac{\lambda^x}{(x+1)!} \]

\[ + \sum_{i=1}^{n} \left[ \binom{x}{N_i-1} \prod_{j=1}^{n} e(N_j, M_j; \lambda) \frac{\lambda^x}{(x+1)!} \right] \]

\[ - \binom{x}{M_i} \prod_{j=1}^{n} e(N_j, M_j; \lambda) \frac{\lambda^x}{(x+1)!} \]

\[ = \frac{n \lambda}{(x+1)!} p(x, n; \mathcal{G}, \lambda) \]

\[ + \sum_{i=1}^{n} \left[ \frac{x!}{(N_i-1)! (x-N_i+1)!} \frac{p(x-N_i+1, n-1; \mathcal{G}_{(i)}, \lambda)}{e(N_i, M_i; \lambda)} \frac{(x-N_i+1)!}{\lambda^{x-N_i+1}} \right] \]

\[ - \frac{x!}{M_i! (x-M_i)!} \frac{p(x-M_i, n-1; \mathcal{G}_{(i)}, \lambda)}{e(N_i, M_i; \lambda)} \frac{(x-M_i)!}{\lambda^{x-M_i}} \frac{\lambda^{x+1}}{(x+1)!} \]
\[
\frac{n\lambda}{(x+1)} p(x, n; \lambda, \lambda) \\
+ \sum_{i=1}^{N} \left[ \frac{\lambda}{(x+1)(N-i-1)!} \sum_{\lambda_i} p(x-N_i+1, n-1; \lambda_{(i)}, \lambda) \right. \\
- \frac{\lambda^{M_i+1}}{(x+1) M_i! \prod_{\lambda_i}} p(x-M_i, n-1; \lambda_{(i)}, \lambda) \left. \right] .
\]

**THEOREM 3.**

If a random variable \( X \sim D\{(n, M, \lambda) \} \), then the p.f. \( p(x, n; (N, MD, \lambda)) \) in (5.4) satisfies the recurrence relation

\[
p(x+1, n; (N, MD, \lambda)) = \frac{n}{(x+1)} \lambda p(x, n; (N, MD, \lambda) \\
- \frac{\lambda^{M+1}}{M! \prod_{\lambda_i}} p(x-M, n-1; (N, MD, \lambda) \\
+ \frac{\lambda^N}{(N-1)! \prod_{\lambda_i}} p(x-N+1, n-1; (N, MD, \lambda))
\]

where \( x = nN, nN+1, \ldots, nM-1 \).

**(5.6)**

**PROOF.**

We use the recurrence relation (2.10) for \( D \)-numbers

\[
D_i(x+1, n; (N, MD)) = nD_i(x, n; (N, MD)) - (\begin{array}{c} x \\ M \end{array}) D_i(x-M, n-1; (N, MD)) \\
+ n(\begin{array}{c} x \\ N-1 \end{array}) D_i(x-N+1, n-1; (N, MD)),
\]

where \( x = nN, nN+1, \ldots, nM-1 \).

Then

\[
p(x, n; (N, MD, \lambda)) = e(N, M; \lambda)^{-n} \frac{\lambda^x}{x!}
\]

(1)
\[ p(x-M, n-1; CN, MD, \lambda) = \frac{\lambda^{x-M}}{(x-M)!} e(N, M; \lambda)^{-(n-1)} \frac{\lambda^{x-N+1}}{(x-N+1)!} \]

\[ p(x-N+1, n-1; CN, MD, \lambda) = \]

\[ e(N, M; \lambda)^{-(n-1)} \frac{\lambda^{x-N+1}}{(x-N+1)!} \]

\[ = p(x+1, n; CN, MD, \lambda) = \text{left hand side}. \]

**THEOREM 4.**

If a random variable \( X \sim \text{DD}(n; CN, MD, \lambda) \), then the p.f. \( p(x, n; CN, MD, \lambda) \) in (5.4) for the special cases of \( x = nM \) and \( nM-1 \) satisfies

(i) \( p(nM, n; CN, MD, \lambda) = \frac{\lambda^{nM}}{n!} p(nM-1, n; CN, MD, \lambda), \)

\[ n = 1, 2, \ldots \] \hfill (5.7)

(ii) \( p(nM-1, n; CN, MD, \lambda) \)

\[ = \frac{n}{n-1} \frac{\lambda^{n}}{n! e(N, M; \lambda)^{(n-1)}} p((n-1)M-1, n-1; CN, MD, \lambda), \]

\[ n = 2, 3, \ldots \] \hfill (5.8)

**PROOF.**

(i) \( p(nM-1, n; CN, MD, \lambda) = e(N, M; \lambda)^{-n} \frac{\lambda^{nM-1}}{(nM-1)!} \)
\[ p(nM, n; (N, MD, \lambda)) = e(N, M; \lambda)^{-n} D_i^n (nM, n; (N, MD)) \frac{\lambda^{nM}}{nM!}, \]

When we compare the above formulas, and use property (2.12) for \( D_i \) numbers, we get

\[ p(nM, n; (N, MD, \lambda)) = \frac{\lambda^{nM}}{nM!} p(nM-1, n; (N, MD, \lambda)). \]

(ii) \( p(nM-1, n; (N, MD, \lambda)) \)

\[ = e(N, M; \lambda)^{-n} D_i^n (nM-1, n; (N, MD)) \frac{\lambda^{nM-1}}{(nM-1)!} \]

\[ = e(N, M; \lambda)^{-n} \frac{n}{n-1} \frac{(nM-1)!}{M! (nM-1-M)!} D_i^n (nM-1, n-1; (N, MD)) \frac{\lambda^{nM-1}}{(nM-1)!} \]

(by property (2.13) for \( D_i \) numbers)

\[ = \frac{n}{n-1} \frac{\lambda^M}{M! e(N, M; \lambda)} . \]

\[ e(N, M; \lambda)^{-n} D_i^n (nM-1, n-1; (N, MD)) \frac{\lambda^{(nM-1)-n}}{(nM-1)!} \]

\[ = \frac{n}{n-1} \frac{\lambda^M}{M! e(N, M; \lambda)} p(nM-1, n-1; (N, MD, \lambda)). \]

**THEOREM 5.**

If a random variable \( X \sim DDX(n; (N, MD, \lambda)) \), then the p.f. \( p(x, n; (N, MD, \lambda)) \) in (5.4) satisfies

(i) for fixed \( N, n > 1 \)

\[ p(nN+j, n; (N, MD, \lambda)) = \left( \frac{e(N, M+i; \lambda)}{e(N, M; \lambda)} \right)^n p(nN+j, n; (N, M+i), \lambda), \quad (5.9) \]

where
\[ j = 0, 1, \ldots, (M-N); \quad i = 1, 2, \ldots \]

(ii) for fixed \( M, \ n > 1 \)

\[
p(nM-j, n; (N, M), \lambda) = \left( \frac{e(N-i, M; \lambda)}{e(N, M; \lambda)} \right)^n p(nM-j, n; (N-i, M), \lambda), (5.10)
\]

where

\[ j = 0, 1, \ldots, (M-N); \quad i = 1, 2, \ldots \]

PROOF.

(i) \( p(nN+j, n; (N, M), \lambda) \)

\[
e(N, M; \lambda) J^{-n} D_i (nN+j, n; (N, M)) \frac{\lambda^{nN+j}}{(nN+j)!}
\]

\[
e(N, M; \lambda) J^{-n} D_i (nN+j, n; (N, M+i)) \frac{\lambda^{nN+j}}{(nN+j)!}
\]

(by property (2.14) for \( D_i \) numbers)

\[
= \left( \frac{e(N, M+i; \lambda)}{e(N, M; \lambda)} \right)^n p(nN+j, n; (N, M+i), \lambda).
\]

(ii) \( p(nM-j, n; (N, M), \lambda) \)

\[
e(N, M; \lambda) J^{-n} D_i (nM-j, n; (N, M)) \frac{\lambda^{nM-j}}{(nM-j)!}
\]

\[
e(N, M; \lambda) J^{-n} D_i (nM-j, n; (N-i, M)) \frac{\lambda^{nM-j}}{(nM-j)!}
\]

(by property (2.15) for \( D_i \) numbers)

\[
= \left( \frac{e(N-i, M; \lambda)}{e(N, M; \lambda)} \right)^n p(nM-j, n; (N-i, M), \lambda).
\]
THEOREM 6.

If a random variable $X \sim \text{DDX}_n; (N, M, \lambda)$, then the p.f. $p(x, n; (N, M, \lambda))$ in (5.4) satisfies

(i) $p(x, n; (N+1, M, \lambda)) =
\begin{align*}
&= \left( \frac{e^{(N+1, M, \lambda)}}{e^{CN+1, M, \lambda}} \right)^n \sum_{s=0}^{A} (-1)^s \binom{n}{s} \left( \frac{\lambda^N}{N! e^{CN, M, \lambda}} \right)^s p(x-Ns, n-s; (N, M, \lambda)),
\end{align*}
(5.11)
where

$A = \left\lfloor \frac{nM-x}{M-N} \right\rfloor \leq n; \quad x = n(N+1), \ldots, nM$,

\begin{align*}
p(x, n; (N, M, \lambda)) =
&= \left( \frac{e^{(N+1, M, \lambda)}}{e^{CN, M, \lambda}} \right)^n \sum_{s=0}^{A} (-1)^s \binom{n}{s} \left( \frac{\lambda^N}{N! e^{CN+1, M, \lambda}} \right)^s p(x-Ns, n-s; (N+1, M, \lambda)),
\end{align*}
(5.12)
where \quad $x = nN, \ldots, nM$.

(ii) $p(x, n; (N, M-1, \lambda)) =
\begin{align*}
&= \left( \frac{e^{(N, M, \lambda)}}{e^{CN, M-1, \lambda}} \right)^n \sum_{s=0}^{B} (-1)^s \binom{n}{s} \left( \frac{\lambda^M}{M! e^{CN, M, \lambda}} \right)^s p(x-Ms, n-s; (N, M, \lambda)),
\end{align*}
(5.13)
where

$B = \left\lfloor \frac{x-nN}{M-N} \right\rfloor \leq n; \quad x = nN, \ldots, n(M-1)$.

\begin{align*}
p(x, n; (N, M, \lambda)) =
&= \left( \frac{e^{(N, M-1, \lambda)}}{e^{CN, M, \lambda}} \right)^n \sum_{s=0}^{B} (-1)^s \binom{n}{s} \left( \frac{\lambda^M}{M! e^{CN, M-1, \lambda}} \right)^s p(x-Ms, n-s; (N, M-1, \lambda)),
\end{align*}
(5.14)
where \quad $x = nN, \ldots, nM$. 
PROOF.

(i) \( p(x, n; (N+1, MD, \lambda)) \)

\[ = [e(N+1, M; \lambda)]^{-n} \sum_{s=0}^{A} \left( -1 \right)^s \frac{\binom{N}{s} p(x)}{(N-s)!} D_i (x-Ns, n-s; (N, MD), \frac{\lambda}{x!}) \]

(by property (2.16) of \( D_i \) numbers)

\[ = \left[ e(N, M; \lambda) \right]^{-n} \sum_{s=0}^{A} \left( -1 \right)^s \frac{\binom{N}{s} p(x)}{(N-s)!} e(N, M; \lambda)^s \]

\[ \cdot \left[ e(N, M; \lambda) \right]^{-\left( n-s \right)} \frac{\lambda^{x-Ns}}{(x-Ns)!} \]

\[ = \left[ e(N, M; \lambda) \right]^{-n} \sum_{s=0}^{A} \left( -1 \right)^s \frac{\binom{N}{s} \lambda^N}{(N-s)! e(N, M; \lambda)^s} \]

\( p(x-Ns, n-s; (N, MD, \lambda)) \).

This proves (5.11).

Similarly, we can use properties (2.17), (2.18) and (2.19) for \( D_i \) numbers to prove the rest of the theorem.

5.4. MODAL PROPERTIES

PROPERTY 1.

If a random variable \( X \sim DDXn; (N, MD, \lambda) \), then

(i) when \( \lambda < \frac{N+1}{n} \), the p.f. \( p(x, n; (N, MD, \lambda)) \) has a unique mode at \( x = nN; \)

(ii) when \( \lambda = \frac{N+1}{n} \), the p.f. \( p(x, n; (N, MD, \lambda)) \) is bimodal, the modes being at \( x = nN \) and \( nN+1; \)
(iii) when $\lambda > \frac{N+1}{n}$, the p.f. $p(x, n; (N, MD, \lambda))$ increases first with $x$ and then decreases with $x$, except perhaps when the p.f. assumes equal values at two consecutive $x$'s at the time of its change from an increasing to a decreasing function.

**Proof.**

For $x = nN+1, \ldots, nM$,

$$\frac{p(x-1, n; (N, MD, \lambda))}{p(x, n; (N, MD, \lambda))} = \frac{xD_1(x-1, n; (N, MD))}{\lambda D_1(x, n; (N, MD))} = \frac{h(x, n; (N, MD))}{\lambda}.$$ 

From property 2 of Chapter 2, Section 2.5, we know that for $n \geq 1$, $h(x, n; (N, MD))$ is an increasing function of $x$ and $h(x, n; (N, MD)) \geq \frac{N+1}{n}$ for $nN < x \leq nM$, with equality for $x = nN+1$.

(i) For $\lambda < \frac{N+1}{n}$, $\frac{p(x-1, n; (N, MD, \lambda))}{p(x, n; (N, MD, \lambda))} > 1$ for $nN < x \leq nM$.

Hence, $p(x, n; (N, MD, \lambda))$ is a decreasing function here and has a unique mode at $x = nN$.

(ii) For $\lambda = \frac{N+1}{n}$, $\frac{p(x-1, n; (N, MD, \lambda))}{p(x, n; (N, MD, \lambda))} = 1$.

Hence, $p(x, n; (N, MD, \lambda))$ is bimodal, the modes being at $x = nN$ and $nN+1$.

(iii) For $\lambda > \frac{N+1}{n}$, $\frac{p(x-1, n; (N, MD, \lambda))}{p(x, n; (N, MD, \lambda))} < 1$ when $h(x, n; (N, MD)) < \lambda$.

$$\frac{p(x-1, n; (N, MD, \lambda))}{p(x, n; (N, MD, \lambda))} = 1$$ when $h(x, n; (N, MD)) = \lambda$.
\[
p(x-1, n; (N, MD, \lambda)) > 1 \text{ when } h(x, n; (N, MD) > \lambda}.
\]

Hence, \( p(x, n; (N, MD, \lambda) \) increases first with \( x \) and then decreases except perhaps when the p.f. assumes equal values at two consecutive \( x \)'s at the time of its change.

5.5 RECURSION AND DECOMPOSITION

**THEOREM 7. (RECURSION)**

Let \( T_1 \) and \( T_2 \) be independent r.v.'s, and \( T = T_1 + T_2 \). Then \( T \sim DD(n; \mathfrak{L}, \Lambda) \) if \( T_1 \sim DD(n_1; \mathfrak{L}_1, \Lambda_1) \) and \( T_2 \sim DD(n_2; \mathfrak{L}_2, \Lambda_2) \) for \( n = n_1 + n_2 \), where \( n_i \)'s are nonnegative integers.

**PROOF.**

Suppose \( T_1 \sim DD(n_1; \mathfrak{L}_1, \Lambda_1) \); \( T_2 \sim DD(n_2; \mathfrak{L}_2, \Lambda_2) \).

Let

\[
\mathfrak{L}_1 = \langle (N_1, M_1), \ldots, (N_{n_1}, M_{n_1}) \rangle, \quad \mathfrak{L}_2 = \langle (N_{n_1+1}, M_{n_1+1}), \ldots, (N_n, M_n) \rangle;
\]

\[
\Lambda_1 = \langle \lambda_1, \lambda_2, \ldots, \lambda_{n_1} \rangle, \quad \Lambda_2 = \langle \lambda_{n_1+1}, \lambda_{n_1+2}, \ldots, \lambda_n \rangle;
\]

\[
\mathfrak{L} = \mathfrak{L}_1 \cup \mathfrak{L}_2; \quad \Lambda = \Lambda_1 \cup \Lambda_2;
\]

and let
\[
L_1 = \sum_{i=1}^{n_1} N_i, \quad L_2 = \sum_{i=n_1+1}^{n} N_i; \quad Q_1 = \sum_{i=1}^{n_1} M_i, \quad Q_2 = \sum_{i=n_1+1}^{n} M_i.
\]

Then for \(0 < z < 1\), by Lemma 1 and Theorem 1, \(T_1\) and \(T_2\) have probability generating functions

\[
\psi_{T_1}(z) = \prod_{i=1}^{n_1} \frac{e(N_i, M_i; z \lambda_i)}{e(N_i, M_i; \lambda_i)}, \quad \text{and} \quad \psi_{T_2}(z) = \prod_{i=n_1+1}^{n} \frac{e(N_i, M_i; z \lambda_i)}{e(N_i, M_i; \lambda_i)},
\]

respectively.

Since \(T_1\) and \(T_2\) are independent, \(T = T_1 + T_2\) has p.g.f.

\[
\psi_T(z) = \psi_{T_1}(z) \psi_{T_2}(z)
\]

\[
= \prod_{i=1}^{n} \frac{e(N_i, M_i; z \lambda_i)}{e(N_i, M_i; \lambda_i)}, \quad 0 < z < 1.
\]  \hspace{1cm} (5.15)

By Lemma 1 and Theorem 1, (5.15) is the p.g.f. of the D distribution with parameters \(n, \lambda, \Lambda\), and the p.g.f. determines the distribution function uniquely (Fisz, 1963, p.125). Therefore

\[T \sim DD(n; \lambda, \Lambda).\]
COROLLARY 3. (DECOMPOSITION)

If \( X \sim \text{DD}(n; \Sigma, \Lambda) \), then \( X \) can be expressed as the sum of \( n \) independently distributed r.v.'s, each having \( \text{DD}(1; (N_i, M_i), \lambda_i) \) (i.e., \( \text{DTPD}(N_i, M_i; \lambda_i) \)), \( i = 1, 2, \ldots, n \).

5. \( \hat{c} \) MOMENTS

In this section, we proceed to find the moments of the D distribution. It is interesting to see that the moments depend only on the incomplete exponential function \( e(N_i, M_i; \lambda_i) \), and not on the D numbers.

THEOREM 8.

If \( X \sim \text{DD}(n; \Sigma, \Lambda) \), then the rth factorial moment is given by

\[
m_{(r)} = \sum_{0 \leq k \leq r} \binom{k}{k_1, k_2, \ldots, k_n} \frac{\prod_{i=1}^{k_i} e(N_i - k_i, M_i - k_i; \lambda_i)}{e(N_i, M_i; \lambda_i)} \tag{5.18}
\]

where

\[
\min_{i} \langle N_i \rangle > r ; \quad K = 0, 1, \ldots, r ; \quad r = 1, 2, \ldots
\]

PROOF.

By Lemma 1, the p.g.f. of \( X \) is
\[
\psi(x) = \prod_{i=1}^{n} e(N_i, M_i; s\lambda_i) \div e(N_i, M_i; \lambda_i), \quad 0 < s < 1,
\]

Using Leibniz's formula for higher order derivatives of products (see Wong, 1979, p. 197-198), we get

\[
m_{(r)}(s) = \left. \frac{\partial^r \psi(s)}{\partial s^r} \right|_{s=1}
\]

\[
= \prod_{i=1}^{n} e(N_i, M_i; \lambda_i)^{-1}.
\]

\[
\sum_{0 \leq k_i \leq r} \left( \begin{array}{c} r \\ k_1, k_2, \ldots, k_n \end{array} \right) \prod_{i=1}^{n} \lambda_i^{k_i} e(N_i - k_i, M_i - k_i; s\lambda_i) \bigg|_{s=1}
\]

\[
\sum_{i=1}^{n} k_i = r
\]

(by property (1.18) for the IEF)

\[
= \sum_{0 \leq k_i \leq r} \left( \begin{array}{c} r \\ k_1, k_2, \ldots, k_n \end{array} \right) \prod_{i=1}^{n} \frac{\lambda_i^{k_i} e(N_i - k_i, M_i - k_i; \lambda_i)}{e(N_i, M_i; \lambda_i)}.
\]

\[
\sum_{i=1}^{n} k_i = r
\]

**COROLLARY 4.**

If \( X \sim DD(n; \lambda, \lambda) \), then
\[ E(X) = \sum_{i=1}^{n} \lambda_i \frac{e(N_i - 1, M_i - 1; \lambda_i)}{e(N_i, M_i; \lambda_i)} \quad ; \quad (5.19) \]

\[ \text{Var}(X) = \sum_{i=1}^{n} \lambda_i \left( \frac{e(N_i - 2, M_i - 2; \lambda_i)}{e(N_i, M_i; \lambda_i)} + \frac{e(N_i - 1, M_i - 1; \lambda_i)}{e(N_i, M_i; \lambda_i)} \right) \]

\[ - \lambda_i \left( \frac{e(N_i - 1, M_i - 1; \lambda_i)}{e(N_i, M_i; \lambda_i)} \right)^2 \quad . \quad (5.20) \]

\[ \text{COROLLARY 5.} \]

If \( X \sim \text{DD}(n; (N, M, \lambda)) \),

\[ E(X) = n\lambda \frac{e(N-1, M-1; \lambda)}{e(N, M; \lambda)} \quad ; \quad (5.21) \]

\[ \text{Var}(X) = \]

\[ n\lambda \left( \lambda - \frac{e(N-2, M-2; \lambda)}{e(N, M; \lambda)} + \frac{e(N-1, M-1; \lambda)}{e(N, M; \lambda)} - \lambda \left( \frac{e(N-1, M-1; \lambda)}{e(N, M; \lambda)} \right)^2 \right) \quad . \quad (5.22) \]

We use the relationship of \( m_r \) and \( m_{(r)} \) to obtain the \( r \)th moment (see Johnson and Kotz, 1969).
COROLLARY 6.

If $X \sim \text{DD}(n; \lambda, \Delta)$, then the $r$th moment is given by

$$m_r = \sum_{k=0}^{r} S(r, k) m_{(k)} \quad (5.23)$$

where

$$\min \langle N_i \rangle > r; \quad k_i = 0, 1, \ldots, r, \quad \text{and} \quad (i)$$

$S(r, k)$ is the Stirling number of the second kind (Jordan, 1965).

Using the recurrence relationship for Stirling numbers of the second kind, we can prove the following recurrence relationship for $m_r$.

THEOREM 9.

If $X \sim \text{DD}(n; \lambda, \Delta)$, (i.e., $\lambda_i = \lambda$, $i = 1, 2, \ldots, n$), then

$$m_r = \lambda \frac{d}{d\lambda} m_{r-1} + m_{r-1} m. \quad (5.24)$$
PROOF.

Use the recurrence relationship for Stirling numbers of the second kind (Jordan, 1965), formula (1.40)

\[ S(r, k) = kS(r-1, k) + S(r-1, k-1). \]

Thus

\[ m_r = \sum_{k=0}^{r} S(r, k) \lambda^k \sum_{k_1, \ldots, k_n} \left( \prod_{i=1}^{n} \frac{e(N_i, M_i; \lambda)}{e(N_i, M_i; \lambda)} \right) \]

\[ = \sum_{k=0}^{r} kS(r-1, k) \lambda^k \sum_{k_1, \ldots, k_n} \left( \prod_{i=1}^{n} \frac{e(N_i, M_i; \lambda)}{e(N_i, M_i; \lambda)} \right) \]

\[ + \sum_{k=0}^{r} S(r-1, k-1) \lambda^k \sum_{k_1, \ldots, k_n} \left( \prod_{i=1}^{n} \frac{e(N_i, M_i; \lambda)}{e(N_i, M_i; \lambda)} \right). \]

Since

\[ m_{r-1} = \sum_{k=0}^{r-1} S(r-1, k) \lambda^k \sum_{k_1, \ldots, k_n} \left( \prod_{i=1}^{n} \frac{e(N_i, M_i; \lambda)}{e(N_i, M_i; \lambda)} \right), \]

then
\[
\frac{dm_{r-1}}{d\lambda} = \sum_{k=0}^{r-1} kS(r-1,k)\lambda^k \sum_{k_{i}, \ldots, k_n} \frac{\sum_{i=1}^{n} e(N_i - k_i, M_i - k_i; \lambda)}{\prod_{i=1}^{n} e(N_i, M_i; \lambda)}
\]

\[
+ \sum_{k=0}^{r-1} S(r-1,k)\lambda^k \sum_{k_{i}, \ldots, k_n} \frac{\sum_{i=1}^{n} e(N_i - k_i - 1, M_i - k_i - 1; \lambda) \prod_{j \neq i} e(N_i - k_i, M_i - k_i; \lambda)}{\prod_{i=1}^{n} e(N_i, M_i; \lambda)}
\]

\[
- \sum_{k=0}^{r-1} S(r-1,k)\lambda^k \sum_{k_{i}, \ldots, k_n} \frac{\prod_{i=1}^{n} e(N_i - k_i, M_i - k_i; \lambda)}{\prod_{i=1}^{n} e(N_i, M_i; \lambda)}
\]

Note that \(S(r-1,r) = 0\), and by comparing \(m_r\) and \(\frac{dm_{r-1}}{d\lambda}\), we have

\[
\frac{dm_{r-1}}{d\lambda} = \frac{1}{\lambda} m_r - \frac{1}{\lambda} m_{r-1} m_i
\]
Hence

\[ m_r = \lambda \frac{d m_{r-1}}{d \lambda} + m_{r-1} m_1. \]

We also notice that if all \( \lambda_i = \lambda \), the D distribution becomes a generalized power series distribution (GPSD) (Patil, 1963). The result of Theorem 9 is the same as the result of Gupta (1974). Also, we get the following recurrence relationships:

(i) recurrence relation among central moments,

\[ \mu_{r+1} = \lambda \frac{d \mu_r}{d \lambda} + \mu_r \mu_{r-1}, \] \hspace{1cm} (5.25)

(ii) recurrence relation among factorial moments,

\[ m_{r+1}^{(r)} = \lambda \frac{d m_r^{(r)}}{d \lambda} + m_r^{(r)} m_1 - r m_r^{(r)}, \] \hspace{1cm} (5.26)

(iii) recurrence relation among cumulants,

\[ K_{r+1} = \lambda \sum_{j=1}^{r-1} m_r^{(r-1)} \frac{d k_j}{d \lambda} - \sum_{j=2}^{r} \sum_{j=2}^{r-1} m_{r+1-j} K_j, \] \hspace{1cm} (5.27)

where \( K_r \) is the rth cumulant.

5.7. ESTIMATION

In this section, we wish to find the minimum variance unbiased (MVU) estimate of the p.f. of DD(k; \( \mathcal{L} \), \( \lambda \)). Here, we only consider \( \lambda_i = \lambda \), \( i = 1, \ldots, n \).
Recall \[ d(x, n; \lambda) = GD(x, n; \lambda) \lambda^x. \] (5.28)

Now, (5.2) can be rewritten as

\[ p(x, n; \lambda) = \prod_{i=1}^{n} e(N_i, M_i; \lambda) \lambda^{-i} d(x, n; \lambda) \frac{\lambda^x}{x!} \] (5.29)

where \[ x = \sum_{i=1}^{n} N_i, \sum_{i=1}^{n} N_i + 1, ..., \sum_{i=1}^{n} M_i; \]

\[ \lambda \in \Lambda^*, \quad \Lambda^* = \{ \lambda | \lambda > 0, \lambda \in \mathbb{R} \}. \]

**Lemma 2.**

The family of D distribution \( \langle p_\lambda : \lambda \in \Lambda^* \rangle \), where \( p_\lambda = p(x, n; \lambda) \) in (5.29) is an exponential family of discrete distribution.

**Proof.**

Suppose \( X \) has a D distribution \( GX(n; \lambda) \), \( \lambda > 0 \). Then the distribution function (5.29) can be rewritten as

\[ p(X, n; \lambda) \]

\[ = \left[ \exp \left( X \log \lambda + \log \left( \prod_{i=1}^{n} e(N_i, M_i; \lambda) \right) \right) + \log d(x, n; \lambda) \right] \cdot \mathcal{I}_\Lambda(X) \]

\[ = \left[ \exp \left( c(\lambda) T(X) + d(\lambda) + SC(X) \right) \right] \cdot \mathcal{I}_\Lambda(X), \]

where \( \Lambda = \left\{ \sum_{i=1}^{n} N_i, \sum_{i=1}^{n} N_i + 1, ..., \sum_{i=1}^{n} M_i \right\} \subset \mathbb{R}^n \).

Therefore the family of distribution of \( X \) is a one parameter exponential family with
\[ c(\lambda) = \log \lambda ; \]
\[ d(\lambda) = \log \left( \prod_{i=1}^{n} e^{c(N_i, M_i; \lambda)^{-1}} \right) ; \]
\[ T(X) = X ; \]
\[ S(X) = \log \frac{d(X, n; \varnothing)}{\varnothing!} ; \]

where \( c(\lambda), d(\lambda) \) are real-valued functions on \( \Lambda^* = \langle \lambda \mid \lambda > 0 \rangle \),
\( T(X), S(X) \) are real-valued functions on \( \Lambda \subset \mathbb{R}^n \).

Let \( Z_1, Z_2, \ldots, Z_m \) be a random sample from \( DD(X; n, \varnothing, \lambda) \) with
known parameters \( n, \varnothing, \) and unknown parameter \( \lambda, \) and
\( \{ P_{\lambda}^{(m)}, \lambda \in \Lambda^* \} \) be the family of distribution of \( Z = (Z_1, Z_2, \ldots, Z_m) \) in \( \mathbb{R}^{mn} \).

**Lemma 3.**

\[ Z = \sum_{i=1}^{m} Z_i \] is a complete sufficient statistic for the
family of distribution.

**Proof.**

Since \( Z_1, Z_2, \ldots, Z_m \) are independent and identically
distribution with the common distribution \( p_\lambda \) in (5.29) and
\( p_\lambda \) is an exponential family of discrete distribution by
Lemma 2, we have

\[ p(Z, \lambda) = \prod_{i=1}^{m} p(Z_i, \lambda) \]
\[ = \prod_{i=1}^{m} \left[ \exp \left( c(\lambda) T(Z_i) + d(\lambda) + S(Z_i) \right) \right] \cdot I_A(Z_i) \]
\[
= \exp \left[ (c(\lambda) \sum_{i=1}^{m} T(Z_i) + m\delta(\lambda) + \sum_{i=1}^{m} S(Z_i) ) \right] \cdot I_{\Lambda^{(m)}} (Z, Z_2, \ldots, Z_m)
\]

where \( \Lambda^{(m)} = \{ (Z_1, Z_2, \ldots, Z_m) | Z_i \in A, 1 \leq i \leq m \} \).

Therefore, the \( p_x^{(m)} \) form a one parameter exponential family.

Then \( \sum_{i=1}^{m} T(Z_i) = \sum_{i=1}^{m} Z_i = Z \) is the natural sufficient and complete statistic of the family of distribution (Bickel and Doksum, 1977, p.69, p.123, also, see Appendix B.1).

Next, we proceed to find a MVU estimate.

**Theorem 10.**

The MVU estimate of (5.29) is

\[
\hat{p}_x (Z, m) = \frac{\binom{z}{x} d(x, n; \mathcal{L}) d(z-x; (m-1)n; \mathcal{L}^{(m-1)})}{d(z, mn; \mathcal{L}^{(m)})} \tag{5.30}
\]

where \((m-1)L + x \leq z \leq (m-1)Q + x; \quad x = L, L+1, \ldots, Q; \)

\[
L = \sum_{i=1}^{m} N_i, \quad Q = \sum_{i=1}^{m} M_i;
\]

\(z\) is the observed value of \( Z = \sum_{i=1}^{m} Z_i; \)

\( Z_i \sim \text{i.i.d. } \text{DD}(n; \mathcal{L}, \lambda), \quad i = 1, \ldots, m; \)

\( \mathcal{L}^{(m)} = m \text{ identical sets of } \mathcal{L}. \)

**Proof.**

Let \( \hat{p}_x (Z) \) be an unbiased estimate of \( p(x, n; \mathcal{L}, \lambda) \) in (5.29). By using Theorem 7, \( Z = \sum_{i=1}^{m} Z_i \sim \text{DD}(mn; \mathcal{L}^{(m)}, \lambda). \)
Then

\[ \sum_{z=mL}^{mQ} \hat{p}_x (Z) \left( \prod_{i=1}^{n} e(N_i, m_i; \lambda_i) \right)^{-m} d(z, mn; \mathcal{L}^{(m)}) \frac{\lambda^z}{z!} = \prod_{i=1}^{n} e(N_i, m_i; \lambda_i)^{-1} d(x, n; \mathcal{L}) \frac{\lambda^x}{x!}, \]

where \( x = L, L+1, \ldots, Q. \)

Hence

\[ \sum_{z=mL}^{mQ} \hat{p}_x (Z) \frac{d(z, mn; \mathcal{L}^{(m)}) \lambda^z}{z!} = \frac{1}{\left( \prod_{i=1}^{n} e(N_i, m_i; \lambda_i) \right)^{1-m}} d(x, n; \mathcal{L}) \frac{\lambda^x}{x!}, \]

\[ = \frac{\lambda^x}{x!} d(x, n; \mathcal{L}) \sum_{k=(m-1)L}^{(m-1)Q} \frac{\lambda^k}{k!} d(k, (m-1)n; \mathcal{L}^{(m-1)}) \]

\[ = \frac{\lambda^x}{x!} d(x, n; \mathcal{L}) \sum_{k=(m-1)L+Q}^{mQ} \frac{1}{(k-Q)!} d(k-Q, (m-1)n; \mathcal{L}^{(m-1)}). \]

Comparing coefficients of \( \lambda^z \) on both sides with

\( z = x + k - Q, \quad k-Q = z-x, \quad \) and

\( mL \leq z \leq mQ; \)

\( (m-1)L + Q \leq z-x \leq mQ; \)

\( (m-1)L + x \leq z \leq (m-1)Q + x, \)

\[ \hat{p}_x (Z) \frac{1}{z!} d(z, mn; \mathcal{L}^{(m)}) \]

\[ = \frac{1}{x!} d(x, n; \mathcal{L}) \frac{1}{(z-x)!} d(z-x, (m-1)n; \mathcal{L}^{(m-1)}). \]
Since $\hat{p}_x(Z)$ is an unbiased estimate of $p(x,n;\mathcal{L},\lambda)$ and is a function of a complete sufficient statistic $Z$, by Lehmann-Scheffe Theorem (Bickel and Doksum, 1977, p.122, also see Appendix B.1), the MVU estimate of (5.28) is (5.30). 

**COROLLARY 7.**

The MVU estimate of $p(x,n;(N,MD),\lambda)$ defined in (5.4) is

$$\hat{p}_x(Z, m) = \frac{\binom{z}{x} D_i(x, n; (N, MD)) D_i(z-x; (m-1)n; (N, MD))}{D_i(z, mn; (N, MD))}$$

(5.31)

where $(m-1)n + x \leq z \leq (m-1)nM + x$; $x = nN, nN+1, \ldots, nM$;

$z$ is the observed value of $Z = \sum_{i=1}^{m} Z_i$,

$Z_i \sim \text{i.i.d. } DD(n, \mathcal{L}, \lambda)$, $i = 1, \ldots, m$.

The estimates (5.30) and (5.31) are easily obtainable using tables of $D_i(x, n; (N, MD))$ and $d(x, n; \mathcal{L})$ (See Appendix A.1, Table I and II).

**THEOREM 11.**

The MVU estimate of the variance of $\hat{p}_x(Z, m)$ in (5.30) is

$$\text{Var}(\hat{p}_x(Z, m)) = \left\{ \left( \sum_{x} D(x, n; \mathcal{L}) d(z-x; (m-1)n; \mathcal{L}^{(m-1)}) \right)^2 \right\}$$

$$\frac{d(z, mn; \mathcal{L}^{(m-1)})}{d(z, mn; \mathcal{L}^{(m)})}$$

$$- \left( \sum_{x} d(x, n; \mathcal{L}) \right)^2 \frac{d(z-2x; (m-2)n; \mathcal{L}^{(m-2)})}{d(z, mn; \mathcal{L}^{(m)})}$$

(5.32)
PROOF.

Since (5.29) is a p.f. of the generalized power series distribution, we can apply the results of GPSD given by Patil (1963). The MVU estimate of the variance is

\[
\hat{\text{Var}}(\hat{p}_x(z, m)) = [\hat{p}_x(z, m) - \hat{p}_x(z-x, m-1)] \hat{p}_x(z, m)
\]

\[
= [\hat{p}_x(z, m)]^2 - \hat{p}_x(z-x, m-1) \hat{p}_x(z, m).
\]

Substituting (5.30) into the above formula, we obtain

\[
\hat{\text{Var}}(\hat{p}_x(z, m)) = \left(\frac{\binom{Z}{x} \text{d}(x, n; \mathcal{L}) \text{d}(z-x; (m-1)n; \mathcal{L}^{(m-1)})}{\text{d}(z, mn; \mathcal{L}^{(m)})}\right)^2
\]

\[
- \frac{\binom{Z}{x} \text{d}(x, n; \mathcal{L}) \text{d}(z-x; (m-1)n; \mathcal{L}^{(m-1)})}{\text{d}(z, mn; \mathcal{L}^{(m)})}
\]

\[
\cdot \frac{\binom{Z-x}{x} \text{d}(x, n; \mathcal{L}) \text{d}(z-2x; (m-2)n; \mathcal{L}^{(m-2)})}{\text{d}(z-x, (m-1)n; \mathcal{L}^{(m-1)})}.
\]

Then we get (5.32). □

REMARK.

The results of the power series distribution (Patil, 1963) are identical to those presented in this section.

5.8. APPLICATIONS

EXAMPLE 1. ($M^X/G^X/\infty$ queue)

Suppose groups of customers arrive at times
corresponding to a Poisson process with parameter \( \lambda \). Let \( Y_i \) be the group size of the \( i \)th arrival group, where \( Y_i \sim \text{DD}\left(N_i, M_i, \beta_i \right) \) are independent.

Assume that there are infinitely many servers so that there is no waiting time for customers. Service is in bulk for each group that comes, and the service time for groups are independent and identically distributed with distribution function \( H(x) \).

The problem we are interested in is the number of customers being served at time \( t \). If there are initially no customers present, we have the following results (Karlin and Taylor, 1981, p. 521-522).

(i) \( P_k(t) = P(\text{there are exactly } k \text{ groups being served at time } t) \)

\[
= (\exp(-\lambda \int_0^t (1-H(x)) dx))(\lambda \int_0^t (1-H(x)) dx)^k / k! \quad (5.33)
\]

(ii) \( P_k = \lim_{t \to \infty} P_k(t) = e^{-\alpha \lambda} (\alpha \lambda)^k / k! \quad (5.34) \)

(iii) \( D_n(t) = P(\text{n group departures by time } t) \)

\[
= \frac{1}{n!} (\lambda \int_0^t H(u) du)^n \exp(-\lambda \int_0^t H(u) du) \quad (5.35)
\]

(iv) \( \tilde{F}(t, T) = P(\text{no departures in time } (t, t+T)) \)

\[
= \exp(-\lambda \int_t^{t+T} H(x) \, dx) \quad (5.36)
\]

(v) \( \tilde{F}_n(t, T) = P(\text{n departures in time } (t, t+T)) \)
\[ \frac{1}{n!} \left[ \frac{\lambda}{t+\lambda} \right]^{n} t^{n} \exp(-\lambda \int_{t}^{t+T} H(x) \, dx) \exp(-\lambda \int_{t}^{t+T} H(x) \, dx). \]  

(5.37)

Using compound distribution theory, we find

(i) \( C(x,t) = P( \text{there are exactly } x \text{ customers being served at time } t) \)

\[ = \sum_{n=0}^{\infty} P( \sum_{i=1}^{\infty} Y_i = x \mid N(t) = n \} \times \sum_{n=0}^{\infty} P(N(t) = n \} \times P(t). \]

By (5.2) and (5.33), we get

\[ C(x,t) = (\exp(-\lambda \int_{0}^{t} [1-H(x)] \, dx)) \]

\[ \times \sum_{n=0}^{\infty} \frac{(\lambda \omega)^{n}}{n!} \prod_{i=1}^{n} [e(N_{i}, M_{i}; \beta_{i})]^{-1} G(x, n; \mathcal{L}, \mathcal{S}) \times !. \]

(5.38)

Similarly

(ii) \( C(x) = \lim_{t \to \infty} C(x,t) \)

\[ = e^{-\lambda \alpha} \sum_{n=0}^{\infty} \frac{(\lambda \omega)^{n}}{n!} \prod_{i=1}^{n} [e(N_{i}, M_{i}; \beta_{i})]^{-1} G(x, n; \mathcal{L}, \mathcal{S}) \times !. \]

(5.39)

where \( \alpha = \int_{0}^{\infty} x \, dH(x). \)

(iii) \( C_D(x,t) = P(x \text{ customer departures by time } t) \)

\[ = \sum_{n=0}^{\infty} P(x, n; \mathcal{L}, \mathcal{S}, \mathcal{D}) \]

(5.40)
\[ \exp(-\lambda \int_0^t H(x) \, dx) \sum_{n=0}^{\infty} \frac{(\lambda \int_0^t H(x) \, dx)^n}{n!} \]

\[ \cdot \prod_{i=1}^{n} \left[ e(N_i, M_i ; \beta_i ) \right]^{-i} \text{D}(x, n; \mathcal{L}, \mathcal{B}) / x! \]  

(iv) \( C_\phi(x; t, T) = \mathbb{P}(x \text{ customer departures in time } (t, t+T)) \)

\[ = \sum_{n=0}^{\infty} \mathbb{P}(x, n; \mathcal{L}, \mathcal{B}) \phi_n (t, T) \]

\[ = \exp(-\int_t^{t+T} H(x) \, dx) \sum_{n=0}^{\infty} \frac{(\int_t^{t+T} H(x) \, dx)^n}{n!} \]

\[ \cdot \prod_{i=1}^{n} \left[ e(N_i, M_i ; \beta_i ) \right]^{-i} \text{D}(x, n; \mathcal{L}, \mathcal{B}) / x! \]  

In the special case of an \( M^\lambda / M^\lambda / \infty \) queue, where \( H(x) = 1 - e^{-\mu x} \), (5.38) - (5.41) reduce to

(i) \( C(x, t) = \exp\left(\frac{\lambda}{\mu} (1 - e^{-\mu t})^n\right) \sum_{n=0}^{\infty} \left(\frac{\lambda (1 - e^{-\mu t})}{\mu}\right)^n \]

\[ \cdot \prod_{i=1}^{n} \left[ e(N_i, M_i ; \beta_i ) \right]^{-i} \text{D}(x, n; \mathcal{L}, \mathcal{B}) / x! \]  

(ii) \( C(x) = e^{-\lambda / \mu} \sum_{n=0}^{\infty} \frac{\lambda / \mu} {n!} \prod_{i=1}^{n} \left[ e(N_i, M_i ; \beta_i ) \right]^{-i} \text{D}(x, n; \mathcal{L}, \mathcal{B}) / x! \)  

(5.43)
(iii) \( C_D(x,t) = \exp\left(-\frac{\lambda t - \lambda}{\mu}(1-e^{-\mu t})\right) \sum_{n=0}^{\infty} \frac{(\lambda t - \lambda)(1-e^{-\mu t})^n}{n!} \cdot \prod_{i=1}^{n} \left[ e(N_i, M_i; \beta_i) \right]^{-1} \text{GD}(x, n; \mathcal{E}, \infty) / x! \). \hspace{1cm} (5.44)

(iv) \( C_{\Phi}(x; t, T) = \exp\left(-\frac{\lambda T - \lambda}{\mu} e^{-\mu t}(1-e^{-\mu T})\right) \sum_{n=0}^{\infty} \frac{(-\lambda T - \lambda)(1-e^{-\mu T})^n}{n!} \cdot \prod_{i=1}^{n} \left[ e(N_i, M_i; \beta_i) \right]^{-1} \text{GD}(x, n; \mathcal{E}, \infty) / x! \). \hspace{1cm} (5.45)

Given values of \( \mathcal{E} = \{N_i, M_i\}, \mathcal{S} = \{\beta_i\}, \lambda, \mu, t, \) we can use tables of D numbers to get \( C(x,t), \ c(x), \ C_D(x, t) \) and \( C_{\Phi}(x; t, T) \) easily.

For instance, suppose \( (N_i, M_i) = (1, 4), \ \beta_i = 3, \ \lambda = 2, \ \mu = 4 \) in an \( M^X/M^X/\infty \) queue. We want to find the probability that 5 customers are being served at time \( t \). Using (5.42), we have

\[
C(5, t) = e^{-0.5(1-e^{-4t})} \cdot \frac{3^5}{5!} \left( \frac{0.5(1-e^{-4t})}{e(1,4;3)} \right)^2 D_I(5,2;(1,4))/2!
+ \left( \frac{0.5(1-e^{-4t})}{e(1,4;3)} \right)^3 D_I(5,3;(1,4))/3!
+ \left( \frac{0.5(1-e^{-4t})}{e(1,4;3)} \right)^4 D_I(5,4;(1,4))/4!
+ \left( \frac{0.5(1-e^{-4t})}{e(1,4;3)} \right)^5 D_I(5,5;(1,4))/5!
\]

From the IEF table in Appendix A.4, Table VII, we get
$e(1,4;3) = 15.375$ and from tables of $D$ numbers (Appendix A.1, Table I), we get $D_1(5,2;(1,4)) = 30$, $D_1(5,3;(1,4)) = 150$, $D_1(5,4;(1,4)) = 240$, $D_1(5,5;(1,4)) = 120$.

If $t = 2$, $e^{-0.5+e^{-4i}} = e^{-0.4998} = 0.6066$.

Hence,

$$C(5,2) = 0.6066 \cdot \frac{3^5}{5!} \left[ \left( \frac{0.4998}{15.375} \right)^2 \cdot 30/2! + \left( \frac{0.4998}{15.375} \right)^2 \cdot 150/3! \\
+ \left( \frac{0.4998}{15.375} \right)^2 \cdot 240/4! + \left( \frac{0.4998}{15.375} \right)^2 \cdot 120/5! \right]$$

$$= 0.02054.$$ 

Similarly, we can find $C(x), C_D(x,t)$ and $C_{D}(x,t,T)$.

Usually, $\beta_i$'s are unknown. In this case, we need to take a random sample $Z_1, Z_2, ..., Z_m$ from the $D$ distribution, where $Z_i$ is the total number of customer departures by time $t$. Then we can use (5.32) to get an estimate for $C(x,t)$, i.e.,

$$\hat{C}(x,t) = \exp\left( -\frac{\lambda}{\mu} (1-e^{-\mu t}) \right) \sum_{n=0}^{\infty} \frac{\lambda^n}{\mu^n n!}$$

$$\cdot \frac{\binom{Z}{x} \binom{Z}{x}}{d(z, mn; \L^{(m)})}, \quad (5.46)$$

where $z$ is the observed value of $z = \sum_{i=1}^{m} Z_i$.

For instance, in the above example, assume $\beta$ is unknown, we have observed values of $z_1 = 3, z_2 = 5, z_3 = 8$ and $m = 2$. Then
\[ \hat{c}(5,2) = e^{-0.5(1-e^{-8})} \cdot \left\{ \frac{(0.5(1-e^{-8}))^2}{2!} \cdot \frac{5 \text{DX}(5,2; (1,4)) \text{DX}(3,2; (1,4))}{\text{DX}(8,4; (1,4))} + \frac{(0.5(1-e^{-8}))^3}{3!} \cdot \frac{5 \text{DX}(5,3; (1,4)) \text{DX}(3,3; (1,4))}{\text{DX}(8,6; (1,4))} \right\} \]

\[ = 0.6066 \left( \frac{0.4998)^2}{2!} \cdot \frac{56 \cdot 30 \cdot 6}{39480} + \frac{(0.4998)^3}{3!} \cdot \frac{56 \cdot 150 \cdot 6}{191520} \right) \]

\[ = 0.02267. \]

**EXAMPLE 2.** (Applications in the NBA Basketball World Championship Series)

Suppose the number of games played by any two teams in the World Championship Series can be treated as a doubly truncated Poisson variate with truncated parameters \(N, MD = (4,7)\). Then the total number of games played in every year’s World Championship Series has a D distribution.

The following data are the number of games played by the Boston Celtics in the World Championship Series, 1983-1987 (Sachare and Sloan, 1988).

<table>
<thead>
<tr>
<th>Year</th>
<th>Semi Final 1</th>
<th>Final 2</th>
<th>Championship 3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1983-1984</td>
<td>7</td>
<td>5</td>
<td>7</td>
<td>19</td>
</tr>
<tr>
<td>1984-1985</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>17</td>
</tr>
<tr>
<td>1985-1986</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>15</td>
</tr>
<tr>
<td>1986-1987</td>
<td>7</td>
<td>7</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>(\Sigma)</td>
<td></td>
<td></td>
<td></td>
<td>71</td>
</tr>
</tbody>
</table>
Now we estimate the distribution of the total number of games played in every year's World Championship Series.

To illustrate, by using (5.32), the probability of playing 12 games in the series with \( z = 71, n = 3, m = 4 \) is given by

\[
\hat{p}_{12}(71, 4) = \frac{\binom{71}{12} \cdot D(12, 3; (4, 7)) D(59, 9; (4, 7))}{D(71, 12; (4, 7))} \\
= \frac{\binom{71}{12} \cdot 34650 \cdot 6.543182147 \cdot 10^{52}}{0.7786068892 \cdot 10^{24}} \\
= 0.000372 .
\]

The following table gives estimates of the p.f. based on 1983-1987 data. The probabilities sum to more than 1 due to round off error.

<table>
<thead>
<tr>
<th>Total No. of games Played in a year</th>
<th>( \hat{p}_x(z, m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.000372</td>
</tr>
<tr>
<td>13</td>
<td>0.003551</td>
</tr>
<tr>
<td>14</td>
<td>0.017728</td>
</tr>
<tr>
<td>15</td>
<td>0.059842</td>
</tr>
<tr>
<td>16</td>
<td>0.132773</td>
</tr>
<tr>
<td>17</td>
<td>0.211860</td>
</tr>
<tr>
<td>18</td>
<td>0.245569*</td>
</tr>
<tr>
<td>19</td>
<td>0.191196</td>
</tr>
<tr>
<td>20</td>
<td>0.104683</td>
</tr>
<tr>
<td>21</td>
<td>0.032531</td>
</tr>
<tr>
<td>( \Sigma )</td>
<td>1.000106</td>
</tr>
</tbody>
</table>

* is the highest probability
These data reveal that it is most likely that Boston has to play 18 games in the World Championship Series, if Boston wins the eastern championship.

5.9 TABLES OF THE P.F. OF THE D DISTRIBUTION

Using Theorems 2 and 3 (formulas (5.5) and (5.6)), we can now tabulate the p.f. of the D distribution (see Appendix A.5, Table VIII). As the results of the above properties are independent of D numbers which can get very large in value, the calculations now are much more accurate and efficient. This alternative approach of tabulating the p.f. of the D distribution is therefore recommended.
Chapter 6

THE R DISTRIBUTION

6.1 INTRODUCTION

In this chapter, we introduce the R distribution. Its p.f. can be expressed in terms of an R number and an incomplete exponential function which are defined in Chapter 3 and Chapter 1, respectively. The Poisson distribution and the right truncated Poisson distribution are all special cases of this R distribution. We prove the R distribution as the distribution of the sum of n independent but not identical right truncated Poisson variables. The R distribution can also be treated as a special case of the D distribution which was introduced in Chapter 5.

We derive some properties of the R distribution, such as recurrence relations, modal properties, etc. They are based on the properties of the R numbers. These cannot be derived from the general properties of the truncated power series distribution.

In this chapter a MVU estimate of the p.f. of the R distribution is given. It can easily be calculated from tables of the incomplete exponential function (Appendix A.4, Table VII) and R numbers (Appendix A.2, Table III, IV). We
also provide the computational method for the p.f. of the R distribution without R numbers. Tables of the p.f. of the R distribution are included.

Finally, several examples are included at the end to illustrate the use of the R distribution.

6.2 R DISTRIBUTION

A random variable $X$ is said to have a right truncated Poisson distribution (RTPD) with parameters $\lambda$, $M$, if

$$P(X=k) = \left( \sum_{j=0}^{M-1} \frac{\lambda^j}{j!} \right) \frac{\lambda^k}{k!}, \quad \lambda > 0, \; M \text{ is an integer} \quad (6.1)$$

where $k = 0, 1, \ldots, M$.

**DEFINITION 1.** A random variable $X$ is said to have a R distribution $(\text{R}(n; M, \Lambda))$, if its p.f. is

$$p(x, n; M, \Lambda) = P(X=x) = \prod_{i=1}^{n} e_i(M_i, \lambda_i)^x \text{GR}(x, n; M, \Lambda)/x! \quad (6.2)$$

where $x = 0, 1, \ldots, \sum_{i=1}^{n} M_i$.

$\text{GR}(x, n; M, \Lambda)$ is the R number (see Chapter 3, Section 3.2), and

$e_i(M_i, \lambda_i)$ is the incomplete exponential function.

(see Chapter 1, Section 1.4)

**LEMMA 1.** If $Y_i \sim \text{RTPD}(M_i, \lambda_i)$ are independent, $i = 1, \ldots, n$,

then the probability generating function of $X = \sum_{i=1}^{n} Y_i$ is
\[ \psi_{x}(s) = \prod_{i=1}^{n} \frac{e_{i}(M_{i}, s\lambda_{i})}{e_{i}(M_{i}, \lambda_{i})}, \quad 0 < s < 1. \] (6.3)

**Proof.**

For \( 0 < s < 1 \),

\[ \psi_{x}(s) = \sum_{x} P(X=x) s^{x} = E(s^{X}) = \prod_{i=1}^{n} E(s^{Y_{i}}) \]

\[ = \prod_{i=1}^{n} \sum_{y_{i}=0}^{M_{i}} P(Y_{i}=y_{i}) s^{y_{i}} \]

\[ = \prod_{i=1}^{n} e_{i}(M_{i}, \lambda_{i}) \left( \sum_{y_{i}=0}^{M_{i}} \frac{\lambda_{i}^{y_{i}}}{y_{i}!} s^{y_{i}} \right) \]

\[ = \prod_{i=1}^{n} e_{i}(M_{i}, s\lambda_{i}) \left( \sum_{y_{i}=0}^{M_{i}} \frac{(s\lambda_{i})^{y_{i}}}{y_{i}!} \right) \]

\[ = \prod_{i=1}^{n} e_{i}(M_{i}, s\lambda_{i}). \]

**Theorem 1.** If \( Y_{i} \sim \text{RTP}(M_{i}, \lambda_{i}) \) are independent, \( i = 1, \ldots, n \), then \( X = \sum_{i=1}^{n} Y_{i} \sim \text{RD}(n; \pi, \Lambda) \) with p.f. (6.2).

**Proof.**

From the probability generating function of \( X \) given in Lemma 1, we can get \( P(X=x) \) by collecting the coefficients of \( s^{x} \) such that \( \sum_{i=1}^{n} y_{i} = x \).
COROLLARY 1. If $M_i \equiv M$, $\lambda_i \equiv \lambda$ in Theorem 1 (i.e., $Y_i$'s are i.i.d.), then the p.f. of $X$ reduces to

$$p(x, n; M, \lambda) = e_1(M; \lambda)^{-n} R_1(x, n; M; \lambda)^{x/x!}$$

(6.4)

where $x = 0, 1, \ldots, nM$.

COROLLARY 2. Special cases:

(i) $RD(X; \alpha, \lambda)$ is the ordinary Poisson distribution,
(ii) $RD(X; M, \lambda)$ is the right truncated Poisson distribution,
(iii) $DD(X; (0, M), \lambda)$ is the $R$ distribution.

6.3 Recurrence Relations of P.F.

The properties of the $R$ distribution depend on the individual properties of the $R$ numbers and incomplete exponential function. The properties of the $R$-numbers are all given in Chapter 3 (see also Huang and Fung, 1989b) and the readers may refer to them.

But interestingly, the following results in this section for the $R$ distribution are independent of the $R$ numbers. Since the $R$ numbers become very large with increasing arguments, results of this section provide a more convenient alternative in the calculations of the p.f. of the $R$ distribution.

Let $r(x, n; M) = GR(x, n; M, \lambda)^{x/x!}$. Using the recurrence relation for $r$-numbers, we obtain

THEOREM 2. (Recurrence Relations for the P.F. of $RD(X; M, \lambda)$

If a random variable $X \sim RD(X; M, \lambda)$, then the p.f. $p(x, n; M, \lambda)$ in (6.2) satisfies the recurrence relation
\[ p(x+1, n; \mathbb{M}, \lambda) = \frac{n!}{(x+1)!} p(x, n; \mathbb{M}, \lambda) \]

\[ = \sum_{i=1}^{n} \left( \frac{\lambda^{x-M_i}}{(x+1)(x-M_i)!} \right) \left( \mathbb{M}_i \right) \left( \mathbb{M}_i \right) p(x-M_i, n-1; \mathbb{M}_i, \lambda) \]  \( (6.5) \)

where
\[ \mathbb{M}_i = \{ M_j | j \neq i \}; \]
\[ x = 0, 1, \ldots, \sum_{i=1}^{n} M_i - 1. \]

**Proof.**

Consider
\[ p(x-M_i, n-1; \mathbb{M}_i, \lambda) = \prod_{j \neq i}^{n} \left( \frac{\lambda^{x-M_i}}{(x-M_i)!} \right) r(x-M_i, n-1; \mathbb{M}_i, \lambda), \]

and
\[ p(x, n; \mathbb{M}, \lambda) = \prod_{i=1}^{n} \left( \frac{\lambda^{x}}{x!} \right) r(x, n; \mathbb{M}, \lambda). \]

Using (3.14)
\[ r(x+1, n; \mathbb{M}) = nr(x, n; \mathbb{M}) - \sum_{i=1}^{n} \left( \binom{x}{M_i} r(x-M_i, n-1; \mathbb{M}_i) \right), \]

where
\[ \mathbb{M}_i = \{ M_j | j \neq i \}; \quad x = 0, 1, \ldots, \sum_{i=1}^{n} M_i - 1. \]

We have
\[ p(x+1, n; \mathbb{M}, \lambda) = \prod_{i=1}^{n} \left( \frac{\lambda^{x+1}}{(x+1)!} \right) r(x+1, n; \mathbb{M}, \lambda). \]
\[
\prod_{i=1}^{n} \binom{M}{i}^{-1} \left( n \gamma(x, n; M) - \sum_{i=1}^{n} \binom{x}{M_i} r(x-M_i, n-1; \pi_i) \right) \frac{\lambda^{x+1}}{(x+1)!} 
\]

\[
\prod_{i=1}^{n} \binom{M}{i}^{-1} n \gamma(x, n; M) \frac{\lambda^{x+1}}{(x+1)!} 
\]

\[
- \sum_{i=1}^{n} \left( \binom{x}{M_i} \prod_{j=1}^{n} \binom{M_j}{i}^{-1} r(x-M_i, n-1; \pi_i) \frac{\lambda^{x+1}}{(x+1)!} \right) 
\]

\[
= \frac{n \lambda}{(x+1)} p(x, n; M, \lambda) 
\]

\[
- \sum_{i=1}^{n} \left( \frac{x!}{M_i! (x-M_i)!} \frac{p(x-M_i, n-1; \pi_i, \lambda)}{e_i(M_i, \lambda)} \frac{r(x-M_i, n-1; \pi_i, \lambda)}{\lambda^{x+1}} \right) 
\]

\[
= \frac{n \lambda}{(x+1)} p(x, n; M, \lambda) - \sum_{i=1}^{n} \left( \frac{\lambda^{M+1}}{(x+1) e_i(M, \lambda)} p(x-M_i, n-1; \pi_i, \lambda) \right) 
\]

Similarly, using the recurrence relation of \( R_i \) numbers, we obtain

**Theorem 3.**

If a random variable \( X \sim RD(n; M, \lambda) \), then the p.f. \( p(x, n; M, \lambda) \) in (6.4) satisfies the recurrence relation

\[
p(x+1, n; M, \lambda) = \frac{n}{(x+1)} \left( \lambda p(x, n; M, \lambda) 
\right.
\]

\[
- \frac{\lambda^{M+1}}{M! e_i(M, \lambda)} p(x-M, n-1; M, \lambda) \right) 
\]

(6.6)

where \( x = 0, 1, \ldots, nM-1 \).
PROOF.

We use the recurrence relation for $R_i$-numbers (3.15)

$$R_i(x+1, n; MD) = nR_i(x, n; MD) - \binom{x}{M} R_i(x-M, n-1; MD)$$

where $x = 0, 1, \ldots, nM-1$.

Consider

$$p(x, n; M, \lambda) = e_i(M, \lambda)^{-n} R_i(x, n; MD) \frac{\lambda^x}{x!} \quad (i)$$

$$p(x-M, n-1; M, \lambda) = e_i(M, \lambda)^{-(n-1)} R_i(x-M, n-1; MD) \frac{\lambda^{x-M}}{(x-M)!} \quad (ii)$$

If we substitute (i) and (ii) into the right hand side of (6.6), and we get

$$e_i(M, \lambda)^{-n} R_i(x+1, n; MD) \frac{\lambda^{x+1}}{(x+1)!}$$

$$= p(x+1, n; M, \lambda) = \text{left hand side.} \quad \blacksquare$$

THEOREM 4.

If a random variable $X \sim RD(n; M, \lambda)$, then the p.f. $p(x, n; M, \lambda)$ in (6.4) for the special cases of $x = nM$ and $nM-1$ satisfies

\begin{align*}
(i) & \quad p(nM, n; M, \lambda) = \frac{\lambda^n}{n^M} p(nM-1, n; M, \lambda), \quad n = 1, 2, \ldots \quad (6.7) \\
(ii) & \quad p(nM-1, n; M, \lambda) = \frac{n}{n-1} \frac{\lambda^M}{M! e_i(M, \lambda)} p((n-1)M-1, n-1; M, \lambda), \\
& \quad n = 2, 3, \ldots \quad (6.8)
\end{align*}
PROOF.

(i) \( p(nM, n; M, \lambda) = e_i (M, \lambda)^{-n} R_i(nM, n; MD) \frac{\lambda^{nM}}{nM!} \),

\( p(nM, n; M, \lambda) = e_i (M, \lambda)^{-n} R_i(nM, n; MD) \frac{\lambda^{nM}}{nM!} \).

Using property (3.17) for \( R_i \) numbers, it is easy to get

\( p(nM, n; M, \lambda) = \frac{\lambda}{nM} p(nM-1, n; M, \lambda) \).

(ii) \( p((n-1)M-1, n; M, \lambda) \)

\[ = e_i (M, \lambda)^{-n} R_i((n-1)M-1, n; MD) \frac{\lambda^{nM-1}}{(nM-1)!} \]

\[ = e_i (M, \lambda)^{-n} \frac{n}{n-1} \frac{(nM-1)!}{M!(nM-1-M)!} R_i((n-1)M-1, n-1; MD) \frac{\lambda^{nM-1}}{(nM-1)!} \]

(by property (3.18) for \( R_i \) numbers)

\[ = \frac{n}{n-1} \frac{\lambda^M}{M! e_i (M, \lambda)} p((n-1)M-1, n-1; M, \lambda) \].

\[ \square \]

THEOREM 5.

If a random variable \( X \sim RD(n; M, \lambda) \), then the p.f. \( p(x, n; M, \lambda) \) in (6.4) for fixed \( M \) satisfies

\[ p(x, n; M, \lambda) = \left( \frac{e_i (M+i, \lambda)}{e_i (M, \lambda)} \right)^n p(x, n; M+i, \lambda), \text{ for } n = 2, 3, \ldots \ (6.9) \]

\( x = 0, 1, \ldots, M; \quad i = 1, 2, \ldots \).

PROOF.
\[ p(x, n; M, \lambda) = e_1^{(M, \lambda)} \frac{\lambda^x}{x!} \]

\[ = e_1^{(M, \lambda)} \frac{\lambda^x}{x!} \]

(by property (3.19) for \( R_I \) numbers)

\[ = \left( \frac{e^{(M+i, \lambda)}}{e_1^{(M, \lambda)}} \right)^n p(x, n; M+i, \lambda). \]

**THEOREM 6.**

If a random variable \( X \sim RD(n; M, \lambda) \), then the p.f. \( p(x, n; M, \lambda) \) in (6.4) satisfies

(i) \( p(x, n; M-1, \lambda) = \)

\[ = \left( \frac{e_1^{(M, \lambda)}}{e_1^{(M-1, \lambda)}} \right)^n \sum_{s=0}^{B} (-1)^s \binom{n}{s} \left( \frac{\lambda^M}{M! e_1^{(M, \lambda)}} \right)^s p(x-Ms, n-s; M, \lambda), \quad (6.10) \]

where

\[ B = \left\lfloor \frac{n}{M} \right\rfloor \leq n; \quad x = 0, 1, \ldots, n(M-1). \]

(ii) \( p(x, n; M, \lambda) \)

\[ = \left( \frac{e_1^{(M-1, \lambda)}}{e_1^{(M, \lambda)}} \right)^n \sum_{s=0}^{B} \binom{n}{s} \left[ \frac{\lambda^M}{M! e_1^{(M-1, \lambda)}} \right]^s p(x-Ms, n-s; M-1, \lambda), \quad (6.11) \]

where \( x = 0, 1, \ldots, nM. \)

**PROOF.**

(i) \( p(x, n; M-1, \lambda) \)

\[ = e_1^{(M-1, \lambda)} \frac{\lambda^x}{x!} . \]
\[ e_i (M-1, \lambda)^{-n} \sum_{s=0}^{B} (-1)^s \frac{\binom{n}{s} (x)}{(M!)^s} R_1 (x-Ms, n-s; M) \frac{\lambda^x}{x!} \]

(by property (3.20) for \( R_1 \) numbers)

\[ = \left( \frac{e_i (M, \lambda)}{e_i (M-1, \lambda)} \right)^n \sum_{s=0}^{B} (-1)^s \frac{\binom{n}{s} (x)}{(M!)^s} \frac{\lambda^s}{s!} \frac{p(x-Ms, n-s; M, \lambda)}{e_i (M, \lambda)} \]

\[ = \left( \frac{e_i (M, \lambda)}{e_i (M-1, \lambda)} \right)^n \sum_{s=0}^{B} (-1)^s \binom{n}{s} \left( \frac{\lambda^s}{M! e_i (M, \lambda)} \right)^s p(x-Ms, n-s; M, \lambda) . \]

Similarly, we can use property (3.21) for \( R_1 \) numbers to prove part (ii) of the theorem. 

6.4 MODAL PROPERTIES

PROPERTY 1.

If a random variable \( X \sim \text{RD}(n; M, \lambda) \), then for each \( n = 1, 2, ... \)

(i) when \( \lambda < \frac{1}{n} \), the p.f. \( p(x, n; M, \lambda) \) has a unique mode at \( x = 0 \);

(ii) when \( \lambda = \frac{1}{n} \), the p.f. \( p(x, n; M, \lambda) \) is bimodal, the modes being at \( x = 0 \) and 1;

(iii) when \( \lambda > \frac{1}{n} \), the p.f. \( p(x, n; M, \lambda) \) increases first with \( x \) and then decreases with \( x \), except perhaps when the p.f. assumes equal values at two consecutive \( x \)'s at the time of its change from an increasing to a decreasing function.

The proof is similar to the one given in Chapter 5,
Section 5.4, Property 1.

6.5 RECURSION AND DECOMPOSITION

**THEOREM 7. (RECURSION)**

Let $T_1$ and $T_2$ be independent r.v.'s, and $T = T_1 + T_2$. Then $T \sim RD(n; \mathbb{M}, \Lambda)$ if $T_1 \sim RD(n_1; \mathbb{M}_1, \Lambda_1)$ and $T_2 \sim RD(n_2; \mathbb{M}_2, \Lambda_2)$ for $n = n_1 + n_2$, where $n_1$'s are nonnegative integers.

**PROOF.**

Suppose $T_1 \sim RD(n_1; \mathbb{M}_1, \Lambda_1)$; $T_2 \sim RD(n_2; \mathbb{M}_2, \Lambda_2)$.

Let

$$\mathbb{M} = \{M_1, \ldots, M_n\}, \quad \mathbb{M}_1 = \{M_{n_1+1}, \ldots, M_n\};$$

$$\Lambda = \{(\lambda_1, \ldots, \lambda_n)\}, \quad \Lambda_1 = \{(\lambda_{n_1+1}, \ldots, \lambda_n)\};$$

$$\mathbb{M} = \mathbb{M}_1 \cup \mathbb{M}_2; \quad \Lambda = \Lambda_1 \cup \Lambda_2;$$

$$Q_1 = \sum_{i=1}^{n_1} M_i, \quad Q_2 = \sum_{i=n_1+1}^{n} M_i.$$

Then for $0 < z < 1$, by Lemma 1 and Theorem 1, $T_1$ and $T_2$ have probability generating functions
\[
\Psi_T(z) = \prod_{i=1}^{n_1} \frac{e^{n_1 \left(M_i, z\lambda_i \right)}}{e^{\left(M_i, \lambda_i \right)}} , \text{ and } \Psi_T(z) = \prod_{i=n_1+1}^{n} \frac{e^{\left(M_i, z\lambda_i \right)}}{e^{\left(M_i, \lambda_i \right)}} ,
\]
respectively.

Since \( T_1 \) and \( T_2 \) are independent, \( T = T_1 + T_2 \) has p.g.f.

\[
\Psi_T(z) = \Psi_{T_1}(z)\Psi_{T_2}(z)
\]

\[
= \prod_{i=1}^{n} \frac{e^{\left(M_i, z\lambda_i \right)}}{e^{\left(M_i, \lambda_i \right)}} , \quad 0 < z < 1. \tag{6.12}
\]

By Lemma 1 and Theorem 1, (6.12) is the p.g.f. of the distribution with parameters \( n, \mu, \lambda \), i.e.

\( T \sim \text{RD}(n; \mu, \lambda) \). \( \blacksquare \)

**Corollary 3. (Decomposition)**

If \( X \sim \text{RD}(n; \mu, \lambda) \), then \( X \) can be expressed as the sum of \( n \) independently distributed r.v.'s, each having \( \text{RD}(1; M_i, \lambda_i) \) (i.e., \( \text{RTPD}(M_i, \lambda_i) \)), \( i = 1, 2, \ldots n \).
6.6 MOMENTS

In this section, we proceed to find the moments of the $R$ distribution. It is interesting to see that the moments depend only on the incomplete exponential function $e_i(M, \lambda)$ and not on $R$ numbers.

**Theorem 8.** If $X \sim RD(n; M, \lambda)$, then the $r$th factorial moment is given by

$$m^{(r)} = \sum_{\sum k_i = r} \left\{ \binom{r}{k_1, k_2, \ldots, k_n} \prod_{i=1}^{n} \frac{\lambda_i^{k_i} e_{i}(M-k_i, \lambda_i)}{e_{i}(M, \lambda_i)} \right\}, \quad (6.15)$$

where

$$\min_{(i)} \langle M_i \rangle > r; \quad k_i = 0, 1, \ldots, r; \quad r = 1, 2, \ldots$$

**Proof.**

By Lemma 1, the p.g.f. of $X$ is

$$\psi_X(s) = \prod_{i=1}^{n} \frac{e_{i}(M_i, s\lambda_i)}{e_{i}(M_i, \lambda_i)}, \quad 0 < s < 1.$$ 

and, using Leibniz formula for higher order derivatives of products (see Wong, 1979, p.197-198), we get
\[ m^{(r)} = \frac{\partial^r \psi_i(s)}{\partial s^r} \bigg|_{s=1} \]

\[ = \prod_{i=1}^{n} e_i(M_i, \lambda_i)^{-1} \cdot \sum_{0 \leq k_1 \leq r} \binom{r}{k_1, \ldots, k_n} \prod_{i=1}^{n} \lambda_i e_i(M_i-k_i, s \lambda_i) \bigg|_{s=1} \]

(by property (1.18) for the IEF)

\[ = \sum_{0 \leq k \leq r} \binom{r}{k_1, k_2, \ldots, k_n} \prod_{i=1}^{n} \lambda_i e_i(M_i-k_i, \lambda_i) \cdot e_i(M_i, \lambda_i). \]

\[ \sum_{k_i = r} \]

**COROLLARY 4.**

If \( X \sim RX(n; M, \lambda) \), then

\[ \text{E}(X) = \sum_{i=1}^{n} \frac{\lambda_i e_i(M_i-1, \lambda_i)}{e_i(M_i, \lambda_i)} \quad \text{E}^{(6.16)} \]

\[ \text{Var}(X) = \sum_{i=1}^{n} \lambda_i \left( \frac{\lambda_i e_i(M_i-2, \lambda_i)}{e_i(M_i, \lambda_i)} + \frac{e_i(M_i-1, \lambda_i)}{e_i(M_i, \lambda_i)} \right) - \lambda_i \left( \frac{e_i(M_i-1, \lambda_i)}{e_i(M_i, \lambda_i)} \right)^2 \]

\[ \text{Var}^{(6.17)} \]
COROLLARY 5.

If \( X \sim RD(n; M, \lambda) \),

\[
E(X) = n\lambda \frac{e_i(M-1, \lambda)}{e_i(M, \lambda)} ;
\]

\[
\text{Var}(X) = n\lambda \left( \frac{e_i(M-2, \lambda)}{e_i(M, \lambda)} + \frac{e_i(M-1, \lambda)}{e_i(M, \lambda)} - \lambda \frac{e_i(M-1, \lambda)}{e_i(M, \lambda)} \right)^2.
\]

We use the relationship of \( m_r \) and \( m_{(r)} \) to obtain the rth moment (see Johnson and Kotz, 1969).

COROLLARY 7.

If \( X \sim RD(n; \overline{M}, \lambda) \), then the rth moment is given by

\[
m_r = \sum_{k=0}^{r} \sum_{\mathbf{k}} \frac{r!}{k_1! k_2! \cdots k_n!} \frac{n \lambda}{e_i(M, \lambda)} \prod_{i=1}^{k} \frac{e_i(M-k_i, \lambda)}{e_i(M, \lambda)} \]

where

\[
\min_{(i)} \langle M_i \rangle > r ; \quad k_i = 0,1,\ldots,r.
\]

and \( \sum_{k=0}^{r} \) is the Stirling number of the second kind (Jordan, 1965).
Using the recurrence relationship of Stirling numbers of the second kind, we can prove the following recurrence relationship of \( m_r \).

**Theorem 9.**

If \( X \sim RD(n; \pi, \lambda) \), i.e., \( \lambda_i \equiv \lambda \), \( i = 1, 2, \ldots, n \), then

\[
m_r = \lambda \frac{d m_{r-1}}{d\lambda} + m_{r-1} \lambda_i.
\]  

**Proof.**

Use the recurrence relationship of Stirling numbers of the second kind (Jordan, 1965), formula (1.41)

\[
S(r, k) = kS(r-1, k) + S(r-1, k-1).
\]  

Thus

\[
m_r = \sum_{k=0}^{r} S(r, k) \lambda^k \sum_{k_1, \ldots, k_n} \frac{n}{e_i(M_{i}, \lambda) \prod_{i=1}^{n}}
\]

\[
= \sum_{k=0}^{r} kS(r-1, k) \lambda^k \sum_{k_1, \ldots, k_n} \frac{n}{e_i(M_{i}, \lambda) \prod_{i=1}^{n}}
\]

\[
+ \sum_{k=0}^{r} S(r-1, k-1) \lambda^k \sum_{k_1, \ldots, k_n} \frac{n}{e_i(M_{i}, \lambda) \prod_{i=1}^{n}}.
\]
Since
\[
m_{r-1} = \sum_{k=0}^{r-1} S(r-1, k) \lambda^k \sum_{k_1, \ldots, k_n}^{k} \prod_{i=1}^{n} \frac{e_i(M_i - k_i, \lambda)}{e_i(M_i, \lambda)},
\]
we obtain
\[
\frac{dm_{r-1}}{d\lambda} = \sum_{k=0}^{r-1} kS(r-1, k) \lambda^{k-1} \sum_{k_1, \ldots, k_n}^{k} \prod_{i=1}^{n} \frac{e_i(M_i - k_i, \lambda)}{e_i(M_i, \lambda)} + A - B,
\]
where
\[
A = \sum_{k=0}^{r-1} S(r-1, k) \lambda^k \sum_{k_1, \ldots, k_n}^{k} \prod_{i=1}^{n} \frac{e_i(M_i - k_i, \lambda)}{e_i(M_i, \lambda)} \sum_{j \neq i}^{n} e_i(M_i - k_i, \lambda) \prod_{i=1}^{n} e_i(M_i, \lambda)
\]
\[
B = \sum_{k=0}^{r-1} S(r-1, k) \lambda^k \sum_{k_1, \ldots, k_n}^{k} \prod_{i=1}^{n} e_i(M_i, \lambda)
\]
Note that \( S(r-1, r) = 0 \), and by comparing \( m_r \) and \( \frac{dm_{r-1}}{d\lambda} \), we have

\[
\frac{dm_{r-1}}{d\lambda} = \frac{1}{\lambda} m_r - \frac{1}{\lambda} m_{r-1} m_1.
\]

Hence

\[
m_r = \lambda \frac{dm_{r-1}}{d\lambda} + m_{r-1} m_1.
\]

We also notice that if all \( \lambda_i = \lambda \), the R distribution becomes a Generalized Power Series Distribution (GPSD) (Patil, 1963). The result of Theorem 9 is the same as the result of Gupta (1974). Also, we get the following recurrence relationship as in Chapter 5 (5.25), (5.26), (5.26):

(i) recurrence relation between the central moments,

\[
\mu_{r+1} = \lambda \frac{d\mu_r}{d\lambda} + r \mu_1 \mu_{r-1}.
\]

(ii) recurrence relation between the factorial moments,

\[
m_{(r+1)} \sim \lambda \frac{dm_{(r)}}{d\lambda} + m_{(r)} m_{(1)} = rm_{(r)}.
\]

(iii) recurrence relation between cumulants,
\[ K_{r+1} = \lambda \sum_{j=1}^{r} \left( \begin{array}{c} r-1 \\ j-1 \end{array} \right) m_{r-j} \frac{dK}{d\lambda} + \sum_{j=2}^{r} \left( \begin{array}{c} r-2 \\ j-2 \end{array} \right) m_{r+1-j} K_j \] (6.24)

where \( K_r \) is the \( r \)th cumulant.

6.7. ESTIMATION

In this section, we wish to find the minimum variance unbiased (MVU) estimate of the p.f. of \( RD(n; \Pi, \lambda) \). Here, we only consider \( \lambda_i = \lambda, \ i = 1, \ldots, n \).

Recall \( r(x, n; \Pi) = GR(x, n; \Pi, \lambda)^x / \lambda^x \).

Now, (6.2) can be rewritten as

\[ p(x, n; \Pi, \lambda) = \prod_{i=1}^{n} \left( \begin{array}{c} M_i \lambda_i \\ \lambda \end{array} \right) \frac{r(x, n; \Pi)}{x!} \frac{\lambda^x}{x!} \] (6.25)

where \( x = 0, 1, \ldots, \sum_{i=1}^{n} M_i \).

Let \( Z_1, Z_2, \ldots, Z_m \) be a random sample from \( RD(n; \Pi, \lambda) \) with known parameters \( n, \Pi, \) and unknown parameter \( \lambda \). Since \( p(x, n; \Pi, \lambda) \) in (6.25) is an exponential family discrete distribution, then \( Z = \sum_{i=1}^{m} Z_i \) is a complete sufficient statistic for the family of distribution (Bickel and Doksum, 1977, p. 69, p. 123, also, see Appendix B.1).
THEOREM 10.

The MVU estimate of (6.25) is

\[
\hat{p}_x(Z, m) = \frac{\binom{Z}{x} r(x, n; M^{(m-1)}_0) r(z-x; (m-1)n; M^{(m-1)}_0)}{r(z, mn; M^{(m)})}
\]  

(6.26)

where \( x \leq z \leq (m-1)Q + x \); \( x = 0, 1, ..., Q \);

\[
Q = \sum_{i=1}^{m} M_i;
\]

\( z \) is the observed value of \( Z = \sum_{i=1}^{m} Z_i \);

\( Z_i \sim \text{i.i.d. RD}(n; M, \lambda), \quad i = 1, ..., m; \)

\( M^{(m)} = m \) identical sets of \( M \).

PROOF.

Let \( \hat{p}_x(Z) \) be an unbiased estimate of \( p(x, n; M, \lambda) \) in (6.25);

Then

\[
\sum_{z=0}^{mQ} \hat{p}_x(Z) \left( \prod_{i=1}^{n} e_i(M_i, \lambda) \right)^{-m} r(z, mn; M^{(m)}) \frac{\lambda^z}{z!}
\]

\[
= \prod_{i=1}^{n} \left[ e_i(M_i, \lambda) \right]^{-i} r(x, n; M) \frac{\lambda^x}{x!},
\]

where \( x = 0, 1, ..., Q \).

Hence

\[
\sum_{z=0}^{mQ} \hat{p}_x(Z) \ r(z, mn; M^{(m)}) \frac{\lambda^z}{z!}
\]
\[
\frac{1}{n! \prod_{i=1}^{n} \lambda_i^{x_i}} r(x, n; \prod_{i=1}^{n} \lambda_i^{x_i}) 
= \frac{\lambda^x}{x!} r(x, n; \prod_{i=1}^{n} \sum_{k=0}^{(m-1)a} \frac{\lambda^k}{k!} r(k, (m-1)n; \prod_{i=1}^{n} \lambda_i^{x_i}) 
= \frac{\lambda^x}{x!} r(x, n; \prod_{i=1}^{n} \sum_{k=0}^{mQ} \frac{\lambda^{k-Q}}{(k-Q)!} r(k-Q, (m-1)n; \prod_{i=1}^{n} \lambda_i^{x_i}) 
\]

Comparing coefficients of \(\lambda^z\) on both sides with

\[z = x + k - Q, \quad k-Q = z-x, \quad \text{and} \]

\[0 \leq z \leq mQ; \quad Q \leq z-x \leq mQ; \quad x \leq z \leq (m-1)Q + x,
\]

we obtain

\[
\hat{p}_x(Z) \frac{1}{z!} r(z, mn; \prod_{i=1}^{m}) \]

\[
= \frac{1}{x!} r(x, n; \prod_{i=1}^{n} \frac{1}{(z-x)!} r(z-x, (m-1)n; \prod_{i=1}^{n} \lambda_i^{x_i})
\]

By the Lehmann-Scheffe Theorem (Bickel and Doksum, 1977, p.122, also, see Appendix B.1), we see that (6.26) is the MVU estimate of (6.25).

**COROLLARY 7.**

The MVU estimate of \(p(x, n; M, \lambda)\) (6.4) is

\[
\hat{p}_x(Z, m) = \frac{\left(\sum_{x} R_i(x, n; \prod_{i=1}^{n} \lambda_i^{x_i}) R_i(z-x, (m-1)n; \prod_{i=1}^{n} \lambda_i^{x_i})\right)}{R_i(z, mn; \prod_{i=1}^{n} \lambda_i^{x_i})}
\]

(6.27)

where \(x \leq z \leq (m-1)nM + x; \quad x = 0, 1, \ldots, nM;\)
\( z \) is the observed value of \( Z = \sum_{i=1}^{m} Z_i \),

\( Z_i \sim \text{i.i.d. } RD(n; \lambda) \), \( i = 1, \ldots, m \);

The estimates (6.26) and (6.27) are easily obtainable using of tables of \( R_i(x, n; (N, M)) \) and \( r(x, n; \Pi) \) (see Appendix A.2, Tables III and IV).

**Theorem 11.**

The MVU estimate of the variance of \( \hat{p}_x(Z, m) \) in (6.26) is

\[
\text{Var}(\hat{p}_x(Z, m)) = \left( \frac{R_i(x, n; \Pi) r(z-x; (m-1)n; \Pi^{(m-1)})}{r(z, mn; \Pi^{(m)})} \right)^2 - \left( \frac{Z}{x} \right)^2 \frac{r(x, n; \Pi)}{r(z, mn; \Pi^{(m)})}^2 \frac{r(z-2x; (m-2)n; \Pi^{(m-2)})}{r(z, mn; \Pi^{(m)})}.
\]  

(6.28)

**Proof.**

Since (6.25) is a p.f. of the generalized power series distribution, we can apply results of GPSD given by Patil (1963). The MVU estimate of the variance is

\[
\text{Var}(\hat{p}_x(z, m)) = \left( \hat{p}_x(z, m) - \hat{p}_x(z-x, m-1) \right) \hat{p}_x(z, m)
\]

\[= (\hat{p}_x(z, m))^2 - \hat{p}_x(z-x, m-1) \hat{p}_x(z, m). \]

By substituting (6.25) into the above formula, we obtain
\[
\text{Var}(\hat{p}_x(Z, m)) = \left( \frac{\left( \frac{z}{x} \right) r(x, n; M) r(z-x; (m-1) n; \mathbb{M}^{(m-1)})}{r(z, mn; \mathbb{M}^{(m)})} \right)^2
\]

\[
- \frac{\left( \frac{z}{x} \right) r(x, n; M) r(z-x; (m-1) n; \mathbb{M}^{(m-1)})}{r(z, mn; \mathbb{M}^{(m)})}
\]

\[
\cdot \frac{\left( \frac{z-x}{x} \right) r(x, n; M) r(z-2x; (m-2) n; \mathbb{M}^{(m-2)})}{r(z-x, (m-1) n; \mathbb{M}^{(m-1)})}
\]

Then we easily get (6.28).

6.8. APPLICATIONS

**EXAMPLE 1.** (n-channel of M/M/c/c queues)

Assume that we have \(n\) independent different M/M/c/c queues in a system with arrival rate \(\lambda_i\), mean service time \(1/\mu_i\), \(c_i\) servers and maximum capacity \(c_i\) for the \(i\)th queue, \(i = 1, 2, \ldots, n\).

We know the limiting probability of an M/M/c/c queue exists for all \(\lambda, \mu < \infty\),

\[
p_k = P(k \text{ customers in queue})
\]

\[
= \begin{cases} 
\sum_{j=0}^{c} \frac{(\lambda/\mu)^j}{j!} k! \left( \frac{\lambda/\mu}{k} \right)^k, & k \leq c \\
0, & k > c
\end{cases}
\]
\[ p_c = P(\text{all } c \text{ servers are busy}) \]
\[ = \frac{(\lambda/\mu)^c/c!}{\sum_{j=0}^{c} (\lambda/\mu)^j/j!} = B(c, \frac{\lambda}{\mu}) \]

where \( B(c, \frac{\lambda}{\mu}) \) is Erlang's loss formula (see Kleinrock, 1975, Vol. I, p.106).

Now we consider the total number of customers in the system, \( N \), which is distributed as \( RDC(n; \pi, \mathcal{B}) \) where

\[ \pi = \{ c_i | i = 1, \ldots, n \}, \quad \mathcal{B} = \{ \beta_i = \lambda_i/\mu_i | i = 1, \ldots, n \}. \]

By (6.4), we have

\[ P(N=k) = \prod_{i=1}^{n} \frac{e(c_i, \beta_i)^{-1} R(k, n; \pi, \mathcal{B}) / k!}{c_i} \]

for \( k = 0, 1, \ldots, \sum_{i=1}^{n} c_i \).

(6.29)

\[ P(\text{all servers are busy}) = \prod_{i=1}^{n} \frac{e(c_i, \beta_i)^{-1} R(c, n; \pi, \mathcal{B}) / c!}{c_i} \]

\[ = \tilde{B}(c, \mathcal{B}; n), \quad (6.30) \]

where \( c = \sum_{i=1}^{n} c_i \).

We can call \( \tilde{B}(c, \mathcal{B}; n) \) the \( n \)-channel Erlang's formula.

Note that we can also use

\[ P(\text{all servers are busy}) = \prod_{i=1}^{n} P(c_i \text{ customers in queue}) \]
to get (6.30).

By (6.16) and (6.17), the expected number and variance of the number of customers in the system can be found as

\[
E(N) = \sum_{i=1}^{n} \frac{\lambda_i / \mu_i}{e_i(c_i, \lambda_i / \mu_i)} \quad (6.31)
\]

\[
\text{Var}(N) = \sum_{i=1}^{n} \left\{ \frac{(\lambda_i / \mu_i) e_i(c_i, -2, \lambda_i / \mu_i)}{e_i(c_i, \lambda_i / \mu_i)} + \frac{e_i(c_i - 1, \lambda_i / \mu_i)}{e_i(c_i, \lambda_i / \mu_i)} \right\} - \left( \frac{e_i(c_i - 1, \lambda_i / \mu_i)}{e_i(c_i, \lambda_i / \mu_i)} \right)^2 \quad (6.32)
\]

For instance, let

\[
\lambda_1 = 3, \mu_1 = 6, c_1 = 2; \quad \lambda_2 = 2, \mu_2 = 4, c_2 = 3; \quad \lambda_3 = 4, \mu_3 = 8, c_3 = 4.
\]

Using Table VII and IV, we get

\[
P(N = k) = \frac{e_i(2, 0.5) e_i(3, 0.5) e_i(4, 0.5)}{k!} r(k, 3; (2, 3, 4)) \frac{0.5^k}{k!}
\]

\[
P(2 \text{ customers in the system})
\]

\[
= (1.625 \cdot 1.64583 \cdot 1.64844)^{-1}(9) \frac{0.5^2}{2!} = 0.255177.
\]

\[
P(\text{all servers are busy})
\]

\[
= (1.625 \cdot 1.64583 \cdot 1.64844)^{-1}(1260) \frac{0.5^9}{9!} = 0.000001538.
\]

\[
E(N) = 0.5 \left( \frac{e_i(1, 0.5)}{e_i(2, 0.5)} + \frac{e_i(2, 0.5)}{e_i(3, 0.5)} + \frac{e_i(3, 0.5)}{e_i(4, 0.5)} \right) = 1.45.
\]
\[
\text{Var}(ND) = 0.5 \left[ \frac{e_i(0, 0.5)}{e_i(2, 0.5)} + \frac{e_i(1, 0.5)}{e_i(2, 0.5)} - 0.5 \left( \frac{e_i(1, 0.5)}{e_i(2, 0.5)} \right)^2 \right] \\
+ \left[ 0.5 \frac{e_i(1, 0.5)}{e_i(3, 0.5)} + \frac{e_i(2, 0.5)}{e_i(3, 0.5)} - 0.5 \left( \frac{e_i(2, 0.5)}{e_i(3, 0.5)} \right)^2 \right] \\
+ \left[ 0.5 \frac{e_i(2, 0.5)}{e_i(4, 0.5)} + \frac{e_i(3, 0.5)}{e_i(4, 0.5)} - 0.5 \left( \frac{e_i(3, 0.5)}{e_i(4, 0.5)} \right)^2 \right] \\
= 1.22277.
\]

Now, assume that we do not know \( \lambda_i \) and \( \mu_i \), but assume the quotients \( \frac{\lambda_i}{\mu_i} \), \( i = 1, 2, \ldots, n \) are equal. For estimation purposes, we can take a random sample \( T_1, T_2, \ldots, T_m \). If the observed values of the total number in each system are \( t_1 = 2, \ t_2 = 4 \) for \( m = 2 \), and \( t = 2 + 4 = 6 \), then by (6.25) we get

\[
\hat{p}_x(T) = \frac{\binom{6}{x} r(x, 3; (2, 3, 4)r(6-x, (2-1)3; (2, 3, 4)^{(2-1)})}{r(6, (2)(3); (2, 3, 4)^{(2)})}.
\]

If \( x = 2 \),

\[
\hat{p}_2(T) = \frac{(15)(9)(71)}{39990} = 0.239685.
\]

**EXAMPLE 2.**

Four 9-seat mini-van buses are scheduled to arrive simultaneously at a checkpoint where all passengers have to be checked. Assume that the number of passengers in each bus is distributed as a right truncated Poisson \( \text{RTPDM}(\lambda) \),
and that they are independent from one bus to another. The person in charge at the checkpoint would be interested in the number of people passing through and not so much in the number of people from each bus. The total number of people passing through, \( N \), is distributed as an \( RD(4; M, \lambda) \).

Suppose \( \lambda=5 \), and \( M=9 \). Using (6.4), Table III and the values of \( R_i(N,4;9) \), we get

\[
P(N=16) = P(16 \text{ people are passing through})
\]

\[
= (e(9,5))^{-4} R_i(16, 4; 9) \left( \frac{5^{16}}{16!} \right)
\]

\[
= (143.68946)^{-4}(4266715596)\left( \frac{5^{16}}{16!} \right)
\]

\[
= 0.072995.
\]

The expected value of the number passing through (6.18) is

\[
E(N) = (4)(5) \frac{e_i(8,5)}{e_i(9,5)} = (20) \frac{138.30717}{143.68946} = 19.25084,
\]

and the standard deviation of the number passing through (6.19) is

\[
S.D.(N) = \left[ (4)(5) \left( \frac{e_i(7,5)}{e_i(9,5)} + \frac{e_i(8,5)}{e_i(9,5)} - (5) \left( \frac{e_i(8,5)}{e_i(9,5)} \right)^2 \right) \right]^{1/2}
\]

\[
= (16.1139)^{1/2} = 4.0142.
\]
6.9 TABLES OF THE P.F. OF R DISTRIBUTION

Using Theorems 2 and 3 (formulas (6.5) and (6.6)), we can now tabulate the p.f. of the R distribution (see Appendix A.6, Table IX, and Huang, 1990). As the results of the above properties are independent of the R numbers which can get very large in value, the calculations now are much more accurate and efficient. This alternative approach to tabulating the p.f. of the R distribution is therefore recommended.
CHAPTER 7

A MORE GENERALIZED STIRLING DISTRIBUTION
OF THE SECOND KIND

7.1 INTRODUCTION

In this chapter, we introduce a more generalized Stirling distribution of the second kind (MGSDSK). Its p.f. can be expressed in terms of a more generalized Stirling number of the second kind and an incomplete exponential function which are defined in Chapter 4 and Chapter 1. The Poisson distribution, the left truncated Poisson distribution, ITPD, the Stirling distribution of the second kind and its generalized cases (GSDSK), are all special cases of this MGSDSK distribution. We proved the MGSDSK distribution is the distribution of the sum of n independent but not identically distributed left truncated Poisson variables. MGSDSK can also be treated as a special case of the D distribution.

We derive some properties of MGSDSK distribution, like recursion and decomposition relations, modal properties, etc. They are based on the properties of the more generalized Stirling numbers of the second kind. These properties cannot be derived from the general properties of
the truncated power series distribution.

In this chapter a MVU estimate of the p.f. of the MGSDK is given. It can easily be calculated from tables of the incomplete exponential function (Appendix A.4, Table VII) and the more generalized Stirling numbers of the second kind (Appendix A.3, Tables V and VI). We also provide the computational method of the p.f. of the MGSDK without the more generalized Stirling numbers of the second kind. Tables of the p.f. of the MGSDK are included.

7.2 A MORE GENERALIZED STIRLING DISTRIBUTION OF THE SECOND KIND

A random variable $X$ is said to have a left truncated Poisson distribution (LTPD) with parameters $\lambda$, $N$, if

$$P(X=k) = \left( \sum_{j=N}^{\infty} \frac{\lambda^j}{j!} \right) \frac{\lambda^k}{k!}, \quad \lambda > 0, \quad 0 \leq N,$$  \hspace{1cm} (7.1)

where $k = N, N+1, \ldots$.

**Definition 1.** A random variable $X$ is said to have a more generalized Stirling distribution of the second kind (MGSDK$(n; \Pi, \Lambda)$), if its p.f. is

$$p(x, n; \Pi, \Lambda) = P(X=x) = \prod_{i=1}^{n} e_{2}(N_{i}, \lambda_{i})^{-1} GSC(x, n; \Pi, \Lambda) / x!$$ \hspace{1cm} (7.2)

where $x = \sum_{i=1}^{n} N_{i}, \sum_{i=1}^{n} N_{i}+1, \ldots$

$GSC(x, n; \Pi, \Lambda)$ is the more generalized Stirling number of the second kind (see Chapter 2,
Section 2.2.

\( e_2(N_i, \lambda_i) \) is the incomplete exponential function.
(see Chapter 1, Section 1.4).

**Lemma 1.** If \( Y_i \sim \text{LTPD}(N_i, \lambda_i) \) are independent, \( i = 1, \ldots, n \), then the probability generating function of \( X = \sum_{i=1}^{n} Y_i \) is

\[
\psi_X(s) = \prod_{i=1}^{n} \frac{e_2(N_i, s \lambda_i)}{e_2(N_i, \lambda_i)} , \quad 0 < s < 1. \quad (7.3)
\]

**Proof.**

For \( 0 < s < 1 \),

\[
\psi_X(s) = \sum_{x} P(X=x) s^x = E(s^X) = \prod_{i=1}^{n} E(s^{Y_i})
\]

\[
= \prod_{i=1}^{n} \sum_{y_i=N_i}^{\infty} P(Y_i = y_i) s^{y_i}
\]

\[
= \prod_{i=1}^{n} e_2(N_i, \lambda_i)^{-1} \left( \sum_{y_i=N_i}^{\infty} \frac{\lambda_i}{y_i} s^{y_i} \right)^{-1}
\]

\[
= \prod_{i=1}^{n} e_2(N_i, \lambda_i)^{-1} \left( \sum_{y_i=N_i}^{\infty} \frac{s \lambda_i}{y_i} s^{y_i} \right)
\]

\[
= \prod_{i=1}^{n} \frac{e_2(N_i, s \lambda_i)}{e_2(N_i, \lambda_i)}. \quad \blacksquare
\]

**Theorem 1.** If \( Y_i \sim \text{LTPD}(N_i; \lambda_i) \) are independent, \( i = 1, \ldots, n \), then \( X = \sum_{i=1}^{n} Y_i \sim \text{MGSDSK}(n; \mathcal{N}, \Lambda) \) with p.f. \( (7.2) \).
PROOF.
From the probability generating function of $X$ given in Lemma 1, we can get $P(X=x)$ by collecting the coefficients of $s^x$ such that $\sum_{i=1}^{n} y_i = x$. 

COROLLARY 1. If $N_i \equiv N$, $\lambda_i \equiv \lambda$ in Theorem 1 (i.e., $Y_i$'s are i.i.d.), then the p.f. of $X$ reduces to

$$p(x,n;N,\lambda) = e^x (N,\lambda)^{-n} S(x,n;N,\lambda^x/\lambda) \quad (7.4)$$

where $x = nN$, $nN+1, \ldots$

DEFINITION 2. A random variable is said to have a generalized Stirling distribution of the second kind ($GSDSK(n;N,\lambda)$), if its p.f. is (7.4).

COROLLARY 2. Special cases:
(i) $MGSDSK(1;0,\lambda)$ is the ordinary Poisson distribution.
(ii) $MGSDSK(1;N,\lambda)$ is the left truncated Poisson distribution.
(iii) $MGSDSK(n;1,\lambda)$ is the Stirling distribution of the second kind.
(iv) $MGSDSK(2;(0,N),\lambda)$ is ITPD.
(v) $DDX(n;(N,\infty),\lambda)$ is GSDSK. When all $N_i > 0$, $M_i \equiv 0$, $DDX(n;\infty,\lambda)$ is MGSDSK.

7.3 RECURRENCE RELATIONS OF P.F.

The properties of $MGSDSK$ depend on the individual properties of the MG-Stirling numbers of the second kind and
the incomplete exponential function. The properties of the MG-Stirling numbers of the second kind are all given in Chapter 4 and the readers may refer to them.

Let \( g(x, n; \mathcal{M}) = GS(x, n; \mathcal{M}, \lambda) \lambda^x \). Using the recurrence relation for \( g \)-numbers, we obtain the following theorem.

**Theorem 2. Recurrence Relation for the P.F. of MGSDSK(\( n; \mathcal{M}, \lambda \))**

If a random variable \( X \sim MGSDSK(n; \mathcal{M}, \lambda) \), then the p.f. \( p(x, n; \mathcal{M}, \lambda) \) in (7.2) satisfies the recurrence relation

\[
p(x+1, n; \mathcal{M}, \lambda) = \frac{n \lambda}{(x+1)} p(x, n; \mathcal{M}, \lambda) \\
+ \sum_{i=1}^{n} \left[ \frac{\lambda^n_{i}}{(x+1)(N_{i}-1)! \epsilon_{2}(N_{i}, \lambda) \lambda} p(x-N_{i} +1, n-1; \mathcal{M}_{(i)}, \lambda) \right]
\]

(7.5)

where

\[
\mathcal{M}_{(i)} = \{ N_{j} | j \neq i \};
\]

\[
x = \sum_{i=1}^{n} N_{i}, \quad \sum_{i=1}^{n} N_{i} +1, \ldots .
\]

**Proof.**

Consider

\[
p(x-N_{i} +1, n-1; \mathcal{M}_{(i)}, \lambda) \]

\[
= \prod_{j \neq i} \epsilon_{2}(N_{j}, \lambda)^{-1} g(x-N_{i} +1, n-1; \mathcal{M}_{(i)}) \lambda^{x-N_{i} +1} \]

and \( p(x, n; \mathcal{M}, \lambda) = \prod_{i=1}^{n} \epsilon_{2}(N_{i}, \lambda)^{-1} g(x, n; \mathcal{M}_{\lambda} \lambda^x) \).

Using (4.8)
\[ g(x+1, n; \mathcal{D}) = ng(x, n; \mathcal{D}) + \sum_{i=1}^{n} \binom{x}{N_i -1} g(x-N_i +1, n-1; \mathcal{M}_i), \]

where

\[ \mathcal{M}_i = \{ N_j | j \neq i \}; \]

\[ x = \sum_{i=1}^{n} N_i, \sum_{i=1}^{n} N_i +1, \ldots, \]

we can derive

\[ p(x+1, n; \mathcal{M}, \lambda) = \prod_{i=1}^{n} \frac{e_2(N_i, \lambda)^{-1} g(x+1, n; \mathcal{M}) \lambda^{x+1}}{(x+1)!} \]

\[ = \prod_{i=1}^{n} \frac{e_2(N_i, \lambda)^{-1} \left( ng(x, n; \mathcal{M}) + \sum_{i=1}^{n} \binom{x}{N_i -1} g(x-N_i +1, n-1; \mathcal{M}_i) \right) \lambda^{x+1}}{(x+1)!} \]

\[ = \prod_{i=1}^{n} \frac{e_2(N_i, \lambda)^{-1} \left( ng(x, n; \mathcal{M}) \lambda^{x+1} \right)}{(x+1)!} \]

\[ + \left( \sum_{i=1}^{n} \binom{x}{N_i -1} \prod_{j=1}^{n} e_2(N_j, \lambda)^{-1} g(x-N_i +1, n-1; \mathcal{M}_j) \right) \frac{\lambda^{x+1}}{(x+1)!} \]

\[ = \frac{n!}{(x+1)!} p(x, n; \mathcal{M}, \lambda) \]

\[ + \sum_{i=1}^{n} \frac{x!}{(N_i -1)! (x-N_i +1)!} \frac{p(x-N_i +1, n-1; \mathcal{M}_i), \lambda)}{e_2(N_i, \lambda)} \frac{\lambda^{x+1}}{(x+1)!} \frac{\lambda^{x+1}}{(x+1)!} \]

\[ = \frac{n!}{(x+1)!} p(x, n; \mathcal{M}, \lambda) \]

\[ + \sum_{i=1}^{n} \left( \frac{\lambda^{N_i}}{(x+1)! (N_i -1)! e_2(N_i, \lambda)} p(x-N_i +1, n-1; \mathcal{M}_i), \lambda) \right). \]
Similarly, using the recurrence relation for S numbers, we can obtain the following theorem.

**THEOREM 3. (RECURSION RELATION FOR THE P.F. OF GSĐSK(n; N, λ))**

If a random variable $X \sim \text{GSĐSK}(n; N, \lambda)$, then the p.f. $p(x, n; N, \lambda)$ in (7.4) satisfies the recurrence relation

$$p(x+1, n; N, \lambda) = \frac{n}{(x+1)} \left[ \lambda p(x, n; N, \lambda) \right. $$

$$+ \left. \frac{\lambda^N}{(N-1)! e_2(N, \lambda)} p(x-N+1, n-1; N, \lambda) \right]$$

where $x = nN, nN+1, \ldots$. \hspace{1cm} (7.6)

**PROOF.**

We use the recurrence relation of S numbers (4.9),

$$S(x+1, n; N) = nS(x, n; N) + n \binom{x}{N-1} S(x-N+1, n-1; N),$$

where $x = nN, nN+1, \ldots$.

Consider

$$p(x, n; N, \lambda) = e_2(N, \lambda)^{-n} S(x, n; N) \frac{\lambda^x}{x!}, \hspace{1cm} (i)$$

$$p(x-N+1, n-1; N, \lambda) = e_2(N, \lambda)^{-(n-1)} S(x-N+1, n-1; N) \frac{\lambda^{x-N+1}}{(x-N+1)!}. \hspace{1cm} (ii)$$

Substituting (i) and (ii) into the right side of (7.6), we get

$$e_2(N, \lambda)^{-n} S(x+1, n; N) \frac{\lambda^{x+1}}{(x+1)!}$$

$$= p(x+1, n; N, \lambda) = \text{left hand side.}$$
THEOREM 4.
If a random variable $X \sim \text{GSDSK}(n; N, \lambda)$, then the p.f. $p(x, n; N, \lambda)$ in (7.4) satisfies

(i) $p(x, n; N+1, \lambda)$

$$
= \left( \frac{e_2(N, \lambda)}{e_2(N+1, \lambda)} \right)^n \sum_{s=0}^{n} (-1)^s \binom{n}{s} \left( \frac{\lambda^N}{N! e_2(N, \lambda)} \right)^s p(x-Ns, n-s; N, \lambda),
$$

(7.7)

where $x = n(N+1), n(N+1)+1, \ldots$.

(ii) $p(x, n; N, \lambda)$

$$
= \left( \frac{e_2(N+1, \lambda)}{e_2(N, \lambda)} \right)^n \sum_{s=0}^{n} (-1)^s \binom{n}{s} \left( \frac{\lambda^N}{N! e_2(N+1, \lambda)} \right)^s p(x-Ns, n-s; N+1, \lambda),
$$

(7.8)

where $x = nN, nN+1, \ldots$.

PROOF.

(i) $p(x, n; N+1, \lambda)$

$$
= e_2(N+1, \lambda)^{-n} S(x, n; N+1) \frac{\lambda^x}{x!}
$$

$$
= e_2(N+1, \lambda)^{-n} \sum_{s=0}^{n} (-1)^s \frac{\binom{n}{s}}{s!} S(x-Ns, n; N) \frac{\lambda^x}{x!}
$$

(by property (4.10) for $S$ numbers)

$$
= \left( \frac{e_2(N, \lambda)}{e_2(N+1, \lambda)} \right)^n \sum_{s=0}^{n} (-1)^s \frac{\binom{n}{s}}{s!} \frac{(x-Ns)^{Na}}{e_2(N, \lambda)^s} \frac{\lambda^{Na}}{x^s} p(x-Ns, n-s; N, \lambda).
$$
\[ = \left( \frac{\lambda^N}{n! \cdot e^2(N+1, \lambda)} \right)^n \sum_{s=0}^{n} (-1)^s \binom{n}{s} \left( \frac{\lambda^N}{n! \cdot e^2(N, \lambda)} \right)^s p(x-Ns, n-s; N, \lambda). \]

Similarly, we can use property (4.11) for \( S \) numbers to prove part (ii) of the theorem.

### 7.4. MODAL PROPERTIES

**PROPERTY 1.**

If a random variable \( X \sim \text{GSDSK}(n; N, \lambda) \), then

1. for \( N > 0 \)
   - (i) if \( \lambda < \frac{N+1}{n} \), the p.f. \( p(x, n; N, \lambda) \) has a unique mode at \( x = nN; \)
   - (ii) if \( \lambda = \frac{N+1}{n} \), the p.f. \( p(x, n; N, \lambda) \) is bimodal, the modes being at \( x = nN \) and \( nN+1; \)
   - (iii) if \( \lambda > \frac{N+1}{n} \), the p.f. \( p(x, n; N, \lambda) \) increases first with \( x \) and then decreases with \( x \), except perhaps when the p.f. assumes equal values at two consecutive \( x \)'s the time of its change from an increasing to a decreasing function;

2. for \( N = 0 \), \( \text{GSDSK}(n; 0, \lambda) \) is Poisson(\( n\lambda \)).

The proof is similar to the one given in Chapter 5, Section 5.4, Property 1.
7.5. RECURSION AND DECOMPOSITION

THEOREM 5. (RECURSION)

Let $T_i$ and $T_2$ be independent r.v.'s, and $T = T_1 + T_2$. Then $T \sim MGSDSK(n; \mathcal{M}, \Lambda)$ if $T_i \sim MGSDSK(n_i; \mathcal{M}_i, \Lambda_i)$ and $T_2 \sim MGSDSK(n_2; \mathcal{M}_2, \Lambda_2)$ for $n = n_1 + n_2$, where $n_i$'s are nonnegative integers.

PROOF.

Suppose $T_i \sim MGSDSK(n_i; \mathcal{M}_i, \Lambda_i)$; $T_2 \sim MGSDSK(n_2; \mathcal{M}_2, \Lambda_2)$,
where

$\mathcal{M}_1 = \langle N_1, \ldots, N_{n_1} \rangle$, $\mathcal{M}_2 = \langle N_{n_1+1}, \ldots, N_n \rangle$;

$\Lambda_1 = \langle \lambda_1, \lambda_2, \ldots, \lambda_{n_1} \rangle$, $\Lambda_2 = \langle \lambda_{n_1+1}, \lambda_{n_1+2}, \ldots, \lambda_n \rangle$;

$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$; $\Lambda = \Lambda_1 \cup \Lambda_2$.

Let

$L_1 = \sum_{i=1}^{n_1} N_i$, $L_2 = \sum_{i=n_1+1}^{n} N_i$;

Then for $0 < z < 1$, by Lemma 1 and Theorem 1, $T_1$ and $T_2$ have probability generating functions.
\[
\Psi_T(z) = \prod_{i=1}^{n_1} \frac{e^{(N_i, z\lambda_i)}}{e^{(N_i, \lambda_i)}}, \quad \text{and} \quad \Psi_T(z) = \prod_{i=n_1+1}^{n} \frac{e^{(N_i, z\lambda_i)}}{e^{(N_i, \lambda_i)}}.
\]

Since \( T_1 \) and \( T_2 \) are independent, \( T = T_1 + T_2 \) has p.g.f.

\[
\Psi_T(z) = \Psi_{T_1}(z) \Psi_{T_2}(z)
\]

\[
= \prod_{i=1}^{n} \frac{e^{(N_i, z\lambda_i)}}{e^{(N_i, \lambda_i)}}, \quad 0 < z < 1. \tag{7.9}
\]

By Lemma 1, this is the p.g.f. of MGSSDK distribution with parameters \( n, \Pi, \Lambda \); hence

\[ T \sim \text{MGSSDK}(n; \Pi, \Lambda). \]

**COROLLARY 3. (DECOMPOSITION)**

If \( X \sim \text{MGSSDK}(n; \Pi, \Lambda) \), then \( X \) can be expressed as the sum of \( n \) independently distributed random variables, each having MGSSDK(1; \( N_i, \lambda_i \)) (i.e., \( \text{LTPD}(N_i, \lambda_i) \)), \( i = 1, 2, \ldots, n \).

### 7.6 MOMENTS

We proceed to find the moments of the D distribution. It is an interesting thing to see that the moments depend only on the incomplete exponential function \( e^{(N, \lambda)} \), and
not on the generalized Stirling numbers of the second kind.

**Theorem 6.**

If \( X \sim \text{MGSDSK}(n; \lambda, \Lambda) \), then the \( r \)th factorial moment is given by

\[
m_{(r)} = \sum_{0 \leq k \leq r} \binom{r}{k_1, k_2, \ldots, k_n} \prod_{i=1}^{n} \frac{\lambda_i}{e^{\lambda_i} (N_i - k_i)} \right) \frac{n}{2} \mathbf{e}^{\lambda_i (N_i - k_i)} \right),
\]

where

\[
\min_{i} \langle N_i \rangle > r ; \quad k_i = 0, 1, \ldots, r; \quad r = 1, 2, \ldots
\]

**Proof.**

By Lemma 1, the p.g.f. of \( X \) is

\[
\psi_X(s) = \prod_{i=1}^{n} \frac{e^{\lambda_i (N_i + s N_i)}}{e^{\lambda_i (N_i)}}, \quad 0 < s < 1.
\]

Using Leibniz's formula for higher order derivatives of products (see Wong, 1979, p.197-198), we get

\[
m_{(r)} = \frac{\partial^r \psi_X(s)}{\partial s^r} \bigg|_{s=1}
\]
\[
= \prod_{i=1}^{n} \lambda_{i}^{-1} \cdot \sum_{\sum_{i=1}^{r} k_{i} = r} \prod_{i=1}^{n} \lambda_{i}^{-1} e_{2}(N, k_{i}, \lambda_{i}, \lambda) \bigg|_{s=1}
\]
(by property (1.18) for the IEF)

\[
= \sum_{\sum_{i=1}^{r} k_{i} = r} \prod_{i=1}^{n} \frac{\lambda_{i}^{-1} e_{2}(N, k_{i}, \lambda_{i}, \lambda)}{e_{2}(N, \lambda_{i}, \lambda)}
\]

**COROLLARY 4.**

If \( X \sim MGSDSK(n; \Pi, \Lambda) \), then

\[
EC(x) = \sum_{i=1}^{n} \frac{\lambda_{i}^{-1} e_{2}(N, -1, \lambda_{i}, \lambda_{i})}{e_{2}(N, \lambda_{i}, \lambda_{i})}
\]

\[
(7.13)
\]

\[
Var(x) = \sum_{i=1}^{n} \left( \frac{\lambda_{i}^{-1} e_{2}(N, -2, \lambda_{i}, \lambda_{i})}{e_{2}(N, \lambda_{i}, \lambda_{i})} + \frac{e_{2}(N, -1, \lambda_{i}, \lambda_{i})}{e_{2}(N, \lambda_{i}, \lambda_{i})} - \lambda_{i} \left( \frac{e_{2}(N, -1, \lambda_{i}, \lambda_{i})}{e_{2}(N, \lambda_{i}, \lambda_{i})} \right)^{2} \right).
\]

\[
(7.14)
\]

**COROLLARY 5.**

If \( X \sim GSDSK(n; N, \lambda) \),
\[ E(X) = \frac{e_2(N-1, \lambda)}{n \lambda e_2(N, \lambda)}; \]  \hspace{1cm} (7.15)

\[ \text{Var}(X) = n \lambda \left\{ \frac{e_2(N-2, \lambda)}{e_2(N, \lambda)} + \frac{e_2(N-1, \lambda)}{e_2(N, \lambda)} - \lambda \left( \frac{e_2(N-1, \lambda)}{e_2(N, \lambda)} \right)^2 \right\}. \]  \hspace{1cm} (7.16)

We use the relationship of \( m_r \) and \( m_r \) to obtain the rth moment (see Johnson and Kotz, 1969).

**COROLLARY 6.**

If \( X \sim \text{MGSDSK}(n; \Omega, \Lambda) \), then the rth moment is given by

\[ m_r = \sum_{k=0}^{r} S(r, k) m_{(k)} \]

\[ = \sum_{k=0}^{r} S(r, k) \prod_{j=1}^{k} \left( \frac{\lambda_j e_2(N-k_j, \lambda_{<j})}{e_2(N, \lambda_{<j})} \right) \]  \hspace{1cm} (7.17)

where

\[ \min_{(1)} < N > r; \quad k = 0, 1, \ldots, r, \quad \text{and} \]

\[ S(r, k) \text{ is the Stirling number of the second kind} \]

(Jordan, 1965).

**THEOREM 7.**

If \( X \sim \text{MGSDSK}(n; \Omega, \Lambda) \) (i.e., \( \lambda_i \equiv \lambda \), \( i = 1, 2, \ldots, n \)), then
\[
m_r = \frac{\frac{d m_{r-1}}{d \lambda}}{m_{r-1}} + m_{r-1} \cdot m_r.
\]

(7.18)

**Proof.**

Use the recurrence relationship of Stirling number of the second kind (Jordan, 1965), formula (1.41)

\[
S(r, k) = kS(r-1, k) + S(r-1, k-1).
\]

Thus

\[
m_r = \sum_{k=0}^{r} S(r, k) \lambda^k \sum_{k_1, \ldots, k_n} \left( \prod_{i=1}^{k} \frac{e_{\lambda}(N-k_i, \lambda)}{e_{\lambda}(N, \lambda)} \right)
\]

\[
= \sum_{k=0}^{r} kS(r-1, k) \lambda^k \sum_{k_1, \ldots, k_n} \left( \prod_{i=1}^{k} \frac{e_{\lambda}(N-k_i, \lambda)}{e_{\lambda}(N, \lambda)} \right)
\]

\[
+ \sum_{k=0}^{r} S(r-1, k-1) \lambda^k \sum_{k_1, \ldots, k_n} \left( \prod_{i=1}^{k} \frac{e_{\lambda}(N-k_i, \lambda)}{e_{\lambda}(N, \lambda)} \right).
\]

Then

\[
m_{r-1} = \sum_{k=0}^{r-1} S(r-1, k) \lambda^k \sum_{k_1, \ldots, k_n} \left( \prod_{i=1}^{k} \frac{e_{\lambda}(N-k_i, \lambda)}{e_{\lambda}(N, \lambda)} \right).
\]

Hence
\[ \frac{dm_{r-1}}{d\lambda} = \sum_{k=0}^{r-1} \frac{m_{r-k-1} \lambda^{r-k} \sum_{k_1, \ldots, k_n} K \prod_{i=1}^{n} e_2(N_i, \lambda)}{\prod_{i=1}^{n} e_2(N_i, \lambda)} + A - B \]

where

\[ A = \sum_{k=0}^{r-1} S(r-1, k) \lambda^k \sum_{k_1, \ldots, k_n} k \prod_{i=1}^{n} e_2(N_i, \lambda) \prod_{j \neq i} e_2(N_j, \lambda) \]

\[ B = \sum_{k=0}^{r-1} S(r-1, k) \lambda^k \sum_{k_1, \ldots, k_n} K \prod_{i=1}^{n} e_2(N_i, \lambda) \prod_{i=1}^{n} e_2(N_i, \lambda) \]

Note \( S(r-1, r) = 0 \), and by comparing \( m_r \) and \( \frac{dm_{r-1}}{d\lambda} \), we have

\[ \frac{dm_{r-1}}{d\lambda} = \frac{1}{\lambda} m_r - \frac{1}{\lambda} m_{r-1} m \]

Hence

\[ m_r = \lambda \frac{dm_{r-1}}{d\lambda} + m_{r-1} m \]

We also notice that if all \( \lambda_i = \lambda \), the D distribution becomes a Generalized Power Series distribution (GPSD) (Patil, 1963). The result of Theorem 7 is the same as the result of Gupta (1974). We also get the following recurrence relationships:

(i) recurrence relation between the central moments,
\[ \mu_{r+1} = \lambda \frac{d\mu_r}{d\lambda} + r \mu_r \mu_{r-1} \]  
(7.19)

(ii) recurrence relation between the factorial moments,

\[ m_r^{(r+1)} = \lambda \frac{d m_r^{(r)}}{d\lambda} + m_r^{(r)} m_{r-1}^{(1)} - r m_r^{(r)} \]  
(7.20)

(iii) recurrence relation between cumulants,

\[ K_{r+1} = \lambda \sum_{j=1}^{r} \binom{r-1}{j-1} m_{r-j} \frac{dK_j}{d\lambda} + \sum_{j=2}^{r} \binom{r-2}{j-2} m_{r-1-j} K_j \]  
(7.21)

where \( K_r \) is the \( r \)-th cumulant.

7.7. ESTIMATION

In this section, we wish to find the minimum variance unbiased (MVU) estimate of the p.f. of MGSDSK\( (n; \Pi, \lambda) \). Here, we only consider \( \lambda_i = \lambda, \ i = 1, \ldots, n \).

Recall \( g(x, n; \Pi) = GSK(x, n; \Pi, \lambda) / \lambda^x \).  
(7.22)

Now, (7.2) can be rewritten as

\[ p(x, n; \Pi, \lambda) = \prod_{i=1}^{n} e \left( \frac{N_i \lambda}{2} \right) - g(x, n; \Pi, \lambda) \frac{\lambda^x}{x!} \]  
(7.23)

where \( x = \sum_{i=1}^{n} N_i, \sum_{i=1}^{n} N_i + 1, \ldots. \)

Let \( Z_1, Z_2, \ldots, Z_m \) be a random sample from MGSDSK\( (n; \Pi, \lambda) \) with known parameters \( n, \Pi, \) unknown parameter \( \lambda \). Since \( p(x, n; \Pi, \lambda) \) in (7.23) is an exponential family discrete distribution, then \( Z = \sum_{i=1}^{m} Z_i \) is a complete sufficient statistic for the family of distribution (Bickel and Doksum,
THEOREM 8.

The MVU estimate of (7.23) is

\[ \hat{p}_x(z, m) = \frac{\binom{z}{x} g(x, n; N) g(z-x; (m-1)n; \mathcal{G}^{(m-1)})}{g(z, mn; \mathcal{G}^{(m)})} \]  

(7.24)

where \( z \geq (m-1)L + x; \quad x = L, L+1, \ldots; \quad L = \sum_{i=1}^{m} N_i; \)

\( z \) is the observed value of \( Z = \sum_{i=1}^{m} Z_i, \)

\( Z_i \sim \text{i.i.d. MGSDSK}(n, \mathcal{G}, \lambda), \quad i = 1, \ldots, m; \)

\( \mathcal{G}^{(m)} = m \) identical sets of \( \mathcal{G}. \)

PROOF.

Let \( \hat{p}_x(Z) \) be an unbiased estimate of \( p(x, n; \mathcal{G}, \Lambda) \) in (7.23).

Then

\[ \sum_{z=mL}^{\infty} \hat{p}_x(Z) \left( \prod_{i=1}^{n} e_{(N_i, \lambda)} \right)^{-m} g(z, mn; \mathcal{G}^{(m)}) \frac{\lambda^z}{z!} \]

\[ = \prod_{i=1}^{n} \left( e_{N_i, \lambda} \right)^{-1} \hat{g}(x, n; N) \frac{\lambda^x}{x!}, \]

where \( x = L, L+1, \ldots. \)

Hence

\[ \sum_{z=mL}^{\infty} \hat{p}_x(Z) g(z, mn; \mathcal{G}^{(m)}) \frac{\lambda^z}{z!} \]
\[ \begin{align*}
&= \frac{1}{\prod_{i=1}^{n} \frac{e(x_i, \lambda)}{x_i!}^{1-m}} g(x, n; \mathcal{G}^x) \\
&= \frac{\lambda^x}{x!} g(x, n; \mathcal{G}) \sum_{k=(m-1)\mathcal{L}}^{\infty} \frac{\lambda^k}{k!} g(k, (m-1)n; \mathcal{G}^{(m-1)}) \\
&= \frac{\lambda^x}{x!} g(x, n; \mathcal{G}) \sum_{k=(m-1)\mathcal{L}+x}^{\infty} \frac{\lambda^{k-x}}{(k-x)!} g(k-x, (m-1)n; \mathcal{G}^{(m-1)}) .
\end{align*} \]

Comparing coefficients of \( \lambda^z \) on both sides with \( z = x + k \), \( k = z-x \), and
\( mL \leq z \); \( (m-1)L \leq z-x \); \( (m-1)L + x \leq z \),

\[ \hat{p}_x(Z) \frac{1}{z!} g(z, mn; \mathcal{G}^{(m)}) \]

\[ \begin{align*}
&= \frac{1}{x!} g(x, n; \mathcal{G}) \frac{1}{(t-x)!} g(t-x, (m-1)n; \mathcal{G}^{(m-1)}) .
\end{align*} \]

Hence, by the Lehmann-Scheffe Theorem (Bickel and Doksum, 1977, p.122, also, see Appendix B.1), we see that (7.24) is the MVU estimate of (7.23). \( \blacksquare \)

**Corollary 7.**

The MVU estimate of \( p(x, n; N, \lambda) \) (7.4) is

\[ \hat{p}_x(Z, m) = \frac{\binom{Z}{x} S(x, n; \mathcal{G}) S(z-x; (m-1)n; \mathcal{G})}{S(z, mn; \mathcal{G})} \]  

(7.25)

where \( z \geq (m-1)nN + x \); \( x = nN, nN+1, \ldots \);

\( z \) is the observed value of \( Z = \sum_{i=1}^{m} Z_i \),

\( Z_i \sim \text{i.i.d. MGS} \mathcal{G} \text{SK}(n; \mathcal{G}, \lambda), \quad i = 1, \ldots, m . \)
The estimates (7.24) and (7.25) are easily obtainable with the availability of tables of \( S(x,n; N) \) and \( g(x,n; N) \) (see Appendix A.3, Tables V and VI).

**Theorem 9.**

The MVU estimate of the variance of \( \hat{p}_x(Z,m) \) in (7.24) is

\[
\text{Var}(\hat{p}_x(Z,m)) = \left( \frac{\binom{Z}{x} g(x,n; N) g(z-x; (m-1)n; N^{(m-1)})}{g(z, mn; N^{(m)})} \right)^2 \]

\[
- \frac{\binom{Z}{x} g(x,n; N) \binom{z-x}{x} g(z-2x; (m-2)n; N^{(m-2)})}{g(z, mn; N^{(m)})} \quad (7.26)
\]

**Proof.**

Since (7.23) is a p.f. of the generalized power series distribution, we can apply the results of GPSD given by Patil (1963). The MVU estimate of the variance is

\[
\text{Var}(\hat{p}_x(Z,m)) = \left[ (\hat{p}_x(Z,m)) - \hat{p}_x(z-x, m-1) \right] \hat{p}_x(Z,m)
\]

\[
= \left( \hat{p}_x(Z,m) \right)^2 - \hat{p}_x(z-x, m-1) \hat{p}_x(Z,m).
\]

If we substitute (7.24) into the above formula, we obtain
\[
\text{Var}(p_x(Z, m)) = \left( \frac{\binom{z-x}{x} g(x, n; TD g(z-x; (m-1)n; \mathcal{N}^{(m-1)})}{g(z, mn; \mathcal{N}^{(m)})} \right)^2 - \left( \frac{\binom{z-x}{x} g(x, n; TD g(z-2x; (m-2)n; \mathcal{N}^{(m-2)})}{g(z-x, (m-1)n; \mathcal{N}^{(m-1)})} \right).
\]

With this, we can easily get (7.26). 

7.8 TABLES OF THE P.F. OF MG-STIRLING DISTRIBUTION

Using Theorems 2 and 3 (formulas (7.5) and (7.6)), we can now tabulate the p.f. of the MG-Stirling distribution (See Appendix A.7, Table X). As the properties given in the above results do not involve MG-Stirling numbers which can get very large in value, the calculations now are much more accurate and efficient. This alternative approach of tabulating the p.f. of the MG-Stirling distribution is therefore recommended.
Chapter 8

INTERVENED TRUNCATED POISSON DISTRIBUTION

8.1 INTRODUCTION

Shanmugam (1985) proposed an intervened Poisson distribution (IPD) which is derived from the sum of a zero truncated Poisson variate and an independent Poisson variate. He assumed that there is an intervention during the observational period and its effect is to change the Poisson parameter $\theta$ to $\rho\theta$. Let $X = Y + Z$, where $Y$ is the number of cases before intervention and $Z$ is the number of cases after intervention. The observational apparatus is activated only when $Y > 0$. Hence, $Y$ is assumed to be a zero truncated Poisson variate with parameter $\theta$, and $Z$ is an independent Poisson variate with parameter $\rho\theta$. Assume also that a record of $X$ is kept only, and not $Y$ and $Z$ individually. Shanmugam obtained the distribution of $X$ and its statistical properties, and gave an example to illustrate its application.

In this Chapter, we generalize Shanmugam's result and derive the distribution of $X = Y + Z$, where $Y$ and $Z$ are independent, $Z$ is an ordinary Poisson variate, and $Y$ is a
Left (right, doubly) truncated Poisson variate. $X$ is then said to have an intervened left (right, doubly) truncated Poisson distribution, respectively.

The intervened truncated Poisson distribution (ITPD) can be applied to problems in which events occur in different rates in two time intervals. An example is one in which a minimum number of customers is needed before a certain program will start. This kind of decision is usually based on cost-effective reasons. In course offerings, a minimum number of students have to pre-register first before the course will actually be offered. This is often the case in universities as well as in continuing education classes. Let $Y$ be the number of students that pre-register (minimum $N$), and $Z$ be the number of students that enroll in the course after the decision has been made that the course will be offered, then $X$ is the total number of students in the course and $X$ has an intervened left truncated Poisson distribution. Similarly, in organizing a tour group, a minimum number of tourists ($Y$) have to join by a certain date, otherwise, the tour will be cancelled. $X$ is the total number of tourists in the group that the organizer is ultimately interested in and it is composed of $Y$ and $Z$, the number of tourists that join before and after the decision date.

Another type of application may involve an agency providing service to two kinds of customers. For example: let $Y$ be the number of students living in the dormitory that eats in the dining room for a particular meal (maximum $M$), and $Z$ be the number of visitors that eat there. Then the kitchen has to prepare a total of $X$ meals. If $Y$ is
assumed to be right truncated Poisson and $Z$ is Poisson, then $X$ is intervened right truncated Poisson.

If a store gives discounts to customers who buy more than a certain quantity $(N)$ of an item but limits them to a maximum number $(M)$, then we can assume that the number of discounted items a customer buys is doubly truncated Poisson. The total number of items a customer buys, $X$, includes those that are discounted $(Y)$ and those that are regular-priced $(Z)$ which has no restrictions at all. In this case, we say that $X$ is an intervened doubly truncated Poisson.

For each of the cases above, the probability distribution function, probability generating function and the rth moment are derived. Parameters are estimated by the method of moments and maximum likelihood. To avoid repetitions, only the case of the intervened doubly truncated Poisson distribution will be given in detail here and the other two will be treated as special cases of it. Some numerical examples are included at the end.

Also, ITPD is a special case of the D distribution, Here, we only illustrate certain properties of ITPD. Other general properties are omitted.

8.2 INTERVENED DOUBLY TRUNCATED POISSON DISTRIBUTION

(IDTPD)

Consider a random variable $X = Y + Z$, where $Y$ is a doubly truncated Poisson random variable with p.f.

$$P(Y=k) = e(N,M;\theta)^{-1}\theta^k \frac{\theta^k}{k!}, \quad (8.1)$$

where
\( k = N, N+1, \ldots, M; \quad \theta > 0; \quad N \) is a nonnegative integer.

and \( Z \) is a Poisson random variable with mean \( \rho \theta \), \( Y \) and \( Z \) are stochastically independent. As in Shanmugam (1985), we assume that the observational apparatus has kept a record of only \( X \), and not \( Y \) and \( Z \) individually.

**Theorem 1.**

If \( Y \sim \text{DTPDC}(N, M; \theta) \), \( Z \sim \text{Poisson}(\rho \theta) \), are independent, then \( X = Y + Z \) has probability function

\[
P(X = x) = \begin{cases} 
\left( e^{\rho \theta} e(N, M; \theta) \right)^{-1} \frac{\theta^x}{x!} \left( (1+\rho)^x \sum_{y=0}^{N-1} \binom{x}{y} \rho^{x-y} - \sum_{y=M+1}^{x} \binom{x}{y} \rho^{x-y} \right) & , \quad x > M \\
\left( e^{\rho \theta} e(N, M; \theta) \right)^{-1} \frac{\theta^x}{x!} \left( (1+\rho)^x \sum_{y=0}^{N-1} \binom{x}{y} \rho^{x-y} \right) & , \quad x \leq M
\end{cases}
\]

where \( x = N, N+1, \ldots; \quad \theta > 0; \quad 0 \leq \rho < \infty \),

\( N, M \) are nonnegative integers; \( N < M \).

**Proof.**

(i) When \( x > M \),

\[
P(X = x) = \sum_{y=N}^{M} P(Y = y) P(Z = x-y | Y = y)
= \sum_{y=N}^{M} \left( e(N, M; \theta) \right)^{-1} \frac{\theta^y}{y!} \left( e^{-\rho \theta} \frac{(\rho \theta)^{x-y}}{(x-y)!} \right)
= \left( e^{\rho \theta} e(N, M; \theta) \right)^{-1} \sum_{y=N}^{M} \frac{\theta^y (\rho \theta)^{x-y}}{y! (x-y)!}
\]

(ii) When \( N \leq x \leq M \),


\[ P(X=x) = \sum_{y=N}^{x} P(Y=y)P(Z=x-y|Y=y) \]

\[ = [e^{\rho \theta} e(N, \theta)]^{-1} \sum_{y=N}^{x} \frac{\theta^y (\rho \theta)^{x-y}}{y!(x-y)!}. \]

It can be verified that the above is a probability distribution.

Definition 1.

If a random variable \( X \) has p.f. (8.2), then \( X \) has an intervened doubly truncated Poisson distribution (IDTPD).

Remark.

(1) When \( M = \infty \), IDTPD becomes an intervened left truncated Poisson distribution (ILTPD). The probability function (8.2) can be rewritten as

\[ P(X=x) = [e^{\rho \theta} e(N, \theta)]^{-1} \frac{\theta^x}{x!} \left[(1+\rho)^x - \sum_{y=0}^{N-1} \binom{x}{y} \rho^{x-y}\right] \quad (8.3) \]

where \( x = N, N+1, \ldots \); \( \theta > 0 \); \( 0 \leq \rho < \infty \).

Specifically, for \( N = 1 \),

\[ P(X=x) = [e^{\rho \theta} (e^{\theta} - 1)]^{-1} \frac{\theta^x}{x!} \left[(1+\rho)^x - \rho^x\right], \quad x = 1, 2, \ldots \quad (8.4) \]

This is the IPD proposed by Shanmugam (1985).

(2) When \( 0 = N < M < \infty \) in (8.2), IDTPD becomes the intervened right truncated Poisson distribution (IRTPD). The probability function (8.2) can be written as
\[ P(X=x) = \begin{cases} \frac{(e^{P\theta} e_1(M; \theta))^{-1} \theta^x}{x!} \left( (1+P)^x - \sum_{y=M+1}^{x} \binom{x}{y} P^{x-y} \right), & x > M \\ \frac{(e^{P\theta} e_1(M; \theta))^{-1} \theta^x}{x!} (1+P)^x, & x \leq M \end{cases} \] (8.5)

where \( x = 0, 1, \ldots; \theta > 0; 0 \leq P < \infty; \)

\( M \) is a positive integer.

### 8.3 Moments

Using the probability generating functions of \( Y \) and \( Z \), we obtain the following lemma.

**Lemma 1.**

The probability generating function of an IDTPD random variable is

\[ \Psi_X(s) = e^{P\theta(s-1)} \frac{e(N, M; s\theta)}{e(N, M; \theta)}, \quad 0 < s < 1. \] (8.6)

**Proof.**

Since \( Y \) and \( Z \) are independent, then for \( 0 < s < 1 \),

\[
\Psi_X(s) = \Psi_Y(s) \Psi_Z(s) = [e^{P\theta(s-1)} e(N, M; s\theta)] e^{P\theta(s-1)} e(N, M; s\theta) = e^{P\theta(s-1)} \frac{e(N, M; s\theta)}{e(N, M; \theta)}.
\]

Next, we proceed to find the \( r \)th moment, \( m_r \), and \( r \)th factorial moment, \( m_{(r)} \), of \( X \). By Lemma 1, we obtain

**Property 1.**

If \( X \sim \text{IDTPD}(N, M; \theta) \), then the \( r \)th factorial moment is
\[ m_{(r)} = E[X^{(r)}] \]
\[ = \Psi^{(r)}_X(1) \]
\[ = \left\{ \begin{array}{ll}
[\rho^{N-r} e(N,M;\theta)]^{-1} \sum_{i=0}^{r} \rho^{r-i} \binom{r}{i} e(N-i,M-i;\theta), & N \geq r \\
[\rho^{N-r} e(N,M;\theta)]^{-1} \sum_{i=0}^{N} \rho^{r-i} \binom{r}{i} e(N-i,M-i;\theta), & N < r \end{array} \right. \quad (8.7) \]

Then, we use the relationship between \( m_r \) and \( m_{(r)} \)
\[ m_r = \sum_{k=0}^{r} S(r,k) m_{(k)} \]
where \( S(r,k) \) is the Stirling number of the second kind (see Chapter 1, section 1.3). Then we have \( m_r \).

**PROPERTY 2.**

If \( X \sim IDTPD(N,M;\theta) \), then the \( r \)th moment is
\[ m_r = E[X^r] \]
\[ = \left\{ \begin{array}{ll}
[\rho^{N-r} e(N,M;\theta)]^{-1} \sum_{k=0}^{r} S(r,k) \theta^k \sum_{i=0}^{k} \rho^{k-i} \binom{k}{i} e(N-i,M-i;\theta), & N \geq r \\
[\rho^{N-r} e(N,M;\theta)]^{-1} \left[ \sum_{k=0}^{N} S(r,k) \theta^k \sum_{i=0}^{k} \rho^{k-i} \binom{k}{i} e(N-i,M-i;\theta) \right], & N < r \\
+ \sum_{k=N+1}^{r} S(r,k) \theta^k \sum_{i=0}^{N} \rho^{k-i} \binom{k}{i} e(N-i,M-i;\theta) \right\}, & N < r \end{array} \right. \quad (8.8) \]

**PROPERTY 3.**

If \( X \sim IDTPD(N,M;\theta) \), then the mean and variance of \( X \) are
\[ E(X) = \mu = m_1 \]
\[ = e(N,M;\theta)^{-1} \sum_{i=0}^{r} \rho e(N,M;\theta) + e(N-1,M-1;\theta) \]
\[ = e(N,M;\theta)^{-1} \left[ \rho e(N,M;\theta) + e(N-1,M-1;\theta) \right] \quad (8.9) \]
\[ \text{Var}(X) = \sigma^2 = m_{(2)} - [m_{(1)}]^2 \]
\[ = e(N, M; \theta^{-1}) e^2 \{ \rho^2 e(N, M; \theta) + 2\rho e(N-1, M-1; \theta) + e(N-2, M-2; \theta) \} + \mu - \mu^2. \]  

(8.10)

**Remark.**

From now on in this chapter, whenever the variable \( k \) is less than 0, we set \( \theta^k/k! = 0 \).

### 8.4 Estimation

**(1) Method of Moments**

In terms of estimating the parameters, we can use the method of moments. By equating the first two theoretical moments with the sample moments, we can solve for \( \theta \) and \( \rho \) by numerical techniques

\[
\begin{align*}
\tilde{\theta} \left[ \rho + 1 + \frac{\tilde{\theta}^{(N-1)} / (N-1)! - \tilde{\theta}^M / M!}{e(N, M; \tilde{\theta})} \right] &= \bar{x}, \\
\tilde{\theta}^2 \left[ (1+\rho)^2 \right. \\
&\left. + \frac{(2\rho+1)(\tilde{\theta}^{(N-1)} / (N-1)! - \tilde{\theta}^M / M!)+(\tilde{\theta}^{(N-2)} / (N-2)! - \tilde{\theta}^{(M-1)} / (M-1)!)}{e(N, M; \tilde{\theta})} \right]
\end{align*}
\]

(8.11)

\[
\begin{align*}
&= S_x^2 - \bar{x} + \bar{x}^2
\end{align*}
\]

where \( \bar{x} \) and \( S_x^2 \) are the sample mean and variance respectively.
REMARK.

(1) In the ILTPD case, the pair of equations that need to be solved for the moment estimates is

\[
\begin{align*}
\tilde{\theta} \left( \rho + 1 + \frac{\tilde{\theta}^{(N-1)}(N-1)!}{e_2(N, \tilde{\theta})} \right) &= \bar{X} \\
\theta^2 \left( (1 + \rho)^2 + \frac{(2\rho+1)\tilde{\theta}^{(N-1)}(N-1)! + \tilde{\theta}^{(N-2)}(N-2)!}{e_2(N, \tilde{\theta})} \right) &= S_x^2 - \bar{X} + \bar{X}^2.
\end{align*}
\]

(8.12)

(2) In the IRTPD case, the pair of equations that need to be solved for the moment estimates is

\[
\begin{align*}
\tilde{\theta} \left( \rho + 1 - \frac{\tilde{\theta}^M/M!}{e_1(M, \tilde{\theta})} \right) &= \bar{X} \\
\theta^2 \left( (1 + \rho)^2 - \frac{(2\rho+1)\tilde{\theta}^M/M! + \tilde{\theta}^{(M-1)}/(M-1)!}{e_1(M, \tilde{\theta})} \right) &= S_x^2 - \bar{X} + \bar{X}^2.
\end{align*}
\]

(8.13)

(2) MAXIMUM LIKELIHOOD METHOD

Alternatively, we can use the maximum likelihood method. Let \( X_1, X_2, \ldots, X_{n^*} \) be observations greater than \( M \) and \( X_{n^*+1}, X_{n^*+2}, \ldots, X_n \) be observations less than or equal to \( M \). To maximize the logarithm of the likelihood, the following equations have to be solved iteratively for \( \tilde{\theta} \) and \( \hat{\rho} \).
\[
\bar{X} = \hat{\theta} \left(1 + \hat{\rho} + \frac{\hat{\theta}^{(N-1)/2(N-1)!} - \hat{\theta}^{(N-1)/2(N-1)!}}{e^{(N, \hat{\theta})}}\right)
\]

(8.14)

\[
\hat{\theta} = \frac{1}{n} \left\{ \sum_{i=1}^{n} \sum_{y=N}^{\infty} \binom{\hat{x}_i}{y} \rho^{\hat{x}_i-y} \frac{\sum_{y=N}^{\infty} \binom{\hat{x}_i}{y} \rho^{\hat{x}_i-y}}{\sum_{y=N}^{\infty} \binom{\hat{x}_i}{y} \rho^{\hat{x}_i-y}} \right\}
\]

(8.15)

**REMARK.**

(1) In the ITPD case, the set of equations that needs to be solved for the maximum likelihood estimates is

\[
\bar{X} = \hat{\theta} \left(1 + \hat{\rho} + \frac{\hat{\theta}^{(N-1)/2(N-1)!}}{e^{2(N, \hat{\theta})}}\right)
\]

(8.15)

\[
\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} \frac{\sum_{y=N}^{\infty} \binom{\hat{x}_i}{y} \rho^{\hat{x}_i-y}}{\sum_{y=N}^{\infty} \binom{\hat{x}_i}{y} \rho^{\hat{x}_i-y}}
\]

(2) In the IRTPD case, the set of equations that needs to be solved for the maximum likelihood estimates is

\[
\bar{X} = \hat{\theta} \left(1 + \hat{\rho} - \frac{\hat{\theta}^{M/2M!}}{e^{1(M, \hat{\theta})}}\right)
\]

(8.16)

\[
\hat{\theta} = \frac{1}{n} \left\{ \sum_{i=1}^{n} \sum_{y=0}^{\infty} \binom{\hat{x}_i}{y} \rho^{\hat{x}_i-y} \frac{\sum_{y=0}^{\infty} \binom{\hat{x}_i}{y} \rho^{\hat{x}_i-y}}{\sum_{y=0}^{\infty} \binom{\hat{x}_i}{y} \rho^{\hat{x}_i-y}} + \sum_{i=n+1}^{\infty} \frac{\hat{x}_i}{1+\hat{\rho}} \right\}
\]
One can use the IMSL subroutine ZSPOW (1982) to solve these equations.

Asymptotic variances and covariance of $\hat{\theta}$ and $\hat{\rho}$ can also be found by the usual method involving the second derivatives. One can then form approximate confidence intervals for these parameters.

8.5 NUMERICAL EXAMPLES

EXAMPLE 1.

Data on student enrolment of graduate courses in the English Department of the University of Windsor were gathered from winter semester, 1986 to winter semester, 1988. It is customary that at least 2 students have to preregister in the course before the department decides to offer it. Additional students may also register in the course after school starts. The total number of students in each course $X$ is recorded for 37 courses the department offered during these two years.

The data is given in Table 1. We assume that $X$ has an intervened left truncated Poisson distribution. The parameters $\theta$ and $\rho$ were estimated by both the maximum likelihood method as well as the method of moments. Expected frequencies computed from each of these estimates are also given in Table 1 for comparison purposes. It can be seen that the method of moments gives a slightly better fit than MLE in this case. Both $\chi^2$ values are much smaller than the critical value of 9.49 ($\alpha = 0.05$).
EXAMPLE 2.

Another comparison is made of the moment estimators and the maximum likelihood estimators using the cholera epidemic example given by Shanmugam (1985). In Table 2, \( x \) is the number of cholera cases in a household and \( f_x \) is the number of houses with \( x \) cases of cholera. The mean \( \bar{x} \) and variances \( s^2_x \) are 1.56 and 0.58. The maximum likelihood estimates, \( \hat{\theta} = 0.48 \) and \( \hat{\rho} = 0.62 \) are obtained by us with \( N = 1 \) in (8.4) via the IMSL subroutine ZSPOW. The expected frequencies calculated from these MLE are displayed in Table 2 together with those from the moment estimates as given by Shanmugam (1985). It seems that the results are almost identical but the MLE gives a slightly better fit to the data than the moment estimates.

The data in Examples 3 and 4 are computer generated to illustrate applications of the intervened truncated Poisson distribution.

EXAMPLE 3.

A tourist agency is offering different tours to customers throughout the year. In order to be cost effective, they require a minimum of 6 tourists to join before they decide to actually conduct the tour. Tourists may also join after the date of decision. The total number of people in each tour group \( X \) is recorded for 200 different tours they offer during the year.

The data are given in Table 3. We assume that \( X \) has an intervened left truncated Poisson distribution. The parameters \( \theta \) and \( \rho \) were estimated by both the maximum
likelihood method as well as the method of moments. Expected frequencies computed from each of these estimates are given in Table 3 for comparison purposes. It can be seen that the method of moments gives a slightly better fit than the MLE in this case. Both \( \chi^2 \) values are much smaller than the critical value of 15.5 (\( \alpha = 0.05 \)).

**Example 4.**

Another example concerns the number of items each customer buys from a grocery store. There is a sale on eggs but each customer is limited to buying a maximum of 4 cartons. Other regular-priced items have no limits. The total number of items each customer buys is presented in Table 4. We assume that \( X \) has an intervened right truncated Poisson distribution. Again, the parameters were estimated by both methods and expected frequencies computed. MLE gives a slightly better fit than the method of moments in this set of data. In any case, both \( \chi^2 \) values are much smaller than \( \chi^2_{0.05} = 18.3 \), indicating that the fit is not bad.
TABLE 1

English Courses Enrolment Data

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$f_i$</th>
<th>Sample Frequencies</th>
<th>Expected Frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Moment Estimates</td>
<td>M. L. E.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\hat{\theta} = 5.96395$</td>
<td>$\hat{\theta} = 5.87998$</td>
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<tr>
<td></td>
<td></td>
<td>$\hat{\theta} \hat{\rho} = 0.07175$</td>
<td>$\hat{\theta} \hat{\rho} = 0.12883$</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1.60255</td>
<td>1.60237</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.90336</td>
<td>4.94943</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>3.30081</td>
<td>3.34706</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>4.98274</td>
<td>5.03400</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>6.01489</td>
<td>6.05062</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>6.05068</td>
<td>6.05951</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>5.21714</td>
<td>5.20149</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>3.93613</td>
<td>3.90684</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
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<td>2.60838</td>
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<td>1.56733</td>
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<tr>
<td></td>
<td>6</td>
<td>5.10715</td>
<td>5.03187</td>
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<tr>
<td>11</td>
<td>2</td>
<td>0.87421</td>
<td>0.85616</td>
</tr>
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</table>

\[ \sum \frac{\text{Obs} - \text{Exp}}{\text{Exp}}^2 \]

|       |       | 2.82765            | 2.85949             |
### Table 2

Epidemic of cholera in a village in India

<table>
<thead>
<tr>
<th>Sample Frequencies</th>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>(\sum) (Obs-Exp)^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected Frequencies</td>
<td>32</td>
<td>16</td>
<td>6</td>
<td>1</td>
<td>Exp</td>
<td></td>
</tr>
<tr>
<td>M.E.</td>
<td>(\hat{\theta} = 0.48)</td>
<td>31.71</td>
<td>17.16</td>
<td>4.94</td>
<td>1.19</td>
<td>0.3387</td>
</tr>
<tr>
<td>(\hat{\rho} = 0.63)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M.L.E.</td>
<td>(\hat{\theta} = 0.48)</td>
<td>31.64</td>
<td>17.15</td>
<td>4.99</td>
<td>1.22</td>
<td>0.3236</td>
</tr>
<tr>
<td>(\hat{\rho} = 0.62)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 3

Tour group data

<table>
<thead>
<tr>
<th>Sample Frequencies</th>
<th>Expected Frequencies</th>
<th>Moment estimates</th>
<th>M.L.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_i)</td>
<td>(f_i)</td>
<td>(\hat{\theta} = 7.7644)</td>
<td>(\hat{\theta} \sim \hat{\rho} = 2.3196)</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>3.2313</td>
<td>14.3109</td>
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<td>7</td>
<td>8</td>
<td>11.0796</td>
<td>11.6546</td>
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<tr>
<td>8</td>
<td>22</td>
<td>20.4855</td>
<td>20.7432</td>
</tr>
<tr>
<td>9</td>
<td>23</td>
<td>27.4337</td>
<td>27.1541</td>
</tr>
<tr>
<td>10</td>
<td>33</td>
<td>30.0027</td>
<td>29.4104</td>
</tr>
<tr>
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<td>23</td>
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</tr>
<tr>
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<td>29</td>
<td>24.2910</td>
<td>23.9374</td>
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<td>13</td>
<td>23</td>
<td>18.9492</td>
<td>18.8362</td>
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<td>5.9549</td>
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<td>3.5725</td>
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<td>19</td>
<td>1</td>
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<tr>
<td>20+</td>
<td>0</td>
<td>0.9636</td>
<td>1.0432</td>
</tr>
</tbody>
</table>

\[ \sum \frac{(\text{Obs-Exp})^2}{\text{Exp}} = 7.3794 \]

\[ \sum \frac{(\text{Obs-Exp})^2}{\text{Exp}} = 7.8559 \]
Table 4

Grocery store data

<table>
<thead>
<tr>
<th>$X_i$</th>
<th>$f_i$</th>
<th>Sample Frequencies</th>
<th>Expected Frequencies</th>
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</thead>
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<tr>
<td></td>
<td></td>
<td></td>
<td>Moment estimates</td>
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<tr>
<td></td>
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<td>$\tilde{\theta} = 2.5490$</td>
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<td>$\tilde{\theta} \parallel = 9.6552$</td>
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<td>$\tilde{\rho} = 4.7866$</td>
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</tr>
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<tr>
<td>22+</td>
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<td></td>
<td>0.2686</td>
</tr>
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</table>

$\sum \frac{(\text{Obs}-\text{Exp})^2}{\text{Exp}}$ | 4.6679 | 4.7130 |
PART III

THE COMPOUND D DISTRIBUTION
CHAPTER 9

THE D-POISSON COMPOUND DISTRIBUTION

9.1 INTRODUCTION

The Neyman Type A distribution is a rather well-known generalized Poisson which has been widely used in the studying of biological phenomena by, for example, Skellam (1952), Beall and Rescia (1953), Evans (1953), Gurland (1963); and in the theory of accident proneness, Leiter and Hamdan (1973), Cacoullos and Papageorgiou (1980).

DEFINITION 1. Assume random variables \( X_i \sim \text{Poisson}(\lambda) \), \( i=1,2,...,n \), are i.i.d. Let

\[
Z = \sum_{i=1}^{Y} X_i, \quad \text{where } Y \sim \text{Poisson}(\beta).
\]

Then we call the joint distribution of \((Z,Y)\) the Neyman Type A distribution with p.f.

\[
p(z,y) = e^{-\beta y} \frac{\lambda^y (y!)^z}{z!} \frac{\beta^y}{y!}, \quad \lambda, \beta > 0, \quad z, y = 1, 2, .... \quad (9.1)
\]

Symbolically this can be represented by

Neyman Type A \( \sim \) Poisson \( \vee \) Poisson.
In parts I and II of this dissertation, we investigated the D numbers and D distribution and their special cases. We call the distribution of the sum of n independent doubly truncated Poisson random variables the D distribution (DD) which depends on the D numbers and an incomplete exponential function.

A new extension of the D distribution in this Chapter is the case when n is a random variable, specifically \( n \sim \text{Poisson} \). We call this new distribution the D-Poisson compound distribution (D-PCD). Clearly it is also an extension of the Neyman Type A distribution. Symbolically

\[
\text{D-PCD} \sim \text{Doubly truncated Poisson} \vee \text{Poisson}.
\]

We will study the properties of the D-PCD in this chapter. Maximum likelihood estimates of the parameters and the MVU estimates are obtained. The calculation is easy since they only depend on the incomplete exponential function and the D numbers that have all been tabulated (see Appendix A.1, Tables I and II).

The D-PCD can be applied to various problems in queuing theory such as bulk service. For an example, consider the total number of books checked out at the circulation desk of the library in a fixed period of time. Most libraries have rules that each person cannot check out more than M books at a time. If we assume the number of people who comes to the counter to check out books in this period of time is distributed as Poisson and the number of books each person checked out is doubly truncated Poisson with a minimum of 1 and a maximum of M, and the number of books each person borrowed is independent of each other,
then the total number of books checked out is D-PCD.

At the end of this chapter, we will give another example called the Ambassador Bridge Problem.

For convenience, we define the following notation.

**DEFINITION 2.** We define

\[ A_k(z; (N, M), \lambda, \beta) = \sum_{y=0}^{[z/N]} y^k \frac{D(z, y; (N, M))}{\left( \frac{\beta}{e(N, M; \lambda) \lambda} \right)^y} y!, \quad (9.2) \]

where \( z = 0, 1, \ldots \); \( \lambda, \beta > 0 \); \( k = 0, 1, \ldots \);
\( 0 \leq N < M \), \( N, M \) are integers.

**DEFINITION 3.** A rational exponential function \( R_k(N, M; \lambda) \) is defined by

\[ R_k(N, M; \lambda) = \frac{e(N-k, M-k; \lambda)}{e(N, M; \lambda)}, \quad (9.3) \]

where \( \lambda > 0 \); \( 0 \leq N < M \), \( N, M \) are integers.
\( e(N, M; \lambda) \) is the incomplete exponential function given by (1.9) (See Chapter 1, Section 1.4).

9.2 THE D-POISSON COMPOUND DISTRIBUTION

**DEFINITION 4.** Let a random variable \( Y \sim \text{Poisson}(\beta), \beta > 0 \), and random variables \( X_i \)'s \( \sim \text{DTPD}(N, M; \lambda) \) be i.i.d., \( i = 1, 2, \ldots \), \( \lambda > 0 \), \( 0 \leq N < M \), where \( N, M \) are integers. The two-dimensional random vector \((Z, Y)\) where

\[ Z = \sum_{i=1}^{Y} X_i \]

is said to have a D-Poisson compound distribution D-PCDC\((N, M; \lambda, \beta)\).
THEOREM 1. If a random vector \( (Z, Y) \sim \text{D-PDC}(N, M; \lambda, \beta) \), then \( Z \) and \( Y \) have a joint distribution

\[
p(z, y; (N, M), \lambda, \beta) = P(Z=z, Y=y) = e^{-\beta} D_{i}^{(z, y; (N, M), \lambda)} \frac{\beta}{e^{(N, M; \lambda)}} \frac{1}{y!} \quad (9.4)
\]

where \( y = 0, 1, 2, \ldots, \lfloor z/N \rfloor; \quad z = yN, yN+1, \ldots, yM. \)

PROOF.

\[
P(Z=z, Y=y) = P(\sum_{i=1}^{y} X_i = z | Y=y) P(Y=y)
\]

\[
= \left( e^{(N, M; \lambda)} \right)^{-y} D_{i}^{(z, y; (N, M), \lambda)} \frac{\lambda^{z}}{z!} \left( e^{-\beta} \frac{\beta^{y}}{y!} \right)
\]

\[
= e^{-\beta} D_{i}^{(z, y; (N, M), \lambda)} \frac{\beta^{y}}{e^{(N, M; \lambda)}} \frac{1}{z!} \frac{1}{y!}. \quad \blacksquare
\]

REMARK.

The p.f. of the Neyman Type A distribution (9.1) is a special case of (9.4) when \( N=0 \), and \( M=\infty \). In that case

\[
D(z, y; (0, \infty)) = y^{2}; \quad e(0, \infty; \lambda) = e^{\lambda}.
\]

From Theorem 1, we can obtain

COROLLARY 1. The marginal probability function of \( Z \) is given by

\[
f_{Z}(z) = e^{-\beta} \frac{\lambda^{z}}{z!} \sum_{y=0}^{\lfloor z/N \rfloor} D_{i}^{(z, y; (N, M), \lambda)} \frac{\beta}{e^{(N, M; \lambda)}} \frac{1}{y!}
\]
\[ e^{-\beta \frac{\lambda^z}{z!}} A_0(z; CN, MD, \lambda, \beta), \quad (9.5) \]

where \( z = 0, 1, 2, \ldots; \)

\[ A_0(z; CN, MD, \lambda, \beta) = \sum_{y=0}^{\lfloor z/N \rfloor} D_1(z, y; CN, MD) \left( \frac{\beta}{e^{CN+1} \lambda^y} \right)^y / y! . \]

**COROLLARY 2.** The conditional probability function \( f(y|z) \) is given by

\[ f(y|z) = \frac{D_1(z, y; CN, MD) \left( \frac{\beta}{e^{CN+1} \lambda^y} \right)^y / y!}{A_0(z; CN, MD, \lambda, \beta)} \quad (9.6) \]

where \( z = 0, 1, 2, \ldots; \quad y = 0, 1, \ldots, \lfloor z/N \rfloor. \)

### 9.3 Recurrence Relations of the P.F.

Using the recurrence relations of D numbers (see Chapter 2, section 2.3, formulae (2.12) - (2.19)), we can get the following properties of \( p(z, y; CN, MD, \lambda, \beta) \) in (9.4). All proofs will be omitted here since they are similar to the recurrence relations of the p.f. of the D distribution.

**THEOREM 2.** If a random vector \((Z, Y) \sim D-PCDZ(CN, MD, \lambda, \beta)\), then the p.f. \( p(z, y; CN, MD, \lambda, \beta) \) in (9.4) satisfies the recurrence relation

\[
p(z+1, y; CN, MD, \lambda, \beta) = \frac{1}{(z+1)} \left[ y! \ p(z, y; CN, MD, \lambda, \beta) - \frac{\lambda^{z+1} \beta}{M! e^{CN+1} \lambda^y} p(z-M, y-1; CN, MD, \lambda, \beta) \\
+ \frac{\lambda^N \beta}{(N-1)! e^{CN+1} \lambda^y} p(z-N+1, y-1; CN, MD, \lambda, \beta) \right]
\]

where \( z = yN, yN+1, \ldots, yM-1 \); \( y = 0, 1, 2, \ldots, \lfloor z/N \rfloor. \) \( (9.7) \)
THEOREM 3.

If a random vector \((Z, Y) \sim D\text{-PCD}(z, y; (N, M), \lambda, \beta)\), then the p.f. \(p(z, y; (N, M), \lambda, \beta)\) in (9.4) for the special values \(z = yM\) and \(yM - 1\) satisfies

(i) \(p(yM, y; (N, M), \lambda, \beta) = \frac{\lambda}{yM} p(yM - 1, y; (N, M), \lambda, \beta)\), \quad (9.8)

(ii) \(p(yM - 1, y; (N, M), \lambda, \beta)\)

\[= \frac{y}{y - 1} M! e(N, M; \lambda) \frac{\lambda^{M - 1}}{(N - 1)!} p(y - 1, M - 1, y; (N, M), \lambda). \quad (9.9)\]

THEOREM 4.

If a random vector \((Z, Y) \sim D\text{-PCD}(N, M), \lambda, \beta)\), then the p.f. \(p(z, y; (N, M), \lambda, \beta)\) in (9.4) satisfies

(i) for fixed \(N, y > 1\)

\(p(yN + j, y; (N, M), \lambda, \beta)\)

\[= \left(\frac{e(N, M + i; \lambda)}{e(N, M; \lambda)}\right)^n p(yN + j, y; (N, M + i), \lambda, \beta), \quad (9.10)\]

where

\(j = 0, 1, \ldots, (M - N)\); \quad \(i = 1, 2, \ldots\)

(ii) for fixed \(M, y > 1\)

\(p(yM - j, y; (N, M), \lambda, \beta)\)

\[= \left(\frac{e(N - i, M; \lambda)}{e(N, M; \lambda)}\right)^n p(yM - j, y; (N - i, M), \lambda, \beta), \quad (9.11)\]

where

\(j = 0, 1, \ldots, (M - N)\); \quad \(i = 1, 2, \ldots\)

THEOREM 5.

If a random vector \((Z, Y) \sim D\text{-PCD}(N, M), \lambda, \beta)\), then
the p.f. \( p(z, y; (N, MD, \lambda, \beta) \) in (9.4) satisfies

(i) \( p(z, y; (N+1, MD, \lambda, \beta) = \)

\[
= \sum_{s=0}^{A} (-1)^{s} \frac{1}{s!} \left( \frac{\lambda^{N} \beta}{N! eCN, N; \lambda} \right)^{s} p(z-Ns, y-s; (N, MD, \lambda, \beta), \tag{9.12}
\]

where

\[
A = \left[ \frac{yM - z}{M - N} \right] \leq y; \quad z = y(N+1), \ldots, yM.
\]

p(z, y; (N, MD, \lambda, \beta) =

\[
= \sum_{s=0}^{A} \frac{1}{s!} \left( \frac{\lambda^{N} \beta}{N! eCN+1, N; \lambda} \right)^{s} p(z-Ns, y-s; (N+1, MD, \lambda, \beta), \tag{9.13}
\]

where \( z = yN, \ldots, yM. \)

(ii) \( p(z, y; (N, M-1), \lambda, \beta) = \)

\[
= \sum_{s=0}^{B} \frac{1}{s!} \left( \frac{\lambda^{M} \beta}{M! eCN, M; \lambda} \right)^{s} p(z-Ms, y-s; (N, MD, \lambda, \beta), \tag{9.14}
\]

where

\[
B = \left[ \frac{z-yN}{M-N} \right] \leq y; \quad z = yN, \ldots, y(M-1).
\]

p(z, y; (N, MD, \lambda, \beta) =

\[
= \sum_{s=0}^{B} \frac{1}{s!} \left( \frac{\lambda^{M} \beta}{M! eCN, M-1; \lambda} \right)^{s} p(z-Ms, y-s; (N, M-1), \lambda, \beta), \tag{9.15}
\]

where \( z = yN, \ldots, yM. \)

**Remark.**

If we compare (9.4) with the definition of the
multivariate generalized power series distribution (GPSD) (Patil, 1966), we can say the D-PCD is a multivariate GPSD. Some properties of GPSD are presented:

(i) (Recursion relation)

\[
p(z+a, y+b; (N, MD, \lambda, \beta)) = \frac{D_{1}(z+a) y!}{D_{1}(z, y; (N, MD)(z+a)!(y+b)!} \lambda^{a} \left(\frac{\beta}{e(N, M; \lambda)}\right)^{b} p(z, y; (N, MD, \lambda, \beta)).
\]

(ii) (Property of proportions)

\[
\frac{p(z+a, y+b; (N, MD, \lambda, \beta))}{p(z, y; (N, MD, \lambda, \beta)} = p(z, y; (N, MD, \lambda, \beta) \lambda^{a} \left(\frac{\beta}{e(N, M; \lambda)}\right)^{b}.
\]

Note that (i) and (ii) involve D numbers, but the recurrence relations that we presented in (9.8) - (9.15) do not depend on D numbers. The formulae that we provided are simpler to calculate.

9.4 MOMENTS

**Lemma 1.** The probability generating function of $Z$ in Theorem 1 is given by

\[
\psi_{Z}(s) = e^{\frac{e(N, M; s\lambda)}{e(N, M; \lambda)} - 1}, \quad 0 < s < 1.
\]  
\[\tag{9.16}\]

**Proof.**

The probability generating function of $Y$ in Theorem 1 is

\[
\psi_{Y}(s) = e^{\beta(s-1)}, \quad 0 < s < 1,
\]
and the p.g.f. of $X_i$ in Theorem 1 is (see Chapter 5, (5.3))

$$
\Psi_{X_i}(s) = \frac{e(N, M; s\lambda)}{e(N, M; \lambda)}, \quad 0 < s < 1.
$$

Then

$$
\Psi_Z(s) = \Psi_{Y^{-1}}(\Psi_{X_i}(s)) = e^\beta \frac{e(N, M; s\lambda)}{e(N, M; \lambda)} - 1, \quad 0 < s < 1.
$$

(See Karlin and Taylor, 1981, p.13).

**THEOREM 6.** If a random vector $(Z, Y) \sim D\text{-PCDX}(N, MD, \lambda, \beta)$, then

(i) \( E(Z) = \lambda\beta R_1(N, M; \lambda) \) \hspace{1cm} (9.17)

(ii) \( \text{Var}(Z) = \lambda\beta(\lambda R_2(N, M; \lambda) + R_1(N, M; \lambda)) \) \hspace{1cm} (9.18)

where \( R_k(N, M; \lambda) \) is the rational exponential function in (9.3).

**PROOF.**

Using \( E(Z) = E(Y)E(X) \)

$$
\text{Var}(Z) = [E(X)]^2\text{Var}(Y) + E(Y)\text{Var}(X)
$$

(see Karlin and Taylor, 1981, p.13),

and, using

\( E(Y) = \text{Var}(Y) = \beta, \)

\( E(X) = \lambda R_1(N, M; \lambda), \)

\( \text{Var}(X) = \lambda(\lambda R_2(N, M; \lambda) + R_1(N, M; \lambda)) - \lambda(R_1(N, M; \lambda))^2 \)

(see Chapter 5, Section 5.6, (5.21), (5.22)), then we get (9.17) and (9.18).
THEOREM 7. If a random vector \((Z, Y) \sim D-\text{PCD}(CN, MD; \lambda, \beta)\), then the kth factorial moment of \(Z\) is given by

\[
m_{z(k)} = \sum_{\sum i_j = k} \frac{k!}{i_1! \cdots i_l!} \beta^i \left( \frac{\lambda^j}{j!} \right)^i \prod_{j=1}^l \left( \frac{R_j(CN, M_j; \lambda)}{j!} \right)^i.
\]

where \(k = 1, 2, \ldots\)

(9.19)

PROOF.

The kth factorial moment of \(Z\) is

\[
m_{z(k)} = E[Z(Z-1) \cdots (Z-k+1)] = \Psi_z^{(k)}(1).
\]

Using Leibniz's formula for higher order derivatives of a composite function (see Wong, 1979, p.197-198), we have

\[
m_{z(k)} = \frac{\partial^k \Psi_y(\Psi_x(s))}{\partial s^k} \bigg|_{s=1}
\]

\[
= \sum_{\sum i_j = k} \frac{k!}{i_1! i_2! \cdots i_l!} \frac{\psi_y^{(1)}(1)}{1!} \left( \frac{\psi_x^{(1)}(1)}{1!} \right)^i_1 \frac{\psi_y^{(2)}(1)}{2!} \left( \frac{\psi_x^{(1)}(1)}{2!} \right)^i_2 \cdots \frac{\psi_y^{(l)}(1)}{l!} \left( \frac{\psi_x^{(1)}(1)}{l!} \right)^i_l
\]

\[
= \sum_{\sum i_j = k} \frac{k!}{i_1!} m_{y(i)} \left( \frac{i}{1}, i_2, \ldots, i_l \right) \prod_{j=1}^l \left( \frac{m_{x(j)}}{j!} \right)^{i_j}
\]
But, we have

\[ m_y = \beta^j, \quad \text{(see Johnson and Kotz, 1969, p. 90)} \]

and

\[ m_{x(i)} = \lambda^j R(N, M; \lambda), \quad \text{(see Chapter 5, Section 5.6, (5.18))} \]

So, we get (9.19).

Next, we use the relationship of \( m_r \) and \( m_{(r)} \) to obtain the \( r \)-th moment.

**COROLLARY 3.** If a random vector \((Z, Y) \sim D-PCDX(N, MD; \lambda, \beta)\), then the \( r \)-th moment is given by

\[
m_r = \sum_{k=0}^{r} S(r, k) m_{(k)}
\]

\[
= \sum_{k=0}^{r} S(r, k) \sum_{\substack{i_1 \leq i_2 \leq k \ldots \leq i_l \leq n \sum_{j=1}^{l} j i_j = k \sum_{j=1}^{l} j i_j = k \sum_{j=1}^{l} j i_j = k \sum_{j=1}^{l} j i_j = k}} \frac{k!}{i_1! i_2! \ldots i_l!} \beta^{i_1} R(N, M; \lambda)^{i_1} \beta^{i_2} R(N, M; \lambda)^{i_2} \ldots \beta^{i_l} R(N, M; \lambda)^{i_l}
\]

where \( S(r, k) \) is the Stirling number of the second kind (Jordan, 1965, also, Chapter 1, section 1.5).

Next, we use the p.f. of D-PCD and the conditional p.f. \( f(y \mid z) (9.4) \) to find the regression and correlation of \( Z \) and \( Y \).

**THEOREM 8.** If a random vector \((Z, Y) \sim D-PCDX(n, MD; \lambda, \beta)\), then the regression of \( Z \) on \( Y \) and the regression of \( Y \) on \( Z \)
are

\[ E(Z|Y=y) = \lambda R_1(N, M; \lambda) y \quad \text{with} \]

\[ \text{Var}(Z|Y=y) = \gamma(y R_2(N, M; \lambda) + R_1(N, M; \lambda) - \gamma[R_1(N, M; \lambda)])^2 \]

\[ \text{Var}(Y|Z=z) = \frac{A(z; C(N, MD, \lambda, \beta))}{A_0(z; C(N, MD, \lambda, \beta))} \quad \text{with} \]

\[ \text{Cov}(Y|Z=z) = \frac{A_2(z; C(N, MD, \lambda, \beta))}{A_0(z; C(N, MD, \lambda, \beta))} - \left(\frac{A(z; C(N, MD, \lambda, \beta))}{A_0(z; C(N, MD, \lambda, \beta))}\right)^2 \]

where \( A_k(z; C(N, MD, \lambda, \beta)) \) and \( R_k(N, M; \lambda) \) are given by (9.2) and (9.3), respectively.

**Theorem 9**. If a random vector \((Z, Y) \sim D-PCD(N, MD, \lambda, \beta)\), then

\[ E(ZY) = \lambda \beta (\beta + 1) R_1(N, M; \lambda) \]

\[ \text{Cov}(Z, Y) = \lambda \beta R_1(N, M; \lambda) \]

\[ \text{Corr}(Z, Y) = \frac{\lambda R_1(N, M; \lambda)}{\sqrt{\lambda^2 R_2(N, M; \lambda) + \lambda R_1(N, M; \lambda)}}. \]

9.5 ESTIMATION

9.5.1 MAXIMUM LIKELIHOOD ESTIMATION OF PARAMETERS

Let \((Z_i, Y_i), \ i = 1, 2, \ldots, m\) be a bivariate random sample from the \( D-PCD(N, MD; \lambda, \beta) \).

**Theorem 10**. The maximum likelihood estimators of \( \lambda \) and \( \beta \) in
(9.4) are obtained from

\[ \hat{\lambda} R_i(N, M; \hat{\lambda}) = \hat{\lambda} \frac{e(N-1, M-1; \hat{\lambda})}{e(N, M; \hat{\lambda})} = \bar{z} / \bar{y} \]  
\[ (9.27) \]

\[ \hat{\beta} = \bar{y} \]  
\[ (9.28) \]

where \( \bar{z} \) and \( \bar{y} \) are the mean of \((Z_1, Z_2, \ldots, Z_m)\) and \((Y_1, Y_2, \ldots, Y_m)\), respectively.

**PROOF.**

The likelihood function is

\[ L((Z_1, Y_1), (Z_2, Y_2), \ldots, (Z_m, Y_m); \lambda, \beta) \]

\[ = \prod_{i=1}^{m} \left( e^{-\beta D_i(z_i, y_i; (N, M))} \left( \frac{\beta}{e(N, M; \beta)} \right)^{z_i} \frac{1}{z_i! y_i!} \right) \]

\[ \ln L = -m \beta + \sum_{i=1}^{m} \left( \ln D_i(z_i, y_i; (N, M)) + z_i \ln \lambda \right) \]

\[ + y_i \left( \ln \beta - \ln e(N, M; \lambda) \right) - \ln z_i! - \ln y_i! \]

To maximize \( \ln L \), we set

\[ \frac{\partial \ln L}{\partial \lambda} = \frac{1}{\lambda} \bar{m} \bar{z} - R_i(N, M; \lambda) \bar{m} \bar{y} = 0 \]

\[ \frac{\partial \ln L}{\partial \beta} = -m + \frac{1}{\beta} \bar{m} \bar{y} = 0 . \]

Then we get (9.27) and (9.28).

**REMARK.**

(i) Equations (9.27) and (9.28) are also the equations that need to be solved if we want to use the method of moments.

(ii) The asymptotic variance-covariance matrix of \( \hat{\lambda} \) and \( \hat{\beta} \) is given by
\[ v = \begin{bmatrix} \hat{\lambda}^2/m^2 & 0 \\ 0 & \hat{\beta}/m \end{bmatrix}. \] (9.29)

9.5.2 THE MVU ESTIMATE FOR THE P.F. OF D-PCD

Let \((Z_1, Y_1), (Z_2, Y_2), \ldots, (Z_m, Y_m)\) be a random sample from \(D\)-PCD\((N,M,D; \lambda, \beta)\), where \((N,M)\) are known, and \(\lambda, \beta\) are unknown. Let

\[ T = \sum_{i=1}^{m} Z_i \quad \text{and} \quad Q = \sum_{i=1}^{m} Y_i. \]

Using Definition 7 and the theory of complete sufficient statistic (Bickel and Doksum, 1977, p.123), we can say that \((T, Q)\) is a complete sufficient statistic for the family of \(D\)-PCD\((N,M; \lambda, \beta)\).

Now, we are going to prove the following important theorem.

**THEOREM 11.** The minimum variance unbiased (MVU) estimate of \((9.4)\) is

\[ \hat{\omega}_{(z,y)}(T, Q; m) = \frac{(m-1)^{2m} \binom{t}{z} \binom{q}{y} D_{(z, y; (N,M), D, (t-z, q-y; (N,M))}}{m^q D_{(t, q; (N,M))}}. \] (9.30)

where

\((Z_1, Y_1) \sim D\)-PCD\((N,M; \lambda, \beta)\) are i.i.d., \(i = 1, \ldots, m\).

t, q are the observed values of \(T = \sum_{i=1}^{m} Z_i, \quad Q = \sum_{i=1}^{m} Y_i\), respectively,
\((t,q) \in \mathbb{W}^*_m\),

\[
\mathbb{W}^*_m = \mathbb{W}_{m-1} + \{z, y\}
\]

\[
= \left\{ (t,q) \begin{array}{l}
q = y, y+1, \ldots, \lfloor t/(m-1)N \rfloor; \\
t = (m-1)Nq+z, (m-1)Nq+z+1, \ldots, (m-1)Nq+z; \\
y = 0, 1, \ldots, \lfloor z/N \rfloor; \\
z = Ny, Ny+1, \ldots, yM
\end{array} \right\}
\]

\[
\mathbb{W}_{m-1} = \left\{ (v,u) \begin{array}{l}
u = 0, 1, \ldots, \lfloor v/(m-1)N \rfloor; \\
v = (m-1)Nu, (m-1)Nu+1, \ldots, (m-1)Mu
\end{array} \right\}
\]

\(y = 0, 1, \ldots, \lfloor z/N \rfloor; \quad z = yN, yN+1, \ldots, yM.\)

**PROOF.**

Let \(\hat{p}_{(z,y)}(T,Q;m)\) be an unbiased estimate of 

\(p(z,y;CN,MD,\lambda,\beta)\) in (9.4). Then

\[
\sum_{q=0}^{\infty} \sum_{t=qN}^{\omega} \hat{p}_{(z,y)}(T,Q;m) e^{-m/\beta} D_{1}^{*}(t,q;CN,MD) \lambda^{t} \left( \frac{m^\beta}{e^{CN, M; \lambda t}} \right)^{q} \frac{1}{t! q!}
\]

\[
= e^{-\beta D_{1}(z,y;CN,MD) \lambda^{y}} \left( \frac{\beta}{e^{CN, M; \lambda y}} \right)^{y} \frac{1}{z! y!}
\]

(9.31)

where \(y = 0, 1, \ldots, \lfloor z/N \rfloor; \quad z = yN, yN+1, \ldots, yM.\)

Note that

\[
e^{(m-1)\beta} = \sum_{u=0}^{\omega} \sum_{v=uN}^{uM} D_{1}^{*}(v,v;CN,MD) \frac{\lambda^{v}}{v!} \left( \frac{(m-1)\beta}{e^{CN, M; \lambda v}} \right)^{u} \frac{1}{u!}, \quad (v,u) \in \mathbb{W}_{m-1}.
\]
Substituting this result to (9.31), we have

\[
\sum_{q=0}^{\infty} \sum_{t=q}^{q^M} \hat{p}_{(z,y)}(T, Q; mD) D_{(t, q; (N, MD)} \frac{\lambda^t}{t!} \left( \frac{m\beta}{e(N, M; \lambda)} \right)^q
\]

\[
= D_{(z, y; (N, MD)} \frac{\lambda^z}{z!} \left( \frac{\beta}{e(N, M; \lambda)} \right)^y
\]

\[
\times \sum_{u=0}^{\infty} \sum_{v=u}^{u^M} D_{(v, u; (N, MD)} \frac{\lambda^v}{v!} \left( \frac{e(N, M; \lambda)}{e(N, M; \lambda)} \right)^u
\]

Let \( \theta = \frac{\beta}{e(N, M; \lambda)} \), and comparing the coefficients of \( \lambda^t \) and \( \theta^q \) on both sides with \( t = z+y, \quad q = y+u, \quad (t, q) \in \mathbb{N}_{m-1} \)

\([z, y] = \mathbb{N}_m^* \), we get

\[
\hat{p}_{(z,y)}(T, Q; mD) \frac{1}{t!} \frac{m^q}{q!} D_{(t, q; (N, MD)}
\]

\[
= D_{(z, y; (N, MD)} \frac{1}{z!} \frac{1}{y!} D_{(t-z, q-y; (N, MD)} \frac{1}{(t-z)!} \frac{(m-1)^{q-y}}{(q-y)!}
\]

Then we get (9.30).

Since \((T, Q)\) is a complete sufficient statistic for the family of D-PCD, by the Lehmann-Scheffe Theorem (Bickel and Doksum, 1977, p.122, also, see Appendix B.1), we see that \( \hat{p}_{(z,y)}(T, Q; m) \) in (9.30) is a MVUE of the p.f. of D-PCD\((N, MD; \lambda, \beta)\)
REMARK.

Patil (1966) has given the MVU estimate of the p.f. for a multivariate GPSD which is consistent with the result of Theorem 12.

Further, using the results of Patil (1966), we can get the MVU estimate of the variance of the MVU estimate. Under the conditions of Theorem 12, it always exists and is given by

\[
\text{Var}(\hat{p}_{(z,y)}(T, Q; m))
= \frac{\left((m-1)^q y^t \binom{t}{2}\binom{q}{2} D_{I}(z, y; (N, \infty)) D_{I}(t-z, q-y; (N, \infty))\right)^2}{m^q D_{I}(t, q; (N, \infty))}
\]

\[
= \frac{(m-2)^q (2z)! (2y)!}{m^q (z!)^2 (y!)^2} \frac{t}{2z} \binom{q}{2y} \left[D_{I}(z, y; (N, \infty)) \right]^2 D_{I}(t-2z, q-2y; (N, \infty))
\]

(9.32)

9.6 EXAMPLE

EXAMPLE. (Ambassador Bridge Problem)

The Ambassador Bridge is an international bridge between the U.S. (Detroit) and Canada (Windsor). The traffic is very heavy with thousands of cars and trucks driving over the bridge every day. It is important to investigate the number of cars and trucks passing through the border.
One interesting aspect involves the fact that there are automobile manufacturing companies on both sides of the border. Special trucks transport new cars over the bridge from one side to the other. These trucks can carry a maximum capacity of 10 cars, and if they are carrying a load, they have to stop at customs. Empty trucks can just drive through and they are not considered in our study here. The problem is to estimate how many new cars are carried over the bridge in a given period of time. These results could help in the management or systems design of the bridge.

The data were collected on Feb. 2-5, 1988 from 1:00 \(\text{P.M.} - 1:10 \text{P.M.}\) (see Table 1).

<table>
<thead>
<tr>
<th>Date</th>
<th>Time</th>
<th>Data set 1 Canada (\rightarrow) U.S.A.</th>
<th>Data set 2 U.S.A. (\rightarrow) Canada</th>
</tr>
</thead>
<tbody>
<tr>
<td>1988</td>
<td></td>
<td>Number of Cars (Z_i) (Y_i)</td>
<td>Number of Cars (Z_i) (Y_i)</td>
</tr>
<tr>
<td>Feb. 2</td>
<td>1:00 - 1:10</td>
<td>5, 7, 8, 8 (28) 4</td>
<td>6, 9 (15) 2</td>
</tr>
<tr>
<td>Feb. 3</td>
<td>1:00 - 1:10</td>
<td>8, 9, 7 (24) 3</td>
<td>8, 9 (15) 2</td>
</tr>
<tr>
<td>Feb. 4</td>
<td>1:00 - 1:10</td>
<td>8, 5 (13) 2</td>
<td>9, 8 (15) 2</td>
</tr>
<tr>
<td>Feb. 5</td>
<td>1:00 - 1:10</td>
<td>8, 7 (15) 2</td>
<td>7, 7 (14) 2</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>(\sum) 80 (11)</td>
<td>(\sum) 59 (8)</td>
</tr>
</tbody>
</table>
Let $Y$ be the number of trucks passing over the bridge from 1:00 p.m. - 1:10 p.m. Assume that $Y$ is a Poisson variable with parameter $\beta$, i.e.

$$Y \sim \text{Poisson}(\beta).$$

Let $X_i$ denote the number of cars being carried by the $i$th truck. Assume that $X_i$ is a doubly truncated Poisson variable with parameters $\lambda$, and $(N,M) = (1,10)$, i.e.

$$X_i \sim \text{DTPD}(1,10;\lambda).$$

Then the total number of cars carried over the bridge from 1:00 p.m. - 1:10 p.m. has a D-PCDK$(1,10;\lambda,\beta)$, where $\lambda$, $\beta$ are unknown.

(1) **DATA SET 1.** (Canada $\rightarrow$ U.S.A.)

In this example, we have $t = 80$, $q = 11$, and $m = 4$. Using (9.30) we have

$$\hat{p}_{(z,y)}(80,11;4)$$

$$= \frac{3^{11-y}(80 \choose 11)(11 \choose y)D_1(z,y;(1,10))D_1(80-z,11-y;(1,10))}{D_1(80,11;(1,10))}.$$

(2) **DATA SET 2.** (U.S.A. $\rightarrow$ Canada)

We have $t = 59$, $q = 8$, $m = 4$, and from (9.30) we have
\[
\hat{p}_{(z,y)}(59,8;4) = \frac{\binom{59}{3} \binom{8}{y} D_{(z,y);(1,10)} D_{(59-z,8-y);(1,10)}}{D_{(59,8);(1,10)}}.
\]

Tables 2 and 3 give the MVU estimates of the p.f. using Data set 1 and Data set 2, respectively. The calculations need only three D numbers that have been tabulated (see Appendix A.1, Table I). For instance, an estimate of the probability of having 3 trucks \((y=3)\) carrying 18 cars \((z=18)\) over the bridge during 1:00 p.m. - 1:10 p.m. is

\[
\hat{p}_{(18,3)}(80,11;4)
\]

\[
= \frac{\binom{3}{11-3} \binom{80}{18} \binom{11}{3} D_{(18,3);(1,10)} D_{(62,8);(1,10)}}{D_{(80,11);(1,10)}}
\]

\[
= \frac{\binom{3}{4} \binom{80}{18} \binom{11}{3} (370236438)(1.737124910 \times 10^{55})}{3.95345084 \times 10^{82}}
\]

\[
= \frac{(6561)(3.5521420 \times 10^{17})(165)(370236438)(1.737124910 \times 10^{55})}{(4194304)(3.95345084 \times 10^{82})}
\]

\[
= 0.014915.
\]

where \(D_{(80,11);(1,10)} = 3.953454 \times 10^{82}\);

\(D_{(18,3);(1,10)} = 370236438\);

\(D_{(62,8);(1,10)} = 1.737125 \times 10^{55}\);
Table 2  ESTIMATION OF THE P.F. BY USING DATA SET 1

(Canada → U.S.A.)

\( \hat{p}(z, y) \)

<table>
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<tr>
<th>z</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>( \hat{p}(z, \cdot) )</th>
</tr>
</thead>
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<td>-</td>
<td>-</td>
<td>-</td>
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<tr>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>0.001165</td>
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<td>0.000002</td>
<td>0.000000</td>
<td>-</td>
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<td>-</td>
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<td>0.000000</td>
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<td>0.000000</td>
<td>0.000000</td>
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<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
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</tr>
<tr>
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<td>0.002038</td>
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\( \hat{p}(\cdot, y) \) 0.154862 0.258103 0.258103 0.172068 0.163937 (y≥5)
Table 3  ESTIMATION OF THE P.F. BY USING DATA SET 2

(U.S.A. → Canada)

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\[ \hat{p}(z, y) = 0.266967 0.311462 0.207541 0.087409 0.126521 \] (y ≥ 5)
Table 4 shows the maximum estimated probability of cars corresponding to the number of trucks.

Table 4. MAXIMUM ESTIMATED PROBABILITY OF CARS CORRESPONDING TO THE NUMBER OF TRUCKS

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Finally, from the estimates of the marginal probabilities \( \hat{p}(z, \cdot) \) and \( \hat{p}(\cdot, y) \) from Tables 2 and 3, we could say that it is most likely that 2 or 3 trucks pass over the bridge from Canada to the U.S.A., and 2 trucks pass over the bridge from the U.S.A. to Canada. The maximum probability is for a total of 14 cars to be carried over the bridge from Canada to the U.S.A.; but only 8 cars carried over the bridge from U.S.A. to Canada.
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by censored samples of grouped observation in the estimation of statistical parameters", *Biometrika*, 49, 245-249.


# APPENDIX A.1 D Numbers

\[
\rho_{x,n}(N,M) = \min_{1 \leq i < \max(N,k+n-1)} \left( \sum_{i=1}^{n} y_{x}^{i} \right)
\]

## Table 1

| \(N, M\) = (1, 4) |
|------------------|------------------|------------------|------------------|------------------|
| \(x\)            | 1                | 2                | 3                | 4                | 5                | 6                | 7                |
| 1                | 1                | 1                | 1                | 1                | 1                | 1                | 1                |
| 2                | 1                | 2                | 2                | 2                | 2                | 2                | 2                |
| 3                | 1                | 6                | 6                | 6                | 6                | 6                | 6                |
| 4                | 1                | 14               | 14               | 14               | 14               | 14               | 14               |
| 5                | 1                | 30               | 150              | 240              | 120              | -                | -                |
| 6                | 1                | 50               | 540              | 1560             | 1800             | -                | -                |
| 7                | 1                | 70               | 1600             | 2400             | 16800            | 320              | -                |
| 8                | 1                | 90               | 4620             | 39440            | 12600            | 191520           | 141120           |
| 9                | 1                | 110              | 16620            | 643200           | 37152000         | 123908400       | 134248400        |
| 10               | 1                | 130              | 230500           | 21714000         | 125616000        | 132084000       | 296352000        |
| 11               | 1                | 150              | 344500           | 60660000         | 5555555555       | 5545040000      | 2609000000       |
| 12               | 1                | 170              | 54717000         | 5757575000       | 53895462400      | 272353545000    | 260900000000     |
| 13               | 1                | 190              | 7683175000       | 8363800000       | 813465330000     | 813465330000    | 260900000000     |
| 14               | 1                | 210              | 17691750000      | 1933801669800    | 1933801669800    | 1933801669800   | 260900000000     |
| 15               | 1                | 230              | 305402350000     | 35470919725000   | 35470919725000   | 35470919725000  | 260900000000     |

## Table 2

| \(N, M\) = (1, 5) |
|------------------|------------------|------------------|------------------|------------------|
| \(x\)            | 1                | 2                | 3                | 4                | 5                | 6                | 7                |
| 1                | 1                | 1                | 1                | 1                | 1                | 1                | 1                |
| 2                | 1                | 2                | 2                | 2                | 2                | 2                | 2                |
| 3                | 1                | 6                | 6                | 6                | 6                | 6                | 6                |
| 4                | 1                | 14               | 14               | 14               | 14               | 14               | 14               |
| 5                | 1                | 30               | 150              | 240              | 120              | -                | -                |
| 6                | 1                | 50               | 540              | 1560             | 1800             | -                | -                |
| 7                | 1                | 70               | 1600             | 2400             | 16800            | 320              | -                |
| 8                | 1                | 90               | 4620             | 39440            | 12600            | 191520           | 141120           |
| 9                | 1                | 110              | 16620            | 643200           | 37152000         | 123908400       | 134248400        |
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211
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\[
\min_{\{x_j \neq 1 \}} \min_{\mathcal{L}} \left\{ \sum_{j \neq 1} x_j \right\},
\]

\[
d(x, n, \mathcal{L}) = \min_{\{x_j \neq 1 \}} \min_{\mathcal{L}} \left\{ y_1 x_1 + y_2 x_2 + \ldots + y_n x_n \right\},
\]

where \( y_1 = \max_{\{x_j \neq 1 \}} x_j \).

### Table II

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## APPENDIX A.2 R Numbers

### Table III. $R_2$ Numbers $R_2(x,n;M)$

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APPENDIX A.3 MG-Stirling Numbers

### TABLE V

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* in case \(N=1\), \(S(x,n;N)\) is Stirling number of the second kind.
# APPENDIX A.4 TABLE VII

Incomplete Exponential Function \( u(x, a) = \frac{1}{x!} \int_0^\infty e^{-t} a^t t^{-x} dt \)

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APPENDIX A.5

TABLE VIII  D DISTRIBUTION

\[ p(x; \nu) = \frac{1}{\Gamma(\nu/2)} \left( \frac{\nu}{2\pi} \right)^{1/2} \frac{\nu^{\nu/2}}{2^\nu} \frac{1}{\Gamma(\nu/2)} \int_{0}^{x} t^{\nu/2 - 1} e^{-\nu t/2} dt \]

\[ \lambda = 1.5 \]

\( (\nu, \lambda) = (2, 7) \)

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\[ x = 0.5, 0.2, 0.1, 0.05, 0.025 \]

\[ \nu = 1, 2, 3, 4, 5, 6, 7 \]
### Table VII: D Distribution

\[ p(x; n, k; n) = \frac{e^{-x} x^{n+k-1}}{k! (n-k)!} \]

\[ \lambda = 5 \]

\( (n, k) = (2, 7) \)

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### Appendix A.7

**Table X** General 17th Stirling Distribution of the Second Kind

\[ p(x|x_k) = \frac{1}{x!} (e^{x} - 1)^{x_k} \]

\[ \lambda = 0.5 \]

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### Generalized Stirling Distribution of the Second Kind

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\lambda = 3 \\
N = 5
\end{align*}
\]

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### Generalized Stirling Distribution of the Second Kind

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N = 3
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### Additional information

- The table provides values for the generalized Stirling distribution of the second kind for different values of \(\lambda\) and \(N\).
- The values are given for specific ranges of \(n\).
### APPENDIX A.8 STIRLING NUMBERS

#### TABLE XI

STIRLING NUMBERS OF THE FIRST KIND, \( s(n, k) \)

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APPENDIX B  SOME BASIC THEORIES

B.1 COMPLETE SUFFICIENT STATISTICS

(Bickel and Duksun, 1977)

We are given a random experiment with sample space \( \Omega \). On this sample space we have defined a random vector \( \mathbf{X} = (X_1, \ldots, X_n) \), where \( X_i \) are i.i.d., \( i = 1, 2, \ldots, n \). When \( \omega \) is the outcome of the experiment, \( \mathbf{X}(\omega) \) is referred to as the observations or data.

We consider \( \mathbf{X} \)'s probability distribution. This distribution is assumed to be a member of a family \( \mathcal{P} \) of probability distributions on \( (\mathbb{R}^n, \mathcal{B}^n) \). \( \mathcal{P} \) is referred to as the model.

We are usually interested in various "parameters" of the family \( \mathcal{P} \). The parameter \( \theta \) ranges over a known parameter space \( \Theta \). Thus we can write the model as \( \mathcal{P} = \{ \mathcal{P}_\theta : \theta \in \Theta \} \).

We call any real or vector-valued function of observations a statistic and denote it by \( \mathbf{T}(\mathbf{X}) \) or \( T \).

(1) SUFFICIENT STATISTICS

A statistic \( \mathbf{T}(\mathbf{X}) \) is called sufficient for a parameter \( \theta \), if and only if, the conditional distribution of \( \mathbf{X} \) given \( \mathbf{T}(\mathbf{X}) = t \) does not involve \( \theta \). Thus, once the value of a sufficient statistic \( T \) is known, the sample \( \mathbf{X} = (X_1, \ldots, X_n) \) does not contain any further information about \( \theta \).
THEOREM B.1. (FACTORIZATION THEOREM) (Bickel and Doksum, 1977, p. 65)

In a regular model, a statistic \( T(X) \) with range \( I \) is sufficient for \( \theta \) if, and only if, there exists a function \( g(t, \theta) \) defined for \( t \) in \( I \) and \( \theta \) in \( \Theta \) and a function \( h \) defined on \( \mathbb{R}^n \) such that

\[
p(X, \theta) = g(T(X), \theta) \ h(X) \tag{B.1}
\]

for all \( X \in \mathbb{R}^n, \ \theta \in \Theta \).

(2) COMPLETE STATISTICS

A statistic \( T \) is said to be complete, if the only real-valued function \( g \) defined on the range of \( T \) which satisfies

\[
E_{\theta} [g(T)] = 0 \quad \text{for all } \theta \tag{B.2}
\]

is the function \( g(T) = 0 \) (Bickel and Doksum, 1977, p. 121).

(3) LEHMANN-SCHETTE THEOREM

THEOREM (Bickel and Doksum, 1977, p. 122)

If \( T(X) \) is a complete sufficient statistic and \( S(X) \) is an unbiased estimate of \( q(\theta) \), then \( T^*(X) = E(S(X) | T(X)) \) is an uniformly minimum variance unbiased (U.M.V.U.) estimate of \( q(\theta) \). If \( \text{Var}_{\theta} (T^*(X)) < \infty \) for all \( \theta \), \( T^*(X) \) is the unique U.M.V.U. estimate of \( q(\theta) \).

REMARK

If we can find a statistic of the form \( h(T(X)) \), such that \( h(T(X)) \) is an unbiased estimate of \( q(\theta) \), then \( h(T(X)) \) is U.M.V.U.
(4) EXPONENTIAL FAMILIES

The family of distributions of a model \( \{ \mathcal{P}_\theta : \theta \in \Theta \} \), is said to be a \( k \) parameter exponential family, if there exist real-valued functions \( c_1, ..., c_k \), and \( d \) of \( \theta \), real-valued functions \( T_1, ..., T_k \), \( S \) on \( \mathbb{R}^n \) and a set \( A \subset \mathbb{R}^n \) such that the density (frequency) function \( p(x, \theta) \) of the \( \mathcal{P}_\theta \) may be written,

\[
p(x, \theta) = \left( \exp \left( \sum_{i=1}^{k} c_i(\theta)T_i(x) + d(\theta) + S(x) \right) \right) I_A(x) \quad (B.3)
\]

where \( I_A \) is the indicator of the set \( A \). Note that the functions \( c_i, d, S, \) and \( T_i \) are not unique. The set \( A \) cannot depend on \( \theta \).

We shall refer to the vector \( T(x) = (T_1(x), ..., T_k(x)) \) as the natural sufficient statistic of the family (see Bickel and Doksum, 1977, p.72)

THEOREM (Bickel and Doksum, 1977, p.123)

Let \( \{ \mathcal{P}_\theta : \theta \in \Theta \} \) be a \( k \) parameter exponential family as given by (B.3). Suppose that the range of \( \mathcal{C} = (c_1(\theta), ..., c_k(\theta)) \) has a nonempty interior (contains an open \( k \) rectangle). Then, \( T(x) = (T_1(x), ..., T_k(x)) \) is complete as well as sufficient.

B.2 THE POWER SERIES DISTRIBUTION

(1) THE POWER SERIES DISTRIBUTION

Noack (1950) defined a power series distribution as

\[
P(X = x) = a(x) \frac{\theta^x}{f(\theta)}, \quad \text{for } x = 0, 1, ...
\]
where \( a(x) \theta^x \geq 0 \), \( a(x) \) is a function of \( x \) or constant, and

\[
f(\theta) = \sum_{x=0}^{\infty} a(x) \theta^x \text{ is convergent for } |\theta| \leq r.
\]

(2) THE GENERALIZED POWER SERIES DISTRIBUTION (GPSD)

Let \( T \) be a subset of the set \( I \) of non-negative integers. Define

\[
f(\theta) = \sum a(x) \theta^x
\]

where the summation extends over \( T \) and \( a(x) > 0 \), \( \theta \geq 0 \) with \( \theta \in \Theta \), the parameter space, such that \( f(\theta) \) is finite and differentiable. One has \( \Theta = (\theta : 0 \leq \theta < R) \) where \( R \) is the radius of convergence of the power series of \( f(\theta) \). Then a random variable \( X \) with probability function

\[
P(X = x) = a(x) \frac{\theta^x}{f(\theta)}, \quad \text{for } x \in T
\]

is said to have the generalized power series distribution (GPSD) with range \( T \) and the series function \( f(\theta) \) (Patil, 1963).

B.3 THE PRINCIPLE OF INCLUSION AND EXCLUSION

(Riordan, 1958)

Suppose there are \( N \) objects, and \( N(a) \) (using function notation) have the property \( a \); then, if \( a' \) denotes the absence of property \( a \),

\[
N(a') = N - N(a)
\]

for each object either has or has not property \( a \). If two properties \( a \) and \( b \) are in question, the number without both
is given by

\[ N(a'b') = N - N(a) - N(b) + N(ab) \]

for, in subtracting \( N(a) \) and \( N(b) \) from the total, \( N(ab) \) has been subtracted twice and must be restored. This justifies the term "inclusion and exclusion"; the process is one of including everything, excluding those not required, including those wrongly excluded, and so on, alternately including and excluding.

The general result may be stated as:

**THEOREM (Riordan, 1958, p.51)**

If of \( N \) objects, \( N(a) \) have property \( a \), \( N(b) \) property \( b, \ldots \), \( N(ab) \) both \( a \) and \( b \), \( N(abc) \) \( a, b, \) and \( c \), and so on, the number \( N(a'b'c'\ldots) \) with none of these properties is given by

\[ N(a'b'c'\ldots) = N - N(a) - N(b) - \ldots \\
+ N(ab) + N(ac) + \ldots \\
- N(abc) - \ldots \\
+ \ldots \]

The proof of this theorem by mathematical induction is simple once it is noted that the formula \( N(a') = N - N(a) \) can be applied to any collection of the objects which is suitably defined.
ABBREVIATIONS

DD..........D distribution
D-PCD......D-Poisson compound distribution
GPSD.......Generalized power series distribution
IEF........Incomplete exponential function
i.i.d.......Independently identically distributed
ITPD........Intervened Truncated Poisson distribution
IDTPD......Intervened doubly Truncated Poisson distribution
IRTPD......Intervened right Truncated Poisson distribution
ILTPD......Intervened left Truncated Poisson distribution
MVUE........Minimum variance unbiased estimate
MLE.........Maximum likelihood estimate
p.f.........Probability function
p.g.f.......Probability generating function
RD..........R distribution
r.v..........Random variable
SDSK........Stirling distribution of the second kind
            MGSDK......More generalized Stirling distribution of the
            second kind
            MSDK.......Generalized Stirling distribution of the second
            kind
TPD...........Truncated Poisson distribution
            ZTPD.......Zero Truncated Poisson distribution
            DTPD......Doubly Truncated Poisson distribution
            RTPD......Right Truncated Poisson distribution
            LTPD......Left Truncated Poisson distribution
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