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DYNAMIC ANALYSIS OF STRUCTURES WITH UNCERTAIN PROPERTIES

BY

SUDHAN SAMPAD BANIK

A Thesis
Submitted to the Faculty of Graduate Studies and Research through the Department of Civil and Environmental Engineering in Partial Fulfillment of the Requirements for the Degree of Master of Applied Science at the University of Windsor

Windsor, Ontario, Canada
2003
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ABSTRACT

The properties of engineering structures are variable by nature. This variation affects considerably the dynamic response of structures and this is overlooked in traditional deterministic analysis. The incorporation of statistical variability of structural properties in analysis has been a topic of considerable research for the last thirty years. The determination of statistical measures of a desired response variable, such as the natural frequencies, maximum displacement, or stresses has been a primary goal of this study.

The objective of this research work is to determine the quantitative effect of uncertainties on the dynamic response of a structure with uncertain parameters through different methods. Three different approaches are used in this work: (i) the perturbation method, which addresses the problem analytically by assuming small variation in the structural properties; (ii) the Monte Carlo simulation, which treats the problem numerically and predicts accurate stochastic response (iii) the mixed method, which is a compromise between the Monte Carlo simulation and the perturbation method. These three types of methods are implemented within a deterministic finite element code (CALFEM) to solve a stochastic eigenvalue problem associated to structural dynamics. In addition, the mixed method is applied to the dynamic analysis of a multistory building with uncertain stiffness subjected to an earthquake excitation. The response statistics obtained from this method are compared with the Monte Carlo simulation results.
From this investigation, it is found that there may be a significant variation in the response variables for the associated uncertainties with the structural properties. The main concern of this study is to explore the various techniques for the treatment of uncertainties of dynamic systems such as material and geometric variation.
To My Father

Who inspired me the most for higher education
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$A_n = \text{acceleration response of } n^{\text{th}} \text{ mode}$

$[A] = \text{system matrix}$

CDF = cumulative distribution function

COV = coefficient of variation

$[\text{cov}] = \text{covariance matrix}$

$[C] = \text{damping matrix}$

$[C]_m = \text{modal damping matrix}$

$D_n = \text{displacement response of } n^{\text{th}} \text{ mode}$

$E[.] = \text{expected value}$

$F_\lambda(x) = \text{cumulative distribution function}$

$k = \text{mean value of stiffness}$

$[K] = \text{structural stiffness matrix}$

$[K]_m = \text{modal stiffness matrix}$

$m = \text{mean value of mass}$

$[M] = \text{structural mass matrix}$

$[M]_m = \text{modal mass matrix}$

$N_{\text{sim}} = \text{number of simulations}$

$P[.] = \text{probability}$

PDF = probability distribution function
\( \{q\} = \text{displacement vector in normal coordinates} \)

\( \{\ddot{q}\} = \text{acceleration vector in normal coordinates} \)

\( r_n = \text{dynamic response of } n^{\text{th}} \text{ mode} \)

\( r_n'' = \text{static response of } n^{\text{th}} \text{ mode} \)

\( r_{no} = \text{peak response of } n^{\text{th}} \text{ mode} \)

\( r_o = \text{total response.} \)

\( \{u\} = \text{displacement vector} \)

\( \{\ddot{u}\} = \text{acceleration vector} \)

\( \ddot{u}_g = \text{ground acceleration} \)

\( \text{Var} = \text{variance} \)

\( X = \text{random variable} \)

\( [\delta A] = \text{small variation in system matrix} \)

\( \delta k = \text{small variation in stiffness} \)

\( \delta m = \text{small variation in mass} \)

\( \delta \{\phi\} = \text{small variation in eigenvector} \)

\( \delta \lambda = \text{small variation in eigenvalue} \)

\( \{\phi\}^i = i^{\text{th}} \text{ eigenvector} \)

\( [\Phi] = \text{modal matrix} \)

\( \Gamma_n = \text{modal participation factor of } n^{\text{th}} \text{ mode.} \)

\( \lambda^i = i^{\text{th}} \text{ eigenvalue} \)

\( \mu = \text{mean} \)
\[ \sigma = \text{standard deviation} \]

\[ \rho = \text{correlation coefficient} \]

\[ \omega = \text{natural circular frequency} \]
Chapter One

INTRODUCTION

1.1 General

Most of the current design codes are based on the deterministic principle. However, the random variability in the material strength and loads are recognized and are implicitly considered by specifying a safety factor between nominal strengths and nominal loads.

The traditional structural analysis is based on the assumption that the material properties and the geometric layout of a structure are deterministic and therefore completely known. Once an analysis is performed the uncertainties in the input as well as in the loading are addressed by including the concept of safety factors. These safety factors are a practical way to take into account the variability of the structural properties without addressing the problem from a stochastic point of view.

Code-based designs are in general, acceptable for the vast majority of structures. However, there are situations in which minimum weight designs or other type of optimal designs are important. Special structures such as space stations, which are not governed by any design code, may be designed considering different scenarios where uncertainties play a major role.
Probabilistic methods of structural analysis are now used in a broad spectrum of industries. In some areas, the probabilistic concepts are integrated into the design codes whereas in many others probabilistic methods are used to solve special problems. The initial goal for probabilistic structural mechanics application has been safety concerns. With increasing public demand for safer structures and products, the use of probabilistic methods in this direction should increase. For example, the use of uniform hazard spectra, which have the same probability of exceedance at different frequencies, has been proposed for the future version of the National Building Code of Canada (Hong and Wang, 2002).

Besides the random properties, modern engineering structures are quite complex to analyze. Such complexity results from the intricate interaction among the different parts of the structure, as well as the interaction of the structure with the surrounding medium. Addressing such complexity requires recourse to accurate and efficient numerical algorithms. The finite element method has proven to be well suited for a large class of engineering problems. Finite element algorithms have a sound and well-developed theoretical basis and their efficiency has long been proven and tested on a variety of problems. However, most of the available finite element packages are based on deterministic models and have no provision for analysis of structures with uncertain properties.
Over the past three decades, probabilistic structural mechanics has been an expanding field within structural engineering. Many methods dealing with structural uncertainties were proposed. Among these methods the statistical simulations are the most successful, because they are able to predict the actual random behavior of the structural response. These simulation techniques require large number of repetitive analysis of a structural model. If a finite element analysis is performed in each repetition, the entire process becomes extremely computer-intensive and expensive and not suited for practical analysis. Therefore, an effort has to be made to modify such simulation techniques for analyzing practical problems. This contribution is about the analysis of different strategies dealing with structural uncertainties.

1.2 Research Objectives

The main objectives of this research work are to:

1. Get insight into the effect of uncertain material and geometric properties of a structure on its dynamic response.

2. Incorporate uncertainty within a deterministic finite element code for dynamic analysis of structures.

3. Investigate existing methods of analysis for stochastic eigenvalue problem using an existing deterministic finite element code.

4. Select an efficient method for evaluating the response variability of large dynamic models.
1.3 Thesis Outline

This thesis includes five chapters. A background study and literature review related to the dynamic analysis of structures with uncertain properties is presented in Chapter Two. It includes a brief overview of uncertain structural properties and their modeling. The effects of uncertain properties on the dynamic response and an overview of existing methods dealing with uncertain systems is also presented. The methods are studied in the context of the solution of stochastic eigenvalue problem. The last part of the second chapter discusses the different techniques for the evaluation of the dynamic response of a multi-degree of freedom system.

Chapter Three deals with the evaluation of variation of natural frequencies of structures with uncertain properties. The Monte Carlo simulation and the perturbation method are employed to evaluate the statistical characteristics of natural frequencies of truss structures. First, their implementations in a deterministic finite element code (CALFEM) are discussed and then the applications of these methods to stochastic eigenvalue analysis of two different truss structures are presented. An analytical study related to the sensitivity of the fundamental frequency with respect to structural properties is also presented to verify the trend of the results.

The fourth chapter deals with a mixed formulation for the dynamic analysis of structures with uncertain properties. This chapter includes the basic formulation of the mixed method, its implementation in CALFEM and presents some applications. A comparative
study of this formulation with the perturbation method and the Monte Carlo simulation is also presented in this chapter regarding the selection of an efficient method for large dynamic problems.

The fifth chapter of this thesis summarizes this investigation and presents the conclusions based on the investigation. Recommendations for future research are also proposed.
Chapter Two

BACKGROUND STUDY AND LITERATURE REVIEW

2.1 Introduction

Uncertainty, in the scope of this research, is defined as a lack of knowledge about any process or measurement. All the engineering systems are uncertain in the sense that their design parameters are not completely known. As a result of this uncertainty, the actual response of the structure does not match with the predicted response. In the case of structural dynamics, the uncertainties in design parameters may lead to considerable variation from the predicted behavior, which can be the origin of many problems. In some particular applications, it is necessary to include the uncertain behavior of its properties in the dynamic analysis.

Before the 1960's it was always assumed that structural systems have well defined properties and if any variation exist, they are negligible compared to variation in the forcing function. Soong and Bogdanoff (1963) were among the first who considered the problem of random properties of a structural system on dynamic response. A significant amount of research has been performed to develop methods dealing with the randomness in structural properties. The goal of these methods is that given statistically characterized input quantities, a desired output response quantity can also be characterized statistically. All the methods that have been developed are based on two basic techniques: (i) Perturbation method and (ii) Monte Carlo simulation. The perturbation method deals with
the small statistical variations in the structural properties, whereas the Monte Carlo simulation can handle large statistical variations of those quantities. Both techniques have some advantages and disadvantages depending on the nature of the problem.

In this chapter, a brief discussion on the identification of uncertainties in structural dynamic problems and different approaches of their modeling are presented first. The effect of uncertain properties on a dynamic problem will lead to random eigenvalues and eigenvectors. A detailed discussion of the perturbation method and the Monte Carlo simulation is presented in the context of the solution of this random eigenvalue problem. At the end of this chapter, a brief review of the evaluation of dynamic response for earthquake induced ground motion is presented.

2.2 Sources of uncertainty in structural dynamics

Uncertainties in structural dynamics arise from two main sources: (i) statistical source and (ii) non-statistical source (Prasthofer and Beadle, 1975).

(i) Statistical source: This source of uncertainties is due to the stiffness, mass, or damping fluctuations caused by random variations in material and geometric properties such as Young's modulus, cross sectional area, length, Poisson's ratio, randomness in boundary condition and variations caused by manufacturing and assembly techniques. In addition, the physical properties can experience variations over time as a result of wear and tear or just inherent deterioration. The applied loads can also be considered as stochastic
phenomena. In fact, many structural excitations (due to earthquake, wind etc.) encountered in practice exhibit a stochastic nature.

(ii) Non statistical source: The second source of uncertainties is due to the inaccuracies and the assumptions introduced in the mathematical modeling of the geometry, boundary conditions and the materials constitutive behavior.

When a structural model such as a finite element model is considered, the sources of uncertainty can be element stiffness, modal damping coefficients, spring constants modeling the boundary condition, or soil stiffness when accounting soil-structure interaction.

2.3 Modeling of parameter uncertainty

There are two approaches to model the stochastic nature of a structure (i) random variable approach and (ii) random field approach. Random variable approach deals with the variability of one or more input design variables of a structure irrespective of space and time. The random field approach deals with the spatial and temporal variability of the structural properties. The stochastic finite element approach is one method available so far to deal with random field phenomena.

Modeling of structures with uncertain parameters starts by representing the parameters as random variables distributed according to a given distribution. For random input
variables the response of the system is necessarily random and the central problem will be to relate the statistics of the parameters to those of the response quantities. From an engineering point of view, it is of particular interest to relate the dispersion of the parameter distribution to the dispersion of the system characteristics' distribution. For linear systems, the dispersion in natural frequencies and frequency responses due to the uncertainties in the structural parameters would be of central interest. The terminologies related to random variable approach that are frequently used in this thesis are defined in the following section.

**Random variable**

A random variable is a function that assigns a number to each outcome of a random experiment. For example, rolling a die and recording the outcome yields a random variable with range \{1, 2, 3, 4, 5, and 6\}. Each element in this set has an equal probability of occurrence. There are situations where the values that a random variable can take, have different probability of occurrence. For example, the heights of a class of students can take values that have different probabilities. Thus, a random variable can be characterized by the probability distribution of the values that it can take from a fixed range.

**Probability distribution function**

The probability distribution function (PDF) is a function that associates a probability to a given value of a random variable. If \( X \) is a continuous random variable, then the
\[ P(a \leq X \leq b) = \int_a^b f(x)dx \]

That is, the probability that \( X \) takes on a value in the interval \([a, b]\) is the area under the graph of the distribution function, as illustrated in Figure 2.1. The graph of \( f(x) \) is called probability distribution curve.

![Probability distribution curve](image)

**Figure 2.1:** Probability distribution curve.

**Gaussian distribution**

Among various types of probability distribution, Gaussian or normal distribution is the most widely used in engineering. When the sample size become large, many numerical populations have distributions that can be fit very closely by an appropriate normal curve.

A continuous random variable \( X \) is said to have a Gaussian or normal distribution with mean \( (\mu) \) and standard deviation \( (\sigma) \), where \(-\infty < \mu < \infty\) and \(0 < \sigma\), if the probability distribution function of \( X \) is given by the following expression:

\[ f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]
Figure 2.2: Gaussian distribution

Figure 2.2 illustrates the shape of the normal distribution. It is seen from this figure that the distribution is symmetric with the numerical population more concentrated in the middle than in the tails. Most of the statistical variations encountered in material property and the geometry of a structure can be represented by this distribution.

2.4 Effect of uncertainties in structural dynamics

The effect of uncertainties in the structural system may be the cause of significant changes in its dynamic behavior. For example, periodic structures such as honeycomb structures having small variations in their periodicity can exhibit a phenomenon known as normal mode localization. This phenomenon takes place in a manner that vibration in the structure induced by an external source cannot propagate along large distance; instead it remains confined to a region close to the source. The effect of these irregularities has a
similar consequence on damping. It limits the propagation of vibration at large distance from the excitation source. This effect is mainly caused by the confinement of the energy close to the source.

In general, structural systems are designed in such away that the natural frequencies of the system and the forcing frequency are well separated. Due to parameter uncertainties, the frequencies of these systems vary randomly. Thus a reduction in frequency separation could occur which may lead to dangerously high amplification. In the case of earthquake response of a multi-story building, small changes in the story masses and stiffness may cause large variations in the response if the modal frequencies lie close to the peaks of the power spectral density of the earthquake acceleration spectrum.

2.5 Eigenvalue problem for deterministic structural dynamics

Linear dynamic analysis of multi-degree of freedom (MDOF) systems requires the solution of a standard eigenvalue problem. For undamped free vibration of a MDOF system, the equations of motion can be expressed as

\[ [M] \dddot{u} + [K] \ddot{u} = 0 \]  \hspace{1cm} (2.1)

where \([M]\) and \([K]\) are mass and stiffness matrices, respectively and \(\dddot{u}\) and \(\ddot{u}\)are the acceleration and displacement vectors. For simple harmonic free vibration motion, the displacement vector can be expressed as

\[ \{u\} = \{\phi\} \sin \omega t \]  \hspace{1cm} (2.2)

Therefore, the acceleration vector is given by
\{ii\} = -\omega^2 \{\phi\} \sin \omega t = -\lambda \{\phi\} \sin \omega t \quad 2.3

where $\lambda = \omega^2$ and $\omega$ is the natural circular frequency of the system. Substituting equations 2.2 and 2.3 into equation 2.1 and dropping the sine term the following system is obtained

$$[K]\{\phi\}^i = \lambda^i [M] \{\phi\}^i \quad 2.4$$

where is $\{\phi\}^i$ the $i^{th}$ mode shape corresponding to $i^{th}$ frequency $\lambda^i$. The above equation represents the eigenvalue problem in structural dynamics.

2.6 Stochastic eigenvalue problem

The value of the natural frequency of an undamped single degree of freedom system is given by the square root of stiffness to mass ratio. For identical systems this value is assumed to be constant. However, experiments have shown that this value varies randomly (Mok and Murray, 1965), because in reality the physical properties of a structural system cannot be measured exactly. Thus, the eigenvalues are random variables whose statistical properties are determined by the random coefficients of inertia and stiffness terms of equations of motion. The natural frequency $\lambda$ of a simple mass spring system,

$$\lambda = \omega^2 = \frac{k}{m} \quad 2.5$$

The variation of $\lambda$ due to variations in stiffness $k = \bar{k} + \delta k$ and mass $m = \bar{m} + \delta m$, may be expressed as a Taylor series:
\[ \delta \lambda = \lambda - \bar{\lambda} = \frac{\partial \lambda}{\partial k} \delta k + \frac{\partial \lambda}{\partial m} \delta m + \frac{1}{2} \frac{\partial^2 \lambda}{\partial k^2} (\delta k)^2 + \frac{1}{2} \frac{\partial^2 \lambda}{\partial m^2} (\delta m)^2 + \ldots \ldots \]  \hspace{1cm} 2.6

where the over bar quantities refer to mean values and \( \bar{\lambda} = \frac{k}{m} \). When the variations \( \delta k \) and \( \delta m \) are random variables the natural frequency will be a random variable. The mean and variance of \( \lambda \) can be evaluated as follows:

\[ E[\lambda] = \bar{\lambda} + \frac{1}{2} \frac{\partial^2 \lambda}{\partial k^2} E[\delta k^2] + \frac{1}{2} \frac{\partial^2 \lambda}{\partial m^2} E[\delta m^2] + \ldots \ldots \]  \hspace{1cm} 2.7

and

\[ E[(\lambda - \bar{\lambda})^2] = \left( \frac{\partial \lambda}{\partial k} \right)^2 E[\delta k^2] + \left( \frac{\partial \lambda}{\partial m} \right)^2 E[\delta m^2] + 2 \frac{\partial \lambda}{\partial k} \frac{\partial \lambda}{\partial m} E[\delta k \delta m] + \ldots \]  \hspace{1cm} 2.8

For multi-degrees of freedom system, the solutions for natural frequencies and mode shapes require the solution of the general eigenvalue problem. If the stiffness and the mass matrices are random due to the randomness in structural properties, the eigenvalues and eigenvectors are also random. Let \( X \) denote a vector, which contains all random parameters. Hence, the stiffness matrix \([K(X)]\) and the mass matrix \([M(X)]\) respectively are functions of the random vector \( X \). The generalized eigenvalue problem can be written as follows:

\[ [K(X)] \{ \phi(X) \}^i = \lambda^i(X)[M(X)] \{ \phi(X) \}^i \]  \hspace{1cm} 2.9

where \( \{ \phi(X) \}^i \) is the \( i^{th} \) eigenvector corresponding to the \( i^{th} \) eigenvalue \( \lambda^i(X) \). Equation 2.9 is called random or stochastic eigenvalue problem because the random values of \( X \) will generate random values of \( \lambda^i(X) \) and \( \{ \phi(X) \}^i \). The main concern for a stochastic
eigenvalue problem is the statistical characterization of the random eigenvalues and the eigenvectors.

2.7 Methods for solution of stochastic eigenvalue problem

There are two fundamental ways to solve the stochastic eigenvalue problem (i) analytical approach and (ii) numerical approach. Among analytical approaches, the perturbation method is widely used because of its simplicity. Numerical method such as Monte Carlo simulation is generally applicable to all types stochastic problems and is often used to verify the results obtained from analytical methods. A detailed discussion of these methods is presented below:

2.7.1 Perturbation method

General

The perturbation method is the most widely used technique for analyzing uncertain system. This method consists of expanding all the random variables of an uncertain system around their respective mean values via Taylor series and deriving analytical expression for the variation of desired response quantities such as natural frequencies and mode shapes of a structure due to small variation of those random variables. Since the 1960’s, the perturbation method has been considered as an efficient method for solving stochastic eigenvalue problem (Brown, 1998). Boyce and Goodwin (1964) used the perturbation method for the solution of the eigenvalue problem of random strings and beams. They used the integral equation approach to invert the governing differential
operator, and consider randomness caused by material properties. Soong and Bogdanoff (1963) used the transfer matrix method to investigate the behavior of disordered linear chain. They expressed the frequency response of the chain as a perturbation type expansion in terms of the small deviation of the mass. They concluded that even for small levels of disorder in the system, there would be considerable variation in the higher frequencies. Collins and Thomson (1969) used the perturbation approach to evaluate the eigenvalue problem of the system with statistical properties. They used a matrix approach along with statistical correlation of the elements, which opened the way to apply the perturbation approach to a much broader class of dynamic problems. Astill and Shinozuka (1972) used the perturbation approach to treat the general problem of computing the eigenvalues and eigenvectors statistics of a system with random parameters defined by their covariance matrix. A second order Taylor series expansion about the mean value was used to represent the eigenvalue and eigenvector of the system. Hasselman and Hart (1972) extended the work to develop a perturbation scheme compatible with the finite element method. Nakagiri and Hisada (1982) and Hisada and Nakagiri (1985) investigated again the random eigenvalue problem allowing for random boundary conditions. They concluded that the second order perturbation was too intractable to be of any practical interest in solving real physical problem. Using the perturbation technique Zhang and Chen (1991) derived a simple formula for the evaluation of the standard deviations of the random eigensolutions.
perturbation technique Zhang and Chen (1991) derived a simple formula for the evaluation of the standard deviations of the random eigensolutions.

**Basic approach for the perturbation method**

The basic idea behind the perturbation method is to express the stiffness and mass matrices and the responses in terms of Taylor series expansion with respect to the parameters centered at the mean values. Generally, the Taylor series is expanded only to the first order. That is why this method is often referred to as first-order perturbation method. A response quantity $\lambda$, depending on random variables $X_i$ is expressed as

$$\lambda(X) = \lambda(\mu_X) + \sum_{i=1}^{N} \frac{\partial \lambda}{\partial X_i} (X_i - \mu_{X_i})$$  \hspace{1cm} (2.10)

where $\mu_{X_i}$ denotes the mean value of a random variables $X_i$. We can derive the simple approximations of the expected value $(\mu_\lambda)$ and the standard deviation $(\sigma_\lambda)$ of the response quantity from the above expression. The expected value of the second term of equation 2.10 is zero. Therefore, the expected value can be expressed as

$$\mu_\lambda = E[\lambda(X)] \approx \lambda(\mu_X)$$  \hspace{1cm} (2.11)

Similarly, the variance of the first term of equation 2.10 is zero. The standard deviation can be expressed as

$$\sigma_\lambda = \sqrt{E[(\lambda(X) - \mu_\lambda)^2]} \approx \sqrt{\sum_{i=1}^{N} \left( \frac{\partial \lambda}{\partial X_i} \right)^2 \sigma_{X_i}^2}$$  \hspace{1cm} (2.12)
\[ [A](\phi)^i = \lambda^i (\phi)^i \]

where \([A] = [M]^{-1} [K] \). Now, let \([A]\) be perturbed by a small variation such that

\[ [A]_p = [A] + [\delta A] \]

where

\[ [\delta A] = \text{small change in } A \]

\[ [A]_p = \text{perturbed matrix} \]

Similarly, the eigenvalues and eigenvectors can be expressed as follows

\[ \lambda^i_p = \lambda^i + \delta \lambda^i \]

\[ (\phi)^i_p = (\phi)^i + (\delta \phi)^i \]

where \( \delta \lambda^i \) and \( (\delta \phi)^i \) are the perturbation of the eigenvalues and eigenvectors, respectively. The perturbed eigenvalue problem is written as:

\[ ([A] + [\delta A])(\phi)^i_p + (\delta \phi)^i_p) = (\lambda^i + \delta \lambda^i) (\phi)^i + (\delta \phi)^i \]

Simplifying both sides of the above equation and assuming that the second order effects are negligible compared to the first order, we obtain the direct expression for the perturbation of \( \lambda \) and \( (\phi) \) without resolving the above eigenvalue problem. The result is given by

\[ \delta \lambda^i = (\phi)^i^T [\delta A] (\phi)^i \]

\[ (\delta \phi)^i = \sum_{k=1}^{n} \frac{\{\phi\}^k^T [\delta A] (\phi)^i}{\lambda^i - \lambda^k} (\phi)^k, \quad i, k = 1, \ldots, n; i \neq k \]
These first order terms can be substituted into equations 2.16 and 2.17 to obtain the new eigenvalues and eigenvectors.

Collins and Thompson (1969), Kiefling (1970), Collins et al. (1970), Zhang and Chen (1991) used perturbation method to derive the statistics of random eigensolution. They obtained the derivatives of \( \lambda_i \) and \( \{ \phi_i \} \) with respect to each random variable \( X \), which can be expressed in the following form:

\[
\frac{\partial \lambda_i}{\partial X} = \{ \phi_i \}^T \left( \frac{\partial K}{\partial X} - \frac{\partial \lambda_i[M]}{\partial X} \right) \{ \phi_i \}
\]

2.21

\[
\frac{\partial \{ \phi \}^{(i)}}{\partial X} = -\frac{1}{2} \left( \{ \phi \}^{(i)} \right)^T \frac{\partial [M]}{\partial X} \{ \phi \}^{(i)} + \sum_{j=1 \& j \neq i}^n \frac{1}{\lambda_i - \lambda_j} \times \{ \phi \}^{(j)} \left( \frac{\partial [K]}{\partial X} - \frac{\partial \lambda_i[M]}{\partial X} \right) \{ \phi \}^{(j)}
\]

2.22

The standard deviations of \( \lambda_i \) and \( \{ \phi_i \} \) are then obtained using these eigenvalues and eigenvectors sensitivities in equation 2.12 which resulted from the first order Taylor series expansion. The mean was obtained by directly applying equation 2.11 for \( \lambda \). The partial derivatives of the elements of the stiffness and mass matrices can be obtained by taking the derivative of the analytical expression for each element of the matrix with respect to each random property.

Pierre (1985) performed an extensive study of the analysis of structural systems with parameter uncertainties. He formulated the "Statistical perturbation method" which incorporates the correlation between an input random variable and the response variables.

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(here the eigenvalues and eigenvectors) in the perturbation method. He gave the following expression for the covariance matrix of $\lambda$, which is a function of the covariance matrix of $X$.

$$
[cov_{\lambda}] = \left[ \frac{\partial \lambda}{\partial X} \right] [cov_X] \left[ \frac{\partial \lambda}{\partial X} \right]^T
$$

2.23

where $[cov_X]$ and $[cov_{\lambda}]$ is the covariance matrix for random variables and eigenvalues, respectively.

In a structural system, where the input random variables are statistically uncorrelated, $[cov_{\lambda}]$ and $[cov_X]$ contain only the variance terms in the diagonal and the results from equation 2.23 are identical with those from equation 2.12.

Mahadevan (1994) proposed a simple method for using the perturbation in stochastic eigenvalue problem. This method is referred to as the perturbation-sensitivity analysis. The basic formulation is relatively simple, and it is presented in the following steps:

1. The eigenvalue problem for mean system is solved first. The resulting eigenvalues and eigenvectors can be written as $\{\lambda\}_{\text{mean}}$ and $[\Phi]_{\text{mean}}$ respectively.

2. The random variable $X_i$ is varied by $\Delta X_i$, where $i = 1, 2, \ldots$, number of random variables.

3. The eigenvalue problem is computed for the new system. The resulting eigenvalues and eigenvectors can be denoted as $\{\lambda\}_{\text{new}}$ and $[\Phi]_{\text{new}}$ respectively.
4. Now the eigenvalue and the eigenvector sensitivities can be computed from the following equations:

\[
\frac{\partial \{\lambda\}}{\partial X_i} \approx \frac{\{\lambda\}_\text{new} - \{\lambda\}_\text{mean}}{X_i^\text{new} - X_i^\text{mean}} \tag{2.24}
\]

and

\[
\frac{\partial [\Phi]}{\partial X_i} \approx \frac{[\Phi]_\text{new} - [\Phi]_\text{mean}}{X_i^\text{new} - X_i^\text{mean}} \tag{2.25}
\]

5. The standard deviations of the eigenvalues and the eigenvectors can be obtained by substituting these values in equation 2.12.

### 2.7.2 Monte Carlo Simulation

**General**

It is apparent from the previous discussion that the perturbation method uses the variability of the random parameters from their mean value and does not take into account the actual probability distribution of those parameters. To account for the actual random behavior of the input parameters, a numerical simulation scheme is needed. The most widely used numerical simulation technique is the Monte Carlo simulation. This technique was originally developed for Manhattan Project during World War II (Rubinstein, 1981). However, it is now applied to a wide range of problems. In this technique the values of the uncertain parameters are randomly selected to create scenarios of a problem. These values are taken from a fixed range and selected to fit a probability distribution. In the Monte Carlo simulation, the random selection process is repeated many times to create multiple scenarios. Each time a value is randomly selected, it forms one possible scenario and the solution to the problem is obtained. Together, these
scenarios give a range of possible solutions, some of which are more probable and some are less probable.

The Monte Carlo method is a quite versatile mathematical tool capable of handling situations where all other methods fail to succeed. In structural dynamics it has attracted intense attention only recently following the widespread availability of inexpensive computational systems. This computational availability has triggered an interest in developing sophisticated and efficient simulation algorithms. Shinozuka (1972) had a pioneering role in introducing the method to the field of structural dynamics. He used the Monte Carlo simulation for simulating a random process as the superposition of a large number of sinusoids having a uniformly distributed random phase angle. This approach has been successfully used in a variety of problems for the simulation of earthquake records, sea-wave elevations, and various other random phenomena. Astill and Shinozuka (1972), Chang and Yang (1991) and Zhang and Ellingwood (1995) used this method to obtain the effects of random material properties. However, in most of the studies, the Monte Carlo simulation was used to verify the results obtained from approximate methods.

Monte Carlo method for solving stochastic eigenvalue problem

The general eigenvalue problem for a dynamic system can be written in the following form:

\[ [K]\phi_i = \lambda_i [M]\phi_i \]  

2.26
If the structural properties or geometry of the dynamic system are uncertain then the resulting stiffness and mass matrices are random. The randomness in mass and stiffness matrices creates a stochastic eigenvalue problem. Let $X$ be a random parameter. The structural eigenvalue problem can be written as:

$$[K(X)]\{\phi(X)\}^T = \lambda^i(X)[M(X)]\{\phi(X)\}^T$$  \hspace{1cm} 2.27

The Monte Carlo simulation generates a set of random values of $X$ according to its probability distribution function. The set can be written as $X = \{x_1, x_2, \ldots, x_N\}$, where $N$ is the number of simulation. For each values of $X$, the stiffness and mass matrices are computed and the resulting eigenvalue problem is solved. At the end of $N$ simulations, we have a random set of values $\{\lambda^i_1, \lambda^i_2, \ldots, \lambda^i_j, \ldots, \lambda^i_N\}$ for $\lambda^i$. From this finite set of solutions the expected value and the variance are computed using the following formulas:

$$\mu_\lambda = E\{\lambda\} = \frac{1}{N} \sum_{j=1}^{N} \lambda_j$$  \hspace{1cm} 2.28

and

$$\sigma^2_\lambda = E\{(\lambda - E\{\lambda\})^2\} = \frac{1}{N} \sum_{j=1}^{N} (\lambda_j - E\{\lambda\})^2$$  \hspace{1cm} 2.29

**Generation of random variables**

In the Monte Carlo simulation, the random variables need to be generated according to their respective probability distribution. Such random variable generations require

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---
random numbers that are uniformly distributed between 0 and 1. The theory behind the generation of random numbers is discussed in the following:

A random number generator is provided as a built-in function in modern digital computers. In these generators, a sequence of integers $X_1, X_2, X_3...$ is defined by the following recursive equation

$$X_i = (aX_{i-1} + b) \mod c; \quad (for \ i = 1, 2, 3...$$

where Int operator extracts the integer part of a float number, $a$ is the multiplier, $b$ is the increment and $c$ is the modulus. These model constants $a$, $b$, and $c$ should be nonnegative integers. The starting value $x_0$ is called the seed, which should be provided by the user of the model. The value $X_i$ is obtained by dividing $(aX_{i-1} + b)$ and letting $X_i$ be the remainder of the division. The random number $U_i$ is defined as

$$U_i = X_i / c \ (for \ i = 1, 2, 3..........)$$

The value $X_i$ is normalized by the division $c$, because $0 \leq X_i \leq c$. The parameters of this recursive model should satisfy the following conditions: $0 < c$, $a < c$, $b < c$ and $X_0 < c$. It is evident from this recursive model that the process is not truly random, because it can be repeated with the same results all the time. For this reason, the process is commonly called pseudorandom number generation.

**Generation of random numbers from non-uniform distribution**

To generate random variables from a given probability distribution, the inverse transformation method is widely used, because of its simplicity and directness. Figure 2.3
shows the process of this generation. Let $X$ be the continuous random variable to be generated and let $F_X(x)$ be its cumulative distribution function (CDF). A random number, $u$, which is uniformly distributed between 0 and 1, is first generated. Then a value of the generated continuous random variable, $X$, is determined as follows:

$$X = F_X^{-1}(u)$$

where $F_X^{-1}$ is the inverse of the cumulative distribution function of the random variable $X$. This procedure is repeated as many times as required using a different value of $u$. Since the range of $F_X(x)$ is [0,1], a unique value for $x$ is obtained for all the time.

![Diagram](image)

**Figure 2.3:** Inverse transformation method for generation of random numbers

### 2.8 Response of MDOF system

In practice, most structural models have several degrees of freedom and an equal number of vibration modes when subjected to dynamic loading. Therefore, it is necessary to
compute the response of each mode in order to obtain the actual response of MDOF system. The most popular method for evaluation of the response of MDOF system is the mode superposition method (or modal analysis). This method is very useful when the system response can be accurately evaluated by considering only a relatively small subset of all the vibration modes for the system. For most linear systems, the response is dominated by few lower frequencies. The major limitation of this method is that it is valid for linear systems with classical (viscous) damping. Using mode superposition method, there are two ways to evaluate the dynamic response: (i) time history analysis and (ii) response spectrum analysis. These two methods are discussed briefly in the following:

2.8.1 Time history analysis

The time history analysis is a straightforward technique to obtain the response of a MDOF system subjected to dynamic loading. The differential equations governing the response of an MDOF system to earthquake induced ground motion can be written as:

\[
[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = -[M]\{I\}\ddot{u}_g (t)
\]

where \(\ddot{u}_g (t)\) is the ground acceleration and \(\{I\}\) is a unit vector.

The above equation can be written in the following form using the normal coordinates

\[
[M]_m \{\dddot{\bar{q}}\} + [C]_m \{\ddot{\bar{q}}\} + [K]_m \{q\} = -[\Phi]^T [M]\{I\}\ddot{u}_g (t)
\]

where \([M]_m\), \([C]_m\) and \([K]_m\) are the modal mass, the damping and the stiffness matrix, respectively and \(\{q\}\) represents the normal coordinates, which is defined by the matrix

\[
[\Phi] = [\Phi]_1 \ldots [\Phi]_n
\]

[\(\Phi\)]
transformation, \( \{u\} = [\Phi] \{q\} \), \([\Phi]\) represents the modal matrix. Due to the orthogonality conditions, the equation of motion of the structure is written in the normal coordinates as a set of uncoupled equations. After solving for each of the normal coordinates, the displacement \( \{u(t)\} \) of an N degree of freedom system can be expressed, as the superposition of modal contributions:

\[
\{u(t)\} = \sum_{n=1}^{N} \{\phi_n\} q_n(t)
\]

where \( \{\phi_n\} \) is the mode shape vector corresponding to \( n^{th} \) mode of the structure and \( N \) is the total number of modes. Equation 2.32 represents the displacement of a system at any instance of time history.

### 2.8.2 Response spectrum analysis

The evaluation of dynamic response at every instant of time during an earthquake time history requires significant computational effort, even for a relatively simple structural system. However, for many engineering applications, we are mostly interested with the maximum absolute values of response quantities experienced by the structures during an earthquake. A plot of the maximum peak response to a specific input such as earthquake induced ground motion versus the natural period or the natural frequency is called the response spectrum. If an MDOF system is decomposed into uncoupled N single degrees of freedom system, then the total response of that system can be found using the response spectrum. The response spectrum analysis uses the values from a response spectrum to calculate the maximum modal response rather than lengthy time domain solution to
obtain the response maxima. The response spectrum analysis is discussed briefly in the following.

Equation 2.32 can be expanded in the following form:

\[
\{u(t)\} = \sum_{n=1}^{N} \{\phi_n\} \Gamma_n D_n(t)
\]

where \( \Gamma_n \) is modal participation factor, which is given by

\[
\Gamma_n = \frac{\{\phi_n^T\} [M] \{I\}}{\{\phi_n^T\} [M] \{\phi_n\}}
\]

and \( D_n(t) \) is the displacement of a SDOF system having \( n^{th} \) frequency of the MDOF system and subjected to same ground acceleration. For any response quantity \( r(t) \), such as base shear, bending moment, equation 2.33 can be rewritten in the following form

\[
r(t) = \sum_{n=1}^{N} r_n(t) = \sum_{n=1}^{N} r_n^{\text{st}} A_n(t)
\]

where \( A_n(t) \) is the \( n^{th} \) mode acceleration response of the structure, which is equal to \( \omega^2 D_n(t) \) and \( r_n^{\text{st}} \) is the static value of \( r_n \) due to external forces. The value of peak response of \( n^{th} \) natural mode of an MDOF system can be obtained from the earthquake response spectrum. The peak modal response \( r_{no} \) can be defined \( r_{no} = r_n^{\text{st}} A_n \), where \( A_n \) is the peak acceleration of the structure when vibrating at \( n^{th} \) mode. Now, to proceed with the response spectrum analysis, a combination of peak responses is necessary.
Three methods are available to combine the responses: (i) Absolute sum (ABSSUM) method (ii) Square root of sum of squares (SRSS) method and (iii) Complete quadratic combination (CQC) method.

The most straightforward way to combine the responses is to add them algebraically. This is called absolute sum procedure. However, adding the responses in this manner will produce very conservative results. In reality, all the modes do not reach their peak at the same time.

The square root of sum of squares (SRSS) rule for modal combination has the following form:

\[ r_o = \left( \sum_{n=1}^{N} r_{no}^2 \right)^{1/2} \]  \hspace{1cm} (2.36)

This combination rule provides excellent response estimate for structures with well-separated frequencies. For systems with closely spaced frequencies this rule provides poor estimates of the spectral response. The complete quadratic combination (CQC) rule for modal combination is applicable to wider class of structures as it overcomes the limitation of the SRSS rule. According to the CQC rule, the peak response can be calculated by the following expression:

\[ r_o \equiv \left( \sum_{i=1}^{N} \sum_{n=1}^{N} \rho_{io} r_{io} r_{no} \right)^{1/2} \]  \hspace{1cm} (2.37)
Each of the $N^2$ terms on the right side of the equation is the product of peak responses in the $i$th and $n$th modes and $\rho_{in}$ is the correlation coefficient for these modes which varies between 0 and 1 and $\rho_{in} = 1$ for $i = n$.

2.9 Summary

This chapter reviews the necessary ingredients for dynamic analysis of structures with uncertain properties. The perturbation method and the Monte Carlo simulation are presented in context of a stochastic eigenvalue problem. Both methods are discussed in their simplest form for implementation within a deterministic finite element code. The application of these methods is carried out in the next chapter for dynamic analysis of truss structures having uncertain properties.
Chapter Three

EVALUATION OF THE VARIATION OF NATURAL FREQUENCIES OF STRUCTURES WITH UNCERTAIN PROPERTIES

3.1 Introduction

As stated in Chapter Two, that the Monte Carlo simulation and the perturbation method are available to deal with uncertainties in dynamic structural system. In this chapter, the implementation of the Monte Carlo simulation and the perturbation method using CALFEM is presented first. Then these two methods are used to evaluate the variation of the natural frequencies due to uncertainties in structural properties. An analytical study relating to the variation of natural frequencies with the variation of structural properties is taken as a basis to verify the trend followed by the results obtained from stochastic analysis. At the end, a comparative study between the Monte Carlo simulation and the perturbation method is presented.

3.2 Modification of deterministic finite element code for stochastic analysis

Most of the finite element based structural analysis software such as ABAQUS, ADINA, and NASTRAN are capable of very detailed deterministic analysis of complex structures. However, these software packages are not suitable for stochastic analysis as there is no provision for such analysis. On the other hand, the technical computational programming language MATLAB has a powerful statistics toolbox, which is capable of generating random numbers from any type of probability distribution. Thus it offers a way to perform stochastic analysis, if the deterministic structural analysis code is based on
MATLAB functions. Researchers from Lund Institute of Technology, Sweden have developed a MATLAB based finite element toolbox, CALFEM (1993), for deterministic structural analysis. In this thesis, MATLAB based computer programs are developed for stochastic analysis of structures using CALFEM and the statistics toolbox of MATLAB.

3.2.1 Implementation of the Monte Carlo simulation in CALFEM

The basic steps for the implementation of the Monte Carlo simulation in CALFEM are presented in Figure 3.1. A stochastic eigenvalue problem is chosen for this scheme.

![Diagram showing the implementation process]

**Figure 3.1**: Basic steps for implementation of the Monte Carlo simulation in CALFEM
3.2.2 Implementation of the perturbation method in CALFEM

A computational scheme for perturbation method is also developed using CALFEM. The method presented by Mahadevan (1994) is taken as the basis for this development. The basic steps regarding the implementation of perturbation method are shown in Figure 3.2.

A stochastic eigenvalue problem is selected to describe the steps.

![Diagram showing the implementation steps]

**Figure 3.2:** Basic steps for implementation of the perturbation method in CALFEM

\[
\text{Mean}(\lambda) = \lambda_{\text{original}}
\]

\[
\text{Var}(\lambda) = \sum_{i=1}^{N} \left( \frac{\partial \lambda}{\partial X_i} \right)^2 \text{Var}(X_i)
\]

*Evaluation of the variation of natural frequencies of structures with uncertain properties*
3.3 Number of simulations for Monte Carlo technique:

Theoretically, the distribution and statistics of a response variable obtained from the Monte Carlo simulation will represent the actual random behavior of the structural response, if the number of simulation is infinite. However, for practical application, this is not feasible. It is shown in the literature (Melchers, 1999) that the error associated with the estimates of the Monte Carlo simulation is inversely proportional to the square root of the number of simulations. Due to this relationship, the convergence of the error will be achieved at very high number of simulations.

Another way to show the convergence of error of the Monte Carlo estimates with number of simulations is to measure the coefficients of variation of the mean and the standard deviation of the generated samples with the number of simulations. If the generated samples do not represent the actual distribution then one would expect variation in the statistical estimates (mean, standard deviation etc.) of the response variables. In this thesis, only a Gaussian probability distribution is considered to characterize the random behavior of the design variables. A study related to the uncertainty associated with the number simulations is performed in the context of the uncertainty in the generated samples from a Gaussian distribution.

Figures 3.3 and 3.4 show the coefficient of variation (COV) of the mean and the standard deviation of the generated samples from a Gaussian distribution with the number of simulations. A Gaussian distribution having mean value equal to 5.0e-4 and standard
deviation equal to 1.25e-4 is selected for this purpose and the studies are performed using a MATLAB based computer program.

Figure 3.3: Uncertainty in mean value associated with number of simulations

Figure 3.4: Uncertainty in standard deviation associated with number of simulations

Evaluation of the variation of natural frequencies of structures with uncertain properties
From Figures 3.3 and 3.4, it can be concluded that the rate of convergence of error decreases exponentially with the number of simulations. It is also noticed from these figures that at least 1000 simulations are needed to obtain a reliable Monte Carlo estimation, if a Gaussian or normal distribution is selected for the random variables.

3.4 The variation of natural frequencies due to the variation of structural property (Analytical approach):

Many engineering structures such as highway bridges, aerospace and offshore structures are frequently subjected to dynamic loads and thus dynamic analysis is necessary to determine the vibration response of these structures. A usual requirement is that the natural frequencies should be far away from the frequency of the exciting force. To ensure such separation between frequencies a significant amount of research work was conducted towards developing a relationship between the structural modification and natural frequencies. Djoudi and Bahai (2001) derived a formulation, which is able to compute the required modification in the material and geometric properties in order to obtain desired natural frequencies. They applied their formulation for two types of truss structures: (i) plane truss cantilever (ii) space truss tower. In both cases they obtained the required modification in bar cross sectional areas to achieve desired frequency. The description of these structures along with the results obtained by the authors are presented below:
3.4.1 Plane truss example

The first example is a twelve bar truss cantilever as shown in Figure 3.5. The mean values of the material properties (Young's modulus, \( E \) and mass density, \( \rho \)) and the cross-sectional area, \( A \) of the bars are also shown in that figure.

\[
\begin{align*}
E &= 2 \times 10^{11} \text{N/m}^2 \\
\rho &= 7860 \text{ kg/m}^3 \\
A &= 5 \times 10^{-4} \text{ m}^2 \text{ for all bars}
\end{align*}
\]

![Plane truss structure](image)

**Figure 3.5:** Plane truss structure (After Djoudi and Bahai, 2001)

The sensitivity of the fundamental frequency with respect to variation of each bar cross-sectional area is presented in Figure 3.6.
Figure 3.6: Variation of first frequency with required modification on bars area. (Adapted from Djoudi and Bahai, 2001)

3.4.2 Space truss example

The second example is a space truss tower as shown in figure 3.7. The dimensions and material properties are shown in the same figure.

Figure 3.7: Space truss structure (After Djoudi and Bahai, 2001)
The mean values of the cross sectional areas for each bar are given in Table 3.1.

**Table 3.1: Cross-sectional areas of different bar groups**

<table>
<thead>
<tr>
<th>Bar group</th>
<th>Designation</th>
<th>Cross-sectional area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corner members in bottom level</td>
<td>C1</td>
<td>$3 \times 10^{-4}$ m$^2$</td>
</tr>
<tr>
<td>Corner members in top level</td>
<td>C2</td>
<td>$3 \times 10^{-4}$ m$^2$</td>
</tr>
<tr>
<td>Horizontal members in bottom level</td>
<td>B1</td>
<td>$1.5 \times 10^{-4}$ m$^2$</td>
</tr>
<tr>
<td>Horizontal members in top level</td>
<td>B2</td>
<td>$0.8 \times 10^{-4}$ m$^2$</td>
</tr>
<tr>
<td>Diagonal members in bottom level</td>
<td>T1</td>
<td>$3 \times 10^{-4}$ m$^2$</td>
</tr>
<tr>
<td>Diagonal members in top level</td>
<td>T2</td>
<td>$3 \times 10^{-4}$ m$^2$</td>
</tr>
</tbody>
</table>

The sensitivity of the fundamental frequency with respect to variation of cross-sectional area of each bar group is presented in Figure 3.8.

**Figure 3.8:** Variation of first frequency with the required modification in the cross-sectional area of bars. (Adapted from Djoudi and Bahai, 2001)
The above examples are analyzed in this thesis for the evaluation of the stochastic variation of the natural frequencies due to the uncertainty in bar cross-sectional area. The Monte Carlo simulation and the perturbation method are used for these analyses.

3.5 Evaluation of the stochastic variation of natural frequencies using Monte Carlo simulation

The plane truss and the space truss examples of the preceding section are considered for a stochastic analysis using Monte Carlo simulation. Only the stochastic variations of natural frequencies due to uncertainties in structural properties are evaluated. The considerations for analysis and the results for both examples are discussed below:

Plane truss

The following considerations are made while analyzing the plane truss structure of Figure 3.5:

- The cross-sectional areas of different bars are considered as random variables and are assumed to be normally distributed with the mean values shown in Figure 3.5.

- Several analyses are performed using MATLAB based computer programs. In each analysis the cross-sectional area of one specific bar is considered as a random variable while the other bar areas are kept constant at their mean values.

- Results are obtained for six different coefficients of variations of each random variable. These are 0.05, 0.1, 0.15, 0.2, 0.25 and 0.3.

- One thousand simulations are used for each analysis.
- Only the mean and the standard deviation of the first two frequencies are calculated at the end of each analysis.

Along with the analysis, the first two mode shapes of the plane truss for the mean values of the random variables are also obtained using CALFEM. These mode shapes are presented in Figures 3.9 and 3.10.

**Figure 3.9:** First mode shape of the plane truss
Figure 3.10: Second mode shape of the plane truss

Figures 3.11 and 3.12 show the effect of the stochastic variation of different bar areas on first two natural frequencies. It is seen from Figure 3.11 that bar 3 has a major effect on first frequency. This fact can be verified from the results (Figure 3.6) of the analytical study, where the first frequency is more sensitive to any changes in cross-sectional area of bar 3. It is realized from the first mode shape (Figure 3.9) of the structure that if the stiffness and the mass of bar 3 is too low or too high compared to the other bars, the first frequency can be very low or very high. It is expected that, bar 3 has a major effect on the uncertainty of the first frequency. For the second frequency, bar 2 has the most effect, which is seen in Figure 3.12. The contribution of the other bars is insignificant compared to that of bar 2.
**Figure 3.11:** Effect of stochastic variation of bar area on first frequency

**Figure 3.12:** Effect of stochastic variation of bar area on second frequency
Space truss:

The following considerations are made while analyzing the space truss tower of Figure 3.7:

- The cross-sectional areas of different bar groups are considered as random variables and are assumed to be normally distributed with the mean values shown in table.
- Several analyses are performed using MATLAB based computer programs. In each analysis the cross-sectional area of one specific bar group is considered as a random variable while the cross-sectional areas of other bar groups are kept constant at their mean values.
- Results are obtained for six different coefficients of variation of each random variable. These are 0.05, 0.1, 0.15, 0.2, 0.25 and 0.3.
- One thousand simulations are used for each analysis.
- Only the mean and the standard deviation of the first two frequencies are calculated at the end of each analysis.

Along with the analysis, the first two mode shapes of the space truss for the mean values of the random variables are also obtained using CALFEM. These mode shapes are presented in Figures 3.13 and 3.14.
Figure 3.13: First mode shape of the space truss

Figure 3.14: Second mode shape of the space truss
Figures 3.15 and 3.16 show the effect of stochastic variation of bar cross-sectional areas on first two natural frequencies. It is seen that bar groups C1 and C2 have the major effect on the first frequency. The impact of other bar groups is not significant compared to C1 and C2 bar groups. This fact is verified from the first mode shape (Figure 3.13) of the space truss structure where the corner columns control the mode shape. These results are in good agreement with the results found from analytical study (Figure 3.8) For the stochastic variation of second frequency all the bars except the diagonal members have major contribution as seen from Figure 3.16.

![Graph](image)

**Figure 3.15:** Effect of stochastic variation of bar area on first frequency
3.6 Evaluation of the stochastic variation of natural frequencies using perturbation method

The space truss example is taken for the stochastic analysis using the perturbation method. Only the stochastic variation of fundamental frequency due to uncertainties is evaluated. The following considerations are made during analysis:

- The perturbation method presented by Mahadevan (1994) is used for the analysis.
- Several analyses are performed using a MATLAB based computer program. In each analysis the cross-sectional area of one specific bar group is considered as a random variable while the cross-sectional areas of other bar groups are kept constant at their mean values.
• Results are obtained for six different coefficients of variations of each random variable. These are 0.05, 0.1, 0.15, 0.2, 0.25 and 0.3.

• Only the mean and the standard deviation of the fundamental frequency are calculated at the end of each analysis.

3.7 Comparison of the Monte Carlo simulation and the perturbation method

The results obtained from the perturbation method are compared with those obtained from the Monte Carlo simulation. These comparisons are presented in Figures 3.17 to 3.22. In addition to this, a comparison of computational time required by each method with respect to the number of random variables for space truss example is presented in Table 3.2.

![Graph showing comparison between Monte Carlo and Perturbation methods](image)

**Figure 3.17**: Comparative study for C1 bar Group
Figure 3.18: Comparative study for C2 bar Group

Figure 3.19: Comparative study for B1 bar Group
**Figure 3.20**: Comparative study for B2 bar Group

**Figure 3.21**: Comparative study for T1 bar Group
Figure 3.22: Comparative study for T2 bar Group

Table 3.2: Comparison of Computational time

<table>
<thead>
<tr>
<th>Number of Random Variables</th>
<th>Computation time (sec) in Perturbation technique</th>
<th>Computation time (sec) in Monte Carlo simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.19</td>
<td>68.75</td>
</tr>
<tr>
<td>2</td>
<td>0.26</td>
<td>68.76</td>
</tr>
<tr>
<td>3</td>
<td>0.33</td>
<td>68.81</td>
</tr>
<tr>
<td>4</td>
<td>0.4</td>
<td>68.84</td>
</tr>
<tr>
<td>5</td>
<td>0.47</td>
<td>68.87</td>
</tr>
<tr>
<td>6</td>
<td>0.54</td>
<td>68.88</td>
</tr>
</tbody>
</table>

It is observed from Figures 3.17 to 3.22 that the results of the perturbation method are comparable to those of the Monte Carlo simulation for small range of uncertainties.
(coefficient of variation up to 10%). When the variation becomes large, the perturbation method gives poor results compared to the Monte Carlo simulation, because it calculates the sensitivities of the random parameters around their mean values. On the other hand, the Monte Carlo simulation provide the actual behavior of the variation as compared with the analytical study, although from Table 3.2 it is seen that the computational time is much higher compared to that of the perturbation method. The computational time increases in the perturbation method, when the number of random variables increases. On the other hand, there is no significant increase in computational time in case of the Monte Carlo simulation.

3.8 Summary and conclusion

In this chapter, the evaluation of the stochastic variation of natural frequencies of structures with uncertain properties has been presented using the Monte Carlo simulation and the perturbation method. For this purpose, MATLAB based computer programs are developed so that a stochastic analysis can be performed using a deterministic finite element code. The results obtained from different methods are then compared.

The results show that the statistical variation in the structural properties can have a significant effect on the natural frequencies of a structure, which cannot be perceived through an analytical study.
The Monte Carlo simulation handles the variability in structural models in a very straightforward way and evaluates the distribution of the response variables. The obtained distribution can be considered exact if the number of simulations is very high. From the process of the Monte Carlo, it is evident that it can deal with any type of random variable. Therefore, this technique can be used in variety of structural analysis problems. However, the process involves many repetitions of a full deterministic model and thereby it is computationally intensive.

The perturbation method can be considered useful if the statistical variation of the random variable is low and their distribution is normal. As seen from the discussions of this chapter, perturbation method is not suitable for evaluation of response variability that is caused by large amount of statistical variation of the random parameters, if the response variables and the random parameters have a nonlinear relationship. Another drawback of this method is that it is unable to predict the distribution type of the response variable. However, from computational point of view, the perturbation method uses few repetitions depending on the number of random variables.
4.1 Introduction

The mixed formulation is a combination of the perturbation method and the Monte Carlo simulation. This formulation is introduced in the field of stochastic analysis to overcome the limitations of the perturbation method and the Monte Carlo simulation. As stated in the previous chapter, the perturbation method depends on the mean values of the design parameters and does not give satisfactory results in the high variability level of these parameters if the response variables have a nonlinear relation with input random variables or the distributions of the input random variables are different from the normal distribution. In contrast, the Monte Carlo simulation handles this nonlinearity by evaluating the response for a random set of design variables. From the computational point of view, the Monte Carlo simulation involves a large number of repetitions of a full deterministic finite element analysis to evaluate the statistical characteristics of response variables whereas the perturbation method enables the response variability by using a few repetitions of the analysis. So an efficient combination of both methods would, by large extent, reduce the computational cost that one would expect for a direct Monte Carlo analysis.

Although different attempts has been made over the last several years on how to combine these two methods efficiently so that the response statistics can be derived at an
affordable cost, still there is a lack of agreement about the application of such combination in practical problems. Székely et al. (1995) presented a mixed approach, where they perform a perturbation of the mass and the stiffness matrices prior to the simulations. In this way, they were able to reduce the computational time required to setup these matrices within each simulation. However, this type of combination does not reduce the computational time significantly and therefore not suitable for practical application. Nieuwenhof and Coyette (2002) made an attempt to assess the response statistics by sampling the eigenfrequencies and eigenmodes according to the perturbation estimation of their probability density functions. They applied this method for evaluating the frequency response function variability of a simply supported plate subjected to a vertical point load. The concept of their formulation is taken in this chapter to evaluate the response statistics of a practical problem.

In this chapter, the mixed method is used for the dynamic analysis of structures with uncertain properties. The first part of this chapter deals with the basic formulation of the mixed method and its implementation in CALFEM for the evaluation of stochastic variation of dynamic response. The plane truss example of Chapter Three is taken to ensure that this method works well for the stochastic eigenvalue problem. The second part of this chapter deals with the stochastic variation of the response of a multi-story building subjected to an earthquake ground motion using mixed method. The variation of response obtained using the Monte Carlo simulation and the perturbation method are included along with the mixed formulation for comparison. The computational speed-up
obtained using the mixed method over the Monte Carlo simulation is also presented in this section.

4.2 Formulation of the mixed method for the stochastic eigenvalue problem

As stated in Chapter Two, the stochastic eigenvalue problem can be written in the following form:

\[
[K(X)]^i[\phi(X)] = \lambda^i(X)[M(X)][\phi(X)]^i
\]

where the stiffness and the mass matrices are functions of random variables \( X \). The solution of the above eigenproblem for a random set of values \( X \) will produce a random set for each eigenvalue and eigenvector. The statistical means and the standard deviations can be calculated from the respective random sets of the eigenvalues and the eigenvectors. The procedure of the mixed method is described below:

- The solution of the eigenproblem for the mean values of random parameters \( X \). The resulting eigenvalue and eigenvector can be denoted \( \lambda_{\text{mean}} \) and \( \{\phi\}_{\text{mean}} \) respectively.

- The evaluation of the eigenvalue and the eigenvector sensitivities with respect to the mean values of \( X \) using the following equations:

\[
\frac{\partial\{\lambda\}}{\partial X_i} = \frac{\{\lambda\}_{\text{new}} - \{\lambda\}_{\text{mean}}}{X_i^{\text{new}} - X_i^{\text{mean}}} \quad \text{and} \quad \frac{\partial\{\Phi\}}{\partial X_i} = \frac{[\Phi]_{\text{new}} - [\Phi]_{\text{mean}}}{X_i^{\text{new}} - X_i^{\text{mean}}}
\]

where the new values of the respective parameters can be obtained by solving the eigenproblem for a small deviation in the mean values of \( X \).

- Employment of the Monte Carlo process to generate a random set of values \( \{X_1, X_2, ..., X_p, ..., X_{N_{\text{run}}}\} \) for \( X \).
• Generation of a random set of the eigenvalues and the eigenvectors for the generated random set of values of $X$ using $N_{sim}$ number of repetitions of the following process:

$$\{\lambda\}^p = \{\lambda\}_{mean} + \sum_{i=1}^{N} \frac{\partial \{\lambda\}}{\partial X_i} (X_i^p - X_{i_{mean}})$$

$$[\Phi]^p = [\Phi]_{mean} + \sum_{i=1}^{N} \frac{\partial [\Phi]}{\partial X_i} (X_i^p - X_{i_{mean}})$$

where $P$ denotes any repetition among $N_{sim}$ simulations.

It is evident from the above formulation that unlike the perturbation method this mixed formulation will generate the distribution of the random eigenvalues and eigenvectors. This distribution is generated within a reasonable computational cost compared to that of the Monte Carlo simulation. In addition to this, the mixed method takes into account the type of distribution of the random variables, whereas the perturbation method assumes the random parameters are normally distributed. The random samples of the eigenvectors and the eigenvalues can be used to obtain the statistics of the response quantities such as displacement, shear and bending moment.

4.3 Implementation in CALFEM

The formulation described in the previous section can easily be implemented in CALFEM. For such implementation an algorithm is developed which can take advantage of the statistical functions of MATLAB and combine these functions with CALFEM. The flowchart of this implementation is presented in Figure 4.1.
Figure 4.1: Flowchart for the mixed method
4.4 Evaluation of the stochastic variation of natural frequency using mixed formulation

The plane truss example of Chapter Three is considered for the evaluation of the stochastic variation of the natural frequency using mixed method. Only the cross-sectional area of bar 3 is considered as a random variable and assumed to be normally distributed. Figure 4.2 shows the effect of the stochastic variation of the bar cross-sectional area on the fundamental frequency. The results are compared with those obtained from the Monte Carlo simulation and the perturbation method.

![Graph showing the effect of COV of bar area on COV of frequency.]

**Figure 4.2:** Stochastic variation of fundamental frequency

It is noticed from Figure 4.2 that the mixed method gives better results than those of the perturbation method in high variability level of structural parameters. This behavior is expected, because the mixed method uses Taylor’s series expansion for a set of values of
the random parameters whereas the perturbation method attempts that expansion only one time.

4.5 Evaluation of the stochastic variation of earthquake response using mixed formulation

The major advantage of the mixed method over the perturbation method is that it generates a set of random eigenvalues and eigenvectors, which can be used for further calculations such as displacement, shear and bending moment within a simulation. Thus a distribution is generated for these response variables, which can be different from normal distribution assumed by the perturbation method.

A practical application of this method in the context of earthquake response of a multi-storied building is considered. The description of the problem and the results are discussed in the following sections.

4.5.1 Problem description

A seven-story shear frame structure (Figure 4.3) is considered to evaluate the response variability using mixed method. The results are then compared with the Monte Carlo simulation results. The mean values of the story masses and stiffnesses are shown in the figure. It is assumed that the modal damping ratio is 5% for all modes and the structure is excited by earthquake ground motion only along x-direction and has translational degrees of freedom only along that direction. It is also assumed that only the mean values of the story stiffnesses are known deterministically and the uncertainties in structural stiffnesses
are defined by their associated coefficients of variations. The story masses and the modal damping ratios are assumed to have insignificant variations. The excitation input is in the form of acceleration response spectrum and is shown in Figure 4.4.

<table>
<thead>
<tr>
<th>Stiffnesses $k_i$ (10^6 N/m)</th>
<th>Masses $m_i$ (tons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>$m_7 = 85$</td>
</tr>
<tr>
<td>6</td>
<td>$m_6 = 85$</td>
</tr>
<tr>
<td>5</td>
<td>$m_5 = 85$</td>
</tr>
<tr>
<td>4</td>
<td>$m_4 = 85$</td>
</tr>
<tr>
<td>3</td>
<td>$m_3 = 85$</td>
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<td>2</td>
<td>$m_2 = 85$</td>
</tr>
<tr>
<td>1</td>
<td>$m_1 = 85$</td>
</tr>
</tbody>
</table>

$\bar{k}_1 = 150$
$\bar{k}_2 = 150$
$\bar{k}_3 = 165$
$\bar{k}_4 = 135$
$\bar{k}_5 = 135$
$\bar{k}_6 = 110$
$\bar{k}_7 = 110$

**Figure 4.3:** Seven-story structure
Figure 4.4: Acceleration response spectra with damping ratio of 0.05

The following considerations are made while analyzing the above problem using the mixed method and the Monte Carlo simulation:

- The story stiffnesses are modeled as Gaussian (normal) random variables having mean values indicated in Figure 4.3 and assumed to be uncorrelated.
- Based on the analysis of Chapter Three, one thousand simulations are used.
- Results are obtained for different coefficient of variations (5%, 10%, 15% and 20%) in the story stiffnesses.
- The response spectrum analysis is employed to obtain the structural response quantities.
- Only mean and standard deviation of the response quantities such as the story displacement and the story shear are computed after the simulations.

4.5.2 Comparison of the stochastic variation of modal frequencies

Before using the mixed formulation to evaluate earthquake induced response variability of the above seven story structure, an investigation is carried out to compare the stochastic variation of the modal frequencies obtained by the Monte Carlo simulation, the mixed method and the perturbation method. The results are presented in Tables 4.1 and 4.2. Percentage of error is calculated for mixed method and perturbation with respect to the results obtained from the Monte Carlo simulation. The coefficients of variations of modal frequencies obtained from different methods are compared in Figures 4.5 to 4.8.

It is noticed from the Tables 4.1 and 4.2 and Figures 4.5 to 4.8 that the variability of modal frequencies obtained from mixed method are in good agreement with that obtained from the Monte Carlo simulation even in the high level of uncertainty in structural properties. Thus we have a good indication that the mixed formulation can be used for assessing the response variability of structures.
Table 4.1: Comparison of statistics of natural frequencies computed by Monte Carlo, perturbation and mixed method for different levels of uncertainties in story stiffness

<table>
<thead>
<tr>
<th>Modal undamped frequencies</th>
<th>Monte Carlo</th>
<th>Perturbation</th>
<th>Mixed method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean value</td>
<td>Standard deviation</td>
<td>Mean value</td>
</tr>
<tr>
<td>( \omega_1 )</td>
<td>8.7218</td>
<td>0.0947</td>
<td>8.7249</td>
</tr>
<tr>
<td>( \omega_2 )</td>
<td>24.2516</td>
<td>0.2813</td>
<td>24.2633</td>
</tr>
<tr>
<td>( \omega_3 )</td>
<td>39.2735</td>
<td>0.4468</td>
<td>39.2868</td>
</tr>
<tr>
<td>( \omega_4 )</td>
<td>53.3332</td>
<td>0.6477</td>
<td>53.3562</td>
</tr>
<tr>
<td>( \omega_5 )</td>
<td>62.5080</td>
<td>0.7146</td>
<td>62.4980</td>
</tr>
<tr>
<td>( \omega_6 )</td>
<td>71.5724</td>
<td>0.8474</td>
<td>71.5006</td>
</tr>
<tr>
<td>( \omega_7 )</td>
<td>80.8822</td>
<td>1.0514</td>
<td>80.7291</td>
</tr>
<tr>
<td>10% COV in story stiffness</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \omega_1 )</td>
<td>8.6951</td>
<td>0.2014</td>
<td>8.7249</td>
</tr>
<tr>
<td>( \omega_2 )</td>
<td>24.1651</td>
<td>0.5590</td>
<td>24.2633</td>
</tr>
<tr>
<td>( \omega_3 )</td>
<td>39.1511</td>
<td>0.9001</td>
<td>39.2868</td>
</tr>
<tr>
<td>( \omega_4 )</td>
<td>53.1267</td>
<td>1.2689</td>
<td>53.3562</td>
</tr>
<tr>
<td>( \omega_5 )</td>
<td>62.4063</td>
<td>1.4314</td>
<td>62.4980</td>
</tr>
<tr>
<td>( \omega_6 )</td>
<td>71.5234</td>
<td>1.7209</td>
<td>71.5006</td>
</tr>
<tr>
<td>( \omega_7 )</td>
<td>81.0520</td>
<td>2.1991</td>
<td>80.7291</td>
</tr>
<tr>
<td>15% COV in story stiffness</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \omega_1 )</td>
<td>8.6343</td>
<td>0.3056</td>
<td>8.7249</td>
</tr>
<tr>
<td>( \omega_2 )</td>
<td>24.0401</td>
<td>0.8777</td>
<td>24.2633</td>
</tr>
<tr>
<td>( \omega_3 )</td>
<td>38.9399</td>
<td>1.4369</td>
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<td>( \omega_4 )</td>
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<tr>
<td>( \omega_5 )</td>
<td>62.3837</td>
<td>2.1479</td>
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</tr>
<tr>
<td>( \omega_6 )</td>
<td>71.5657</td>
<td>2.4151</td>
<td>71.5006</td>
</tr>
<tr>
<td>( \omega_7 )</td>
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<td>80.7291</td>
</tr>
<tr>
<td>20% COV in story stiffness</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \omega_1 )</td>
<td>8.5694</td>
<td>0.4254</td>
<td>8.7249</td>
</tr>
<tr>
<td>( \omega_2 )</td>
<td>23.8013</td>
<td>1.2595</td>
<td>24.2633</td>
</tr>
<tr>
<td>( \omega_3 )</td>
<td>38.7171</td>
<td>1.8904</td>
<td>39.2868</td>
</tr>
<tr>
<td>( \omega_4 )</td>
<td>52.5334</td>
<td>2.6113</td>
<td>53.3562</td>
</tr>
<tr>
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<td>62.1196</td>
<td>2.8239</td>
<td>62.4980</td>
</tr>
<tr>
<td>( \omega_6 )</td>
<td>71.5733</td>
<td>3.2905</td>
<td>71.5006</td>
</tr>
<tr>
<td>( \omega_7 )</td>
<td>81.9181</td>
<td>4.0560</td>
<td>80.7291</td>
</tr>
</tbody>
</table>

The mixed formulation for dynamic analysis of structures with uncertain properties
Table 4.2: Comparison of the perturbation method and the mixed method with respect to % error in coefficients of variation.

<table>
<thead>
<tr>
<th>Modal undamped frequencies</th>
<th>Monte Carlo</th>
<th>Perturbation</th>
<th></th>
<th>Mixed method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5% COV in story stiffness</td>
<td>% COV</td>
<td>% Error</td>
<td>% COV</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>1.0858</td>
<td>1.1072</td>
<td>1.9701</td>
<td>1.1363</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>1.1599</td>
<td>1.3704</td>
<td>18.1442</td>
<td>1.1648</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>1.1377</td>
<td>1.4786</td>
<td>29.9694</td>
<td>1.1680</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>1.2144</td>
<td>1.2836</td>
<td>5.6978</td>
<td>1.2518</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>1.1432</td>
<td>1.2730</td>
<td>11.3528</td>
<td>1.1571</td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>1.1840</td>
<td>1.0087</td>
<td>14.8071</td>
<td>1.2397</td>
</tr>
<tr>
<td>$\omega_7$</td>
<td>1.2999</td>
<td>0.9614</td>
<td>26.0441</td>
<td>1.3925</td>
</tr>
<tr>
<td></td>
<td>10% COV in story stiffness</td>
<td>% COV</td>
<td>% Error</td>
<td>% COV</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>2.3162</td>
<td>2.2155</td>
<td>4.3497</td>
<td>2.2096</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>2.3133</td>
<td>2.7408</td>
<td>18.4810</td>
<td>2.3012</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>2.2990</td>
<td>2.9572</td>
<td>28.6287</td>
<td>2.2796</td>
</tr>
<tr>
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<td>2.3884</td>
<td>2.5675</td>
<td>7.4953</td>
<td>2.4520</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>2.3884</td>
<td>2.5460</td>
<td>11.0008</td>
<td>2.2068</td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>2.2937</td>
<td>2.0173</td>
<td>16.1567</td>
<td>2.4791</td>
</tr>
<tr>
<td>$\omega_7$</td>
<td>2.4061</td>
<td>1.9229</td>
<td>29.1297</td>
<td>2.7530</td>
</tr>
<tr>
<td></td>
<td>15% COV in story stiffness</td>
<td>% COV</td>
<td>% Error</td>
<td>% COV</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>3.5394</td>
<td>3.3227</td>
<td>6.1225</td>
<td>3.3998</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>3.6510</td>
<td>4.1111</td>
<td>12.6038</td>
<td>3.4705</td>
</tr>
<tr>
<td>$\omega_3$</td>
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<td>4.4356</td>
<td>20.2041</td>
<td>3.4770</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>3.7500</td>
<td>3.8511</td>
<td>2.6951</td>
<td>3.5095</td>
</tr>
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<td>10.9193</td>
<td>3.4467</td>
</tr>
<tr>
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</tr>
<tr>
<td></td>
<td>20% COV in story stiffness</td>
<td>% COV</td>
<td>% Error</td>
<td>% COV</td>
</tr>
<tr>
<td>$\omega_1$</td>
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<td>4.4227</td>
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<tr>
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<td>3.5868</td>
<td>4.6048</td>
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<tr>
<td>$\omega_3$</td>
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<td>5.9142</td>
<td>21.1282</td>
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</tr>
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<td>3.2992</td>
<td>4.8667</td>
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</tr>
<tr>
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</tr>
<tr>
<td>$\omega_7$</td>
<td>4.9513</td>
<td>3.8457</td>
<td>22.3293</td>
<td>5.3199</td>
</tr>
</tbody>
</table>

The mixed formulation for dynamic analysis of structures with uncertain properties
Figure 4.5: Comparison among different methods for variability of modal frequencies (COV of story stiffness = 5%)

Figure 4.6: Comparison among different methods for variability of modal frequencies (COV of story stiffness = 10%)
Figure 4.7: Comparison among different methods for variability of modal frequencies (COV of story stiffness = 15%)

Figure 4.8: Comparison among different methods for variability of modal frequencies (COV of story stiffness = 20%)
4.5.3 Comparison of the response variability

Figures 4.9 to 4.12 show the comparison of the Monte Carlo simulation and the mixed method with respect to the coefficient of variations (COV) of the story displacement. It is very clear from the figures, that the mixed method gives almost the same results of the Monte Carlo simulation and follows the similar trends even for uncertainties equal to 15% of the mean structural properties. But it also shows that if uncertainties are more than 15%, the response variations calculated by the mixed method tend to deviate from those obtained using the Monte Carlo simulation.

The comparisons of the COV of the story shear obtained from mixed formulation for different level of uncertainties in the structural stiffness with that of the Monte Carlo simulation are shown in Figures 4.13 to 4.16. It is also noted from these comparisons that the results of the mixed formulation are in good agreement with those of the Monte Carlo simulation even in high range of uncertainties (such as COV 15%) in the input parameters.

From Figures 4.9 to 4.16, two interesting points are noted:

- The mixed method underestimates the response variability in high variability level of structural properties.
- The displacement variability is high in the lower story levels whereas uncertainties in structural properties affect the story shears in upper story levels. This phenomenon was occurred due to the smaller values of the mean response amplitude at these story levels.
**Figure 4.9:** Comparison of methods for story displacement (Coefficient of variation of stiffness = 5%)

**Figure 4.10:** Comparison of methods for story displacement (Coefficient of variation of stiffness = 10%)
Figure 4.11: Comparison of methods for story displacement (Coefficient of variation of stiffness = 15%)

Figure 4.12: Comparison of methods for story displacement (Coefficient of variation of stiffness = 20%)
Figure 4.13 Comparison of methods for story shear
(Coefficient of variation of stiffness = 5%)

Figure 4.14: Comparison of methods for story shear
(Coefficient of variation of stiffness = 10%)
Figure 4.15: Comparison of methods for story shear (Coefficient of variation of stiffness = 15%)

Figure 4.16: Comparison of methods for story shear (Coefficient of variation of stiffness = 20%)
4.6 Comparison of the computational speed-up

As stated in Chapter Three, the crucial issue of stochastic dynamic analysis of large structures is the computational time. A comparative study is performed to investigate the computational speed-up of the mixed method over the Monte Carlo simulation.

Figure 4.17 illustrates a simple frame structure, which is taken as an ideal example for this purpose. Using the finite element discretization, this frame structure is meshed six times to increase the number of active degrees of freedom and thereby exploring the computational efficiency of the mixed method for structures with large degrees of freedom. Several considerations are made while performing the analysis using the mixed method and the Monte Carlo simulation:

- 2D beam elements are used for finite element analysis.
- Initially the structure is discretized into four elements. These four elements are discretized in successive meshing to increase the number of active degrees of freedom.
- Young’s modulus is considered as a random variable for this problem.
- One thousand simulations are used for each model.
- Eigenvalue solutions are computed for whole system using the general eigen solver and for reduced system using the Lanczos solver.
- The computational time is evaluated based on the generation of 1000 samples of eigenvalues and eigenvectors.
- Computational efficiency is measured as a ratio of time taken by the Monte Carlo simulation to that of the mixed method.
Figure 4.17: Simple frame structure

Table 4.3 shows that the mixed method is computationally faster than the Monte Carlo simulation. Figure 4.18 shows the variation of the speed-up with number of active degrees of freedom when the eigensolutions are obtained using the general eigen solver and the Lanczos method. It is noticed that the Lanczos method gives better performance, because it attempts to find a few lowest frequencies. Also it is noticed that as the number of active degrees of freedom increases the corresponding increase in efficiency becomes low after certain range.
Table 4.3: Comparison of computational time

<table>
<thead>
<tr>
<th>No. of active degrees of freedom</th>
<th>General eigen solution</th>
<th></th>
<th></th>
<th></th>
<th>Lanczos method</th>
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</thead>
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<tr>
<td></td>
<td>Computational time (sec)</td>
<td>Efficiency</td>
<td>Computational time (sec)</td>
<td>Efficiency</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Monte Carlo</td>
<td>Mixed method</td>
<td></td>
<td>Monte Carlo</td>
<td>Mixed method</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>11.657</td>
<td>0.15</td>
<td>77.71333</td>
<td>23.294</td>
<td>0.161</td>
<td>144.6832</td>
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<tr>
<td>23</td>
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<td>110.6783</td>
<td>33.608</td>
<td>0.21</td>
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<tr>
<td>47</td>
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<td>120.6409</td>
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<td>0.35</td>
<td>172.2486</td>
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</tr>
<tr>
<td>95</td>
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<td>0.551</td>
<td>245.9256</td>
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</tr>
<tr>
<td>191</td>
<td>1969.5</td>
<td>12.732</td>
<td>154.689</td>
<td>636.886</td>
<td>1.773</td>
<td>359.2138</td>
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<tr>
<td>383</td>
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<td>197.0374</td>
<td>5277</td>
<td>11.266</td>
<td>468.4005</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.18: Computational speed-up of mixed method over Monte Carlo simulation

4.7 Summary and conclusion:

A mixed formulation has been presented in this chapter for the solution of the stochastic eigenvalue problem. This formulation has also been extended for evaluation of
earthquake induced response variability of structures with uncertain properties. In the first step of this formulation, the method proposed by Mahadevan (1994) is used to compute the eigenvalue and the eigenvector sensitivities. Then the formulation proposed by Nieuwenhof and Coyette is taken for the generation of random samples of the eigenvalues and the eigenvectors. The variability of earthquake-induced response is computed using the sample values of the eigenvalues and the eigenvectors.

Three different numerical examples are presented in this chapter to investigate the performance of the mixed formulation. The first example is a stochastic eigenvalue problem for a plane truss structure. This example shows the mixed formulation gives good results compared to the perturbation method even in the high variability level of structural design parameters. The second example is a practical application of the mixed formulation. It shows that the mixed formulation provides satisfactory results for response variability estimates. The third example is presented to show the computational speed-up achieved over the Monte Carlo simulation using the mixed formulation. It is seen from this numerical application that the mixed formulation is a very time efficient method for the evaluation of response variability of large dynamic models.
Chapter Five
SUMMARY AND CONCLUSIONS

5.1 Summary

This investigation is devoted to the implementation of various techniques for the
eigenvalue problem of structures with uncertain properties. In addition, observing the
quantitative effects of the uncertain properties on the dynamic response of a structure is
aimed.

The dynamic analysis of structures having uncertain material and geometric properties is
based on the statistical knowledge of these properties in the form of the distributions they
follow. Once these distributions are known, the uncertainties are included in the
evaluation of dynamic response. In practical applications, most of the structural
properties follow Gaussian distribution.

Among methods developed for the probabilistic analysis of structures with uncertainties,
the Monte Carlo simulation and the perturbation method are well established. A major
part of this investigation is devoted to their application in the context of stochastic
eigenvalue problem. Two types of truss structures are analyzed using both methods and
the results are compared. These examples are selected in order to compare the results
obtained by the stochastic finite element approach with an analytical investigation
presented by Djoudi and Bahai (2001).
The Monte Carlo simulation is suited for incorporation within the existing finite element code. However, it was found that computational time is a crucial issue for this method to deal with practical dynamic problems. The perturbation method is presented in the simplest form for implementation into an existing finite element program. Despite its simplicity, this method is applicable only for low variability of the structural properties. However, the computational efficiency is a major advantage to consider it for large problems where variability is low.

A coupling of the Monte Carlo simulation and the perturbation method in the form of mixed method is used to obtain the response statistics of a multistory building subjected to earthquake ground motion. This technique is validated by comparing the results obtained using the established Monte Carlo simulation. A study related to computational speed-up of the mixed method over the Monte Carlo simulation is presented and showed that the mixed method leads to encouraging results.

5.2 Conclusions

The following conclusions can be drawn from the present study:

- The uncertainties in the structural properties have a significant effect on the natural frequencies and other responses such as displacement and shear.

- The Monte Carlo simulation is well suited to deal with uncertainties in dynamic systems. Its implementation in a deterministic finite element code is straightforward. This technique appears to be the only universal method that can provide accurate solutions of
stochastic problems. The disadvantage of the Monte Carlo simulation is that it is computationally time consuming due to the required large number of repetitions of the finite element deterministic solution.

- The perturbation method is suited for problems with low variability of the structural parameters. For higher variability of the structural parameters this method performs poorly.

- The perturbation method can be used to select the most sensitive random parameters for stochastic analysis at low computational cost.

- The mixed formulation is compromise between computational efficiency and accuracy. It offers an efficient way to perform stochastic analysis for large dynamic model. Unlike the perturbation method, this technique also gives the distributions of the response variables. This formulation is readily useable in existing deterministic finite element code.

5.3 Recommendations for future research

The following topics can be recommended for future research:

- Throughout the present study the correlation among the random variables is ignored. Future work can extend this investigation to include the correlation in the dynamic model for stochastic analysis. For that case, the formulation presented for perturbation method should be updated.
- Many structural parameters such as Young's modulus and mass density may vary spatially. The random field approach can be investigated for dynamic analysis of uncertain systems, where structural properties vary in space.

- The Monte Carlo simulation is very time consuming in its standard form. In the years to come, however, the continued evolution of digital computers will further enhance the usefulness of Monte Carlo simulation technique in the area of engineering mechanics. The development of various reduction techniques such as the importance sampling is also promising direction to enhance the applicability of this method for structural engineering problems.
REFERENCES


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