Design and realization of two-dimensional recursive digital filters.

Amir. Mazinani

University of Windsor

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DESIGN AND REALIZATION OF TWO-DIMENSIONAL RECURSIVE DIGITAL FILTERS

by

Amir MAZINANI

A Dissertation submitted to the Faculty of Graduate Studies and Research through the Department of Electrical Engineering in Partial Fulfilment of the requirements for the Degree of Doctor of Philosophy at the University of Windsor

Windsor, Ontario, Canada 1991
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ABSTRACT

Digital processing of two dimensional signals is becoming increasingly important, and is finding applications in diverse scientific disciplines. Of the two classes of 2-D digital filters, namely recursive and nonrecursive, the first class is probably the most efficient in terms of hardware costs. However, it suffers from the stability problem as well as phase linearity, which is inherent in nonrecursive filters.

This thesis will review past work on the stability problem and methods for stabilizing filters. It has been shown that only a 2-D analog transfer function with a 2-variable Very Strictly Hurwitz Polynomial -VSHP- in its denominator can produce a 2-D stable recursive digital filter upon the application of double bilinear transformation. This thesis examines the existing methods of generating a 2-variable VSHP and presents a new one.

Existing design techniques for 2-D filters are briefly outlined, and a number of new methods suggested. These methods are generally iterative in nature and use linear and/or non-linear optimization techniques to determine the coefficients of the filter transfer function by minimizing an objective function which is a measure of difference between the magnitude and/or phase response of the designed and desired 2-D filter respectively. These design methods are for the general class of 2-D filters as well as a special class; separable denominator, non-separable numerator transfer function with or without circular cutoff boundaries.
The design procedures presented for the special class of 2-D filters, enjoy low computation and implementation costs. This was made possible by a considerable reduction of the number of coefficients in the transfer function and the choice of two step optimization approach.

This thesis also presents various strategies for the design of 2-D filters with non circular cutoff boundary, using a cascade of several 2-D filter building blocks. These building blocks are formed by a transformation of 1-D filters. This approaches is very simple and yields a good approximation to the desired response. It requires a 1-D realization structure. Several modifications to these techniques are also presented to improve the accuracy of the design.

A review of various realization structures for 2-D filters is presented in this thesis along with a new realization technique for a class of 2-D filters. Also, a variety of design examples are presented to illustrate the effectiveness of the proposed design techniques.
To Fakhri, Sanaz, Mani,
Fatemeh & Ali Asghar
ACKNOWLEDGEMENTS

I would like to express my deep gratitude and sincere appreciation to my supervisor Dr. M. Ahmadi for his advice, help and guidance through the course of this research. I also wish to thank Dr. M. Shridhar my co-supervisor for his guidance, support and helpful remarks during this course of study.

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<td>$A^T$</td>
<td>transpose of $A$</td>
</tr>
<tr>
<td>$A^{-1}$</td>
<td>inverse of $A$</td>
</tr>
<tr>
<td>det$A$</td>
<td>determinant of $A$</td>
</tr>
<tr>
<td>$</td>
<td>A</td>
</tr>
<tr>
<td>BIBO</td>
<td>bounded input bounded output</td>
</tr>
<tr>
<td>$a \cup b$</td>
<td>$a$ or $b$</td>
</tr>
<tr>
<td>$\cap_{i=1}^{2} \</td>
<td>z_i</td>
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<td>BHP</td>
<td>broad sense Hurwitz polynomial</td>
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<tr>
<td>CAD</td>
<td>computer aided design</td>
</tr>
<tr>
<td>$C_i^j$</td>
<td>combination of $i$ items taken $j$ at a time</td>
</tr>
<tr>
<td>$C_{r,t}$</td>
<td>matrix $C$ with $r$ rows and $t$ columns</td>
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<td>n-D</td>
<td>n-dimensional</td>
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<td>DHT</td>
<td>discrete Hilbert transform</td>
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<td>$\frac{\partial}{\partial z_i}$</td>
<td>partial derivative with respect to $z_i$</td>
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<td>$E_{Mag}$</td>
<td>error of the magnitude response</td>
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<td>FIR</td>
<td>finite impulse response</td>
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<td>$</td>
<td>H</td>
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<td>HP</td>
<td>Hurwitz polynomial</td>
</tr>
<tr>
<td>$H(s_1,s_2)$</td>
<td>2-variable analog transfer function</td>
</tr>
<tr>
<td>$H(z_1,z_2)$</td>
<td>2-D digital transfer function</td>
</tr>
<tr>
<td>$-$</td>
<td>complex transformation of $H$</td>
</tr>
<tr>
<td>Im</td>
<td>imaginary part</td>
</tr>
<tr>
<td>IIR</td>
<td>infinite impulse response</td>
</tr>
<tr>
<td>$I_n$</td>
<td>identity matrix of $n \times n$</td>
</tr>
<tr>
<td>$I_p$</td>
<td>set of all discrete frequency points along $\omega_1$ and $\omega_2$ axis in pass-band only</td>
</tr>
<tr>
<td>$I_{PS}$</td>
<td>set of all discrete frequency points along $\omega_1$ and $\omega_2$ axis in pass-band and stop-band</td>
</tr>
<tr>
<td>$\ell_p$</td>
<td>least $p^{th}$ square norm</td>
</tr>
<tr>
<td>$\lim$</td>
<td>limit</td>
</tr>
<tr>
<td>$\lambda_p(A)$</td>
<td>eigenvalue of $A$</td>
</tr>
<tr>
<td>Abbreviation</td>
<td>Description</td>
</tr>
<tr>
<td>-------------</td>
<td>--------------------------------------------------</td>
</tr>
<tr>
<td>LD</td>
<td>large diagonal</td>
</tr>
<tr>
<td>MSE</td>
<td>mean square error</td>
</tr>
<tr>
<td>NHP</td>
<td>narrow sense Hurwitz polynomial</td>
</tr>
<tr>
<td>NTT</td>
<td>number transfer theory</td>
</tr>
<tr>
<td>$\prod_{k=1}^{n} z_k$</td>
<td>product of $z_1, z_2, \ldots, z_n$</td>
</tr>
<tr>
<td>DPLSI</td>
<td>double planar least squares inverse</td>
</tr>
<tr>
<td>$\otimes$</td>
<td>Kronecker product of two matrices</td>
</tr>
<tr>
<td>Re</td>
<td>real part</td>
</tr>
<tr>
<td>$R_l(\cdot)$</td>
<td>region includes all the points of $\cdot$</td>
</tr>
<tr>
<td>RHS</td>
<td>right hand side</td>
</tr>
<tr>
<td>RNS</td>
<td>residue number systems</td>
</tr>
<tr>
<td>SHP</td>
<td>strict Hurwitz polynomial</td>
</tr>
<tr>
<td>$\sum_{i=1}^{N} a_i$</td>
<td>$a_0 + a_2 + \ldots + a_N$</td>
</tr>
<tr>
<td>SD</td>
<td>small diagonal</td>
</tr>
<tr>
<td>$\tau$</td>
<td>group delay</td>
</tr>
<tr>
<td>VSHP</td>
<td>very strict Hurwitz polynomial</td>
</tr>
<tr>
<td>$z^*$</td>
<td>complex conjugate of $z$</td>
</tr>
<tr>
<td>$z^{-1}$</td>
<td>unit delay in $z$-domain</td>
</tr>
</tbody>
</table>
INTRODUCTION

1-1 AN OVERVIEW

The last two decades have witnessed tremendous advances in the technology of digital systems, with a dramatic drop in the cost of basic hardware elements used in implementing such systems. As a result, many applications of digital signal processing have become feasible, and this in turn has stimulated further development of the theory. During the years 1965 to 1975, research was generally confined to one-Dimensional (1-D) systems, and since 1975 the development of Two-Dimensional (2-D) and then more general N-dimensional systems have been intensified.

2-D data, in the form of numbers regularly or irregularly spread in a plane, are commonly encountered in a variety of engineering disciplines. In some cases, 1-D signal processing techniques can be adapted to deal with 2-D data, but there are situations where such adaptations prove to be inadequate. This is the reason behind the need for
a general theory for 2-D and N-D systems on the line of the well developed 1-D system theory.

The manipulation of 2-D data is commonly referred to as 2-D signal processing. In signal processing, a dimension can mean any physical domain in which a signal is defined. Time, space and frequency are examples of such domains, and any combination of these is allowed as a co-ordinate system. This provides some insight into how N-D signals can exist in spite of the human mind's inability to comprehend anything beyond 3-dimensions.

There are three general branches of image processing. The most important of these, digital filtering, is the main theme of this thesis. The remaining two aspects, image encoding, and pattern recognition, will not be considered.

2-D digital filters are computational algorithms that transform a 2-D input sequence of numbers into a 2-D output sequence according to pre-specified rules, hence yielding some desired modifications to the characteristics of the input sequences.

Applications of 2-D digital filters cover a wide spectrum, the objective usually being either enhancement of an image to make it more acceptable to the human eye, or removal of the effects of some degradation mechanisms, or separation of features to facilitate identification and measurement by human or machine.

Many important applications of 2-D digital filters have been in the field of space technology. Here, digitally processed satellite images are used in monitoring environmental effects, earth resources and urban land use [78,129]. In such applications, 2-D digital filters enhance or reduce boundaries, remove low frequency shading effects,
reduce noise, and correct for distortion inherent to the imaging systems employed [154,155].

There are also applications in medicine and biology. 2-D X-ray films are digitally processed to reduce the low spatial frequency components, and by doing so feature lines and other features with large high frequency components become easier to detect. This procedure is sometimes followed or preceded by contrast enhancement and noise reduction.

In addition, biomedical uses include removal of scan lines in radio-isotope scanning, low frequency background noise reduction in photo-micrographs, and digital processing of acoustical holograms, the latter rapidly establishing themselves as replacements for x-ray [7,150].

Seismic prospecting is one area in which data is acquired as a 2-D sequence that is not an image in the conventional sense. Seismic detectors are placed at intervals both along and/or across an area (for two or three dimensional data acquisition considering the time interval as a dimension). The digitized output of the detectors form a 2/3-D data array that is processed to minimize the effect of multiple reflections and wind-induced noise. Information about the subsurface structure of the locality is then readily obtained from the output of the filtering operations [1,153].

The above is only a brief review of some of the many possible applications of 2-D filters. However, it is evident that any area in which 2-D data is encountered is a possible field of application of 2-D digital filters.
1-2 CHARACTERIZATION OF 2-D DIGITAL FILTERS

Linear digital filters are generally classified into two groups, namely non-recursive, that is known as Finite Impulse Response -FIR- and recursive, which is known as Infinite Impulse Response -IIR- digital filters. The emphasis of this work will be on the linear 2-D IIR digital filters.

A 2-D IIR digital filter can be characterized by its difference equation as

\[
y(m,n) = \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} \ x(m-i,n-j) - \sum_{i=0}^{M} \sum_{j=0}^{N} b_{ij} \ y(m-i,n-j)
\]

or by its z transfer function as

\[
H(z_1,z_2) = \frac{\sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} \ z_1^{-i} z_2^{-j}}{\sum_{i=0}^{M} \sum_{j=0}^{N} b_{ij} \ z_1^{-i} z_2^{-j}} = \frac{A(z_1,z_2)}{B(z_1,z_2)}
\]

The output of the IIR filter is a function of present and past inputs as well as the past outputs of the filter. Therefore, it is possible for the output to become increasingly large despite the size of the input signal. As a result, these filters can become unstable. It should be noted that stability means the response to a bounded input data array should be a bounded array. This is commonly known as BIBO stability. To ensure stability the denominator polynomial must have the following constraints:
\[ B(z_1, z_2) \neq 0 \quad \text{for} \quad \prod_{i=1}^{2} |z_i| \geq 1 \]

(1.3)

The details of the stability constraints are presented in section (1.4).

1-3 SUB-CLASSES OF 2-D IIR FILTERS

The transfer function of the IIR filter in equation (1.2) shows that the impulse response of the filter is spread over the first quadrant of 2-D plane. Two subclasses of IIR filters are separable product and separable denominator non-separable numerator transfer function. They will be discussed in the following sections.

1-3.1 SEPARABLE PRODUCT 2-D IIR FILTERS

The transfer function for this particular class is shown in equation (1.4). The resulting filter is a cascade of two 1-D filters in \( \omega_1 \) and \( \omega_2 \) axes [124-127].

\[ H(z_1, z_2) = H_1(z_1) H_2(z_2) \]

(1.4)

The transfer function of each individual filter in this case can be written as follows:

\[ H_k(z_k) = \frac{\sum_{i=0}^{M} a_i \ z_k^{-i}}{\sum_{i=0}^{N} b_i \ z_k^{-i}} \quad \text{for} \quad k = 1, 2 \]

(1.5)
The advantages of this class of filters are:

i) The stability problem is reduced to that of the 1-D filter, which is easy to handle.

ii) Design and realization of 1-D filters can be used for their designs and implementations.

The disadvantage of this sub-class of filters is the shape of cutoff boundary, which is restricted to a rectangular one.

The number of grid points that is required for the calculation of the cost function in design procedure of a filter, for general and separable product filters, is shown in figure (1.1). In this case with \( M = N \), the reduction in number of coefficients -Multipliers-, with a great saving in implementation cost, and reduction in number of function evaluations are shown in table (1.1)
Figure 1.1

2-D and 1-D grid points

Table 1.1

<table>
<thead>
<tr>
<th>No. of Coefficients</th>
<th>General class</th>
<th>Separable Product</th>
<th>Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2(M+1)^2</td>
<td>4(M+1)</td>
<td>2(M^2-1)</td>
</tr>
</tbody>
</table>

| No. of grid points  | 2n^2          | n                 | n(2n-1)   |

Reduction of coefficients and grid points of separable filter
1-3.2 SEPARABLE DENOMINATOR NON-SEPARABLE
NUMERATOR 2-D IIR FILTERS

The transfer function of this class of filters is

\[ H(z_1, z_2) = \frac{\sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} z_1^{-i} z_2^{-j}}{\sum_{i=0}^{M} b_{1i} z_1^{-i}} \left( \sum_{j=0}^{N} b_{2j} z_2^{-j} \right) \]

(1.6)

This class of filters has the advantages of the separable product filter but not their disadvantage, that is, circular, elliptical and in general non-symmetrical cutoff boundaries can be designed using this class of filter [2,45,84,104,128,130,142-3,145]. It should be noted that with this class of filters, circular symmetric, Quadrantal symmetric and octagonal symmetric magnitude response is achievable. The definition of these types of symmetries will be given in the next sections.

1-3.2.1 CENTRAL SYMMETRY

In this sub-class of 2-D filters the magnitude specification of equation (1.6) is symmetric with respect to the origin [2,45,104,131,144]. It means that the magnitude response of the filter is the same in the first and third quadrants; also they are equal in second and fourth quadrants.

If we consider \( M = N \) in equation (1.6) a reduction of \((M^2 - 1)\) multipliers is achieved over the general class of 2-D filters, as shown in table (1.2). Figure (1.2) illustrates the required number of grid points to evaluate the cost function during the design procedure.
Central symmetry grid points

![Figure 1.2: Central symmetry grid points](image)

### Table 1.2

<table>
<thead>
<tr>
<th>No. of Coefficients</th>
<th>General class</th>
<th>Central Symmetry</th>
<th>Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>2(M+1)^2</td>
<td>(M+1)(M+3)</td>
<td>M^2-1</td>
<td></td>
</tr>
<tr>
<td>2n^2</td>
<td>2n^2</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

Reduction of coefficients and grid points of central symmetry filter
1-3.2.2 QUADRANTAL SYMMETRY

A quadrantal symmetric 2-D filter [2,107] is obtained if the following constraints are imposed on the transfer function defined in equation (1.6)

\[ a_i = a_{M-i} = a_{i,M-j} = a_{M-i,M-j} \]  \hspace{1cm} (1.7)

\[ b_i = b_{2j} \]  \hspace{1cm} (1.8)

In this case we can rewrite equation (1.6) as:

\[ H(z_1, z_2) = \frac{z_1^{-Mz} z_2^{-Mz} \sum_{i=0}^{Mz} \sum_{j=0}^{Mz} a_{i,j} \cos(i \omega_1) \cos(j \omega_2)}{\left( \sum_{i=0}^{Mz} b_i z_1^{-i} \right) \left( \sum_{i=0}^{Mz} b_i z_2^{-i} \right)} \]  \hspace{1cm} (1.9)

where \( \cos \omega_i = (z_i^{1/2} + z_i^{-1/2})/2 \) for \( i=1,2 \) or alternatively as:

\[ H(z_1, z_2) = \frac{z_1^{-Mz} z_2^{-Mz} \sum_{i=0}^{Mz} \sum_{j=0}^{Mz} a'_{i,j} (\cos \omega_1)^i (\cos \omega_2)^j}{\left( \sum_{i=0}^{Mz} b_i z_1^{-i} \right) \left( \sum_{i=0}^{Mz} b_i z_2^{-i} \right)} \]  \hspace{1cm} (1.10)

It has been proven by Rajan and Swamy [2] that a circular symmetric filter is possible only through this type of transfer function. The magnitude response of this class of filters has symmetry in all four quadrants. Therefore, it requires only \( n^2 \) number of function evaluations for the design of a filter in this class, as shown in figure (1.3). Table (1.3) shows that a reduction of \( (7M^2/4+M-1) \) multipliers is obtained using this type of transfer function.
Figure 1.3

Quadrantal symmetry grid points

Table 1.3

<table>
<thead>
<tr>
<th>General class</th>
<th>Quadrantal Symmetry</th>
<th>Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Coefficients</td>
<td>$2(M+1)^2$</td>
<td>$M^3/4+3M+3$</td>
</tr>
<tr>
<td>No. of grid points</td>
<td>$2n^2$</td>
<td>$n^2$</td>
</tr>
</tbody>
</table>

Reduction of coefficients and grid points of quadrantal symmetry filter
1-3.2.3 OCTAGONAL SYMMETRY

If in addition to the above constraints for the circular symmetric filter, we include an additional constraint

\[ a_{i,j} = a_{j,i} \]  

(1.11)

then an octagonal symmetric filter is realized [2,132]. The magnitude response of this class of filters is similar in all octants. The number of the grid points needed for the calculation of the cost function to design such a filter is shown in figure (1.4).

Table (1.4) shows a further reduction of coefficients. Also, the reduction in the number of grid points, required to design the filter, which is nearly 75% less than the general case.
Octagonal symmetry grid points

![Diagram of octagonal symmetry grid points]

Table 1.4

<table>
<thead>
<tr>
<th></th>
<th>General class</th>
<th>Octagonal Symmetry</th>
<th>Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Coefficients</td>
<td>$2(M+1)^2$</td>
<td>$(M^2+22M+24)/8$</td>
<td>$(15M^2+10M-8)/8$</td>
</tr>
<tr>
<td>No. of grid points</td>
<td>$2n^2$</td>
<td>$n(n+1)/2$</td>
<td>$3n^2/4$</td>
</tr>
</tbody>
</table>

Reduction of coefficients and grid points of octagonal symmetry filter
1-4 STABILITY

A 2-D filter is said to be stable if its response to a bounded input sequence is a bounded sequence (BIBO). This is the most commonly used definition of stability. It is applicable for one, two or higher dimensional systems [3,4].

An absolutely bounded sequence is

\[ |x_{m,n}| \leq P < \infty \quad \text{for all } m,n \]  \hspace{1cm} (1.11)

where \( P \) is a finite positive real number.

A 2-D system is BIBO stable if and only if its impulse response is absolutely summable:

\[ \sum_{m=0}^{M} \sum_{n=0}^{N} |h_{m,n}| = S < \infty \]  \hspace{1cm} (1.12)

and \( S \) is a finite positive real number.

Consider the transfer function of a 2-D filter as:

\[ H(z_1,z_2) = \frac{A(z_1,z_2)}{B(z_1,z_2)} \]  \hspace{1cm} (1.13)

The stability of a 2-D recursive digital filter is determined by the coefficients of the denominator polynomial \( B(z_1,z_2) \) of the \( z \)-transfer function of the filter [5,6]. However, testing for stability is difficult because the fundamental theorem of algebra does not permit factorization of a 2-variable polynomial as a product of first and second order polynomials. It is therefore difficult, if not impossible, to determine the poles' locations.
1-4.1 STABILITY THEOREMS

A direct extension of the condition for stability in the 1-D case to the 2-D case was first proposed by Shanks et al. [5] and can be stated as follows:

Theorem 1.1

Given that $B(z_1, z_2)$ is a polynomial in $z_1$ and $z_2$, a necessary and sufficient condition for the coefficients of the expansion of $H(z_1, z_2) = 1 / B(z_1, z_2)$ to converge absolutely, and hence for $h_{m,n}$ to be absolutely summable, is

$$B(z_1, z_2) \neq 0 \quad \text{for} \quad \bigcap_{i=1}^{2} |z_i| \geq 1$$

(1.14a)

where

$$B(z_1, z_2) = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} b_{ij} z_1^i z_2^j$$

(1.14b)

The above condition suggests a test procedure for checking the stability of the filter by finding the poles of the z-transfer function, as in the 1-D case. This test is very tedious to apply, since it involves mapping an infinite number of points from the $z_1$ plane into the $z_2$ plane. However, a considerably simplified version has been derived by Huang [8]:

Theorem 1.2

A causal 2-D recursive filter with a z-transfer function (1.13) is stable if and only if:

1. The map of $R_i \equiv (z_i, \mid z_i \mid = l)$ in the $z_2$ plane according to $B(z_1, z_2) = 0$ lies outside $d_2 \equiv (z_2, \mid z_2 \mid \geq l)$,

2. No point in $d_i \equiv (z_i, \mid z_i \mid \geq l)$ maps into the point $z_2 = 0$ by the relation $B(z_1, z_2) = 0$.

Ansell [12] has shown that theorem 1.2 can be reduced to a stability test involving
a finite number of steps. However, it has been shown that this can still be very tedious.

Consider the change of variable in the z-transfer function as:

\[ H(z_1, z_2) = \frac{A(s_1, s_2)}{B(s_1, s_2)} \quad \text{for} \quad i = 1, 2 \]

where \( A(s_1, s_2) \) and \( B(s_1, s_2) \) are polynomials in \( s_1 \) and \( s_2 \). Theorem 1.2 can now be restated as follows:

**Theorem 1.3**

The causal 2-D recursive filter \( H(z_1, z_2) \) is stable if and only if:

1. In all real finite \( \omega_i \), the complex polynomial in \( s_2, B(j \omega_i, s_2) \), has no zeros in \( \text{Re}(s_2) \geq 0 \), and

2. The real polynomial in \( s_1, B(s_1, 1) \), has no zeros in \( \text{Re}(s_1) \geq 0 \).

3. Also, \( B(s_1, s_2) \) should be free of non essential singularities of the second kind in the region \( \text{Re}(s_1) \geq 0 \) and \( \text{Re}(s_2) \geq 0 \).

Most implementations of the stability tests have been based on Huang’s [8] stability conditions. DeCarlo et al. [10] and Strintzis [11] proposed another theorem, which further simplified Huang’s theorem and has been used for some testing procedures:

**Theorem 1.4**

Let \( H(z_1, z_2) = 1 / B(z_1, z_2) \) be a first quadrant recursive filter. This filter is stable if and only if \( B(z_1, z_2) \) satisfies the following conditions:
1. \( B(z_1, z_2) \neq 0 \) for \( |z_1| = 1, |z_2| = 1 \):
2. \( B(a, z_2) \neq 0 \) for \( |z_2| \geq 1 \) and \( a \) such that \( |a| = 1 \); and
3. \( B(z_1, b) \neq 0 \) for \( |z_1| \geq 1 \) and \( b \) such that \( |b| = 1 \).

**1.4.2 STABILITY TESTS**

The conditions of Huang’s theorem (1.2) for stability can be stated as follows:

1. \( B(z_1, 0) \neq 0 \) for \( |z_1| \geq 1 \) (1.16)
2. \( B(z_1, z_2) \neq 0 \) for \( |z_1| = 1 \cap |z_2| \geq 1 \) (1.17)

The Anderson and Jury [13] stability test is divided into two parts: first, condition (1.16) is checked for the value of \( B(z_1, z_2) \) with \( z_1 \) restricted, and second, condition (1.17) is checked for the value of \( B(z_1, z_2) \) with both \( z_1 \) and \( z_2 \) restricted. This method of testing the stability can be summarized as follows [44]:

1. For checking condition (1.16):
   i) Use the bilinear transformation of \( B(z_1, 0) = 0 \) and apply the Hurwitz method [44] to the transformed polynomial.
   ii) An alternative test involves forming the Schur-Cohn matrix from \( B(z_1, 0) = 0 \) and testing this for positiveness [14].

2. For checking condition (1.17) two successive tests as follows are needed:
   i) Apply a Schur-Cohn test to \( B(z_1, z_2) = 0 \) to get the self-inverse polynomials and check the positiveness of these polynomials.
ii) Second, the positiveness of a number of polynomials of \( B(z_1, z_2) = 0 \) on \( |z| = 1 \) should be checked.

A local positivity method has been introduced by Bose [15] for checking stability of 2-D filters. Consider the stability conditions (1.16) and (1.17). Condition (1.16) is a 1-D stability test and there exist several methods [16,17,18,19] for checking it. Condition (1.17) is more difficult and it appears in most of the 2-D stability tests articles.

A new table form of testing of the Huang's stability condition has been presented by Karan and Srivastava [20]. This method turns out to be simpler than those proposed in [13,15,21], due to the fact that there is no need of obtaining either all the leading principal minors of the Schur-Cohn matrix or the determinant of it.

A modified form of the Maria and Fahmy table has been presented by Jury [22]. In this table, the appropriate entries of the first column are equivalent to the appropriate minors of the Hermitian Schur-Cohn matrix. The positivity of the latter matrix requires the positivity checking of the \((n-1)\) entries for only one point, say, \( z=1 \), instead of all \( |z| = 1 \), and the last entry, which is equivalent to Schur-Cohn determinant, is to be checked for positivity of all \( |z| = 1 \), where \( n \) is the degree of the polynomial.

Let

\[
F(z) = \sum_{k=0}^{n} a_k z^k
\]  

(1.18)

where \( a_n \neq 0 \), and \( a_k^* \) is the complex conjugate of \( a_k \).

Form the following table:
\[
F(z) = \begin{bmatrix} a_0 & a_1 & a_{n-2} & \cdots & a_2 & a_1 & a_0 \\ a_0^* & a_1^* & a_{n-2}^* & \cdots & a_2^* & a_1^* & a_0^* \end{bmatrix} \\
F^*(z) = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} & b_n \\ b_0^* & b_1^* & b_2^* & \cdots & b_{n-2}^* & b_{n-1}^* & b_n^* \end{bmatrix} \\
\Delta_1 = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} & b_n \\ b_0^* & b_1^* & b_2^* & \cdots & b_{n-2}^* & b_{n-1}^* & b_n^* \end{bmatrix} \\
\Delta_2 = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} & c_n \\ c_0^* & c_1^* & c_2^* & \cdots & c_{n-2}^* & c_{n-1}^* & c_n^* \end{bmatrix} \\
\Delta_3 = \begin{bmatrix} d_0 & d_1 & d_2 & \cdots & d_{n-2} & d_{n-1} & d_n \\ d_0^* & d_1^* & d_2^* & \cdots & d_{n-2}^* & d_{n-1}^* & d_n^* \end{bmatrix} \\
\vdots \\
\Delta_{n-3} = \begin{bmatrix} s_0 & s_1 & s_2 & \cdots & s_{n-2} & s_{n-1} & s_n \\ s_0^* & s_1^* & s_2^* & \cdots & s_{n-2}^* & s_{n-1}^* & s_n^* \end{bmatrix} \\
\Delta_{n-2} = \begin{bmatrix} r_0 & r_1 & r_2 & \cdots & r_{n-2} & r_{n-1} & r_n \\ r_0^* & r_1^* & r_2^* & \cdots & r_{n-2}^* & r_{n-1}^* & r_n^* \end{bmatrix} \\
\Delta_{n-1} = \begin{bmatrix} t_0 & t_1 & t_2 & \cdots & t_{n-2} & t_{n-1} & t_n \\ t_0^* & t_1^* & t_2^* & \cdots & t_{n-2}^* & t_{n-1}^* & t_n^* \end{bmatrix} \\
\Delta_n = u_0
\]  

(1.19)

where

\[
F^*(z) = z^* F \left( \frac{1}{z} \right)
\]  

(1.20)

and

\[
b_k = \begin{bmatrix} a_k & a_{n-k} \\ a_{n-k}^* & a_k^* \end{bmatrix}, \quad \text{for } k=0,1,\ldots,n-1
\]

\[
c_k = \begin{bmatrix} b_k & b_{n-k} \\ b_{n-k}^* & b_k^* \end{bmatrix}, \quad \text{for } k=0,1,\ldots,n-2
\]

\[
d_k = \frac{1}{b_0} \begin{bmatrix} c_k & c_{n-k} \\ c_{n-k}^* & c_k^* \end{bmatrix}, \quad \text{for } k=0,1,\ldots,n-3
\]

\[
\vdots
\]

\[
t_0 = \frac{1}{s_0} \begin{bmatrix} r_0 & r_2 \\ r_2^* & r_0^* \end{bmatrix}
\]

\[
u_0 = \frac{1}{r_0} \begin{bmatrix} t_0 & t_1 \\ t_1^* & t_0^* \end{bmatrix}
\]  

(1.21)
The necessary and sufficient conditions for the roots of $F(z)$ to be inside the unit circle are

$$\Delta_k > 0, \quad \text{for } k=0,1,...,n$$

(1.22)

where $\Delta_k$ are the appropriate entries of the first column in the table.

Another stability testing procedure, based on a circuit theory approach, was presented by Shentov et al. [24]. The key part of this test involves successive reduction of the order of certain all-pass sections, whose stability depends on the stability of the original filter. This method is shown to be equivalent to the Maria and Fahmy method [21], but has the advantage of an additional insight gained from the circuit interpretation.

**EXAMPLE 1.1**

As an example for testing stability of a 2-D analog filter using Ansell's test [12] consider $A(s_1,s_2) = 1$, $B(s_1,s_2) = 1 + s_1 + s_2 + s_1 s_2$. Following the test procedures outlined by Ansell, one can express $B(j \omega, j \omega_2)$ for real $\omega_1$ and $\omega_2$ as;

$$B(j \omega_1, j \omega_2) = \left[ b_0(\omega_1) \omega_2^n + b_1(\omega_1) \omega_2^{n-1} + ... + b_n(\omega_1) \right]$$

$$+ j \left[ a_0(\omega_1) \omega_2^n + a_1(\omega_1) \omega_2^{n-1} + ... + a_n(\omega_1) \right]$$

(1.23a)

where $a_i(\omega_1)$ and $b_i(\omega_1)$ for $i=1,2,...,n$ are real polynomials in $\omega_1$ and $\omega_2$, where neither $a_0(\omega_1)$ nor $b_0(\omega_1)$ are identically zero. Also, we define $\gamma_k l(\omega_1)$ as

$$\gamma_{kl} = a_h b_l - a_l b_h$$

(1.24a)

for $0 \leq l, k \leq n$, equating to zero as and bs not present in function $B(j \omega, j \omega_2)$.
Quantity $C(\omega_i)$ denotes the $n \times n$ symmetrical polynomial matrix whose typical element $C_{ij}(\omega_i)$ $(0 \leq i, j \leq n)$ is the sum of all those $\gamma_{kl}(\omega_i)$ $(0 \leq k, l \leq n)$ for which both

$$k+l = i+j-l \quad \text{and} \quad \mid l-k \mid > \mid i-j \mid$$

(1.24b)

are satisfied. Then the $n$ successive principal minors of $C(\omega_i)$ must be positive for all real $\omega_i$.

In this example the polynomial can be expressed as follows:

$$B(j \omega_i, j \omega_2) = 1 + j \omega_i + j \omega_2 + j \omega_i j \omega_2$$

$$= 1 - \omega_i \omega_2 + j (\omega_i + \omega_2)$$

where

$$b_0(\omega_i) = -\omega_i$$
$$b_1(\omega_i) = j$$
$$a_0(\omega_i) = 1$$
$$a_1(\omega_i) = \omega_i$$

Sturm's method [43] can be used for positivity test of the minors.

$$C_{i,j} = a_k b_i - a_i b_k \quad \text{for} \quad \left\{ \begin{array}{l}
0 \leq k \\
l \leq n
\end{array} \right.$$

(1.25)

$$C_{o,1} = (1)(1) - (-\omega_i \omega_i) = 1 + \omega_i^2$$

for all real $\omega_i$, $C_{o,1} > 0$; therefore the first condition (1) of Ansell is satisfied. For the second condition we can write

$$B(s_i,1) = 1 + s_i + 1 + s_i = 2 + 2 s_i$$

$$s_i = -1$$

(1.26)

There is no root when $Re[s_i] \geq 0$, thus the condition (2) also is satisfied. As a result the polynomial is stable.
1-4.3 STABILITY IN THE PRESENCE OF NONESSENTIAL SINGULARITIES OF THE SECOND KIND

In the preceding methods of the stability testing we were considering

\[ H(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)} \]  \hspace{1cm} (1.27)

with \( A(z_1, z_2) = 1 \) to avoid nonessential singularities of the second kind.

1-4.3.1 EFFECT OF THE NUMERATOR POLYNOMIAL ON STABILITY

In the case of 2-D filters, the numerator can affect the stability of the filter. As an example, consider the following three transfer functions due to Goodman [25]:

\[ H_1(z_1, z_2) = \frac{1}{2 - z_1 - z_2} \]  \hspace{1cm} (1.28)

\[ H_2(z_1, z_2) = \frac{(1 - z_1)(1 - z_2)}{2 - z_1 - z_2} \]  \hspace{1cm} (1.29)

\[ H_3(z_1, z_2) = \frac{(1 - z_1)^8 (1 - z_2)^8}{2 - z_1 - z_2} \]  \hspace{1cm} (1.30)

\( H_1 \) is unstable, \( H_2 \) and \( H_3 \) have the same denominator polynomial as \( H_1 \), yet \( H_3 \) is stable and \( H_2 \) is not.

This situation is illustrated graphically in figure (1.5). If we look carefully we see that the only point on or inside the bicircle where the denominator polynomials are zero is at the point \( z_1 = z_2 = 1 \); the numerator polynomial of both \( H_2 \) and \( H_3 \) are also zero at
Nonessential singularity of the second kind on the unit bicircle
this point. Thus $H_2$ and $H_3$ both possess a nonessential singularity of the second kind on the unit bicircle. One of the pole images and one of the zero images of the pole and zero root maps are tangent at a point which is on the unit bicircle. Heuristically, filter stability is related to the degree of tangency [64].

Alexander and Woods [26] proposed a sufficient condition for BIBO stability when $H(z_1,z_2)$ has nonessential singularity of the second kind for $\{(z_1,z_2); |z_i|=1, \ i=1,2\}$. Still the problem of BIBO stability of 2-D digital filters in the presence of nonessential singularity of the second kind is left open. However, recently several contributions [27,28,31,32,33,34] have been made on this problem and some progress has been achieved.

A simple criterion giving a necessary and sufficient condition for BIBO stability of 2-D digital filter transfer functions having simple nonessential singularities of the second kind on $\{(z_1,z_2); |z_i|=1, \ i=1,2\}$ has been presented by Roytman et al. [28].

Algebraic stability tests usually require considerable storage space and computational time for implementation on a digital computer. In contrast, numerical stability tests, though inexact, often yield correct information concerning stability at low computational cost.

A possible approach to testing the stability of 2-D filters based on the Lyapunov equations has been presented by Agathoklis [35,36]. The complexity and a high computational time is still there. However, these can be further simplified by considering the solution of the Lyapunov equation using Kronecker products as employed by Fernando [37], Sendaula [39] and Agathoklis et al. [40,41] for the discrete case.
Most recently, an algebraic stability test for the Very Strictly Hurwitz Polynomial -VSHP- has been developed by Agathoklis [42]. Based on the formulation of the frequency dependent Lyapunov equation using Kronecker products, three new necessary and sufficient conditions -theorem 4- for the VSHP property are presented.

Consider the 2-variable characteristic polynomial of the form

$$ C(s_1, s_2) = \det \begin{bmatrix} s_1 I_n - A_{11} & -A_{12} \\ -A_{21} & s_2 I_n - A_{22} \end{bmatrix} $$  \hspace{1cm} (1.31)

which is equation (1) of [42], is a VSHP if and only if the following conditions are satisfied;

i) \hspace{1cm} Re \left[ \lambda_i \{ A_{22} \} \right] < 0 \hspace{1cm} (1.32)

ii) \hspace{1cm} Re \left[ \lambda_i \{ A_{11} \} \right] < 0 \hspace{1cm} (1.33)

iii) \hspace{1cm} Re \left[ \lambda_i \left\{ X_4 + X_3 X_1^{-1} X_2 \right\} \right] \neq 0 \hspace{1cm} (1.34)

where

$$ X_1 = I_n \otimes A_{11}^T + A_{11}^T \otimes I_n $$  \hspace{1cm} (1.35)

$$ X_2 = \begin{bmatrix} I_n \otimes A_{21}^T & -A_{21}^T \otimes I_n \end{bmatrix} $$  \hspace{1cm} (1.36)

$$ X_3 = \begin{bmatrix} I_n \otimes A_{12}^T \\ A_{12}^T \otimes I_n \end{bmatrix} $$  \hspace{1cm} (1.37)

$$ X_4 = \begin{bmatrix} -I_n \otimes A_{22}^T & 0 \\ 0 & A_{22}^T \otimes I_n \end{bmatrix} $$  \hspace{1cm} (1.38)

and \( \lambda_i(A) \) denotes the eigenvalues of \( A \), \( Re[s] \) is the real part of \( s \) and \( \otimes \) is the
Kronecker product of two matrices.

This approach requires testing the eigenvalues of three constant matrices only and thus is simpler than existing polynomial tests.

**EXAMPLE 1.2**

In this example the same polynomial of example (1.1) will be tested using theorem 4 of the Agathoklis method in [42].

\[
C(s_1,s_2) = \det \begin{bmatrix} s_1 - a_{11} & -a_{12} \\ -a_{21} & s_2 - a_{22} \end{bmatrix} \\
= \det \begin{bmatrix} s_1 + 1 & 1 \\ 0 & s_2 + 1 \end{bmatrix} = 1 + s_1 + s_2 + s_1 s_2
\]

(i) \(\text{Re} \left[ \lambda_i \{A_{11}\} \right] = a_{11} = -1 < 0\) \hfill (1.40)

(ii) \(\text{Re} \left[ \lambda_i \{A_{22}\} \right] = a_{22} = -1 < 0\) \hfill (1.41)

(iii) \(\text{Re} \left[ \lambda_i \{X_4 + X_1 X_1^{-1} X_2\} \right] = \text{Re} \left( \frac{a_{22} \det A}{a_{11}^2} \right)^{1/2} \)

\[= \text{Re} \left[ \left( \frac{-1}{-1} \right)^{1/2} \right] = 1 \neq 0\] \hfill (1.42)

The necessary and sufficient conditions of theorem 4 of [42] are satisfied, therefore the above polynomial is a VSHP.
1-5 STABILIZATION

Unlike the 1-D case, where stabilization is straightforward, in the 2-D case the problem is as yet unsolved. Checking for stability is possible, but not an easy task. Therefore, algorithms that stabilize a known unstable filter without any or considerable change in the magnitude of the transfer function become very useful. Stabilization can be accomplished using:

1. Double Planar Least Square Inverse -DPLSI- method
2. Discrete Hilbert transform method
3. Complex cepstrum method -Spectral Factorization-

The first two are the best known and more popular methods for stabilization. The DPLSI method was presented by Shanks [5]. This method is an extension of the well known theorem on 1-D least square inverse polynomials [47].

Hilbert transform method, proposed by Read and Treitel [48], is also an extension of a 1-D method. It is known that the log magnitude and the phase of a minimum phase 1-D sequence are related through Hilbert transform. Therefore, a stable version of a non-minimum phase sequence can be obtained by replacing its phase with the Hilbert transform of its log magnitude. The method of Read and Treitel is basically a direct extension of the 1-D case.

The third method is computationally more efficient and practical than the other two methods. Also, it is suitable for CAD filtering application. It has a drawback, that is, an infinite series has to be truncated and this may cause distortion on filters' specification. In the following sections, an overview of the stabilization methods will be presented.
1-5.1 DPLSI STABILIZATION METHOD

The objective of the stabilization is to obtain a stable filter by altering the coefficients of an unstable filter without changing its magnitude response. Consider the transfer function:

\[ H(z_1, z_2) = \frac{1}{B(z_1, z_2)} \]  \hspace{1cm} (1.43)

as a first quadrant plane recursive filter which is found BIBO unstable. Consider another polynomial \( C(z_1, z_2) \) of specified degree \( M, N \) in \( z_1 \) and \( z_2 \):

\[ C(z_1, z_2) = \sum_{i=0}^{M} \sum_{j=0}^{N} c_{ij} z_1^i z_2^j \]  \hspace{1cm} (1.44)

Form the product polynomial as:

\[ B(z_1, z_2) C(z_1, z_2) = D(z_1, z_2) \]

\[ = 1 + \sum_{i=0}^{M} \sum_{j=0}^{N} d_{ij} z_1^i z_2^j \]  \hspace{1cm} (1.45)

Definition: Among all polynomials of specified order \( M \) and \( N \) in \( z_1 \) and \( z_2 \), \( C(z_1, z_2) \) is the planar least-square inverse or Wiener inverse of \( B(z_1, z_2) \) provided the coefficients \( c_{i,j} \) are such that:

\[ \sum_{i=0}^{M} \sum_{j=0}^{N} |d_{ij}|^2 \]  \hspace{1cm} (1.46)

is minimized.

Note that the convolution of a polynomial with its inverse in the space domain results approximately in a unit impulse. This is equivalent to minimization of:
\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |1 - B(\omega_1, \omega_2) C(\omega_1, \omega_2)|^2 \, d\omega_1 \, d\omega_2
\]  

(1.47)

Now, suppose \( B' \) is the PLSI of \( C \), or double PLSI of \( B \). Consider the filter:

\[
H'(z_1, z_2) = 1 / B'(z_1, z_2)
\]  

(1.48)

**Shanks' Conjecture:** The filter \( H'(z_1, z_2) \) is BIBO stable and the magnitude response \( |H'(\omega_1, \omega_2)| \) is approximately equal to \( |H(\omega_1, \omega_2)| \).

Note that the size of array \( C \) is arbitrary in this method. This conjecture was made based on the fact shown earlier by Robinson [47], that the PLSI of a 1-D polynomial is stable -minimum delay.

Shanks presented a number of numerical examples based on the above technique. He showed the stability of these examples by mapping of \( z_2 \) unit circle onto the \( z_1 \) plane using the transformation \( B(z_1, z_2) = 0 \). From Shanks' or Huang's stability theorems, the filter would be stable if the unit circle in one variable and the transformation of the unit circle in the other complex variable are strictly exclusive.

**EXAMPLE 1.3**

Consider a recursive filter with the following denominator coefficient array [5]:

\[
B = \begin{bmatrix}
1.0 & -1.20002759 & 0.40002239 \\
-1.00003018 & 1.70007079 & -0.65005088 \\
0.40002035 & -0.7005488 & 0.25004387 \\
\end{bmatrix}
\]  

(1.49)

Figure (1.6) shows the unit impulse response of \( F(z_1, z_2) = 1 / B(z_1, z_2) \). The unit impulse response tends to grow larger and larger with increasing values of the spatial
coordinates \( x \) and \( y \). The instability can be verified by plotting the root map of \( B \), which is shown in figure (1.7). Since a portion of the mapping of the \( z_2 \) unit circle -dark area- falls inside the \( z_1 \) unit circle, \( B \) is not minimum phase.

The unit impulse response and root map of \( B' \) is shown in figures (1.8) and (1.9) respectively. The \( z_1 \) unit circle does not have any intersection with the mapping of the \( z_2 \) unit circle, and hence \( B' \) is minimum phase. The unit impulse response also decays very fast. In order to produce a stable filter, first an 8x8 PLSI of \( B \) is computed. Then, a 3x3 PLSI of the first PLSI is derived. The evaluation of this double PLSI yields:

\[
B' = \begin{bmatrix}
1.11567 & -1.23469 & 0.38431 \\
-1.0932 & 1.68765 & -0.66821 \\
0.37577 & -0.65178 & 0.29767
\end{bmatrix}
\] (1.50)

The amplitude spectra of \( F=1/B \) and \( F'=1/B' \) are given in figure (1.10). These contour plots show that the amplitude spectra of the original and PLSI filter, although not identical, are very similar. One of the factors that govern the quality of the approximate inversion is the size of the intermediate array \( C \). The larger the size of \( C \), the better the resemblance of the magnitude of the frequency response of \( F' \) to that of \( F \).

Although Shanks' conjecture holds for many useful cases, it is generally not correct. This was pointed out for the first time by Genin and Kamp [49] by a counter example. Jury introduced a new conjecture with additional constraints [95]:

Jury's Conjecture: If the original 2-D polynomial and inverse are of the same order, then the reciprocal of PLSI is a stable filter.

This conjecture holds for some lower order polynomials, but Kayran and King [96] presented a counter example to it.
Figure 1.6

Unit impulse response of the unstable filter

Figure 1.7

Root map of unstable filter
Unit impulse response of the stabilized filter

Root map of stabilized filter
Amplitude spectra of $B$ (-----) and $B'$ (———)
1.5.2 DISCRETE HILBERT TRANSFORM STABILIZATION METHOD

In 1-D, the real and imaginary parts of the Fourier transform of a minimum phase transfer function are related by the discrete Hilbert transform.

\[
DHT\{H\} = \frac{1}{N} \sum_{k=0}^{N-1} (\log |H(n)|) \left( 1 - (-1)^{k-n} \cotan \frac{\pi}{N} (k-n) \right) \tag{1.51}
\]

Given the denominator \(z\)-transform of an originally unstable 1-D recursive filter, the discrete Hilbert transform enables the designer to stabilize the filter without any distortion to the amplitude spectrum [44].

It is reasonable to ask whether the same technique can be used in 2 or more dimensions. Read and Treitel [48] have investigated this possibility and outlined a procedure to obtain a 2-D minimum phase array from a mixed or maximum phase counter-part as follows:

Step 1. Given a finite discrete 2-D array, augment the coefficient array with zeros to satisfy the causality condition. The number of zeros should normally increase the size of the array so that it becomes amenable to FFT analysis.

Step 2. Calculate the natural logarithm of the amplitude spectrum of the augmented 2-D array.

Step 3. Apply the 2-D discrete Hilbert transform to this 2-D log magnitude array. Thus the log magnitude is treated as the real part and the discrete Hilbert transform then yields the imaginary parts.

Step 4. Use the imaginary part as the phase spectrum corresponding to the given...
amplitude spectrum. These two spectral characteristics completely describe
the transform of the minimum phase array.

Step 5. After conversion from amplitude and phase to real and imaginary parts
-i.e. polar to cartesian-, compute the inverse transform and truncate to
obtain the same dimensions as the original array. This yields the minimum
phase version of the original array.

Note that the DHT stabilization is sensitive to the number of points in the discrete Hilbert
transform. The higher the number -i.e. resolution-, the better the chance of success.

Woods [50] showed via a counter example that it is not possible in general to find
a stable 2-D recursive filter with an amplitude response exactly matching that of the
original unstable filter. Bose [51] discussed the pitfalls in applying the Hilbert transform
to 2-D continuous transfer functions.

1-5.3 SPECTRAL FACTORIZATION STABILIZATION METHODS

The concept of spectral factorization is extended to 2-D in such a way as to
preserve the analytic characteristics of the factors. The factorization makes use of a
homomorphic transform procedure due to Wiener [163]. Difficulties with factoring 2-D
spectra have limited the generalization of many 1-D signal processing formalism to 2-D
applications. This has led to the reformulation of many 2-D problems in 1-D analytical
design formulations [164,165].

The approach used by Pistor [166] is to decompose the array representing the
impulse response of the 2-D filter into four single quadrant arrays, each recursing in a
different direction. Then, the output array may be obtained by convolving the input successively with the four quadrant array in the appropriate directions. The resulting factors are shown to be recursively computable and stable in agreement with 1-D spectral factorization. Ekstrom and Woods [167] have shown that the factors are generally not 2-D polynomials, but can be approximated as such.

Spectral factorization in the cepstral domain [52] has been as widely used as the DPLSI or DHT method. Ready [56] remarked that the cepstrum method, as well as the Hilbert transform method, make use of the log magnitude. The 2-D DHT approach is a special case of the cepstrum method, if no zeros on the unit hypercircle are allowed.

For one example in [52], the error of the spectral factorization method with respect to the chosen error criterion was found to be less than that of DHT and DPLSI methods. However, Bose [63] calls for more evidence before a comparison between these methods can be made.

1-6 **A SURVEY OF VERY STRICTLY HURWITZ POLYNOMIAL**

In the preceding sections, different procedures for testing the stability of 2-D filters were discussed. Most of these test procedures were found to be complex for computer implementation. In addition, there is a deficiency of a solid theoretical background and procedures for stabilizing a 2-D filter. This has inhibited the widespread application of recursive filters, despite their significant advantages over non-recursive filters.

A straightforward and successful method for designing a 2-D digital filter is to
assign a 2-variable stable polynomial in the denominator of an analog transfer function [61,67,109-10]. Then, by application of double bilinear transformation, the 2-D digital transfer function can be formed.

For many years researchers had used this idea, until the difficulties in employing this approach were discovered by Goodman [58]. Based on Koga’s result [65], an \( n \times n \) multi-variable positive real matrix can be realized as a multi-variable finite lumped passive network. Dubois and Blostein [97], later proposed a method for the design of 2-D IIR filter using the transfer function of an analog 2-variable passive 2-port lossy network. The 2-D discrete version of this analog filter was obtained through the double bilinear transformation.

A class of 2-variable reactance functions was introduced by Ahmadi et al. [80], as a transformation applied to a 1-D analog filter to obtain a first quadrant stable analog filter. The discrete version of the filter was obtained by the application of double bilinear transformation. The method was very simple, but it was restricted in the shape of the cutoff boundary that it could produce.

Ramamoorthy and Bruton [60] presented a technique based on Koga’s idea and decomposition of an \( n \times n \) multi-variable positive real admittance matrix to generate stable analog transfer functions. Prasad and Reddy [98] have modified the method of [60] to generate Strictly Hurwitz Polynomial -SHP- using a passive and lossy \( n \)-port network which is terminated with capacitors, resistors and ideal gyrators, to design a 2-D filter with the above polynomial. Then, by applying the double bilinear transformation to the
transfer function [99], the discrete version of the transfer function is obtained.

Goodman [58] showed that not all 2-variable analog transfer functions with SHP denominators yield stable 2-D digital transfer functions upon bilinear transformation. Rajan et al. [59] have classified various types of Hurwitz polynomials and derived conditions for 2-variable polynomials to be free of any non-essential singularities of the second kind. For details see section (2-4). Now the problem is that of generating Very SHP -VSHP- and assigning it to the denominator of the analog transfer function.

Ramachandran and Ahmadi [62] have recently shown how a 2-variable VSHP could be generated directly. They imposed conditions on an n-port lossless frequency independent network where each of its terminals are terminated with a \( s_1 \) or \( s_2 \) type capacitor, to become a VSHP [62]. Another approach to the solution is given in [61] by using the properties of the even or odd parts of a Hurwitz polynomial. They also introduced another technique for the generation of VSHP through application of the positive definite matrices [109-10].

An alternative approach for generating a 2-variable VSHP is due to Abiri et al. [67]. In this method, the properties of symmetric positive-definite or positive-semi-definite matrix are used to generate a 2-variable VSHP.

1-7 DESIGN OF 2-D FILTERS

By designing a 2-D filter we mean determination of the coefficients of the transfer function of the filter in such a way that the magnitude and/or group delay response of the designed filter approximates a desired one. This process can be carried out through:
i) Iterative approach using non-linear programming

ii) Iterative approach using linear programming

iii) Transformation techniques.

1-7.1 SURVEY OF NON-LINEAR ITERATIVE APPROACH

There are many techniques for the design of 2-D filters satisfying prescribed magnitude and/or group delay response using non-linear optimization techniques [60-1,109-10,133-36]. These techniques are implemented by minimizing some measure of the difference between the actual response of the filter and the desired response. This generally leads to a nonlinear optimization problem in the unknown parameters which cannot be solved exactly and requires iterative methods.

The earliest technique for the design of 1-D recursive filters was proposed by Kalman [100], in 1958. Maria and Fahmy [105-6] used minimization of the sum of squares of the differences between the desired and actual response by a nonlinear optimization algorithm. In this algorithm stability of the designed filter was checked at the end of each iteration during the optimization process.

A 2-D design technique proposed by Bertran [102] is essentially an iterative one which starts from an arbitrary initial set of numerator and denominator coefficients and attempts to improve or these values to give a closer approximation to the desired impulse response. Nowrouzian et al. [101] have extended many of the 1-D iterative design approaches to N-D. Delcaro and Sicuranza [147] presented a design method to formulate the problem as minimizing the sum of the magnitude squares of the complex errors.
between actual and ideal filters.

Aly and Fahmy [108] proposed a technique to design 2-D recursive digital filters with specified magnitude and group delay characteristics. This approach is an improved version of [105-6] by incorporating the stability requirements of [58]. Minimax technique (in \(\ell_p\)-norm sense when \(p \to \infty\)) rather than least-squares technique (\(p = 2\)) has been used by Charalambous [45] for the design of 2-D IIR filters. Minimax is also used to approximate both magnitude and group delay of the filter by Shimizu and Hirata [141].

Lampropoulous and Fahmy [148] proposed a generalized performance index which can be minimized by iteration based on Newton's method. Hinamoto and Maekawa [149] introduced a method which designs the denominator and the numerator of the transfer function separately. Many circularly symmetric filters were designed using iterative approaches that can be found in the literature [2,45,79,104,131,139-40].

Rajan and Swamy [2,107] have also proposed methods in which the denominator polynomial of the transfer function is separable in \(z_1\) and \(z_2\), to approximate quadrantal symmetric filters. The advantage of these methods is that once the filter characteristic has been determined, a single check is required to determine whether or not stability requirements are met.

The repeated stability checking will increase the computational time of calculating the coefficients of the filters using direct iterative method. Yet there is another promising approach which solves the stability problem explained in chapter 2. Ahmadi and Ramachandran [61,67,109-10] have extensively used the advantages of 2-variable VSHP
to design various types of 2-D digital filters, in which the stability of the designed filters is guaranteed.

1-7.2 SURVEY OF LINEAR ITERATIVE APPROACH

Linear programming has been widely used in the design of both analog and digital filters. The design of analog filters, using linear programming, was considered by Mathews et al. [111]. Later on, this approach was used by Rabiner and Hu [112-13] in the design of FIR digital filters. This design method included the approximation of both magnitude and linear phase specifications and the designs were carried out in one and two dimensions.

Another approach is to restructure the problem so that linear optimization techniques may be used to minimize an error function. A technique based on this approach was presented by Dudgeon [114], for 2-D recursive digital filters. Chottera and Jullien [115] have modified the linear programming approach of Rabiner et al. [116] by including the stability constraint in the process of optimization, as well as including the constant group delay response in the design process, since these filters are of better value in image processing applications, if they have linear phase characteristic [117].

1-7.3 SURVEY OF TRANSFORMATION TECHNIQUE

In a single variable, the fundamental theorem of algebra states that any polynomial may be factorized into the product of a number of first and second order factors with real coefficients. This permits the designer to factorize the given specified transfer function
in either the continuous s-domain or the discrete z-domain into a number of first or second order functions, which may be cascaded to obtain the required specification.

A corresponding theorem for 2-variable polynomials does not exist and therefore such simple design techniques are not possible. Thus the designer is forced to use techniques which involve the direct design of higher order systems. It is for this reason that a number of techniques have evolved for transforming stable 1-D filters into 2-D filters whose cutoff boundaries have a prescribed shape and whose amplitude spectrum in some given cross section is determined from the prototype filter. In frequency-domain transformations one can map both 1-D and 2-D IIR filters into 2-D IIR filters.

One of the most significant problems is that of designing a 2-D filter having a circular cutoff boundary. A few of the attempts at this approximation will be summarized below. The earliest attempt at a solution to this problem was put forward by Hall [118]. He suggested that a 2-D filter could be formed from a cascade of two 1-D filters, each of which varied in one dimension only. The cutoff boundary of the resulting filter will be almost rectangular with rounded corners.

Shanks et al. [5] proposed a technique which transforms a 1-D low-pass filter, designed in the continuous domain to a 2-D filter. These filters are called rotated filters because they are obtained by rotating 1-D filters. Costa and Venetsanopoulos [79] have modified the rotated filter approach by cascading a number of rotated filters whose angles of rotation are uniformly distributed over 180°, to achieve a magnitude response which approximates a circular symmetric cutoff boundary by a polygon.
McClellan's transformation [119-20] is a direct application of a spectral transformation to 2-D design techniques. This transformation was later extended to the design of 2-D IIR digital filters by Bernabo et al. [121].

Another technique has been presented by Ahmadi et al. [80] for the design of stable 2-D recursive digital filters. The stability of the resulting filters is guaranteed. The transfer function of the filter is obtained from a 1-D prototype by applying a new transformation technique in the frequency domain.

A higher order 2-variable reactance function has been used by Kayran and King [81] to arrive at approximately circularly symmetric 2-D filter. To ensure the stability of the resulting filter, both the 1-D prototype and transfer function must be stable [122]. It has been shown that the only admissible transformation for mapping 1-D IIR filter into 2-D IIR filter [123] has the form of \( z_1^{p_1} z_2^{q_1} \). A technique for designing recursive 2-D fan filter based on complex transformation has been presented by King and Kayran [46].

1-8 SURVEY OF REALIZATION METHODS

Implementation of a digital filter is the process of converting the transfer function of a designed filter into a filter network. The realization of 2-D digital filters can be obtained through the transfer function [5,85], the state-space model [87-89] and transformation techniques [90].

Direct realization of a 2-D transfer function was presented by Shanks et al. [5], Mitra et al. [85] and Dabbagh et al. [86]. Lodge and Fahmy [87] have presented a state-space realization in which the multiplier coefficients are functions of the entries of the
state-space matrices. Chakrabarti and Mitra [88] proposed a realization structure using
decision and graph theory for minimum of delay elements.

A systolic array realization based on the local state-space model has been
presented by Lampropoulos and Fahmy [89]. Erfani and Peikari [90] used frequency
transformation techniques to realize 2-D digital filters. It has been shown that the bilinear
transformation can be applied to implement 2-D digital building blocks without delay-free
loops.

Kwan and Hirano [91] used a delayed multipath realization for 2-D recursive
filters, to overcome the disadvantages of those techniques which involve the application
of the decomposition theorems to the 2-D transfer function [168], and those using LU
decomposition of the matrix coefficients of the 2-D polynomials of the transfer function
[92]. Their technique does not have the disadvantage of requiring the transfer function
of a digital filter to be transformed into another form with different coefficient values, but
allows the original transfer function to be used directly with the same coefficients for
realization.

A family of 2-D structures is presented by Mecklenbräuker and Mersereau [94]
for implementing 2-D FIR digital filters designed by means of a transformation of a 1-D
design. These implementations are computationally efficient for orders up to 50 \times 50
and are straightforward to program or build in hardware. With this technique a zero-
phase filter can be implemented by a zero-phase operator of the form \((z+z')/2\).
1-9 ORGANIZATION OF THE DISSERTATION

The outline of this dissertation is as follows:

Beginning with Chapter 2, a review of some definitions along with relative examples and a number of different approaches to the generation of VSHPs will be presented. Then a new and computationally efficient method for generating 2-variable VSHP will be given. Application of this technique will be illustrated with design examples.

Chapter 3 discusses the design of 2-D recursive digital filters using iterative techniques and a systematic procedure to formulate the design problem will be given. In these formulations the filters are designed to achieve a desired magnitude and/or constant group delay response. Linear, non-linear and a combination of both types of programming will be used to design the 2-D filters. The usefulness of these techniques will be illustrated by several examples. The sensitivity analysis for the linear programming will also be presented. At last, design of 2-D filters with integer coefficients will be presented along with the examples.

Chapter 4 is about the transformation techniques for the design of 2-D filters. By using these techniques we will design non-symmetric 2-D filters as well as symmetric and quadrantal fan filters. A comparison will be made to compare the new techniques with the existing methods. All the design methods will be illustrated with examples.

In Chapter 5, different types of realization structures will be given. Then a realization structure for a class of 2-D recursive digital filters that has low sensitivity will
be presented. In this method we will use the predistorted version of the bilinear transformation and the low sensitive ladder structure.

In the final chapter, Chapter 6, relevant conclusions will be derived from this study including suggestions for future work.
2-1 INTRODUCTION

For many years researchers in the area of 2-D recursive digital filter design were relying on the transformation of a 2-variable passive analog network, using double bilinear transformation to avoid the stability problem associated with recursive filters. Goodman [58] showed that not all stable 2-D analog transfer functions will yield 2-D stable recursive digital filters upon the application of the double bilinear transformation.

Rajan et al. [59] have shown that only a special class of 2-D analog filters can be transformed by the application of the double bilinear transformation to obtain stable recursive digital filters. This special class of 2-D analog transfer functions should have denominators which are the 2-variable Very Strictly Hurwitz Polynomials. In this chapter a new method is presented for the generation of 2-variable Very Strictly Hurwitz
Polynomial -VSHP- which are free of any non-essential singularities of the second kind, in the region \( \text{Re}[s_i] \geq 0 \text{ for } i=1,2 \) including the points at infinity. 

First we shall review some definitions and a survey of previous work, then proceed to present the new method for generating 2-variable VSHP.

### 2-2 THE VALUE OF 2-VARIABLE FUNCTION AT INFINITY

The 2-D biplane \((s_1,s_2)\) consists of two complex planes \(s_1\) and \(s_2\) and an infinite distant point which can have infinite coordinates in any one or both of those planes, and so there exist an infinite number of points at infinity [59]. They can be classified into three categories as follows:

- **Category 1.** \(s_1 = \infty\) and \(s_2 = \text{finite}\)
- **Category 2.** \(s_1 = \text{finite}\) and \(s_2 = \infty\)
- **Category 3.** \(s_1 = \infty\) and \(s_2 = \infty\)

By applying the transformation method to each variable, the value of the function at each of the above points is defined as:

1. \[ f(\infty,s_2) = \lim_{s_1 \to 0} f\left(\frac{1}{s_1},s_2\right) \] (2.1)
2. \[ f(s_1,\infty) = \lim_{s_2 \to 0} f\left(s_1,\frac{1}{s_2}\right) \] (2.2)
3. \[ f(\infty,\infty) = \lim_{s_1 \to 0} \lim_{s_2 \to 0} f\left(\frac{1}{s_1},\frac{1}{s_2}\right) \] (2.3)
2-3 SINGULARITIES

It is well known that a two-variable rational function of the form:

\[ H(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)} \]  \hspace{1cm} (2.4)

may possess two types of singularities and they may be defined as follows:

1. **Nonessential singularities of the first kind.** \( H(z_1, z_2) \) is said topossess a nonessential singularity of the first kind at \( (z_1^*, z_2^*) \)
   if
   \[ A(z_1^*, z_2^*) \neq 0 \]
   and
   \[ B(z_1^*, z_2^*) = 0 \]  \hspace{1cm} (2.5)

2. **Nonessential singularities of the second kind.** \( H(z_1, z_2) \) is said
to possess a nonessential singularity of the second kind at \( (z_1^*, z_2^*) \)
   if
   \[ A(z_1^*, z_2^*) = 0 \]
   and
   \[ B(z_1^*, z_2^*) = 0 \]  \hspace{1cm} (2.6)

**EXAMPLE 2.1**

Here are two examples of ambiguity in the transfer functions. For the two-variable case, the nonessential singularity shown is of the second kind.

1. In the 1-D case, the transfer function of \( H(z) \) has a pole and a zero at \( z = 0.5 \).

   \[ H(z) = \frac{(z - 0.5) A'(z)}{(z - 0.5) B'(z)} \]  \hspace{1cm} (2.7a)

   This creates ambiguity at \( z = 0.5 \) which is:
\[
H(z) \bigg|_{z = 0.5} = \frac{0}{0} \tag{2.7b}
\]

To remove this, \((z - 0.5)\) is cancelled from numerator and denominator of the transfer function.

2. In the 2-D case, the transfer function of \(H(z_1, z_2)\) has a pole with a nonessential singularity of the second kind at \(z_1 = z_2 = 0.5\), which cannot be removed through pole zero cancellation.

\[
H(z_1, z_2) = \frac{(z_1 - 0.5)(z_2 - 0.5) A'(z_1, z_2)}{(1 - 4z_1z_2) B'(z_1, z_2)} \tag{2.8}
\]

Therefore, this may cause ambiguity in the transfer function at this point.

2-4 DEFINITION OF VARIOUS TYPES OF HURWITZ POLYNOMIALS

There are four types of Hurwitz polynomials. Their definitions are stated below in a slightly different form, in terms of singularities rather than zeros, as has been the common practice [59]. This has been done so as to facilitate a uniform definition for all the four types of polynomials, differing only in the region of analyticity. In the following definitions \(B(s_1, s_2)\) is a polynomial in \(s_1\) and \(s_2\) and \(Re(s)\) is the real part of \(s\).

Definition 1.

Polynomial \(B(s_1, s_2)\) is a Broad sense Hurwitz Polynomial -BHP- if
The polynomial $B(s_1, s_2)$ does not possess any singularities in the region

$$\{ (s_1, s_2) \mid Re(s_1) > 0, Re(s_2) > 0, |s_1| < \infty, |s_2| < \infty \} \quad \text{(2.9)}$$

**Definition 2.**

Polynomial $B(s_1, s_2)$ is a Narrow sense Hurwitz Polynomial -NHP- if $1/B(s_1, s_2)$ does not possess any singularities in the region

$$\{ (s_1, s_2) \mid Re(s_1) > 0, Re(s_2) > 0, |s_1| < \infty, |s_2| < \infty \}$$

$$\cup \{ (s_1, s_2) \mid Re(s_1) = 0, Re(s_2) > 0, |s_1| < \infty, |s_2| < \infty \}$$

$$\cup \{ (s_1, s_2) \mid Re(s_1) > 0, Re(s_2) = 0, |s_1| < \infty, |s_2| \leq \infty \} \quad \text{(2.10)}$$

**Definition 3.**

Polynomial $B(s_1, s_2)$ is a Strict Hurwitz Polynomial -SHP- if $1/B(s_1, s_2)$ does not possess any singularities in the region

$$\{ (s_1, s_2) \mid Re(s_1) \geq 0, Re(s_2) \geq 0, |s_1| < \infty, |s_2| < \infty \} \quad \text{(2.11)}$$

**Definition 4.**

Polynomial $B(s_1, s_2)$ is a Very Strictly Hurwitz Polynomial -VSHP- if $1/B(s_1, s_2)$ does not possess any singularities in the region

$$\{ (s_1, s_2) \mid Re(s_1) \geq 0, Re(s_2) \geq 0, |s_1| \leq \infty, |s_2| \leq \infty \} \quad \text{(2.12)}$$
It may be noted that each successive condition includes the previous conditions, so that each successive class of Hurwitz polynomials is a subclass of the earlier classes. For 2-D digital filters, a denominator polynomial must be SHP as a necessary and VSHP as a sufficient condition to ensure stability. A few examples will show the various classes of Hurwitz polynomials.

**EXAMPLE 2.2**

\[ B_1(s_1, s_2) = s_1 + s_1 s_2. \]  
As can be seen \( 1/(s_1 + s_1 s_2) \) does not have any singularity in the region specified in definition (1). However, at \( s_1 = 0 \) it has a first kind singularity. Therefore \( s_1 + s_1 s_2 \) is a BHP.

**EXAMPLE 2.3**

\[ B_2(s_1, s_2) = 1 + s_1 s_2. \]  
Polynomial \( B_2(s_1, s_2) \) does not have any singularity in the region of Definition 2. There is a relation between the poles \( s_2 = -1/s_1 \), for \( s_1 = 0, s_2 = \infty \) and vice-versa. Therefore the polynomial is not SHP, but it is NHP.

**EXAMPLE 2.4**

Polynomial \( B_3(s_1, s_2) = 1 + s_1 + s_2 \) is a SHP. However, it is not a VSHP because

\[
\frac{1}{B_3(\infty, \infty)} = \left| \begin{array}{c} s_1 s_2 \\ s_1 s_2 + s_1 + s_2 \end{array} \right| = \left| \begin{array}{c} 0 \\ 0 \end{array} \right|
\]

Also \( 1 + s_1 + s_1 s_2 \) and \( 1 + s_2 + s_1 s_2 \) are SHPs, but not VSHPs.
EXAMPLE 2.5

Polynomial \( B_d(s_1,s_2) = 1 + s_1 + s_2 + s_1 s_2 \) can be shown to be VSHP in the following steps:

Step 1. Test the polynomial \( B_d(s_1,s_2) \) to see whether it is a SHP. For this purpose, the procedure suggested by Ansell [12] and modified by Huang [8] can be used. Example (1.1) or (1.2) shows the proof of this problem, which is a SHP. If the given polynomial is a SHP, then proceed to the next step to test if the polynomial is VSHP or not.

Step 2. Checking the condition of VSHP [59]:

\[
\lim_{s_1 \to 0} \left[ \frac{1}{B_d(1/s_1, s_2)} \right] = \frac{s_1}{s_1 + 1 + s_1 s_2 + s_2} \neq \frac{0}{0} \tag{2.14}
\]

\[
\lim_{s_1 \to 0} \left[ \frac{1}{B_d(s_1, 1/s_2)} \right] = \frac{s_2}{s_2 + 1 + s_1 s_2 + s_1} \neq \frac{0}{0} \tag{2.15}
\]

\[
\lim_{s_1 \to 0} \left[ \frac{1}{B_d(1/s_1, 1/s_2)} \right] = \frac{s_1 s_2}{s_1 s_2 + s_2 + s_1 + 1} \neq \frac{0}{0} \tag{2.16}
\]

Therefore the polynomial \( B_d(s_1,s_2) \) is a VSHP.
2-5 GENERATION OF VSHP

In this section a review of some of the previous methods for generating VSHP are given.

2-5.1 GENERATION OF 2-VARIABLE VSHP USING DERIVATIVES OF ODD OR EVEN PARTS OF HURWITZ POLYNOMIALS

This method, which is a refinement of the work presented by Ramamoorthy and Bruton [60], generates a 2-variable VSHP [61] using the following steps:

Step 1. A suitable even or odd part of an n-variable Hurwitz polynomial is generated.

Step 2. The corresponding derivative giving the odd or even part is associated with it.

Step 3. The resulting n-variable Hurwitz polynomial is converted to a two-variable VSHP [59].

With this method, a large number of possibilities exist to generate VSHPs. One such possibility is discussed below. Let us consider the even part of a Hurwitz polynomial as the starting point:

Consider the polynomial $M_{2n}$ given by:

$$M_{2n} = \det \left| \mu I_{2n} + A_{2n} \right|$$  \hspace{1cm} (2.17a)

where $\mu$ is a diagonal matrix of order $2n$ given by:
\[ \mu = \text{diag} \{ \mu_1, \mu_2, \mu_3, \ldots, \mu_{2n} \} \]  

(2.17b)

and \( A_{2n} \) is a skew-symmetric matrix of order 2n given by:

\[
A_{2n} \Delta A = \begin{bmatrix}
0 & a_{1,2} & a_{1,3} & \ldots & a_{1,2n} \\
-a_{1,2} & 0 & a_{2,3} & \ldots & a_{2,2n} \\
-a_{1,3} & -a_{2,3} & 0 & \ldots & a_{3,2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{1,2n} & a_{2,2n} & a_{3,2n} & \ldots & 0
\end{bmatrix}
\]

(2.17c)

and \( I_{2n} \) is the identity matrix of order 2n.

From the diagonal expansion of the determinant of a matrix, \( M_{2n} \) can be written as

\[
M_{2n} = \det A - \sum_{1 \leq i_1 < i_2 \leq 2n} \mu_{i_1} \mu_{i_2} A_{i_1 i_2} \\
+ \sum_{1 \leq i_1 < i_2 < i_3 < \ldots < i_{2n} \leq 2n} \mu_{i_1} \mu_{i_2} \mu_{i_3} \ldots A_{i_1 i_2 i_3 \ldots i_{2n}} + \ldots + \mu_{i_1} \mu_{i_2} \mu_{i_3} \ldots \mu_{2n}
\]

(2.18)

where \( A_{i_1 i_2} \) is the determinant of the sub-matrix of \( A \) obtained by removing both the \( i_1 \)th and \( i_2 \)th rows and columns and is of order 2n-2; \( A_{i_1 i_2 i_3 i_4} \) is the determinant of the sub-matrix of \( A \) obtained by removing both the \( i_1 \)th, \( i_2 \)th, \( i_3 \)th and \( i_4 \)th rows and columns and is of order 2n-4; and so on. The following properties should be noted:

1. All odd-order terms of the type \( \mu_1 \) and \( \mu_1 \mu_2 \mu_3 \), etc., are absent, since the determinant of an odd-order skew-symmetric matrix is zero.

2. The degree of any \( \mu_i \) (\( i = 1, 2, \ldots, 2n \)) is unity.

3. The quantities \( \det A, A_{i_1 i_2}, A_{i_1 i_2 i_3 i_4}, \) etc., are non-negative numbers, since the determinant of an even-order skew-symmetric matrix is a perfect square.
Since the matrix $|\mu^2 + A_2|_n$ is always physically realizable, $M_{2n}$ represents the even part of a 2n-variable Hurwitz polynomial. Therefore, $(\partial M_{2n} / \partial \mu_i)$ is a reactance function. As a consequence

$$M' = M_{2n} - \sum_{i=1}^{2n} K_j \frac{\partial M_{2n}}{\partial \mu_j}$$

(2.19)

is a 2n-variable Hurwitz polynomial, where $K_j$ is a real non-negative constant. From equation (2.19), a two-variable VSHP can be generated by equating some of the $\mu$'s to $s_1$ and the rest to $s_2$, while ensuring that the conditions of a two-variable VSHP are satisfied.

2-5.2 GENERATION OF 2-VARIABLE VSHP USING PROPERTIES OF POSITIVE DEFINITE MATRICES

Ramachandran and Ahmadi [62] have recently shown how a 2-variable VSHP could be directly generated. They imposed conditions on an n-port lossless frequency independent network where each of its terminals are terminated with a $s_1$ or $s_2$ type capacitor, to become a VSHP. Figure (2.1) shows this network. They generalized the formulation through the application of the positive definite matrices, which is explained as follows.
n-port frequency independent network

Figure 2.1

n-port frequency independent network
A symmetric positive-definite matrix is always physically realizable [65]. It is known that any positive definite matrix $P$ can always be decomposed into a product of two matrices, $Q$ and $Q^T$, where $Q$ is either an upper or lower triangular matrix [66]. Matrix $D$ is defined as

$$D = A \Gamma A^T s_1 + B \Delta B^T s_2 + G$$

(2.20)

where $A$ and $B$ in this case are upper-triangular matrices of the form

$$A = \begin{bmatrix}
1 & a_{12} & a_{13} & \ldots & a_{1m} \\
0 & 1 & a_{23} & \ldots & a_{2m} \\
0 & 0 & 1 & \ldots & a_{3m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix}$$

(2.21a)

$$B = \begin{bmatrix}
1 & b_{12} & b_{13} & \ldots & b_{1m} \\
0 & 1 & b_{23} & \ldots & b_{2m} \\
0 & 0 & 1 & \ldots & b_{3m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix}$$

(2.21b)

$\Gamma$ and $\Delta$ are diagonal matrices with non-negative elements;

$$\Gamma = \begin{bmatrix}
\gamma_1^2 & 0 & 0 & \ldots & 0 \\
0 & \gamma_2^2 & 0 & \ldots & 0 \\
0 & 0 & \gamma_3^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \gamma_m^2
\end{bmatrix}$$

(2.21c)
\[
\Delta = \begin{bmatrix}
\delta_1^2 & 0 & 0 & \ldots & 0 \\
0 & \delta_2^2 & 0 & \ldots & 0 \\
0 & 0 & \delta_3^2 & \ldots & 0 \\
0 & 0 & 0 & \delta_m^2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \delta_m^2 \\
\end{bmatrix}
\]

(2.21d)

and \(G\) is a real skew-symmetric matrix given by

\[
G = \begin{bmatrix}
0 & g_{12} & g_{13} & \ldots & g_{1m} \\
-g_{12} & 0 & g_{23} & \ldots & g_{2m} \\
-g_{13} & -g_{23} & 0 & \ldots & g_{3m} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-g_{1m} & -g_{2m} & -g_{3m} & \ldots & 0 \\
\end{bmatrix}
\]

(2.21e)

A 2-variable HP can be obtained by associating the corresponding partial derivatives of \(\det D\) with respect to each variable

\[
D(s_1, s_2) = \det D + K_1 \frac{\partial (\det D)}{\partial s_1} + K_2 \frac{\partial (\det D)}{\partial s_2}
\]

(2.22a)

where \(K_1\) and \(K_2\) are non-negative constants. Similarly, other HPs can be formed using higher-order partial derivatives. The required VSHP is generated using higher-order partial derivatives. The necessary steps are given below, assuming that \(\det D=M\).
1. Form

\[ M_1 = \mathcal{M} - K_{11} \frac{\partial \mathcal{M}}{\partial S_1} - K_{12} \frac{\partial \mathcal{M}}{\partial S_2} \tag{2.22b} \]

This is known to be a two-variable HP.

2. Now form

\[ M_2 = M_1 - K_{21} \frac{\partial M_1}{\partial S_1} - K_{22} \frac{\partial M_1}{\partial S_2} \tag{2.22c} \]

which involves higher-order partial derivatives of \( \mathcal{M} \). This will also be a two-variable HP.

3. This process can be continued until we form

\[ M_n = M_{n-1} + K_{1n} \frac{\partial M_{n-1}}{\partial S_1} + K_{2n} \frac{\partial M_{n-1}}{\partial S_2} \tag{2.22d} \]

It has been proven by Ahmadi and Ramachandran that (2.22d) gives a VSHP.

An alternative approach for generating a 2-variable VSHP is due to Abiri et al. [67]. They used the properties of symmetric positive-definite or positive-semi-definite matrix:

\[ C = A \mu A^T + G \tag{2.23} \]

where \( \mu \) is a diagonal matrix with complex elements given by:

\[ \mu = \text{diag} \left[ \mu_1, \mu_2, \mu_3, \ldots, \mu_n \right] \tag{2.23a} \]

and \( A \) is a square matrix of the form:
\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \ldots & a_{1m} \\
a_{21} & a_{22} & a_{23} & \ldots & a_{2m} \\
a_{31} & a_{32} & a_{33} & \ldots & a_{3m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mm}
\end{bmatrix}
\]  

where \( G \) is a real skew symmetric matrix given by:

\[
G = \begin{bmatrix}
0 & g_{12} & g_{13} & \ldots & g_{1m} \\
-g_{12} & 0 & g_{23} & \ldots & g_{2m} \\
-g_{13} & -g_{23} & 0 & \ldots & g_{3m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-g_{1m} & -g_{2m} & -g_{3m} & \ldots & 0
\end{bmatrix}
\]  

Now, consider

\[
M_{e(o)} = |C_a| = \det C_a = \det [A^\mu A^T + G]
\]  

where subscript \( e(o) \) stands for even (odd). If \( A \) is a non-singular matrix (i.e., \( |A| \neq 0 \)), then we can alternatively construct the following operation:

\[
\det [A^{-1}C_a(A^{-1})^T] = \det [A^{-1}(A^\mu A^T + G)(A^{-1})^T]
\]  

or

\[
|A^\mu| \cdot |C_a| \cdot (A^{-1})^T = \det [\mu + A^{-1}G(A^{-1})^T] = \det (\mu + G')
\]

where \( G' = A^\mu G(A^\mu)^T \). Since \( \det A^{-1} = 1/\det A \) we have
\[ |C_n| = |A|^2 \det(\mu + G^\ast) = |A|^2 \det(W) \quad (2.24c) \]

where \( W = \mu + G^\ast \). But

\[ (G^\ast) = [A^{-1}G(A^{-1})^T]^T = A^{-1}G^T(A^{-1})^T = [-A^{-1}G(A^{-1})^T] = -G^\ast \quad (2.24d) \]

as \( G \) is a skew-symmetric matrix (i.e., \( G^T = -G \)). The equality in equation (2.24d) implies that \( G^\ast \) is also a skew-symmetric matrix.

Since \( W \) is already known to be a reactance matrix, it follows from equation (2.24c) that \( C_n \) is also a reactance matrix (for a reactance matrix \( W \), \( W + W^T = 0 \), \( W^T \) being the transpose of \( W \) with \( \mu_i \) changed to \(-\mu_i\); thus \( C_n \) is always physically realizable. The determinant of \( C_n \) then constitutes either the even part or the odd part of an \( n \)-variable HP, depending on whether the order of the matrix \( A \) is even or odd, with the condition that \( \det A \neq 0 \). It can now be easily proved that \( (\partial \det C / \partial \mu_i)/\det C \) is a reactance function and therefore

\[ M = M(\mu_1, \mu_2, \mu_3, \ldots, \mu_n) = M_{s(0)} + \sum_{j=1}^{n} K_j \frac{\partial M_{s(0)}}{\partial \mu_j} \quad (2.25) \]

is an \( n \)-variable HP with \( K_j \) being positive constants.

Similarly, other HPs can be formed using higher-order derivatives. From equation (2.25), a two-variable HP is generated by putting some \( \mu_i \) equal to \( s_i \), and the rest equal
$s_2$. It should be noted that one of the properties of $M_{e_0}$ (det $C_\alpha$) is that the quantities $\mu_i$ for $i=1,2, \ldots, n$ are of degree unity. This means that the terms $\mu_1^k, \mu_2^k, \mu_3^k, \ldots, \mu_n^k$ for $k=2,3, \ldots, n$ are absent in det $C_\alpha$. In fact, this is one of the conditions required for the generation of a VSHP from det $C_\alpha$. The required VSHP is generated using higher-order partial derivatives as outlined in [68].

The following steps would lead to the generation of a VSHP. First, we form

$$M_1 = M + k_{11} \frac{\partial M}{\partial s_1} + k_{21} \frac{\partial M}{\partial s_2}$$

(2.26a)

where $M$ is a 2-variable HP obtained from equation (2.25). This is known to be a two-variable HP. We now form

$$M_2 = M_1 + k_{12} \frac{\partial M_1}{\partial s_1} + k_{22} \frac{\partial M_1}{\partial s_2}$$

(2.26b)

which involves higher-order partial derivatives of $M$. This will also be a two-variable HP. This process can be continued until we form:

$$M_n = M_{n-1} + k_{1,n-1} \frac{\partial M_{n-1}}{\partial s_1} + k_{2,n-1} \frac{\partial M_{n-1}}{\partial s_2}$$

(2.26c)

It should be noted that quantities $k_{ij}$ in equation (2.26a)-(2.26c) are all positive constants. Equality (2.26c) can easily be shown to be a VSHP.
2-5.3 A COMPUTATIONALLY EFFICIENT METHOD FOR GENERATING 2-VARIABLE VSHP

In this proposed technique the concept of resistive matrices is added to equation (2.20) to avoid the lengthy process of calculating the higher order partial derivatives. Details of this method are given as follows:

Consider the matrix \( D_m \) defined as:

\[
D_m = AA^T s_1 + BB^T s_2 + CC^T + G
\]  

(2.27)

Where \( A \) is a \([m \times m]\) matrix and is of the form:

\[
A = \begin{bmatrix}
ad_{11} & a_{12} & a_{13} & \ldots & a_{1m} \\
o & a_{22} & a_{23} & \ldots & a_{2m} \\
o & 0 & a_{33} & \ldots & a_{3m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
o & 0 & 0 & \ldots & a_{mm}
\end{bmatrix}
\]  

(2.28)

\( a_{ii} \) are non-negative for \( i = 1,2,\ldots,m \) and \( B \) is a \([m \times m]\) matrix as below:

\[
B = \begin{bmatrix}
b_{11} & b_{12} & b_{13} & \ldots & b_{1m} \\
o & b_{22} & b_{23} & \ldots & b_{2m} \\
o & 0 & b_{33} & \ldots & b_{3m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
o & 0 & 0 & \ldots & b_{mm}
\end{bmatrix}
\]  

(2.29)
\(b_i\) are non-negative for \(i = 1, 2, ..., m\). Also \(C\) is a \([m \times m]\) matrix shown as:

\[
C = \begin{bmatrix}
c_{11} & c_{12} & c_{13} & \cdots & c_{1m} \\
0 & c_{22} & c_{23} & \cdots & c_{2m} \\
0 & 0 & c_{33} & \cdots & c_{3m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & c_{mm}
\end{bmatrix}
\]

(2.30)

\(c_{ii}\) are non-negative for \(i = 1, 2, ..., m\) and \(G\) is a real skew-symmetric matrix of the form:

\[
G = \begin{bmatrix}
0 & g_{12} & g_{13} & \cdots & g_{1m} \\
g_{12} & 0 & g_{23} & \cdots & g_{2m} \\
g_{13} & g_{23} & 0 & \cdots & g_{3m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
g_{1m} & g_{2m} & g_{3m} & \cdots & 0
\end{bmatrix}
\]

(2.31)

We can prove that under certain conditions, \(\text{det}D_m\) gives a 2-variable VSHP. To show this, let us partition Matrix \(A\) as:

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
\vdots & \vdots \\
0 & A_{22}
\end{bmatrix}
\]

(2.32)

where \(A_{11}\) is an upper triangular matrix \([k \times k]\), \(A_{12}\) is a \([k \times (m-k)]\) matrix and \(A_{22}\) is an upper triangular matrix \([m-k \times m-k]\). Similarly for matrices \(B\), \(C\) and \(G\) one can write:
\[ B = \begin{bmatrix} B_{11} & B_{12} \\ -- & -- \\ 0 & B_{22} \end{bmatrix} \]  

(2.33)

\[ C = \begin{bmatrix} C_{11} & C_{12} \\ -- & -- \\ 0 & C_{22} \end{bmatrix} \]  

(2.34)

\[ G = \begin{bmatrix} G_{11} & G_{12} \\ -- & -- \\ -G_{12}^T & G_{22} \end{bmatrix} \]  

(2.35)

Now the product of \( A \) and \( A^T \) can be written as

\[
AA^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} A_{11}^T & 0 \\ A_{12}^T & A_{22}^T \end{bmatrix}
\]

\[
= \begin{bmatrix} A_{11}A_{11}^T + A_{12}A_{12}^T & A_{12}A_{22}^T \\ \cdots & \cdots \\ A_{22}A_{12}^T & A_{22}A_{22}^T \end{bmatrix}
\]  

(2.36)

The determinant of \( AA^T \) contains \( s_i^m \) as a factor. Therefore, \( A_{11} \) can be made equal to zero to reduce the power of \( s_i \) to \((m-k)\). Similarly, \( B_{11} \) can be made equal to zero, but \( C_{11} \) need not be zero.
Now \( A \) and \( B \) are positive semi-definite matrices of rank \((m-k)\) and \( C \) is a positive definite matrix of rank \( m \). In this case by equating \( A_{ij} = 0 \) in equation (2.36) one can write:

\[
AA^T = \begin{bmatrix}
A_{12}A_{12}^T & A_{12}A_{22}^T \\
A_{22}A_{12}^T & A_{22}A_{22}^T
\end{bmatrix}
\]

(2.37)

The value of \( AA^T \) determinant is zero. Therefore \( D_m \) can become:

\[
D_m = \begin{bmatrix}
A_{12}A_{12}^T & A_{12}A_{22}^T \\
A_{22}A_{12}^T & A_{22}A_{22}^T
\end{bmatrix} s_1 - \begin{bmatrix}
B_{12}B_{12}^T & B_{12}B_{22}^T \\
B_{22}B_{12}^T & B_{22}B_{22}^T
\end{bmatrix} s_2 + \begin{bmatrix}
C_{11}C_{11}^T - C_{12}C_{12}^T & C_{12}C_{22}^T \\
C_{22}C_{12}^T & C_{22}C_{22}^T
\end{bmatrix} + \begin{bmatrix}
G_{11} & G_{12} \\
-G_{12}^T & G_{22}
\end{bmatrix}
\]

(2.38)

or

\[
D_m = \begin{bmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}
\]

where

\[
\begin{align*}
D_{11} &= A_{12}A_{12}^T s_1 - B_{12}B_{12}^Ts_2 + C_{11}C_{11}^T + C_{12}C_{12}^T + G_{11} \\
D_{12} &= A_{12}A_{22}^T s_1 - B_{12}B_{22}^Ts_2 + C_{12}C_{22}^T + G_{12} \\
D_{21} &= A_{22}A_{12}^T s_1 - B_{22}B_{12}^Ts_2 + C_{22}C_{12}^T - G_{12}^T \\
D_{22} &= A_{22}A_{22}^T s_1 - B_{22}B_{22}^Ts_2 + C_{22}C_{22}^T - G_{22}
\end{align*}
\]

(2.39)

The determinant of (2.39) can be evaluated in several ways, one way [169] is
to write it as:

$$\det D_m = \begin{vmatrix} D_{12} & D_{11} - D_{12} D_{21}^{-1} D_{21} \end{vmatrix}$$

(2.40)

Now let $D_{22}$ be of the order of $2 \times 2$. Consider the $s_1$ portion, it will be:

$$AA^T = \begin{bmatrix} a_{m-1,m-1} & a_{m-1,m} \\ a_{m,m} & a_{m,m} \end{bmatrix} \begin{bmatrix} a_{m-1,m-1} & 0 \\ a_{m-1,m} & a_{m,m} \end{bmatrix}$$

$$= \begin{bmatrix} a_{m-1,m-1}^2 - a_{m-1,m}^2 & a_{m-1,m}^2 \\ a_{m-1,m} a_{m,m} & a_{m,m}^2 \end{bmatrix}$$

(2.41)

This determinant always contains $s_1^2$ terms unless $a_{m-1,m} = 0$, which is required, because in a VSHP, $s_1^2$ by itself can not appear.

A similar situation is for $s_2^2$-term, therefore $b_{m-1,m} = 0$. Now $D_{22}$ can be written as:

$$D_{22} = \begin{bmatrix} a_{m-1,m}^2 & a_{m-1,m} a_{m,m} \\ a_{m-1,m} a_{m,m} & a_{m,m}^2 \end{bmatrix} s_1$$

$$- \begin{bmatrix} b_{m-1,m}^2 & b_{m-1,m} b_{m,m} \\ b_{m-1,m} b_{m,m} & b_{m,m}^2 \end{bmatrix} s_2$$

$$+ \begin{bmatrix} c_{m-1,m-1}^2 - c_{m-1,m}^2 & c_{m-1,m} c_{m,m} \\ c_{m-1,m} c_{m,m} & c_{m,m}^2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & g_{m-1,m} \\ -g_{m-1,m} & 0 \end{bmatrix}$$

(2.42a)
$D_{22} = \begin{bmatrix}
    d_{11} & d_{12} \\
    d_{21} & d_{22}
\end{bmatrix}$

where

\[
\begin{align*}
    d_{11} &= a_{m-1,m}^2 s_1 - b_{m-1,m}^2 s_2 - c_{m-1,m-1}^2 - c_{m-1,m}^2 \\
    d_{12} &= a_{m-1,m} a_{m,m} s_1 - b_{m-1,m} b_{m,m} s_2 - c_{m-1,m} c_{m,m} - g_{m-1,m-1} \\
    d_{21} &= a_{m-1,m} a_{m,m} s_1 - b_{m-1,m} b_{m,m} s_2 + c_{m-1,m} c_{m,m} - g_{m-1,m-1} \\
    d_{22} &= a_{m,m}^2 s_1 - b_{m,m}^2 s_2 + c_{m,m}^2
\end{align*}
\] (2.42b)

It can be easily verified that the factors of $s_1^2$-term and $s_2^2$-term are zero.

Therefore, the $D_{22}$ determinant can be written as

\[
\det D_{22} = \alpha s_1 - \beta s_2 - \gamma s_1 s_2 + \delta
\] (2.43)

where

\[
\begin{align*}
    \alpha &= (a_{m,m} c_{m-1,m} - a_{m-1,m} c_{m,m})^2 - a_{m,m}^2 c_{m-1,m-1}^2 \\
    \beta &= (b_{m,m} c_{m-1,m} - b_{m-1,m} c_{m,m})^2 + b_{m,m}^2 c_{m-1,m-1}^2 \\
    \gamma &= (a_{m,m} b_{m-1,m} - a_{m-1,m} b_{m,m})^2 \\
    \delta &= c_{m-1,m}^2 c_{m,m}^2 + g_{m-1,m-1}^2
\end{align*}
\] (2.44)

The determinant now can be written as:

\[
\begin{align*}
\det D_{22} &= \left[(a_{m,m} c_{m-1,m} - a_{m-1,m} c_{m,m})^2 - a_{m,m}^2 c_{m-1,m-1}^2\right] s_1 \\
&\quad + \left[(b_{m,m} c_{m-1,m} - b_{m-1,m} c_{m,m})^2 + b_{m,m}^2 c_{m-1,m-1}^2\right] s_2 \\
&\quad + (a_{m,m} b_{m-1,m} - a_{m-1,m} b_{m,m})^2 s_1 s_2 \\
&\quad + c_{m-1,m}^2 c_{m,m}^2 + g_{m-1,m-1}^2
\end{align*}
\] (2.48)
As can be seen from Eq. (2.48), all the coefficients are positive and the condition of VSHP is preserved. This means that in the partitioning, it is required to put $a_{m-1,m-1} = 0$ and $b_{m-1,m-1} = 0$, and make sure that rank of the matrix

$$
\begin{bmatrix}
  a_{m-1,m} & b_{m-1,m} \\
  a_{m,m} & b_{m,m}
\end{bmatrix}
$$

is two. In this case determinant of $D_{22}$ constitutes a VSHP. It is assumed that the rank of the matrix $G_{22}$

$$
\begin{bmatrix}
  0 & g_{m-1,m-1} \\
  -g_{m-1,m-1} & 0
\end{bmatrix}
$$

(2.49b)

can be either zero or two.

Another possible partitioning of $A$ and $B$ is as follows:

$$
A = \begin{bmatrix}
  A_{11} & 0 \\
  0 & A_{22}
\end{bmatrix}
$$

$$
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \ldots & a_{1,m-2} \\
  0 & a_{22} & a_{23} & \ldots & a_{2,m-2} \\
  0 & 0 & a_{33} & \ldots & a_{3,m-2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & a_{m-2,m-2}
\end{bmatrix}
\begin{bmatrix}
  A_{11} & 0 \\
  0 & A_{22}
\end{bmatrix}
\begin{bmatrix}
  O \\
  0 & a_{m-1,m} \\
  0 & a_{m,m}
\end{bmatrix}
$$

(2.50)
The process can be continued for the subsequent sub-matrices \( A_{ij} \) and \( B_{ij} \), each of which are of order \([m-2 \times m-2]\).

There is another way of partitioning which is presented as:

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & O \\
  0 & a_{22} & O \\
  \vdots & \vdots & \vdots \\
  O & \vdots & A_{22}
\end{bmatrix}
\]  

(2.52)

where \( A_{ij} \) is an upper triangular matrix of order 2 and \( A_{22} \) is upper triangular matrix of order \((m-2)\). Similarly,

\[
B = \begin{bmatrix}
  b_{11} & b_{12} & O \\
  0 & b_{22} & O \\
  \vdots & \vdots & \vdots \\
  O & \vdots & B_{22}
\end{bmatrix}
\]  

(2.53)
where $B_{22}$ is upper triangular of order $(m-2)$.

Now we can write:

$$AA^T = \begin{bmatrix}
  a_{11}^2 - a_{12}^2 & a_{12}a_{22} & O \\
  a_{12}a_{22} & a_{22}^2 & O \\
  \cdots & \cdots & \cdots \\
  O & O & A_{22}A_{22}^T
\end{bmatrix}$$

(2.54)

If $s_{i'}$ is to be absent, $a_{11}$ should be zero. Similarly, in $BB^T$, $b_{11}$ should be zero.

$$BB^T = \begin{bmatrix}
  h_{11}^2 - h_{12}^2 & b_{12}b_{22} & O \\
  b_{12}b_{22} & b_{22}^2 & O \\
  \cdots & \cdots & \cdots \\
  O & B_{22}B_{22}^T & \cdots
\end{bmatrix}$$

(2.55)

The matrix $D_m$ can be written as:

$$D_m = \begin{bmatrix}
  a_{12}^2s_1 + b_{12}^2s_2 + c_{11}^2 + c_{12}^2 - b_{12}a_{22}s_1 + b_{12}b_{22}s_2 - c_{12}c_{22} - g_{12} & a_{12}a_{22}s_1 + b_{12}b_{22}s_2 - c_{12}c_{22} - g_{12} & O \\
  a_{12}^2s_1 + b_{12}^2s_2 + c_{11}^2 + c_{12}^2 - b_{12}a_{22}s_1 + b_{12}b_{22}s_2 - c_{12}c_{22} - g_{12} & a_{12}a_{22}s_1 + b_{12}b_{22}s_2 - c_{12}c_{22} - g_{12} & O \\
  \cdots & \cdots & \cdots \\
  O & O & D_{m-2}
\end{bmatrix}$$

(2.56)

where:

$$D_{m-2} = A_{22}A_{22}^T + B_{22}B_{22}^T + C_{22}C_{22}^T + G_{22}$$

(2.57)

In this case, the coefficient of $s_1s_2$-term, $s_1$-term, $s_2$-term and constant-term are:
\[ s_{1s_2 \text{-term}} \Rightarrow (a_{12} b_{12} - a_{12} b_{22})^2 \]  
\[ s_1 \text{-term} \Rightarrow (a_{12} c_{22} - a_{22} c_{12})^2 + a_{22} c_{12}^2 \]  
\[ s_2 \text{-term} \Rightarrow (b_{12} c_{22} - b_{22} c_{12})^2 + b_{22} c_{12}^2 \]  
\[ Constant \text{-term} \Rightarrow c_{11}^2 c_{22}^2 + g_{12}^2 \]

This constitutes a VSHP provided \[ \begin{vmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{vmatrix} \neq 0 \].

The following observations can be made from (2.58-2.61).

a) \( c_{12} \) does not affect the constant-term.
b) \( c_{12} \) affects the \( s_i \) and \( s_2 \)-term only.
c) \( a_{12} \) influences \( s_{1s_2} \)-term and \( s_i^* \)-term only.
d) \( b_{12} \) influences \( s_{1s_2} \), and \( s_2 \) terms only.
b) \( g_{12} \) affects the constant-term only.

Similar observations can be made about \( a_{22} \), \( b_{22} \), \( c_{11} \), and \( c_{22} \).

It is also noted that all the terms are positive, and that \( s_i^m \) and \( s_2^m \) terms are absent.

Therefore, the polynomial produced by using a resistive matrix is a VSHP. The proof is similar to that of example (2.5).
EXAMPLE 2.6

For verification of the above method, an example will be presented to generate a second order 2-variable VSHP using matrix $D_2$.

$$D_2 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1^2 & 0 \\ 0 & \alpha_2^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} s_1 + \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1^2 & 0 \\ 0 & \beta_2^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} s_2$$

\[ + \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1^2 & 0 \\ 0 & \gamma_2^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} + \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} \]

(2.62)

$$D_2 = \begin{bmatrix} \alpha_1^2 + a^2\alpha_2^2 & a\alpha_2^2 \\ a\alpha_2^2 & \alpha_2^2 \end{bmatrix} s_1 + \begin{bmatrix} \beta_1^2 + b^2\beta_2^2 & b\beta_2^2 \\ b\beta_2^2 & \beta_2^2 \end{bmatrix} s_2$$

\[ + \begin{bmatrix} \gamma_1^2 - c^2\gamma_2^2 & c\gamma_2^2 \\ c\gamma_2^2 & \gamma_2^2 \end{bmatrix} + \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} \]

(2.63)

Taking the determinant of $D_2$ and setting $\alpha_i = \beta_i = 0$ yields:

$$D(s_1, s_2) = [\alpha_1^2\beta_2^2(b-a)^2] s_1 s_2 + [\alpha_1^2\gamma_1^2 + \alpha_2^2\gamma_2^2(c-a)^2] s_1$$

\[ + [\beta_1^2\gamma_1^2 + \beta_2^2\gamma_2^2(c-b)^2] s_2 + [g^2 + \gamma_1^2\gamma_2^2] \]

(2.64)

As can be seen from (2.64), all $s_1$, $s_2$ and $s_1 s_2$ factors exist and are positive. $s_1^2$ and $s_2^2$ terms are absent. Therefore, the polynomial is a VSHP - refer to example (2.5).

A higher order VSHP can be generated either by cascading several second order polynomials or by starting with higher order matrices in equation (2.27), then it is
required to follow the same procedure described in this section. The latter has the disadvantage of causing sub-optimality in the formation of 2-D analog filter transfer function.

If Ahmadi's method [109] is utilized to generate the same order VSHP (Appendix A), its second order partial derivative is a VSHP:

\[
D_2(s_1,s_2) = \left( (a_1-b_2)^2 \alpha^2 \beta^2 \right)
\cdot \left[ s_1s_2 + (k_2+k_1)s_1 
+ (k_4+k_3)s_2 
+ k_2k_3 + k_4k_4 \right] + s_2^2
\] (2.65)

In comparison with the method that we presented for a second order VSHP (2.64), generation of polynomials using the proposed technique allows a greater degree of freedom than the existing ones, since for the same order polynomial, it has more independent parameters.

Also, the coefficients of \(s_1, s_2, \) and \(s_1s_2\) are different, which is not the case for the method presented by Ahmadi and Ramachandran [62].

2-7 CONCLUSIONS

In this chapter definitions of various types of Hurwitz polynomials, value of the function at infinity and nonessential singularities of the first and second kind were discussed. An efficient method for generating a 2-variable VSHP was presented. In this
method, properties of positive definite matrices along with resistive matrices were utilized to generate VSHPs. The order of the matrices determined the order of the generated VSHPs. Higher order VSHPs can therefore be obtained by either increasing the order of the matrices or by cascading lower order VSHPs. The latter has the disadvantage of causing sub-optimality in the formation of 2-D analog filter transfer functions.

The method presented here does not require the calculation of higher order derivatives and therefore it is computationally more efficient. We also gave various partitioning techniques which can be used by the designer to obtain a VSHP. Application of this technique to the design of 2-D recursive filters was also presented.
DESIGN OF 2-D RECURSIVE DIGITAL FILTERS USING ITERATIVE TECHNIQUES

3-1 INTRODUCTION

This chapter deals with the design of 2-D recursive digital filters satisfying a prescribed magnitude with or without group-delay specifications. The method to be presented here deals with the general class of 2-D filters as well as a special class of 2-D filters namely separable denominator non-separable numerator transfer functions.

The proposed methods are iterative in nature and use non-linear optimization techniques or a combination of both linear and non-linear optimization techniques to determine the coefficients of the designed filter.
3-2 DESIGN OF 2-D RECURSIVE DIGITAL FILTERS USING NON-LINEAR PROGRAMMING

To design a 2-D recursive digital filter using the proposed method the following steps must be followed:

(i) Generate a two-variable VSHP using the concept of resistance matrices in conjunction with the properties of the positive definite matrices as described in chapter two and assign this to the denominator of a 2-D analog filter expressed as:

\[
H(s_1, s_2) = \frac{A(s_1, s_2)}{B(s_1, s_2)} = \frac{\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} n_{ij} s_1^i s_2^j}{\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} d_{ij} s_1^i s_2^j}
\]

(3.1)

(ii) The analog 2-D transfer function is then discretized by applying the double bilinear transformation

\[
H(z_1, z_2) = H(s_1, s_2) \left| \begin{array}{c}
    s_i = \frac{2}{T} \frac{1-z_i^{-1}}{1+z_i^{-1}} \\
    i=1,2
\end{array} \right.
\]

(3.2a)

\[
H(z_1, z_2) = \frac{\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} a_{ij} z_1^{-i} z_2^{-j}}{\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} b_{ij} z_1^{-i} z_2^{-j}}
\]

(3.2b)
(iii) Now the error of the magnitude response can be calculated using equation

\[
E_{Mag}(e^{j\alpha n_1}, e^{j\alpha n_2}, \Psi) = |H_i(e^{j\alpha n_1}, e^{j\alpha n_2})| - |H_D(e^{j\alpha n_1}, e^{j\alpha n_2}, \Psi)| \tag{3.3}
\]

where \(E_{Mag}\) is the error of the magnitude response and \(\Psi\) is the coefficient vector to be calculated while \(|H_i|\) and \(|H_D|\) are the magnitude responses of the ideal and designed filters, respectively. The group delay response \(E_{\tau_i}\) is

\[
E_{\tau_i} = \tau_i T - \tau_{D_i}(\omega_{1m}, \omega_{2n}, \Psi) \quad i=1,2 \tag{3.4}
\]

where \(\tau_i\) is a constant representing the ideal group delay response of the filter. \(\tau_{D_i}(i=1,2)\) is the group delay response of the designed filter.

(iv) By using equations (3.3) and (3.4) the general mean square error can be formulated, for approximation of magnitude and group delay response of the filter, as follows

\[
E_{\ell_i}(j\omega_{1m}, j\omega_{2n}, \Psi) = \sum_{m,n \in \ell_{rs}} E_{Mag}^2(j\omega_{1m}, j\omega_{2n}, \Psi) + \sum_{m,n \in \ell_{P}} E_{\tau_1}^2(j\omega_{1m}, j\omega_{2n}, \Psi) + \sum_{m,n \in \ell_{P}} E_{\tau_2}^2(j\omega_{1m}, j\omega_{2n}, \Psi) \tag{3.5}
\]
where $I_{ps}$ is a set of all discrete frequency pairs along the $\omega_1$ and $\omega_2$ axis in the pass-band and stop-band of the filter and $I_p$ is a set of all discrete frequency pairs along the $\omega_1$ and $\omega_2$ axis in the pass-band of the filter only.

(v) To design a stable 2-D filter satisfying a prescribed magnitude and constant group delay response, $\Psi$ should be calculated in such a way that $E_t$ in equation (3.5) is minimized subject to constraints needed for the denominator polynomial to be VSHP. It should be noted that, if only magnitude specification is required, calculation of the sum of squares of group delay error in the last two terms of equation (3.5) will be dropped. This is a nonlinear optimization problem, which can be solved by using the Fletcher and Powell technique [57].

Also, it should be noted that since the optimization procedure is actually carried out in the $z$-domain, the inherent warping effect of bilinear transformation will be eliminated by this process.

**DESIGN EXAMPLE 3.1**

In this example a 2-variable VSHP is generated using the method discussed earlier and assigned to the denominator of an analog transfer function of the equation (3.1). Then by the application of bilinear transformation, the discrete version of the filter is obtained from equation (3.1). The new parameters of the 2-D digital filter, $a_y$ and $b_y$ are used as the variables of the optimization, in such a way that the desired objective function is minimized. The objective function is defined in equation (3.5).
The cost function, which contains the error of magnitude between the ideal and the designed filter and $E_r$, group delay error in both directions, can be calculated using equation (3.5). In the example to be considered here a fourth order digital filter is designed with the following specifications:

$$
|H_f(e^{j\omega_m},e^{j\omega_n})| = \begin{cases} 
1 & 0.0 \leq \sqrt{\omega_{1m}^2 + \omega_{2n}^2} \leq 1.0 \\
0 & 1.5 \leq \sqrt{\omega_{1m}^2 + \omega_{2n}^2} \leq \pi 
\end{cases}
$$

(3.6)

With constant group delay characteristic, optimization technique [69] is used to minimize the objective function of equation (3.5). Table(3.1) shows the values of the coefficients of the designed filter, while Fig.(3.1) shows the magnitude plot of the designed filter.

**Table 3.1**

<table>
<thead>
<tr>
<th>Numerator Coefficients</th>
<th>Denominator Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{0,0} = 0.24250460E+01$</td>
<td>$b_{0,0} = 0.16435898E+03$</td>
</tr>
<tr>
<td>$a_{0,1} = 0.40828800E+01$</td>
<td>$b_{0,1} = -0.10108646E+03$</td>
</tr>
<tr>
<td>$a_{0,2} = 0.66948986E+01$</td>
<td>$b_{0,2} = 0.29635269E+02$</td>
</tr>
<tr>
<td>$a_{1,0} = 0.52808161E+01$</td>
<td>$b_{1,0} = -0.64718079E+02$</td>
</tr>
<tr>
<td>$a_{1,1} = 0.83481598E+01$</td>
<td>$b_{1,1} = -0.13981224E+02$</td>
</tr>
<tr>
<td>$a_{1,2} = 0.75414143E+01$</td>
<td>$b_{1,2} = 0.15327198E+02$</td>
</tr>
<tr>
<td>$a_{2,0} = 0.57536539E+01$</td>
<td>$b_{2,0} = -0.36191940E+01$</td>
</tr>
<tr>
<td>$a_{2,1} = 0.71639252E+01$</td>
<td>$b_{2,1} = 0.32147812E+02$</td>
</tr>
<tr>
<td>$a_{2,2} = 0.37494564E+01$</td>
<td>$b_{2,2} = -0.10551155E+02$</td>
</tr>
</tbody>
</table>

The values of the coefficients of 2-D low-pass filter.
Figure (3.2) shows the group delay response of the designed filter with respect to the $\omega_1$ direction. Figure (3.3) shows the group delay response of the designed filter with respect to the $\omega_2$ direction.

Figure 3.1

Magnitude plot of the 2-D low-pass designed filter
Figure 3.2

Group delay response of the designed filter with respect to the $\omega_r$ direction
Figure 3.3

Group delay response of the designed filter with respect to the $\omega_2$ direction
3-3 DESIGN OF 2-D CIRCULAR SYMMETRIC RECURSIVE DIGITAL FILTER USING SEPARABLE DENOMINATOR AND NON-SEPARABLE NUMERATOR

Recently many researchers have focused on the design of 2-D recursive digital filters with separable denominator and non-separable numerator transfer function [2,45,55,107,128]. In order to reduce the computational burden of the design process of separable denominator and non-separable numerator transfer function, the optimization process is broken into two steps. In the first step, denominator coefficients are determined through either a nonlinear optimization technique or an analytical approach for the design of 1-D all pole or pole, zero filters.

After proper discretization process the coefficients of the numerator are calculated in the second phase of optimization to equalize the magnitude or magnitude and phase of the 2-D filter response. This technique, though sub-optimal, has been shown by Kwan et al. [29] and Ahmadi et al. [55] to be very efficient.

The transfer function of these filters has the following form:

\[
H(z_1,z_2) = \frac{\sum_{i=0}^{M_1} \sum_{j=0}^{N_1} a_{ij} z_1^{-i} z_2^{-j}}{\left( \sum_{i=0}^{M_2} b_i z_1^{-i} \right) \left( \sum_{j=0}^{N_2} b_j z_2^{-j} \right)} = \frac{A(z_1,z_2)}{B_1(z_1)B_2(z_2)}
\]

(3.7)

To ensure stability
\[ B_i(z_i) \neq 0 \quad |z_i| \geq 1 \quad \text{for} \quad i = 1, 2 \quad (3.8) \]

Without loss of generality one can assume \( M_1 = N_1 = M_2 = N_2 = M \). The advantages of the above transfer function can be cited as follows:

(i) Stability problem associated with this class of filters is that of a 1-Dimensional one which is easy to check and if needed to stabilize.

(ii) The transfer function of equation (3.7) requires \((M^2 - 1)\) less multiplier coefficients than the general class of non-separable numerator and denominator transfer functions.

(iii) Circular symmetric cut-off boundary is obtained if in equation (3.7)

\[ a_{i,j} = a_{M-i,j} = a_{i,M-j} = a_{M-i,M-j} \]

\[ b_{1i} = b_{2j} \quad (3.9) \]

In this case the transfer function of equation (3.7) can be written as (with \( M \) even)

\[ H(z_1, z_2) = \frac{\sum_{i=0}^{M/2} \sum_{j=0}^{M/2} a'_{i,j} \cos \omega_1 \cos j \omega_2}{B_1(z_1) B_2(z_2)} \quad (3.10.a) \]

where:

\[ \cos \omega_i = \frac{z_i^{-1} + z_i}{2} \quad \text{and} \quad z_i = e^{i\omega_i T} \quad \text{for} \quad i = 1, 2 \quad (3.10.b) \]
or alternatively

\[
H(z_1, z_2) = \frac{\sum_{i=0}^{M/2} \sum_{j=0}^{M/2} a''_{i,j} (\cos \omega_1)^i (\cos \omega_2)^j}{B_1(z_1) B_2(z_2)}
\]

(3.11)

As can be seen from equations (3.10.a) and (3.11), a further reduction of multiplier coefficients is obtained through the constraints of equation (3.9).

(iv) Octagonal symmetric filter is also achieved if extra constraint is added to equation (3.9), i.e.

\[
a_{i,j} = a_{j,i}
\]

(3.12)

(v) It has also been shown in [55] that non-symmetrical cutoff boundary filter is obtained using this type of transfer function.

By designing a 2-D filter we mean calculation of the coefficients of the transfer function \(a, a'\) or \(a''\) in equations (3.7), (3.10.a) and (3.11) in such a way that the magnitude response of the designed filter approximates the desired one. In the next section we present an efficient algorithm for the design of this class of filters.
3.3.1 DESIGN METHOD UTILISING NON-LINEAR ITERATIVE APPROACH

In the method to be presented here a 1-D analog filter $H(s)$ is designed using Butterworth, Chebyshev or Elliptic filter [9] so that the magnitude response of the 2-D filter along $\omega_1$ and $\omega_2$ axes is approximated. By application of bilinear transformation in variables $z_1$ and $z_2$ to $H(s)$ and cascading them together a 2-D filter with rectangular cutoff boundary is generated.

$$H_1(z_1, z_2) = H(z_1) \cdot H(z_2)$$

$$H(z_i) = H(s)$$

$$s_i = \frac{2}{T} \cdot \frac{1-t_i^4}{1+t_i^4} \quad i=1,2$$

(3.13)

(3.14)

Now $H_i(z_1, z_2)$ is cascaded by a 2-D FIR filter with either quadrantal or octagonal symmetry property which is

$$H_2(z_1, z_2) = \sum_{i=0}^{M/2} \sum_{j=0}^{M/2} a'_{ij} \cos i\omega_1 \cos j\omega_2$$

(3.15)

-with $a'_{ij} = a'_{ji}$ for octagonal symmetry - or alternatively
\[ H_2(z_1, z_2) = \sum_{i=0}^{M/2} \sum_{j=0}^{M/2} a''_{i,j} (\cos \omega_1)^i (\cos \omega_2)^j \]  

(3.16)

- with \( a''_{y} = a''_{\mu} \) for octagonal symmetry.

This will yield

\[ H(z_1, z_2) = H_I(z_1, z_2) \cdot H_S(z_1, z_2) \]  

(3.17)

The magnitude error at different grid points along \( \omega_1 \) and \( \omega_2 \) axes are calculated using

\[ E_{Mag}(e^{j\omega_{1}T}, e^{j\omega_{2}T}, \Psi) = |H_I(e^{j\omega_{1}T}, e^{j\omega_{2}T})| \]  

(3.18)

\[ - |H_D(e^{j\omega_{1}T}, e^{j\omega_{2}T}, \Psi)| \]

where \( \Psi \) is the coefficient vector and \(|H_I|\) and \(|H_D|\) are the magnitude response of the ideal and designed filter respectively. The mean square error can be calculated using the relationship

\[ E_t(e^{j\omega_{1}T}, e^{j\omega_{2}T}, \Psi) = \sum_{m} \sum_{n} E^2_{Mag}(e^{j\omega_{1}T}, e^{j\omega_{2}T}, \Psi) \]  

(3.19)

Since \( b_{II} \) and \( b_{I} \) are calculated using the analytical method for designing 1-D
Butterworth / Chebyshev / Elliptic filter these coefficients will be all constants in the second phase of our design process where $a'_g$ or $a''_g$ are to be determined by minimizing $E_{i2}$ in equation (3.19) using a suitable nonlinear optimization method. In this section we have used the Fletcher and Powell technique [69] to calculate the above coefficients. It should be noted that, the solution obtained using this algorithm is mostly sub-optimal. This is because of the nature of the objective function which is a nonlinear function of the optimization parameters and finding global minima can not be guaranteed through the application of this algorithm. However, this technique offers high speed processing and the results obtained are satisfactory.

A comparison of computation costs of the proposed algorithm to that of [30] indicates a 50% reduction on the number of grid points for $E_{i2}$ evaluation of quadrantal symmetric filters and 75% reduction for octagonal symmetric filters. Also, there is some reduction on the number of coefficients' calculation through optimization, which are $2(M+1)$ for central symmetric and $(M+1)$ for quadrantal and octagonal symmetric filters.

**DESIGN EXAMPLE 3.2**

To illustrate the usefulness of the proposed method a 2-D filter is designed with the following magnitude specification
\begin{equation}
H_f(e^{j\omega_1 T}, e^{j\omega_2 T}) = \begin{cases} 
1 & 0.0 \leq \sqrt{\frac{\omega_1^2}{\omega_{1m}^2} + \frac{\omega_2^2}{\omega_{2n}^2}} \leq 1.0 \\
0 & 1.5 \leq \sqrt{\frac{\omega_1^2}{\omega_{1m}^2} + \frac{\omega_2^2}{\omega_{2n}^2}} \leq \pi 
\end{cases}
\end{equation}

\textit{H}(s) is a fourth order Chebyshev analog filter. Figure (3.4) shows the magnitude plot of the ideal filter. Figure (3.5) shows the magnitude response and contour plot of a 2-D Chebyshev filter while figure (3.6) shows the magnitude and contour plot of the resulting filter. Table (3.2) shows the coefficients of the Chebyshev filter along with the coefficients of the FIR filter. Figure (3.7) shows the magnitude plot of the FIR filter.

<table>
<thead>
<tr>
<th>Numerator Coefficients</th>
<th>Denominator Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{00} = 0.68700123$</td>
<td>$b_{10} = 1.0592537$</td>
</tr>
<tr>
<td>$a_{01} = 3.70566654$</td>
<td>$b_{11} = 2.8656217$</td>
</tr>
<tr>
<td>$a_{02} = 4.22173405$</td>
<td>$b_{12} = 4.7977681$</td>
</tr>
<tr>
<td>$a_{10} = -1.96264458$</td>
<td>$b_{13} = 3.3460839$</td>
</tr>
<tr>
<td>$a_{11} = 3.72037792$</td>
<td>$b_{14} = 2.7944915$</td>
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<tr>
<td>$a_{12} = -2.18535328$</td>
<td>$b_{20} = 2.2220230$</td>
</tr>
<tr>
<td>$a_{20} = 0.17578387$</td>
<td>$b_{21} = 2.19898033$</td>
</tr>
<tr>
<td>$a_{22} = -2.19898033$</td>
<td>Chebyshev</td>
</tr>
</tbody>
</table>

The coefficients of the Chebyshev filter along with the coefficients of the FIR filter.
Figure 3.4

The magnitude plot of the ideal filter.
Figure 3.5

The magnitude response and contour plot of 2-D Chebyshev filter
Figure 3.6

The magnitude and contour plot of resulting filter with Chebyshev prototype
DESIGN EXAMPLE 3.3

The magnitude specification in this example is the same as in the previous example, but $H(s)$ is a fourth order Butterworth analog filter. Figure (3.8) shows the magnitude response and contour plot of a 2-D Butterworth filter and figure (3.9) shows the magnitude and contour plot of the resulting filter. Table (3.3) shows the coefficients of the Butterworth and FIR filter.

<table>
<thead>
<tr>
<th>Numerator Coefficients</th>
<th>Denominator Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{00} = 2.08909512$</td>
<td>$b_{10} = 1.000000$</td>
</tr>
<tr>
<td>$a_{01} = -5.41446209$</td>
<td>$b_{20} = 1.000000$</td>
</tr>
<tr>
<td>$a_{02} = 3.62768555$</td>
<td>$b_{11} = 2.613000$</td>
</tr>
<tr>
<td>$a_{10} = -1.64257622$</td>
<td>$b_{21} = 2.613000$</td>
</tr>
<tr>
<td>$a_{11} = 3.41228390$</td>
<td>$b_{12} = 3.414000$</td>
</tr>
<tr>
<td>$a_{12} = -3.34343910$</td>
<td>$b_{22} = 3.414000$</td>
</tr>
<tr>
<td>$a_{20} = -0.54663271$</td>
<td>$b_{13} = 2.613000$</td>
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<td>$a_{21} = 0.87786937$</td>
<td>$b_{23} = 2.613000$</td>
</tr>
<tr>
<td>$a_{22} = 2.68427849$</td>
<td>$b_{14} = 1.000000$</td>
</tr>
</tbody>
</table>

The coefficients of the Butterworth and FIR filter
Figure 3.7

The magnitude plot of the FIR filter
Figure 3.8

The magnitude response and contour plot of 2-D Butterworth filter
Figure 3.9

The magnitude and contour plot of resulting filter associated with Butterworth filter
3-3.2 DESIGN OF 2-D FILTERS USING A COMBINATION OF LINEAR AND NON-LINEAR PROGRAMMING

In the design method to be presented here, the following steps are taken.

(i) Design two 1-D digital filters satisfying the magnitude specification of the desired 2-D filter along \( \omega_1 \) and \( \omega_2 \) axes with or without constant group-delay response using the technique given in [55]. Note that if the phase specification is not of any concern two 1-D analog filters -Butterworth, Chebyshev or Elliptic- can be designed and then discretized for this purpose. In this step

\[
H_1(z_1, z_2) = H(z_1) \cdot H(z_2)
\]

(3.21)

is a separable product 2-D filter with rectangular cutoff boundary.

(ii) Cascade the designed \( H_1(z_1, z_2) \) with a 2-D nonrecursive filter of the form

\[
H_2(z_1, z_2) = \sum_{i=0}^{M/2} \sum_{j=0}^{M/2} a''_{ij} \cos i\omega_1 \cos j\omega_2
\]

or

\[
H_2(z_1, z_2) = \sum_{i=0}^{M/2} \sum_{j=0}^{M/2} a''_{ij} (\cos i\omega_1 \gamma)(\cos j\omega_2 \gamma)
\]

(3.23)

where

\[
\cos \omega_i = \frac{z_i^{-1} + z_i}{2} \quad \text{and} \quad z_i = e^{j\omega_i \tau} \quad \text{for} \quad i = 1, 2
\]

(3.23.b)
Note that cascading $H_1$ by either of $H_2$ above will yield a 2-D filter with the same phase characteristics of $H_1$ since $H_2$ is a linear phase 2-D filter.

(iii) Calculate the error of the magnitude response as

$$E_{Mag}(j \omega_1, j \omega_2) = |H_1| - |H_1| \cdot |H_2| \leq \varepsilon \quad (3.24)$$

where $|H_1|$ is the magnitude response of the ideal filter. By dividing both sides of the inequality by $|H_1|$, which is known at this stage and keep $|H_2|$ on one side

$$|H_2| \geq \frac{|H_1|}{|H_1|} - \frac{\varepsilon}{|H_1|} \quad (3.25)$$

inequality (3.25) can be written as

$$\begin{cases} 
H_2 \geq \frac{|H_1|}{|H_1|} - \frac{\varepsilon}{|H_1|} \\
H_2 \leq -\frac{|H_1|}{|H_1|} + \frac{\varepsilon}{|H_1|} 
\end{cases} \quad (3.26)$$

or alternatively one can write

$$\begin{cases} 
H_2 - \frac{\varepsilon}{|H_1|} \geq \frac{|H_1|}{|H_1|} \\
H_2 - \frac{\varepsilon}{|H_1|} \leq -\frac{|H_1|}{|H_1|} 
\end{cases} \quad (3.27)$$
(iv) Calculate $a_{ij}$ or $a_{ij}^*$ by minimizing $e$ subject to the two sets of constraints in (3.27)

\[
\begin{align*}
\text{Minimize} & \quad e \\
\text{S. to} & \quad \begin{cases} 
H_2 - \frac{e}{|H_1|} \geq \frac{|H_2|}{|H_1|} \\
H_2 - \frac{e}{|H_1|} \leq -\frac{|H_1|}{|H_1|}
\end{cases}
\end{align*}
\]  
(3.28)

A linear programming can be used for the determination of the coefficients of the $H_2(z_1, z_2)$.

**DESIGN EXAMPLE 3.4**

To illustrate the usefulness of the proposed method a 2-D filter is designed with the following magnitude specification (arbitrary phase response):

\[
|H_1(e^{j\omega_1}, e^{j\omega_2})| = \begin{cases} 
e^{-j\sqrt{\omega_1^2 + \omega_2^2}} & \text{for } 0.0 \leq \sqrt{\omega_1^2 + \omega_2^2} \leq 0.8 \\
e^{-j\sqrt{\frac{1}{2}(\omega_1^2 + \omega_2^2)}} & \text{for } 0.8 < \sqrt{\omega_1^2 + \omega_2^2} \leq \pi
\end{cases}
\]  
(3.29)

where $\omega_c$ is representing the cutoff frequency of the filter. In this example, the method of [149] was used to derive 1-D all pole filter with $z_1$ and $z_2$ variables. Coefficients of these two identical 1-D filters are shown in table (3.4). $H_2(z_1, z_2)$ is of the form
\[ H_z(z_1, z_2) = \sum_{n_1} \sum_{n_2} a_{n_1 n_2} (\cos \omega_1)^{n_1} (\cos \omega_2)^{n_2} \]  

(3.30)

In order to reduce the size of the linear programming problem, the number of constraints, which is twice the number of grid points, must be reduced. To do this, appropriate points are chosen to satisfy the specification of the filter in one direction. Then using the relationship:

\[
\begin{align*}
\omega_{1m} &= \omega_n \cos \phi \\
\omega_{2n} &= \omega_n \sin \phi
\end{align*}
\]  

(3.31)

the 2-D sample points are formed, which are shown in figure (3.10).

In this example 400 grid points are chosen. The constraint set in equation (3.28) represents 800 functions. In this case, we have to evaluate 9 variables, which may assume negative or positive values. However, the simplex method cannot handle negative variables. The remedy is to replace each variable with two non-negative variables. Thus in the final formulation of the problem, we have a total of 19 variables (18 variables resulting from the replacement and the variable \( e \)), and 800 constraints. It is easier to solve the dual of this problem.

The standard linear programming algorithm uses the values of the Right Hand Sides -RHS- as the starting point. These values, represented by \( |H_i|/|H_i| \), depend on the form of the ideal filter. Various experimentations revealed that exponential approximation of the ideal filter for pass-band, stop-band and transient region, lead to the
faster convergence of the solution. Figure (3.10) shows the section of the ideal filter with chosen grid points. Figure (3.11) illustrates the three-Dimensional plot of ideal filter.

The values of $\alpha$ and $\beta$ in equation (3.29) are, $\alpha \geq 1$ and $\beta \leq 1$.

Figure (3.12) shows the magnitude and contour plot of the all pole 2-D filter $H_j(z_1,z_2)$. Table (3.5) shows the values of the coefficients of $H_2(z_1,z_2)$, which are found using linear programming with $\alpha = 6.0$ and $\beta = 0.2$. Figure (3.13) shows its magnitude and contour plot. Figures (3.14) and (3.15) show the magnitude and contour plot of the designed 2-D filter in absolute value and DB respectively.

### Table 3.4

<table>
<thead>
<tr>
<th>Denominator</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_{10} = 5.563914$</td>
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<td>$b_{11} = -11.47563$</td>
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<td>$b_{12} = 11.87721$</td>
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</tr>
<tr>
<td>$b_{13} = -6.616661$</td>
<td>$b_{23} = -6.616661$</td>
</tr>
<tr>
<td>$b_{14} = 1.651168$</td>
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</tr>
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</table>

Values of the coefficients of all pole filter

### Table 3.5

<table>
<thead>
<tr>
<th>Numerator</th>
<th>Coefficients</th>
</tr>
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<tbody>
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<td>$a_{00} = 0.25633241$</td>
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</tr>
<tr>
<td>$a_{11} = 0.57964734$</td>
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</tr>
</tbody>
</table>

Values of the coefficients of $H_j(z_1,z_2)$
Figure 3.10

2-D sample points
Figure 3.11

Ideal filter
Figure 3.12

Magnitude and contour plot of $H_f(z_1, z_2)$
Figure 3.13

Magnitude and contour plot of $H_2(z_1, z_2)$
Figure 3.14

Magnitude and contour plot of designed 2-D filter
Figure 3.15

Magnitude and contour plot of the designed filter in DB
3-3.3 SENSITIVITY ANALYSIS

For analyzing the sensitivity of a linear programming, the effect of changes in the
coefficients of variables and the RHSs on the solution of the problem will be examined.
Usually in real life by some measurement we can collect the data of a problem. It is
possible that the data are collected with a certain physical tolerance. This may cause
small changes in the coefficients of the objective function and the RHSs of the
constraints. As a result, these changes will effect the solution of the problem [38]. Thus,
we should investigate this effect on our problem.

In filter design the cost function has only one variable \( c \). Any changes in the
coefficient of this variable does not affect the solution as long as the problem involves
minimizing a single variable. Also, the solution to this problem is sensitive to the small
changes in the RHS values. Any changes in \( u_2 \) values of \( |H_c|/|H_1| \) -RHS- means
a different set of specifications for the ideal filter. Therefore, after optimization we may
obtain the coefficients of a different filter.

In example (3.4), the optimal objective function value was found to be equal to
0.1770316747. The sensitivity measures show that the range of the objective function
coefficient is between zero to infinity. This means that, for an objective function of the
form:

\[
\min \quad k \epsilon \quad (3.28.1)
\]

the design of the filter remains unchanged as long as \( k > 0 \). The sensitivity measures
for the 800 RHSs values show that nine of them are restricted within a short tolerance range, indicating that they are very critical and the program is sensitive to the changes in the RHSs coefficients. Table (3.6) shows those critical RHSs with their corresponding tolerance ranges.

<table>
<thead>
<tr>
<th>RHS</th>
<th>MINIMUM</th>
<th>MAXIMUM</th>
</tr>
</thead>
<tbody>
<tr>
<td>NO.7</td>
<td>0.70327996</td>
<td>0.70328009</td>
</tr>
<tr>
<td>NO.20</td>
<td>0.00004274</td>
<td>0.00004465</td>
</tr>
<tr>
<td>NO.330</td>
<td>0.70328002</td>
<td>0.70328014</td>
</tr>
<tr>
<td>NO.400</td>
<td>0.00004465</td>
<td>0.00004656</td>
</tr>
<tr>
<td>NO.401</td>
<td>-1.02916930</td>
<td>-0.95122150</td>
</tr>
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<td>NO.418</td>
<td>-0.56019060</td>
<td>-0.56018710</td>
</tr>
<tr>
<td>NO.494</td>
<td>-0.56018760</td>
<td>-0.56018700</td>
</tr>
<tr>
<td>NO.532</td>
<td>-0.56018720</td>
<td>-0.56018680</td>
</tr>
<tr>
<td>NO.589</td>
<td>-0.56018810</td>
<td>-0.56018690</td>
</tr>
</tbody>
</table>

Maximum and minimum values of RHSs
3-4 QUANTIZATION OF THE FILTER COEFFICIENTS

In the realization of IIR filters in hardware on a VLSI chip or software on a general purpose computer, the accuracy with which filter coefficients can be specified is limited by the word length of the computer or the length of the register that is provided to store the coefficients. Since the coefficients used in implementation of a given filter are not exact, the poles and zeros of the resulting system function will in general be different from the desired poles and zeros. Consequently, we obtain a filter that has a frequency response which is different from the frequency response of the filter with unquantized coefficients. However, when these coefficients are quantized by means of rounding or truncation, the frequency response of the digital filter may fail to meet the desired specifications [70,71].

3-4.1 SENSITIVITY

Sensitivity of a 2-D transfer function with respect to its coefficients has been presented by Lampropoulos and Fahmy [78]. The transfer function of 2-D IIR filter can be defined as:

$$H(z_1, z_2) = \frac{\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} a_{ij} z_1^{-i} z_2^{-j}}{1 + \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} b_{ij} z_1^{-i} z_2^{-j}}$$ (3.32)

and the sensitivity is
\[ S(z_1, z_2) = \sum_i \left[ \frac{\partial H(z_1, z_2)}{\partial C_i} \right]^2 \]  

(3.33)

where \( C \) is the coefficients vector, \((a_0, a_0, \ldots, a_{N,1}, b_0, b_0, \ldots, b_{N,1})\).

### 3.4.2 Design of 2-D Filter with Integer Coefficients

Among the many articles [72-5] in the area of the design of filters with finite coefficients length, Kodek [72] has presented a technique to design an optimal finite word-length FIR digital filter using integer programming. Jing and Fam [73] have used a technique based on discretization and re-optimization to minimize the performance error caused by truncation of coefficients.

An improved form of the algorithm proposed by Wan and Fahmy [74], has been presented by Kim [75]. Their algorithm starts by varying the most sensitive element of the coefficient vector, the variation is carried out by integer quantization step of \( q=2^b \), where \( b \) is the number of bits used in representing the coefficients, in a direction such that decreases the objective function. After fixing this coefficient, the optimization program will be repeated for the rest of the coefficients. The sensitivity is measured to find the next sensitive coefficients and the procedure is continued until all the coefficients are covered.

Two examples will be presented here to design filters with integer coefficients.
EXAMPLE 3.5

In this example the 2-D IIR filter with infinite precision coefficients of example (3.4) is used as a prototype filter. Then by using the algorithm given in [75], the integer coefficient version of the filter is obtained.

First the most sensitive coefficient of the filter is selected. Then, the cost function is calculated by rounding the coefficient up and down. The rounded coefficient corresponding to the lowest cost function is fixed and a new set of coefficients is obtained by re-optimizing the cost function. The procedure is repeated for the remaining coefficients.

Table (3.7) presents the new coefficients of the 2-D IIR filter. Figure (3.16) shows the three-dimensional magnitude plot and two dimensional contour plot of the filter with the new coefficients. As can be seen from figure (3.16) the magnitude of the filter is acceptable.
Table 3.7

<table>
<thead>
<tr>
<th>Numerator</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{00} = 0$</td>
<td>$a_{12} = 40$</td>
</tr>
<tr>
<td>$a_{01} = -10$</td>
<td>$a_{20} = 11$</td>
</tr>
<tr>
<td>$a_{02} = 10$</td>
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<td>$a_{22} = -60$</td>
</tr>
<tr>
<td>$a_{11} = -10$</td>
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</table>

<table>
<thead>
<tr>
<th>Denominator</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_{10} = 560$</td>
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<tr>
<td>$b_{11} = 115$</td>
<td>$b_{21} = 115$</td>
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<td>$b_{12} = 120$</td>
<td>$b_{22} = 120$</td>
</tr>
<tr>
<td>$b_{13} = 66$</td>
<td>$b_{23} = 66$</td>
</tr>
<tr>
<td>$b_{14} = 16$</td>
<td>$b_{24} = 16$</td>
</tr>
</tbody>
</table>

New coefficients of IIR filter
Figure 3.16

Magnitude and contour plot of the filter with integer coefficients
EXAMPLE 3.6

In this example a symmetric fan filter is designed with integer coefficients. The coefficients of a 1-D low-pass filter is obtained using VSHP for the denominator of its transfer function as shown in Chapter 2. Integer coefficients for this filter are obtained in the same manner as shown in example 3.5. Then by using the complex transformation of Kayran and King [46] the 2-D symmetric fan filter is formed.

Table 3.8 shows the coefficients of the 1-D low-pass filter. Figure 3.17 illustrates the magnitude and the contour plot of the resulting 2-D symmetric fan filter.
Table 3.8

<table>
<thead>
<tr>
<th>Numerator</th>
<th>Denominator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0 = 15$</td>
<td>$b_0 = 308$</td>
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<tr>
<td>$a_1 = -24$</td>
<td>$b_1 = -531$</td>
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<tr>
<td>$a_2 = 20$</td>
<td>$b_2 = 698$</td>
</tr>
<tr>
<td>$a_3 = 30$</td>
<td>$b_3 = -515$</td>
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<td>$a_4 = 34$</td>
<td>$b_4 = 365$</td>
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<tr>
<td>$a_5 = 33$</td>
<td>$b_5 = -162$</td>
</tr>
<tr>
<td>$a_6 = 5$</td>
<td>$b_6 = 62$</td>
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<tr>
<td>$a_7 = 56$</td>
<td>$b_7 = -14$</td>
</tr>
<tr>
<td>$a_8 = 49$</td>
<td>$b_8 = 2$</td>
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</tbody>
</table>

Integer coefficients of the 1-D low-pass filter
Figure 3.17

Magnitude and contour plot of 2-D symmetric fan filter
3-5 CONCLUSIONS

In this chapter two efficient methods for the design of circularly symmetric 2-D recursive digital filters with separable denominator and non separable numerator transfer function have been presented:

I) A two step approach to the design of a class of 2-D filters with separable denominator and non separable numerator transfer function was given. In this approach two 1-D analog filters are designed and discretized by the application of bilinear transformation to obtain a 2-D filter with a rectangular cutoff boundary. This filter is cascaded with a FIR filter which acts as equalizer for the magnitude response to make it more circular. Computational burden for determining the coefficients of the filter's transfer function is minimal in comparison to many of the existing techniques.

II) In the second method coefficients of the denominator polynomials are calculated through the application of a non-linear optimization technique while the coefficients of the numerator are determined by the minimization of the cost function using a linear programming approach. This method though sub-optimal in nature yields the best local minimum solution.

Sensitivity of the filter coefficients has been discussed. Two methods for designing 2-D filters with integer coefficients have been presented. To implement a 2-D filter on hardware, it is convenient to have them with integer coefficients. The given examples illustrated the utility of the proposed techniques.
4

DESIGN OF 2-D RECURSIVE DIGITAL FILTERS WITH NON-CIRCULAR CUTOFF BOUNDARY USING TRANSFORMATION TECHNIQUES

4-1 INTRODUCTION

In this chapter various transformation techniques for the design of 2-D recursive digital filters with non-circular symmetric cutoff boundary are reviewed. Depending on the shape of the cutoff boundary and the type of frequency response, various procedures are presented. In these procedures 1-D filters are used as the main building block and by various types of transformations other building blocks are formed. It is shown that cascading an appropriate number of these building blocks will yield a 2-D filter approximating the desired shape of cutoff boundary.
4-2 TRANSFORMATION TECHNIQUES FOR THE DESIGN OF 2-D FILTERS

In this section various techniques are presented for the design of 2-D filters with non-circular symmetric cutoff boundary. Most of these techniques use a 1-D filter and a suitable transformation, applied to it, to obtain a desirable 2-D transfer function. Descriptions of various 2-D transformations are presented in the following sections.

4-2.1 ROTATED FILTERS

The design technique proposed by Shanks et al. [5] consists of mapping 1-D into 2-D filters with arbitrary direction in a 2-D frequency response plane. These filters are called rotated filters because they are obtained by rotating 1-D filters.

Suppose a 1-D continuous filter, whose impulse response is real, is given in its factored form

\[ H_1(s) = K_0 \frac{\prod_{i=0}^{m} (s-q_i)}{\prod_{i=0}^{n} (s-p_i)} \]  \hspace{1cm} (4.1)

where \( K_0 \) is a scalar gain constant. The zero locations \( q_i \) and pole locations \( p_i \) may be complex, in which case their conjugates are also present in the corresponding product. The cutoff frequency for this filter is assumed to be unity.

The filter given in equation (4.1) can also be viewed as a 2-D filter that varies in one dimension only and could be written as follows:
\[ H_2(s_1, s_2) = H_1(s_2) = K_0 \frac{\prod_{i=0}^{m} (s_2 - q_i)}{\prod_{i=0}^{n} (s_2 - p_i)} \] (4.2)

Clockwise rotation of the \( s_1, s_2 \) axes through an angle \( \beta \) by means of the transformation

\[ s_1 = s'_1 \cos \beta + s'_2 \sin \beta \] (4.3)

\[ s_2 = -s'_1 \sin \beta + s'_2 \cos \beta \] (4.4)

will result in a filter whose frequency response is rotated by an angle \( -\beta \) with respect to the frequency response of filter (4.2). Thus

\[ H_2(s'_1, s'_2) = K_0 \frac{\prod_{i=0}^{m} \left[ (s'_2 \cos \beta - s'_1 \sin \beta) - q_i \right]}{\prod_{i=0}^{n} \left[ (s'_2 \cos \beta - s'_1 \sin \beta) - p_i \right]} \] (4.5)

The above function describes a continuous 2-D filter in the new coordinate system of \( s'_1 \) and \( s'_2 \). The corresponding digital version of the above filter is obtained through application of the double bilinear transformation

\[ s'_i = \frac{2}{T_i} \frac{1 - z_i^{-1}}{1 + z_i^{-1}} \quad i = 1, 2 \] (4.6)

It is also frequently assumed that \( T_1 = T_2 = T \), namely the sample interval is the same in both directions. Unfortunately, despite its simplicity, this technique as it stands neither
guarantees the stability of the designed filter nor achieves a circular symmetric cutoff boundary.

Costa and Venetsanopoulos [79] have modified the rotated filter approach by cascading a number of rotated filters whose angles of rotation are uniformly distributed over 180°, to achieve a magnitude response which approximates a circular symmetric cutoff boundary by a polygon. This polygon has an even number of sides, since each filter contributes two opposite sides of the polygon. The stability of the designed filter is ensured if the following two conditions hold:

1. \(270° \leq \beta \leq 360°\), where \(\beta\) is the angle of rotation.

2. \(C_i < 0\) for \(i = 1,2,\ldots,M\), where \(C_i = \text{Re} \left\{ (T/2)p_i \right\}\) and \(p_i\) represents the location of a pole.

**EXAMPLE 4.1**

A 2-D low-pass filter is designed by transforming a 1-D third order Butterworth filter possessing transfer function

\[
H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}
\]  

(4.7)

into a 2-D analog filter using the transformation of (4.3) by cascading two rotated filters with \(\beta = 300°\). Figure (4.1) shows the magnitude response of the resulting 2-D filter.
Figure 4.1

Magnitude and contour plot of rotated 2-D filter
2-VARIABLE REACTANCE FUNCTION

The design of 2-D recursive zero phase low-pass digital filters can be accomplished using a method described by Ahmadi et al. [80]. In this method, a 1-D low pass analog transfer function

\[ H(s) = \sum_{i=0}^{\infty} c_i s^i \]

is transformed into a 2-D first quadrant filter by using the second order 2-variable reactance function

\[ s \Rightarrow \frac{a_1 s_1 + a_2 s_2}{b_1 + b_2 s_1 s_2} \]

This transformation will yield a stable first quadrant 2-D filter. Cutoff frequencies along the two frequency axes determined by the prototype filter characteristics and the coefficients \( a_1, a_2, b_1 \), and \( b_2 \) are all positive real constants to ensure a low-pass to low-pass property for equation (4.9). It can be easily shown that, without loss of generality, \( b_1 \) can be set equal to unity so as to reduce the number of transformation coefficients to a minimum of three. Thus, the transformation to be considered is

\[ s \Rightarrow \frac{a_1 s_1 + a_2 s_2}{1 + b s_1 s_2} \]

A zero-phase 2-D low-pass analog filter can be obtained by forming the transfer function
\[ H (s_1s_2) = H(s_1s_2) H(-s_1s_2) H(s_1,-s_2) H(-s_1,-s_2) \quad (4.11) \]

The discrete version of the zero-phase filter can be obtained by application of double bilinear transformation. Despite the advantages of this technique, which are stability and simplicity of design, the shape of the cutoff boundary is not circular [44].

Higher order 2-variable reactance function has been used, by King and Kayran [81], to arrive at approximately circularly symmetric cutoff boundary 2-D filter.

**EXAMPLE 4.2**

A 2-D low-pass filter is designed by transforming the 1-D low-pass prototype filter of example 4.1 into a 2-D first quadrant analog filter using the transformation of (4.10). The magnitude response and the contour plot of the 2-D filter is shown in figure (4.2).

For the transformation coefficients \( a_1 = a_2 = 1 \) and \( b_1 = 0.1 \).
Figure 4.2

Magnitude response and the contour plot of 2-D filter
4-2.3 COMPLEX TRANSFORMATION

A technique for designing recursive 2-D fan filters based on a complex transformation applied to a 1-D low-pass filter has been presented by Kayran and King [46]. Also a set of transformed filters with their appropriate combination form, a zero-phase fan filter, is discussed.

A causal and stable 2-D filter can be generated according to

\[ H(z_1, z_2) = H_f(z) \]  (4.12)

via the transformation

\[ z = e^{j\phi} \frac{\alpha_1}{\beta_1} z_1 + \frac{\alpha_2}{\beta_2} z_2 \]  (4.13.a)

the corresponding frequency transformation is

\[ \omega \Rightarrow \phi + \frac{\alpha_1}{\beta_1} \omega_1 + \frac{\alpha_2}{\beta_2} \omega_2 \]  (4.13.b)

If the prototype filter is a low-pass filter with cutoff frequency \( \omega_c = \pi/2 \) the amplitude plot of the frequency response after transformations with \( \alpha_1/\beta_1 = 1/3 \), \( \alpha_2/\beta_2 = 1/3 \), \( \phi = \pi/2 \) is shown in figure (4.3). There are three effects of transformation of equation (4.13.a) on the resulting filter [44]:

1. The frequency response of the resulting filter will be shifted by \( \phi \) along \( \omega \), axis.

2. The angle of rotation of frequency response is \( \text{arc tan} \ (\alpha_2/\beta_2) \). Since the original filter is 1-D and a function of \( z_1 \), the angle of rotation will be defined by the fractional power of \( z_2 \).

3. The fractional power of \( z_1 \) will scale the frequency response by a factor \( \alpha_1/\beta_1 \).
However, the periodicity of the frequency response will be \( (\alpha_r/\beta_r)2\pi \) instead of \( 2\pi \).

The other effects of the transformation on the resulting filter may be specified as follows:

4. When \( \alpha_r/\beta_r > 0 \) and \( \alpha_2/\beta_2 > 0 \), the transformation is causal; otherwise it is non-causal. However, the transformation \( \alpha_r/\beta_r < 0 \) or/and \( \alpha_2/\beta_2 < 0 \) may be implemented in the spatial time domain, for a finite array, by reorienting the input signal array.

5. When \( \phi = 0 \), the resulting filter alone cannot be implemented directly in the spatial time domain. On the other hand, such transformed filters may be combined in appropriate ways so that complex values do not exist in the final transfer function.

6. Both rotation and frequency scaling are equivalent to the rotation of the recursion direction with a new sampling interval.

7. The stability of the resulting filter is unaffected by \( \alpha_r/\beta_r > 0 \) and \( \alpha_2/\beta_2 > 0 \). For \( \alpha_r/\beta_r < 0 \) or \( \alpha_2/\beta_2 < 0 \), the transformed filter will be unstable for the casual recursion direction. However, if the direction of the finite input array is changed, there are always four non-causal recursion directions where the filter function is stable.

4.2.4 SEPARABLE PRODUCT FILTER

A 2-D filter can be obtained by cascading two 1-D filters. It should be noted that filters designed with a separable product transfer function will have their pass-band and stop-band contours rectangular in shape. Figure (4.4) shows the magnitude response and contour plot of a 2-D filter designed by cascading of two 1-D Butterworth low-pass filters.
Figure 4.3

1-D low-pass filter after complex transformation
Figure 4.4

Magnitude response and contour plot of a 2-D separable product filter
4-3 DESIGN OF ROTATED ELLIPTICALLY SYMMETRIC 2-D FILTERS

In the method to be presented here a 2-D separable product filter is generally used as the principle building block. The form of this type of transfer function is as follows:

\[ H(z_1, z_2) = H(z_1) \cdot H(z_2) \]  

\[ \text{(4.14)} \]

where

\[ H(z_i) = \frac{\sum_{j=0}^{M-1} a_j z_i^{-j}}{\sum_{j=0}^{M-1} b_j z_i^{-j}} \quad i = 1, 2 \]  

\[ \text{(4.15)} \]

The derived 2-D separable product transfer function of equation (4.14) is cascaded by an appropriate 2-D filter to arrive at the desired magnitude response. Since the desired cutoff boundary has the shape of a rotated ellipse, the following steps are followed:

Step (1) Design a 1-D low-pass digital filter \( H(z) \) with the cutoff frequency \( \omega_c = \frac{LD}{2\sqrt{2}} \) where LD is the magnitude of the large diagonal of the ellipse in rad/sec.

Step (2) Cascade \( H(z_1) \) and \( H(z_2) \) to obtain a 2-D low-pass filter with rectangular cutoff boundary.

Step (3) Design another 1-D low-pass digital filter \( H'(z) \) with the cutoff
frequency $\omega_{z} = \frac{SD \sqrt{2}}{2}$ where SD is the value of the small diagonal of the ellipse in rad/sec.

Step (4) Apply the complex transformation of Kayran and King [46] which is

$$z = e^{j\phi} \frac{z_1^{\alpha_1}}{\beta_1} \frac{z_2^{\alpha_2}}{\beta_2}$$  

(4.16)

to $H(z)$ to obtain $H(z_1, z_2)$.

Step (5) Cascade $H(z_1)$, $H(z_2)$ and $H(z_1, z_2)$ to obtain the desired response.

Step (6) Realize the designed 2-D filter using the predistorted technique of Erfani, Ahmadi and Ramachandran [82].

It is worth noting that this design yields the filter coefficients without any elaborate optimization procedure.

**DESIGN EXAMPLE 4.3**

We wish to design a $-45^\circ$ rotated elliptical symmetry cutoff boundary 2-D low-pass filter with the following specifications: LD = 2.824 rad/sec. and SD = 1.414 rad/sec.

In this example, since the cutoff frequency for $H(z_1)$, $H(z_2)$ and $H(z)$ are identical, only one filter needs to be designed. Table (4.1) shows the value of the coefficients of $H(z)$ which is a fourth order all pole filter. The transformation used for $-45^\circ$ rotated ellipse is obtained from (4.16) with $\alpha_i/\beta_i$ set to $+1$ for $i = 1, 2$. Thus

$$z = z_1 z_2$$  

(4.17)

Figure (4.5) shows the magnitude response of the $H(z_1) H(z_2)$ as well as its contour plot.
Figure (4.6) shows the magnitude response and the contour plot of the $H(z_1,z_2)$.

The magnitude response of the designed filter is shown in Figure (4.7) along with its contour plot.

A careful look at the contour plot of the designed filter reveals that the technique yields a good approximation to the desired response. Modifications can be done to improve the results further. This can be in either of these two forms:

(i) Choose as the first block a 2-D filter with circular cutoff boundary as in [83] or [44], [53] then go through steps 4 - 6. Different degrees of rotation can be obtained by using general form of equation (4.16) and for realization of $z^a b$ use the technique given in [54].

(ii) In the second approach for obtaining a better approximation the designed 2-D filter is cascaded by a 2-D FIR filter which acts as a magnitude equalizer. The coefficients of the 2-D FIR filter can be determined by minimizing the least mean square error between the response of the desired filter and the designed filter using any suitable optimization technique.
Table 4.1

<table>
<thead>
<tr>
<th>Denominator</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_{16} = b_{20} = b_{36} = 5.563914$</td>
<td></td>
</tr>
<tr>
<td>$b_{11} = b_{21} = b_{31} = -11.47563$</td>
<td></td>
</tr>
<tr>
<td>$b_{12} = b_{22} = b_{32} = 11.87721$</td>
<td></td>
</tr>
<tr>
<td>$b_{13} = b_{23} = b_{33} = -6.616661$</td>
<td></td>
</tr>
<tr>
<td>$b_{14} = b_{24} = b_{34} = 1.651168$</td>
<td></td>
</tr>
</tbody>
</table>

Values of the coefficients of three 1-D low-pass IIR fourth order all-pole filters -note all three filters have the same specifications-
Figure 4.5

Magnitude response and the contour plot of the $H(z_1) H(z_2)$
Figure 4.6

Magnitude response and the contour plot of the $H(z_1, z_2)$
Figure 4.7

The magnitude response of the designed filter along with its contour plot
DESIGN EXAMPLE 4.4

In this example we designed another -45° rotated elliptical filter with the same specifications as in example (4.3). First two 1-D low-pass filters are designed to form \( H(z_1) \) and \( H(z_2) \). But for the third filter \( H(z) \) a low-pass FIR filter is designed, then the complex transformation of Kayran and King [46] applied to it. The resulting 2-D filter is formed by cascading all these three filters.

Table (4.2) shows the value of the coefficients of the \( H(z) \) which is a fourth order all pole filter. The transformation used for -45° rotated ellipse is obtained from equation (4.16) with \( \alpha/\beta_i \) set to +1 for \( i = 1, 2 \). Therefore, the form of transformation is \( z = z_1 z_2 \). Figure (4.8) shows the magnitude response and contour plot of the resulting filter.
Table 4.2

<table>
<thead>
<tr>
<th>Denominator Coefficients</th>
</tr>
</thead>
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<td>$b_{10} = b_{20} = 5.563914$</td>
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<tr>
<td>$b_{13} = b_{23} = -6.616661$</td>
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<tr>
<td>$b_{14} = b_{24} = 1.651168$</td>
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<table>
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<td>$a_4 = 0.20033956$</td>
</tr>
<tr>
<td>$a_5 = 0.31278229$</td>
</tr>
<tr>
<td>$a_6 = 0.31695855$</td>
</tr>
<tr>
<td>$a_7 = 0.21935773$</td>
</tr>
<tr>
<td>$a_8 = 0.08511233$</td>
</tr>
</tbody>
</table>

Values of the coefficients of two 1-D low-pass IIR all-pole filters and the coefficients of FIR filter
Figure 4.8

Magnitude response and the contour plot of the resulting filter
4-3.1 COMPARISON WITH OTHERS TECHNIQUES

A comparison has been made between the proposed technique and those reported in [55,84]. To examine the performance of the designed filter the Mean Square Error -MSE- is calculated in three different regions namely pass-band, transient and stop-band regions.

We are introducing two cost functions; the first one involves the sum of square of errors in the pass-band and the stop-band of the designed filters, while the second one uses the sum of square of errors in transition region only. In this case we can examine the closeness of the desired cutoff boundary to the designed one as well as the closeness of the magnitude response of the designed filter to the desired one.

Here we used the technique given in [55,84] with the following specifications:

\[
|H(z_1,z_2)| = \begin{cases} 
1 & \text{for } 0 \leq \omega_g \leq 1 \\
2 - \omega_g & \text{for } 1 < \omega_g < 2 \\
0 & \text{for } 2 \leq \omega_g \leq \pi
\end{cases}
\]

where

\[
\omega_g = \sqrt{\frac{1}{2}(\omega_1 - \omega_2)^2 + \frac{1}{8}(\omega_1 + \omega_2)^2}
\]  

(4.18)

The magnitude and contour plot of the ideal filter is shown in figure (4.9), while figures (4.10) and (4.11) are showing the results of the filters designed using 2-term and 3-term separable denominator 2-D transfer functions, which we are using for comparison.

The error has been calculated by considering a set of 41x41 grid points for both of the 2-term and 3-term separable denominator as well as the proposed technique. Table
(4.3) illustrates the performance measures of the proposed and the technique given in [55].

The performance of the cutoff boundary with respect to the ideal filter for the new method is better than the 2-term separable denominator design method and it is close to the one with 3-term. Furthermore, the performance of the new technique in the pass-band and stop-band regions is much better than the existing techniques.

<table>
<thead>
<tr>
<th>MSE</th>
<th>2-Term</th>
<th>3-Term</th>
<th>New Technique</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pass-band and stop-band</td>
<td>0.67813 E+01</td>
<td>0.255108 E+02</td>
<td>0.998942 E+00</td>
</tr>
<tr>
<td>region</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Transient region</td>
<td>0.16738 E+03</td>
<td>0.516445 E+02</td>
<td>0.613306 E+02</td>
</tr>
</tbody>
</table>

The error of the proposed and existing techniques
Figure 4.9

Magnitude and contour plot of ideal filter
Figure 4.10

Magnitude and contour plot of 2-term filter
Figure 4.11

Magnitude and contour plot of 3-term filter
4-4 DESIGN OF NON-SYMMETRICAL 2-D BAND-PASS FILTERS

This type of filter, first introduced by Taguchi and Hamada [84] and later used by Ahmadi et al. [55], has the following magnitude specification:

\[
|H(e^{j\omega_1}, e^{j\omega_2})| = \begin{cases} 
1 & \text{for } 0 \leq \sqrt{(\omega_1 - 1.5)^2 + (\omega_2 - 1.5)^2} \leq R \\
& \text{where } \omega_1, \omega_2 \geq 0 \\
0 & \text{elsewhere} \end{cases}
\]

\[
\text{for } 0 \leq \sqrt{(\omega_1 + 1.5)^2 + (\omega_2 + 1.5)^2} \leq R \\
\text{where } \omega_1, \omega_2 \leq 0
\] (4.19)

To design a 2-D filter with the above specification, two design methods can be formulated.

4-4.1 METHOD I

This method, which requires no optimization, is very simple to implement and can be summarized as follows:

Step (1) Design a 1-D band-pass filter with the following specification:

\[
|H(e^{j\omega})| = \begin{cases} 
1 & \text{for } 1.5 - R \leq |\omega| \leq 1.5 + R \\
0 & \text{elsewhere} \end{cases}
\text{rad/sec}
\] (4.20)
Step (2) Cascade $H(z_1)$ and $H(z_2)$ to obtain a 2-D band-pass filter with rectangular cutoff boundary.

Step (3) Design a 1-D low-pass filter $H'(z)$ with the following specification:

$$| H'(e^{j\omega}) | = \begin{cases} 
1 & \text{for } 0 \leq |\omega| \leq 2R \\
0 & \text{elsewhere}
\end{cases} \text{ rad/sec} \quad (4.21)$$

Step (4) Form $H'(z_1,z_2)$ as follows:

$$H'(z_1,z_2) = H'(z) \quad \begin{bmatrix}
\frac{z_1}{p_1} \\
\frac{z_2}{p_2}
\end{bmatrix} \quad (4.22)$$

Step (5) Form the final transfer function as follows:

$$H(z_1,z_2) = H(z_1) H(z_2) H'(z_1,z_2) \quad (4.23)$$

**DESIGN EXAMPLE 4.5**

As an example we design a 2-D filter with the magnitude specification of (4.19) with $R = 0.5$ rad/sec. Figure (4.12a) shows the magnitude response as well as the contour plot of the desired filter. Tables (4.4) and (4.5) show the value of the coefficients of the two fourth order 1-D band-pass and low-pass filters respectively. Figure (4.12b) shows the magnitude response and the contour plot of the cascade of $H(z_1) H(z_2)$ filter. In this example the transformation used is of the form $z = z_1 z_2$ and figure (4.12c) shows
the 3-D plot of the magnitude response and its 2-D contour plot. Figure (4.12d) shows the magnitude response and the contour plot of the designed filter.

A careful examination of the magnitude response of the designed filter reveals the close proximity of the designed filter response to that of the desired one. It is worth noting that the filter designed in this example consists of three fourth order all-pole 1-D filters, only one of which is transformed using $z = z_1 z_2$. Essentially the number of multipliers required for the implementation of the filter in this example is 15 as compared to 35 of [55] and 40 of [84]. The overall structure is therefore simple and efficient.
Table 4.4

<table>
<thead>
<tr>
<th>Denominator</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>b_{10} = b_{20} = 3.27042961</td>
</tr>
<tr>
<td></td>
<td>b_{11} = b_{21} = -0.59318829</td>
</tr>
<tr>
<td></td>
<td>b_{12} = b_{22} = 3.83931541</td>
</tr>
<tr>
<td></td>
<td>b_{13} = b_{23} = -0.41144586</td>
</tr>
<tr>
<td></td>
<td>b_{14} = b_{24} = 1.55635929</td>
</tr>
</tbody>
</table>

The value of the coefficients of the two fourth order 1-D band-pass filter

Table 4.5

<table>
<thead>
<tr>
<th>Denominator</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>b_{30} = 5.563914</td>
</tr>
<tr>
<td></td>
<td>b_{31} = -11.47563</td>
</tr>
<tr>
<td></td>
<td>b_{32} = 11.87721</td>
</tr>
<tr>
<td></td>
<td>b_{33} = -6.616661</td>
</tr>
<tr>
<td></td>
<td>b_{34} = 1.651168</td>
</tr>
</tbody>
</table>

The value of the coefficients of the low-pass filter respectively
Figure 4.12a

The magnitude response and the contour plot of the desired band-pass filter
Figure 4.12b

Cascaded $H(z_1)H(z_2)$ filter
Figure 4.12c

The magnitude response and the contour plot 1-D filter after transformation
Figure 4.12d

The magnitude response and the contour plot of the designed band-pass filter
4-4.2 METHOD II

If a more accurate approximation to the desired 2-D response is required the following modifications can be carried out.

Step (1) Design a low-order 2-D FIR filter satisfying the following magnitude response:

\[ |H(e^{j\omega_1}, e^{j\omega_2})| = \begin{cases} 
1 & \text{for } 0 \leq \sqrt{(\omega_1 - 1.5)^2 + (\omega_2 - 1.5)^2} \leq R \\
& \text{where } \omega_1, \omega_2 \geq 0 \\
& \text{for } 0 \leq \sqrt{(\omega_1 + 1.5)^2 + (\omega_2 + 1.5)^2} \leq R \\
& \text{where } \omega_1, \omega_2 \leq 0 \\
& \text{for } 0 \leq \sqrt{(\omega_1 - 1.5)^2 + (\omega_2 + 1.5)^2} \leq R \\
& \text{where } \omega_1 \geq 0, \omega_2 \leq 0 \\
& \text{for } 0 \leq \sqrt{(\omega_1 + 1.5)^2 + (\omega_2 - 1.5)^2} \leq R \\
& \text{where } \omega_1 \leq 0, \omega_2 \geq 0 \\
0 & \text{elsewhere} 
\end{cases} \] (4.24)

Using any of the iterative techniques reported in [44],
Step (2)  Go to Steps (1) - (5) of the previous method.

**DESIGN EXAMPLE 4.6**

As an example, the 2-D filter with the magnitude specification of equation (4.19) has been designed. Figure (4.13) shows the magnitude and contour plot of the designed 2-D FIR filter in step (1) which is a fourth order filter. Table (4.6) shows the values of the coefficients of the 2-D FIR filter. It should be noted that the overall transformation of the designed filter is the same as the previous example (4.5) cascaded with a 2-D FIR filter. Therefore, the rest of the coefficients can be found out in tables (4.4) and (4.5). Figure (4.14) shows the magnitude response and contour plot of the final 2-D filter. A close look at the responses in Figure (4.14) reveals the closeness of the 2-D designed filter to that of the desired one.

<table>
<thead>
<tr>
<th>Numerator</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{40}$</td>
<td>-1.03015327</td>
</tr>
<tr>
<td>$a_{61}$</td>
<td>-0.13759875</td>
</tr>
<tr>
<td>$a_{92}$</td>
<td>1.42955303</td>
</tr>
<tr>
<td>$a_{11}$</td>
<td>-0.13738745</td>
</tr>
<tr>
<td>$a_{11}$</td>
<td>-0.01882632</td>
</tr>
<tr>
<td>$a_{12}$</td>
<td>0.19731450</td>
</tr>
<tr>
<td>$a_{30}$</td>
<td>1.43036652</td>
</tr>
</tbody>
</table>

The values of the coefficients of the 2-D FIR filter
Figure 4.13

The magnitude and contour plot of the designed 2-D FIR filter
Figure 4.14

The magnitude response and contour plot of the final 2-D band-pass filter
4-4.3 COMPARISON WITH OTHER TECHNIQUES

In this section a comparison has been made between the existing method reported in [55,84] and the proposed technique. The method described in section (4-3.1) is used to examine the performance of the designed filter. The mean square error, with respect to the ideal filter, is calculated in three different regions of pass-band, transient and stop-band.

The ideal filter has the following specifications:

\[ | H(z_1, z_2) | = \begin{cases} 
1 & \text{for } 0.0 \leq \omega_x \leq 0.5 \\
1.5 - \omega_x & \text{for } 0.5 < \omega_x < 1.5 \\
0 & \text{for } 1.5 \leq \omega_x \leq \pi 
\end{cases} \]

where:

\[
\begin{align*}
\begin{align*}
\text{for } & \begin{cases} 
\omega_1 > 0 \\
\omega_2 < 0
\end{cases} \Rightarrow & \omega_x = \sqrt{(\omega_1 - 1.5)^2 + (\omega_2 + 1.5)^2} \\
\text{for } & \begin{cases} 
\omega_1 < 0 \\
\omega_2 > 0
\end{cases} \Rightarrow & \omega_x = \sqrt{(\omega_1 + 1.5)^2 + (\omega_2 - 1.5)^2}
\end{align*}
\end{align*}
\]

(4.25)

The magnitude and contour plot of the ideal filter is shown in figure (4.14a). Figure (4.14b) shows the results of the filter designed with the technique given in [55]. The errors have been calculated by considering a set of 4141 grid points for both techniques. Table (4.6a) shows the performance measures of the proposed method and the technique of [55].

It is shown that the closeness of the cutoff boundary as well as the magnitude
response of the designed filter in transient, pass-band and stop-band regions of the proposed method are better than the existing techniques.

Table 4.6a

<table>
<thead>
<tr>
<th>MSE</th>
<th>Technique of [55]</th>
<th>New Technique</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pass-band and stop-band region</td>
<td>0.431120 E+02</td>
<td>0.365789 E+02</td>
</tr>
<tr>
<td>Transient region</td>
<td>0.495967 E+02</td>
<td>0.240951 E+02</td>
</tr>
</tbody>
</table>

The error of the proposed and existing techniques of 2-D non-symmetric band-pass filter
Figure 4.14a
Magnitude and contour plot of ideal 2-D non-symmetric band-pass filter
Figure 4.14b
Magnitude and contour plot of 2-D non-symmetric band-pass filter [55]
4-5 FAN-FILTER DESIGN

Most of the design techniques considered so far have been for filters having cutoff boundaries which are approximately circular. In this section we shall consider a different type of 2-D profile, one having a fan or wedge shape. The practical significance of such filters originates in the field of geophysical prospecting. As we mentioned earlier the technique of King and Kayran [46] can be used to design a 2-D fan filter.

Consider a stable 1-D low-pass filter \( H(z) \), by using the complex transformation technique:

\[
H(z_1, z_2) = H(z) \quad \left| \begin{array}{c}
\alpha_1 & \alpha_2 \\
\beta_1 & \beta_2 \\
z & e^{\phi}z_1 \beta_1 z_2 \end{array} \right. \quad (4.26)
\]

a 2-D filter will be produced. For example if the prototype low-pass filter has cutoff frequency \( \omega_c = \pi/2 \) the amplitude plot of the frequency response after transformation \( \alpha_1/\beta_1 = 1/3 \), \( \alpha_2/\beta_2 = -1/3 \), \( \phi = 0 \) is shown in figure (4.15).

By changing the parameters to \( \alpha_1/\beta_1 = -1/3 \), \( \alpha_2/\beta_2 = -1/3 \), \( \phi = 0 \) the direction of 2-D transformed filter will change to the second and fourth quadrants. Figure (4.16) shows this effect on the transformed filter. The effect of shifting by choosing \( \phi = \pi/2 \) can be observed from figure (4.17).
Figure 4.15

The magnitude and contour plot of transformed low-pass filter
Figure 4.16

The effect of the sign on transformed filter
Figure 4.17

The shifting effect by $\phi = \pi/2$ on transformed filter
4-5.1 SYMMETRIC FAN-FILTER

A symmetric fan filter will be designed with the specification:

\[
|H(e^{j\omega_1}, e^{j\omega_2})| = \begin{cases} 
1 & \text{for } |\omega_1| \geq |\omega_2| \\
0 & \text{elsewhere}
\end{cases}
\]

(4.27)
in the following steps:

Step (1) Design a low-pass prototype filter with a cutoff frequency at \(\omega_c = \pi/2\).

Step (2) To obtain the shifted, scaled and rotated characteristics in frequency domain, apply the complex transformation of:

\[
H(z_1, z_2; \frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \phi) = H(z) \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \frac{\alpha_2}{\beta_2} \\ \frac{\phi}{\beta_1} \end{bmatrix} \quad z \rightarrow e^{j\phi} z_1 \frac{\alpha_2}{\beta_1} z_2
\]

(4.28)

In general, the filter coefficients in function (4.28) will be complex and the variables \(z_1\) and \(z_2\) will have rational non-integer powers.

Step (3) By appropriate combinations of transformed filters the zero phase fan filter transfer function can be written as:

\[
H_f(z_1, z_2) = \tilde{H}_1(z_1, z_2) \tilde{H}_2(z_1, z_2) \tilde{H}_3^*(z_1, z_2) \tilde{H}_4^*(z_1, z_2)
\]

\[
+ \tilde{H}_1^*(z_1, z_2) \tilde{H}_2^*(z_1, z_2) \tilde{H}_3(z_1, z_2) \tilde{H}_4(z_1, z_2)
\]

(4.29a)
where:

\[ h_1(z_1, z_2) = \tilde{H}(z_1, z_2; \frac{1}{3}, \frac{1}{3}, \frac{\pi}{2}) \]

\[ h_2(z_1, z_2) = \tilde{H}(z_1, z_2; -\frac{1}{3}, \frac{1}{3}, \frac{\pi}{2}) \]

\[ h_3(z_1, z_2) = \tilde{H}(z_1, z_2; \frac{1}{3}, -\frac{1}{3}, \frac{\pi}{2}) \]

\[ h_4(z_1, z_2) = \tilde{H}(z_1, z_2; -\frac{1}{3}, -\frac{1}{3}, \frac{\pi}{2}) \] \hspace{1cm} (4.29b)

where \( \tilde{H}^* \) denotes the complex conjugate of those of \( \tilde{H} \).

**DESIGN EXAMPLE 4.7**

In this example a symmetrical fan filter with the following specifications:

\[
| H_f(e^{j\omega_1}, e^{j\omega_2}) | = \begin{cases} 
1 & \text{for } |\omega_1| \geq |\omega_2| \\
0 & \text{otherwise}
\end{cases} \hspace{1cm} (4.30)
\]

will be designed using the complex transformation technique given in [46].

Consider an eighth order low-pass 1-D filter with the transfer function of the form:

\[
H(z) = \frac{\sum_{i=0}^{8} a_i z^{-i}}{\sum_{j=0}^{8} b_j z^{-j}} \hspace{1cm} (4.31)
\]

By using a 2-variable VSHP for the denominator of the above transfer function given in chapter 2, we can design this 1-D low-pass filter with cutoff frequency at \( \omega_c = \pi/2 \). The
coefficients of the designed filter are shown in table (4.7). Figure (4.18) shows the magnitude and contour plot of the designed filter. The values of transformation parameters in this example are $\alpha, \beta_1 = \pm 1.3^\circ, \alpha_2 \beta_2 = \pm 1.3^\circ, \phi = \pi/2$.

<table>
<thead>
<tr>
<th>Numerator Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0 = 0.53332424 \ E+00$</td>
</tr>
<tr>
<td>$a_1 = 0.10337353 \ E+01$</td>
</tr>
<tr>
<td>$a_2 = 0.53644800 \ E+00$</td>
</tr>
<tr>
<td>$a_3 = 0.46120137 \ E+00$</td>
</tr>
<tr>
<td>$a_4 = 0.44944412 \ E+00$</td>
</tr>
<tr>
<td>$a_5 = 0.45273215 \ E+00$</td>
</tr>
<tr>
<td>$a_6 = 0.48578310 \ E+00$</td>
</tr>
<tr>
<td>$a_7 = 0.443266d1 \ E+00$</td>
</tr>
<tr>
<td>$a_8 = 0.45028806 \ E+00$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Denominator Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_0 = 0.94492559 \ E+01$</td>
</tr>
<tr>
<td>$b_1 = 0.24347443 \ E+02$</td>
</tr>
<tr>
<td>$b_2 = 0.43023773 \ E+02$</td>
</tr>
<tr>
<td>$b_3 = 0.51218002 \ E+02$</td>
</tr>
<tr>
<td>$b_4 = 0.47096680 \ E+02$</td>
</tr>
<tr>
<td>$b_5 = 0.30853607 \ E+02$</td>
</tr>
<tr>
<td>$b_6 = 0.15684788 \ E+02$</td>
</tr>
<tr>
<td>$b_7 = 0.48516045 \ E+01$</td>
</tr>
<tr>
<td>$b_8 = 0.86507702 \ E+00$</td>
</tr>
</tbody>
</table>

Coefficients of 1-D prototype filter
Figure 4.18

Magnitude and contour plot of symmetric fan filter
4-5.2 QUADRANTAL FAN-FILTER

A quadrantal fan filter will be design with the frequency characteristic:

\[
| H_f(e^{j\omega_1}, e^{j\omega_2}) | = \begin{cases} 
1 & \text{for } \omega_1 \omega_2 \geq 0 \\
0 & \text{for } \omega_1 \omega_2 < 0 
\end{cases} 
\] (4.32)

using the following steps:

Step (1) Design a low-pass prototype filter, the same as the one in symmetric fan filter design, with a cutoff frequency at \( \omega_c = \pi/2 \).

Step (2) The shifted, scaled and rotated filter is produced using the complex transformation of equation (4.28). With \( (\alpha, \beta, \alpha', \beta') \) equal to \((1,0), (0,1), (-1,0), (0,-1)\) and \( \phi = \pi/2 \).

Step (3) The zero phase quadrantal fan filter will be produced, in the same manner as the symmetrical fan filter, using equation (4.29.a).

**DESIGN EXAMPLE 4.8**

We want to design a quadrantal fan filter with the specification given in equation (4.32), in which the pass-bands are in first and third quadrants. The low-pass prototype filter has the characteristic as:

\[
| H(z) | = \begin{cases} 
1 & \text{for } 0 \leq \omega \leq \frac{\pi}{2} \\
0 & \text{for } \frac{\pi}{2} < \omega \leq \pi 
\end{cases} 
\] (4.33)

This specification is the same as the one in example 4.7 thus the coefficients table is 4.7.

Figure (4.19) shows the magnitude and contour plot of the resulting quadrantal fan filter.
Figure 4.19
The magnitude and contour plot of the quadrantal fan filter
4-6 CONCLUSIONS

In this chapter several strategies have been presented for the design of 2-D filters with non-circular symmetric magnitude response. In these strategies 1-D filters are used to generate appropriate 2-D filters with rectangular cutoff boundaries. These filters are then cascaded with 2-D filters derived from their 1-D counterparts through complex transformation of [46]. The advantages of these proposed techniques can be listed as follows:

(i) The design methods are very simple to implement and if high accuracy in terms of the shape of the cutoff boundary is not an important issue, optimization procedures for computing the coefficients can be avoided entirely.

(ii) Most of these design methods can utilize the powerful realization structure of 1-D filters which can be chosen such that the sensitivity problem is minimized. One such realization structure can be that of [84].

(iii) These filters are inherently stable since they are derived mainly from 1-D stable filters.

(iv) Modifications made to those techniques to make them more accurate are very straightforward and often only require an approximation problem to that of a 2-D FIR filter.

(v) For different angles of rotation transformation of Kayran and King [46] applied to 1-D recursive filter may result in non-causal filters. To avoid this problem 1-D prototype can be chosen to be of FIR type. Then a simple translation can convert
the non-causal 2-D FIR filter to a causal one.

(vi) The choice of the building blocks is unlimited and so is the type of the 2-D filters that can be designed with the proposed techniques.

(vii) The proposed design techniques have been compared with the existing techniques. If the shape of the cutoff boundary is not restricted to the perfect circular, it has been shown, by means of comparison, that the proposed techniques are more accurate and simple to design.

(viii) The complex transformation, with the rational non-integer powers of 1/3 for the variables, has been used to design symmetric and quadrantal zero phase fan filters.
5

REALIZATION STRUCTURES
FOR 2-D RECURSIVE DIGITAL FILTERS

5.1 INTRODUCTION

Realization is the process of converting the transfer function of the designed filter into a filter network. Implementations of digital filters both in software and hardware are greatly indebted to the fast software algorithms and great advances in the area of VLSI technology.

Digital filters can be realized using shift registers as the delay units, digital multipliers and adders. By interconnection of these elements in an appropriate manner, we may implement a network to obtain a desired specification. The realization of 2-D digital filters can be performed through the transfer function [5,85], the state space model [87,89] and using transformation techniques [90].

In this chapter we review some of the fundamental realization structures which exist in the literature along with a new realization structure for a class of 2-D filters.
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5.2 REALIZATION FROM THE TRANSFER FUNCTION

The simplest form of implementation is that derived from the recursion equation in two dimensions.

\[ y(m,n) = \sum_{i=0}^{M} \sum_{j=0}^{M} a_{ij} x(m-i,n-j) - \sum_{i=0}^{M} \sum_{j=0}^{M} b_{ij} y(m-i,n-j) \]  

(5.1)

As we mentioned earlier, a 2-D filter configuration can be represented in a block diagram form, the basic elements of which are multipliers and adders and two types of delay elements with transfer functions \( z_{1}^{-1} \) and \( z_{2}^{-1} \). Figure (5.1) shows the block elements to multiply input signal by a constant and add input signals. Figure (5.2) is presenting the \( z_{1} \) and \( z_{2} \) delay elements.

It should be noted that synthesis is always non unique. Different realizations of the same transfer function offer different figures of merit in terms of round-off error caused by quantization or truncation, coefficient sensitivity and memory requirements [63].

The equation (5.1) is interpreted graphically in figure (5.3). An input mask or window of finite area, the shape of which is determined by the array \( a_{ij} \), is positioned over the input array at a position that depends on \( (m,n) \). Only a finite number of input samples are covered and multiplied by the appropriate coefficients \( a_{ij} \) and the resulting products are summed. Similarly and synchronously, the output mask, determined by the array \( b_{ij} \), is swept over the output array. All of the output samples which are covered by this mask, except the one at \( (m,n) \), are weighted by the coefficients \( b_{ij} \), summed and subtracted
Figure 5.1
Fundamental multiplier and adder

Figure 5.2
Fundamental delay blocks
from the sum derived from the input mask to produce the value $y(m,n)$. This number is stored in the output array, the two masks are moved to new location, and the process is repeated [64].

**Figure 5.3**

Illustration of calculation of output graphically
5-2.1 DIRECT REALIZATION

A direct realization algorithm is given by Shanks et al. [5]. It has been shown that

the equation (5.1) can be rewritten as:

\[
Y(z_1, z_2) = \left[ \sum_{i=0}^{M} \sum_{j=0}^{M} a_{ij} z_1^{-i} z_2^{-j} \right] X(z_1, z_2) \\
- \left[ \sum_{i=0}^{M} \sum_{j=0}^{M} b_{ij} z_1^{-i} z_2^{-j} \right] Y(z_1, z_2)
\]

\[(5.2)\]

An example of the direct form realization for a second order case is illustrated in figure (5.4).

Another direct form realization is the 2-D canonical form, which was first introduced by Mitra et al. [85]. One can rewrite the equation (5.2) as a set of two equations (5.3) and (5.4).

\[
W(z_1, z_2) = X(z_1, z_2) - \sum_{i=0}^{M} \sum_{j=0}^{M} b_{ij} z_1^{-i} z_2^{-j} W(z_1, z_2)
\]

\[(5.3)\]

\[
Y(z_1, z_2) = \sum_{i=0}^{M} \sum_{j=0}^{M} a_{ij} z_1^{-i} z_2^{-j} W(z_1, z_2)
\]

\[(5.4)\]

\(W(z_1, z_2)\) is an intermediate variable. A second order network corresponding to equations (5.3) and (5.4) is shown in figure (5.5). In this realization the number of delay blocks used is reduced by two with respect to the direct realization of figure (5.4) [86].
Figure 5.4

Direct form realization for a second order IIR filter
Figure 5.5

Realization structure of a second order 2-D transfer function.
5.2.2 CONTINUED FRACTION EXPANSION REALIZATION

Depending on the relationships between the coefficients of transfer function, it is possible to obtain a corresponding realization structure based on continued fractions. A number of types of continued fraction expansion were presented in [85], along with their corresponding realizations. The first type transfer function can be expanded as:

\[ H(z_1, z_2) = C_1 + \frac{1}{A_1 z_1^{-1} + \frac{1}{C_2 + \frac{1}{B_1 z_2^{-1} + \frac{1}{C_3 + \frac{1}{A_2 z_1^{-1} + \frac{1}{\ddots + \frac{1}{C_k}}}}}}} \]

(5.5)

Then the realization structure is shown in figure (5.6).

The next type of realization in this group is the result of expansion of \( 1/H(z_1, z_2) \), which is shown in figure (5.7).
Figure 5.6

First type of continued fraction expansion realization
Figure 5.7

Second type of continued fraction expansion realization
If we expand $H(z_1, z_2)$ with $C_1 \neq 0$ the result is as follows:

$$H(z_1, z_2) = \frac{1}{C_1 + \frac{1}{A_1 z_1^{-1} + \frac{1}{C_2 + \frac{1}{B_1 z_2^{-1} + \ldots}}}}$$

(5.6)

Figure (5.8) illustrates the realization structure of this type of expansion.

It is easy to prove that for an arbitrary transfer function of a 2-D IIR filter in general the realization by continued fraction expansion is not possible.

Figure 5.8

Third type of continued fraction expansion realization
5.2.3 PARTIAL FRACTION EXPANSION REALIZATION

The partial fraction expansion realization can only be used for a special class of separable denominator transfer function. A separable denominator transfer function can be shown as:

\[ H(z_1, z_2) = \frac{A(z_1, z_2)}{B_1(z_1) B_2(z_2)} \]

\[ = \frac{\sum_{i=0}^{M} \sum_{j=0}^{N} a_{i,j} z_1^{-i} z_2^{-j}}{\left( \sum_{i=0}^{M} b_{1,i} z_1^{-i} \right) \left( \sum_{j=0}^{N} b_{2,j} z_2^{-j} \right)} \]

(5.7)

It has been shown in [93] that a partial fraction expansion can be written as:

\[ H(z_1, z_2) = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \frac{K_{ij}}{(z_1 - \gamma_i)(z_2 - \lambda_j)} + \sum_{i=0}^{M-1} \frac{K_{iN}}{z_1 - \gamma_i} \]

\[ + \sum_{j=0}^{N-1} \frac{K_{Mj}}{z_2 - \lambda_j} + K_{MN} \]

(5.8)

A highly parallel structure of the transfer function (5.8) is shown in figure (5.9). For the realization of the branches, Kung [93] suggested a state space realization.
Figure 5.9

A 2-D realization of partial fraction expansion
5-3 REALIZATION STRUCTURE FOR A CLASS OF 2-D RECURSIVE DIGITAL FILTERS

In this section a realization structure will be presented for a 2-D circular symmetric recursive filter with separable denominator and non-separable numerator, using the design method given in section 3-3. In this method, two 1-D analog filters are designed in $s_1$ and $s_2$, using analog filter theory - Butterworth, Chebyshev or Elliptic-[9], to meet the specifications along $\omega_1$ and $\omega_2$ axes. Then, the two 1-D analog filters are discretized by the application of bilinear transformation. The resulting 2-D filter has a rectangular cutoff boundary. This separable product 2-D filter is then cascaded by a 2-D FIR filter to produce a circular cutoff boundary, as shown in example 3.3.

The resulting transfer function is realized using a least-sensitive structure described in the next section.

5-3.1 LOW SENSITIVE REALIZATION STRUCTURE

Since the starting point in our procedure is the design of an analog filter, one may wish to utilize one of the analog realizations which are less sensitive to coefficient variations and transform the structure into the digital domain. Unfortunately, the direct application of bilinear transformation to an analog realization generates delay free loops, which are undesirable. To overcome this problem, the analog transfer function is predistorted using the technique described in [82]. The details of the procedure are listed below:
Step (1) The numerator and denominator of the analog transfer function are
predistorted to obtain a new transfer function as:

\[
B(s) = \sum_{i=0}^{M} b_i s^i = \sum_{j=0}^{M} b'_j (s-1)^j = \sum_{j=0}^{M} b'_j v^j
\]

(5.9)

where

\[
b'_j = \sum_{i=0}^{M} C'_j b_i \quad i > j
\]

(5.10)

and \(C'_j\) is combination of \(i\) items taken \(j\) at a time. Also,

\[
A(s) = \sum_{i=0}^{M} a_i s^i = \sum_{j=0}^{M} a'_j (s-1)^j = \sum_{j=0}^{M} a'_j v^j
\]

(5.11)

where

\[
a'_j = \sum_{i=0}^{M} C'_j a_i \quad i > j
\]

(5.12)

and

\[
H(s) = \frac{A(s)}{B(s)}
\]

(5.13)

It should be noted that this operation shifts all the roots of the polynomial

\(B(s)\), the denominator of \(H(s)\), one unit to the left. Therefore, the
stability of the filter remains unchanged.

Step (2) Realize \(H(v)\) using any conventional method. The ladder realization
structure is preferred because of its low sensitivity to coefficient variations.

Step (3) Discretize \(H(v)\) using bilinear transformation. The result of this
predistorted function will be of the form:
\[(x - 1) = v \Rightarrow \frac{-2}{1 + z^{-1}} \quad (5.14)\]

Figure (5.10) shows the implementation of this bilinear transformation.

Note that the network in this figure contains no delay-free forward paths, thus it can be placed within delay-free structures without creating delay-free loops.

**Step (4)** Replace \( v \) in realization structure by its digital equivalent. By cascading the two 1-D filters, the 2-D structure of the first part will be realized. This realization can be extended to N-dimensional separable denominator systems. A ladder realization of the 4\(^{th}\) order Butterworth filter is shown in figure (5.11).

**Step (5)** Cascade these two 1-D filters with the realization structure given by Mecklenbräuker and Mersereau [94] for a non-recursive filter. The implementation of the 2-D zero phase FIR filter with zero phase operator \((z + z^{'})/2\) is shown in figure (5.12).
Figure 5.10
Digital network implementation

Figure 5.11
Ladder realization of 4th order Butterworth filter
Figure 5.12

Implementation of the 2-D zero phase filter
5-4 CONCLUSIONS

A number of realization techniques for 2-D digital filters have been discussed. Wherever possible, the concepts have been derived in relation to those of 1-D digital filters. Examples have been used to illustrate the nature of the structures resulting from the techniques discussed.

We have used the 1-D cascade realization structure to minimize the sensitivity problem. The predistorted version of the bilinear transformation contains no delay free forward path. Therefore, it can be replaced within delay free structures. The extension of this approach to N-dimensions is straightforward.
CONCLUSIONS, CONTRIBUTIONS AND SUGGESTIONS FOR FUTURE WORK

6-1 SUMMARY AND CONCLUSIONS

This chapter consists of a summary of the contributions of this thesis along with concluding remarks and suggestions for further research.

In chapter two, various types of 2-variable Hurwitz polynomials were discussed. It is known that only Very Strictly Hurwitz Polynomials yield a 2-variable stable polynomial in $z_1$ and $z_2$ upon the application of the double bilinear transformation. This chapter also presented an overview of various methods of generating 2-variable VSHP.

Most of these techniques use the properties of positive definite matrices to generate the even or odd part of a 2-variable Hurwitz polynomial [61,67]. Then, through calculation of higher order derivatives, a 2-variable VSHP is obtained. We have shown that with the introduction of the resistive matrix to the formulation presented by Ahmadi and Ramachandran [109], one can eliminate the cumbersome calculation of the higher order derivatives, which is mandatory in the existing techniques.
We have also shown that for a 2-variable VSHP of a given order, the number of parameters of the VSHP in our proposed technique is higher than the number of parameters of the 2-variable VSHP derived using other techniques [61,109]. This would generate extra degrees of freedom for the optimization process.

Chapter three is concerned with various iterative techniques for the design of 2-D, stable, recursive digital filters. The first technique deals with the design of a class of 2-D IIR filters with a VSHP at the denominator. The VSHP is generated in analog domain. Then, by the application of bilinear transformation, the discrete transfer function is obtained. By the appropriate choice of a nonlinear optimization technique, the coefficients of the 2-D filters are determined by minimizing an objective function, which is a measure of the difference between the spectrum of the desired and designed 2-D filters. Even though this technique is robust and yields satisfactory results, it is computationally intensive.

It is known that many useful 2-D filters can be designed using a 2-D transfer function with a separable denominator [2,45,55,84,104,128,130,142,143,145]. This choice of transfer function offers a great reduction in the number of coefficients of the 2-D filters, hence saving computation time. Also, by a proper choice of coefficients in the numerator and denominator, quadrantal and octagonal symmetric filters can be obtained. This in turn reduces the number of unknown coefficients even further. This reduces computation time for the optimization process. For the quadrantal and octagonal symmetric filters a further reduction of computations can be achieved by the restriction of the region in the \( \omega_1-\omega_2 \) plane over which the objective function is calculated. This
reduction amounts to 50% and 75% for quadrantal and octagonal symmetry respectively.

Other strategies for the efficient design of this special class of 2-D filters are also outlined in this chapter. In these methods, the numerator and denominator of the 2-D transfer function are calculated separately to reduce the computation time even further. In the first phase of the optimization, two 1-D all pole filters are designed to satisfy the specification along $\omega_1$ and $\omega_2$ axes. Then, by cascading this rectangular cutoff boundary 2-D filter with a 2-D non-recursive filter, the desired shape of cutoff boundary is obtained.

Moreover, in the second phase of the optimization, the numerator is essentially used as a magnitude equalizer to achieve the desired amplitude specification. We have shown that in the second phase, both linear and non-linear optimization techniques can be employed and the former has the advantage of yielding a better local minimum. This two step optimization, although sub-optimal in nature, is a powerful tool for the design of circular symmetric 2-D filters.

Over the past few years, attempts have been made to develop various design methodologies for the design of 2-D filters with non-circular cutoff boundary. Among this class of 2-D filters, rotated elliptical filters, with magnitude response in 1st and 3rd quadrants, and fan filters are worth mentioning. Chapter 4 presents some design strategies for these filters. The review of various transformation techniques in 2-D filter design is presented. All these transformations enjoy one important property; they are applicable to 1-D analog or digital filters. The disadvantage they all have is the various types of cutoff boundaries they present, which cannot be easily changed. It is shown that
by cascading an appropriate number of transformed 1-D filters, an approximation to the
desired cutoff boundary can be achieved.

A number of examples are presented to show the use of these techniques. The
most notable advantage of these transformation strategies is the simplicity of the design
procedure and the absence of any need for optimization. The results are generally
acceptable within certain tolerance limits, which can be set by the designer. However,
modifications to the above techniques have also been presented if greater accuracy is
needed.

Chapter five reviews major realization structures for the 2-D digital transfer
function and also presents a new realization technique for a class of 2-D filters. The
proposed structure has low sensitivity and can easily be extended to include N-D filters
of the same class.

6-2 CONTRIBUTIONS

The contributions of this thesis can be summarized as follows;

1- Development of a novel method for generating 2-variable VSHP using the
properties of positive definite matrices and the concept of resistive matrix
[156,162].

2- Development of a design algorithm for the general class of 2-D filters satisfying
a prescribed magnitude and group-delay specification [156,162].

3- Presentation of a new algorithm for the design of 2-D filters with separable
denominator and non-sparable numerator using a multicriterion optimization
technique, combined with linear and non-linear optimization [157-8].

4- Development of various procedures for the design of 2-D filters with non-circular cutoff boundary using a cascade of several 2-D filter building blocks, along with appropriate modifications to improve the accuracy of the design process [159-60].

5- Development of a new realization structure for a class of 2-D filters with low sensitivity properties [161].

6-3 FUTURE DIRECTIONS

In the development of any new technique, more questions are raised than are answered. The various possibilities which come to mind and deserve consideration are as follows:

1- The high computational cost and labour involved in checking the stability of 2-D recursive digital filters with the existing methods make them uneconomical to use for filters of the higher orders. The time has therefore come for the development of an efficient algorithm for checking the stability of 2-D filters.

2- Most of the existing stabilization algorithms lack the reliability needed to ensure the stability of the designed filter. This is due to the approximation and truncation needed for practical implementation of each technique. The time has come for the development of new stabilization algorithms. This may require a modification to the existing techniques of truncation and smoothing by a weighting sequence to overcome these difficulties as well as other possible answers.

3- Realization techniques need to be developed for a general class of 2-D filters with
low sensitivity and high throughput rate. With recent advances in the area of systolic structures, the time has come to fully utilize this concept along with various arithmetics, which are based on Number Transfer Theory -NTT- and Residue Number Systems -RNS-, to develop an efficient implementation structure for IIR filters.

4- In 1-D filters, an allpass filter can be used as a universal building block. It has been shown [75] that by appropriately adding or subtracting these filters, one can obtain low-pass, band-pass, and high-pass, as well as complementary filters. The question may arise whether similar arrangement can be made for 2-D filters.

5- In light of the many applications 3-D filters are finding, the question arises as to whether direct extension or modification of some of the design, stabilization, and realization techniques of 2-D filters to 3-D is possible.
APPENDIX A

Generation of second order 2-variable VSHP, using properties of positive-definite matrices of Ahmadi et al. [109], can be formulated as follows:

Matrix $D$ is defined in equation (A.1a), where $A$ and $B$ in this case are upper-triangular matrices. $\Gamma$ and $\Delta$ are diagonal non-negative elements matrices. $G$ is a real skew-symmetric matrix.

$$D = A \Gamma A^T s_1 + B \Delta B^T s_2 + G$$  \hspace{1cm} (A.1a)

$$D_2 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1^2 & 0 \\ 0 & \alpha_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} s_1 + \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1^2 & 0 \\ 0 & \beta_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} s_2 + \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix}$$  \hspace{1cm} (A.1b)

$$D_2 = \begin{bmatrix} \alpha_1^2+a^2\alpha_2^2 & a\alpha_2^2 \\ a\alpha_2^2 & \alpha_2^2 \end{bmatrix} s_1 + \begin{bmatrix} \beta_1^2+b^2\beta_2^2 & b\beta_2^2 \\ b\beta_2^2 & \beta_2^2 \end{bmatrix} s_2 + \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix}$$  \hspace{1cm} (A.1c)

Taking the determinant of $D_2$ and setting $\alpha_i = \beta_i = 0$ yields:

$$D(s_1,s_2) = [\alpha_2^2\beta_2^2(b-a)^2 \cdot s_1s_2 + g^2$$  \hspace{1cm} (A.2)

A 2-variable HP can be obtained by associating the corresponding partial derivatives of
\( \det D \) with respect to each variable.

\[
D_1(s_1, s_2) = \det D + K_1 \frac{\partial (\det D)}{\partial s_1} + K_2 \frac{\partial (\det D)}{\partial s_2}\tag{A.3a}
\]

\[
D_1(s_1, s_2) = |\alpha_2^2 \beta_2^2(h-a)^2| \left[ s_1 s_2 + K_1 s_2 + K_2 s_1 \right] + g^2\tag{A.3b}
\]

Where \( K_1 \) and \( K_2 \) are non-negative constants. In this case the second order VSHP is obtained using partial derivatives.

\[
D_2(s_1, s_2) = \det D + K_3 \frac{\partial (\det D_1)}{\partial s_1} + K_4 \frac{\partial (\det D_1)}{\partial s_2}\tag{A.4a}
\]

in this equation \( K_3 \) and \( K_4 \) are non-negative constants and for this case we have:

\[
D_2(s_1, s_2) = [(a_1-b_2)^2 \alpha_2^2 \beta_2^2 ]
\]

\[
\cdot \left[ s_1 s_2 + (k_2 + k_3) s_1 
\quad + (k_1 + k_2) s_2 
\quad + k_2 k_3 + k_1 k_2 \right] + g_2^2 \tag{A.4b}
\]

which is a 2-variable VSHP.
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VITA AUCTORIS

1947 Born on May 31st, Tehran, Iran.

1967 High school diploma in Mathematics and Applied Science, Marvi high school, Tehran, Iran.

1971 B.Sc. Electronics Engineering, Polytechnic, Tehran, Iran.

71-76 Serving National Iranian Radio & TV -NIRT- as a design engineer, technical manager and Head of the technical organization of the 7th Asian Games, Iran.

1976 Management and System Analyst, Industrial Management Institute, Tehran, Iran.

76-85 Serving NIRT as a senior engineer, deputy director and manager of engineering Department of Educational Broadcasting, NIRT, Iran.

1986 M.Sc. Communications Engineering, IRIB Graduate School, Tehran, Iran.

86-87 Serving NIRT as a senior engineer and technical advisor.

1987 Candidate for the degree of Doctor of Philosophy in Electrical Engineering, University of Windsor, Windsor, Ontario, Canada.