FLOW PROBLEMS OF THE FLUIDS OF DIFFERENTIAL TYPE.

A. M. SIDDIQUI

University of Windsor

Follow this and additional works at: https://scholar.uwindsor.ca/etd

Recommended Citation
https://scholar.uwindsor.ca/etd/2127

This online database contains the full-text of PhD dissertations and Masters' theses of University of Windsor students from 1954 forward. These documents are made available for personal study and research purposes only, in accordance with the Canadian Copyright Act and the Creative Commons license—CC BY-NC-ND (Attribution, Non-Commercial, No Derivative Works). Under this license, works must always be attributed to the copyright holder (original author), cannot be used for any commercial purposes, and may not be altered. Any other use would require the permission of the copyright holder. Students may inquire about withdrawing their dissertation and/or thesis from this database. For additional inquiries, please contact the repository administrator via email (scholarship@uwindsor.ca) or by telephone at 519-253-3000ext. 3208.
NOTICE
The quality of this microfiche is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Previously copyrighted materials (journal articles, published tests, etc.) are not filmed.

Reproduction in full or in part of this film is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30.

THIS DISSERTATION HAS BEEN MICROFILMED EXACTLY AS RECEIVED

AVIS
La qualité de cette microfiche dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopy de qualité inférieure.

Les documents qui font déjà l'objet d'un droit d'auteur (articles de revue, examens publiés, etc.) ne sont pas microfilmés.

La reproduction, même partielle, de ce microfilm est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30.

LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L'AVONS REÇUE
FLOW PROBLEMS OF THE FLUIDS
OF DIFFERENTIAL TYPE

by
A. M. Siddiqui

A dissertation
Submitted to the Faculty of Graduate Studies
through the Department of Mathematics and Statistics
in Partial Fulfillment of the Requirements for the
Degree of Doctor of Philosophy at
the University of Windsor

Windsor, Ontario, Canada
1986
Permission has been granted to the National Library of Canada to microfilm this thesis and to lend or sell copies of the film.

The author (copyright owner) has reserved other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without his/her written permission.

L'autorisation a été accordée à la Bibliothèque nationale du Canada de microfilmer cette thèse et de la prêter ou de vendre des exemplaires du film.

L'auteur (titulaire du droit d'auteur) se réserve les autres droits de publication; ni la thèse ni de longs extraits de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation écrite.

ABSTRACT

This dissertation deals with various flow problems in graded (ordered) Non-Newtonian fluids.

1. Solutions for the equations of motion of an incompressible second-grade fluid are obtained by hodograph-transformation method. By introducing a suitable-Legendre-transform function the basic equations are recast in terms of this function, and the conditions which this function should satisfy are stated. Several illustrations of the method are considered and the results for streamlines, velocities and pressure distribution are compared with the corresponding results for viscous fluid.

2. Inverse solutions of the equations of motion of an incompressible second grade fluid are obtained by assuming certain forms for the stream functions a priori. The equations considered are in plane polar coordinates, axisymmetric polar coordinates and in axisymmetric spherical polar coordinates. Expressions for streamlines, velocity components and pressure distributions are given explicitly, in each case, and compared with the corresponding results of the viscous fluid.
3. A method for studying plane steady flows of a thermodynamically compatible third grade fluid is discussed. Using differential geometry, the equations are recast into a new curvilinear coordinate system and the partial differential equations for the coefficients $E$, $F$, and $G$ of the first fundamental form of the metric are derived. By placing restrictions on the coefficients, a priori, flows are determined which have such a property. Some further illustrations are given in detail which reflect the use of the method.

4. The problem of the helical flow of a simple fluid is investigated by employing the perturbation scheme. The explicit expressions for the velocity and pressure fields are determined. Expressions for volume discharge, torque and normal stress difference are also derived. The results are compared with the corresponding results of the viscous fluid.
Respectfully dedicated to my beloved parents
ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation and gratitude to Dr. P. N. Kaloni for his excellent guidance and encouragement throughout the course of research and preparation of this dissertation. It has been a great privilege and pleasure for the author to work with Dr. Kaloni.

The author is thankful to Dr. O. P. Chandna, Dr. R. M. Barron and Dr. A. C. Smith for their help and personal involvement throughout the tenure of his graduate studies.

Financial assistance from the National Research Council of Canada and the teaching assistantship from the Department of Mathematics at the University of Windsor are gratefully acknowledged.

Thanks are extended to Mr. R. K. Naeem, I.U.H. Mian and K. A. Khan for their invaluable help and constant encouragement.

To Mrs. Ella Mae-Bunt, Mrs. V. A. Allard and Mrs. A. Rowland, sincere thanks for their moral support.

The author also acknowledges the understanding, moral support, continuous encouragement and patience of his wife throughout the entire course of his work.

Finally, he wishes to thank Mrs. Zeleney for her careful and excellent typing of this dissertation.
TABLE OF CONTENTS

ABSTRACT ....................................................... iv
DEDICATION .................................................... vi
ACKNOWLEDGEMENTS .......................................... vii
LIST OF FIGURES ............................................... x

CHAPTER

I. INTRODUCTION .............................................. 1
   1.1 Introduction ........................................... 1
   1.2 Basic Equations ........................................ 11
   1.3 Outline of the Present Work ............................. 13

II. TRANSFORMATION METHOD IN THE FLOWS OF A
    SECOND GRADE FLUID ..................................... 16
    2.1 Introduction .......................................... 16
    2.2 Equations of Motion .................................... 17
    2.3 Equations in the Hodograph Plane ...................... 20
    2.4 Equations for the Legendre Transform Function ....... 22
    2.5 Illustrations .......................................... 27

III. CERTAIN INVERSE SOLUTIONS OF A SECOND
     GRADE FLUID ............................................. 41
    3.1 Introduction .......................................... 41
    3.2 Equations of Motion and Compatibility
           Equations .......................................... 43
    3.3 Solutions ............................................. 51
       (a) Motions where ψ(r,θ)=r^nψ(θ) ...................... 52
       (b) Motions where ψ(r,z)=r^nψ(z) ...................... 57
       (c) Motions where Φ'w=f(r) .......................... 61
       (d) Motions where ψ(R,Θ)=R^nψ(Θ) .................. 65
    3.4 Discussion ............................................. 74

IV. PLANE STEADY FLOWS OF A THIRD GRADE FLUID .......... 76
    4.1 Introduction .......................................... 76
    4.2 Equations of Motion .................................... 78
    4.3 Variant Forms of Equations of Motion ................ 84
4.4 Illustrations .................................................. 94
  4.4.1 Straight Streamlines .................................. 94
  4.4.2 Streamlines as Involutes of a Curve ................. 98

4.5 Further Illustrations ....................................... 102
  (a) Radial Flows ............................................. 102
  (b) Circular Flows .......................................... 104
  (c) Parallel Flows ......................................... 107

V. HELICAL FLOW OF A SIMPLE FLUID ......................... 110

  5.1 Introduction .............................................. 110
  5.2 Preliminaries ............................................ 113
  5.3 Problem Statement ...................................... 114
  5.4 Perturbation Solution ................................... 117
    5.4.1 First Order Problem ............................... 117
    5.4.2 Second Order Problem .............................. 122
    5.4.3 Third Order Problem ............................... 127
    5.4.4 Fourth Order Problem .............................. 134
  5.5 Torque, Normal Thrust, Volume Flux ................... 147
    (a) Torque ................................................ 148
    (b) Normal Thrust Difference ........................... 150
    (c) Volume Flux ......................................... 152
  5.6 Discussion ................................................ 154

REFERENCES ....................................................... 157

VITA AUCTORIS ..................................................... 162
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Stream Line Pattern for Example #2</td>
<td>39</td>
</tr>
<tr>
<td>2.</td>
<td>Stream Line Patterns for Example #4</td>
<td>40</td>
</tr>
<tr>
<td>3.</td>
<td>Velocity Profiles for Example C</td>
<td>109</td>
</tr>
<tr>
<td>4.</td>
<td>Helical Flow</td>
<td>112</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

1.1 Introduction

The classical theory of incompressible viscous fluid is based upon the constitutive assumption

\[ T = -pI + 2\mu D \]  \hspace{1cm} (1.1)

with

\[ \text{tr}D = 0. \]  \hspace{1cm} (1.2)

where \( T \) is the symmetric Cauchy stress tensor, \( D \) is the stretching tensor, \( \mu \) is the viscosity, and \( p \) is the indeterminate pressure. The above equations, along with the dynamical equations of motion lead to the Navier-Stokes equations. A fluid obeying (1.1) and (1.2) is called a linearly viscous or an incompressible Newtonian fluid.

Experimental research shows that the mechanical behaviour of many real fluids, specially those of low molecular weight, appears to be accurately described by (1.1) and (1.2). There are, however, many incompressible materials of great technical and industrial importance whose behaviour cannot be described, satisfactorily, by means of equations.
(1.1) and (1.2). Examples encountered are solutions and
melts of polymers, soap, paints, ceramics, suspensions,
honey, etc. These are considered as non-Newtonian or
viscoelastic fluids.

Roughly speaking, some of the features that are com-
monly observed in the non-Newtonian fluids are:

(i) the occurrence of shear-rate dependence or
apparent viscosity; the viscosity decreases
with increasing shear rate;

(ii) the normal stress effects, unequal normal
stresses in the different directions;

(iii) the high elastic recovery in shear or stress
relaxation.

The necessity for finding the appropriate generalization
of classical equation (1.1) has, therefore, been felt by a
number authors.

Because of the great diversity in the physical structure
of non-Newtonian fluids it is not possible to describe their
mechanical behaviour by a single constitutive equation. For
this reason, a great variety of constitutive equations have
been proposed. The monographs by Truesdell and Noll [1965, 1],
Huilgol [1975, 1], Bird, Armstrong and Hassager [1977, 1],
Schowalter [1978, 1], Zahorski [1982, 1] and the review
article by Truesdell [1974, 1] give extensive coverage of
such constitutive equations.
Two models that have gained considerable support from the experimentalist and theorist are: the Rivlin-Ericksen fluids of differential type of complexity n [1955, 1] and the simple incompressible fluids, initiated by Noll [1958, 1].

Rivlin-Ericksen [1955, 1], proposed their theory of viscoelastic fluids, where the stress tensor was a function of the first n Rivlin-Ericksen tensors, $A_1, \ldots A_n$. For an incompressible fluid, this class is thus characterized by the constitutive equation

$$T = -pI + f(A_1, A_2, A_3, \ldots, A_n)$$  \hspace{1cm} (1.3)

where $T$ is the Cauchy-stress tensor and $-pI$ the spherical stress due to constraint of incompressibility. The function $f$ is an isotropic function.

A few years later Noll [1958, 1], proposed a theory of materials for which the stress was a tensor-valued functional of the deformation gradient history. It was shown that, an incompressible simple fluid may be characterized by a constitutive equation of the form

$$T = -pI + \bar{H}(C_{\tau}(t-s))_{s=0}$$  \hspace{1cm} (1.4)

where $\bar{H}_{s=0}$ is the functional operator, $C_{\tau}(t-s)$ is the relative deformation tensor, i.e., the history of Cauchy-Green tensor and $p$ denotes the indeterminate pressure.
The theories of Rivlin-Ericksen and Noll are quite general but, apart from a class of flows, cannot be used to solve complicated flow problems. As a matter of fact, the exact solutions, using these constitutive equations, involve only motions which are highly constrained (viscous flows and motions with constant stretch-history). As a result several approximate constitutive equations have been proposed in the past three decades.

Our principal concern in this thesis is the approximate constitutive equations based on results of the Coleman and Noll [1960, 1; 1961, 1]. These authors recognized that the essential feature for the construction of various approximate constitutive equations of a simple incompressible fluid is the principle of fading memory.

We note that $C_t(t-s)$ occurring in (1.4) can be expanded in Taylor series

$$C_t(t-s) = I - sA_1 + \frac{1}{2!}s^2A_2 \ldots + (-1)^n \frac{s^n}{n!}A_n + o(s^n)$$

(1.5)

Let $0 < \alpha \leq 1$ be a real number. If one replaces $s$ by $\alpha s$, it implies the performance of the original history at the slower rate. In such a case (1.5) becomes

$$C_t(t-\alpha s) = I - \alpha sA_1 + \frac{1}{2} \alpha^2 s^2A_2 \ldots$$

$$+ (-1)^n \frac{\alpha^n s^n}{n!}A_n + o(\alpha^n s^n).$$

(1.6)
Introducing the notation

\[ A_n^a = \sigma^n A_n \]

in the above one gets

\[ C_t(t-a) = I - sA_1^a + \frac{s^2}{2!}A_2^a - \ldots + (-1)^n \frac{s^n A_n^a}{n!} \]

\[ + O(a^n s^n) \]  

(1.8)

On substituting (1.8) in (1.4) and using an approximation theorem due to Coleman and Noll [1960, 1], one obtains

\[ T = -pI + \sum_{j=1}^n \hat{G}_{j; k_1 \leq \ldots \leq k_j} A_{k_1}^a \ldots A_{k_j}^a \]

\[ + O(a^n), \]  

(1.9)

where \(|O(a^n)|\) goes to zero faster than \(a^n\) as \(a \to 0\). Here the terms \(\hat{G}_{j; k_1 \leq \ldots \leq k_j} (A_{k_1}^a \ldots A_{k_j}^a)\) are linear in each of the variables and \(\hat{G}_{j; k_1 \leq \ldots \leq k_j}\) is an isotropic function.

By assigning different values to \(n\) and then using the representations for various multilinear isotropic tensor functions along with the condition of incompressibility, the fluids of different grade or order are defined. In order to explain the procedure we choose \(n = 3\). The equation (1.9) then takes the form.
\[ T = -pI + \mathcal{G}_{1;1}(A_1^a) + \mathcal{G}_{1;2}(A_2^a) + \mathcal{G}_{1;3}(A_3^a) + \mathcal{G}_{2;1,1}(A_1^a, A_1^a) \\
+ \mathcal{G}_{2;1,2}(A_1^a, A_2^a) + \mathcal{G}_{3;1,1,1}(A_1^a, A_1^a, A_1^a) + O(a^3) \]  
\hspace{1cm} (1.10)

On using (1.7) in the above, (1.10) may be rewritten as

\[ T = -pI + a\mathcal{G}_{1;1}(A_1) + a^2[\mathcal{G}_{1;2}(A_2) + \mathcal{G}_{2;1,1}(A_1, A_1)] \\
+ a^3[\mathcal{G}_{1;3}(A_3) + \mathcal{G}_{2;1,1}(A_1, A_2) \\
+ \mathcal{G}_{3;1,1,1}(A_1, A_1, A_1)] + O(a^3). \]  
\hspace{1cm} (1.11)

We remark that the tensor functions in (1.11) must have the following form:

\[ \mathcal{G}_{1;1}(A_1) = [\lambda_1 trA_1]I + \lambda_2 A_1, \]

\[ \mathcal{G}_{1;2}(A_2) = [\lambda_3 trA_2]I + \lambda_4 A_2, \]

\[ \mathcal{G}_{2;1,1}(A_1, A_1) = [\lambda_5 (trA_1)^2 + \lambda_6 trA_1^2]I \\
+ [\lambda_7 trA_1]A_1 + \lambda_8 A_1^2, \]

\[ \mathcal{G}_{1;3}(A_3) = [\lambda_9 trA_3]I + \lambda_{10} A_3, \]

\[ \mathcal{G}_{2;1,2}(A_1, A_2) = [\lambda_{11} (trA_1)(trA_2) + \lambda_{12} tr(A_1 A_2)]I. \]
\[
\hat{G}_{3;1,1,1}(A_1, A_1, A_1) = [\lambda_{16}(\text{tr}A_1)^3 + \lambda_{17}(\text{tr}A_1^2)(\text{tr}A_1)]^2 + \lambda_{18}(\text{tr}A_1^3)I + [\lambda_{19}(\text{tr}A_1^2)A_1 + [\lambda_{20}(\text{tr}A_1)^2]A_1^2 + [\lambda_{21}(\text{tr}A_1)]A_1^2 + \lambda_2 A_1^3, \quad (1.12)
\]

where the coefficients \( \lambda_1, \ldots, \lambda_{22} \) are constants. Using the condition of incompressibility, i.e., \( \text{tr}A_1 = 0 \), the terms involving \( \lambda_1, \lambda_5, \lambda_7, \lambda_11, \lambda_14, \lambda_16, \lambda_17, \lambda_20 \) and \( \lambda_21 \) disappear automatically. Furthermore, we note that

\[
\text{tr}A_2 = \text{tr}A_1^2,
\]

\[
A_1^3 = \frac{1}{2}(\text{tr}A_1^2)A_1 + \frac{1}{3}(\text{tr}A_1^3)I, \quad (1.13)
\]

\[
\hat{G}_{1,1}(A_1) = \nu_1 A_1, \quad \hat{G}_{1,2}(A_2) = [\nu_2 \text{tr}A_1^2]I + \nu_3 A_2,
\]

\[
\hat{G}_{2,1,1}(A_1, A_1) = [\nu_4 \text{tr}A_1^2]I + \nu_5 A_1^2,
\]

\[
\hat{G}_{1,3}(A_3) = [\nu_6 \text{tr}A_3]I + \nu_7 A_3,
\]

\[
\hat{G}_{2,1,2}(A_1, A_2) = [\nu_8 \text{tr}(A_1 A_2)]I + [\nu_9 \text{tr}A_1^2]A_1
\]
\[ G_{3:1,1,1}(A_1, A_1, A_1) = \left[ u_2 \text{tr} A_1^3 \right] I + \left[ u_2 \text{tr} A_1^2 \right] A_1, \quad (1.14) \]

where the coefficients \( u_i \), being just the coefficients \( a_i \), are numbered in a more convenient order. On substituting (1.14) into (1.11), we find that

\[
T = -pI + au_1A_1 + \alpha^2 \left[ (u_2 + u_4)(\text{tr} A_1^2)I + u_3A_2 + u_5A_1^2 \right] \\
+ \alpha^3 \left[ (u_6 \text{tr} A_3 + u_5 \text{tr}(A_1 A_2) + u_1 \text{tr} A_1^3)I \right] \\
+ u_7A_3 + (u_9 + u_{12})(\text{tr} A_1^2)A_1 + u_{10}(A_1 A_2 + A_2 A_1) \\
+ o(\alpha^3) 
\]

Now, absorbing all the scalar multiples of \( I \) in \( p \), we write

\[
T = -pI + au_1A_1 + \alpha^2 u_3A_2 + \alpha^2 u_5A_1^2 + \alpha^3 u_7A_3 \\
+ \alpha^3 (u_9 + u_{12})(\text{tr} A_1^2)A_1 + \alpha^3 u_{10}(A_1 A_2 + A_2 A_1) \\
+ o(\alpha^3) 
\]

Motivated by (1.15), we define an incompressible fluid
of third order by the constitutive equation

\[ T = -pI + uA_1 + a_1A_2 + a_2A_1^2 + \beta_1A_3 \]

\[ + \beta_2(A_1A_2^2+A_2A_1^2) + \beta_3(\text{tr}A_1^2)A_1 \quad (1.16) \]

where \( u, a_1, a_2, \beta_1, \beta_2, \beta_3 \) are material constants.

Following the procedure stated above, we write down the constitutive equations for \( n = 1, 2, 3, 4 \), respectively \[ 1965, 1 \] as

\[ T = -pI + uA_1, \]

\[ T = -pI + uA_1 + a_1A_2 + a_2A_1^2, \]

\[ T = -pI + uA_1 + a_1A_2 + a_2A_1^2 + \beta_1A_3 + \beta_2(A_1A_2^2+A_2A_1^2) \]

\[ + \beta_3(\text{tr}A_1^2)A_1, \]

\[ T = -pI + uA_1 + a_1A_2 + a_2A_1^2 + \beta_1A_3 + \beta_2(A_1A_2^2+A_2A_1^2) \]

\[ + \beta_3(\text{tr}A_1^2)A_1 + \gamma_1A_4 + \gamma_2(A_3A_1+A_1A_3) \]

\[ + \gamma_3A_2 + \gamma_4(A_2A_1^2+A_1A_2^2) + \gamma_5(\text{tr}A_2)A_2 \]

\[ + \gamma_6(\text{tr}A_2)A_1^2 + [\gamma_7 \text{tr}A_3 + \gamma_8 \text{tr}(A_2A_1^2)]A_1, \]

\[ (1.17) \]
where $\nu$, $\alpha_i$, $\beta_i$ and $\gamma_i$ are material constants. $A_\Gamma$ represents the Rivlin-Ericksen tensors defined by

$$A_1 = \text{grad } \nu + (\text{grad } \nu)^T$$

$$A_{\Gamma+1} = A_{\Gamma t} + (\text{grad } A_{\Gamma}) \nu + A_{\Gamma}(\text{grad } \nu) + (A_{\Gamma}(\text{grad } \nu))^T$$

(1.18)

Here, $\nu$ is velocity field; the subscript $t$ represents the partial derivative with respect to time and superposed $T$ denotes the transposition.

We emphasize that the above constitutive equations (1.17), can also be obtained from the Rivlin-Erickson fluid of complexity $n$. If one observes that the physical dimension of the $A_i$, $i=1, \ldots, n$ are $t^{-1}$, where $t$ denotes time, then, a sequence of approximations can be developed to (1.3) correct to the order $t^{-1}, t^{-2}, t^{-3}, t^{-4}, \ldots$, and structure of the constitutive equations so obtained, remains the same as that of (1.17).

Rivlin [1965, 2] has also shown that if one considers the steady flow and replaces the velocity field $\nu$ by $\varepsilon \nu$ where $\varepsilon$ is constant, then the expansion of (1.3) to an order $\varepsilon$, $\varepsilon^2$, $\varepsilon^3$ and $\varepsilon^4$, respectively, again leads to results (1.17)$_1$, (1.17)$_2$, (1.17)$_3$ and (1.17)$_4$. 
1.2 Basic Equations

The basic equations of the grade or order fluids are the field equations, the constitutive equations and boundary conditions.

(i) Field Equations. The field equations governing these fluids are, the balance of linear momentum

\[
\text{div} \, T + \rho f = \rho \dot{\mathbf{v}}. \tag{1.19}
\]

and the constraint of incompressibility

\[
\text{div} \, \mathbf{v} = \text{tr} A_1 = 0 \tag{1.20}
\]

where \( T \) is Cauchy stress tensor, \( \rho \) is the density, \( f \) is the body force per unit mass and the dot signifies material differentiation with respect to time.

(ii) Constitutive Equations. The constitutive equations for the first, second, third and fourth grade* or order* are respectively given by \((1.17)_1\), \((1.17)_2\), \((1.17)_3\) and \((1.17)_4\).

(iii) Boundary Conditions. If the grade or order fluids are considered to be exact models then in general it turns out that governing equations become of higher order, of third order in a second grade (order) fluid

*From this point onward we will continue to use the word "grade" to preserve the notion of "exactness."
and of fourth order in a third grade (order) fluid as compared to the second order in the first grade (order) Navier-Stokes equations. Thus, in order to obtain a determinate solution of a problem, one usually requires some additional initial and/or boundary conditions.

This requirement, however, may not be necessary in certain flow problems. In general, as long as the grade (order) fluids are considered as exact models, some additional conditions shall be required to solve a well-posed problem. On the other hand, if one considers the order fluids to be successive approximations of a simple fluid, then such additional boundary conditions are not necessary. Since in this latter case the starting first order equations become the Stokes equations (creeping viscous flow equations) the usual no-slip boundary conditions of the linearly viscous fluid suffices to completely solve the problem.
1.3 Outline of the Present Work

The purpose of the present work is to investigate the solutions of several flow problems in the fluids of second, third and fourth grades. Following Truesdell [1974, 1], we take up the point of view that either "the fluid of grade \( n \), like its special case, the Navier-Stokes fluid, may be regarded as a fluid in its own right, not necessarily an approximation to any other one, and so studied" or may be considered as successive approximations of the simple fluid.

Accordingly, in Chapters II, III and IV we have considered the grade fluid constitutive equations to be exact in their own right while in Chapter V we have considered them as approximations.

Chapter II deals with the study of plane steady flow problems of a second grade fluid by using hodograph-transformation methods. By interchanging the dependent and independent variables, we transform the equations of motion to the hodograph plane. On introducing the Legendre-transform function of a stream function, we then, write down these equations of motion in terms of Legendre-transform function. The equation this function must satisfy is then obtained and several examples to illustrate the use of the method are considered.

In Chapter III, we study the flow problems of a second grade fluid in plane polar, axisymmetric polar and axisym-
ometric spherical coordinates. We write down the equations of motion in convenient form and then derive compatibility equations by eliminating the pressure distribution in each of these coordinates. Solutions are obtained by selecting different forms of the stream function. These include Jeffery-Hamel [1915, 1; 1916, 1] and Squire [1951, 1] problems in the context of a second grade fluid. In each case, expressions for streamlines, velocity components and pressure distribution are given explicitly, and compared with the corresponding results in the linearly viscous fluid.

Fosdick and Rajagopal [1980, 1] have studied the thermodynamics of a third grade fluid in detail.

In Chapter IV we generalize the Martin's approach [1971, 1] to investigate the plane steady flow problems of a third grade fluid which is thermodynamically compatible. Using differential geometry, the flow equations are developed in a new coordinate system. In order to illustrate the usefulness of the method certain simple flows are examined.

In Chapter V we consider order fluid as an approximation of a simple incompressible fluid. Here we study the problem of helical flow of a fourth order fluid by using the perturbation method. We give the complete analytical solution of the problem and determine the
explicit expressions for velocity and pressure fields. Expressions for torque, volume discharge and normal stresses are derived and compared with the corresponding expressions of the linear viscous fluid case.
CHAPTER II

TRANSFORMATION METHOD IN THE FLOWS
OF A SECOND GRADE FLUID

2.1 Introduction

In recent years, transformation techniques have become some of the powerful methods for solving non-linear partial differential equations. Amongst many, the hodograph transformations have gained considerable success in gas-dynamics problems. Ames [1965, 3] has given an excellent survey of this method together with application in various other fields. Recently, Chandna, Barron and Smith [1982, 2] have used this method to study plane steady viscous incompressible flow problems. Chandna, Barron and Chew [1982, 3] have applied hodograph transformation to obtain the solutions in variably inclined MHD plane flows.

In this chapter we also employ hodograph transformation to study the flow problems of a second-grade fluid. In section 2.2 we write down the equations of motion in a convenient form. Section 2.3 contains the transformation of these equations to the hodograph plane by interchanging the dependent and independent variables. In section 2.4 we introduce a Legendre-transform function of the stream-
function and recast all the transformed equations in terms of this transformed function. The equation this function must satisfy is then determined. Several illustrations to display the use of the method are considered in section 2.5.

With regard to the streamlines and velocity components we find that some of the results of the viscous fluid hold also for the second-grade fluid. In some cases we find that the non-Newtonian nature of the fluid eliminates certain flows which are otherwise possible in Newtonian fluids. The pressure distribution, in almost all cases, appears to be different to that obtained for linearly viscous fluid.

We point out that our approach is an inverse method in the sense that we select a form for the Legendre-transform function and then find conditions when such a function will be possible for physically meaningful situation. We then determine the stream function, velocity components and pressure distribution, via certain suitable relations, for such possible cases.

2.2 Equations of Motion

An incompressible, homogenous fluid of second grade is characterized by the Cauchy stress tensor $T$ in the following form (c.f., (1.17)2):

\[ T = -pI + \mu A_1 + a_1 A_2 + a_2 A_1^2 \]  

(2.1)

where $\mu$ is the viscosity, $a_1$ and $a_2$ are material constants commonly referred to as the normal stress moduli, $p$ is the indeterminate pressure.
Substituting (2.1) into the balance of linear momentum (1.19) and making use of the following identities [1977, 2]:

\[
div[(\text{grad } v)^T A_1] = (\text{grad } v)^T \text{ div } A_1 + A_1 \cdot (\text{grad}(\text{grad } v)^T),
\]

(2.2)

\[
div [(\text{grad } A_1)v] = (\text{grad div } A_1)v + \text{ div } [A_1(\text{grad } v)^T],
\]

(2.3)

we get

\[
\rho \ddot{v} = \rho f - \text{ grad } p + \mu \text{ div } A_1 + a_1 [\text{ div } A_{1t} + (\text{grad div } A_1)v \\
+ (\text{grad } v)^T \text{ div } A_1 + A_1 \cdot (\text{grad}(\text{grad } v)^T)] \\
+ (a_1 + a_2) \text{ div } A_1^2,
\]

(2.4)

where subscript \(t\) denotes partial derivative with respect to time.

In the case of steady plane flow, when body forces are absent, the equations (1.20) and (2.4) reduce to

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

(2.5)

\[
\rho [u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}] + \frac{\partial p}{\partial x} = \mu \nabla^2 u + a_1 [u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \nabla^2 u + \frac{\partial u}{\partial x} \\
+ \frac{\partial v}{\partial x} \nabla^2 v] + \frac{1}{4} (3a_1 + 2a_2) \frac{\partial}{\partial x} |A_1|^2,
\]

(2.6)

\[
\rho [u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}] + \frac{\partial p}{\partial y} = \mu \nabla^2 v + a_1 [u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \nabla^2 v + \frac{\partial u}{\partial y} \\
+ \frac{\partial v}{\partial y} \nabla^2 v] + \frac{1}{4} (3a_1 + 2a_2) \frac{\partial}{\partial y} |A_1|^2,
\]

(2.7)
where $\nabla^2$ denotes the two-dimensional Laplacian operator and

$$|A_1|^2 = \text{tr} A_1^T = [4\left(\frac{\partial u}{\partial x}\right)^2 + 4\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2].$$

Equations (2.5)-(2.7) are three partial differential equations for three unknowns: $u$, $v$ and $p$.

We introduce the two-dimensional vorticity function and a generalized pressure $h$ as

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

$$h = \frac{1}{2p}(u^2 + v^2) - a_1(u^2v + uv^2) - \frac{1}{4}(3a_1 + 2a_2)|A_1|^2 + p.$$  \hspace{1cm} (2.9)

When (2.8) and (2.9) are employed in (2.6) and (2.7) we find that (2.5)-(2.7) are replaced by a system of four partial differential equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega,$$

$$\frac{\partial h}{\partial x} = \rho \omega - \mu \frac{\partial \omega}{\partial y} - a_1 v \frac{\partial \omega}{\partial y},$$

$$\frac{\partial h}{\partial y} = -\rho u \omega + \mu \frac{\partial \omega}{\partial x} + a_1 u v \frac{\partial \omega}{\partial x}. \hspace{1cm} (2.10)$$

for the four unknown functions $u$, $v$, $\omega$, $h$ of $(x,y)$. Once a solution for these is determined, the pressure $p$ is obtained from the generalized pressure expression (2.9).
2.3 Equations in the Hodograph Plane

Let the flow variables $u(x,y), v(x,y)$ be such that, in the flow region under consideration, the Jacobian

$$ J = \frac{\partial (u,v)}{\partial (x,y)} = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \neq 0, \quad (2.11) $$

satisfies $0 < |J| < \infty$. In such cases we may take $x$ and $y$ as functions of $u$ and $v$, i.e.,

$$ x = x(u,v), \quad y = y(u,v). \quad (2.12) $$

We, therefore, have

$$ \frac{\partial u}{\partial x} = J \frac{\partial y}{\partial v}, \quad \frac{\partial u}{\partial y} = -J \frac{\partial x}{\partial v}, $$

$$ \frac{\partial v}{\partial x} = -J \frac{\partial y}{\partial u}, \quad \frac{\partial v}{\partial y} = J \frac{\partial x}{\partial u}, \quad (2.13) $$

$$ J(x,y) = \frac{\partial (u,v)}{\partial (x,y)} = \left( \frac{\partial (x,y)}{\partial (u,v)} \right)^{-1} = j(u,v) \quad (2.14) $$

$$ \frac{\partial f}{\partial x} = \frac{\partial (f,v)}{\partial (x,y)} = J \frac{\partial (f,v)}{\partial (u,v)} = j \frac{\partial (f,v)}{\partial (u,v)}, $$

$$ \frac{\partial f}{\partial y} = -\frac{\partial (f,x)}{\partial (x,y)} = j \frac{\partial (x,f)}{\partial (u,v)} = j \frac{\partial (x,f)}{\partial (u,v)}, \quad (2.15) $$

where $f = f(x,y)$ is any continuously differentiable function and $f(u,v)$ is the transformed function in the $(u,v)$ plane.

Now we take up the four equations (2.10) and employ the above transformations in these equations. We find that transformed system of equations in the hodograph plane
(u,v) is given as

\[ \frac{\partial v}{\partial v} + \frac{\partial x}{\partial u} = 0 \]  \hspace{1cm} (2.16)

\[ j(\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u}) = \omega \]  \hspace{1cm} (2.17)

\[-j \frac{\partial (h, y)}{\partial (u, v)} = -\rho \omega + ujw_1 + \alpha_1v\frac{\partial (x, jw_1)}{\partial (u, v)} + \frac{\partial (-y, jw_2)}{\partial (u, v)} \]  \hspace{1cm} (2.18)

\[ j \frac{\partial (x, h)}{\partial (u, v)} = -\rho \omega + ujw_2 + \alpha_1u\frac{\partial (x, jw_1)}{\partial (u, v)} + \frac{\partial (-y, jw_2)}{\partial (u, v)} \]  \hspace{1cm} (2.19)

where

\[ j = \left( \frac{\partial (x, y)}{\partial (u, v)} \right)^{-1} \]

\[ w_1 = w_1(u, v) = \frac{\partial (x, \omega)}{\partial (u, v)} \]

\[ w_2 = w_2(u, v) = \frac{\partial (-y, \omega)}{\partial (u, v)} \]  \hspace{1cm} (2.20)

This is a system of four partial differential equations in the four unknown functions x, y, \omega, h, of (u,v).
Once a solution
\[ x = x(u,v), \quad y = y(u,v), \quad \omega = \omega(u,v), \quad h = h(u,v) \]
is determined, we are lead to the solutions
\[ u = u(x,y), \quad v = v(x,y) \]
and, therefore,
\[ \omega = \omega(x,y), \quad h = h(x,y) \]
for the system (2.10)

2.4 Equations for the Legendre Transform Function

The equation (2.10) implies the existence of a stream function \( \psi(x,y) \) such that
\[ d\psi = -vdx + udy \]
or
\[ \frac{\partial \psi}{\partial x} = -v, \quad \frac{\partial \psi}{\partial y} = u. \quad (2.21) \]

Likewise, equation (2.16) implies the existence of a function \( L(u,v) \), called the Legendre-transform function of the stream function \( \psi(x,y) \), such that
\[ dL = -ydu + xdv \]
or
\[ \frac{\partial L}{\partial u} = -y, \quad \frac{\partial L}{\partial v} = x, \quad (2.22) \]
and the two functions \( \psi(x,y) \), \( L(u,v) \) are related by
\[ L(u,v) = vx - uy + \psi(x,y). \quad (2.23) \]

On introducing \( L(u,v) \) as defined by (2.22), we eliminate \( x(u,v) \) and \( y(u,v) \) from (2.16)-(2.19). We find
that (2.16) is identically satisfied and the other equations take the form

\[ j[L_{uu} + L_{vv}] = \omega(u,v) \]  
\[ (2.24) \]

\[ \frac{\partial (h, L_u)}{\partial (u,v)} = -\rho v \omega + \mu j W_1 + a_1 v j \left( \frac{\partial (L_v,j W_1)}{\partial (u,v)} + \frac{\partial (L_u,j W_2)}{\partial (u,v)} \right) \]  
\[ (2.25) \]

\[ \frac{\partial (L_v,h)}{\partial (u,v)} = -\rho u \omega + \mu j W_2 + a_1 u j \left( \frac{\partial (L_v,j W_1)}{\partial (u,v)} + \frac{\partial (L_u,j W_2)}{\partial (u,v)} \right) \]  
\[ (2.26) \]

where now

\[ W_1 = \frac{\partial (L_v,\omega)}{\partial (u,v)}, \quad W_2 = \frac{\partial (L_u,\omega)}{\partial (u,v)} \]  
\[ (2.27) \]

and

\[ j = [L_{uu}L_{vv} - L_{uv}^2]^{-1} \]  
\[ (2.28) \]

Now we make use of the integrability condition

\[ (jL_{uv} \frac{\partial}{\partial v} - jL_{vv} \frac{\partial}{\partial u}) \left[ j \frac{\partial (L_u,h)}{\partial (u,v)} \right] \]  
\[ (2.29) \]

\[ = (jL_{uu} \frac{\partial}{\partial u} - jL_{uv} \frac{\partial}{\partial u}) \left[ j \frac{\partial (L_v,h)}{\partial (u,v)} \right], \]

i.e.,

\[ \frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x} \]

to eliminate \( h(u,v) \) from (2.25) and (2.26) and obtain
Collecting the results, we have thus established

**THEOREM 1:** If \( L(u,v) \) is the Legendre-transform function of a stream function of the equations of motion governing the plane steady flow of an incompressible fluid of the second grade, then \( L(u,v) \) must satisfy

\[
\alpha_1 \left[ \frac{\partial \left( L_v, \frac{\partial (L_v, jW_1)}{\partial (u,v)} + \frac{\partial (L_u, jW_2)}{\partial (u,v)} \right)}{\partial (u,v)} \right] \\
+ \mu \left[ \frac{\partial (L_v, jW_1)}{\partial (u,v)} + \frac{\partial (L_u, jW_2)}{\partial (u,v)} \right] - \rho (vW_1 + uW_2) = 0.
\]  

(2.30)

where \( W_1, W_2, j \) and \( \omega \) are given by (2.27), (2.28) and (2.24). Given a solution

\[ L = L(u,v) \]
of (2.31), we can find the velocity components as functions of \((x, y)\) from (2.22). Vorticity, generalized pressure and pressure are then obtained from (2.9) and (2.10).

It is also of some interest to develop the flow equations in polar coordinate \((q, \theta)\) in the hodograph plane. On writing

\[
u + iv = qe^{i\theta},
\]

we note the following transformations:

\[
\frac{\partial}{\partial u} = \cos \theta \frac{\partial}{\partial q} - \frac{\sin \theta}{q} \frac{\partial}{\partial \theta},
\]

\[
\frac{\partial}{\partial v} = \sin \theta \frac{\partial}{\partial q} + \frac{\cos \theta}{q} \frac{\partial}{\partial \theta},
\]

\[
\frac{\partial \mathcal{F}, \mathcal{G}}{\partial (u, v)} = \frac{\partial \mathcal{F}^*, \mathcal{G}^*}{\partial (q, \theta)} - \frac{1}{q} \frac{\partial \mathcal{F}^*, \mathcal{G}^*}{\partial (u, v)},
\]

where \(\mathcal{F}(u, v) = \mathcal{F}^*(q, \theta), \mathcal{G}(u, v) = \mathcal{G}^*(q, \theta)\) are continuously differentiable functions.

Defining \(L^*(q, \theta), \omega^*(q, \theta), j^*(q, \theta)\) to be the Legendre transform, vorticity and Jacobian functions, respectively, in \((q, \theta)\) co-ordinates and using these relations (2.32) and (2.33), the expressions for \(j, \omega, W_1\) and \(W_2\) in the \((q, \theta)\) plane, become

\[
j^* = q^4 [q^2 L^*_{qq} (qL^*_{q} + L^*_{\theta}) - (L^*_q - qL^*_q)^2]^{-1},
\]
\( \omega^* = j^* [L_{qq}^* + \frac{1}{q^2} L_{q\theta}^* + \frac{1}{q} L_{\theta}^*], \)  
(2.35)

\[ W_1^* = W_1(q \cos \theta, q \sin \theta) = \frac{1}{q} \frac{3(\sin \theta L_{qq}^* + \cos \theta L_{q\theta}^*, \omega^*)}{3(q, \theta)} \]
(2.36)

\[ W_2^* = W_2(q \cos \theta, q \sin \theta) = \frac{1}{q} \frac{3(\cos \theta L_{qq}^* - \sin \theta L_{q\theta}^*, \omega^*)}{3(q, \theta)} \]
(2.37)

The terms involving the viscosity \( \nu \) and \( a_1 \) are similarly transformed in the \((q, \theta)\) plane as

\[
\frac{1}{q} \frac{3(\sin \theta L_{qq}^* + \cos \theta L_{q\theta}^*, j^* W_1^*)}{3(q, \theta)} + \frac{3(\cos \theta L_{qq}^* - \sin \theta L_{q\theta}^*, j^* W_2^*)}{3(q, \theta)} = X^*(q, \theta) \]  (say)  
(2.38)

and

\[
q \sin \theta \left[ \frac{1}{q} \frac{3(\sin \theta L_{qq}^* + \cos \theta L_{q\theta}^*, j^* X^*)}{3(q, \theta)} + q \cos \theta \left[ \frac{1}{q} \frac{3(\cos \theta L_{qq}^* - \sin \theta L_{q\theta}^*, j^* X^*)}{3(q, \theta)} \right] \right] \]

(2.39)

Summarizing the results, we have:

**COROLLARY:** If \( L^*(q, \theta) \) is the Legendre transform function of a stream function of the equations of motion for the
plane steady incompressible flow of a second-grade fluid, then \( L^*(q, \theta) \) must satisfy

\[
\begin{align*}
\alpha_1 [ \sin \theta \left( \frac{\partial}{\partial q} L^* + \frac{\cos \theta}{q} L^*_\theta \right) \hat{\mathbf{e}}(q, \theta) ] \\
+ \cos \theta \left( \frac{\partial}{\partial q} L^* - \frac{\sin \theta}{q} L^*_\theta \right) \hat{\mathbf{e}}(q, \theta) \\
+ \mu x^* - \rho q (\sin \theta W^*_1 + \cos \theta W^*_2 ) = 0,
\end{align*}
\]

(2.40)

where \( j^*, \mu^*, W^*_1, W^*_2 \) and \( X^* \) are respectively given by (2.34), (2.35), (2.36), (2.37) and (2.38).

Given a solution \( L^* = L^*(q, \theta) \) of (2.40), we can determine \( u, v \) by making use of (2.32) and \((x, y)\) are expressible as

\[
x = \sin \theta L^*_q + \frac{\cos \theta}{q} L^*_\theta,
\]

\[
y = \frac{\sin \theta}{q} L^*_\theta - \cos \theta L^*_q,
\]

(2.41)

and the remaining flow variables are obtained from the system (2.10).

2.5 Illustrations

In this section we consider some of the applications of Theorem 1 and its Corollary.
(1) As a first application we let
\[ L(u,v) = Au^m + Bv^n \]  \hspace{1cm} (2.42)
be the Legendre transform function where \( m \neq 0, n \neq 0, m \neq 1, n \neq 1 \) and where \( A, B \) are non-zero constants and \( m, n \in \mathbb{R} \). If we substitute (2.42) in (2.24), (2.27) and (2.28), we obtain

\[ j = [nm(m-1)(n-1)ABu^{m-2}v^{n-2}]^{-1}, \]

\[ \omega = \frac{1}{Bn(n-1)v^{n-2}} + \frac{1}{Am(m-1)u^{m-2}}, \]

\[ W_1 = \frac{Bn(n-1)(m-2)u^{-m+1}v^{n-2}}{Am(m-1)}, \]

\[ W_2 = \frac{Am(m-1)(2-n)u^{m-2}v^{-n+1}}{Bn(n-1)}. \]  \hspace{1cm} (2.43)

On employing (2.42) and (2.43) in (2.31) we find that

\[ L(u,v) = Au^m + Bv^n \]

can be the Legendre transform of a stream function for a plane steady flow of a second-grade fluid provided that (for all \( u \) and \( v \)) \( m \) and \( n \) satisfy

\[ a_1 \left[ \frac{Am(m-1)(2-n)(3-2n)(4-3n)u^{m-1}v^{3-3n}}{B^3n^3(n-1)^3} \right. \]

\[ + \frac{Bn(n-1)(m-2)(2m-3)(3m-4)u^{3-3m}v^{n-1}}{A^3m^3(m-1)^3} \]

\[ + \frac{mu[B(n-1)n(m-2)(2m-3)u^{2-2m}v^{n-2}}{A^2m^2(m-1)^2} \]
\[ + \frac{\text{Am}(m-1)(2-n)(3-2n)u^{m-2}v^{2-2n}}{B^n n^2 (n-1)^2} \]
\[ + \rho \frac{\text{Bn}(n-1)(2-m)u^{1-m}v^{n-1}}{\text{Am}(m-1)} + \frac{\text{Am}(m-1)(n-2)}{\text{Bn}(n-1)} u^{m-1} v^{1-n} = 0. \]

(2.44)

Equation (2.44) is satisfied only for \( m = n = 2 \), and (2.42) and (2.43) then become

\[ L(u,v) = Au^2 + Bv^2, \]

\[ j = \frac{1}{4AB}, \quad \omega = \frac{A+B}{2AB}, \]

\[ W_1(u,v) = W_2(u,v) = 0 \]

(2.45)

Substituting (2.45) in (2.22), we find

\[ u(x,y) = -\frac{y}{2A}, \quad v(x,y) = \frac{x}{2B}, \]

(2.46)

and streamlines and the pressure turn out to be, respectively,

\[ \frac{x^2}{4B} + \frac{y^2}{4A} = \text{constant}, \]

(2.47)

\[ p(x,y) = \frac{\rho}{8AB} (x^2 + y^2) + \frac{1}{8} (3a_1 + 2a_2) \frac{(A-B)^2}{A^2 B^2} + p_0. \]

Clearly, the streamlines are circles, if \( A=B \) and ellipses or hyperbolae otherwise.
We remark that the streamlines are similar to those obtained for the viscous fluid but the pressure function is different from the viscous-fluid case.

(2) In the next example we consider

\[ L(u,v) = u^m v^n, \]  

(2.48)

to be the Legendre-transform function with \( m \neq 0 \), \( n \neq 0 \) and \( m + n \neq 1 \). As before, using (2.48) in (2.24, 2.27 and 2.28), we find

\[ j = \left[ u^{2-2n} v^{2-2n} \right] / [m n (1-m-n)] , \]

\[ \omega = \left[ \frac{(m-1)}{n (1-m-n)} u^2 + \frac{(n-1)}{m (1-m-n)} v^2 \right] u^{-m} v^{-n} , \]

\[ W_1 = \frac{m(m-1)}{(1-m-n)} u^{-1} - \frac{n(n-1)(2n+m-2)}{m(1-m-n)} u v^{-2} \]

(2.49)

\[ W_2 = \frac{m(m-1)(2m+n-2)}{n(1-m-n)} u v^{-2} - \frac{n(n-1)}{(1-m-n)} v^{-1} \]

On substituting the above expressions in (2.31) we note that (2.48) can be the Legendre-transform of a stream function for a plane steady flow of second-grade fluid provided (for all \( u \) and \( v \)) \( m \) and \( n \) satisfy

\[ 4 \rho_1 \frac{(m-1)(1-n)(n-m)}{mn(1-m-n)^2} u^{1-2m} v^{1-2n} \]
\[
+ \frac{n(n-1)(2n+m-2)(3n+2m-3)}{m^3(1-m-n)^2} u^{3-2m,v-2n-1} \\
- \frac{m(m-1)(2m+n-2)(3m+2n-3)}{n^3(1-m-n)^2} u^{2m-1,v^3-2n} \\
+ \mu [ \frac{2(2m-1)(1-n)}{1-m-n} u - m - n \right] u + \frac{n(n-1)(2n+m-2)(3n+2m-3)}{m^2(1-m-n)^2} u^{2-m,v-2} - m - n - 2 \\
+ \frac{2m(m-1)}{n} u - n - 2 - m - n \\
+ \phi \left[ \frac{2m(m-1)}{n} u - n - 2 - m - n \right] = 0. \quad (2.50)
\]

Clearly, (2.50) is satisfied if \( m = n = 1 \), and (2.48), (2.49) then reduce to

\[
L(u,v) = uv, \\
j = -1, \quad \omega = 0, \quad W_1 = W_2 = 0, \quad (2.51)
\]

where \( j, \omega, W_1, W_2 \) are functions of \((u,v)\). On proceeding as in the previous example, we now find

\[
u(x,y) = x, \quad v(x,y) = -y
\]

and

\[
\Phi(x,y) = -\frac{1}{2} \rho (x^2 + y^2) + 2(3\alpha_1 + 2\alpha_2) + C, \quad (2.52)
\]

where \( C \) is an arbitrary constant. The streamlines are
given by

\[ \gamma x = C_2 \]  

(2.53)

which are rectangular hyperbolae shown in Fig. 1.

We point out that the presence of the normal stress modulus \( a_1 \), that is the consideration of non-Newtonian nature of the fluid, eliminates two other possible solutions, namely:

\[ m = 1, \quad n = -1, \quad 6\mu = \rho, \]

and

\[ m = -1, \quad n = 1, \quad 6\mu = \rho, \]

which are possible for viscous fluids [1982, 2].

(3) In the remainder of this section, we investigate the solutions of flow problems in \((q, \theta)\) coordinates. Let

\[ L*(q, \theta) = F(q) \]  

(2.54)

be the Legendre-transform function such that \( F'(q) \neq 0 \), \( F''(q) \neq 0 \). Using (2.54) in (2.34) - (2.37), we get

\[ j* = \frac{q}{F'(q)F''(q)} = j*(q), \]

\[ \omega* = \frac{qF''(q) + F'(q)}{F'(q)F''(q)} = \omega*(q), \]
\[ W_1^* = \frac{F'(q) \cos \theta}{q} \left[ \frac{qF''(q) + F'(q)}{F'(q)F''(q)} \right]' = \frac{F'}{q} \cos \theta \omega^*(q) \]

\[ W_2^* = \frac{F'(q) \sin \theta}{q} \left[ \frac{qF''(q) + F'(q)}{F'(q)F''(q)} \right]' = \frac{F'}{q} \sin \theta \omega^*(q) \]

(2.55)

When these relations are employed in (2.40), we find that the terms involving \( a_1 \), the normal-stress modulus, and the terms involving \( \rho \), the density, both become identically zero, and we obtain the condition

\[ \omega^* + \frac{F'(q) \frac{\omega^*}{F''}}{q} = 0. \]

(2.56)

For \( \omega^*(q) \neq 0 \), the above equation, after integrating twice with respect to \( q \), yields

\[ \omega^*(q) = C \ln F' + D \]

(2.57)

where \( C \) and \( D \) are arbitrary constants. Using (2.55) and (2.41) we find

\[ q = k_1 r \ln r + k_2 r + k_3 / r, \]

(2.58)

where

\[ r = \sqrt{x^2 + y^2} \text{ and } k_1 = \frac{1}{2} C, \quad k_2 = \frac{1}{4} (2D - C) \]

and \( k_3 \) are arbitrary constants. With the help of (2.58) and (2.41) we can find that

\[ u(x, y) = -y \left[ \frac{1}{2} k_1 \ln(x^2 + y^2) + k_2 + \frac{k_3}{x^2 + y^2} \right] , \]
\[ v(x,y) = x\left(\frac{1}{2}k_1 \ln(x^2 + y^2) + k_2 + \frac{k_3}{x^2 + y^2}\right), \]

\[ \omega(x,y) = k_1 \ln(x^2 + y^2) + k_1 + 2k_2. \quad (2.59) \]

Since \( \omega(x,y) \) is harmonic, the terms involving \( a_1 \) in the linear momentum equations in system (2.10) are identically zero, and after considerable simplification, the pressure distribution turns out to be

\[
p = 2\left(\frac{1}{4}k_1 k_2 + \frac{1}{8}k_1^2(x^2 + y^2)\right)\left(\ln(x^2 + y^2)\right)^2
+ \left(\frac{1}{2}k_1^2 - \frac{1}{4}k_1^2\right)(x^2 + y^2) + k_2 k_3 \ln(x^2 + y^2)
+ \left(\frac{1}{4}k_1^2 + \frac{1}{2}k_2^2 - \frac{1}{2}k_1 k_2\right)(x^2 + y^2)
- \frac{k_3^2}{2(x^2 + y^2)} - k_2 k_3
+ 2a_1 k_1 \left(\frac{1}{2}k_1 \ln(x^2 + y^2) + k_2 + \frac{k_3}{x^2 + y^2}\right)
- 2\omega k \tan^{-1}\left(\frac{x}{y}\right)
+ \frac{3a_1 + 2a_2}{2}k_1^2 + \frac{4k_2^2}{(x^2 + y^2)},
- \frac{4k_1 k_3}{x^2 + y^2} + \rho_0. \quad (2.60)\]
The stream function \( \psi(x,y) \) is given by
\[
\psi(x,y) = (x^2 + y^2) \left[ \frac{1}{4} k_1 \ln(x^2 + y^2) + \frac{1}{2} k_2 - \frac{1}{4} k_1 \right] \\
+ \frac{1}{2} k_3 \ln(x^2 + y^2) + \mathcal{C}.
\]

We point out that the pressure (2.60) is considerably different from the viscous-fluid case.

We note that (2.56) is also satisfied for \( \omega^* = \text{constant} = \omega_0 \). In this case the velocity components and vorticity are given as
\[
u(x,y) = -\frac{ky}{x^2 + y^2} - \frac{1}{2}\omega_0 y, \\
v(x,y) = \frac{kx^2}{x^2 + y^2} + \frac{1}{2}\omega_0 x, \\
-\nabla^2 \psi = \omega_0.
\]

and the stream function is given by
\[
\psi(x,y) = f(x,y) - \frac{1}{4}\omega_0 (x^2 + y^2),
\]

where \( f(x,y) \) is a harmonic function.

(4) Now we investigate the solution of a flow problem when \( L^*(q,\theta) \) is a function of \( \theta \) only. We assume
\[
L^*(q,\theta) = G(\theta)
\]
to be the Legendre transform for the system of equations (2.40) such that \( G'(\theta) \neq 0 \). On employing (2.62) in (2.34) - (2.37) we find that

\[
\begin{align*}
   j^*(q, \theta) &= -\frac{q^4}{G'(\theta)^2}, \\
   \omega^*(q, \theta) &= -\frac{q G''(\theta)}{G'(\theta)^2}, \\

   W_1^*(q, \theta) &= \frac{G'''(\theta) \cos \theta - 2G''(\theta) \sin \theta}{qG'(\theta)}, \\
   W_2^*(q, \theta) &= \frac{-G''(\theta) \sin \theta + 2G''(\theta) \cos \theta}{qG'(\theta)} \\
\end{align*}
\]

(2.63)

Using these relations in (2.40), we obtain the condition

\[
4\alpha_1 q^2 [G^{iV} + 4G''] = \mu G'[G^{iV} + 4G''] + 2\rho G'G''.
\]

(2.64)

The above equation is satisfied if

\[
G^{iV} + 4G'' = 0 \text{ and } \mu(G^{iV} + 4G'') + 2\rho G'G'' = 0.
\]

(2.65)

thus

\[
G(\theta) = A\theta + B,
\]

(2.66)

where \( A, B \) are arbitrary constants, is the solution of (2.64).

Proceeding as before, we find

\[
\begin{align*}
u(x, y) &= \frac{Ax}{x^2 + y^2}, \\
v(x, y) &= \frac{Ay}{x^2 + y^2}
\end{align*}
\]

(2.67)
The stream function $\psi(x,y)$ and pressure $p(x,y)$ take the form, respectively,

$$
\psi(x,y) = A \tan^{-1} \left( \frac{y}{x} \right) + C,
$$

$$
p(x,y) = C - \frac{2 A^2}{2(x^2 - y^2)} + \frac{3a_1 + 2 a_2}{4} \frac{A}{(x^2 - y^2)^2} \left( \frac{\partial^2}{\partial x^2} \left( \frac{Ax}{x^2 - y^2} \right) \right)^2
$$

$$
+ \frac{32 A^2 x^2 y^2}{(x^2 + y^2)^4}.
$$

(2.68)

The streamlines pattern is shown in Fig. 2.

(5) Finally, we consider the case when

$$
L^*(q, \theta) = q^2 G(\theta). \tag{2.69}
$$

Following the previous examples, we note that

$$
j^* = \left( 4G^2 + 2GG'' - G'^2 \right)^{-1} = j^*(\theta),
$$

$$
\omega^* = \frac{4G + G''}{4G^2 + 2GG'' - G'^2} = \omega^*(\theta),
$$

$$
W^*_1 = \frac{\omega^*}{q} (2G \sin \theta + G' \cos \theta),
$$

$$
W^*_2 = \frac{\omega^*}{q} (2G \cos \theta - G' \sin \theta).
$$

Substituting these relations in (2.40), we get
\[ 2a_1[Gj^*((4G^2 + G^2)j\omega^*)']' + \mu((4G^2 + G^2)j\omega^*)' - 2G\omega^*q^2 = 0. \quad (2.70) \]

Equation (2.70) is satisfied if

\[ 2a_1[Gj^*((4G^2 + G^2)j\omega^*)']' + \mu((4G^2 + G^2)j\omega^*)' = 0 \quad (2.71) \]

and

\[ 2\rho G\omega^* = 0. \]

Since \( G \neq 0, \rho \neq 0; \) therefore, \( \omega^* = \text{constant} = \omega_0. \)

Hence, the solution and the rest of the analysis is the same as in the viscous-fluid case [1982, 2], except that the pressure is now given by

\[ p(x, y) = -\frac{\rho}{2k_3}(x^2 + y^2) + 8(3a_1 + 2a_2) \frac{k_1^2 + k_2^2}{k_3^2} + p_0. \]
Fig. 1. Stream Line Pattern for Example #2
Fig. 2. Streamline Patterns for Example #4
CHAPTER III
CERTAIN INVERSE SOLUTIONS OF A SECOND GRADE FLUID

3.1 Introduction

There are a very few exact solutions of the Navier-Stokes equations and these become even fewer if non-Newtonian constitutive equations are considered in the equations of motion. As pointed out in Chapter I, the governing equations, in general, in the case of second grade fluid are of third order as compared to second order Navier-Stokes equations. Thus, one requires an additional boundary (initial) condition over and above the boundary conditions used to the Navier-Stokes equations. This, however, may not be necessary in a specific problem in which the kinematical assumptions make the particular terms in $A_2$, the second Rivlin-Ericksen tensor, disappear automatically; see Rajagopal [1982, 4]. It is partly for this reason that, in some situations, the inverse method (indirect method*) becomes attractive in the studies of non-Newtonian fluids.

Usually, in the inverse method, the boundary conditions are not prescribed at the outset and solution of the

*Berker [1963, 2] used the terminology "indirect method" when boundary conditions are not prescribed at the outset.
differential equations is sought by assuming certain geometrical or physical properties of the field. Nemenyi [1951, 2] has given an excellent survey of this method. Recently, Kaloni and Huschilt [1984, 1] gave the solutions to the equations of motion of a second grade fluid by employing the inverse method. By employing a slightly different inverse method steady plane flow of such fluids were considered by Kaloni and Siddiqui [1983, 1]. In both of the above papers most of the problems studied were in steady plane flow. In the present work, we study the related steady flow problems in plane polar coordinates, in axisymmetric polar coordinates and in axisymmetric spherical polar coordinates. In Section 3.2 we first write the equations of motion in suitable form and then, by eliminating the pressure, derive the compatibility equation in each of the coordinate system. In Section 3.3, by assuming certain forms of the stream function a priori, we determine flows which are consistent with the compatibility equations. We then determine velocity and pressure field by using the equations derived in Section (3.2).

The various characteristic features of the flow of a second grade fluid have been studied by several authors. Ting [1963, 1] has investigated a class of non-steady flows which include, channel flows, pipe flows under
constant pressure gradient and flow between infinite parallel planes. Morkovitz and Coleman [1964, 1] have studied the viscometric flows. Tanner [1966, 1] and Huilgol [1973, 1] have each considered the steady plane flows in which inertia terms were neglected. Fosdick and Rajagopal [1978, 2; 1979, 3] have studied certain anomalous features as well as the uniqueness and drag in steady motion. Dunn and Poskick [1974, 2] have discussed the thermodynamics and stability of these fluids. These authors also gave extensive references to other related works up to that time. Recently, Rajagopal [1981, 1] and Rajagopal and Gupta [1981, 2, 3] have found certain exact solutions in such fluids. Rajagopal [1984, 2] has also found some interesting results when inertia terms are neglected in these fluids.

We also mention here, the related work of Fosdick and Truesdell [1977, 2] and Morris [1979, 1] in the Rivlin-Ericksen fluids. We remark that, our results, though closely related with the work of above authors, do not fail in their framework. In comparing with the viscous fluid results we have frequently referred to the article of Berker [1963, 2].

3.2 Equations of Motion and Compatibility Equations

We write down the equations of motion and compatibility equations for second grade fluid in different coordinate systems.
For steady plane flows in polar coordinates we take

\[ \mathbf{V} = [u(r, \theta), v(r, \theta), 0] \]  \hspace{1cm} (3.1)

If we introduce the vorticity function \( \omega(r, \theta) \) and a generalized pressure \( h(r, \theta) \) as

\[ \omega = \frac{3v}{r} + \frac{\nu}{r} - \frac{1}{r} \frac{\partial u}{\partial \theta} \]  \hspace{1cm} (3.2)

\[ h = \frac{\rho}{2}(u^2 + v^2) - a_1 \left( v \frac{\partial u}{\partial r} - \frac{u}{r} \frac{\partial v}{\partial \theta} \right) \omega \]

\[ - \frac{(3a_1 + 2a_2)}{4} |A_1|^2 + \mathbf{p}, \]  \hspace{1cm} (3.3)

where

\[ |A_1|^2 = [4(\frac{\partial u}{\partial r})^2 + 4(\frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta})^2 \]

\[ + 2(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \theta})^2], \]

and employ (3.1) - (3.3) in (1.20) and (2.4), we find

that, in the absence of body forces, the latter become

\[ \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} = 0, \]  \hspace{1cm} (3.4)

\[ \frac{\partial h}{\partial r} + \rho v \omega + \frac{u}{r} \frac{\partial \omega}{\partial \theta} + a_1 v^2 \omega = 0, \]  \hspace{1cm} (3.5)

\[ \frac{1}{r} \frac{\partial h}{\partial \theta} + \rho u \omega - \frac{u \omega}{r} - a_1 u v \omega = 0, \]  \hspace{1cm} (3.6)

where
\[
\n\psi^2 = \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2},
\]

denotes the two-dimensional Laplacian operator. On defining the stream function \(\psi(r, \theta)\) through

\[
u = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad \psi = -\frac{\partial \psi}{\partial r}, \quad (3.7)
\]

the equation (3.4) is satisfied identically and (3.5) and (3.6) take the form

\[
\frac{\partial h}{\partial r} - \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial \theta} \right)^2 - \frac{\mu}{r} \frac{\partial \psi}{\partial \theta} \left( \psi^2 \right) + \alpha_1 \frac{\partial}{\partial r} \left( \psi^4 \right) = 0, \quad (3.8)
\]

\[
\frac{1}{r} \frac{\partial h}{\partial \theta} - \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} \right)^2 + \mu \frac{\partial \psi}{\partial r} \left( \psi^2 \right) + \alpha_1 \frac{\partial}{\partial \theta} \left( \psi^4 \right) = 0, \quad (3.9)
\]

where

\[
\psi^4 = \psi^2 \psi^2.
\]

On eliminating \(h\) in the above equations we find the compatibility equation to be

\[
-\rho \left[ \frac{\partial}{\partial \mathbf{r}} \left( \frac{\partial \psi}{\partial \mathbf{r}} \right) \right] + \alpha_1 \left[ \frac{\partial}{\partial \mathbf{r}} \left( \psi^4 \right) \right] = r \mu \nu \psi^4, \quad (3.10)
\]

where

\[
\frac{\partial}{\partial \mathbf{r}} \left( \frac{\partial \psi}{\partial \mathbf{r}} \right) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial y} \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right),
\]

is ordinary notation for Jacobian.

We remark that in plane polar coordinates the
coefficient $a_2$ does not contribute to the stream function and hence to the velocity distribution. It does, however, manifest in the expression for the pressure.

For steady axisymmetric flows in polar coordinates we assume

$$V = [u_1(r, z), 0, w_1(r, z)]. \quad (3.11)$$

Introducing the vorticity function $\tilde{\Omega}(r, z)$ and generalized pressure $\tilde{h}(r, z)$ defined by

$$\tilde{\Omega} = \left( \frac{\partial w_1}{\partial r} - \frac{\partial u_1}{\partial z} \right), \quad (3.12)$$

$$\tilde{h} = \frac{\rho}{2} (u_1^2 + w_1^2) - a_1 \left[ w_1 \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) - u_1 \frac{\partial}{\partial z} \right] \tilde{\Omega}$$

$$- \frac{(3a_1 + 2a_2)}{4} |A_1|^2 + p, \quad (3.13)$$

where

$$|A_1|^2 = \left[ 4 \left( \frac{\partial u_1}{\partial r} \right)^2 + 4 \left( \frac{\partial w_1}{\partial z} \right)^2 + 4 \left( \frac{1}{r} \right)^2 ight]$$

$$+ 2 \left( \frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial r} \right)^2,$$

and making use of (3.11)-(3.13) we find that the equations (1.20) and (2.4), in the absence of body forces, take the form

$$\frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{\partial w_1}{\partial z} = 0, \quad (3.14)$$
\[ \frac{\partial h}{\partial r} - \rho \omega_1 \Omega + \mu \frac{\partial^2 \Omega}{\partial z^2} + a_1 \omega_1 (v^2 \Omega - \frac{\Omega}{r^2}) \]
\[- (a_1 + a_2) \left[ \frac{2}{r} \frac{\partial}{\partial z} (u_1 \Omega) + \frac{\Omega^2}{r} \right] = 0, \quad (3.15)\]
\[ \frac{\partial h}{\partial z} + \rho u_1 \Omega - \mu \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \Omega - a_1 u_1 (v^2 \Omega - \frac{\Omega}{r^2}) \]
\[+ (a_1 + a_2) \left[ \frac{2}{r} \frac{\partial}{\partial r} (u_1 \Omega) \right] = 0. \quad (3.16)\]

where now
\[ v^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \]

The stream function \( \psi(r,z) \) defined through
\[ u_1 = \frac{1}{r} \frac{\partial \psi}{\partial z}, \quad w_1 = - \frac{1}{r} \frac{\partial \psi}{\partial r}, \quad (3.17)\]

satisfies equation (3.14) identically and equations (3.15) and (3.16) become
\[ \frac{\partial h}{\partial r} - \rho \frac{\partial \psi}{\partial r} \frac{E^2 \psi}{r^2} - \frac{\mu}{r} \frac{\partial}{\partial z} (E^2 \psi) + a_1 \frac{\partial \psi}{\partial r} \frac{E^4 \psi}{r^2} \]
\[+ (a_1 + a_2) \left[ \frac{2}{r} \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial z} \frac{E^2 \psi}{r^2} \right) - \left( \frac{E^2 \psi}{r^2} \right)^2 \right] = 0, \quad (3.18)\]
\[ \frac{\partial h}{\partial z} - \rho \frac{\partial \psi}{\partial z} \frac{E^2 \psi}{r^2} + \frac{\mu}{r} \frac{\partial}{\partial r} (E^2 \psi) + a_1 \frac{\partial \psi}{\partial z} \frac{E^4 \psi}{r^2} \]
\[+ (a_1 + a_2) \left[ \frac{2}{r} \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} \frac{E^2 \psi}{r^2} \right) \right] = 0, \quad (3.19)\]
where

\[
E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},
\]

\[
E^4 = E^2 \cdot E^2.
\]

After eliminating the generalized pressure \( \hat{\n} \) between (3.18) and (3.19), we obtain the compatibility equation as

\[
\rho \left[ \frac{\partial \hat{\sigma} - E^2 \gamma / r^2}{\partial (r, z)} \right] + \frac{\mu}{r} E^4 \gamma \\
= a_1 \left[ \frac{\partial^3}{\partial (r, z)} \left( E^2 \gamma / r^2 \right) \right] + \frac{2(a_1 + a_2)}{r} \frac{\partial^2}{\partial z} E^2 \left( \frac{\partial^2 \gamma}{r^2} \right) \\
+ 2 \left( \frac{\partial^2}{\partial z^2} \right) \left( \frac{\partial \gamma}{z} \right) \left( \frac{\partial^2 \gamma}{r^2} \right)
\]

(3.20)

where

\[
\left( \frac{\partial \hat{\sigma} - E^2 \gamma / r^2}{\partial (r, z)} \right) = \frac{\partial^2}{\partial r \partial z} \left( \frac{E^2 \gamma}{r^2} \right) - \frac{\partial^2}{\partial z \partial r} \left( \frac{E^2 \gamma}{r^2} \right).
\]

Finally, we consider steady axisymmetric flows in spherical polar coordinates and take

\[
\mathbf{v} = [u_z(R, \theta), v_z(R, \theta), 0]
\]

(3.21)

As before, we again, introduce the vorticity function \( \nabla(R, \theta) \) and generalized pressure \( \hat{\n}(R, \theta) \) as,

\[
\hat{\n} = \frac{\partial v_z}{\partial R} + \frac{v_z}{R} - \frac{1}{R} \frac{\partial u_z}{\partial \theta}.
\]

(3.22)
\[ \bar{h} = \frac{\partial}{\partial R} \left( \bar{u}^2 + \bar{v}^2 \right) - a_1 \left[ \bar{v} \left( \frac{\partial}{\partial \theta} \bar{\theta} \right) + \frac{1}{R} \bar{\theta} \right] - \frac{\bar{u}_2}{R \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \bar{\theta} \right) \]
\[ - \frac{1}{4} \left( a_1 + a_2 \right) |A_1|^2 + p \]  
(3.23)

where

\[ |A_1|^2 = \left[ \frac{4}{3} (\frac{\partial \bar{u}_2}{\partial R})^2 + 4 \left( \frac{\bar{u}_2}{R} + \frac{1}{R} \frac{\partial \bar{v}_2}{\partial \theta} \right)^2 + 4 \left( \frac{\bar{u}_2}{R} + \frac{\bar{v}_2}{R \cot \theta} \right)^2 \right] \]

On employing (3.21)-(3.23) in equations (1.20) and (2.4) and assuming that body forces are absent, we obtain

\[ \frac{\partial \bar{u}_2}{\partial R} + \frac{2}{R} \bar{u}_2 + \frac{1}{R} \frac{\partial \bar{v}_2}{\partial \theta} + \frac{\bar{v}_2}{R \cot \theta} = 0, \]  
(3.24)

\[ \frac{\partial \bar{h}}{\partial R} - \bar{v}_2 \bar{\theta} + \frac{\mu}{R \sin \theta} \frac{\partial}{\partial \theta} \left( \bar{\theta} \sin \theta \right) + a_1 \bar{v}_2 \left( \bar{v}^2 \bar{\theta} - \frac{\bar{\theta}^2}{R^2 \sin^2 \theta} \right) \]
\[ - \left( a_1 + a_2 \right) \left[ \frac{2}{R \sin \theta} \left( \frac{\partial \bar{u}_2}{\partial \theta} + \frac{\bar{u}_2}{R \cot \theta} \bar{\theta} \sin \theta \right) \right] \]
\[ + \frac{\bar{h}}{R} = 0, \]  
(3.25)

\[ \frac{1}{R} \frac{\partial \bar{h}}{\partial \theta} + \bar{v}_2 \bar{\theta} - \frac{\mu}{R} \left( \frac{\partial}{\partial \theta} \bar{\theta} + \frac{1}{R} \bar{\theta} \right) - \frac{a_1 \bar{u}_2}{R} \bar{v}^2 \bar{\theta} - \frac{\bar{\theta}^2}{R^2 \sin^2 \theta} \]
\[ + \left( a_1 + a_2 \right) \left[ \frac{2}{R} \frac{\partial}{\partial R} \left( \left( \bar{u}_2 + \bar{v}_2 \cot \theta \right) \bar{\theta} \right) \right] \]
\[ - \frac{\cot \theta}{R} \frac{\partial \bar{h}}{\partial R} = 0, \]  
(3.26)

where
\[ \psi^2 = \frac{1}{R^2} \frac{\partial^2}{\partial R^2} + \frac{1}{R^2 \sin \theta} \frac{\partial^2}{\partial \theta^2} \]  

On setting

\[ \sigma = \cos \theta \]

and defining the stream function \( \bar{\psi}(R, \sigma) \) through

\[ u_2 = \frac{1}{R^2} \frac{\partial \bar{\psi}}{\partial \sigma}, \quad v_2 = \frac{1}{R \sqrt{1-\sigma^2}} \frac{\partial \bar{\psi}}{\partial R}, \]

the equation (3.24) is satisfied identically, and (3.25) and (3.26) may be written as

\[ \frac{\partial \bar{\psi}}{\partial R} - \frac{1}{R} \frac{\partial}{\partial R} \left( \frac{\partial \bar{\psi}}{\partial \sigma} \right) \frac{\partial^2 \bar{\psi}}{\partial R^2} \left( \frac{\partial \bar{\psi}}{\partial \sigma} \right)^2 - \frac{\mu}{R^2} \frac{\partial}{\partial \sigma} \left( \frac{\partial^2 \bar{\psi}}{\partial R^2} \right) \]

\[ - (a_1 + a_2) \left( \frac{2}{R} \frac{\partial \bar{\psi}}{\partial \sigma} \right) \left( \frac{1}{R^2} \frac{\partial^2 \bar{\psi}}{\partial \sigma^2} + \frac{\sigma}{R^3 (1-\sigma^2)} \frac{\partial \bar{\psi}}{\partial R} \right) \]

\[ + \frac{1}{R} \left( \frac{\partial^2 \bar{\psi}}{\partial \sigma^2} \right)^2 = 0, \]

\[ - \frac{1}{R} \frac{\partial \bar{\psi}}{\partial \sigma} \left( \frac{1}{R^3} \frac{\partial^2 \bar{\psi}}{\partial \sigma^2} \right) - \frac{\mu}{R^2} \frac{\partial}{\partial \sigma} \left( \frac{\partial^2 \bar{\psi}}{\partial R^2} \right) \]

\[ - \frac{1}{R^2 (1-\sigma^2)^2} \frac{\partial \bar{\psi}}{\partial \sigma} \frac{\partial^2 \bar{\psi}}{\partial R^2} \left( \frac{\partial \bar{\psi}}{\partial \sigma} \right)^2 \]

\[ + \frac{\sigma}{R^2 (1-\sigma^2)^2} \frac{\partial \bar{\psi}}{\partial \sigma} \frac{\partial^2 \bar{\psi}}{\partial R^2} \left( \frac{\partial \bar{\psi}}{\partial \sigma} \right)^2 \]

where
\[ D^2 \psi = \frac{1}{\sigma R^2} \left( (1 - \sigma^2) \frac{\partial^2 \psi}{\partial \sigma^2} + \frac{1}{R^2} \frac{\partial^2 \psi}{\partial \sigma^2} \right) \]

\[ D^4 = D^2 \cdot D^2. \]

On eliminating \( h \) in above equations we find the compatibility equation to be

\[ \frac{\sigma}{(1 - \sigma^2)} D^4 \psi + \sigma \left\{ \frac{\partial^2 \psi}{\partial \sigma^2} \right\} = a_1 \left\{ \frac{\partial^4 \psi}{\partial R^2 (1 - \sigma^2)} \right\}, \]

\[ + \frac{2(1 + \alpha^2)}{(1 - \sigma^2)} \left\{ \frac{\partial D^2 \psi}{\partial \sigma^2} + \frac{\sigma}{R^2} \frac{\partial D^2 \psi}{\partial \sigma^2} \right\} \frac{\partial}{\partial \sigma} D^2 \psi \]

\[ - \left\{ \frac{\partial^2 \psi}{R^2} \frac{\partial}{\partial \sigma} + \frac{\sigma \partial^2 \psi}{R^2} \frac{\partial}{\partial \sigma} \right\} \frac{\partial D^2 \psi}{\partial R} \right\}, \quad (3.30) \]

where

\[ \frac{\partial^2 \psi}{\partial \sigma^2} \frac{\partial^2 \psi}{\partial R^2 (1 - \sigma^2)} \]

\[ \frac{\partial^2 \psi}{\partial R^2 (1 - \sigma^2)} \frac{\partial}{\partial \sigma} \frac{\partial^2 \psi}{\partial R^2 (1 - \sigma^2)} \]

\[ \frac{\partial^2 \psi}{\partial \sigma^2} \frac{\partial^2 \psi}{\partial R^2 (1 - \sigma^2)} \]

3.3 Solutions

In this section we now consider the inverse solution of the above equations by assuming the form for stream function \( \psi \), a priori.
(a) Motions where $\psi(x, \theta) = r^n F(\theta)$

We seek the motions for which the stream function is of the form

$$\psi(x, \theta) = r^n F(\theta), \quad (3.31)$$

where $F$ is an arbitrary function of its argument and $n$ is an integer. On substituting this expression into equation (3.10) we find the relation

$$\mu H + \sigma r^n [nF'(\theta) - (n-2)F'G] + \alpha \gamma (n-2) [nF'H' - (n-4)F'H'] = 0, \quad (3.32)$$

where

$$G(\theta) = F''(\theta) + n^2 F(\theta),$$

$$H(\theta) = G''(\theta) + (n-2)^2 G(\theta). \quad (3.33)$$

For $n = 0$, (3.32) gives the following two differential equations

$$4F'G = 0,$$

$$\mu H + 2 \sigma F'G = 0. \quad (3.34)$$

In the viscous fluids, this case corresponds to the well-known Jeffery-Hamel flows (c.f. Berber [1963, 2]). On employing (3.33), for $n = 0$, in (3.34), we get

$$4F'(F'' + 4F') = 0.$$
\[ \mu (P^{i}V + 4F'') + 2\rho F'F'' = 0. \] (3.35)

On assuming \( F' \neq 0 \), equations (3.35) yield

\[ 2\rho F'F'' = 0. \] (3.36)

Thus,

\[ F = A\theta + B, \] (3.37)

where \( A \) and \( B \) are arbitrary constants, is the desired solution. We remark that it is a universal solution (c.f. Fosdick and Truesdell [1977, 2]). The stream function and velocity components are given as

\[ \psi(r, \theta) = A\theta + B, \]

\[ u = \frac{A}{r}, \quad v = 0, \] (3.38)

and vorticity is identically zero.

The streamlines are radial lines \( \theta = \) constant and velocity field is purely radial and its magnitude decreases as flow leaves the origin while \( A \) is positive. Thus we have a two dimensional source flow of a second grade fluid. The pressure distribution, by making use (3.38) in (3.8), (3.9) and (3.3), turns out to be

\[ p = -\frac{\rho A}{r^2} + 2(3a_1 + 2a_2)\frac{A^2}{r^4} + C \] (3.39)

where \( C \) is an arbitrary constant.
Next we consider the case when \( n = 2 \). The equations (3.32) and (3.33) take the form, respectively,

\[
\mu H + 2pr^2FG' - 2a_1(F'H + H'F) = 0,
\]

\[
G(\theta) = F'' + 4F,
\]

\[
H(\theta) = G''.
\]  

Equation (3.40), however, gives the following two differential equations

\[
\mu H - 2a_1(FH)' = 0,
\]

\[
G' = 0.
\]

Thus, we require the solution of

\[
F'' + 4F = \lambda_1.
\]

which is applicable to both viscous and second grade fluid. The general solution of this equation is

\[
F = C_1 \cos 2\theta + C_2 \sin 2\theta + \frac{\lambda_1}{4},
\]

where \( C_1 \), \( C_2 \) and \( \lambda_1 \) are arbitrary constants. We point out that this again is a universal solution.

The stream function and velocity components are found to be
\[ \psi = r^2 \left( C_1 \cos 2\theta + C_2 \sin 2\theta + \frac{A_1}{4} \right) \]

\[ u = 2r \left[ -C_1 \sin 2\theta + C_2 \cos 2\theta \right] \]

\[ v = -2r \left[ C_1 \cos 2\theta + C_2 \sin 2\theta + \frac{A_1}{4} \right] \quad (3.45) \]

and the vorticity turns out to be constant. The pressure distribution, after using (3.45) in (3.8), (3.9) and (3.3), may be written as

\[ p = A_1 \psi + \frac{\partial}{\partial z} \left( u^2 + v^2 \right) + 8 \left( 3\alpha_1 + 2\alpha_2 \right) \left( C_1^2 + C_2^2 \right) + p_0. \quad (3.46) \]

where \( p_0 \) is an arbitrary constant.

For other values of \( n \), we note, from (3.32), that we are required to satisfy

\[ H = 0, \quad nF'G' - (n-2)F'G = 0 \]
\[ nF'H' - (n-4)F'H = 0, \quad (3.47) \]

where \( G \) and \( H \) are given by (3.33). On integrating (3.47), we get

\[ G = C_2^\prime \frac{(n-2)}{n} \quad (n \neq 0) \quad (3.48) \]

which along with (3.33) form a non-linear differential equation for \( F \) (except when \( n = 2 \)). In general, solutions in other cases are, therefore, not easily amenable. In the case when \( n = 1 \), we have
\[ G = C_2 F^{-1}. \]

On combining this with (3.33)\textsubscript{1}, we get
\[ FF'' + F^2 = C_2. \]

The first integral of this equation is
\[ F'^2 = 2C_2 \ln F - F^2 + D_2, \]
where \( C_2 \) and \( D_2 \) are arbitrary constants. The above is compatible with (3.47)\textsubscript{1} provided \( C_2 = 0 \) and in that case one easily finds
\[ F = A_1 \cos \theta + B_1 \sin \theta, \quad (3.49) \]
where \( A_1 \) and \( B_1 \) are arbitrary constants. Accordingly,
\[ \psi = r(A_1 \cos \theta + B_1 \sin \theta) \]
\[ u = r(-A_1 \sin \theta + B_1 \cos \theta) \]
\[ v = -(A_1 \cos \theta + B_1 \sin \theta). \quad (3.50) \]

and vorticity turns out to be zero. The pressure is given by
\[ p = p_0 - \frac{\nu}{2}(A_1^2 + B_1^2). \quad (3.51) \]
In passing we note that when \( \psi = \psi(r) \), the equation (3.10) reduces to

\[
\frac{d}{dr}\left[r \frac{d}{dr}\left(\frac{1}{r} \frac{d}{dr} \frac{d\psi}{dr}\right)\right] = 0
\]

The solution of which is

\[
\psi = A_1 r^2 \ln r + B_1 r^2 + C_1 \ln r. \tag{3.52}
\]

In the above \( A_1, B_1, C_1 \) are arbitrary constants.

Using (3.52) in (3.8), (3.9) and (3.3), we find pressure distribution as

\[
P = -4\mu A_1 \theta + A_1 \left[8A_1 \ln r + 4(A_1^2 + 2A_1 B_1) + \frac{4A_1 C_1}{r^2}\right]
\]

\[
+ (3a_1 + 2a_2)\left[A_1 - \frac{C_1}{r^2}\right]^2 + \rho S(r)
\]

where

\[
S(r) = 2A_1 (\ln r)^2 (A_1 r^2 + C_1) + 21n (2A_1 B_1 r^2 + A_1 C_1 + 2B_1 C_1)
\]

\[
+ \frac{1}{2}(A_1^2 + 4B_1^2) r^2 - \frac{1}{2} \frac{C_1^2}{r^2}. \tag{3.53}
\]

(b) Motion where \( \hat{\psi}(r, z) = r^n F(z) \)

On substituting

\[
\hat{\psi} = r^n F(z) \tag{3.54}
\]
into equation (3.20), we find

\[ r^{2n-7} \left\{ 6(a_1 + a_2) \left( n(n-2)^2 \right) + \left( n-4 \right) \right\} \]

\[ + 2r^n \{ 2(2n-2)(n-4)F'F'' \} \]

\[ + 2(2n-2)(n-2)(2n-2)F'F'' \] \( \rho \{ n(n-2)F'F'' \} \}

\[ r^{2n-3} \{ nF'F'' \} \]

\[ - 2r^{2n-3} \{ nF'F'' \} \]

\[ - \mu \{ n(n-2)^2(n-4)F'F'' + 2n(n-2)r^{-3} + r^n \} + n(n-2)F'F'' \] \( = 0 \),

(3.55)

where dashes over \( F \) denotes the derivatives with respect to \( z \).

For \( n = 0 \), the equation (3.55) reduces to

\[ 16(a_1 + a_2)F'F'' + 2r^2 \{ - \rho F'F'' + a_1 F'F''' \}\]

\[ + (a_1 + a_2)(2F'F'' + F'F''') \} - \mu r^4 F'' = 0 \],

(3.56)

and which is satisfied provided

\[ (a_1 + a_2)F'F'' = 0, \]
\[-pF''P''+a_1FP^iv+(a_1+a_2)(2P''P'''+P^iv) = 0,\]

\[\mu F^iv = 0.\]

Since \((a_1+a_2) \neq 0, \mu \neq 0\), we find that

\[F(z) = A_2z + B_2\]  \hspace{1cm} (3.57)

satisfies all the conditions. The other quantities of interest are given as

\[\psi = A_2z + B_2, \quad u_1 = \frac{A_2}{r}, \quad w_1 = 0.\]  \hspace{1cm} (3.58)

Using (3.58) in (3.18), (3.19) and (3.13), the pressure distribution is given by

\[p = p_o - \frac{\rho A_2^2}{2r^2} + \frac{2(3a_1+2a_2)}{r^4} A_2,\]  \hspace{1cm} (3.59)

where \(A_2, B_2, p_o\) are arbitrary constants.

For \(n = 2\) we find that (3.55) reduces to

\[\mu F^iv+2pFP''-2a_1FP^V-2(a_1+a_2)[2P''P'''+P^iv] = 0.\]  \hspace{1cm} (3.60)

On integrating the above equation and setting the integrating constant to be zero we find

\[vF'''+2FP''P''^2 = \frac{a_1}{\rho}[2FP^iv+2P''^2] + \frac{a_2}{\rho}[2FP''+P''^2].\]  \hspace{1cm} (3.61)
where

\[ v = \frac{u}{\rho}. \]

It is not possible to integrate the above equation systematically further but we give below two particular solutions. The first solution is of the form

\[ F(z) = \frac{-Mu}{2(\phi - a_1 \beta^2)} (1 + ke^{\beta z}) \quad (3.62) \]

where

\[ \beta = \sqrt{\frac{\zeta}{4a_1 + 3a_2}}. \]

and \( k \) is a constant. For the second solution we note that left hand side of (3.61) is essentially the contribution because of the viscous part of the equation and a solution of which is (c.f. Berker [1963, 2])

\[ F = 2
\]

On substituting (3.63) on the right hand side of (3.61), we find that it is also satisfied provided \((7a_1 + 2a_2) = 0\). On assuming that a relation of this type holds, equation (3.63) then gives the other solution. For \( n = 2 \), we substitute (3.54) in (3.18), (3.19) and (3.13) and obtain the pressure distribution as
\[ p = -2\mu F' - 2\rho F'' + \frac{(3\alpha_1 + 2\alpha_2)}{2} \left[ 12F'F'' + r^2 F'''^2 \right] \]

\[ + \alpha_1 \left[ \frac{r^2}{2} F'''^2 + 4F''^2 \right] + 2(\alpha_1 + \alpha_2) F''^2 + P_o, \quad (3.64) \]

where \( F \) can be substituted in the above either from (3.62) or (3.63).

For \( n = 4 \) we find that (3.55) reduces to

\[ r^5 \left[ \alpha_1 \left( 4F F'' - 2F' F''' \right) + 2(\alpha_1 + \alpha_2) \left( 2F'' F' + F'' F^{iv} \right) \right] \]

\[ - \rho (2F'''' - 2F'F'') \]

\[ + r^3 \left[ \alpha_1 \left( 32F''' \right) + 2(\alpha_1 + \alpha_2) \left( 4F F'' \right) - \rho (32F') \right] \]

\[ - \mu F^{iv} - 16\mu F'' = 0. \quad (3.65) \]

It appears that the only solution that satisfies the above equation is the trivial solution \( F = \text{constant} \).

(c) Motion where \( E^2 \psi = f(r) \)

For viscous fluids the solution of this problem is given by Berkner [1963, 2]. On substituting

\[ E^2 \psi = f(r) \quad (3.66) \]

in (3.20) we get
\[ \rho r^2 \frac{\partial^2 \phi}{\partial z^2} [rf' - 2f] - \mu r^3 [rf'' - f'] \rho - \rho r \frac{\partial}{\partial z} [r^2 f']' \]

\[ -3rf'' + 3f' + 2(\alpha_1 + \alpha_2) \frac{\partial^2 \phi}{\partial z^2} [r^2 f' + 5rf' + 8f] \]

\[ + 4(\alpha_1 + \alpha_2) r^2 \frac{\partial^2 \phi}{\partial z^2} [rf' - 2f] = 0. \] \hspace{1cm} (3.67)

Since \[ E^2 \phi = f(r) \] corresponds to

\[ \hat{\phi} = (Ar^2 + B)z + F(r), \] \hspace{1cm} (3.68)

where \[ f(r) = (F' - \frac{F'}{r}) \] and A, B are arbitrary constants, we can, on setting

\[ G(r) = rf' - 2f \]

in (3.67) and using the above, replace (3.67) by

\[ (Ar^2 + B)r^2 G^{''} + [\mu r^2 - (Ar^2 + B)(5\alpha_1 + 2\alpha_2)] rg' \]

\[ + [8(\alpha_1 + \alpha_2)(Ar^2 + B) - \rho r^2 (Ar^2 + B)] G \]

\[ - 8(\alpha_1 + \alpha_2)Ar^2 G = 0. \] \hspace{1cm} (3.69)

If we set \[ A = 0 \] in (3.69) and use the substitution

\[ G(r) = \xi^2 \eta(\xi) \]

where \[ r^2 = \xi, \] we find that (3.69) reduces to
\[
\xi \eta'' + \left[ \frac{1}{2B_1} \left( 2a_1 - a_2 \right) \eta' + \left( \frac{4\mu - \rho B}{4a_1 B} \right) \eta \right] = 0. \quad (3.70)
\]

A further change of variable \( \eta(\xi) = \chi(\phi) \) reduces (3.70) to

\[
\phi \chi''(\phi) + \left[ - \frac{1}{a_1} a_2 - \phi \right] \chi'(\phi) - \left[ 2 - \frac{2B}{2\mu} \right] \chi(\phi) = 0.
\]  

(3.71)

The solution of (3.71), when \( \frac{2a_1 - a_2}{a_1} \neq n, n \) an integer, is given by

\[
\chi(\phi) = C_1 \chi_1(\phi) + C_2 \chi_2(\phi), \quad (3.72)
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants and

\[
\chi_1 = F_1 \left( \frac{4\mu - \rho B}{2\mu}, \frac{2a_1 - a_2}{a_1}, \phi \right)
\]

\[
\chi_2 = \phi^{\frac{2a_1 - a_2}{a_1}} F_1 \left( \frac{2\mu(a_1 + a_2) - \rho B a_1}{2\mu a_1}, \frac{a_2}{a_1}, \phi \right)
\]  

(3.73)

and where \( F_1 \) is confluent hypergeometric function. On substituting back the values of the different variables in terms of original variables we get

\[
\xi = C_1 \int r \left( \int r \left( \int f_1(r) \, dr \right) \, dr \right) \, dr + C_2 \int r \left( \int r \left( \int f_2(r) \, dr \right) \, dr \right) \, dr + D r^4 + E r^2 + B z, \quad (3.74)
\]

where \( D \) and \( E \) are integration constants and where
\[ f_1(r) = rF_1\left( \frac{4\mu - \sigma B}{2\mu}, \frac{2a_1 - a_2}{a_1}, \frac{-\mu r^2}{2a_1 B} \right) \]

\[ f_2(r) = r\left( \frac{-\mu r^2}{2a_1 B} \right)^{\frac{2a_1 - a_2}{a_1} \choose 2} \frac{2\mu a_2}{2\mu a_1} \]

\[ \frac{a_2}{a_1}, \frac{-\mu r^2}{2a_1 B} \]

We remark that the above solution is different from the corresponding solution for a viscous fluid. Using (3.68), for \( A = 0 \), in (3.18), (3.19) and (3.13), we obtain pressure distribution in terms of \( F \) as

\[ p = \tilde{h} - \frac{\partial}{r^2} (B^2 + F, F^2) + a_1 \left[ \frac{F''r}{r^2} - \frac{F'''}{r^3} + \frac{F'}{r^4} \right] \]

\[ + \frac{3a_1 + 2a_2}{4} \left[ \frac{8B^2}{r^4} + 2\left( \frac{F'}{r^4} + \frac{F''}{r} \right) \right], \quad (3.75) \]

where

\[ \tilde{h} = \int \left[ \frac{F''}{r^2} \left( \frac{1}{a_1} \frac{d^2}{dz^2} - \frac{1}{r} \frac{d}{dr} \right) \right] (F'' - \frac{F'}{r}) \]

\[ + \left( \frac{a_1 + a_2}{r^3} \right) (F'' - \frac{F'}{r})^2 \] dr

\[ + 2 \left( \frac{\sqrt{B}(F'' - \frac{F'}{r}) - \mu \frac{d}{dr}(F'' - \frac{F'}{r})} {r^2} \right) \] dr
\[-\frac{a_1 B}{r^2} \left\{ \left( \frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} \right) \left( r^2 F' - \frac{F'}{r} \right) \right\} \]

\[+ \frac{2(a_1 + a_2)}{r} \left( \frac{1}{r^2} \left( \frac{d}{dr} \left( r^2 F' \right) - \frac{F'}{r} \right) \right) \left\{ 1 + Q \right\} \]

and \( Q \) is an arbitrary constant.

(d) Motions where \( \psi(R, \sigma) = R^N F(\sigma) \)

We now consider solutions of (3.30) of the form

\[\psi(R, \sigma) = R^N F(\sigma). \tag{3.76}\]

Substituting (3.76) into equation (3.30) we find the relation

\[\mu H_1 + \rho R^{-1} \left[ nFG_1 \right] - \left( n-4 \right) F' G_1 = R^{-3} \left[ a_1 \left( nFH_1 - (n-6) F'H_1 \right) \right] \]

\[+ \frac{2(a_1 + a_2)}{r} \left( F'G + nFG_1 \right)'' \]

\[+ \left( 2n-6 \right) \left( 2n-5 \right) \left[ \frac{nFG_1}{(1-a^2)} \right] - \left( G_1 G' + (n-2) a^2 G_1 \right) \right\}, \tag{3.77}\]

where we have set

\[G(\sigma) = (1-a^2)F'' + n(n-1)F, \]

\[G_1(\sigma) = G(1-a^2)^{-1}, \]

\[H(\psi) = (1-a^2)G'' + (n-2)(n-3)G, \]

\[H_1(\psi) = H(1-a^2)^{-1}. \tag{3.78}\]
If we take \( n = 0 \), the relation (3.77) gives the following three differential equations:

\[
H_1 = 0, \quad F'G_1 = 0, \\
6a_1 F'H_1 + 2(a_1 + a_2) \{ (F'G)' + 30 F'G' \} \\
-G_1G' - 2aG^2 = 0. 
\] (3.79)

On using (3.78) in (3.79), for the case \( n = 0 \), we find that second equation in (3.79) reduce to

\[
F'F'' = 0. 
\]

That is

\[
F = A_3 \psi + B_3, 
\] (3.80)

where \( A_3 \) and \( B_3 \) are arbitrary constants, and first and third equations in (3.79) are satisfied identically.

The stream function and velocity components for this case are, respectively,

\[
\psi = A_3 \psi + B_3 \\
u_2 = \frac{A_3}{R^2}, \quad v_2 = 0. 
\] (3.81)

On making use of (3.81) in (3.26), (3.25)' and (3.23), we find that pressure distribution is given by
\[ p = C_3 - \frac{g A_2^2}{2 R^4} + \left(3a_1 + 2a_2 \right) \frac{6 A_3^2}{R^6} \]  

(3.82)

where, \( A_3, B_3, C_3 \) are arbitrary constants.

For \( n = 1 \), equation (3.77) has the form

\[
R^2 \left( \mu H_1 + \rho (F G_1' + 3 F' G_1') \right) = a_1 \left( F H_1' + 5 F' H_1 \right) \\
+ 2(a_1 + a_2) \left( (F' G + \sigma F G_1')'' + 12(F' G_1' + \frac{\sigma F G_1}{1 - \sigma^2}) \\
- (G' G_1 - \sigma G_1^2) \right),
\]  

(3.83)

where

\[
G = (1 - \sigma^2) F'', \quad G_1 = (1 - \sigma^2)^{-1} G, \\
H = (1 - \sigma^2) G'' + 2G, \quad H_1 = (1 - \sigma^2)^{-1} H.
\]  

(3.84)

Proceeding as before, (3.83) gives the following two differential equations

\[
\mu H_1 + \rho (F G_1' + 3 F' G_1') = 0,
\]  

(3.85)

\[
a_1 \left( F H_1' + 5 F' H_1 \right) + 2(a_1 + a_2) \left( (F' G + \sigma F G_1')'' + 12(F' G_1' + \frac{\sigma F G_1}{1 - \sigma^2}) \\
- (G' G_1 - \sigma G_1^2) \right) = 0.
\]  

(3.86)

We remark that equation (3.85) is also obtained in the case when the fluid is viscous. If we substitute (3.84):
in (3.85) and then integrate it thrice with respect to \( \sigma \), we obtain

\[ 2\mu(1-\sigma^2)F' + 4\mu\sigma F + \rho F^2 = A_4 \sigma^2 + B_4 \sigma + C_4, \quad (3.87) \]

where \( A_4, B_4, C_4 \) are arbitrary constants. Squire [1951,1] has considered various special cases of (3.87) by assigning different values to \( A_4, B_4 \) and \( C_4 \). We shall here consider the most popular case which was also studied by Landau [1959, 1], independently.

On setting \( A_4 = B_4 = C_4 = 0 \), equation (3.87) admits a solution of the form

\[ F = \frac{2\mu(1-\sigma^2)}{(\sigma-a)}, \quad (3.88) \]

where \( a \) is an arbitrary constant. Substitution of (3.88) in (3.86) yields, after considerable simplification,

\[
96\mu^2 \left[ \frac{1}{(\sigma-a)^2} - \frac{2\sigma}{(\sigma-a)^3} + \frac{(1-\sigma^2)}{(\sigma-a)^4} \right] \left[ a_1 \left( -\frac{1+\sigma^2}{\sigma-a} \right)
\right.

\[ + \frac{2a(1-\sigma^2)}{(\sigma-a)^2} + \frac{10(1-\sigma^2)^2}{(\sigma-a)^3} + (a_1 + a_2) \left( 2\sigma + \frac{(1+\sigma^2)}{(\sigma-a)} \right)
\]

\[ + \frac{7a(1-\sigma^2)}{(\sigma-a)^2} + \frac{4(1-\sigma^2)^2}{(\sigma-a)^3} \right] = 0. \quad (3.89) \]

If we set \( a = 1 \), the equation (3.89) is identically satisfied, and the solution (3.88) becomes
\[ F_1 = -2v(1+\sigma) \quad \text{for} \quad a = +1, \]
\[ F_2 = 2v(1-\sigma) \quad \text{for} \quad a = -1. \]  \hspace{1cm} (3.90)

Also, if we take \( a = 0 \), then equation (3.89) is, again satisfied provided \((7a_1+2a_2) = 0\). In this case solution (3.88) has the form
\[ F_3 = \frac{2v}{\sigma} (1-\sigma^2) \]  \hspace{1cm} (3.91)

Using (3.90), (3.91) in (3.76) with \( n = 1 \), we find the stream function in these three cases are given, respectively, as
\[ \overline{\psi} = R[-2v(1+\sigma)], \quad \overline{\psi} = R[2v(1-\sigma)], \quad \overline{\psi} = \frac{2v}{\sigma} (1-\sigma^2)R. \]  \hspace{1cm} (3.92)

The velocity components and vorticity in these three cases, by making use of (3.92) in (3.27) and in (3.22), are obtained, respectively, as
\[ u_2 = -\frac{2v}{R}, \quad v_2 = \frac{2v(1+\sigma)}{R\sqrt{1-\sigma^2}}, \quad \overline{\omega} = 0, \]
\[ u_2 = -\frac{2v}{R}, \quad v_2 = \frac{2v(1-\sigma)}{R\sqrt{1-\sigma^2}}, \quad \overline{\omega} = 0, \]
\[ u_2 = -\frac{2v}{R} \frac{(1+\sigma^2)}{\sigma^2}, \quad v_2 = \frac{2v}{R} \frac{\sqrt{1-\sigma^2}}{\sigma}, \quad \overline{\omega} = \frac{4v}{R^2} \frac{\sqrt{1-\sigma^2}}{\sigma^3}. \]  \hspace{1cm} (3.93)
Employing (3.92)\textsubscript{1} and (3.92)\textsubscript{2} in (3.25), (3.26) and (3.23), we find the pressure distribution, respectively, as

\[
p = C_0 - \frac{4\mu v}{R^2(1-\sigma)} + \frac{8(3\alpha_1 + 2\alpha_2)}{R^4} v^2 \frac{(2-\sigma)}{(1-\sigma)^2},
\]

\[
p = C_0 - \frac{4\mu v}{R^2(1+\sigma)} + \frac{8(3\alpha_1 + 2\alpha_2)}{R^4} v^2 \frac{(2+\sigma)}{(1+\sigma)^2}.
\]

(3.94)

For (3.91)\textsubscript{3}, the pressure, after using the condition \((7\alpha_1 + 2\alpha_2) = 0\), can be easily found to be

\[
p = \frac{\bar{h}}{v} + \frac{8v^2\alpha_1}{R^4} \frac{(1+\sigma^2)}{\sigma^6} - \frac{2\mu v}{R^2} \frac{(1+3\sigma^2)}{\sigma^4} \frac{1}{\sigma^4} \frac{4(3\alpha_1 + 2\alpha_2)}{R^4} v^2 \frac{(1+\sigma^2)}{\sigma^4} + \frac{(1+\frac{1}{\sigma})^2}{\sigma^6} + \frac{2(1-\sigma^2)(1+\sigma^2)^2}{\sigma^6},
\]

(3.95)

where

\[
\bar{h} = \frac{2\mu v}{R^2} \frac{(1+\sigma^2)}{\sigma^4} - \frac{4\alpha_1 v^2}{R^4} \frac{(4+\sigma^2)}{\sigma^6} + C_5,
\]

and where \(C_0\), \(C_5\) are arbitrary constants.

On assuming \(n = 2\), (3.77) reduces to

\[
\mu H_1 + 2\bar{p}R(FG_1)' = R^{-1} [\alpha_1 (2FH_1' + 4F'H_1) + 2(\alpha_1 + \alpha_2)
\]

\[
((FG + 2\sigma FG_1)'' + 2(F'G_1 + \frac{2\sigma FG_1}{1-\sigma^2}) - G_1 G')
\]

(3.96)

where now
\[ G = (1 - \sigma^2)F'' + 2F, \quad G_1 = (1 - \sigma^2)^{-1}G, \]

\[ H = (1 - \sigma^2)G'', \quad H_1 = (1 - \sigma^2)^{-1}H. \quad (3.97) \]

The above relation (3.96) gives rise to the differential equations

\[ H_1 = 0, \quad (FG_1)' = 0 \]

\[ u_1'(2FH_1 + 4F'H_1) + 2\left(\alpha_1 + \alpha_2\right) \left\{ (F'G + 2\sigma FG_1)' \right\} \]

\[ + 2(F'G_1 + \frac{2\sigma FG_1}{(1 - \sigma^2)}) - G_1 G_1' = 0. \quad (3.98) \]

On integrating (3.98) and setting the integration constant to be zero we find the solution as

\[ F = A_0 (1 - \sigma^2) + B_o \sum_{n=0}^{\infty} \frac{\sigma^{2n+1}}{(2n-1)(2n+1)} \quad (3.99) \]

The above solution satisfies the remaining equations in (3.98) provided \( B_o \). Hence, for \( n = 2 \) case, the solution is

\[ F = A_0 (1 - \sigma^2), \quad (3.100) \]

where \( A_o \) is an arbitrary constant. The other quantities of interest are

\[ \psi = A_0 R^2 (1 - \sigma^2) \]

\[ w_2 = -2A_o \sigma, \quad v_2 = 2A_o \sqrt{1 - \sigma^2} \]
\[ p = p_0 - 2\rho A_0^2. \]  \hspace{1cm} (3.101)

Finally, on setting \( n = 4 \), (3.77) takes the form

\[
\mu H_1 + 4\rho R^3 F_1 = R[a_1 (4FH_1 + 2F'H_1) + 2(a_1 + a_2)

\[(F'G + 4\sigma FG_1)^" + 6(F'G_1 + \frac{4\sigma FG_1}{1 - \sigma^2}) - (G_1 G' + 2\sigma G_1^2)]
\]

\hspace{1cm} (3.102)

where

\[ G = (1 - \sigma^2)F" + 12F, \quad G_1 = (1 - \sigma^2)^{-1}G, \]

\[ H = (1 - \sigma^2)G" + 2G \quad H_1 = (1 - \sigma^2)^{-1}H. \] \hspace{1cm} (3.103)

Thus, we have the following three differential equations

\[ H_1 = 0, \quad G_1 = 0, \]

\[ a_1 (4FH_1 + 2F'H_1) + 2(a_1 + a_2) \{(F'G + 4\sigma FG_1)^" + 6(F'G_1 + \frac{4\sigma FG_1}{1 - \sigma^2}) - (G_1 G' + 2\sigma G_1^2)\} = 0. \] \hspace{1cm} (3.104)

The second equation in (3.104) gives

\[ G = A_6 (1 - \sigma^2) \] \hspace{1cm} (3.105)

where \( A_6 \) is an arbitrary constant and \( H_1 = 0 \) is satisfied identically. On employing (3.105) in (3.103)\(_i\), we find
that \( P \) must satisfy the equation

\[
(1-\sigma^2)F'' + 12F = A_6(1-\sigma^2) \tag{3.106}
\]

This is the same equation as that obtained in the viscous fluid case. Accordingly, on adopting the solution given in Berker [1963, 2] we have

\[
F = D_1(1-\sigma^2) + D_2(1-\sigma^2)(1-5\sigma^2) + D_3[(1-\sigma^2)(3-15\sigma^2)\ln(\frac{1+\sigma}{1-\sigma}) + 30\sigma^3 - 26\sigma], \tag{3.107}
\]

where \( D_1, D_2 \) and \( D_3 \) are arbitrary constants. In order to avoid singularity, we assume that \( D_3 = 0 \). Thus, the preceding solution reduces to

\[
F = (1-\sigma^2)(D_1 + D_2(1-5\sigma^2)).
\]

On substituting this solution in the third equation of (3.104) we note that the later is satisfied provided \( D_2 = -D_1 \) and in the case

\[
F = k_1\sigma^2(1-\sigma^2), \text{ where } k_1 = 5D_1 \tag{3.108}
\]

the stream function and velocity components are given by

\[
\dot{\psi} = k_1\sigma^2(1-\sigma^2)R^4,
\]

\[
u_2 = k_1(2\sigma-4\sigma^3)R^2, \quad v_2 = 4k_1\sigma^2\sqrt{1-\sigma^2}R^2. \tag{3.109}
\]
The vorticity and pressure functions turn out to be

\[ \Omega = 2k_1 \sqrt{1-\sigma^2} \cdot R. \]
\[ p = p_0 - 2\sigma R^2 k_1^2(5\sigma^2-1) + (3\sigma_1+2\sigma_2)(16\sigma^2(1-2\sigma^2))^2 \]
\[ + 4\sigma^2 + (8\sigma^3-6\sigma^2)^2 + 2(1-\sigma^2)(8\sigma^2-1)^2 \]

(3.110)

3.4 Discussion

Dunn and Fosdick [1974] considered the thermodynamics and stability of the fluids of second grade and found that, in order to be compatible with thermodynamics and to have free energy take its minimum value in equilibrium, the material constants must satisfy

\[ \mu \geq 0, \quad \sigma_1 \geq 0, \quad \sigma_1 + \sigma_2 = 0. \]

If we set \( \sigma_1 + \sigma_2 = 0 \) in the compatibility equations in different coordinate systems, we observe that all the universal solutions obtained in section 3.3 are also the solutions in the present case. Furthermore, the solutions (3.90), are also valid. However, the following solutions do not hold if the restriction \( \sigma_1 + \sigma_2 = 0 \) is imposed upon them:
(i) \( F(z) = \frac{-\mu \beta}{2(\rho - a_1 \beta^2)} (1 + ka \beta z) \),

where

\( \beta = \sqrt{\rho (4a_1 + 3a_2)^{-1}} \) [c.f. (3.62)]

(ii) \( F = 2v z^{-1} \), provided \( 7a_1 + 2a_2 = 0 \)

[c.f. (3.63)]

(iii) \( F = \frac{2v}{a}(1 - c^2) \), provided \( 7a_1 + 2a_2 = 0 \).

[c.f. (3.96)]
CHAPTER IV

PLANE STEADY FLOWS OF A THIRD GRADE FLUID

4.1 Introduction

Martin [1971, 1] introduced a new method to study the steady plane flows of a viscous fluid, in the absence of external forces. In his work he introduced a curvilinear coordinates system $\phi$, $\psi$ in the plane of flow in which the coordinate lines $\psi=$constant are stream lines of the flow and the coordinate lines $\phi=$ constant are at first, left arbitrary. In this approach, the coefficients $E$, $F$, and $G$ of the first fundamental form of the metric are taken as unknown functions. As a result partial differential equations in terms of $E$, $F$, $G$, the vorticity, etc., are obtained as functions of $\phi$ and $\psi$. These equations are solved by selecting the coordinate lines $\phi=$ constant to be orthogonal trajectories of the streamlines ($F=0$). Martin also gave applications of this approach and obtained several useful results.

Numerous studies have been made on the basis of Martin's approach. The extensive use of this approach has been made in MHD. Magnetohydrodynamic flows when the velocity and magnetic fields are aligned, constantly inclined, or variably inclined have been studied by Chandna and Garg [1976, 1], Garg and Chandna [1976, 2], Chandna, Barron and
Garg [1979, 2], Chandna, Barron and Chew [1983, 2]. The same approach has also been employed by Chandna and Kaloni [1976, 3] to study the plane flow of a Cosserat fluid and Kaloni and Siddiqui [1983, 1] to discuss the steady plane flows of a second grade fluid. Recently, Grossman [1986, 1] has used the above technique to study numerical solutions of the inviscid flow problems.

It may be noted that in the third grade fluid, apart from the work of Rajagopal [1980, 3, 4] so far, very little work has been done.

In this work, we also generalize Martin's [1971, 1] approach to study the steady plane flows of a third grade fluid. As pointed out in Chapter I, that in non-Newtonian fluids not only nonlinearities are increased considerably but, in certain cases, the order of the differential equation is also increased. In general, it turns out that governing flow equations for a third grade fluid become of fourth order. We point out that Martin's approach essentially leads to an inverse method in the sense that one places certain conditions on E, F and G a priori and then determines the flows which have this property.

The plan of this Chapter is as follows: In Section 4.2 we specialize the general equations of a third grade fluid for plane steady flows in a suitable form. In Section 4.3, we employ some results from differential geometry to recast these equations in \((\phi, \psi)\) coordinates. In this section we develop and generalize different forms...
of equations originally proposed by Martin for viscous fluids. In Section 4.4 we give illustrations of the method by considering the coordinate lines $\phi = $ constant to be orthogonal trajectories of the streamlines. In Section 4.5 we consider more specific examples to illustrate the usefulness of the method. In one case we find that results of third grade fluid are the same as that of a second grade fluid (cf., Chapter II and III) while in other cases results obtained are different from the corresponding results of the viscous and the second grade fluids.

4.2 Equations of Motion

An incompressible, homogeneous fluid of third grade is characterized by a Cauchy stress tensor $T$ in the following (cf. (1.17)) form

$$ T = -p I + \mu A_1 + a_1 A_2 + a_2 A_1^2 + B_1 A_3 + B_2 (A_1 A_2 + A_2 A_1) $$

$$ + B_3 (\text{tr} A_1^2) A_1. $$

(4.1)

Here, $A_1, A_2, A_3$ are first three Rivlin-Ericksen tensors defined in Chapter I, $\mu$ is coefficient of viscosity and $a_1, a_2, B_1, B_2$ and $B_3$ are material constants. $p$ is the indeterminate part of stress due to the constraint of incompressibility.

Fosdick and Rajagopal [1980, 1] have shown that if the fluid of third grade are to undergo motions which are compatible with Clausius-Duhem inequality and the assump-
tion that free energy density of the fluid is locally at
rest, then the material constants in (4.1) must meet the
following restrictions:

\[ \mu \geq 0, \quad a_1 \geq 0, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geq 0, \]

\[ -\sqrt{24\mu \beta_3} \leq (a_1 + a_2) \leq \sqrt{24\mu \beta_3} \]

(4.2)

We shall be concerned with the fluid of third grade repre-
sented by (4.1) and in which the material constants satisfy
(4.2). Hence (4.1) reduces to

\[ T = -\pi I + a_1 A_1^2 + a_2 A_1^2 + \beta_3 (\text{tr} A_1^2) A_1 \]

(4.3)

On substituting (4.3) into the balance of linear momentum
(1.19) we obtain,

\[ \rho \ddot{v} = \sigma f - \text{grad} \ p + \mu \text{div} \ A_1 + a_1 [\text{div} \ A_1] + (\text{grad} \ \text{div} \ A_1) v + (\text{grad} \ v)^T \text{div} \ A_1 + A_1 (\text{grad} \ (\text{grad} \ v)^T) + (a_1 + a_2) \text{div} A_1^2 \]

\[ + \beta_3 [A_1 \ \text{grad} \ (|A_1|^2) + |A_1|^2 \text{div} A_1] \]

(4.4)

For plane steady flow, in the absence of body forces,
the equations (1.20) and (4.4) reduce to
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.5)

\rho \left( \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} \right) + \frac{\partial p}{\partial x} = \mu \left[ \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2} \right] + \frac{1}{4} (3a_1 + 2a_2) \left( \frac{\partial}{\partial x} \right)^2 |A_1|^2 + \frac{1}{2} \left( \frac{\partial}{\partial y} \right)^2 |A_1|^2 - \frac{\nu}{2} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} \right)^2 |A_1|^2 \quad (4.6)

\rho \left( \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} \right) + \frac{\partial p}{\partial y} = \mu \left[ \frac{\partial^2 v}{\partial x^2} + \nu \frac{\partial^2 v}{\partial y^2} \right] + \frac{1}{4} (3a_1 + 2a_2) \left( \frac{\partial}{\partial y} \right)^2 |A_1|^2 + \frac{1}{2} \left( \frac{\partial}{\partial x} \right)^2 |A_1|^2 - \frac{\nu}{2} \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \right)^2 |A_1|^2 \quad (4.7)

\text{where}

\nu^2 = \frac{\nu}{\partial x^2} + \frac{\nu}{\partial y^2}

We introduce vorticity function and generalized pressure, respectively, as

\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (4.8)

h = \frac{a_1 + 2a_2}{4} \omega \quad (4.9)

\text{where}
\[ q^2 = u^2 + v^2 \]

\[ M = \text{det} \mathbf{A}_1 \mathbf{A}_1^T = |\mathbf{A}_1|^2 = (4\frac{\partial u}{\partial x})^2 + 4(\frac{\partial v}{\partial y})^2 + 2(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2 \]

\[ (4.10) \]

When (4.8) and (4.9) are employed in (4.6) and (4.7), we find that equations (4.5)-(4.7) are replaced by

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (4.11) \]

\[ \frac{\partial h}{\partial x} + v(-\omega + a_1 v^2 \omega) + u \frac{\partial \omega}{\partial y} + b_3 \frac{\partial}{\partial y}(\omega M) \]

\[ - 2 \left( \left( \frac{\partial u}{\partial x} \frac{\partial}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial}{\partial y} \right) M \right) = 0, \quad (4.12) \]

\[ \frac{\partial h}{\partial y} - u(-\omega + a_1 v^2 \omega) - \frac{\partial \omega}{\partial x} - b_3 \frac{\partial}{\partial x}(\omega M) \]

\[ + 2 \left( \left( \frac{\partial u}{\partial y} \frac{\partial}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial}{\partial y} \right) M \right) = 0. \quad (4.13) \]

With the understanding that (4.8) and (4.10) imply, the equations (4.11) to (4.13) are three partial differential equations for three unknown functions \( u, v \) and \( h \) of \( (x,y) \). Having determined these, the pressure is calculated by using (4.9).

Equation (4.11) implies the existence of the stream function \( \psi(x,y) \) such that
\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (4.14) \]

In place of the rectangular coordinates \( x, y \) in the physical plane as independent variables, we introduce curvilinear coordinates \( \phi, \psi \) in which the curve \( \psi(x,y) = \) constant are streamlines and the curve \( \phi = \) constant are arbitrary family of curves which, with streamlines \( \psi(x,y) = \) constant, generate a curvilinear net \((\phi, \psi)\) in the physical plane. In other words, instead of seeking \( u, v, \omega, M \) and \( h \) as functions of \((x,y)\), we seek \( x, y, u, v, \omega, M \) and \( h \) as functions of \((\phi, \psi)\). Specifically, we seek,

\[
\begin{align*}
  x &= x(\phi, \psi), \quad y = y(\phi, \psi), \quad M = M(\phi, \psi) \\
  u &= u(\phi, \psi), \quad v = v(\phi, \psi), \quad h = h(\phi, \psi) \\
  \omega &= \omega(\phi, \psi)
\end{align*}
\]

Let

\[ x = x(\phi, \psi), \quad y = y(\phi, \psi), \quad (4.15) \]

define a system of curvilinear coordinates in the \((x,y)\) plane. With \((\phi, \psi)\) as curvilinear coordinates, the squared element of arc length along any curve is given by

\[ ds^2 = E d\phi^2 + 2F d\phi d\psi + G d\psi^2, \quad (4.16) \]

where

\[ E = \left( \frac{\partial x}{\partial \phi} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2, \]

and

\[ G = \left( \frac{\partial x}{\partial \psi} \right)^2 + \left( \frac{\partial y}{\partial \psi} \right)^2. \]
\[ F = \frac{\partial x}{\partial \psi} \frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \psi} \frac{\partial y}{\partial \psi}, \]
\[ G = \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2. \]  

(4.17)

Equation (4.15) can be solved to determine 
\[ \phi = \phi(x,y), \quad \psi = \psi(x,y), \]

such that 
\[ \frac{\partial x}{\partial \phi} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial x}{\partial \psi} = -\frac{\partial \phi}{\partial y} \]
\[ \frac{\partial y}{\partial \phi} = -\frac{\partial \psi}{\partial x}, \quad \frac{\partial y}{\partial \psi} = \frac{\partial \phi}{\partial x}. \]  

(4.18)

where \( 0 < |J| < \infty \), and by (4.17)
\[ J = \frac{\partial x}{\partial \psi} \frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial y}{\partial \psi} = \sqrt{E-G-P^2} = \pm W \]  

(4.19)

is the transformation Jacobian.

Let \( \gamma \) be the angle made by the tangent to the coordinate lines \( \phi = \) constant, directed in the sense of increasing \( \phi \), with the x-axis. Then we have
\[ \frac{\partial x}{\partial \phi} = \sqrt{E} \cos \gamma, \quad \frac{\partial y}{\partial \phi} = \sqrt{E} \sin \gamma, \]  

(4.20)

\[ \frac{\partial x}{\partial \psi} = \frac{E}{\sqrt{E}} \cos \gamma - \frac{J}{\sqrt{E}} \sin \gamma, \quad \frac{\partial y}{\partial \psi} = \frac{E}{\sqrt{E}} \sin \gamma + \frac{J}{\sqrt{E}} \cos \gamma, \]
\[ \frac{\partial \gamma}{\partial \phi} = \frac{J}{E} \frac{2}{11} \quad \frac{\partial \gamma}{\partial \psi} = \frac{J}{E} \frac{2}{12}. \]  

(4.21)

The three functions, \( E, F, G \) of \((\phi, \psi)\) must satisfy the Gauss equation
\[
\frac{3}{\psi} \left( \frac{W}{E} \Gamma_{11}^2 \right) - \frac{3}{\phi} \left( \frac{W}{E} \Gamma_{12}^2 \right) = 0 \tag{4.22}
\]

where

\[
\Gamma_{11}^2 = \frac{1}{2W^2} \left[ -F \frac{3}{\phi} E + 2E \frac{3}{\phi} F - E \frac{3}{\phi} \psi \right],
\]

\[
\Gamma_{12}^2 = \frac{1}{2W^2} \left[ E \frac{3}{\phi} G - F \frac{3}{\phi} \psi \right]. \tag{4.23}
\]

For the later use we also record (c.f. Martin [1971,1]).

\[
\frac{3G}{\phi} = \frac{F \Gamma_{11}^2 - E \Gamma_{12}^2}{\sqrt{E} W}, \quad \frac{3g}{\psi} = \frac{F \Gamma_{12}^2 - E \Gamma_{22}^2}{\sqrt{E} W}, \tag{4.24}
\]

\[
\psi^2 = \frac{1}{2} \left[ \frac{3}{\phi} \left( \frac{G_f - F_f}{W} \right) + \frac{3}{\psi} \left( \frac{-F_f + E_f}{W} \right) \right] \tag{4.25}
\]

and where \( q \) is the speed of the flow, and

\[
\Gamma_{22}^2 = \frac{1}{2W^2} \left[ E \frac{3}{\phi} G - 2F \frac{3}{\phi} F + F \frac{3}{\phi} G \right]. \tag{4.26}
\]

Having recorded the above results, we now take up three equations (4.11)-(4.13), along with (4.8)-(4.10) and develop the flow equations in a new form in the new variables \( \phi, \psi \).

In the following work, we assume that the fluid flows towards higher parameter values of \( \phi \) so that \( J = \psi > 0 \).

4.3 Variant Forms of Equations of Motion

We recall that the constraint of incompressibility or equation of continuity (4.11) and the vorticity equation (4.8) have the same form as in the theory of viscous fluid.
Accordingly, we reproduce below the new forms of these equations obtained by Martin [1971, 1].

The necessary and sufficient condition for the flow of a fluid along the coordinate lines $\psi = \text{constant}$ of a curvilinear coordinate system (4.15) with $ds^2$ given by (4.16) to satisfy the principle of conservation of mass is

$$Jq = \sqrt{E}, \quad u + iv = \frac{\sqrt{E} \psi}{J} i\psi, \quad (4.27)$$

where $i = \sqrt{-1}$. Similarly the vorticity equation can be expressed as

$$\omega = \frac{1}{J} \left[ \frac{3}{J} \frac{\partial (E)}{\partial \psi} - \frac{3}{J} \frac{\partial (E)}{\partial \psi} \right]. \quad (4.28)$$

We further recall that the equations (4.9) and (4.10), i.e., the expressions for $h$ and $M$ respectively, have the same form as in the theory of second grade fluid. Thus, we reproduce here, the new form of these equations obtained by Kaloni and Siddiqui [1983, 1].

The expression for $M$, as defined in (4.10) may be transformed in $(\psi, \phi)$ coordinates as

$$M = \left[ 2\omega^2 + \frac{3}{J} \left\{ \Gamma_{11} \frac{3}{J^2} \psi \left( \frac{\partial (E)}{J} \right) - \Gamma_{12} \frac{3}{J^2} \phi \left( \frac{\partial (E)}{J} \right) \right\} \right].$$

However, the above expression may be rewritten as

$$M = \left[ 2\omega^2 + \frac{4}{E} \left\{ \Gamma_{11} \frac{3}{J^2} \left( \frac{E}{J} \right) - \Gamma_{12} \frac{3}{J^2} \left( \frac{E}{J} \right) \right\} \right]. \quad (4.29)$$
Similarly the expression for \( h \), as given in (4.9), can be expressed as

\[
h = \frac{a}{2} \frac{E}{J^2} \left[ \frac{\alpha}{J} \left( \frac{\partial}{\partial \psi} \frac{\alpha}{\partial \psi} \right) - \frac{\alpha}{J} \left( \frac{\partial}{\partial \phi} \frac{\alpha}{\partial \phi} \right) \right] + p
\]

\[
- \frac{1}{4} (3a_1 + 2a_2) [2 \omega^2 + \frac{4}{E} \left( r_{12}^2 \frac{\partial}{\partial \psi} \left( \frac{E}{J^2} \right) - r_{12}^2 \frac{\partial}{\partial \phi} \left( \frac{E}{J^2} \right) \right) ]
\]

(4.30)

**Momentum Equations.** On employing (4.14) in the linear momentum equation (4.12) and (4.13) we have

\[
\frac{\partial h}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial h}{\partial \psi} \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \phi} (-\rho + a_1 v^2 \omega) - \mu \left[ \frac{\partial \omega}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial \omega}{\partial \psi} \frac{\partial \psi}{\partial y} \right]
\]

\[
- \beta \frac{3}{3 \phi} \left( \frac{\partial}{\partial \phi} (Mu) \frac{\partial \phi}{\partial y} \right) + \frac{3}{3 \psi} \left( \frac{\partial}{\partial \psi} (Mu) \frac{\partial \psi}{\partial y} \right)
\]

\[
+ 2 \beta \left[ \frac{\partial u}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{3 u}{3 \phi} \frac{\partial \phi}{\partial x} \right] \left( \frac{\partial \phi}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{3 \phi}{3 \phi} \frac{\partial \phi}{\partial x} \right) M
\]

\[
+ \frac{3 v}{3 \phi} \frac{\partial \phi}{\partial x} + \frac{3 v}{3 \psi} \frac{\partial \psi}{\partial x} \left( \frac{\partial \phi}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{3 \phi}{3 \phi} \frac{\partial \phi}{\partial y} \right) M
\]

(4.31)

\[
\frac{\partial h}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial h}{\partial \psi} \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \phi} (-\rho + a_1 v^2 \omega) + \mu \left[ \frac{\partial \omega}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial \omega}{\partial \psi} \frac{\partial \psi}{\partial x} \right]
\]

\[
+ \beta \left[ \frac{3}{3 \phi} (Mu) \frac{\partial \phi}{\partial x} + \frac{3}{3 \psi} (Mu) \frac{\partial \psi}{\partial x} \right]
\]
\[ + 2\delta_{3}\left[ \left( \frac{\partial u}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial u}{\partial \psi} \frac{\partial \psi}{\partial y} \right) \left( \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial \phi} + \frac{\partial \psi}{\partial \phi} \frac{\partial \phi}{\partial \psi} \right) M \right] \]

\[ + \left( \frac{\partial v}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial v}{\partial \psi} \frac{\partial \psi}{\partial y} \right) \left( \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial \psi} + \frac{\partial \phi}{\partial \psi} \frac{\partial \psi}{\partial \phi} \right) M \right]. \quad (4.32) \]

Making use of transformation equations (4.18) in the above equations we get

\[ \frac{3h}{3\phi} \frac{3y}{3\psi} - \frac{3h}{3\phi} \frac{3y}{3\psi} = \frac{3y}{3\phi}(-\rho \omega + \alpha_1 v^2 \omega) - \mu \left[ \frac{3\omega}{3\phi} \frac{3x}{3\phi} - \frac{3\omega}{3\phi} \frac{3x}{3\phi} \right] \]

\[ - \delta_{3} \left[ \frac{3}{3\psi} (M\omega) \frac{3x}{3\phi} - \frac{3}{3\phi} (M\omega) \frac{3x}{3\phi} \right] \]

\[ + \frac{2\delta_{3}}{3} \left( \frac{3y}{3\phi} \frac{3u}{3\psi} - \frac{3y}{3\phi} \frac{3u}{3\psi} \right) \left( \frac{3x}{3\phi} \frac{3x}{3\phi} - \frac{3x}{3\phi} \frac{3x}{3\phi} \right) \]

\[ + \left( \frac{3y}{3\psi} \frac{3v}{3\phi} - \frac{3y}{3\psi} \frac{3v}{3\phi} \right) \left( \frac{3x}{3\phi} \frac{3x}{3\phi} - \frac{3x}{3\phi} \frac{3x}{3\phi} \right) \right] \quad (4.33) \]

\[ \frac{3h}{3\psi} \frac{3x}{3\phi} - \frac{3h}{3\phi} \frac{3x}{3\phi} = \frac{3x}{3\psi}(-\rho \omega + \alpha_1 v^2 \omega) + \mu \left[ \frac{3\omega}{3\phi} \frac{3y}{3\phi} - \frac{3\omega}{3\phi} \frac{3y}{3\phi} \right] \]

\[ + \delta_{3} \left[ \frac{3}{3\phi} (M\omega) \frac{3y}{3\phi} - \frac{3}{3\phi} (M\omega) \frac{3y}{3\phi} \right] \]

\[ + \frac{2\delta_{3}}{3} \left( \frac{3x}{3\phi} \frac{3u}{3\psi} - \frac{3x}{3\phi} \frac{3u}{3\phi} \right) \left( \frac{3y}{3\phi} \frac{3y}{3\phi} - \frac{3y}{3\phi} \frac{3y}{3\phi} \right) \]

\[ + \left( \frac{3x}{3\phi} \frac{3v}{3\phi} - \frac{3x}{3\phi} \frac{3v}{3\phi} \right) \left( \frac{3y}{3\phi} \frac{3x}{3\phi} - \frac{3y}{3\phi} \frac{3x}{3\phi} \right) \right] \quad (4.34) \]

If we now multiply (4.33) by $\frac{3v}{3\phi}$ and (4.34) by $\frac{3x}{3\phi}$ and subtract we get one equation. Similarly, if we multiply (4.33) by $\frac{3x}{3\phi}$ and (4.34) by $\frac{3x}{3\phi}$ and subtract we get another equation. The equation thus obtained, respectively, are
\[
\begin{align*}
\frac{\partial h}{\partial \phi} - \frac{\partial h}{\partial \psi} &= -E(-\omega + a_1 v^2) - uJ\frac{\partial \omega}{\partial \psi} - \beta_3 J\frac{\partial}{\partial \psi}(M\omega) \\
&+ \frac{2\beta}{J} \left[ \left( \frac{\partial u}{\partial \phi} - \frac{\partial u}{\partial \psi} \right) \left( \frac{\partial v}{\partial \psi} \frac{\partial}{\partial \phi} - \frac{\partial v}{\partial \phi} \frac{\partial}{\partial \psi} \right) \right] M \\
&+ \left( \frac{\partial v}{\partial \phi} - \frac{\partial v}{\partial \psi} \right) \left( \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \phi} \right) M \\
&+ \left( \frac{\partial y}{\partial \phi} - \frac{\partial y}{\partial \psi} \right) \left( \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \phi} \right) M \\
&+ \left( \frac{\partial y}{\partial \phi} - \frac{\partial y}{\partial \psi} \right) \left[ \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \phi} \right] M \\
\end{align*}
\]
(4.35)

\[
\begin{align*}
\frac{\partial h}{\partial \phi} - \frac{\partial h}{\partial \psi} &= -E(-\omega + a_1 v^2) - uJ\frac{\partial \omega}{\partial \psi} - \beta_3 J\frac{\partial}{\partial \psi}(M\omega) \\
&+ \frac{2\beta}{J} \left[ \left( \frac{\partial u}{\partial \phi} - \frac{\partial u}{\partial \psi} \right) \left( \frac{\partial v}{\partial \psi} \frac{\partial}{\partial \phi} - \frac{\partial v}{\partial \phi} \frac{\partial}{\partial \psi} \right) \right] M \\
&+ \left( \frac{\partial v}{\partial \phi} - \frac{\partial v}{\partial \psi} \right) \left( \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \phi} \right) M \\
&+ \left( \frac{\partial y}{\partial \phi} - \frac{\partial y}{\partial \psi} \right) \left( \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \phi} \right) M \\
&+ \left( \frac{\partial y}{\partial \phi} - \frac{\partial y}{\partial \psi} \right) \left[ \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \phi} \right] M \\
\end{align*}
\]
(4.36)

We now use (4.20) to eliminate \( \frac{\partial x}{\partial \phi}, \frac{\partial x}{\partial \psi}, \frac{\partial y}{\partial \phi}, \frac{\partial y}{\partial \psi}, \) and (4.27) to eliminate \( \frac{\partial u}{\partial \phi}, \frac{\partial u}{\partial \psi}, \frac{\partial v}{\partial \phi}, \frac{\partial v}{\partial \psi}, \) in the above equations, we find

\[
\begin{align*}
\frac{\partial h}{\partial \phi} - \frac{\partial h}{\partial \psi} &= -E(-\omega + a_1 v^2) - uJ\frac{\partial \omega}{\partial \psi} - \beta_3 J\frac{\partial}{\partial \psi}(M\omega) \\
&+ \frac{2\beta}{J} \left[ \left( \frac{\partial u}{\partial \phi} - \frac{\partial u}{\partial \psi} \right) \left( \frac{\partial v}{\partial \psi} \frac{\partial}{\partial \phi} - \frac{\partial v}{\partial \phi} \frac{\partial}{\partial \psi} \right) \right] M \\
&+ \left( \frac{\partial v}{\partial \phi} - \frac{\partial v}{\partial \psi} \right) \left( \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \phi} \right) M \\
&+ \left( \frac{\partial y}{\partial \phi} - \frac{\partial y}{\partial \psi} \right) \left( \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \phi} \right) M \\
&+ \left( \frac{\partial y}{\partial \phi} - \frac{\partial y}{\partial \psi} \right) \left[ \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \phi} \right] M \\
\end{align*}
\]
(4.37)

\[
\begin{align*}
\frac{\partial h}{\partial \phi} - \frac{\partial h}{\partial \psi} &= -E(-\omega + a_1 v^2) - uJ\frac{\partial \omega}{\partial \psi} - \beta_3 J\frac{\partial}{\partial \psi}(M\omega) \\
&+ \frac{2\beta}{J} \left[ \left( \frac{\partial u}{\partial \phi} - \frac{\partial u}{\partial \psi} \right) \left( \frac{\partial v}{\partial \psi} \frac{\partial}{\partial \phi} - \frac{\partial v}{\partial \phi} \frac{\partial}{\partial \psi} \right) \right] M \\
&+ \left( \frac{\partial v}{\partial \phi} - \frac{\partial v}{\partial \psi} \right) \left( \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \phi} \right) M \\
&+ \left( \frac{\partial y}{\partial \phi} - \frac{\partial y}{\partial \psi} \right) \left( \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \phi} \right) M \\
&+ \left( \frac{\partial y}{\partial \phi} - \frac{\partial y}{\partial \psi} \right) \left[ \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \phi} \right] M \\
\end{align*}
\]
(4.38)
In order to eliminate \( \frac{\partial \chi}{\partial \phi}, \frac{\partial \chi}{\partial \psi} \), we make use of (4.21) and similarly to eliminate \( \frac{\partial q}{\partial \phi}, \frac{\partial q}{\partial \psi} \) and \( q \) we use (4.24) and (4.27).

The set of two, new form of the momentum equations is

\[
\frac{\partial h}{\partial \phi} - \frac{\partial h}{\partial \psi} = -F (-p\omega + \alpha_1 \nu^2 \omega) - \nu J \frac{\partial \omega}{\partial \psi} - \beta_3 \frac{\partial}{\partial \phi} (M \omega),
\]

\[
+ 2 \beta_3 \left[ \frac{\partial M}{\partial \phi} \frac{\partial \nu^2}{\partial \phi} \right] + \frac{\partial M}{\partial \psi} \frac{\partial \nu^2}{\partial \psi}.
\]

(4.39)

\[
\frac{\partial h}{\partial \psi} - \frac{\partial h}{\partial \phi} = -E (-p\omega + \alpha_1 \nu^2 \omega) - \nu J \frac{\partial \omega}{\partial \phi} - \beta_3 \frac{\partial}{\partial \psi} (M \omega),
\]

\[
+ 2 \beta_3 \left[ \frac{\partial M}{\partial \phi} \frac{\partial \nu^2}{\partial \phi} \right] + \frac{\partial M}{\partial \psi} \frac{\partial \nu^2}{\partial \psi}.
\]

(4.40)

where \( \Gamma_{11}, \Gamma_{12} \) and \( \Gamma_{22} \) are defined in (4.23) and (4.26) and \( \nu^2 \) is defined in (4.25).

Summing up the results, we have

**Theorem 1.** If the streamlines \( \psi(x,y) = \text{constant} \) and an arbitrary family of curves \( \phi(x,y) = \text{constant} \) generate a curvilinear net in the physical plane of a third-grade fluid, the system (4.11)-(4.13) along with (4.8) and (4.10) of five partial differential equations for \( u, v, \omega, M \) and \( h \) as functions of \( (x,y) \) may be replaced by

\[
\omega = \frac{1}{J} \left[ \frac{\partial}{\partial \phi} \left( \frac{F}{J} \right) - \frac{\partial}{\partial \psi} \left( \frac{E}{J} \right) \right].
\]
\[ M = \left( 2 \omega^2 + \frac{4}{E} \left\{ \Gamma_{11} \frac{\partial}{\partial \psi} \left( \frac{E}{J^2} \right) - \Gamma_{12} \frac{\partial}{\partial \phi} \left( \frac{E}{J^2} \right) \right\} \right) \]

\[ \frac{\partial}{\partial \psi} \left( \frac{E}{J} \rho \right)^2 - \frac{\partial}{\partial \phi} \left( \frac{E}{J} \rho \right)^2 = 0 \]

\[ G \frac{\partial h}{\partial \phi} - E \frac{\partial h}{\partial \psi} = -F \left( -\omega^2 + \alpha \theta^2 \right) - J \frac{\partial \omega}{\partial \psi} - \beta \frac{\partial \theta}{\partial \phi} (\omega M) \]

\[ + 2 \beta \left[ \frac{3M}{\partial \phi} \left( \frac{-Gr_{12}^2 + Fr_{22}^2}{J} \right) + \frac{3M}{\partial \psi} \left( \frac{Gr_{11}^2 - Fr_{12}^2}{J} \right) \right] \]

\[ (4.41) \]

of five equations for six unknowns E, F, G, \( \omega \), M, h as functions of \( \phi, \psi \). Given a solution

\[ E = E(\phi, \psi), \quad F = F(\phi, \psi), \quad G = G(\phi, \psi) \]

\[ \omega = \omega(\phi, \psi), \quad M = M(\phi, \psi), \quad h = h(\phi, \psi) \]

of the above, we can find x, y as a function of \( (\phi, \psi) \) from

\[ z = x + iy = \int_0^y \frac{e^{iy}}{\sqrt{F}} \left\{ Ed\phi + (F + iJ)d\psi \right\} \]

\[ (4.42) \]

where

\[ \gamma = \int_0^y \left( \Gamma_{11} d\phi + \Gamma_{12} d\psi \right). \]
and $u$ and $v$ by

$$u + iv = \frac{\sqrt{P}}{J} e^{i\gamma} \quad (4.43)$$

and pressure is determined by (4.30).

It is also of some interest to write the momentum equations in a slightly different form. Thus, multiplying equation (4.33) by $\frac{\partial x}{\partial \phi}$ and (4.34) by $\frac{\partial y}{\partial \phi}$ and adding gives one equation. Again multiplying (4.33) by $\frac{\partial x}{\partial \psi}$ and (4.34) by $\frac{\partial y}{\partial \psi}$ and adding gives second equation. These, respectively, are

$$J \frac{\partial h}{\partial \phi} = \mu \left[ \frac{\partial}{\partial \phi} \frac{\partial w}{\partial \psi} - \frac{\partial w}{\partial \psi} \right] + \beta_3 \left[ \frac{3}{3 \phi} (M \omega) F - \frac{3}{3 \psi} (M \omega) E \right]$$

$$+ 2 \beta_3 \left[ \frac{3M}{3 \phi} \frac{3u}{3 \psi} \frac{3v}{3 \psi} - \frac{3v}{3 \phi} \frac{3x}{3 \psi} + \frac{3M}{3 \phi} \frac{3v}{3 \phi} \frac{3x}{3 \phi} - \frac{3u}{3 \phi} \frac{3y}{3 \phi} \right]$$

$$\quad \quad \quad \quad \quad \quad (4.44)$$

$$J \frac{\partial h}{\partial \psi} = J (\rho \omega + \alpha_1 v^2 \omega) + \mu \left[ \frac{3}{3 \phi} \omega G - \frac{\partial}{\partial \phi} \frac{\partial \omega}{\partial \psi} \right]$$

$$+ \beta_3 \left[ \frac{3}{3 \phi} (M \omega) G - \frac{3}{3 \psi} (M \omega) F \right]$$

$$+ 2 \beta_3 \left[ \frac{3M}{3 \phi} \frac{3u}{3 \psi} \frac{3v}{3 \psi} - \frac{3v}{3 \phi} \frac{3x}{3 \psi} + \frac{3M}{3 \phi} \frac{3v}{3 \phi} \frac{3x}{3 \phi} - \frac{3u}{3 \phi} \frac{3y}{3 \phi} \right]$$

$$\quad \quad \quad \quad \quad \quad (4.45)$$

As before, using (4.20), (4.21), (4.24) and (4.27) to eliminate different variables, we find that the new set of equivalent form of equations is:
\[
\frac{3h}{\phi} = \mu \left[ \frac{3\omega F}{\phi^2} - \frac{3\omega E}{\phi} \right] + \beta_3 \left[ F \frac{3}{\phi} (M\omega) - E \frac{3}{\phi} (M\omega) \right]
\]

\[
+ 2 \beta_3 \left[ \frac{3M}{\phi} \frac{2}{11} - \frac{3M}{\phi} \frac{2}{22} \right].
\]

(4.46)

\[
\frac{3h}{\phi} = J (-\rho + a_1 \nu^2 \omega) + \mu \left[ \frac{3\omega G}{\phi} - \frac{3\omega F}{\phi} \right] + \beta_3 \left[ G \frac{3}{\phi} (M\omega) - F \frac{3}{\phi} (M\omega) \right]
\]

\[
+ 2 \beta_3 \left[ \frac{3M}{\phi} \frac{2}{12} - \frac{3M}{\phi} \frac{2}{22} \right].
\]

(4.47)

Equations (4.46) and (4.47) along with Gauss equation (4.22), vorticity equation (4.28) and equation for M, (4.29), again form an underdetermined system of five equations in six unknowns, F, F, G, \(\omega\), M and h. The reason for it being the arbitrariness inherent in the choice of the coordinate lines \(\phi = \text{constant}\). The system can be made determinate in a number of ways, and one plausible way to assume coordinate lines \(\phi = \text{constant}\) as orthogonal trajectories of the streamlines \(\psi = \text{constant}\).

If we use integrability condition

\[
\frac{\partial^2 h}{\partial \phi \partial \psi} = \frac{\partial^2 h}{\partial \psi \partial \phi}
\]

in above equation (4.46) and (4.47) and set \(F = 0\), we obtain

\[
\omega J \nu^2 \omega + \frac{3}{\phi} (-\rho + a_1 \nu^2 \omega) + \beta_3 J \nu^2 (M\omega)
\]

\[
+ \beta_3 \left[ \frac{3}{G} \left( \frac{1}{J} \frac{3M}{\phi'} \frac{3G}{\phi} - \frac{3M}{\phi} \frac{3G}{\phi} \right) \right].
\]
where now
\[ \nabla^2 f = \frac{1}{J} \left[ \frac{3}{3\Phi} \left( \frac{\partial}{\partial \Phi} \frac{\partial f}{\partial \Phi} \right) + \frac{3}{3\Psi} \left( \frac{\partial}{\partial \Psi} \frac{\partial f}{\partial \Psi} \right) \right] \]

Collecting the results, we have

**Theorem 2.** The equations

\[ \omega J \nabla^2 \omega + \frac{3}{3\Phi} \left( \omega + \frac{1}{J} \nabla^2 \omega \right) + \beta J \nabla^2 (M \omega) \]

\[ + \beta \frac{2}{J} \left( \frac{3}{3\Phi} \left( \frac{3M}{3\Phi} \frac{3G}{3\Phi} - \frac{3M}{3\Phi} \frac{3G}{3\Phi} \right) \right) \]

\[ + \frac{3}{3\Psi} \left( \frac{1}{J} \left( \frac{3M}{3\Psi} \frac{3E}{3\Psi} + \frac{3M}{3\Psi} \frac{3G}{3\Phi} \right) \right) = 0, \]

\[ \omega = -\frac{1}{J} \frac{3}{3\Psi} \left( \frac{E}{J} \right), \]

\[ M = \left[ 2\omega^2 - \frac{2}{J^2} \left( \frac{3E}{3\Psi} \frac{3(1/3)}{3G} + \frac{3G}{3\Phi} \frac{3(1/3)}{3G} \right) \right], \]

\[ \frac{3}{3\Phi} \left( \frac{1}{J} \frac{3E}{3\Phi} \right) + \frac{3}{3\Psi} \left( \frac{1}{J} \frac{3G}{3\Phi} \right) = 0. \]  

(4.49)

Together with \( J = \sqrt{EG} \) form a determinate system of four equations for four unknowns, \( E, G, \omega \) and \( M \) as functions of \( \phi, \psi \). Once a solution

\[ E = E(\phi, \psi), \quad \omega = \omega(\phi, \psi), \quad M = M(\phi, \psi), \quad G = G(\phi, \psi) \]

has been obtained, the function \( h(\phi, \psi) \) is obtained from

\[ h = \int \left( \frac{3h}{3\Phi} \frac{3h}{3\Phi} + \frac{3h}{3\Psi} \frac{3h}{3\Psi} \right) \]
with $F = 0$ and pressure is then obtained by (4.30). The mapping of the $(\zeta, \psi)$ plane upon the physical and hodograph plane is obtained as in Theorem 1.

4.4. Illustrations

In this section we shall make use of the above results to study flows for which certain conditions are placed a priori on the coefficient $E$, $F$, $G$ of the first fundamental form (4.16).

$$ds^2 = Ed\zeta^2 + 2EDd\zeta + Gd\psi^2$$

4.4.1. Straight Streamlines. We raise a question as to what plane flow patterns of a third grade fluid are possible when streamlines are straight lines. We assume that streamlines are not parallel but envelop a curve $C_o$. Taking the tangent lines to $C_o$ and their orthogonal trajectories (the involutes to $C_o$), as a system of orthogonal curvilinear coordinates in the physical plane, we obtain

$$ds^2 = d\xi^2 + (\xi - \alpha)^2 d\sigma^2$$  \hspace{1cm} (4.50)

where $\sigma$ denotes the arc length along $C_o$, $\kappa$ the curvature of $C_o$ and $\xi$ the parameter constant along each involute. If $\eta$ denotes the angle of elevation of a tangent line to $C_o$, we have
\[ \frac{d\eta}{d\sigma} = \kappa \]

and (4.50) becomes

\[ ds^2 = d\xi^2 + (\xi - \sigma)^2 d\eta^2, \quad \sigma = \sigma(\eta) \quad (4.51) \]

In this coordinate system the coordinate curves \( \xi = \text{constant} \) and \( \eta = \text{constant} \) are respectively the involutes of \( C_0 \) and tangent lines to \( C_0 \).

We now proceed to determine the flows for which

\[ \phi = \phi(\xi), \quad \psi = \psi(\eta) \quad (4.52) \]

Using (4.52) in the first fundamental form (4.16), we get

\[ ds^2 = \psi^{-1}(\xi) d\xi^2 + 2\psi'(\xi)\psi'(\eta) d\xi d\eta + \psi'^2(\eta) d\eta^2 \quad (4.53) \]

Comparing (4.53) with (4.51) we have

\[ E = \frac{1}{[\psi'(\xi)]^2}, \quad F = 0, \quad G = [\frac{\xi - \sigma(\eta)}{\psi'(\eta)}]^2 \]

and

\[ J = [\frac{\xi - \sigma(\eta)}{\psi'(\eta)}] \quad (4.54) \]

Using (4.54) in (4.49) and regarding \( \xi, \eta \) as the new independent variables, we find the Gauss equation (4.49) is identically satisfied and the other equations take the form

\[ \nu \left\{ \frac{3}{2} \frac{\partial}{\partial \xi} \left[ (\xi - \sigma) \frac{3}{2} \frac{\partial}{\partial \xi} \right] + \frac{3}{2} \frac{1}{\eta} \frac{\partial}{\partial \eta} \left[ (\xi - \sigma) \frac{3}{2} \frac{\partial}{\partial \eta} \right] \right\} \]
\[ + \psi \frac{\partial}{\partial \xi} \left[ \rho \omega \right] + \frac{\alpha}{(\xi - \sigma)} \left[ \frac{\partial}{\partial \xi} \left( (\xi - \sigma) \frac{\partial \omega}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \frac{1}{(\xi - \sigma)} \frac{\partial \omega}{\partial \eta} \right) \right] \]

\[ + \beta \frac{\partial}{\partial \xi} \left[ \left( (\xi - \sigma) \frac{\partial}{\partial \xi} (M \omega) \right) \right] + \frac{\partial}{\partial \eta} \left[ \frac{1}{(\xi - \sigma)} \frac{\partial}{\partial \eta} (M \omega) \right] \]

\[ + \beta \left[ \frac{\partial}{\partial \xi} \left( \frac{\psi^3}{(\xi - \sigma)} \left( \frac{\partial M}{\partial \xi} \frac{\partial}{\partial \xi} - \frac{\partial M}{\partial \eta} \frac{\partial}{\partial \eta} \right) \frac{(\xi - \sigma)^2}{2} \psi \right) \right] \]

\[ + \frac{\partial}{\partial \eta} \left[ \frac{\psi^3}{(\xi - \sigma)} \frac{\partial M}{\partial \xi} \frac{\partial}{\partial \xi} \frac{(\xi - \sigma)^2}{2} \psi \right] = 0. \quad (4.55) \]

\[ \omega = - \frac{1}{(\xi - \sigma)} \frac{\partial}{\partial \eta} \left( \frac{\psi^4}{(\xi - \sigma)} \right) \quad (4.56) \]

\[ M = \left[ 2 \omega^2 + \frac{8 \psi^2}{(\xi - \sigma)^4} \right] \quad (4.57) \]

On eliminating \( \omega \) and \( M \) in (4.55) with the help of (4.56) and (4.57), we obtain

\[ X_0 + (\xi - \sigma)X_1 + (\xi - \sigma)^2X_2 + (\xi - \sigma)^3X_3 + (\xi - \sigma)^4X_4 \]

\[ + (\xi - \sigma)^5X_5 + (\xi - \sigma)^6X_6 + (\xi - \sigma)^7X_7 + (\xi - \sigma)^8X_8 \]

\[ + (\xi - \sigma)^9X_9 = 0, \quad X_1 = X_1(n) \quad (4.58) \]

where

\[ X_0 = 99d_\omega \beta_3 \]

\[ X_1 = -[9d_\omega \mu + 80c_\sigma \beta_3 + 19c_\omega \beta_3] \]

\[ X_2 = -[(8c_\mu + 63b_\omega \beta_3 + 17c_\omega \beta_3 + \beta_3^1 + D) \beta_3 \]

\[ X_3 = -[9d_\omega \beta_3 + 80c_\sigma \beta_3 + 19c_\omega \beta_3] \]

\[ X_4 = -[(8c_\mu + 63b_\omega \beta_3 + 17c_\omega \beta_3 + \beta_3^1 + D) \beta_3] \]

\[ X_5 = -[9d_\omega \beta_3 + 80c_\sigma \beta_3 + 19c_\omega \beta_3] \]

\[ X_6 = -[(8c_\mu + 63b_\omega \beta_3 + 17c_\omega \beta_3 + \beta_3^1 + D) \beta_3] \]

\[ X_7 = -[9d_\omega \beta_3 + 80c_\sigma \beta_3 + 19c_\omega \beta_3] \]

\[ X_8 = -[(8c_\mu + 63b_\omega \beta_3 + 17c_\omega \beta_3 + \beta_3^1 + D) \beta_3] \]

\[ X_9 = -[9d_\omega \beta_3 + 80c_\sigma \beta_3 + 19c_\omega \beta_3] \]
\[ X_3 = -[(7b + 48a <^1 + 15a b^1 c^1) + C]\beta_3 \]

\[ X_4 = +[105a_1 <^2 - \beta_3 (6a <^1 + 13a' b^1 + b^1 + B)] \]

\[ X_5 = [6a_1 <^3 (15a <^1 2 + 10a' a'^1) - \beta_3 (A + a'') \]

\[ X_6 = [5a_1 <^4 (6a <^1 4 + 4a" <^1 + a' a'^1 + 9a' a'] - \mu (15a <^1 3)] \]

\[ X_7 = [4a_1 <^5 (4a" + 4\psi^1) - \mu (15a <^1 2 + 10a' a'^1)] \]

\[ X_8 = -[\mu (6a <^1 4 + 4a" a'^1 + a' a'^1 + 9a' a'] + \rho \psi' (3a'^1)] \]

\[ X_9 = -[\mu (\psi^1 4 + 4a") + \rho \psi' (2a'' \psi^1)] \quad (4.59) \]

and where

\[ a(n) = 2(\psi^1 3 + 4\psi^1 2 \psi^1) \]

\[ b(n) = 6\psi^1 2 \psi' a'^1 + 8a'^1 3 \]

\[ c(n) = 6a <^1 2 a'^1 3 \]

\[ d(n) = 2a'^1 3 a'^1 3 \]

\[ A(n) = 80a <^1 4 + 8\psi^3 + 288\psi^1 \psi^2 \]

\[ B(n) = 262a <^1 4 + 840a <^1 3 + 96a'^1 2 \psi^1 \]

\[ C(n) = 648a'^1 2 \psi^1 2 + 112a'^1 3 a'^1 \]

\[ D(n) = 378a'^1 3 a'^1 3 \quad (4.60) \]

We note with interest that if we set \( \beta_3 = 0 \) in (4.56), we obtain the results of Kaloni and Siddiqui [1983] and if we
set $\beta_3 = a_1 = 0$ in same (4.58), we obtain the expression of Martin [1971, 1].

Since $\xi, \eta$ are independent variables, for the relation (4.58) to hold identically all the $X_i$ must vanish identically. In particular $X_\circ = 0$ leads to

$$1988 \beta_3 \psi^3 \sigma^5 = 0 \quad (4.61)$$

Now, since $\beta_3 \neq 0$, and $\psi$ cannot vanish identically we find that

$$\sigma^5 = \frac{1}{\kappa} = 0.$$ 

This implies that $C_\circ$ has zero radius of curvature. Summing up we have

**Theorem 3.** If in a steady plane flow of third grade fluid, the streamlines are straight lines, then these lines must be concurrent, or parallel.

**4.4.2. Streamlines as Involutes of a Curve.** Here we investigate flows in which streamlines are involutes of a curve $C_\circ$. As in subsection 4.4.1, we take the curvilinear coordinate system $(\xi, \eta)$ where the coordinate curves $\xi = \text{constant}$ are the involutes and $\eta = \text{constant}$, are tangent lines to curve $C_\circ$.

We now seek flows for which

$$\phi = \phi(\eta), \quad \psi = \psi(\xi) \quad (4.62)$$
Using (4.62) in (4.16), we get

\[ ds^2 = E\phi'^2(\eta)\eta^2 + 2F\phi'(\eta)\phi'(\xi)\xi d\xi d\eta + G\phi'^2(\xi)\xi^2, \]

and comparing it with (4.51) we find that

\[ E = \left[ \frac{(\xi-\sigma)}{\phi'(\eta)} \right]^2, \quad F = 0, \quad G = \left[ \frac{1}{\phi'(\xi)} \right]^2. \]

\[ J = \left[ \frac{\xi-\sigma}{\phi'^2} \right]. \]

(4.63)

Substituting (4.63) in (4.49), we see that the Gauss equation (4.49) is automatically satisfied and the rest of the equations in (4.49) become

\[ \omega \left[ \frac{2}{\delta \xi} \left\{ \left( \xi-\sigma \right) \frac{3 \omega}{\delta \xi} + \frac{3}{\delta \eta} \left( \frac{1}{\xi-\sigma} \frac{3 \omega}{\delta \eta} \right) \right\} \right. \]

\[ + \psi \frac{2}{\delta \eta} \left[ \frac{\alpha_1}{\xi-\sigma} \left( \frac{3}{\delta \xi} \left( \left( \xi-\sigma \right) \frac{3 \omega}{\delta \xi} \right) + \frac{3}{\delta \eta} \left( \frac{1}{\xi-\sigma} \frac{3 \omega}{\delta \eta} \right) \right) \right] \]

\[ + \beta_3 \left[ \frac{3}{\delta \xi} \left\{ \left( \xi-\sigma \right) \frac{3 \omega}{\delta \xi} (M\omega) \right\} + \frac{1}{\delta \eta} \left( \frac{1}{\xi-\sigma} \frac{3 \omega}{\delta \eta} (M\omega) \right) \right] \]

\[ + \beta_3 \left[ \frac{3}{\delta \xi} \left( 2\psi \frac{3M}{\delta \xi} \right) + \frac{3}{\delta \eta} \left( \frac{2\psi''}{\xi-\sigma} \frac{3M}{\delta \eta} \right) \right] = 0, \quad (4.64) \]

\[ \omega = -\frac{1}{(\xi-\sigma) \frac{3}{\delta \xi} \left( \xi-\sigma \right) \psi' \right] \]

\[ M = 2\left[ \omega^2 - \frac{4\psi'\psi''}{(\xi-\sigma)} \right]. \]

(4.65)

On eliminating \( \omega \) and \( M \) between (4.64) and (4.65) we obtain
\[\gamma_0 + (\xi - \sigma)\gamma_1 + (\xi - \sigma)^2\gamma_2 + (\xi - \sigma)^3\gamma_3 + (\xi - \sigma)^4\gamma_4 + (\xi - \sigma)^5\gamma_5 + (\xi - \sigma)^6\gamma_6 + (\xi - \sigma)^7\gamma_7 = 0, \quad (4.66)\]

where

\[\gamma_0 = [\beta_3 \bar{h} - a_1 \psi'(15\psi'\sigma_1^3)],\]
\[\gamma_1 = [\beta_3 g - a_1 \psi'(10\psi'\sigma'\sigma_1^2)],\]
\[\gamma_2 = [\beta_3 f - a_1 \psi'(3\psi'\sigma_1 + \psi'\sigma_1') - \mu(3\psi'\sigma_1^2)],\]
\[\gamma_3 = [\beta_3 e - a_1 \psi'(2\psi''\sigma_1') - \mu\psi' + \rho\psi'\sigma_1] ,\]
\[\gamma_4 = [\beta_3 d - a_1 \psi'(2\psi''\sigma_1') - \mu\psi'_{} + \rho\psi'\sigma_1'],\]
\[\gamma_5 = [\beta_3 c - \mu(-\psi''')],\]
\[\gamma_6 = [\beta_3 b - \mu(2\psi''')],\]
\[\gamma_7 = [\beta_3 a - \mu \psi' \gamma'], \quad (4.67)\]

and where

\[a = -6(\psi''^2\psi''),\]
\[b = 12(\psi'\psi''\psi'''),\]
\[c = -6(\psi''^2\psi'' + \psi'\psi'''),\]
\[d = [18 \psi'\psi''^2 + 12 \psi'\psi'''],\]
\[e = -[18 \psi'\psi''^2 + 6 \psi'\psi'' \sigma'']\]
\[ f = 12\psi'^2\psi'' \sigma'' - 18\psi'\psi'' \sigma'^2 + 6\psi'^3 \]
\[ g = 48\psi'^2\psi'' \sigma'^2 - 6\psi'^3 \sigma'' \]
\[ \ddot{h} = -30\psi'^3 \sigma'^2 \]  \hspace{1cm} (4.68)

We remark that equation (4.66) is consistent with the results of Kaloni and Siddiqui [1983, 1] and Martin (1971, 1) in the various special cases.

The curve \( C_0 \) appears as the curve \( \xi = \sigma(\eta) \) in the plane of variables \( \xi, \eta \). For the relation (4.66) to hold identically, it must hold identically on the curve \( \xi = \sigma(\eta) \) and consequently,

\[ [\beta_3 \dot{h} - \alpha_1(15\psi'\sigma'^3)] = 0. \]

This is equivalent to

\[ 15\sigma'^2\psi'^2[2\beta_3 \psi' + \alpha_1 \sigma'] = 0. \] \hspace{1cm} (4.69)

Now since \( \beta_3 > 0, \alpha_1 \geq 0, \) and \( \psi' = \psi'(\xi) \) cannot vanish identically, we must have \( \sigma'(\eta) = 0 \). That is \( \kappa = 0 \) and, therefore, \( C_0 \) should reduce to a point. We thus have

**Theorem 4.** The streamlines in a plane steady flow of a third grade fluid can be involutes of a curve \( C_0 \) only if \( C_0 \) reduces to a point and streamlines are circles concentric at this point.
4.5 Further Illustrations

In order to demonstrate other applications of the method we now study a few specific flows.

(a) Radial Flows

To study these flows we take \( \phi = \phi(r) \) and \( \psi = \psi(\theta) \) and the expression for \( ds^2 \) in (16.4) becomes

\[
ds^2 = E \phi'^2 dr^2 + 2F \phi' \psi' d\theta + G \psi'^2 d\theta^2 \tag{4.70}
\]

where \((r, \theta)\) are polar coordinates. The squared element of arc length also is

\[
ds^2 = dr^2 + r^2 d\theta^2 . \tag{4.71}
\]

Comparison of (4.70) and (4.71) yields

\[
E = \frac{1}{\phi'^2}, \quad G = \frac{r^2}{\psi'^2}, \quad F = 0, \quad J = \frac{r}{\phi' \psi'} \tag{4.72}
\]

where

\( \phi' > 0, \quad \psi' > 0. \)

Using (4.72) in (4.49) we find

\[
\nu \left( \frac{3}{3r} (r \omega_r) + \frac{3}{3r} \left( \frac{\omega}{r} \right) \right) + \psi \frac{3}{3r} \left( - \rho \omega + \frac{\alpha}{r} \left( \frac{3}{3x} (r \omega_r) + \frac{3}{3y} \left( \frac{\omega_y}{r} \right) \right) \right)
\]

\[
+ \beta_3 \left( \frac{3}{3x} (r \omega_x) \right) + \frac{3}{3y} \left( \frac{(M_r)}{r} \right) \right]
\]

\[
+ \beta_3 \left[ \frac{3}{3r} \left( \frac{\psi}{r^2} (M_0 G_r - M_r G_0) \right) + \frac{3}{3y} \left( \frac{\psi}{r^3} L_r \right) \right] = 0 ;
\]

\[
(4.73)
\]
\[ \omega = \frac{-1}{\xi^2} \psi'' \]

\[ M = \frac{2}{\xi^2} [\psi''^2 + 4\psi'^2 ] \quad (4.74) \]

and the last equation in (4.49) is satisfied identically.

Further simplification of (4.73), using (4.74) and (4.72) results

\[ -\mu \gamma^4 [4\psi''^2 + \psi'^4] - 2\rho \psi'' \psi' \xi^4 + 4\alpha_1 \xi^2 \psi' [4\psi''^2 + \psi'^4] \]

\[ -8\beta_3 [36(2\psi''^3 + 8\psi'' \psi'^2) + \frac{d^2}{d\theta^2}(2\psi''^3 + 8\psi'' \psi'^2)] \]

\[ + 20\psi' \frac{d}{d\theta}(2\psi''^2 + 8\psi' \psi'^2) - 20\psi''(2\psi'' + 8\psi'^2)] = 0. \]

\[ (4.75) \]

Equation (4.75) holds true for all \( r \) if:

\[ \mu [4\psi''^2 + \psi'^4] + 2\rho \psi' \psi'' = 0, \quad (4.76) \]

\[ 4\alpha_1 [4\psi''^2 + \psi'^4] = 0, \quad (4.77) \]

and

\[ \beta_3 [36(2\psi''^3 + 8\psi'' \psi'^2) + \frac{d^2}{d\theta^2}(2\psi''^3 + 8\psi'' \psi'^2)] \]

\[ + 20\psi' \frac{d}{d\theta}(2\psi''^2 + 8\psi' \psi'^2) - 20\psi''(2\psi'' + 8\psi'^2)] = 0. \quad (4.78) \]

Since \( \alpha_1 \neq 0, \Psi > 0 \), equation (4.77) implies that
\[ 4\psi'' + \psi' = 0 \quad (4.79) \]

Using (4.79) in (4.76) we obtain,
\[ \psi'' = 0 \]

and which gives
\[ \psi = \overline{A} \theta + \overline{B} \quad (4.80) \]

where \( \overline{A}, \overline{B} \) are arbitrary constants. On employing (4.80) in (4.75), we find that the latter is automatically satisfied. Thus, \( \psi = \overline{A} \theta + \overline{B} \), is the solution for the radial flows in a third grade fluid. We remark that this solution is also valid in a flow of second grade fluid (c.f. Chapters II-III).

(b) **Circular Flows**

In this example we again use polar coordinates \((r, \theta)\) with squared element of arc length
\[ ds^2 = dr^2 + r^2 d\theta^2 \quad (4.81) \]

and consider those flows for which now \( \psi = \psi(r), \phi = \phi(\theta) \).

For this case (4.16) becomes
\[ ds^2 = E \phi'^2 d\theta^2 + 2F \phi' \psi' d\theta dr + G \psi'^2 dr^2. \quad (4.82) \]

On comparing (4.81) and (4.82) we obtain,
\[ E = \frac{r^2}{\psi'^2}, \quad G = \frac{1}{\psi'^2}, \quad F = 0, \quad J = \frac{r}{\phi' \psi'} \quad (4.83) \]
where \( \psi' > 0, \phi' > 0 \). On employing (4.83) in the system of equations (4.49) we find, as before, that Gauss equation (4.49) is satisfied identically and other equations reduce to

\[
\begin{align*}
\mu \left( \frac{3}{2} \left( r \omega_r \right) + \frac{3}{2} \left( \frac{\omega \theta}{r} \right) + \psi' \frac{3}{2} \left( -\rho \omega + \frac{\alpha}{r} \left( r \omega_r \right) \right) \right. \\
+ \frac{3}{2} \left( -\frac{\omega \theta}{r} \right) \bigg) + \beta_3 \left( \frac{3}{2} \frac{\omega}{r} \left( r \omega_r \left( M \omega \right) \right) + \frac{3}{2} \left( -\frac{\omega \theta}{r} \left( M \omega \right) \right) \bigg) \\
+ \frac{3}{2} \left( \frac{2 \psi'' - \frac{M}{r}}{2} \right) + \frac{3}{2} \left( 2 \psi' \frac{M}{r} \right) \bigg] &= 0, \quad (4.84)
\end{align*}
\]

\[\omega = - \left( \psi' + \frac{\psi'}{r} \right),\]

\[M = 2 \left( \psi'' - \frac{\psi'}{r} \right)^2, \quad (4.85)\]

where we recognize that \( \omega \) and \( M \) are functions of \( r \) only.

Substitution of (4.85) in (4.84) and integration with respect to \( r \) leads to

\[
\begin{align*}
\left[ \mu + 2 \beta_3 \left( \psi'' - \frac{\psi'}{r} \right)^2 \right] \frac{d}{dr} \left( \psi'' + \frac{\psi'}{r} \right) + \left( \psi'' - \frac{\psi'}{r} \right) \frac{d}{dr} \left( 2 \beta_3 \left( \psi'' - \frac{\psi'}{r} \right)^2 \right) \\
= 0. \quad (4.86)
\end{align*}
\]

Equation (4.86) can be rearranged in the form

\[
\begin{align*}
\left[ \mu + 2 \beta_3 \left( \psi'' - \frac{\psi'}{r} \right)^2 \right] \frac{d}{dr} \left( \psi'' \frac{\psi'}{r} \right)' \\
+ 3 \left[ \left( \mu + 2 \beta_3 \left( \psi'' - \frac{\psi'}{r} \right)^2 \right) \left( \frac{\psi'}{r} \right)' \right]' = 0. \quad (4.87)
\end{align*}
\]
On integrating the above, we get
\[ r^3 \left[ (\mu + 2\beta_3 (\psi' - \frac{\mu}{r} \psi')) \psi' \right] = \text{constant} = C_4. \quad (4.88) \]

This equation admits a variety of solutions. If \( C_4 = 0 \), then \( \psi' = kr \), where \( k \) is a constant, is a solution. If \( \beta_3 \neq 0 \), then
\[ \psi' = -\frac{C_4}{2\mu} \frac{1}{r} + C_5r, \quad (4.89) \]
which is a well known solution for viscous fluid. However, when \( \beta_3 \neq 0 \), \( C_4 \neq 0 \), (4.88) is a nonlinear differential equation whose integral is determined in the following manner:

On setting \( \frac{\psi'}{r} = \chi \), \( r \frac{d}{dr} \left( \frac{\psi'}{r} \right) = r \chi' = \lambda \),
the equation (4.88) may be written as
\[ \lambda \left[ \mu + 2\beta_3 \chi^2 \right] = \frac{C_4}{r^2}. \quad (4.90) \]

Now differentiating this with respect to \( \chi \), we have
\[ \frac{\mu + 6\beta_3 \chi^2}{\mu + 2\beta_3 \chi^2} \frac{d\chi}{d\lambda} = -2\chi \lambda' \quad (4.91) \]

An integral of (4.91) gives
\[ 3\chi - 2 \sqrt{\frac{\mu}{2\beta_3}} \tan^{-1} \left( \frac{\chi}{\sqrt{\mu/2\beta_3}} \right) = -2\frac{\psi'}{r} + C_6 \quad (4.92) \]
Elimination of $\lambda$ between (4.92) and (4.90) constitutes the desired solution for a third grade fluid.

(c) Parallel Flows

In this last example we use the rectangular coordinates $(x,y)$ and write

$$ds^2 = dx^2 + dy^2$$

On taking $\phi = \phi(x), \psi = \psi(y)$, (4.16) leads to

$$ds^2 = E\phi''^2dx^2 + 2F\phi''\psi'dx dy + G\psi'^2dy^2.$$  \hspace{1cm} (4.94)

Comparison of (4.93) and (4.94) yields

$$E = \frac{1}{\phi''^2}, \quad G = \frac{1}{\psi'^2}, \quad F = 0, \quad J = \frac{1}{\phi''\psi'}$$  \hspace{1cm} (4.95)

Substituting (4.95) in (4.49), we find

$$\mu \frac{d^2\omega}{dy^2} + \beta_3 \frac{d^2(M\omega)}{dy^2} = 0,$$  \hspace{1cm} (4.96)

$$\omega = \psi'', \quad M = 2(\psi'')^2,$$  \hspace{1cm} (4.97)

and (4.49) is satisfied identically.

On employing (4.97) in (4.96) and then integrating twice, gives

$$\mu\psi'' + 2\beta_3(\psi'')^3 = A_1y + B_1$$  \hspace{1cm} (4.98)

where $A_1, B_1$ are constants.
Clearly, when $\beta_3 = 0$, we get the well known viscous solution, giving the parabolic distribution

$$\mu \psi' = A_1 y^2 + B_1 y + C_7$$  \hfill (4.99)

where $C_7$ is an arbitrary constant. However, when $\beta_3 \neq 0$, we can write (4.98) as

$$\mu \lambda + 2\beta_3 \lambda^3 = A_1 y + B_1$$  \hfill (4.100)

where $\psi'' = \frac{d\lambda}{dy} = \lambda$. On differentiating (4.100) with respect to $\lambda$ and then solving we get integral

$$\mu \lambda^2 + 3\beta_3 \lambda^4 = 2A_1 \psi' + C_8$$  \hfill (4.101)

Here again, equations (4.100) and (4.101) constitute a parametric solution, expressing $y$ and $\psi'$ in terms of the parameter $\lambda$. By eliminating $\lambda$ between them we can easily arrive at the solution in terms of $\psi'$ and $y$.

Using (4.100) and (4.101) we compare the velocity profile for Newtonian fluid with the velocity profiles for a third grade fluid. It is obvious from Figure 4.3 that the presence of material constant $\beta_3$ damps the velocity of the fluid and this effect increases with the increase in the value of $\beta_3$. 
Fig. 3. Velocity Profiles for Example C. The velocity profiles for a third grade fluid with $\beta_3 = .01$, $\beta_3 = .02$ (diamond lines) compared with profile for a Newtonian fluid $u = 1$, full line.
CHAPTER V

HELICAL FLOW OF A SIMPLE FLUID

5.1 Introduction

In this chapter we consider ordered or grade fluids as successive approximation of the simple fluid. Accordingly, we employ the perturbation scheme starting with the Newtonian fluid solution as the first order solution. Joseph and Fosdick [1973, 2] and Fosdick and Kao [1980, 2] have elaborated this method by considering the problems of the flow between the rotating cylinders and the flow around a rotating sphere, respectively. In this chapter we follow their approach to treat the helical flow problem.

The problem of the helical flow of a simple fluid was solved by Coleman and Noll [1959, 2] but in terms of the viscosity and normal stress functions. Rivlin [1956, 1], who first introduced the term helical, gave expressions for the stress components, etc. for Rivlin-Erickson [1955, 1] fluid but never solved the resulting equations. Fredrickson [1960, 2] used the constitutive equation devised by Rivlin [1956, 1] and outlined a trial and error method to solve the non-linear equations. Deirckes and Schowalter [1966, 2] did measurements of pressure drop in
a helical flow of a 3% polysobutylene dissolved in decalin and compared their results with prediction based upon the theory of simple fluids. These authors found close agreement between the measurements and computation.

In the present chapter we give complete analytical solution of the problem of helical flow to the fourth order of approximation. We determine explicit expressions for velocity and pressure fields, torque, volume discharge rate and normal stress difference. We also compare, some of the quantities with the corresponding results of the viscous fluid and point out the effect of the consideration of elasticity of the fluid.

In section 5.2 we write down the constitutive equation for fourth order approximation of a simple fluid in an alternate form. In the next section 5.3, we outline the general problem of concern in this chapter and introduce the series procedure for its solution and set up for study an ordered sequence problems according to the integral powers of the angular velocity of the rotating pipes. In section 5.4, we develop the solutions to this ordered sequence of problems through the fourth order and give a brief summary of our results. In the last two sections, we determine the explicit expressions for torque, volume discharge and normal stresses difference and find that if we take \(c_2 + c_3 > 0\), the volume discharge is decreased and torque is lowered in comparison to the linearly viscous fluid case.
Fig. 4. Helical Flow
5.2 Preliminaries

The fourth order approximation to the general constitutive equation for (Coleman and Noll [1960,1]) simple fluid may be written as (c.f. (1.17))

\[ T = -pI + S \]

where

\[ S = \sum_{i=1}^{4} S_i \]

with

\[ S_1 = \mu A_1, \quad S_2 = \alpha_1 A_2^2 + \alpha_2 A_1^2, \]

\[ S_3 = \beta_1 A_3 + \beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (\text{tr} A_2) A_1, \]

\[ S_4 = \gamma_1 A_4 + \gamma_2 (A_3 A_1 + A_1 A_3) + \gamma_3 A_2^2 + \gamma_4 (A_2 A_1^2 + A_1 A_2^2) \]

\[ + \gamma_5 (\text{tr} A_2) A_2 + \gamma_6 (\text{tr} A_2) A_1^2 + [\gamma_7 \text{tr} A_3 \]

\[ + \gamma_8 \text{tr}(A_2 A_1)] A_1, \quad (5.2) \]

and where the coefficients \( \mu \) is the viscosity and \( \alpha_1, \alpha_2, \beta_1, \ldots, \gamma_8 \) are material constants. For steady motion the Rivlin-Ericksen tensors satisfy the recursion relation

\[ A_{T+1} = (\text{grad} A_T) v + A_T \text{ grad } v + (A_T \text{ grad } v)^T, \]

\[ A_0 = I \quad (5.3) \]

We shall assume that body forces are absent and motion is steady. The field equations then, take the form
\[ \text{div } \mathbf{v} = 0, \quad (5.4) \]

\[ \rho (\nabla \mathbf{v}) \mathbf{v} = -\nabla p + \text{div } \mathbf{S}, \quad (5.5) \]

where

\[ (\nabla \mathbf{v}) \mathbf{v} = (\mathbf{v}, \nabla) \mathbf{v} \]

5.3 Problem Statement

We consider the motion of a simple fluid contained between two concentric circular pipes of radii \(a\) and \(b\) with say, \(a < b\). The pipes rotate about their common axis with constant angular velocities \(\Omega\) and \(\lambda\), respectively. In addition the pipes may translate steadily, parallel to their common axis. We assume that outer pipe translates with a velocity \(\mathbf{U} = \Omega \mathbf{U}\) relative to the inner one (Fig. 4).

We use the cylindrical coordinate system \((r, \theta, z)\), in which \(z\)-axis lies along the common axis of the pipes. Because of the axial symmetry the flow quantities are independent of the angle \(\theta\). The no-slip conditions require that

\[ \mathbf{v} = \begin{cases} \Omega \mathbf{e}_\theta & \text{at } r = a \\ \lambda \Omega \mathbf{e}_\theta + \Omega \mathbf{U} e_2 & \text{at } r = b \end{cases} \quad (5.6) \]

where \(a\) is the radius of inner pipe and \(b\) is the radius of outer pipe.

On assuming that the velocity field and pressure field in the fluid domain are of the form:
\[ \mathbf{v} = \mathbf{v}(r, z; \Omega), \]
\[ p = p(r, z; \Omega), \]  
(5.7)

and, furthermore, (5.7) is sufficiently differentiable with respect to \( \Omega \) yield the series expansions (Fosdick and Kao [1980, 2])

\[ \mathbf{v} = \sum_{k=1}^{n} \frac{1}{k!} \mathbf{v}^{(k)} \Omega^k + O(\Omega^n), \]
\[ p = \sum_{k=0}^{n} \frac{1}{k!} p^{(k)} \Omega^k + O(\Omega^n), \]  
(5.8)

where

\[ (\mathbf{v})^{(k)} = \frac{\partial^k}{\partial \Omega^k} (\mathbf{v}) \big|_{\Omega = 0} \]

and where \( \mathbf{v}^{(k)} \) and \( p^{(k)} \) are functions of position. Because of the axial symmetry it follows that

\[ \mathbf{v}(k) = u^{(k)}(r, z)e_r + v^{(k)}(r, z)e_\theta + w^{(k)}(r, z)e_z, \]  
(5.9)

and that

\[ p^{(k)} = p^{(k)}(r, z). \]

We remark that (5.8), tacitly assumes that when \( \Omega = 0 \), then \( \mathbf{v} = 0 \) and \( p = p^{(0)} \), where \( p^{(0)} \) is the uniform static pressure field of the fluid, which could be set to zero without loss of generality.

Since each of the equations (5.4) and (5.5) is an
identity in $\Omega$, these may be differentiated any number of times with respect to $\Omega$ and evaluated at $\Omega=0$. Thus we obtain

$$\text{div } v^{(k)} = 0 \quad \text{(5.10)}$$

$$\rho (\text{grad } v) v^{(k)} = -\text{grad } p^{(k)} + \text{div } s^{(k)} \quad \text{(5.11)}$$

where for $s^{(k)}$ we will call upon (5.2) and (5.3).

From (5.8) and (5.6), the boundary conditions for $v^{(k)}$ have the form

$$v^{(1)} = \begin{cases} \alpha e_g & \text{at } r = a, \\ \lambda b e_\theta + U e_z & \text{at } r = b, \end{cases}$$

$$v^{(k)} = \begin{cases} 0 & \text{at } r = a, \\ 0 & \text{at } r = b. \end{cases}$$

The above may be written as

$$u^{(1)} = 0, \quad w^{(1)} = 0, \quad v^{(1)} = a, \quad \text{at } r = a,$$

$$u^{(1)} = 0, \quad w^{(1)} = U, \quad v^{(1)} = \lambda b, \quad \text{at } r = b,$$

(5.12a)

and for $k>1$,

$$u^{(k)} = 0, \quad w^{(k)} = 0, \quad v^{(k)} = 0, \quad \text{at } r = a,$$

$$u^{(k)} = 0, \quad w^{(k)} = 0, \quad v^{(k)} = 0, \quad \text{at } r = b,$$

(5.12b)
5.4 Perturbation Solution

Following Fosdick and Kao [1980, 2] we now develop the solution in a perturbation series in powers of Ω (5.8). In order to carry out this construction, we specifically solve the field equation (5.10) and (5.11) for \( v^{(k)} \) and \( p^{(k)} \) at each order \( k = 1, 2, 3, 4 \), and determine their values subject to the boundary conditions (5.12). Since (5.2) represents the fourth-order approximation to the constitutive equation of a simple fluid, the analysis given here is confined up till \( k = 4 \).

5.4.1 First Order Problem

We shall work out the first order problem in detail as the higher order problems will require similar operations. It will turn out that the first order solution for simple fluid is essentially the same as that for the Newtonian fluid.

For \( k = 1 \), (5.10) and (5.11) have the form

\[
\text{div } v^{(1)} = 0, \\
-\text{grad } p^{(1)} + \text{div } S^{(1)} = 0,
\]

where \( S^{(1)} \) is obtained from (5.2) by differentiating once with respect to \( \Omega \) and then setting \( \Omega = 0 \). Using (5.8), (5.3) and (5.2), it follows that
\[ A_1^{(1)} = \text{grad } v^{(1)} + (\text{grad } v^{(1)})^T, \]

\[ A_i^{(1)} = 0, \quad i \geq 2 \quad (5.14) \]

and that

\[ S^{(1)} = S_1^{(1)} + S_2^{(1)} + S_3^{(1)} + S_4^{(1)}, \]

\[ S_1^{(1)} = \mu A_1^{(1)}, \]

\[ S_i^{(1)} = 0, \quad i \geq 2 \quad (5.15) \]

From (5.9), for this case, we note that

\[ v^{(1)} = u^{(1)}(r,z)e_r + v^{(1)}(r,z)e_\theta + w^{(1)}(r,z)e_z, \]

\[ p^{(1)} = p^{(1)}(r,z). \quad (5.16) \]

Now, by the use of (5.16) in (5.14), it readily follows that

\[ A_1^{(1)} = \begin{bmatrix} 2 \frac{\partial u^{(1)}}{\partial r} - \frac{\partial v^{(1)}}{\partial r} & \frac{\partial v^{(1)}}{\partial r} - \frac{v^{(1)}}{r} & 2 \frac{\partial u^{(1)}}{\partial z} + \frac{\partial w^{(1)}}{\partial r} \\ \frac{\partial v^{(1)}}{\partial r} - \frac{v^{(1)}}{r} & 2 \frac{\partial u^{(1)}}{\partial z} & \frac{\partial v^{(1)}}{\partial z} \\ \frac{\partial u^{(1)}}{\partial z} + \frac{\partial w^{(1)}}{\partial r} & \frac{\partial v^{(1)}}{\partial z} & 2 \frac{\partial w^{(1)}}{\partial z} \end{bmatrix} = \text{grad } v^{(1)} + (\text{grad } v^{(1)})^T \quad (5.17) \]

where
\[
\text{grad } v^{(1)} = \begin{bmatrix}
\frac{\partial u^{(1)}}{\partial r}, & -\frac{v^{(1)}}{r}, & \frac{\partial u^{(1)}}{\partial z} \\
\frac{\partial v^{(1)}}{\partial r}, & \frac{u^{(1)}}{r}, & \frac{\partial v^{(1)}}{\partial z} \\
\frac{\partial w^{(1)}}{\partial r}, & 0, & \frac{\partial w^{(1)}}{\partial z}
\end{bmatrix}
\]

Substituting (5.17) into (5.15) to find \(s^{(1)}\), and then inserting these values of \(s^{(1)}\) into (5.13) and making use of (5.16) in (5.13), we find

\[
\frac{\partial u^{(1)}}{\partial r} + \frac{u^{(1)}}{r} + \frac{\partial w^{(1)}}{\partial z} = 0,
\]

\[
-\frac{\partial p^{(1)}}{\partial r} + \mu \left[ \frac{\partial^2 u^{(1)}}{\partial r^2} + \frac{\partial^2 u^{(1)}}{\partial z^2} \right] = 0,
\]

\[
-\frac{\partial p^{(1)}}{\partial z} + \mu \left[ \frac{\partial^2 w^{(1)}}{\partial r^2} + \frac{\partial^2 w^{(1)}}{\partial z^2} \right] = 0,
\]

\[
\mu \left[ \frac{\partial^2 v^{(1)}}{\partial r^2} + \frac{\partial^2 v^{(1)}}{\partial z^2} \right] = 0.
\]

(5.18)

After eliminating the pressure field \(p^{(1)}\) between (5.18)_2 and (5.18)_3, the system (5.18) may be reduced to a set of three partial differential equations

\[
\frac{\partial u^{(1)}}{\partial r} + \frac{u^{(1)}}{r} + \frac{\partial w^{(1)}}{\partial z} = 0,
\]

\[
-\mu \left[ \left( \frac{\partial^2 v^{(1)}}{\partial r^2} + \frac{1}{r} \frac{\partial v^{(1)}}{\partial r} - \frac{1}{r^2} + \frac{\partial^2 v^{(1)}}{\partial z^2} \right) \frac{\partial v^{(1)}}{\partial r} \right] = 0,
\]
\[ \mu \left( \frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} \frac{r}{r} \right) + \frac{\partial^2 v}{\partial z^2} = 0. \quad (5.19) \]

On introducing the streamfunction \( \psi^{(1)} (r,z) \) such that

\[ u^{(1)} = -\frac{1}{r} \frac{\partial \psi^{(1)}}{\partial z}, \quad w^{(1)} = \frac{1}{r} \frac{\partial \psi^{(1)}}{\partial r}, \quad (5.20) \]

we find that (5.19) is satisfied identically and the rest of the equations take the form

\[ -\frac{v}{r} \mathbb{E}^4 \psi^{(1)} = 0, \]

\[ \left[ \frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} \frac{r}{r} \right] + \frac{\partial^2 v}{\partial z^2} = 0, \quad (5.21) \]

where

\[ \mathbb{E}^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \]

\[ \mathbb{E}^4 = \mathbb{E}^2 \cdot \mathbb{E}^2. \]

The boundary conditions (5.12) with the aid of (5.20) reduce to

\[ u^{(1)} = -\frac{1}{r} \frac{\partial \psi^{(1)}}{\partial z} = 0, \quad w^{(1)} = \frac{1}{r} \frac{\partial \psi^{(1)}}{\partial r} = 0, \quad v^{(1)} = a \]

\[ u^{(1)} = -\frac{1}{r} \frac{\partial \psi^{(1)}}{\partial z} = 0, \quad w^{(1)} = \frac{1}{r} \frac{\partial \psi^{(1)}}{\partial r} = U, \quad v^{(1)} = \lambda b, \]

at \( r = a \) and at \( r = b \), respectively. \quad (5.22)

Thus, the solution of (5.21) subject to (5.22) is given as
\[ v^{(1)} = 0e_r + v^{(1)}(r)e_\theta + w^{(1)}(r)e_\phi, \quad (5.23) \]

where

\[ u^{(1)} = 0, \quad w^{(1)} = -\frac{G}{4\mu}r^2 + B_1\ln r + C_1 \]

\[ v^{(1)} = Mr + \frac{N}{r}, \]

and where,

\[ B_1 = \frac{G}{4\mu} (b^2 - a^2) + U \frac{1}{\ln b/a} \]

\[ C_1 = \frac{G}{4\mu} [a^2 \ln b - b^2 \ln a] \frac{1}{\ln b/a} - \frac{U \ln a}{\ln b/a} \quad (5.24) \]

\[ M = \frac{\lambda b^2 - a^2}{b^2 - a^2} \]

\[ N = \frac{(1-\lambda)a^2b^2}{b^2 - a^2} \]

The above solution may also be written in an alternate form as

\[ u^{(1)} = 0, \quad w^{(1)} = \frac{G}{4\mu} [a^2 - r^2 + \frac{(b^2 - a^2)\ln a}{\ln b/a}] + \frac{U \ln r/a}{\ln b/a} \]

\[ v^{(1)} = \frac{\lambda b^2 - a^2}{b^2 - a^2}r + \frac{(1-\lambda)a^2b^2}{b^2 - a^2} \frac{1}{r} \quad (5.25) \]

where in, the constant pressure gradient is given by

\[ \frac{dp^{(1)}}{dz} = -G. \]

The pressure distribution \( p^{(1)} \) for this case, turns out
to be

\[ p^{(1)} = C - Gz, \]  \hfill (5.26)

where \( C \) is an arbitrary constant.

5.4.2 Second Order Problem

For \( k = 2 \), the field equations, i.e., (5.8), (5.10), (5.11), (5.2) and (5.3) yield

\[
\text{div } \mathbf{v}^{(2)} = 0
\]

\[
- \text{grad } p^{(2)} + \text{div } S^{(2)} = 2\rho (\text{grad } v^{(1)}) v^{(1)}
\]  \hfill (5.27)

where

\[
A_1^{(2)} = \text{grad } v^{(2)} + (\text{grad } v^{(2)})^T
\]

\[
A_2^{(2)} = 2[(\text{grad } A_1^{(1)}) v^{(1)} + A_1^{(1)} \text{grad } v^{(1)} + (A_1^{(1)} \text{grad } v^{(1)})^T],
\]

\[
A_i^{(2)} = 0, \quad i \geq 3, \]  \hfill (5.28),

with \( A_1^{(1)} \) and \( \text{grad } v^{(1)} \) given by (5.17)₁,₂ and (5.23) and

where

\[
S^{(2)} = S_1^{(2)} + S_2^{(2)} + S_3^{(2)} + S_4^{(2)}.
\]  \hfill (5.29)
with
\[ \mathbf{s}_1^{(2)} = \mu \mathbf{A}_1^{(2)} \]
\[ \mathbf{s}_2^{(2)} = \alpha_1 \mathbf{A}_2^{(2)} + 2 \alpha_2 (\mathbf{A}_1^{(1)})^2 \]
\[ \mathbf{s}_i^{(2)} = 0, \quad i \geq 3. \]  \hfill (5.30)

Furthermore, for \( k = 2 \), (5.9) has the form,
\[ \mathbf{v}^{(2)} = u^{(2)}(r,z)\mathbf{e}_x + v^{(2)}(r,z)\mathbf{e}_\theta + w^{(2)}(r,z)\mathbf{e}_z \]
\[ \rho^{(2)} = p^{(2)}(r,z). \]  \hfill (5.31)

Now, from the first order solution (5.23), i.e.,
\[ \mathbf{v}^{(2)} = 0\mathbf{e}_x + v^{(1)}(r)\mathbf{e}_\theta + w^{(1)}(r)\mathbf{e}_z, \]
and with the use of (5.17), it follows that,
\[ \mathbf{A}_1^{(1)} = \begin{bmatrix} \frac{d v^{(1)}}{dr} & v^{(1)} & \frac{d w^{(1)}}{dr} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
where
\[
\text{grad } v = \begin{bmatrix}
0 & -\frac{v^{(1)}}{r} & 0 \\
\frac{dv^{(1)}}{dr} & 0 & 0 \\
\frac{dw^{(1)}}{dr} & 0 & 0 \\
\end{bmatrix}
\]

and these results in combination with \((5.28)_2\) give

\[
A_2^{(2)} = 4\left[\left(\frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r}\right)^2 + \left(\frac{dw^{(1)}}{dr}\right)^2\right] \begin{bmatrix}
1, & 0, & 0 \\
0, & 0, & 0 \\
0, & 0, & 0 \\
\end{bmatrix}
\]

\((5.33)\)

Using \((5.32)\), we also find that

\[
(A_1^{(1)})^2 = \begin{bmatrix}
\left(\frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r}\right)^2 + \left(\frac{dw^{(1)}}{dr}\right)^2, & 0, & 0 \\
0, & \left(\frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r}\right)^2, & \frac{dw^{(1)}}{dr} \left(\frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r}\right) \\
0, & \frac{dw^{(1)}}{dr} \left(\frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r}\right), & \left(\frac{dw^{(1)}}{dr}\right)^2 \\
\end{bmatrix}
\]

\((5.34)\)

Thus, from \((5.30)\)_2 and \((5.33)\)--\((5.34)\) we see that
\[
S^{(2)}_2 = \begin{bmatrix}
2(2a_1 + a_2)[(\frac{dv}{dr} - \frac{v(1)}{r})^2 + (\frac{dw}{dr})^2],
0,
0,
0,
2a_2(\frac{dv}{dr} - \frac{v(1)}{r})^2, 2a_2 \frac{dv}{dr} (\frac{dv}{dr} - \frac{v(1)}{r}),
2a_2(\frac{dw}{dr})^2
\end{bmatrix}
\]

(5.35)

On employing (5.31) in (5.28) we obtain analogous to (5.17),

\[
A^{(2)}_1 = \begin{bmatrix}
2 \frac{\partial u}{\partial r} - \frac{v(2)}{r}, \frac{\partial v}{\partial r} - \frac{v(2)}{r}, \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r},
\frac{\partial v}{\partial z}, 2 \frac{u}{r}, \frac{\partial v}{\partial z},
\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \frac{\partial v}{\partial z}, 2 \frac{\partial w}{\partial z}
\end{bmatrix} = \text{grad } v^{(2)} + (\text{grad } v^{(2)})^T
\]

(5.36)

where

\[
\text{grad } v^{(2)} = \begin{bmatrix}
\frac{\partial u}{\partial r}, -\frac{v(2)}{r}, \frac{\partial u}{\partial z},
\frac{\partial v}{\partial r}, \frac{u(2)}{r}, \frac{\partial v}{\partial z},
\frac{\partial v}{\partial r}, 0, \frac{\partial w}{\partial z}
\end{bmatrix}
\]
Substituting (5.36) into (5.30) to find $s_{1}^{(2)}$, using $s_{1}^{(2)}$ and (5.35) in (5.29) to obtain $s^{(2)}$ and then inserting these values of $s^{(2)}$ into (5.27), along with first order solution and (5.31), we find

$$
\frac{\partial u^{(2)}}{\partial r} + \frac{u^{(2)}}{r} + \frac{\partial w^{(2)}}{\partial z} = 0,
$$

$$
-\frac{\partial v^{(2)}}{\partial r} + \mu \left( \frac{\partial^2 u^{(2)}}{\partial r^2} + \frac{\partial v^{(2)}}{\partial r} \right) + \frac{2}{\partial^2 z} = \frac{-2(2n_1 + a_2)}{r} \frac{dP}{dr}
$$

$$
[r \left( \frac{\partial v^{(1)}}{\partial r} - \frac{v^{(1)}}{r} \right)^2 + (\frac{\partial w^{(1)}}{\partial r})^2 + \frac{2a_2}{r} \frac{\partial v^{(1)}}{\partial r} v^{(1)}^2,
$$

$$
-\frac{\partial v^{(2)}}{\partial z} + \frac{2}{\partial^2 z} + \mu \left( \frac{\partial^2 v^{(2)}}{\partial r^2} + \frac{\partial^2 v^{(2)}}{\partial r} \right) + \frac{2}{\partial^2 z} = 0 \quad (5.37)
$$

If we define the streamfunction $\psi^{(2)}(r,z)$ such that

$$
u^{(2)} = -\frac{1}{r} \frac{\partial \psi^{(2)}}{\partial r}, \quad w^{(2)} = \frac{1}{r} \frac{\partial \psi^{(2)}}{\partial r}, \quad (5.38)
$$

then the first equation in (5.37) is satisfied identically and other equations after eliminating pressure $p^{(3)}$ take the form

$$
-\frac{\mu \psi^{(2)}}{E} = 0,
$$
\[ u[\frac{3}{2}v(2) + \frac{3}{2}(\frac{v(2)}{r}) + \frac{3}{2}(\frac{2v(2)}{z^2})] = 0 \]  

(5.39)

For \( k = 2 \), the boundary conditions (5.12), with the aid of (5.38), may be written as

\[ u(2) = -\frac{1}{r} \frac{\partial \psi(2)}{\partial z} = 0, \quad w(2) = \frac{1}{r} \frac{\partial \psi(2)}{\partial r} = 0, \]

\[ v(2) = 0, \text{ at } r = a \text{ and } r = b. \]  

(5.40)

The solution of (5.39) satisfying (5.40) is given by

\[ v(2) = 0, \quad \text{i.e.,} \quad u(2) = v(2) = w(2) = 0. \]  

(5.41)

On employing (5.41) in (5.37), we find that

\[ p(2) = 2(2\alpha_1 + \alpha_2)[(\frac{dv(1)}{dr} - \frac{v(1)}{r})^2 + (\frac{dw(1)}{dr})^2] \]

\[ + \int \frac{1}{r} [4\alpha_1 ((\frac{dv(1)}{dr} - \frac{v(1)}{r})^2 + (\frac{dw(1)}{dr})^2] \]

\[ + 2\alpha_2 (\frac{dw(1)}{dr})^2 + 2\rho v(1)^2 ] dr, \]  

(5.42)

where \( v(1) \) and \( w(1) \) are given by (5.23).

5.4.3 Third Order Problem

For \( k = 3 \), the field equations, i.e., (5.8), (5.10), (5.11), (5.2) and (5.3) yield

\[ \text{div } v(3) = 0, \]  

(5.43)
\[-\operatorname{grad} p^{(3)} + \operatorname{div} S^{(3)} = 3p[(\operatorname{grad} v^{(2)})v^{(1)} + (\operatorname{grad} v^{(1)})v^{(2)}]\]

where

\[A_1^{(3)} = \operatorname{grad} v^{(3)} + (\operatorname{grad} v^{(3)})^T\]
\[A_2^{(3)} = 3[(\operatorname{grad} A_1^{(2)})v^{(1)} + A_1^{(2)} \operatorname{grad} v^{(1)} + (A_1^{(2)})^T \operatorname{grad} v^{(1)} + (\operatorname{grad} A_1^{(1)})v^{(2)} + A_1^{(1)} \operatorname{grad} v^{(2)} + (A_1^{(1)})^T \operatorname{grad} v^{(2)}]\]
\[A_3^{(3)} = 3[(\operatorname{grad} A_2^{(2)})v^{(1)} + A_2^{(2)} \operatorname{grad} v^{(1)} + (A_2^{(2)})^T \operatorname{grad} v^{(1)}]\]
\[A_i^{(3)} = 0, \quad i \geq 4\]

and where

\[S^{(3)} = S_1^{(3)} + S_2^{(3)} + S_3^{(3)} + S_4^{(3)}\]

with

\[S_1^{(3)} = \mu A_1^{(3)}\]
\[S_2^{(3)} = \alpha_1 A_2^{(3)} + 3\alpha_2 [A_1^{(2)} A_1^{(1)} + A_1^{(1)} A_1^{(2)}]\]
\[S_3^{(3)} = \beta_1 A_3^{(3)} + 3[\beta_2 (A_2^{(2)} A_1^{(1)} + A_1^{(1)} A_2^{(2)}) + \beta_3 (\text{tr} A_2^{(2)}) A_1^{(1)}]\]
\[ S_i^{(3)} = 0, \quad i \geq 4 \quad (5.46) \]

For this case, the equations (5.9) may be written as

\[ v^{(3)} = u^{(3)}(r,z)e_r + v^{(3)}(r,z)e_\theta + w^{(3)}(r,z)e_z \]

\[ p^{(3)} = p^{(3)}(r,z) \quad (5.47) \]

The relations (5.44) - (5.46) may be simplified up to some extent by making use of first and second order solutions as follows:

Using the result \( v^{(2)} = 0 \) in (5.36), we find that

\[ A_1^{(2)} = 0, \quad \text{grad} v^{(2)} = 0 \quad \text{and hence} \quad S_1^{(2)} = 0 \quad (5.48) \]

Again, using (5.48) and \( v^{(2)} = 0 \) in (5.44)_2, we find that

\[ A_2^{(3)} = 0 \quad (5.49) \]

Similarly, using first order solution, \( v^{(1)} = 0e_r + v^{(1)}(r)e_\theta + w^{(1)}(r)e_z \) (5.32)_2 and (5.33), it follows that

\[ A_3^{(3)} = 0 \quad (5.50) \]

Furthermore, on employing (5.48) and (5.49) in (5.46)_2, we obtain

\[ S_2^{(3)} = 0 \quad (5.51) \]

Thus, using these information in (5.44) - (5.46), we rewrite these relations as
\[ A_1^{(3)} = \text{grad } \mathbf{v}^{(3)} + (\text{grad } \mathbf{v}^{(3)})^T \] (5.52)

\[ S^{(3)} = S_1^{(3)} + S_3^{(3)} \] (5.53)

with

\[ S_1^{(3)} = \mu A_1^{(3)} \]

\[ S_3^{(3)} = 3[\beta_2 (A_2^{(2)} A_1^{(1)} + A_1^{(1)} A_2^{(2)}) + \beta_3 (\text{tr} A_2^{(2)}) A_1^{(1)}] \ldots (5.54) \]

Now, from (5.54) and (5.32) - (5.33), it readily follows that

\[ S_3^{(3)} = 12(\beta_2 + \beta_3)[(\frac{dv}{dr})^{(1)} - \frac{v}{r}, \frac{dv}{dr}, \frac{dw}{dr}] \begin{bmatrix} 0, & \frac{dv}{dr} - \frac{v}{r}, & \frac{dw}{dr} \\ \frac{dv}{dr} - \frac{v}{r}, & 0, & 0 \\ \frac{dw}{dr}, & 0, & 0 \end{bmatrix} \]

Finally, using (5.47) in (5.52), we may write, analogous to (5.17) and (5.36),
\[ A_{1}^{(3)} = \begin{bmatrix} 2 \frac{\partial u^{(3)}}{\partial r} & \frac{\partial v^{(3)}}{\partial r} - \frac{v^{(3)}}{r} & \frac{\partial u^{(3)}}{\partial z} + \frac{\partial \omega^{(3)}}{\partial r} \\ \frac{3v^{(3)}}{r} & 2 \frac{u^{(3)}}{r} & 2 \frac{\partial v^{(3)}}{\partial z} \\ \frac{\partial u^{(3)}}{\partial z} + \frac{\partial \omega^{(3)}}{\partial r} & \frac{\partial v^{(3)}}{\partial z} & 2 \frac{\partial \omega^{(3)}}{\partial z} \end{bmatrix} = \text{grad } v^{(3)} + (\text{grad } v^{(3)})^{T} \tag{5.56} \]

where

\[ \text{grad } v^{(3)} = \begin{bmatrix} \frac{\partial u^{(3)}}{\partial r} & - \frac{v^{(3)}}{r} & \frac{\partial u^{(3)}}{\partial z} \\ \frac{\partial v^{(3)}}{\partial r} & \frac{u^{(3)}}{r} & \frac{v^{(3)}}{r} \\ \frac{\partial \omega^{(3)}}{\partial r} & 0 & \frac{\partial \omega^{(3)}}{\partial z} \end{bmatrix} \]

Substituting (5.56) into (5.54) to find \( S_{1}^{(3)} \), using \( S_{1}^{(3)} \) and (5.55) in (5.53) to obtain \( S^{(3)} \) and then inserting these values of \( S^{(3)} \) into (5.43), along with first and second order solutions and (5.47), we find

\[ \frac{\partial u^{(3)}}{\partial r} + \frac{u^{(3)}}{r} + \frac{\partial \omega^{(3)}}{\partial z} = 0 \]

\[ -\frac{3\partial}{\partial r} + \mu \left[ \frac{\partial^{2} u^{(3)}}{\partial r^{2}} + \frac{\partial}{\partial r} \left( \frac{u^{(3)}}{r} \right) + \frac{\partial^{2} u^{(3)}}{\partial z^{2}} \right] = 0 \]
\begin{align*}
\frac{\partial^3}{\partial z^3} + \mu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2}\right] = -12(\beta_2 + \beta_3) \frac{d}{dr} \left[ r \frac{dw}{dr} \right]
\end{align*}

\begin{align*}
\left(\frac{dv}{dr} - \frac{v}{r}\right)^2 + \left(\frac{dw}{dr}\right)^2
\end{align*}

\begin{align*}
\mu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2}\right] = -12(\beta_2 + \beta_3) \frac{d}{dr} \left[ r^2 \frac{dv}{dr} \right] - \frac{v}{r}\left(\frac{dv}{dr} - \frac{v}{r}\right)^2 + \left(\frac{dw}{dr}\right)^2
\end{align*}

As before, we introduce the streamfunction \(\psi(3)(r,z)\),

\begin{align*}
u(3) = -\frac{1}{r} \frac{\partial \psi(3)}{\partial z} , \quad \omega(3) = \frac{1}{r} \frac{\partial \psi(3)}{\partial r}
\end{align*}

and find that the first equation in (5.57) is satisfied identically and other equations after eliminating the pressure \(p(3)\) reduce to

\begin{align*}
-\frac{1}{r^4} \frac{\partial^4 \psi(3)}{\partial r^4} = 12(\beta_2 + \beta_3) \frac{d}{dr} \left[ r \frac{dv}{dr} \right] - \frac{v}{r}\left(\frac{dv}{dr} - \frac{v}{r}\right)^2 + \left(\frac{dw}{dr}\right)^2
\end{align*}

\begin{align*}
\mu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2}\right] = -12(\beta_2 + \beta_3) \frac{d}{dr} \left[ r^2 \frac{dv}{dr} \right] - \frac{v}{r}\left(\frac{dv}{dr} - \frac{v}{r}\right)^2 + \left(\frac{dw}{dr}\right)^2
\end{align*}

The boundary conditions for \(k=3\), become
\[ u^{(3)} = -\frac{1}{r} \frac{\partial \psi^{(3)}}{\partial z} = 0, \quad w^{(3)} = \frac{1}{r} \frac{\partial \psi^{(3)}}{\partial r} = 0, \]

\[ v^{(3)} = 0, \text{ at } r = a \text{ and } r = b. \]  

(5.60)

Using \( v^{(1)} \) and \( w^{(1)} \) as given in (5.23) on the right hand side of (5.59), we find the solution of the above problem satisfying (5.60) as

\[ v^{(3)} = 0e_r + v^{(3)}(r)e_\theta + w^{(3)}(r)e_z \]  

(5.61)

where

\[ v^{(3)} = -\frac{\beta_2 + \beta_3}{\mu} \left\{ 16N^3 \left( \frac{1}{r^5} + \frac{b^2 + a^2}{a^4 b^4} \right) r - \frac{b^4 + a^4 + a^2 b^2}{a^4 b^4} \left[ \frac{1}{r} \right] \right\} \]

\[ + 6NB \left[ \frac{1}{r^3} - \frac{1}{a^2 b^2} \right] - \frac{a^2 + b^2}{a^2 b^2} \left[ \frac{1}{r} \right] + 24NB^2 \left( \ln r - \ln a \right) \]

\[ + \frac{b^2 \ln b - a^2 \ln a}{b^2 - a^2} - \frac{a^2 b^2 \ln b}{(b^2 - a^2) r} \]  

(5.62)

\[ w^{(3)} = \frac{\beta_2 + \beta_3}{\mu} \left\{ 12N^2 B_1 \left( \frac{1}{r^4} - \frac{1}{a^4} + \frac{b^4 - a^4}{a^4 b^4} \right) \ln \frac{r}{a} \right\} \]

\[ + 6(B_1^3 - 4N^2 B_0) \left( \frac{1}{r^2} - \frac{1}{a^2} + \frac{b^2 - a^2}{a^2 b^2} \ln \frac{r}{a} \right) \]

\[ - 3B_0^3 \left\{ a^4 r^4 + \frac{(b^4 - a^4) \ln \frac{r}{a}}{\ln b/a} \right\} + 18B_0^2 B_1 \left\{ a^2 r^2 - \frac{(b^2 - a^2) \ln \frac{r}{a}}{\ln b/a} \right\} \]  

(5.63)
and where

\[ B_0 = \frac{G}{2\mu}, \quad \frac{dp^{(3)}}{dz} = -G_1 = 0, \quad (5.64) \]

and \( B_1 \) and \( N \) are defined in (5.23)

The pressure \( p^{(3)} \) turns out to be

\[ p^{(3)} = C_2 \quad (5.65) \]

where \( C_2 \) is an arbitrary constant.

We note with interest that the first term in (5.62) is exactly the same as that reported by Joseph and Posdick [1973, 2] (see Equation (10.11) pp. 363).

5.4.4 Fourth Order Problem

For \( k = 4 \), the field equations, (5.8), (5.10), (5.11), (5.2) and (5.3) yield

\[ \text{div} \ \mathbf{v}^{(4)} = 0 \]

\[-\text{grad} \ p^{(4)} + \text{div} \ \mathbf{s}^{(4)} = \rho [4(\text{grad} \ \mathbf{v}^{(3)})\mathbf{v}^{(1)} \]

\[ + 4(\text{grad} \ \mathbf{v}^{(1)})\mathbf{v}^{(3)} \]

\[ + 6(\text{grad} \ \mathbf{v}^{(2)})\mathbf{v}^{(2)}) \] \quad (5.66)

where

\[ \mathbf{a}_1^{(4)} = \text{grad} \ \mathbf{v}^{(4)} + (\text{grad} \ \mathbf{v}^{(4)})^T, \]
\[ A_2^{(4)} = [4((\text{grad } A_1^{(3)})v^{(1)}) + A_1^{(3)} \text{ grad } v^{(1)}] \]
\[ + (A_1^{(3)} \text{ grad } v^{(1)})^T + 4((\text{grad } A_1^{(1)}))v^{(3)}. \]
\[ + A_1^{(1)} \text{ grad } v^{(3)} + (A_1^{(1)} \text{ grad } v^{(3)})^T] \]
\[ + 6(A_1^{(2)} \text{ grad } v^{(2)} + (A_1^{(2)} \text{ grad } v^{(2)})^T] \]
\[ + (\text{grad } A_1^{(2)})v^{(2)} \} \]

\[ A_3^{(4)} = [4((\text{grad } A_2^{(3)})v^{(1)}) + A_2^{(3)} \text{ grad } v^{(1)}] \]
\[ + (A_2^{(3)} \text{ grad } v^{(1)})^T + 6((\text{grad } A_2^{(2)}))v^{(2)} \]
\[ + A_2^{(2)} \text{ grad } v^{(2)} + (A_2^{(2)} \text{ grad } v^{(2)})^T] \]

\[ A_4^{(4)} = [4((\text{grad } A_3^{(3)})v^{(1)}) + (A_3^{(3)} \text{ grad } v^{(1)})] \]
\[ + (A_3^{(3)} \text{ grad } v^{(1)})^T] \} \]  (5.67)

and where

\[ S^{(4)} = S_1^{(4)} + S_2^{(4)} + S_3^{(4)} + S_4^{(4)}, \]  (5.68)

with
\( S_1^{(4)} = \mu A_1^{(4)} \)

\( S_2^{(4)} = a_1 A_2^{(4)} + a_2 \{ 4(A_1^{(1)} A_1^{(1)} + A_1^{(1)} A_1^{(1)} + 6 A_1^{(2)} A_1^{(2)} \} \),

\( S_3^{(4)} = \beta_1 A_3^{(4)} + \beta_2 \{ 4(A_1^{(1)} A_1^{(1)} + A_2^{(2)} A_1^{(2)} + 6(A_1^{(2)} A_2^{(2)} + A_2^{(2)} A_1^{(2)})) + \beta_3 [4(tr A_2^{(3)}) A_1^{(1)} + 6(tr A_2^{(2)}) A_1^{(2)}] \),

\( S_4^{(4)} = \gamma_1 A_4^{(4)} + 4\gamma_2 A_1^{(2)} A_1^{(2)} A_1^{(1)} + 6\gamma_3 A_2^{(2)} A_2^{(2)} A_1^{(1)} + 12\gamma_4 [A_2^{(2)} A_1^{(1)} A_1^{(1)} + A_1^{(1)} A_1^{(1)} A_2^{(2)}] \)

\( + 6\gamma_5 (tr A_2^{(2)}) A_2^{(2)} + 12\gamma_6 (tr A_2^{(2)}) A_1^{(1)} A_1^{(1)} \)

\( + 4\gamma_7 (tr A_2^{(3)}) A_1^{(1)} + 12\gamma_8 (tr A_2^{(2)}) A_1^{(1)} A_1^{(1)} \).  \quad (5.69) 

For \( k=4 \), the equations (5.9) become

\[ \nu^{(4)} = u^{(4)}(r,z)e_r + v^{(4)}(r,z)e_\theta + w^{(4)}(r,z)e_z \]

\[ p^{(4)} = p^{(4)}(r,z) \quad (5.70) \]

By making use of the results obtained at previous orders, we can reduce the relations in (5.67)-(5.69).

From (5.23), (5.41), (5.33) and previously observed zero tensors in (5.49) and (5.50) we note that
\[ A_3^{(4)} = A_4^{(4)} = 0. \] (5.71)

Similarly, using (5.48), we see that the last term in (5.67) is identically zero, i.e.,

\[ 6(A_1^{(2)} \text{ grad } v^{(2)}) + (A_1^{(2)} \text{ grad } v^{(2)})^T \\
+ (\text{ grad } A_1^{(2)})v^{(2)} = 0. \] (5.72)

Further use of (5.48)-(5.50) together with (5.32)-(5.33) readily shows that

\[ A_1^{(2)}A_1^{(2)} = 0, \quad A_1^{(1)}A_2^{(3)} + A_2^{(3)}A_1^{(1)} = 0, \]

\[ A_1^{(2)}A_2^{(2)} + A_2^{(2)}A_1^{(2)} = 0, \quad (\text{tr}A_2^{(3)})A_1^{(1)} = 0, \]

\[ (\text{tr}A_2^{(2)})A_1^{(2)} = 0, \quad (\text{tr}A_2^{(1)})A_1^{(1)} = 0, \]

\[ (\text{tr}A_2^{(2)}A_1^{(1)})A_1^{(1)} = 0, \quad A_1^{(1)}A_3^{(3)} + A_3^{(3)}A_1^{(1)} = 0. \]

Also, as a consequence of (5.71) and (5.73), we observe that

\[ S_3^{(4)} = 0. \] (5.74)

Thus, in view of (5.71)-(5.74), we rewrite the reduced form of (5.67)-(5.69) as follows:
\[ A_1^{(4)} = \text{grad } v^{(4)} + (\text{grad } v^{(4)})^T, \]

\[ A_2^{(4)} = [4((\text{grad } A_1^{(3)})v^{(1)} + A_1^{(3)})\text{grad } v^{(1)} + (A_1^{(3)}\text{grad } v^{(1)})^T + 4((\text{grad } A_1^{(1)})v^{(3)} + A_1^{(1)}\text{grad } v^{(3)})^T], \]

\[ S^{(4)} = S_1^{(4)} + S_2^{(4)} + S_4^{(4)} \]  

(5.75)

with

\[ S_1^{(4)} = \mu A_1^{(4)} \]

\[ S_2^{(4)} = a_1 A_2^{(4)} + 4a_2(A_1^{(1)} A_1^{(3)} + A_1^{(3)} A_1^{(1)}), \]

\[ S_4^{(4)} = 6\gamma_3 A_2^{(2)} A_2^{(2)} + 12\gamma_4[A_2^{(2)} A_1^{(1)} A_1^{(1)} + A_1^{(1)} A_1^{(1)} A_2^{(2)}] \]

\[ + 6\gamma_5(\text{tr}A_2^{(2)}) A_2^{(2)} + 12\gamma_6(\text{tr}A_2^{(2)}) A_1^{(1)} A_1^{(1)} \]  

(5.77)

Now, from third order solution, \( v^{(3)} = \hat{\rho} e_r + v^{(3)}(r)e_y + v^{(3)}(r)e_z \) and with the use of (5.56), it follows that
\[ A_1^{(3)} = \begin{bmatrix} 0 & \frac{dv^{(3)}}{dr} - \frac{v^{(3)}}{r}, & \frac{dw^{(3)}}{dr} \\ \frac{dv^{(3)}}{dr} - \frac{v^{(3)}}{r}, & 0, & 0 \\ \frac{dw^{(3)}}{dr}, & 0, & 0 \end{bmatrix} \] (5.78)

where, \( \text{grad } v^{(3)} = \begin{bmatrix} \frac{dv^{(3)}}{dr} \\ 0, & 0 \end{bmatrix} \)

and these results in combination with (5.32), (5.23) and (5.75)_2 yield

\[ A_2^{(4)} = 16 \left( \frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r} \right) \left( \frac{dv^{(3)}}{dr} - \frac{v^{(3)}}{r} \right) \\
+ \frac{dv^{(1)}}{dr} \frac{dv^{(3)}}{dr} \begin{bmatrix} 1, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{bmatrix} \] (5.79)
Further use of (5.78) and (5.32) shows that

\[
2\left[ \left( \frac{dv}{dr} - \frac{v}{r} \right) \left( \frac{dv}{dr} - \frac{v}{r} \right) + \frac{dw}{dr} \right], \quad 0, \quad 0
\]

\[
0, \quad 2\left[ \left( \frac{dv}{dr} - \frac{v}{r} \right) \left( \frac{dv}{dr} - \frac{v}{r} \right) \right], \quad \frac{dw}{dr} \left( \frac{dv}{dr} - \frac{v}{r} \right)
\]

\[
A^{(1)} A^{(3)} + A^{(3)} A^{(1)} = \begin{bmatrix}
A^{(1)} A^{(3)} + A^{(3)} A^{(1)} = \\
\frac{dw}{dr} \left( \frac{dv}{dr} - \frac{v}{r} \right) + \frac{dw}{dr} \left( \frac{dv}{dr} - \frac{v}{r} \right)
\end{bmatrix}
\]

(5.80)
Substituting (5.79), (5.80) into (5.77) we find that:

\[ S_2^{(4)} = \begin{bmatrix}
8(2a_1 + a_2) \left( \frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r} \right) \left( \frac{dv^{(3)}}{dr} - \frac{v^{(3)}}{r} \right) + \frac{dw^{(1)}}{dr} \frac{dw^{(3)}}{dr} \\
0, \quad 8a_2 \left( \frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r} \right) \left( \frac{dv^{(3)}}{dr} - \frac{v^{(3)}}{r} \right) + 4a_2 \left( \frac{dw^{(1)}}{dr} \left( \frac{dv^{(3)}}{dr} - \frac{v^{(3)}}{r} \right) + \frac{dw^{(3)}}{dr} \left( \frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r} \right) \right)
\end{bmatrix} \]

(5.81)
Employing (5.32)-(5.33) in (5.77)\textsuperscript{3}, we obtain

\[
S_4^{(4)} = \begin{bmatrix}
96(\gamma_3^4 + \gamma_4^4 + \gamma_5^4 + \frac{\gamma_6^4}{2}[(\frac{d\nu}{dr} - \frac{\nu}{r})^2 + (\frac{d\omega}{dr})^2] \\
0, 48\gamma_6(\frac{d\nu}{dr} - \frac{\nu}{r})^2[(\frac{d\nu}{dr} - \frac{\nu}{r})^2 + (\frac{d\omega}{dr})^2] \\
0, 48\gamma_6(\frac{d\omega}{dr})^2[(\frac{d\nu}{dr} - \frac{\nu}{r})^2 + (\frac{d\omega}{dr})^2] \\
0, 48\gamma_6(\frac{d\nu}{dr} - \frac{\nu}{r})[(\frac{d\nu}{dr} - \frac{\nu}{r})^2 + (\frac{d\omega}{dr})^2] \\
48\gamma_6(\frac{d\omega}{dr})^2(\frac{d\nu}{dr} - \frac{\nu}{r})^2 + (\frac{d\omega}{dr})^2)
\end{bmatrix}.
\]
Finally, using (5.70)\(_1\) in (5.75)\(_1\), we may write, analogous to (5.17), (5.36) and (5.56),

\[
A_1^{(4)} = \begin{bmatrix}
\frac{2\partial u^{(4)}}{\partial r} & \frac{\partial v^{(4)}}{\partial r} - \frac{v^{(4)}}{r} & \frac{\partial u^{(4)}}{\partial z} + \frac{\partial w^{(4)}}{\partial r} \\
\frac{\partial v^{(4)}}{\partial r} - \frac{v^{(4)}}{r} & \frac{2u^{(4)}}{r} & \frac{\partial v^{(4)}}{\partial z} \\
\frac{\partial u^{(4)}}{\partial z} + \frac{\partial w^{(4)}}{\partial r} & \frac{\partial v^{(4)}}{\partial z} & 2\frac{\partial w^{(4)}}{\partial z}
\end{bmatrix}
\]

(5.83)

where

\[
\text{grad } \mathbf{v}^{(1)} = \begin{bmatrix}
\frac{\partial u^{(4)}}{\partial r} & -\frac{v^{(4)}}{r} & \frac{\partial u^{(4)}}{\partial z} \\
\frac{\partial v^{(4)}}{\partial r} & \frac{u^{(4)}}{r} & \frac{\partial v^{(4)}}{\partial z} \\
\frac{\partial w^{(4)}}{\partial r} & 0 & \frac{\partial w^{(4)}}{\partial z}
\end{bmatrix}
\]

Proceeding as before, we substitute (5.83)\(_1\) into (5.77)\(_1\) to obtain \(S_1^{(4)}\), use \(S_1^{(4)}\), (5.81), (5.82) in (5.76) to find \(S^{(4)}\) and then insert these values of \(S^{(4)}\) into (5.66), along with first, second, third order solutions and (5.77), we have
\[
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0
\]

\[
\frac{\partial p}{\partial r} + \nu \left[ \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \frac{\partial}{\partial r} + \frac{\partial^2 u}{\partial z^2} \right]
\]

\[
= -\frac{8(2a_1 + a_2)}{r} \frac{d}{dr} \left[ r \left( \frac{dv}{dr} - \frac{v(1)}{r} \right) \frac{dv(3)}{dr} - \frac{v(3)}{r} \right]
\]

\[
+ \frac{dw(1)}{dr} \frac{dw(3)}{dr} \right] - \frac{8\sigma}{r} v(1)v(3)
\]

\[
+ \frac{8\sigma_2}{r} \left[ \left( \frac{dv(1)}{dr} - \frac{v(1)}{r} \right) \left( \frac{dv(3)}{dr} - \frac{v(3)}{r} \right) \right]
\]

\[
- \frac{96}{r} (\gamma_3 + \gamma_4 + \gamma_5 + \gamma_6/2) \frac{d}{dr} \left[ r \left( \frac{dv(1)}{dr} - \frac{v(1)}{r} \right)^2 + \left( \frac{dw(1)}{dr} \right)^2 \right]
\]

\[
+ \frac{48\gamma_6}{r} \left[ \left( \frac{dv(1)}{dr} - \frac{v(1)}{r} \right)^2 \left( \frac{dv(1)}{dr} - \frac{v(1)}{r} \right)^2 + \left( \frac{dw(1)}{dr} \right)^2 \right],
\]

\[
- \frac{\partial p}{\partial z} + \mu \left[ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right] = 0,
\]

\[
\mu \left[ \frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} \frac{\partial}{\partial r} + \frac{\partial^2 v}{\partial z^2} \right] = 0. \quad (5.84)
\]

As in all earlier cases, we define the streamfunction

\[
\psi^{(4)}(r,z) \text{ through}
\]

\[
u^{(4)} = -\frac{1}{r} \frac{\partial \psi^{(4)}}{\partial z}, \quad w^{(4)} = \frac{1}{r} \frac{\partial \psi^{(4)}}{\partial r}, \quad (5.85)
\]

and observe that \((5.84)\) is satisfied identically. The
rest of the equations in (5.84), after eliminating the pressure \( p^{(4)} \), may be written as

\[- \frac{\mu}{r} E^{(4)} \psi^{(4)} = 0 \]

\[\mu \left[ \frac{\partial^2 \psi^{(4)}}{\partial r^2} + \frac{2}{\partial r} \left( \frac{\psi^{(4)}}{r} \right) + \frac{\partial^2 \psi^{(4)}}{\partial z^2} \right] = 0. \quad (5.86)\]

For this case, the boundary conditions (5.12), with the aid of (5.85) become

\[u^{(4)} = - \frac{1}{r} \frac{\partial \psi^{(4)}}{\partial z} = 0, \quad w^{(4)} = \frac{1}{r} \frac{\partial \psi^{(4)}}{\partial r} = 0,\]

\[v^{(4)} = 0, \text{ at } r = a \text{ and } r = b \quad (5.87)\]

The solution of (5.86) satisfying (5.87) has the form

\[v^{(4)} = 0, \text{ i.e., } u^{(4)} = v^{(4)} = w^{(4)} = 0 \quad (5.88)\]

From (5.88) and (5.84), we get

\[p^{(4)} = 8(2a_1 + a_2) \left[ \left( \frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r} \right) \left( \frac{dv^{(3)}}{dr} - \frac{v^{(3)}}{r} \right) + \frac{dw^{(1)}}{dr} \frac{dw^{(3)}}{dr} \right] + 96(\gamma_3 + \gamma_4 + \gamma_5 + \gamma_6/2) \left[ \frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r} \right]^2 + \left( \frac{dw^{(1)}}{dr} \right)^2 \]
\[ + \int \frac{r}{r} \left[ 16 \alpha_1 \left( \frac{dv^{(1)}}{dr} \right) \left( \frac{dv^{(3)}}{dr} \right) \right] \text{dr} + \int \frac{r}{r} \left[ 96 \gamma_3 + \gamma_4 + \gamma_5 \right] \]

\[ + \int \frac{r}{r} \left[ \left( \frac{dv^{(1)}}{dr} \right)^2 + \left( \frac{dw^{(1)}}{dr} \right)^2 \right] \text{dr} + 48 \gamma_6 \int \frac{d}{dr} \left( \frac{dv^{(1)}}{dr} \right)^2 \left( \frac{dv^{(1)}}{dr} \right)^2 \left( \frac{dw^{(1)}}{dr} \right)^2 \text{dr} \]

\[ + \int \frac{r}{r} \left[ 8 \delta v^{(1)} v^{(3)} \right] \text{dr}, \quad (5.89) \]

where \( v^{(1)} \), \( w^{(1)} \), \( v^{(3)} \) and \( w^{(3)} \) are given by (5.23), (5.62) and (5.63).

Now we summarize our results of the perturbation series through order 4. Recalling (5.8), (5.23), (5.26), (5.41), (5.42), (5.61), (5.65), (5.88) and (5.89), we have shown that

\[ v(r, z, \Omega) = \Omega v^{(1)} + \frac{\Omega^3}{3!} v^{(3)} + O(\Omega^5) \]

\[ f_p(r, z, \Omega) = \Omega (-Gz + C) + \frac{\Omega^2}{2!} p^{(2)}(z) + \frac{\Omega^3}{3!} (C_2) \]

\[ + \frac{\Omega^4}{4!} p^{(4)}(z) + O(\Omega^5). \]

The above may be written as

\[ u(r, z, \Omega) = 0 \]
\[ v(r,z;\Omega) = \Omega v^{(1)} + \frac{\Omega^3}{3!} v^{(3)} + 0(\Omega^5) \]
\[ w(r,z;\Omega) = \Omega w^{(1)} + \frac{\Omega^3}{3!} w^{(3)} + 0(\Omega^5) \]
\[ p(r,z;\Omega) = \Omega (-Gz + \mathcal{C}) + \frac{\Omega^2}{2!} p^{(2)}(r) + \frac{\Omega^3}{3!} (\mathcal{C}_2) + \frac{\Omega^4}{4!} p^{(4)}(r) + 0(\Omega^5) \] (5.90)

where the fields \( v^{(1)}, v^{(3)}, w^{(1)}, w^{(3)} \) and \( p^{(2)}, p^{(4)} \) are known exactly and explicitly.

For the later use, we also record here, by virtue of fourth order solution, that

\[ s^{(4)}_1 = 0. \] (5.91)

5.5 Torque, Normal Thrust, Volume Flux

In order to obtain the formulae for torque and thrust, we substitute the power series expansions of extra stress \( \sigma \) and pressure field into (5.1) and write

\[ T = -i\Omega \left[ \Omega p^{(1)} + \frac{\Omega^2}{2!} p^{(2)} + \frac{\Omega^3}{3!} p^{(3)} + \frac{\Omega^4}{4!} p^{(4)} + 0(\Omega^5) \right] \]
\[ + \left[ \Omega s^{(1)} + \frac{\Omega^2}{2!} s^{(2)} + \frac{\Omega^3}{3!} s^{(3)} + \frac{\Omega^4}{4!} s^{(4)} + 0(\Omega^5) \right]. \]

Collecting the alike terms in \( \Omega \), the above expression has the form,
\[ T = \Omega (-I_P(1) + S^{(1)}) + \frac{\Omega^2}{2!} (-I_P(2) + S^{(2)}) \]
\[ + \frac{\Omega^3}{3!} (-I_P(3) + S^{(3)}) + \frac{\Omega^4}{4!} (-I_P(4) + S^{(4)}) + O(\Omega^5) \]  
(5.92)

On using the definitions of \( S^{(k)} \), \( k = 1, 2, 3, 4 \), from (5.15), (5.29), (5.45) and (5.68), the above expression takes the form

\[ T = \Omega (-I_P(1) + S_1^{(1)}) + \frac{\Omega^2}{2!} (-I_P(2) + S_1^{(2)} + S_2^{(2)}) \]
\[ + \frac{\Omega^3}{3!} (-I_P(3) + S_1^{(3)} + S_2^{(3)} + S_3^{(3)}) \]
\[ + \frac{\Omega^4}{4!} (-I_P(4) + S_1^{(4)} + S_2^{(4)} + S_3^{(4)} + S_4^{(4)}) + O(\Omega^5) \]  
(5.93)

(a) Torque

The torque \( M \) per unit length to maintain the relative motion of bounding pipes is given by

\[ M = r (2 \pi r T_{\tau 0}) \]  
(5.94)

Since \( p^{(k)} \), \( k = 1, 2, 3, 4 \), \( S_1^{(2)}, S_2^{(2)}, S_3^{(3)}, S_4^{(4)} \), \( S_1^{(4)} \), and \( S_4^{(4)} \) have no shearing components, therefore, (5.93) has the form
\[ T_{r\theta} = \Omega S_{1r\theta}^{(1)} + \frac{\Omega^3}{3!} (S_{1r\theta}^{(3)} + S_{3r\theta}^{(3)}) + 0(\Omega^5) \] (5.95)

Substituting the values of \( S_{1r\theta}^{(1)} \), \( S_{1r\theta}^{(3)} \), \( S_{3r\theta}^{(3)} \) into (5.95), we get

\[ T_{r\theta} = \Omega \left[ \mu \left( \frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r} \right) \right] + \frac{\Omega^3}{3!} \left[ \mu \left( \frac{dv^{(3)}}{dr} - \frac{v^{(3)}}{r} \right) \right] 
+ 12(\beta_2 + \beta_3) \left\{ \left( \frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r} \right) \left( \frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r} \right) \right\} 
+ \left( \frac{dw^{(1)}}{dr} \right)^2 \} \] + 0(\Omega^5) \] (5.96)

Thus, (5.96) in combination with (5.94) gives

\[ M = 2\pi r^2 \left[ \Omega \left[ \mu \left( \frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r} \right) \right] + \frac{\Omega^3}{6} \left[ \mu \left( \frac{dv^{(3)}}{dr} - \frac{v^{(3)}}{r} \right) \right] 
+ 12(\beta_2 + \beta_3) \left\{ \left( \frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r} \right) \left( \frac{dv^{(1)}}{dr} - \frac{v^{(1)}}{r} \right) \right\} 
+ \left( \frac{dw^{(1)}}{dr} \right)^2 \} \right] + 0(\Omega^5) \] (5.97)

On inserting the values of \( v^{(1)}, w^{(1)}, v^{(3)} \) from first and third orders solutions into (5.97), we obtain
M = -4πμΩN + \frac{Ω^3π}{3}(β_2 + β_3) \left[ 48NB_0B_0 - 48NB_0^2 \frac{a^2b^2lnb/a}{b^2-a^2} \\
-12NB_1^2 \frac{b^4+a^2}{a^2b^2} \\
-32N^3 \frac{b^4+a^4+a^2b^2}{a^4b^4} + O(Ω^5) \right]

(5.98)

(b) Normal Thrust Difference

We observe from our analysis up to fourth order that

S_{1(2)} = S_{3(3)} = S_{1(4)} = S_{3(4)} = 0

and the normal components of S_{1(1)}, S_{1(3)} and S_{3(3)} are zeros.

Using all these informations in (5.93), we have

T_{\text{rr}} = Ω(-I_{\text{rr}}p^{(1)}) + \frac{Ω^2}{2!}(-I_{\text{rr}}p^{(2)} + S_{2\text{rr}}^{(2)}) + \frac{Ω^3}{3!}(-I_{\text{rr}}p^{(3)}) + \frac{Ω^4}{4!}(-I_{\text{rr}}p^{(4)} + S_{2\text{rr}}^{(4)})

+ S_{4\text{rr}}^{(4)} + O(Ω^5),

(5.99)

If we insert the values of p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)} from (5.26), (5.42), (5.65), (5.89) and the values of S_{2\text{rr}}^{(2)}, S_{2\text{rr}}^{(4)}, S_{4\text{rr}}^{(4)} from (5.35), (5.81), (5.82) into (5.99), we find that...
\[ T_{rr} = \Omega (-C + GZ) + \frac{\Omega^3}{3!} (-C_2) \]

\[-\frac{\Omega^2}{2!} \left[ \int \frac{1}{r} \left[ 4a_1 \left( \frac{dv(1)}{dr} - \frac{v(1)}{r} \right)^2 + \frac{dw(1)}{dr} \right] \right. \]

\[-\frac{\Omega^4}{4!} \left[ \int \frac{1}{r} \left[ 16a_1 \left( \frac{dv(1)}{dr} - \frac{v(1)}{r} \right) \left( \frac{dv(3)}{dr} - \frac{v(3)}{r} \right) \right.ight.

\[+ \frac{dw(1)}{dr} \frac{dw(3)}{dr} \left. \right] + 8a_2 \left[ \frac{dw(1)}{dr} \frac{dw(3)}{dr} \right] \]

\[+ 96(\gamma_3 + \gamma_4 + \gamma_5) \left[ \left( \frac{dv(1)}{dr} - \frac{v(1)}{r} \right)^2 \left( \frac{dw(1)}{dr} - \frac{v(1)}{r} \right)^2 \right. \]

\[+ 8\rho v(1) v(3) + 48\gamma_6 \left( \frac{dw(1)}{dr} \right)^2 \left( \frac{dv(1)}{dr} - \frac{v(1)}{r} \right)^2 \]

\[+ \left( \frac{dw(1)}{dr} \right)^2 \} \right] dr \]

Thus, the expression for the difference

\[ \Delta T_{rr} = T_{rr}|_b - T_{rr}|_a \]

in the normal stresses at outer and inner pipes is given as.
\[ a_{rr} = -\frac{\omega^2}{2r} \left[ a \int \frac{1}{r} \left( 4a_1 \left[ \left( \frac{dv}{dr} - \frac{v}{r} \right)^2 + \left( \frac{dw}{dr} \right)^2 \right] + 2a_2 \left( \frac{dv}{dr} \right)^2 + 2\rho v(1)^2 \right) dr \right] \]

\[ = \frac{\omega^4}{4!} \left[ a \int \frac{1}{r} \left( 96 \left( \frac{dv}{dr} - \frac{v}{r} \right)^2 + \left( \frac{dw}{dr} \right)^2 \right) dr \right] 
+ 8a_2 \frac{dv}{dr} \frac{dw}{dr} \right] \]

\[ + 48 \rho v(1)^2 \left( \frac{dv}{dr} - \frac{v}{r} \right)^2 + \left( \frac{dw}{dr} \right)^2 \]

\[ + 8\rho v(1)^2 \frac{dv}{dr} \frac{dw}{dr} \right] \]

(c) **Volume Flux**

The volume discharge \( Q \) per unit time through a cross-section perpendicular to the pipes is given by

\[ Q = 2\pi \int_a^b r w dr \]

On using \((5.90)\), the above formula may be written as
\[ Q = 2\pi \Omega \left[ a \int_{a}^{b} \omega(1) \, dr + \frac{2\pi \Omega^3}{3} a^{1} \int_{a}^{b} \omega(3) \, dr + 0(\Omega^5) \right] \]

(5.101)

If we substitute \( \omega(1) \), \( \omega(3) \), from first and third order solutions into (5.101) and perform the necessary integration, we get

\[ Q = 2\pi \Omega \left[ \frac{G}{16\mu} \left( \frac{b^4-a^4}{b^2-a^2} \right) + \frac{\left( b^2-a^2 \right)^2}{2} \right] \]

\[ + \frac{2\pi \Omega^3}{3} \left( \frac{B_2 + B_3}{\mu} \right) \left[ \frac{12N^2B_1}{b^2-a^2} \left( 1 - \frac{b^4-a^4}{4b^2a^2lnb/a} \right) \right] \]

\[ + 6\left( B_1^3 - 4N^2B_0 \right) \left( ln \frac{b^2-a^2}{a^2} \right) \]

\[ - 3B_0 \left( \frac{b^6-a^6}{3} - \frac{(b^4-a^4)(b^2-a^2)}{4lnb/a} \right) \]

\[ + 18B_0^2B_1 \left( \frac{b^4-a^4}{4} - \frac{(b^2-a^2)^2}{4lnb/a} \right) + 0(\Omega^5) \]

(5.102)
5.6 Discussion

We now analyze the results of the perturbation technique carried out up to fourth order. The first order velocity field is given as

$$\mathbf{v}^{(1)} = (u^{(1)}, v^{(1)}, w^{(1)}) = [0, M r + \frac{N}{r} \left( \frac{U \ln r/a}{\ln b/a} \right) + \frac{G}{4u} \left( a^2 - r^2 + (b^2 a^2 \ln r/a) \right) \ln b/a]$$

(5.103)

where

$$M = \frac{\lambda b^2 - a^2}{b^2 - a^2}, \quad N = \frac{(1 - \lambda) a^2 b^2}{b^2 - a^2}.$$

The flow described by (5.103) is called Spiral Poiseuille-Couette flow. The following special cases, which are interesting in themselves are covered in the above solutions.

(i) Pressure flow: \( U = 0 \)

(ii) Spiral Couette flow: \( G = 0 \)

(iii) Rotating Couette flow: \( \lambda = 1, \ G = 0 \)

(iv) Circular Couette flow: \( U = 0, \ G = 0 \)

(v) Rotating Poiseuille flow: \( \lambda = 1, \ U = 0 \)

Similarly, we can obtain the special cases at third order velocity field (5.61). For example, if we set \( U = G = 0 \) in (5.61), we find that
\[ \mathbf{v}^{(3)} = (u^{(3)}, v^{(3)}, w^{(3)}) = [0, \frac{-(b_2 + b_3)}{\mu}(16N^3 \left( \frac{1}{z^5} \right) + \frac{b_2^2 + \frac{2}{4} \frac{2}{a b^2}}{r} - \frac{b_4 + a^2 + \frac{2}{a b^2}}{r} \), 0] \]

(5.104)

It is noteworthy that the above expression was also obtained by Joseph and Fosdick [1973,2] in their work. A significant result is that the velocity field is not dependent upon the normal stress functions.

From the second and fourth orders pressure fields, we observe that these expressions heavily depend upon the material constants, \(a_1, a_2, Y_3, Y_3, Y_5, \) and constant density \( \rho \). We further note that the structure of these expressions are somewhat similar to that obtained by Rivlin [1956,1]. However, the velocity components are not known explicitly in the above mentioned case.

We remark that the torque \( M \) is not a function of \( r \), indicating that \( M \) is the torque transmitted to the walls of the pipes. We find that provided \( (b_2 + b_3) > 0 \), the magnitude of torque \( M \) is always smaller than its value in the case of viscous fluid. The same remark also applies for the reduction of the volume discharge in comparison to the linearly viscous fluid. However, if \( (b_2 + b_3) < 0 \), the torque and volume discharge both increase considerably.
Finally, we also point out that in the case of a simple fluid one finds the excess normal stresses on the walls of the pipes, but are completely absent in the linearly viscous fluid case.
REFERENCES

[1915]

[1916]

[1951]


[1955]

[1956]

[1958]

[1959]


[1960]
1. B. D. Coleman and W. Noll: An approximation Theorem for functionals, with applications in continuum


[1961]


[1963]


[1964]


[1965]


[1966]


[1971]

[1973]


[1974]


[1975]

[1976]


[1977]


[1978]

[1979]


[1980]


[1981]


[1983]


[1984]


[1986]

VITA AUCTORIS

1952  Born on 15th of September at Kallar Syedan, Rawalpindi, Pakistan.

1973  Obtained B.Sc. degree from Punjab University, Lahore, Pakistan.

1975  Received M.Sc. degree in Applied Mathematics from the University of Islamabad, Pakistan.

1976  Obtained M.Phil. degree in Mathematics from the University of Islamabad, Pakistan.

1976  Appointed as an Aerodrome Officer at Karachi Airport, Pakistan.

1981  Awarded Teaching and Research Assistantship for Graduate Studies at the University of Windsor.

1982  Graduated with Master of Science in Applied Mathematics from the University of Windsor.

Present  Candidate for the degree of Ph.D. in Mathematics at the University of Windsor.