Forward-recursive adaptive control.

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FORWARD-RECURSIVE ADAPTIVE CONTROL

by

JOHN CHRISTOPHER LUXAT

A Dissertation

Submitted to the Faculty of Graduate Studies through the Department of Electrical Engineering in Partial fulfillment of the requirements for the Degree of Doctor of Philosophy at the University of Windsor

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ABSTRACT

The application of modern control theoretic techniques to solve dynamic optimization problems requires that the equations and parameters characterizing the system dynamics be known. In many cases however, certain system parameters may be of uncertain value, thereby resulting in the need for parameter adaptive control strategies capable of reducing the parameter uncertainty.

The adaptive control problem for a class of nonlinear systems is considered herein. The control problem is posed as a least-squares 'servomechanism' problem coupled with a linear modelling problem. The objective of the linear modelling problem is to generate a sequence of linear approximations to the nonlinear dynamics which will yield both the unknown parameters and inaccessible state variables. It is shown that the control solution must be forward-recursively generated, concurrent with, and based upon, the solution to the modelling problem. A solution to the control problem is obtained through application of the invariant imbedding concept, together with a specific partitioning of the dynamic equations. Conditions for uniqueness of the control vector are obtained and subsequently shown to hold for the derived control algorithm.

A major objective of this work was to obtain a viable approach to real-time optimal adaptive control. This is demonstrated experimentally through application of the control algorithm to the real-time adaptive control of a plasma torch furnace model. Certain limitations of adaptive control, which are inherent in the interaction of the control and parameter identification processes, are shown, and a criterion for assessing the overall efficiency of an adaptive algorithm is proposed.
At the outset of this research the problem of obtaining a valid electrothermal model of a plasma torch heating device was proposed. Such devices have considerable potential in applications requiring high temperature, low contamination thermal sources. As a result of extensive investigation a generalized mathematical model of an A.C. plasma torch has been obtained. The derivation of this model is presented in Appendix I of this dissertation.
ACKNOWLEDGEMENTS

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CHAPTER 1
INTRODUCTION

1.1 The Concept of Adaptive Control

When applying control theory to practical systems the engineer may often encounter difficulties resulting from uncertain knowledge of the characteristics of a system. Specifically, system uncertainty may be divided into two categories; viz. (1) structural uncertainty and (2) parameter uncertainty. In the first category the form, or structure, of a mathematical model representing the system is not completely known; while, in the second category, a prespecified structure (e.g. nth order state equations) is known, but certain of the parameters characterizing the system model are imprecisely known. It is the latter category of uncertain systems which will be considered herein. The problem of controlling systems under conditions of parameter uncertainty has been generally termed the adaptive control problem - the adjective "adaptive" denoting the ability of the control system to modify its behaviour in such a manner as to reduce the effects of system uncertainty.

A number of differing definitions of adaptive control systems have appeared in the literature, e.g. Zadeh [1], Tsypkin [2], Bellman [3]. In this work an adaptive control system will be defined as follows,

Definition 1.1

A control system is adaptive if it assesses, either directly or indirectly, its performance relative to some criterion, J, and subsequently modifies its behaviour in such a manner that, within a finite time of operation, the system performance is forced to, and maintained within, a small region bounded by an extremum value of J.
With the above definition in mind, the structure of an adaptive control system may be decomposed into three fundamental operational processes: (i) performance assessment, (ii) decision and, (iii) system modification, as shown in Figure 1.1.

![Diagram of Adaptive Control System]

**FIG. 1.1 BASIC STRUCTURE OF AN ADAPTIVE SYSTEM**

The manner in which these basic processes are implemented varies greatly in the numerous adaptive schemes which have been proposed in the literature. In some schemes the processes are explicitly identifiable elements in the adaptive control loop, whilst in others the processes are not separately identifiable but are implicitly contained within a computational algorithm.
In the next section a brief survey of adaptive control schemes will be presented. It is not intended that this survey give an exhaustive coverage of the multiplicity of proposed techniques; rather, a general classification will be attempted, and the similarities and differences between the broad categories will be detailed.

1.2 A Survey of Adaptive Control Techniques

1.2 (a) High gain schemes [4-7]

These schemes operate on the principle that, by adjusting the loop gain in a closed loop system, a predefined output response characteristic may be maintained for all operating conditions encountered. The performance is assessed directly in terms of the response characteristic, e.g. frequency or amplitude of a limit cycle. In terms of commercial applications, such as aircraft autopilots, these schemes are probably the most widely implemented techniques to date.

1.2 (b) Parameter perturbation schemes [8-15]

In such schemes an operating characteristic of the controlled plant (e.g. efficiency) is measured directly by suitable instrumentation, and serves as a performance criterion. Continuous or incremental variations are made to one or more parameters of the plant or controller, in order to ascertain a suitable direction of parameter modification such that the performance criterion attains an extremum value. Disadvantages of these schemes are, (a) in multivariable, multiparameter systems difficulties may be encountered in isolating the effects of the various parameter perturbations and, (b) the continuous parameter perturbations produce disturbances in the system variables which, in some cases, cannot be tolerated.
1.2 (c) Model reference techniques [16-20]

The approach adopted in these schemes is based upon matching the response of the system driven by control inputs, with a 'desired' response generated by a model. Typically, the squared error between the actual system response and desired response is utilized as a performance criterion, and the parameters of feedback and/or feedforward controllers are adjusted to minimize the performance criterion. Within this general approach variations may be found in the manner in which suitable parameter adjustment strategies are selected. The methods of steepest descent, conjugate gradients [16, 17, 19], Lyapunov stability theory [18, 19] and a combination of the aforementioned methods [20] have been utilized in designing model reference adaptive systems.

Features common to the previous three general categories are:
(i) design is predominantly performed in the frequency domain (i.e. classical control techniques are utilized), (ii) the structure of the control law is specified a-priori and, (iii) the processes of performance assessment, decision and modification may be explicitly identified in the adaptive loop.

1.2 (d) Optimal-adaptive techniques [21-49]

This category is comprised of adaptive schemes which have evolved as an outgrowth of optimal control theory. As such, it is distinct from the previous three categories in that design is predominantly via time domain techniques; the control law is not specified a-priori, but is obtained as a solution to an optimization problem; and, the three fundamental processes of an adaptive loop are not specifically identifiable. Another major distinction of this class of adaptive techniques is the presence of a parameter identification process in the adaptive
loop—the function of this process being to reduce system uncertainty by estimating the values of unknown parameters.

Owing to the primary importance of parameter identification in these adaptive schemes, a sizable amount of research in this field has been directed toward the study of identification strategies. Proposed approaches include the parameter tracking servo [21], impulse response determination via cross-correlation [22, 23] and the direct methods such as least squares [24, 25], maximum likelihood [26], stochastic approximation [27, 28] and regression analysis [29].

At the present time a general theory of optimal adaptive control has not been developed, nor is there any indication that a general theoretical framework will be forthcoming in the near future. However, the fundamental theoretical results of Fel'dbaum's 'dual control theory' [30] and the work of Sworder [31,32] have highlighted the interaction between control and identification in optimal adaptive systems. It is this interaction which leads to difficulties when applying the mathematical formalism of optimal control theory to adaptive systems, since a separation principle, similar to that of optimal linear stochastic control, is not valid. Furthermore, when nonlinear systems are considered, additional difficulties, of both a theoretical and computational nature, are encountered. In light of these difficulties, it is understandable that the majority of optimal adaptive schemes reported in the literature have been limited to linear systems; with approaches differing in the manner in which suboptimal approximations for the control policy are obtained.

By means of a suitable transformation of variables it is always possible to obtain the solution to a linear optimal control problem
(with a performance criterion quadratic in the control variables) as a backward recursive matrix Ricatti equation, which generates a sequence of control feedback gains. When adaptive features are incorporated, it becomes necessary to completely recompute this sequence of gain matrices following a new estimate of the uncertain system parameters. Strict adherence to the formalism of optimal control theory has led to the open-loop-feedback-optimal approach [33, 34], in which complete recomputation of the control gain sequence is performed over the remaining time-to-go of control operation. Certain suboptimal approaches involve discrete time optimization over either a single interval [35,36], or a fixed subinterval of future control operation [37, 38] — the 'window-shifting' technique. In other suboptimal approaches [39, 40], separation between the identification and control processes is enforced, and a complete gain sequence over the entire operating interval is precomputed and stored. Following an update of the estimated parameters an incremental gain sequence is computed and added to the previously stored gain sequence. Suboptimal approximation to the solution of Bellman's equation of dynamic programming has also been considered [41].

The problem of optimal adaptive control of systems with nonlinear dynamics has received less attention than the counterpart linear adaptive problem. The reasons for this are, most likely, the increased computational complexity of nonlinear optimal control solutions and the increasingly tenuous extension of optimal control theory to nonlinear adaptive problems. The standard nonlinear dynamic optimization techniques, such as the generalized Newton-Raphson method [42, 43], quasilinearization [44], gradient methods [45, 46] and the method of second variations [47] are characterized by time consuming iterative
approaches to the optimal solution, and excessive computer memory
storage requirements. Furthermore, as the order of the system dynamics
increases, the computation time and storage requirements increase
extremely rapidly—factors which effectively inhibit the application
of these techniques in a real time control environment. Approaches to
the nonlinear problem have included; enforced separation of identifica-
tion and control with a quasilinearization solution of the control
optimization problem [48], and a steady-state optimization formulation
[49], among others.

A common feature of the majority of optimal-adaptive schemes is
their adherence to the concept of duality of optimal control and
optimal estimation problems. This duality implies that a forward
recursive optimal estimation solution has a dual backward recursive
solution—which is a solution to an optimal control problem. Utilizing
this duality, some degree of separation between control and estimation
is enforced to obtain a solution to the optimal-adaptive problem.
However, it is exactly this separation which represents the tenuous link
in the extension of optimal control theory to adaptive systems.

1.3 Problem Statement and Procedure

Adaptive control of discrete-time nonlinear systems, with a subset
of uncertain system parameters, is considered in this study. The
approach adopted treats the adaptive problem as a control optimization
problem coupled to an approximate system modelling problem. The control
policy is necessarily suboptimal, since it is evaluated on the basis of
information generated by a sequence of neighbouring linear approxima-
tions to the nonlinear system dynamics. It is shown that the modelling
approximation imposes the necessity for the control policy to be forward-
recursively generated. To this end, the concept of invariant imbedding, together with a suitable partition of state space, are utilized in deriving a solution to the control problem.

A major objective of this work is to obtain an adaptive control algorithm which:

1. is capable of operating in a real-time control environment for non-trivial nonlinear multivariable systems (i.e. system dynamics of order > 3),

2. can be implemented on present-day minicomputers with a minimum (4K) of core storage,

3. does not require observation of all the state variables of the system.

In view of the stringent requirements imposed upon the adaptive algorithm, as given above, it is evident that the ultimate thrust of this work is directed more toward practical application, than it is to the realm of abstract, rigorous, mathematical theory.

1.4 Organization

In Chapter 2 the optimal-adaptive control problem is formulated in a formal manner and a suboptimal approach to the solution of this problem is developed. Basic concepts of nonlinear and linear approximate system representations are presented and criteria for optimality, pertinent to the adaptive control problem, are formulated. The basic canonic equations characterizing the control and modelling optimization problems are obtained through application of the discrete maximum principle.

The adaptive control algorithm is derived in Chapter 3 as a set of forward recursive matrix equations. The concept of invariant imbedding
and a partitioning of state space are utilized in obtaining a forward-recursive solution to the control problem. Conditions for uniqueness and "partition invariance" of the control vector are derived, and a proof is given to demonstrate that these conditions are upheld by the control vector generated by the algorithm.

Application of the algorithm to real-time adaptive control of a plasma torch furnace model is considered in Chapter 4. Extensive experimental results, detailing the performance of the control algorithm under different operating conditions are presented. A discussion of these results appears in Chapter 5.

At an early stage in this research, a study of a.c. plasma torch heating devices was conducted. The outcome of this work was a mathematical model describing the electrothermal characteristics of such devices. The model is detailed in Appendices I and II.
CHAPTER 2

FORMULATION OF AN ADAPTIVE CONTROL STRATEGY

2.1 Description of System Equations

The class of systems considered are those described by a vector set of nonlinear, discrete-time state transition equations. It is assumed that the structure of the state equations is known, whilst a subset of parameters characterizing the system are unknown. Both bias and noise inputs may be present in the system. Not all the state variables are accessible and the available observations are assumed to be, in the general case, nonlinear functions of the state variables. In addition, the observations may be corrupted by measurement noise. The set of equations describing the system dynamics and observations are,

\[ x_{k+1} = f(x_k, a_k, u_k) + d_k + G w_k \]  \hfill (2.1a)

\[ a_{k+1} = a_k \]  \hfill (2.1b)

\[ y_k = h(x_k) + v_k \]  \hfill (2.1c)

Where, \( x_k \) is an \( n \)-dimensional state vector, \( u_k \) is an \( m \)-dimensional control vector, \( a_k \) an \( r \)-dimensional vector of uncertain system parameters, \( d_k \) is an \( n \)-dimensional vector of known bias inputs and \( w_k \) is an \( s \)-dimensional noise vector. The observation and measurement noise vectors, \( y_k \) and \( v_k \) respectively, are of dimension \( e \). It is assumed that the noise vectors are not well defined processes (such as 'white' Gauss-Markov sequences), and that the available information about these processes is,

i) they are zero mean; i.e. \( E[w_k] = 0, E[v_k] = 0 \)

ii) they are bounded by the constraint sets:
\begin{align}
N(0, \tau_1) &= \left\{ w_k \mid \frac{1}{N} \sum_{j=1}^{M+N} w_{ij}^2 \leq x_{1w}^2, \quad i = 1, 2, \ldots, s; \quad \forall w_k, N, M; \quad N \gg 1 \right\} \\
N(0, \tau_2) &= \left\{ v_k \mid \frac{1}{N} \sum_{j=1}^{M+N} v_{ij}^2 \leq x_{1v}^2, \quad i = 1, 2, \ldots, s; \quad \forall v_k, N, M; \quad N \gg 1 \right\}
\end{align}

\subsection*{2.2 Statement of the Control Problem}

Prior to stating the control problem a number of preliminary concepts will be defined. Let \( E^n \) represent \( n \)-dimensional Euclidean state space and \( E^m \) the \( m \)-dimensional Euclidean control space; with \( \Omega_u \subseteq E^m \) the set of admissible control inputs [50, 51]. Let \( \Omega_x \) be a bounded, reachable region in \( E^n \) such that \( \forall u_k \in \Omega_u \) and \( k \in T \), (\( T \) being the set of integers indexing the discrete time points), the system state vector are mapped into \( \Omega_x \) by the state transition equation (2.1a). A number of basic concepts required for the definition of the general control problem will now be defined.

\begin{definition}[Controlled states]
A subset of the states, of dimension \( q \), \( q \ll n \), are required to assume specific values under the influence of the control vector. This subset \( x_k^c \), are the controlled states, and are given by the linear transformation \( x_k^c = \psi x_k \). (Note: if all the states are to be controlled \( \psi = I \), the identity matrix).
\end{definition}

\begin{definition}[Target trajectory]
The vector sequence of values \( \rho_k \) (\( k = 1, 2, \ldots \)) that the controlled states are required to assume is termed the target trajectory of control.

A necessary condition for the existence of an admissible control is that the space of the target trajectory \( \Omega^P \subseteq E^q \) is a bounded region.
such that \( \mathcal{U}^D \cap (\mathbb{R}^q - \Omega_x^C) = [0] \), the null set. That this is a necessary condition is obvious, since \( \mathcal{U}^D \cap (\mathbb{R}^q - \Omega_x^C) \neq [0] \) implies that there are regions of target space which cannot be reached through application of a suitable admissible control. The general control problem will now be defined as an optimal "servomechanism" problem.

**Definition 2.3 (The general control problem)**

The sequence of admissible control vectors \( \hat{u}_k \in \Omega_u \) are required such that the controlled states track the target trajectory, and a performance functional \( J(e_k, u_k) \) is minimized; i.e.

\[
J(e_k, \hat{u}_k) = \min_{u_k \in \Omega_u} \{ J(e_k, u_k) \}
\]

where \( e_k = \rho_k - x_k^C \) is the error between the desired and actual values of the controlled states.

The purpose of introducing the functional, \( J \), in the control problem, is to induce an ordering of the control vector based upon a criterion of 'goodness' of control; i.e. minimum \( J \). In order to obtain a unique optimal ordering of the control vector sequence it is imperative that complete knowledge of the system dynamics be available [52,53]. However, since the system under consideration contains a subset of unknown parameters, it is not possible to directly utilize the extensive results of optimal control theory to induce a unique ordering on the control vector. In fact, as will be seen in Chapter 4, the use of estimates of the unknown parameter vector, \( \hat{\alpha}_k \), will always result in a partial ordering of the control vector; i.e. there are \( u_k^* \) such that,

\[
J(e_k, u_k^*) \leq J(e_k, \hat{u}_k | \hat{\alpha}_k)
\]

Partial ordering of the control vector sequence is inherent in parameter adaptive control systems, and originates from the improvement
in control performance which is forfeited by the system in order to gain
information about the uncertain parameters.

In this case, the uncertain system parameters are not the sole cause
of partially ordered control vectors; the nonlinear system dynamics are
an additional contributory cause. Solutions to nonlinear problems are
invariably based upon simplifying approximations, which in turn intro-
duce errors into the solution — thus resulting in performance degradation,
and a partially ordered control vector sequence. Techniques for obtain-
ing strictly optimal control policies for nonlinear adaptive systems
belong to an 'as-yet-unsolved' class of problems. Even if such techniques
were available, it is highly probable that they would be far too complex
for real-time, on-line control applications in all, but the most trivial,
systems. For these reasons, emphasis will be placed upon suboptimal
control policies from the outset. The approach adopted will be to intro-
duce approximations at the modelling stage, rather than at the solution
stage. A suboptimal control policy based upon the approximate system
representation will be obtained — this is, in essence, a form of 'control
by model', since the control policy will be dependent upon some model
generated by the algorithm.

Firstly, the basic concepts of approximate system representations
will be developed, and secondly, the hierarchical ordering of sub-
optimal control policies will be defined.

2.3 Approximate System Representations

The set of equations (2.1 a–c) describing the system dynamics and
observations comprise a nonlinear system representation, denoted as
$\Sigma_n$. Apart from the fact that certain parameters of $\Sigma_n$ are uncertain,
there is a basic lack of certainty regarding the system states, which
arises from the presence of the noise terms, \( w_k \) and \( v_k \), and the fact that not all the states are accessible. It is necessary, therefore, to form approximations to the system \( F_n \), utilizing the known structure of the system dynamic equations. The process of defining the overall structure of an approximate system representation will be termed 'model structuring'; while the process of generating approximate system state vectors and parameters will be termed 'modelling'.

Owing to the presence of the uncertain parameter vector \( a_k \), any approximate system representation for \( F_n \) will be dependent upon the choice of suitable values for the parameter vector. If \( a \) is continuous, or piecewise continuous, in \( \Omega_a \subset \mathbb{R}^m \) (the bounded space of the parameter vector) then it follows that there are an infinite number of possible approximate system representations. The dependence of the model representations upon the parameter vector will be incorporated at the model structuring stage by defining 'parameter-conditioned' approximate system representations; i.e. the approximate representation is conditioned by the selection of values for the uncertain parameter vector. The manner in which suitable values for the parameter vector are selected is a problem to be solved at the modelling stage.

The first step in the approximation procedure is to select the structure of the system representations; i.e. model structuring. The following approximate system representations are defined.

2.3a The basic approximate representation

Given the nonlinear system \( F_n \) described by equations (2.1 a-c), with the noise terms \( w_k \) and \( v_k \) bounded by the constraint sets of equations (2.2 a-b), an approximate system representation \( \hat{F}_n \) may be defined such that,
i) there is a parameter conditioned sequence,

\[ (\hat{\mathbf{x}}_k, y_k \mid \hat{\mathbf{a}}_k) = \{ (\hat{\mathbf{x}}_j, \hat{\mathbf{y}}_j \mid \hat{\mathbf{a}}_j) \mid j = 0, 1, 2, \ldots, k \} \subseteq \Omega_x \times \Omega_y \times \Omega_a \]

generated according to,

\[ \begin{align*}
\mathbf{q}_{k+1} &= f(\mathbf{q}_k, \hat{\mathbf{a}}_k, u_k) + \mathbf{a}_k + \hat{\mathbf{w}}_k \\
\hat{\mathbf{a}}_{k+1} &= \hat{\mathbf{a}}_k \\
\hat{\mathbf{y}}_k &= h(\mathbf{q}_k) \\
\mathbf{g}_k &= \xi(y_k, \hat{\mathbf{y}}_k)
\end{align*} \]  

(2.3)

with, iii) \((x_{k+1} - \hat{x}_{k+1}) \in N(0, r_{x_{k+1}})\)

\[(y_k - \hat{y}_k) \in N(0, r_{y_k})\]  

(2.4)

where,

\[ N(0, r_{x_{k+1}}) \subseteq N(0, r_x) \cup N(0, h(x_k)) \]

\[ N(0, r_{y_k}) \subseteq N(0, G x_k) \cap N(0, \xi(y_k)) \]  

(2.5)

Note that a number of fundamental concepts are incorporated in the structure of the basic approximate representation \(\hat{\mathbf{y}}_n\). They are:

(a) Information transmission, from the actual system \(\Sigma_n\) to the approximate representation \(\hat{\mathbf{y}}_n\), is present due to the term \(\hat{\mathbf{q}}_k = \xi(y_k, \hat{\mathbf{y}}_k)\). The form of the transmission function \(\xi\) is not defined as yet — it will be defined at the modelling stage.

(b) The information transmission can lead to contraction of the constraint sets, \(N(0, r_{x_{k+1}})\) and \(N(0, r_{y_k})\). This corresponds to improvement in the approximating sequence \((\hat{x}_k, \hat{y}_k \mid \hat{a}_k)\) — see Fig. 2.1.
FIG. 2.1 DEPICTION OF CONSTRAINT SET
CONTRACTION
(c) Without information transmission the constraint sets remain fixed (i.e. there is no decrease in the system uncertainty), and are given by:

\[ N(0, r_x) = N(0, G r_w) \cap N(0, (x_0 - \hat{x}_0)) \]
\[ N(0, r_y) = N(0, r_v) \cup N(0, h(r_x)) \]

(d) Absolute lower limits on the constraint sets are given by \( r_x \rightarrow 0 \) and \( r_y \rightarrow r_v \) as \( \hat{x}_k \rightarrow x_k \) and \( \xi \rightarrow 0 \).

Any trajectory \( (\hat{x}_y, \hat{y}_k | \hat{u}_k) \) generated by the approximate system representation may be considered as a trajectory generated by a family of neighbouring linear systems. This is utilized in the definition of a linear approximate system representation.

2.3b The linear system representation

Given the basic approximate system representation \( \hat{F}_n \), equations (2.3 - 2.5), a linear representation \( \hat{F}_e \) may be defined providing that:

i) there is an indexed sequence of points

\[ \pi^o_{k,x} (x^o_k, u^o_k | a^o_k \in \Omega_x \times \Omega_u \times \Omega_a, \text{ the nominal trajectory}, \]

such that \( \left| \hat{x}_k - x^o_k \right| \leq \varepsilon_1 \), \( \left| u_k - u^o_k \right| \leq \varepsilon_2 \), \( \left| k_k - k^o_k \right| \leq \varepsilon_3 \)

for some \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \)

ii) there are continuous linear maps \( \phi^o : \Omega_x \rightarrow \Omega_x \);

\( \phi^o_a : \Omega_a \rightarrow \Omega_x \); \( \Gamma^o : \Omega_u \rightarrow \Omega_x \); \( H^o : \Omega_x \rightarrow \Omega_y \)

and

iii) continuous maps \( \sigma_x : \Omega_x \times \Omega_u \times \Omega_a \rightarrow \Omega_x \)

and \( \sigma_y : \Omega_x \rightarrow \Omega_y \) such that,

\[ \lim_{\varepsilon \to 0} \left| \sigma_x (\varepsilon \delta x_k, \varepsilon \delta a_k, \varepsilon \delta u_k) \right| / \varepsilon = 0 \]
\[ \lim_{\varepsilon \to 0} \left| \sigma_y (\varepsilon \delta x_k) \right| / \varepsilon = 0 \]
with \( \delta x_k = (\hat{x}_k - x^0_k) \), \( \delta u_k = (u_k - u^0_k) \), \( \delta a_k = (\hat{a}_k - a^0_k) \).

The linear representation is then given by the set of equations:

\[
\begin{aligned}
\hat{x}_{k+1} &= \phi_k \hat{x}_k + \phi_{ak} \delta a_k + \Gamma_k \delta u_k + \gamma_k + \hat{w}_k \\
\hat{a}_{k+1} &= \hat{a}_k \\
\hat{y}_k &= H_k \hat{x}_k + \beta_k 
\end{aligned}
\]  
(2.6)

with,

\[
\begin{aligned}
(x_{k+1} - \hat{x}_{k+1}) &\in N(0, r_{x_{k+1}}) \\
(y_k - \hat{y}_k) &\in N(0, r_{y_k})
\end{aligned}
\]  
(2.7)

and where, \( N(0, r_{x_{k+1}}) \subseteq N(0, G x_w) \cap N(0, \xi(x_{k+1}')) \cup N(0, \sigma_x) \)

\( N(0, r_{y_k}) \subseteq N(0, G y_v) \cup N(0, h(x_{k+1}')) \cup N(0, \sigma_y) \)

The continuous mappings and the vector terms appearing in \( \hat{\Sigma}_e \),

equation (2.6), are defined as,

\[
\begin{aligned}
\phi_k &= \phi(x^0_k) = \frac{\partial f}{\partial \hat{x}} \bigg|_{x^0_k} \\
\phi_{ak} &= \phi_{a}(x^0_k) = \frac{\partial f}{\partial a} \bigg|_{a^0_k} \\
\Gamma_k &= \Gamma(x^0_k) = \frac{\partial f}{\partial u} \bigg|_{u^0_k} \\
H_k &= H(x^0_k) = \frac{\partial h}{\partial x} \bigg|_{x^0_k} \end{aligned}
\]  
(2.8)

\[
\begin{aligned}
\gamma_k &= f(x^0_k, a^0_k, u^0_k) + \delta a_k - \phi_{a} x^0_k - \phi_{ak} a^0_k \\
\beta_k &= h(x^0_k) - H_k x^0_k \\
\sigma_{ix} (\delta x_k, \delta a_k, \delta u_k) &= \frac{1}{2} \left\{ \delta x_k^T \frac{\partial f}{\partial x} \delta x_k \right\}^{\frac{1}{2}} \delta x_k \\
+ \delta a_k^T \frac{\partial f}{\partial a} \delta a_k + \delta u_k^T \frac{\partial f}{\partial u} \delta u_k \\
+ \ldots \quad i = 1, 2, \ldots, n
\end{aligned}
\]  
(2.9)
\[ \sigma_{iY}(\delta x_k) = \frac{1}{2} \left\{ \delta x_k^T \left[ \begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} \end{array} \right] \mid x = x_k^{\circ} \right\}^T \delta x_k \} + \ldots \]

\[ i = 1, 2, \ldots, e \quad (2.10) \]

It is evident that the relationship between neighbouring linear representations, \( \hat{e}_e(x_k^{\circ}) \) and \( \hat{e}_e(x_{k+1}^{\circ}) \), is parametrically imbedded in \( \Omega_x \times \Omega_u \times \Omega_a \) product space through the dependence of the linear mappings, \( \phi_k^e, \phi_k^{a_k}, \Gamma_k^e \) and \( H_k \), on the nominal trajectory sequence. It is further evident that any modelling policy, based upon the linear representation \( \hat{e}_e \), must not only generate suitable values for the \( \hat{e}_e \) and \( \hat{e}_h \) vectors, but must also generate a valid nominal trajectory \( x_k^{\circ} \).

If the nominal trajectory was arbitrarily chosen, solely according to a-priori considerations, then there is no guarantee that the linear representations \( \hat{e}_e \) will, in any manner, correspond to the given nonlinear system \( \hat{e}_n \). This is an important computational consideration to be taken into account when deriving an adaptive strategy for real-time applications. However, the manner in which a suitable nominal trajectory is chosen will be delayed until later in this chapter.

2.4 Optimality and Adaptive Processes

It has been stated in a previous section, (section 2.2), that under conditions of parameter uncertainty, it is not possible to obtain a uniquely ordered control policy for the general control problem (definition 2.3). It would, therefore, appear desirable that in the case of adaptive control systems, some additional means of ordering the selection of suboptimal control policies be incorporated in the definition of the control problem. In this section, an ordering of suboptimal control policies is attempted by means of a hierarchy of optimization problems. The hierarchy is developed such that the suboptimal policy
for a particular optimization problem reflects the available information regarding $\Sigma_n$.

Firstly, the general optimization triplet will be defined.

**Definition 2.4 (The Optimization Triplet)**

An optimization problem may be represented by a triplet $(\Sigma, J, \theta)$, where $\Sigma(\theta)$ is a system representation, $J(\theta)$ is a scalar performance criterion and $\theta$ is a vector of 'optimizing variables'. In other words, the optimization triplet is a shorthand notation for the general optimization problem; i.e. the vector $\theta^* \in \Omega_\theta$ is required such that,

$$J(\theta^*) = \min_{\theta \in \Omega_\theta} \max_{J(\theta)}$$

subject to the constraints imposed by the system representation $\Sigma(\theta)$.

A hierarchy of suboptimal control policies, based upon the linear system representation $\hat{\Sigma}_e$, is now defined.

**Definition 2.5 (The Nominal Optimal Control Policy)**

Given a nominal estimate of the uncertain parameter vector, $\hat{\Sigma}^0$, and the optimization triplet $(\hat{\Sigma}_e, \pi^0_k, J, U)$, the 'nominal optimal control policy' $U_n = \{ U_{nk} | k = 1, 2, \ldots \} \subset \Omega_U$ is that sequence of control vectors which yields the inequality,

$$J(U_n(\pi^0_k)) \leq J(U(\pi^0_k)) \quad U_n \uparrow U$$

subject to,

(a) the equality constraints imposed by $\hat{\Sigma}_e (\pi^0_k)$

and (b) $\forall \{ \xi_k(\xi_n) - \xi_k(\pi^0_k) \}$

$$= \min_{\pi^0_k \in \Omega_x \times \Omega_u} \left[ \forall \{ \xi_k(\xi_n) - \xi_k(\pi^0_k) \} \right]$$

where $\xi_k(\xi)$ is some vector function of the variables of the system $\Sigma$.
(e.g. the observation function) and \( \hat{V}(\cdot) \) is a scalar modelling criterion.

**Definition 2.6 (The Conditional Optimal Control Policy)**

Given the optimization triplet \((\hat{\Sigma}_e (\hat{\pi}_k^0), J, U)\), the 'conditional optimal control policy' \( \hat{U} \) is the sequence of control vectors which yields the inequality,

\[
J \{ \hat{U} \ (\hat{\pi}_k^0) \} < J \{ U \ (\pi_k^0) \}
\]

subject to,

(a) the equality constraints imposed by \( \hat{f}_e (\hat{\pi}_k^0) \)

(b) \( \forall \{ \hat{\xi}_k (\Sigma_n) - \hat{\xi}_k (\hat{\Sigma}_e (\hat{\pi}_k^0)) \} = \min_{\hat{\pi}_k^0 \in \Omega_x \times \Omega_u \times \Omega_a} \left[ \forall \{ \hat{\xi}_k (\Sigma_n) - \hat{\xi}_k (\Sigma (\pi_k^0)) \} \right] \)

and (c) there exists a finite integer \( M \) and an \( \varepsilon > 0 \) such that

\[
|| a_k - \hat{a}_k || < \varepsilon \ \forall_k > M. \text{ Where } a_k \text{ is the true parameter vector of the system } \Sigma_n.
\]

**Definition 2.7 (The Strictly Optimal Control Policy)**

Given the optimization triplet \((\Sigma_e (\pi_k^0), J, U)\) the strictly optimal control policy \( U^* \) is the sequence of control vectors obtained such that,

\[
J(U^* (\pi_k^0) ) < J(U (\pi_k^0) )
\]

\( \forall U^* \), \( U \in \Omega_u \), \( U^* \dagger U \)

\( \forall \pi_k^0 \in \Omega_x \times \Omega_u \times \Omega_a \)

and \( || a_k - \hat{a}_k || = 0 \ \forall_k \)

The control policies defined above may be interpreted as follows.

The nominal optimal control policy (NOCP) is essentially non-adaptive since the parameter vector \( a_k^0 \) is chosen a-priori. The 'best' control
policy \( U_n \) is then generated subject to obtaining a 'best' nominal trajectory \( \hat{\pi}_k^{\pi} \) in \( \Omega_x \times \Omega_u \) product space. The criterion whereby the 'goodness' of \( \hat{\pi}_k^{\pi} \) is judged, is the scalar function \( V(\cdot) \); \( V \) being a measure of the distance between the systems \( \Sigma_n \) and \( \hat{\Sigma}_e \) in the space of the vector \( l \). The conditional optimal control policy (COCP) is defined in a similar fashion to the NOCP; with the exception that \( \hat{e} \) is subject to obtaining a 'best' trajectory in \( \Omega_x \times \Omega_u \times \Omega_a \) product space — i.e. parameter adaptation has been incorporated. The strictly optimal control policy (SOCP) represents the theoretical optimum which would be obtained if all the parameters were known exactly. The ordering induced by the control policies defined above is:

(a) \[ || a_k - \hat{a}_k || < || a_k - \hat{a}_k || \] 
then, \[ J \{ U^{(\pi_k^\pi)} \} \leq J \{ U^{(\pi_k^\pi)} \} \leq J \{ \hat{U}(\pi_k^\pi) \} \] (2.11)

(b) \[ || a_k - \hat{a}_k || \leq || a_k - \hat{a}_k || \] \( k = 1, 2, \ldots, M \)
\[ || a_k - \hat{a}_k || \geq || a_k - \hat{a}_k || \] \( k = M + 1, M + 2, \ldots \)
then, \[ J \{ U_k \} \leq J \{ u_{nk} \} \leq J \{ \hat{U}_k \} \] \( k < M \) \[ J \{ U_k \} \leq J \{ \hat{U}_k \} \leq J \{ u_{nk} \} \] \( k > M \) (2.12)

The adaptive nature of the COCP is evident from the ordering in case b, above. Although the COCP may initially start with worse performance than the NOCP, after a finite number of stages, \( M \), it will become at least as good, if not better, than the NOCP.

2.5 The Conditional Optimal Control Problem With Quadratic Cost

In this section the specific case of the conditional optimal control problem, with quadratic criterion functions \( J(U) \) and \( V(\pi_k^\pi) \), will be examined. The overall adaptive problem is to obtain a solution
to a least squares control optimization problem coupled with a least squares modelling optimization problem. The manner in which the nominal trajectory is selected will be defined, and reasons for this selection will be advanced. Necessary conditions for the minimization of $J$ and $V$ will be obtained and, subsequently reduced to a pair of two point boundary value problems (TPBVP).

2.5a Defining equations

The linear system representation $\hat{\mathbf{L}}_e$ defined by the set of equations (2.6 - 2.10) is to be considered. The control performance criterion is given by,

$$J(\delta u_k) = \frac{1}{2} \sum_{k=1}^{k_f-1} \left\{ \| \varrho_k - \psi \hat{\mathbf{x}}_k \|^2 \Omega_1 + \| \delta u_k \|^2 \Omega_2 \right\}$$  \hspace{1cm} (2.13)

and the modelling performance criterion is given by,

$$V(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k) = \frac{1}{2} \sum_{k=1}^{k_f-1} \left\{ \| y_k - H_k \hat{\mathbf{x}}_k - \beta_k \|^2 \Omega_2 + \| A \hat{\mathbf{u}}_k \|^2 \Omega_2 \right\}$$ \hspace{1cm} (2.14)

The duration of the control process is not fixed (i.e., $k_f$ is free) and can tend to infinity — the infinite interval control problem. The weighting matrices $\Omega_1$, $\Omega_2$, $\Omega_3$ and $\Omega_4$ are ($n \times n$), ($m \times m$), ($e \times e$) and ($s \times s$) symmetric positive definite matrices, respectively. The elements of the weighting matrices associated with $V(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k)$ are defined from the constraint sets $N(0, x_{k+1}^T)$ and $N(0, y_{k+1}^T)$ as,

$$Q_2 = \left[ \text{diag} \left( \sigma_y^x T + \sigma_y^y T \right) \right]^{-1}$$ \hspace{1cm} (2.15)

$$R_2 = \left[ \text{diag} \left( \sigma_x^x T, -\frac{1}{2} \sigma_x^y T \right) \right]^{-1}$$ \hspace{1cm} (2.16)
where \( \sigma'_x \) is the \( s \)-dimensional vector comprised of the non-zero elements of the \( n \)-vector \( \sigma_x \). The noise term in (2.14) has been defined as,

\[
\hat{w}_k = A \hat{v}_k = \begin{bmatrix} G_x^t \tilde{A}_x \end{bmatrix} \begin{bmatrix} w_k \\ \sigma_x^t \end{bmatrix}
\]  

(2.17)

where \( A \) is an \( n \times s + p \) matrix equal to \( (G_x^t \tilde{A}_x) \) and \( \sigma_x = \sigma_x^t \), with \( \sigma_x^t \) as defined above.

2.5b The nominal trajectory, \( \bar{u}_k^0 \)

One approach to generating a nominal trajectory is to consider the non-linear system \( \Sigma_n \) and, by neglecting the noise terms, evaluate a nominal optimal solution over a fixed time interval. Due to the excessive computational burden associated with nonlinear optimization techniques, this computation is usually performed off-line and the precomputed trajectory stored for future use. However, since there are uncertain system parameters, it becomes necessary to repeat these computations as better estimates of the system parameters become available – which obviously creates difficulties when on-line, real-time control is required. For this reason, the approach adopted toward generating a nominal trajectory will not be based upon precomputing and storing some nominal solution to \( \Sigma_n \).

Consider the modelling problem. At any stage \( k \) the 'best' values of \( \hat{X}_k, \hat{a}_k \) are to be selected by minimization of \( V \), with the control vector \( \hat{u}_k \) assumed known. It is obvious that past values of \( \hat{u}_m, \hat{X}_m, \hat{a}_m \), \( m < k \) have already been generated and that, furthermore, the immediate past values \( (m = k-1) \) represent the current a-priori information available to both the control and modelling problems. Therefore, by selecting the nominal trajectory as the available a-priori information,
i.e. \( \pi_k^o = (\mathcal{X}_{k-1}, \mathcal{C}_{k-1} | \mathcal{A}_{k-1}) \), the trajectory will be automatically generated as the overall adaptive solution progresses. It should be noted, however, that this selection of nominal trajectory imposes the restriction that the control vector be generated in a forward recursive manner. This restriction follows from the fact that the nominal trajectory is known only for present and past stages, and not for any future stages.

The advantages of selection \( \pi_k^o \) in this manner are:

(a) no special off-line computations are required,
(b) no large computer memory area is required for storage, and
(c) the problem of selecting a suitable length of trajectory to store is circumvented.

Point (c), above, is of particular importance in processes of variable, or unspecified, operating time.

2.5c Necessary conditions for minimization of \( J \) and \( V \)

Since the optimization problems for control and modelling have been defined in a deterministic sense, application of the discrete maximum principle [54] is possible. Furthermore, since the COCP is specifically based upon a linear representation of the system, \( \mathcal{E}_e(\pi_k^o) \), and since \( \mathcal{E}_e(\pi_k^o) \) is defined at each stage \( k \), by information available from the previous stage, i.e. \( \pi_k^o \), a stage by stage separation of the control and modelling problems may be applied. This separation allows the modelling problem to be treated assuming \( \delta\mathcal{U}_k \) is known, and the control problem to be treated assuming \( (\mathcal{X}_k, \mathcal{A}_k) \) is known. This separation is given further validity if the control is generated forward-recursively, since information about future stages is not required to obtain the
control vector at the present stage.

From equations (2.6), (2.13) and (2.14) the Hamiltonian functions for the control problem $\mathcal{H}_k^C$ and modelling problem $\mathcal{H}_k^m$ are obtained as,

$$\mathcal{H}_k^C = \frac{1}{2} \left\{ \| \phi_k - \psi \hat{z}_k \|^2 + \| \delta \phi_k \|^2 + \lambda_{k+1}^T \left\{ \phi_k \hat{\lambda}_k + \phi_k \hat{\gamma}_k \right\} + \gamma_k \delta \phi_k \right\}$$

$$\mathcal{H}_k^m = \frac{1}{2} \left\{ \| y_k - \Delta_k \hat{x}_k - \beta_k \|^2 + \| \Delta x_k^T \|^2 \right\} + \eta_{k+1}^T \left\{ \phi_k \hat{\eta}_k + \phi_k \hat{\alpha}_k \right\}$$

(2.18)

(2.19)

where, $\vec{x}_k = [\vec{x}_k^T ; \vec{u}_k^T]$; i.e. the state vector has been augmented by the parameter vector,

$$\Delta_k = [H_k ; 0]$$

$$\hat{\phi}_k = \begin{bmatrix} \phi_k \phi_k \end{bmatrix}$$

$$\hat{\gamma}_k = \begin{bmatrix} \gamma_k \\ \gamma_k \end{bmatrix}$$

$$\hat{\alpha}_k = \begin{bmatrix} \alpha_k \\ \alpha_k \end{bmatrix}$$

Application of the discrete maximum principle yields the following necessary conditions for minimization of $J$ and $V$:

Control

$$\hat{z}_{k+1} = \frac{\delta \mathcal{H}_k^C}{\delta \lambda_{k+1}}$$

$$\lambda_k = \frac{\delta \mathcal{H}_k^C}{\delta \phi_k}$$

$$\frac{\delta \mathcal{H}_k^C}{\delta \phi_k} = 0$$

$$u_{k-1} + \delta u_k \in \Omega_u$$

Modelling

$$\hat{z}_{k+1} = \frac{\delta \mathcal{H}_k^m}{\delta \eta_{k+1}}$$

$$\eta_k = \frac{\delta \mathcal{H}_k^m}{\delta \phi_k}$$

$$\frac{\delta \mathcal{H}_k^m}{\delta \phi_k} = 0$$

(2.20)

(2.21)
Boundary conditions:
\[ \hat{x}_o = \hat{x}_o^* ; \text{ a-priori selected} \]
value of initial state vector.
\[ \lambda_{k_f} = 0 ; \hat{x}_{k_f} \text{ unspecified since} \]
\[ \rho_{k_f} \text{ not necessarily known.} \]

Evaluating the necessary conditions (2.20) and (2.21) the following set of canonic equations, defining a pair of two point boundary value problems, is obtained.

(a) **Control**
\[ \hat{x}_{k+1} = \phi_k \hat{x}_k + \phi_k \hat{a}_k + \hat{y}_k + \hat{w}_k - \Gamma_k R_1^{-1} \Gamma_k^T \lambda_{k+1} \]
\[ \lambda_k = \phi_k^T \lambda_{k+1} - \psi^T Q_1 (\rho_k - \psi \hat{x}_k) \]
\[ \delta \hat{u}_k = - R_1^{-1} \Gamma_k^T \lambda_{k+1} \]
\[ \hat{u}_k = u_{k-1} + \delta \hat{u}_k \]
\[ \hat{x}_o = \hat{x}_o^* , \quad \lambda_{k_f} = 0 \]

(b) **Modelling**
\[ \hat{z}_{k+1} = \hat{z}_k + \Gamma_k \delta \hat{u}_k + \hat{y}_k - \Lambda R_2^{-1} \Lambda^T \eta_k + 1 \]
\[ \eta_k = \phi_k^T \eta_{k+1} - \hat{r}_k^T Q_2 (y_k - \hat{r}_k \hat{z}_k - \beta_k) \]
\[ \delta \hat{w}_k = - R_2^{-1} \Lambda^T \eta_{k+1} \]
\[ \hat{z}_o = \hat{z}_o^* , \quad \eta_{k_f} = 0 \]

The normal approach to solving this set of two point boundary value problems is to consider a backward recursive solution to the control problem, with the solution starting at the terminal time. By means of a suitable Riccati transformation connecting the adjoint
vector $\lambda_{k+1}$ to the state vector $\hat{x}_k$, a sequence of backward recursive feedback gains are generated. The modelling problem is generated forward recursively to yield the values of $\hat{x}_k$ required for control.

As has been stated previously, the manner in which the COCP has been defined, together with the manner in which a nominal trajectory is generated, preclude the application of the normal solution approach to the control problem. Specifically, a forward-recursive generated control solution is required. In the next chapter a completely forward-recursive solution for the COCP will be derived.
CHAPTER 3

THE FORWARD-RECURSIVE ADAPTIVE CONTROL ALGORITHM

In the previous chapter consideration was given to approximating nonlinear systems with a linear representation $\hat{\Sigma}_e$. Utilizing $\hat{\Sigma}_e$ a conditional optimal control problem was posed, and the necessity for a forward-recursive control algorithm was shown. In this chapter the concept of invariant imbedding will be used to derive a forward-recursive algorithm from the canonic equations developed in the previous chapter, (2.22a-c) and (2.23 a-c). In addition, a state-space partitioning of the dynamic equations of $\hat{\Sigma}_e$ will be introduced in order to obtain an explicit expression for the suboptimal control vector.

Initially, basic invariant imbedding concepts will be detailed briefly.

3.1 Invariant Imbedding

Invariant imbedding concepts have been applied in many diverse fields by a number of researchers. Known collectively in the field of physics as the invariance principles, these concepts have been utilized to study transport phenomena [55,56], radiative transfer [57], neutron transport theory [58], wave propagation [59] and gas dynamics [60]. Due primarily to the work of Bellman, Kalaba and Wing [61-64] on the mathematical foundations of the invariance principles, these concepts have been generalized and have come to be known as invariant imbedding. Invariant imbedding provides a particularly suitable framework for the solution of boundary value problems [62,65] and eigenvalue problems encountered in modern control theory.

The basic premises of invariant imbedding are:

(a) Any particular process can be considered as a member of a general
family of processes
and, (b) all neighbouring processes in the general family are related
through an underlying invariant relationship.

It may be seen that the concept of a particular process being
imbedded in a family of processes, corresponds directly with the
approach adopted to the linear approximating representation, $\hat{\Sigma}_n$, in the
previous chapter. The nonlinear system $\hat{\Sigma}_n$ was viewed as being imbedded
in a family of neighbouring linear systems $\hat{\Sigma}_e(\pi_k^o)$. The invariant
relationship between the neighbouring linear systems is characterized
parametrically in $\Omega_x \times \Omega_u \times \Omega_q$ product space through the dependence
of the linear mappings of $\hat{\Sigma}_e$ on the points $\pi_k^o \in \Omega_x \times \Omega_u \times \Omega_q$.

The diversity of application of the invariant imbedding concept
is such that, not only can it yield qualitative insights to a problem,
but it can also yield approaches to quantitative solutions of a problem.
It is the quantitative aspects of invariant imbedding which shall be
considered in the sequel.

3.2 The Invariant Imbedding Equation for a Boundary Value Problem

Consider a particular boundary value problem characterized by
the set of equations,

$$x_{k+1} = f(x_k, \lambda_k, k)$$  \hspace{1cm} (3.1)

$$\lambda_{k+1} = g(x_k, \lambda_k, k)$$  \hspace{1cm} (3.2)

with boundary conditions,

$$x_{K_0} = x_0 ; \hspace{1cm} \lambda_{K_f} = 0$$  \hspace{1cm} (3.3)

It should be noted that the boundary value problems of control and
modelling, defined by equations (2.22 a-c) and 2.23 a-c) in the pre-
vious chapter, may be represented in the general form specified above
by (3.1) - (3.3). The initial conditions on the \(x\) vector, i.e. \(x_{k_o}\), may be considered fixed, whilst the terminal conditions on the \(\lambda\) vector may be considered as imbedded in a general family of terminal conditions. Let the imbedded terminal condition be \(\lambda_{k_f} = C\); where \(C \in \Omega_{\lambda}\), the space associated with the \(\lambda\) vector. The unknown terminal condition on the \(x\) vector may be expressed as a functional of the general terminal condition \(C\) and the terminal stage index \(k_f\); i.e.

\[
x_{k_f} = r(C, k_f)
\]

Considering small perturbations, \(\Delta C\) and \(\Delta k = (k_{f+1} - k_f)\), about \(C\) and \(k_f\), the first order perturbation equation for \(r(\cdot)\) may be obtained in the form,

\[
r(C + \Delta C, k_{f+1}) = r(C, k_f) + \frac{\delta r(C, k_f)}{\delta C} \Delta C + \frac{\delta r(C, k_f)}{\delta k_f} \Delta k
\]

\[
+ \frac{\delta^2 r(C, k_f)}{\delta C \delta k_f} \Delta C \Delta k
\]

(3.4)

where,

\[
\frac{\delta r(C, k_f)}{\delta C} = x_j(C + \Delta C_j, k_f) - x_j(C, k_f)
\]

(3.5)

and,

\[
\frac{\delta r(C, k_f)}{\delta k_f} = x_i(C, k_{f+1}) - x_i(C, k_f)
\]

(3.6)

with \(i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, n;\) where \(n\) is the order of the vectors \(x\) and \(\lambda\). In addition, there are the relationships defined by equations (3.1) and (3.2) - the dynamic equations;

\[
\Delta C = \lambda_{k_{f+1}} - \lambda_{k_f} = g(r(C, k_f), C, k_f) - C
\]

(3.7)

\[
r(C + \Delta C, k_{f+1}) = f(r(C, k_f), C, k_f)
\]

(3.8)

Substituting (3.7) and (3.8) into the perturbation equation (3.4), a discrete invariant imbedding equation, relating the family of
terminal conditions \( r(C, k_f) \), may be obtained in the form:

\[
\left\{ \frac{\delta r(C, k_f)}{\delta k_f} \right\} + \left\{ \frac{\delta r(C, k_f)}{\delta C} \cdot \frac{\delta^2 r(C, k_f)}{\delta C \delta k_f} \right\} \cdot \left\{ g(r(C, k_f) - C) \right\}
\]

\[
= f(r(C, k_f), C, k_f) - r(C, k_f)
\]  

(3.9)

The evolution of the family of processes with variable terminal
time is being considered; i.e. \( k_f \in \{ k \mid k \leq k_0 \leq \infty \} \). Consequently,\( k_f \) may be viewed as a running-terminal-time index and will, in all
future development, be replaced by the general stage index \( k \). It
should be noted that a solution obtained from this invariant imbedding
formulation will, at a particular stage \( k \), correspond to a solution
obtained by considering the terminal stage to be \( k \). In the next
section a specific solution to the invariant imbedding equation (3.9)
will be attempted by assuming a specific form for \( r(C, k_f) \).

3.3 Solution of the conditional optimal control problem

Consider the augmented vectors of general terminal conditions for
the modelling and control optimization problems,

\[
\begin{bmatrix}
\lambda_k \\
\eta_k
\end{bmatrix}
= \begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
; \quad \begin{bmatrix}
\frac{\delta x_1(C_1, k)}{\delta k} \\
\frac{\delta x_2(C_1, k)}{\delta k}
\end{bmatrix}
= \begin{bmatrix}
x_1(C_1, k) \\
x_2(C_2, k)
\end{bmatrix}
\]

the general invariant imbedding equation is, from (3.9)

\[
\left\{ \frac{\delta x_1(C_1, k)}{\delta k} \right\} + \left\{ \frac{\delta x_1(C_1, k)}{\delta C_1} + \frac{\delta^2 x_1(C_1, k)}{\delta C_1 \delta k} \right\}
\]

\[
= g_i \{ r_i(C_1, k) - \}
\]

\[
= f_i(r_i(C_1, k), C_i, k) - r_i(C_i, k) ; \ i = 1, 2
\]

(3.10)
Where the functions $f_i^*(\cdot)$ and $g_i^*(\cdot)$ are obtained, from (2.22 a-b) and (2.23 a-b), in the form,
\begin{align*}
f_1(x_1(C_1,k), C_1,k) &= \beta_k x_1(C_1,k) + \delta_k \hat{x}_k + \gamma_k + \hat{\gamma}_k \\
&\quad - R_1^{-1} \Gamma_k g_1(x_1(C_1,k), C_1,k) \\
f_2(x_2(C_2,k), C_2,k) &= \beta_k x_2(C_2,k) + \delta_k \hat{x}_k + \gamma_k \\
&\quad - A R_2^{-1} A^T g_2(x_2(C_2,k), C_2,k)
\end{align*}
(3.11)

\begin{align*}
g_1(x_1(C_1,k), C_1,k) &= \beta_k^{-T} \{ C_1 + \psi^T Q_1 (\rho_k - \psi x_1(C_1,k)) \} \\
g_2(x_2(C_2,k), C_2,k) &= \beta_k^{-T} \{ C_2 + \hat{A}^T Q_2 (y_k - \hat{x}_2(C_2,k) - \beta_k) \}
\end{align*}
(3.12)

(3.13)

(3.14)

A solution for the unknown terminal conditions $x_i(C_i,k)$ is assumed in the form of a linear function of the family of terminal conditions $C_i$, i.e.
\begin{equation}
\begin{bmatrix}
x_1(C_1,k) \\
x_2(C_2,k)
\end{bmatrix} = 
\begin{bmatrix}
\hat{x}_k \\
\hat{x}_k
\end{bmatrix} - 
\begin{bmatrix}
M_k & 0 \\
0 & p_k
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
\end{equation}
(3.15)

Substituting the $f_i^*(\cdot)$ and $g_i^*(\cdot)$ functions given by (3.11) - (3.14) into (3.10), and utilizing the assumed solution for $x_i(C_i,k)$, equation (3.15), the invariant embedding equations may be reduced to the form:
\begin{align*}
\begin{bmatrix}
\hat{s}_{k+1} \\
\hat{z}_{k+1}
\end{bmatrix} &= 
\frac{\Phi_k \hat{x}_k + \phi_k \hat{z}_k + \delta_k \hat{x}_k + \gamma_k + \hat{\gamma}_k + \delta_k \hat{z}_k + \phi_k M_k C_1}{\Phi_k \hat{x}_k + \phi_k \hat{z}_k + \delta_k \hat{x}_k + \gamma_k + \hat{\gamma}_k + \delta_k \hat{z}_k + \phi_k p_k C_2}
\end{align*}
(3.16)
Equating coefficients of like powers of $C_1$ and $C_2$ in the above equation, a set of forward recursive equations defining the solutions of the control and modelling boundary value problems, is obtained as:

**Control:**

\[
\hat{x}_{k+1} = \hat{\phi}_k \hat{x}_k + \hat{\phi}_{ak} \hat{u}_k + \gamma_k + \hat{w}_k + N_k \psi^T Q_1 \left( \rho_k - \psi \hat{x}_k \right) \quad (3.17a)
\]

\[
N_k = (M_{k+1} - \Gamma_k R_2^{-1} \Gamma_k^T) \hat{\phi}_k^{-1} = \hat{\phi}_k M_k \left( I + \psi^T Q_1 \psi \hat{x}_k \right)^{-1} \quad (3.17b)
\]

\[
M_{k+1} = N_k \hat{\phi}_k^T + \Gamma_k R_2^{-1} \Gamma_k^T \quad (3.17c)
\]

**Modelling:**

\[
\hat{z}_{k+1} = \hat{\phi}_k \hat{z}_k + \hat{\Gamma}_k \delta \hat{u}_k + \gamma_k + \hat{K}_k \hat{H}_k^T Q_2 \left( \psi_k - \hat{\psi} \hat{z}_k - \beta_k \right) \quad (3.18a)
\]

\[
\hat{K}_k = \left( P_{k+1} - \hat{\hat{X}}_k^{-1} \hat{\hat{Z}}_k \right) \hat{\phi}_k^{-1} = \hat{\phi}_k P_k \left( I + \hat{\hat{X}}_k^T Q_2 \hat{\hat{X}}_k \right)^{-1} \quad (3.18b)
\]

\[
P_{k+1} = \hat{K}_k \hat{\phi}_k^T + \hat{\hat{Z}}_k R_2^{-1} \hat{\hat{Z}}_k^T \quad (3.18c)
\]

Considering equations (3.17a) and (3.18a) the following observations may be made. In (3.17a) the last term on the right hand side of the equation represents the state vector increment required to drive the controlled states $\psi \hat{x}_k$ to the target trajectory $\rho_k$; i.e. $\Delta \hat{x}_k = N_k \psi^T Q_1 \left( \rho_k - \psi \hat{x}_k \right)$. However, the purpose of the control solution is not to obtain a state vector trajectory such that $\psi \hat{x}_k + \rho_k$, but to obtain the incremental vector $\delta \hat{u}_k$ which, when applied to the actual system $\Sigma_n$, causes $\psi \hat{x}_k + \rho_k$. The incremental control vector is related to a subset of elements of the incremental state vector, $\Delta \hat{x}_k'$, through the equation,

\[
\Gamma' \delta \hat{u}_k = \Delta \hat{x}_k' \quad (3.19)
\]

where, $\Gamma' \delta \hat{u}_k$ is a t-dimensional vector composed of the non-zero elements of the n-vector $\Gamma \delta \hat{u}_k$ and, $\Delta \hat{x}_k'$ is the t-dimensional subvector.
of $\Delta x_k$ corresponding to $\Gamma' \delta u_k$.

With reference to equation (3.18a), the term on the extreme right represents the increments of the augmented state vector $\hat{x}_k$ required to generate the trajectory of states and parameters corresponding to a solution of the modelling problem. It should be noted that this incremental term is the explicit form of the general information transmission function $\xi(y_k, \hat{y}_k)$ which was assumed, but never functionally defined, in the approximate system representations developed in Chapter 2.

The derivation of an explicit relationship for $\xi(y_k, \hat{y}_k)$ has been made possible through the ordering of the modelling problem induced by the criterion $\mathcal{V}$. In other words, the modelling criterion $\mathcal{V}$ serves to order the selection of $\xi(y_k, \hat{y}_k)$ from the set of possible functions — in this case the set of functions are the linear transformations characterized by the general matrices $\Theta$, i.e. $\xi(y_k, \hat{y}_k) = \Theta (y_k - \hat{y}_k)$.

It is obvious, furthermore, that the set of equations (3.18a) — (3.18c) are sufficient to completely specify the solution to the modelling problem, as opposed to the equations for the control problem, which require solution of an additional equation, (3.19), in order to yield the control vector.

Considering equation (3.19), the incremental control vector may be obtained directly, in the form $\hat{\delta u}_k = (\Gamma_k')^{-1} \Delta x_k'$, providing $\Gamma_k'$ is a square, non-singular matrix (i.e. $t = m$). However, in many systems, it is likely that the dimensions of $\hat{\delta u}_k$ and $\Delta x_k'$ will not be the same (i.e. $t \neq m$), in which case $\Gamma_k'$ will be a $t \times m$ matrix, and the inverse $(\Gamma_k')^{-1}$ will not exist. In such cases, the generalized matrix pseudo-inverse [66] may be used to obtain a solution of the form, $\hat{\delta u}_k = (\Gamma_k')^+ \Delta x_k'$, where, $(\Gamma_k')^+$ is the pseudo-inverse of the matrix $\Gamma_k'$. 
However, calculation of the pseudo-inverse can add appreciatively to the overall computational load of the algorithm; which, in turn, can be detrimental to real-time implementation of the algorithm. Therefore, an alternative approach, via state-space partitioning, has been developed. This approach is presented in the next section.

3.4 A Solution for the Control Vector $\hat{\delta u}_k$

Consider the partitioning of the control distribution matrix $\Gamma_k$ given by,

$$
\Gamma_k^T = \begin{bmatrix}
\Gamma_{1k}^T & \Gamma_{2k}^T \\
\Gamma_{3k}^T & \Gamma_{4k}^T
\end{bmatrix}
$$

(3.20)

where $\Gamma_{1k}$ is an $(n-t) \times m$ null matrix (i.e. all the elements of the $(n-t)$ rows of $\Gamma_{1k}$ are zero and $\Gamma_{k}^T$ is a $t \times m$ matrix). $\Gamma_{k}^T$ is obviously the matrix used to connect the incremental control vector to the subset of incremental state vectors $\hat{\Delta x}_{k}^T$ in equation (3.19). Note that, if all the states are directly driven by one or more elements of the control vector, then $\Gamma_{k}^T = \Gamma_{k}$ and $t = n$. The matrix $\Gamma_{k}^T$ may be further partitioned such that $\Gamma_{k}^T = \begin{bmatrix}
\Gamma_{1k}^T & \Gamma_{2k}^T \\
\Gamma_{3k}^T & \Gamma_{4k}^T
\end{bmatrix}$, with either $\Gamma_{2k}$ or $\Gamma_{3k}$ an $m \times m$ nonsingular partition — (it is assumed that $t \geq m$). Let $\Gamma_{3k}$ be the $m \times m$ nonsingular partition of $\Gamma_{k}^T$. Utilizing this partition the incremental control vector may be evaluated from,

$$
\hat{\delta u}_k = \Gamma_{3k}^{-1} \hat{\Delta x}_{3k}
$$

(3.21)

where, from equation (3.17a), $\hat{\Delta x}_{3k}$ is given by:

$$
\hat{\Delta x}_{3k} = \begin{bmatrix}
N_{31k} & N_{32k} & N_{33k}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{1k} \\
\epsilon_{2k} \\
\epsilon_{3k}
\end{bmatrix}
$$
with \[ \begin{bmatrix} e_{1k} \\ e_{2k} \\ e_{3k} \end{bmatrix} = \begin{bmatrix} \psi_1^T \rho_{1k} \psi_1 - \psi_1 \hat{\chi}_k \\ \psi_2^T \rho_{2k} \psi_2 - \psi_2 \hat{\chi}_k \\ \psi_3^T \rho_{3k} \psi_3 - \psi_3 \hat{\chi}_k \end{bmatrix} \]

The three submatrices \( N_{3jk} \) and \( e_{jk} \) \((j = 1, 2, 3)\) are the partitions of the terms in equation (3.17a) which contribute to the partition, \( \Delta \hat{x}_{3k} \), of the incremental state vector. In order that the incremental control vector, obtained from (3.21) above, be unique, it is necessary that the partitions of the control matrix \( N_k \) (i.e. \( N_{ijk} \)) obey certain conditions.

These conditions are stated as follows.

**Theorem 3.1**

Given the general partitioning of the control distribution matrix,

\[ \Gamma_k^T = \begin{bmatrix} \Gamma_{1k}^T & \Gamma_{2k}^T & \cdots & \Gamma_{pk}^T \end{bmatrix} \]

with at least one non-singular \( m \times m \) partition \( \Gamma_{qk}^T \) \((q \in [1, p])\); the control vector \( \Delta \hat{u}_k = \Gamma_q^{-1} \Delta \hat{x}_{qk} \) will be unique providing that,

\[ N_{gjk} = \Gamma_{gk} \Gamma_{qk}^{-1} N_{qjk} \quad g = 1, 2, \ldots, s; \quad j = 1, 2, \ldots, \rho \]

for all non-null partitions \( \Gamma_{gk} \), i.e. \( \Gamma_{gk} \not\rightarrow [0] \).

**Proof:**

Consider the partitions corresponding to the states directly driven by the control vector; i.e. the partitions of the \( \Gamma_k' \) matrix of equations (3.19) and (3.20). Let \((p-i)\) be the number of partitions of \( \Gamma_k' \). From (3.19), the equality relationship between the incremental control vector and the incremental states is such that,
\[
\delta u_k = \begin{bmatrix}
\Sigma N_i + 1 n_k e_n_k \\
\vdots \\
\Sigma N_q n_k e_n_k \\
\Sigma N_p n_k e_n_k \\
\end{bmatrix}
\]

Substituting for \( \delta u_k \) in (3.22), with \( \delta u_k = r^{-1} \sum q_k q_n \Sigma n_k e_n_k \), the following set of equalities are obtained,

\[
\Gamma_{i+1,k} r^{-1} \sum q_k q_n n_k e_n_k = \sum n_k e_n_k \\
\vdots \\
\Gamma_{q,k} r^{-1} \sum q_n e_n_k = \sum q_n e_n_k \\
\vdots \\
\Gamma_{p,k} r^{-1} \sum q_n e_n_k = \sum q_n e_n_k
\]

The above equalities will hold for all \( e_n_k \) providing,

\[
\Gamma_{\ell,k} r^{-1} q_n = N_{\ell,n} \\
\ell = i+1, i+2, \ldots, p
\]

Corollary:

If two nonsingular \( m \times m \) partitions, \( \Gamma_{q,k} \) and \( \Gamma_{r,k} \), exist and the equality conditions of theorem 3.1 hold (i.e. equation (3.21)) then,

\[
\delta u_k = r^{-1} \sum q_k q_n n_k e_n_k = r^{-1} \sum r_k r_n n_k e_n_k
\]

and the control vector is "partition-invariant".
Proof:

Since the equality conditions hold,

$$\Gamma_k \Gamma_k^{-1} N_q k N_k = \Gamma_k^{-1} r_k N_k$$

$$\text{and, } \Gamma_k^{-1} q_n k N_k = \Gamma_k^{-1} r_k N_k$$

Therefore, for all \( n_k \) \((n = 1, 2, \ldots, p)\)

$$\sum_{n=1}^{p} \Gamma_k^{-1} q_n k N_k e_{n_k} = \sum_{n=1}^{p} \Gamma_k^{-1} r_k N_k e_{n_k}$$

But, \( \delta u_k^\wedge = \Gamma_k^{-1} \sum_{n=1}^{p} q_n k N_k e_{n_k} \)

therefore, \( \delta u_k^\wedge = \Gamma_k^{-1} \sum_{n=1}^{p} r_k N_k e_{n_k} = \Gamma_k^{-1} \sum_{n=1}^{p} q_n k N_k e_{n_k} \).

Thus completing the proof that \( \delta u_k^\wedge \) is invariant of the partition used to evaluate the control.

The equality conditions, defined in theorem 3.1, are of great importance if the solution for \( \delta u_k^\wedge \), obtained from the nonsingular partition \( \Gamma_k q_k \), is to be a unique solution. However, considering the defining equations for the control gain matrix \( N_k \), equations (3.17b) and (3.17c), it is not directly evident that such conditions hold. In the next section, a proof is advanced that these conditions hold for the control gain matrix.

3.5 Proof of equality conditions for \( N_k \)

From equation (3.17b) the control gain matrix is given by,

$$N_k = \left( M_{k+1} - \Gamma_k \Gamma_k^{-1} \Gamma_k^T \right) \Phi_k^T$$

and, from (3.17b) and (3.17c), the symmetric \( n \times n \) matrix \( M_{k+1} \), is defined by a Riccati matrix equation,

$$M_{k+1} = \Phi_k \left( I + \Phi_k^T Q_k \Phi_k \right)^{-1} \Phi_k^T + \Gamma_k \Gamma_k^{-1} \Gamma_k^T$$
Initially, consider the basic partition of the control distribution matrix given by equation (3.20), i.e., $\Gamma_k^T = \begin{bmatrix} \Gamma_{1k}^T & \Gamma_{2k}^T \end{bmatrix}$.

where $\Gamma_{1k}$ is the $(n-t) \times n$ null partition, $\Gamma_{1k} = [0]$, and $\Gamma_{2k} = \Gamma_k$.

Define the matrix $\hat{\Gamma}_k$ as,

$$\hat{\Gamma}_k = \begin{bmatrix} 0 & \hat{\Gamma}_k \end{bmatrix} \begin{bmatrix} R_{1k}^{-1} & 0 \\ 0 & R_{2k}^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \hat{\Gamma}_k & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{\Gamma}_k \end{bmatrix}$$

$$\hat{\Gamma}_{2k} = \begin{bmatrix} 0 & \hat{\Gamma}_{2k} \end{bmatrix} \begin{bmatrix} R_{1k}^{-1} & 0 \\ 0 & R_{2k}^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \hat{\Gamma}_{2k} & 0 \end{bmatrix}$$

(3.25)

Matrices $D_k$, $F_k$, and $S_k$, of dimensions $n \times n$, $n \times n$ and $t \times (n-t)$ respectively may be found such that, at each stage $k$, the $n \times n$ symmetric Riccati matrix is given by,

$$M_k = \hat{\Gamma}_k^{-1} \left[ D_k + \hat{\Gamma}_k F_k \hat{\Gamma}_k^T + W_k + W_k^T \right] \hat{\Gamma}_k^{-T}$$

(3.26)

where,

$$D_k = \begin{bmatrix} D_{1k} & 0 \\ 0 & 0 \end{bmatrix} \quad \hat{\Gamma}_k = \begin{bmatrix} 0 & 0 \\ 0 & \hat{\Gamma}_k \end{bmatrix} \quad F_k = \begin{bmatrix} 0 & 0 \\ 0 & F_k \end{bmatrix} \quad W_k = \begin{bmatrix} 0 & 0 \\ 0 & W_k \end{bmatrix}$$

$$S_k = \begin{bmatrix} 0 & 0 \\ 0 & S_k \end{bmatrix}$$

Substituting for $M_k$ from (3.26) into (3.24), $M_{k+1}$ is obtained in the form,

$$M_{k+1} = \left[ D_k + \hat{\Gamma}_k F_k \hat{\Gamma}_k^T + W_k + W_k^T \right] \hat{\Gamma}_k^{-T} A_k \hat{\Gamma}_k^{-T} + \hat{\Gamma}_k$$

(3.27)

where $A_k = [I + \psi F_k \psi^T]^{-1}$

Substituting for $M_{k+1}$ in (3.17b), the control gain matrix is obtained as,

$$N_k = \left[ D_k + \hat{\Gamma}_k F_k \hat{\Gamma}_k^T + W_k + W_k^T \right] B_k$$

(3.28)

with $B_k = \phi^{-1} A_k$. Furthermore, in partitioned form, equation (3.28) may be expressed as,

$$\begin{bmatrix} N_{11k} & N_{12k} \\ N_{21k} & N_{22k} \end{bmatrix} = \begin{bmatrix} D_{1k} & 0 \\ 0 & D_{2k} \end{bmatrix} \begin{bmatrix} \phi^{-1} F_k \phi^{-1} \\ \phi^{-1} S_k \phi^{-T} \end{bmatrix} \begin{bmatrix} B_{11k} & B_{12k} \\ B_{21k} & B_{22k} \end{bmatrix}$$

(3.29)
Now the control vector $\hat{\delta u}_k$ is given by (3.19) as,

$$
\Gamma_k \cdot \hat{\delta u}_k = \Gamma_{2k} \cdot \hat{\delta u}_k = N_{21k} e_{1k} + N_{22k} e_{2k}
$$

Consequently, the partitions $N_{11k}$ and $N_{12k}$ do not contribute to the synthesis of $\hat{\delta u}_k$—although they do contribute to the generation of the $M_{k+1}$ vector. Therefore, attention is focused upon the lower two partitions of $N_k$. From (3.29) the lower partitions may be expressed as,

$$
(N_{21k}, N_{22k}) = \Gamma_{2k} \left( \begin{array}{c}
C_k B_{11k} + F_{2k} T_{2k} B_{21k} \\
\nu_k T_{2k} B_{12k} + F_{2k} T_{2k} B_{22k}
\end{array} \right) = \Gamma_{2k} \nu_k = N_{2k}^d
$$

(3.30)

Having obtained the above general expression for the partitioned control gain matrix, consider the general (p-i) partitions of $\Gamma_{2k}$ as defined in theorem 3.1. As before it is assumed that one of these partitions is an $m \times m$ nonsingular matrix. Let $\Gamma_{2k}$

$$
\begin{bmatrix}
\Gamma_{21k} & \cdots & \Gamma_{2p-1k}
\end{bmatrix}.
$$

Utilizing these partitions of $\Gamma_{2k}$, equation (3.30) may be repartitioned into the form,

$$
\begin{pmatrix}
N_{11k}^d & N_{1p-ik}^d \\
N_{21k}^d & N_{2p-ik}^d
\end{pmatrix}
$$

(3.31)

$$
= \begin{pmatrix}
\begin{pmatrix}
\sum_{j=1}^{p-i} \Gamma_{21jk} V_{j1k} \\
\sum_{j=1}^{p-i} \Gamma_{2pj} V_{j1k}
\end{pmatrix} & \begin{pmatrix}
\sum_{j=1}^{p-i} \Gamma_{21jk} V_{jk} \\
\sum_{j=1}^{p-i} \Gamma_{2pj} V_{jk}
\end{pmatrix} \\
\begin{pmatrix}
\sum_{j=1}^{p-i} \Gamma_{21jk} V_{j1k} \\
\sum_{j=1}^{p-i} \Gamma_{2pj} V_{j1k}
\end{pmatrix} & \begin{pmatrix}
\sum_{j=1}^{p-i} \Gamma_{21jk} V_{jk} \\
\sum_{j=1}^{p-i} \Gamma_{2pj} V_{jk}
\end{pmatrix}
\end{pmatrix}
$$
with,
\[ \Gamma^\nu_{ijk} = \Gamma^\nu_{ik} \Gamma^\nu_{jk}^{-1} \Gamma^T_{2 jk} \]

Consider any two terms \( N^d_{\xi m_k} \) and \( N^d_{\eta m_k} \). From (3.31) they may be expressed as,

(a) \[ N^d_{\xi m_k} = \sum_{j=1}^{p-1} \Gamma^\nu_{2 jk} \Gamma^T_{2 jk} V_{jm_k} \]

(b) \[ N^d_{\eta m_k} = \sum_{j=1}^{p-1} \Gamma^\nu_{2 jk} \Gamma^T_{2 jk} V_{jm_k} \]

Furthermore, let \( \Gamma^\nu_{2 qk} \) be the \( m \times m \), non-singular partition of \( \Gamma^\nu_{2 k} \); therefore, from (3.32) and (3.33), above the following equality is obtained,

\[ \Gamma^\nu_{2 l k} \Gamma^\nu_{2 qk} N^d_{\xi m_k} = \Gamma^\nu_{2 l k} \Gamma^\nu_{2 qk} \Gamma^\nu_{2 qk} \Gamma^\nu_{2 l k} \sum_{j=1}^{p-1} \Gamma^T_{2 jk} V_{jm_k} = N^d_{\xi m_k} \]

Comparison of equation (3.34) with the equality constraint of theorem 3.1, equation (3.21), indicates that they are of exactly the same form. Hence, it has been proved that the gain matrix \( N_k \) generated by the control solution, equations (3.17b) and (3.17c), obeys the required equality constraint. Therefore, any incremental control vector obtained from an \( m \times m \) nonsingular partition of \( \Gamma_k \) will be a unique solution of equation (3.19).
Corollary

The control gain matrix is given by,

\[ A \hat{N}_k = \Gamma_k \sigma^{-1} (P_k - I) \]

where as before \( \Gamma_k = \begin{bmatrix} 0 & 1 \\ 0 & I \end{bmatrix} \)

and \( A = \begin{bmatrix} 0 & 0 \\ 0 & I_t \end{bmatrix} \) where \( I_t \) is the \( t \times t \) identity matrix. The matrix \( P_k \)
corresponds to the backward recursive matrix of the ‘specific optimal
output error servomechanism’ given by the set of equations,

\[ P_k = \psi^T \Omega \psi + \sigma^T P_{k+1} \left[ I + \hat{N} P_{k+1} \right] \sigma^{-1} \sigma \]

\[ \delta \psi_k = -R^{-1} \sigma^T \sigma^{-T} (P_k - I) \psi^T \Omega (P_k - \psi \psi_k) \]

with, \( P_k = P_k' \Omega \)

where \( \Omega = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \)

and \( \psi^T \Omega \psi \) is a nonsingular partition of \( \psi^T \Omega \psi \).

Note, that in partitioned form the matrices \( P_k \) and \( P_k' \) are given by,

\[ \hat{P}_k = \begin{bmatrix} \hat{P}_{11} & 0 \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix} \]

\[ P_k = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \]

Proof

Consider the canonic equations for the control problem as given in

Chapter 2, i.e.

\[ x_{k+1} = a x_k + d_k - \Gamma_k \sigma^{-1} \Gamma_k \lambda_{k+1} \]

\[ \lambda_k = \sigma^T \lambda_{k+1} - \psi^T \Omega (P_k - \psi \psi_k) \]
\[ u_k^* = -E_1 \Gamma_k^T \lambda_{k+1} \]

Consider the specific solution for the control problem of the form,

\[ \lambda_k = -P_k \psi^T Q (\rho_{k}^* - \psi x_k) \]
\[ \rho_k^* = \rho_k - \eta_k \]

i.e. the control is to be a function of the projection of the weighted error of the specifically controlled states onto the state space, i.e.

\[ \psi^T Q (\rho_k - \psi x_k) : \Omega_x^c \rightarrow \Omega_x \]

Substituting, an expression for \( x_{k+1} \) is obtained as,

\[ x_{k+1} = \left[ I + \Gamma_k P_{k+1} \psi^T Q \psi \right]^{-1} \left\{ \Phi x_k + d_k + \Gamma_k P_{k+1} \psi^T Q \rho_{k+1} \right\} \]

where \[ \Gamma_k = \Gamma_k P_{k+1}^{-1} \Gamma_k^T \]

Substituting into the equation for the adjoint variable the following equations are obtained following simplification,

\[ P_k \psi^T Q \rho_{k+1} - \psi^T Q \rho_k = \psi^T \left[ I + \Gamma_k P_{k+1} \psi^T Q \psi \right] \psi^T \]

\[ = \psi^T Q \rho_{k+1} - \psi^T Q \rho_k + \psi^T \left[ I + \Gamma_k P_{k+1} \psi^T Q \psi \right]^{-1} \]

\[ P_k \psi^T Q \psi = \psi^T P_{k+1} \psi^T Q \psi \left[ I + \Gamma_k P_{k+1} \psi^T Q \psi \right]^{-1} \psi + \psi^T Q \psi \]

i.e.

\[ P_k' = \psi^T P_{k+1} \left[ I + \Gamma_k P_{k+1} \right]^{-1} \psi + \psi^T Q \psi \]

\[ \delta u_k^* = P_{k+1}^{-1} \Gamma_k \psi^T (P_k - I) \psi^T Q (\rho_k - \psi x_k) \]

The first two equations define the set of backward recursive equations for the 'specific optimal output error servomechanism', which start from the terminal conditions \( \psi^T Q \rho_k^* = \psi^T Q \rho_f \) and \( P_k = [0] \).

Utilizing the equation for \( \delta u_k^* \) the relationship,

\[ \Gamma_k \delta u_k^* = \Gamma_k \psi^T (P_k - I) \psi^T Q (\rho_k - \psi x_k) \]

is obtained.
Now from the previously obtained control algorithm,

\[ \Gamma \delta \hat{\mu}_k = A \Lambda_k \psi^T \Omega (\phi_k - \psi x_k) \]

where \( \Lambda = \begin{bmatrix} 0 \mid 0 \\ 0 \mid I \end{bmatrix} \)

and therefore \( \delta \hat{\mu}_k = \delta \hat{u}_k \)

with,

\[ A \Lambda_k = \hat{\theta}^T (P_k - I) \]

where \( P_k \psi^T \Omega \psi = P_k' \Lambda_q \)

Let \( \psi^T \Omega \psi = \begin{bmatrix} \Omega_1 \mid 0 \\ 0 \mid \Omega_2 \end{bmatrix} \) \( \Lambda_q = \begin{bmatrix} [I_q \mid 0] \\ 0 \mid 0 \end{bmatrix} \)

\[ \therefore P_k = P_k' \begin{bmatrix} \Omega_1^{-1} \mid 0 \\ 0 \mid 0 \end{bmatrix} = P_k' \Omega \]

**Lemma:**

A suitable choice for the matrices \( F_k \) and \( W_k \) is given by

\[
\begin{bmatrix}
0 & 0 \\
W_k & F_k
\end{bmatrix} = \begin{bmatrix}
\Omega_1^{-1} & 0 \\
0 & \Omega_2^{-1}
\end{bmatrix} \psi^T \begin{bmatrix}
P_k' \mid & I \\
\psi^T \Omega \psi & M_k
\end{bmatrix} \psi^T
\]

In the next chapter the adaptive control algorithm developed here will be applied to the control of a model of a plasma torch furnace. The results of extensive real-time experimentation will be presented.
CHAPTER 4

EXPERIMENTAL STUDY OF REAL-TIME ADAPTIVE CONTROL

The suboptimal adaptive control algorithm, developed as part of this research, has been applied to the control of a model of a plasma torch furnace. Using analog/digital computer facilities, an experimental study of real-time adaptive control was conducted. The experimental arrangement and procedure are outlined in this chapter, and the results of this study are presented.

4.1 The System Equations

A mathematical model of the dynamics of a plasma torch furnace has been obtained as part of this work — details of the model may be found in Appendices I and II. The dynamic equations will be summarized here, since a detailed explanation of their derivation would merely be a repetition of the contents of the aforementioned appendices.

The furnace dynamics are represented by a fifth-order set of normalized state equations; viz.,

\[
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
    \dot{x}_3 \\
    \dot{x}_4 \\
    \dot{x}_5 \\
\end{bmatrix} =
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} & a_{14} & 0 \\
    a_{21} & a_{22} & 0 & a_{24} & 0 \\
    a_{31} & 0 & a_{33} & a_{34} & 0 \\
    0 & 0 & 0 & a_{44} & a_{45} \\
    0 & 0 & 0 & 0 & a_{55} \\
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 \\
    x_5 \\
\end{bmatrix}
\begin{bmatrix}
    y_1 \\
    x_4 \\
    x_5 \\
\end{bmatrix} +
\begin{bmatrix}
    0 \\
    0 \\
    0 \\
    a_{41} & a_{42} \\
    a_{51} & 0 \\
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2 \\
\end{bmatrix}
\begin{bmatrix}
    0 \\
    d_4 \\
    0 \\
\end{bmatrix}
\]

(4.1)
Where \( x_1 \) is the bulk temperature of the gas in the furnace, \( x_2 \) and \( x_3 \) are the temperatures of the exit coolant from the inner and outer electrodes respectively, \( x_4 \) is the average arc current and \( x_5 \) is the magnetic flux density in the electrode gap. The control variables, \( u_1 \) and \( u_2 \), are the voltages applied to exciter windings of the generators supplying power to the arc circuit and field coil. The system parameter \( a_{11} \) is some function of the gas flow through the plasma torch and varies between the values \(-0.2 \rightarrow -0.7\) over the normal operating range of the gas flow. The physical interpretation of the parameters \( a_{ij}, b_{ij}, \gamma_1 \) and \( d_4 \) may be found in Appendix I, with typical numerical values, for the plasma torch considered, given in Appendix II.

The objectives of the control system are to either, (i) maintain the furnace temperature at a set point \( p \) (regulator control), or (ii) follow a sequence of desired values, \( p_k \), (trajectory tracking), while minimizing the electrical losses incurred in the arc and field circuits during control operation. In addition, gas flow variations can occur as a result of external demands generated by the process in which the furnace is being utilized. The concomitant variations induced in the parameter \( a_{11} \) must, therefore, be accommodated by the control system; hence, the need for a parameter adaptive strategy.

The nonlinear system equations of (4.1) are linearized about a nominal trajectory point \( p_k^0 \), as outlined in Chapter 2, and subsequently discretized, using a simple rectangular integration scheme over the sampling interval \( \Delta \). In this manner, discrete time dynamic equations are obtained in the required form; (i.e. a linear system representation \( \tilde{f}_e (p_k^0) \)). The measurement of the furnace temperature is linear and noisy, i.e. \( y_k = x_{1k} + v_k \), with the measurement noise \( v_k \) a Gaussian
process of r.m.s. value \( r_v = 0.1 \).

The control performance index is given by

\[
J(\delta u_k) = \frac{1}{2} \sum_{k=1}^{k_f} \left[ q_1 (s_k - x_{1k})^2 + r_1 \delta u_{1k}^2 + r_2 \delta u_{2k}^2 \right]
\]

with the terminal stage index free (i.e. control over a variable interval is considered). The weighting factors in \( J(\delta u_k) \) have been obtained in a rationalized manner in Appendix I, and are given by,

\[
q_1 = 4.0 \times 10^{-4} \times q_1^1; \quad r_1 = 0.02 \times r_1^1; \quad r_2 = r_2^1
\]

where, \( q_1^1, r_1^1 \) and \( r_2^1 \) are factors, whose values are chosen to reflect the relative importance attached to each of the terms in \( J(\delta u_k) \). In addition, these factors determine the response characteristics of the control system. In the majority of experiments performed, the values used were \( q_1^1 = 25.0, r_1^1 = 1.0 = r_2^1 \), i.e. weighting factors of magnitude,

\[
q_1 = 0.01 \quad R_1 = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} = \begin{bmatrix} 0.02 & 0 \\ 0 & 1.0 \end{bmatrix}
\]

The equations characterizing the discrete-time adaptive control algorithm developed in the previous chapter, are summarized in Table 4.1. Figure 4.1 shows a block diagram of the major computation blocks implemented in the digital controller, together with the plant interfaces.

4.2 The Experimental System

The model of the plasma torch furnace was simulated on an EAI 580 analog computer and was controlled by a minicomputer. The digital computer facility is comprised of a DEC PDP 8/I central processor with 4k core memory, one DF32 disk file of 32k capacity, a 2-channel D/A unit and a DEC AX08 Lab peripheral with 4-channel A/D, 2-channel D/A and two
Modelling

\[
\begin{bmatrix}
\hat{x}_{u1} \\
\hat{\alpha}_{u1}
\end{bmatrix} = \begin{bmatrix} \alpha_k & \alpha_k \\
0 & I
\end{bmatrix} \begin{bmatrix}
\hat{x}_k \\
\hat{\alpha}_k
\end{bmatrix} + \begin{bmatrix} \Gamma_k \\
0
\end{bmatrix} \delta \hat{\alpha}_k + \begin{bmatrix} K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix} \begin{bmatrix} H_k^T \\
0
\end{bmatrix} \alpha_2 \begin{bmatrix} y_k - H_k \hat{\alpha}_k - \beta_k \\
0
\end{bmatrix}
\]

\[+ \begin{bmatrix} \hat{\xi}_k \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix} K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix} = \begin{bmatrix} \alpha_k & \alpha_k \\
0 & I
\end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix} \begin{bmatrix} H_k^T \Delta_k H_k & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}
\]

\[
\Delta_k = \{ Q_2^{-1} + H_k \Delta_k H_k^T \}^{-1}
\]

\[
\begin{bmatrix} P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix} \begin{bmatrix} \phi_\alpha I \\
0
\end{bmatrix} + \hat{\alpha} R_2 A^T
\]

Control

\[
\delta \hat{\alpha}_k = \Gamma_k N_k \psi^T \alpha_1 \{ \alpha_k - \psi \phi_\alpha \}
\]

\[
\hat{\alpha}_k = \hat{\alpha}_{k-1} + \delta \hat{\alpha}_k
\]

\[
N_k = \phi_k \{ M_k - M_k \psi \psi^T [\alpha^T + \psi M_k \psi^T]^{-1} \psi M_k \}
\]

\[
M_{k+1} = N_k \phi_k^T + \Gamma_k R_1 \Gamma_k^T
\]

---

**TABLE 4.1** Summary of Adaptive Algorithm
**FIG. 4.1 IMPLEMENTATION OF THE ADAPTIVE CONTROL ALGORITHM**
real-time clocks. The measurement noise was simulated using a Hewlett-Packard HP1-3722A noise generator. The experimental set-up is shown in Figure 4.2.

A software package implementing the adaptive control algorithm was developed. Owing to the limited core memory available, and the premium placed upon computation times in real-time applications, the programs were written entirely in the DEC machine assembly language, PAL III. The package consists of two sections; an off-line section and an on-line section.

The off-line section performs data handling and assignment operations required for preparation of a system data tape. The system data tape is comprised of information regarding the dimensions of system matrices, the values of known system parameters, initial values of system variables and the mode of operation of the on-line section. The mode of operation of the on-line section may be selected by setting flags in certain memory locations, thereby resulting in the implementation of certain processing options; e.g. state modelling only, state modelling and control, state and parameter modelling and control, etc.

The system data tape serves as the initial input to the on-line section.

The on-line section performs the following operations: (i) calibration of the variable RC clock with reference to the crystal clock, (ii) initial computation, and subsequent updating of system transition matrices and, (iii) real-time processing of the adaptive control algorithm. The on-line section utilizes less than 4k core memory locations, and has sufficient storage capacity to handle systems with up to sixth order state equations, and two observation variables.
FIG. 4.2 THE EXPERIMENTAL SETUP
FIG. 4.3 INTERACTIONS WITHIN THE SOFTWARE PACKAGE
A block diagram, showing the overall interactions within the software package, is given in Figure 4.3. Programming details are presented in Appendix III.

4.3 Procedure

A series of real-time experiments, designed to study a number of aspects of the performance of the adaptive control system, were conducted. Firstly, the system performance, under different operating conditions and with different control requirements, was investigated. These tests included, (a) regulator control with parameter identification,

(b) regulator control with tracking of a continuously varying parameter,

and,

(c) trajectory tracking with parameter identification.

In addition, tests were performed to study,

(a) the response characteristics of regulator control using the COCP (adaptive algorithm) and the NOCP (non-adaptive, nominal parameter algorithm),

(b) the effects of performance index weightings and the sampling interval on the system performance, and,

(c) the factors affecting the interaction between control and identification in adaptive systems.

The procedure adopted in all of these tests was to generate four runs of the system, using the same initial conditions, operating conditions, control requirements, etc. Each run was of fixed, constant duration (256 iterations of the adaptive algorithm) and, from the data accumulated during the four runs, sample ensemble statistics were computed. Quantities obtained in this manner are denoted by a bar over the corresponding symbol variable, e.g. $\bar{x}_1$ is the sample mean of
variable $x_1$.

An initial investigation indicated that, for the fifth order system with one uncertain parameter, the time required to complete one iteration of the algorithm (i.e. calculate new estimates of the states and parameters, evaluate the new control vector and update the modelling, control and system matrices), was $1.25$ sec. Therefore, throughout the majority of experimental runs, a sampling interval of $1.3$ sec. was used.

The initial conditions on the Ricatti matrices of modelling and control, $P_0$ and $M_0$ respectively, were selected as,

$$
P_0(6 \times 6) = \begin{bmatrix}
    0 & \cdots & P_{21}(0) \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & 0 \\
    P_{21}(0) & 0 & 0 \\
    \vdots & \vdots & \vdots \\
    0 & \cdots & P_{22}(0)
\end{bmatrix}; M_0(5 \times 5) = [0]
$$

The values of $P_{21}(0)$ and $P_{22}(0)$ are selected experimentally in order to achieve a trade-off between the time required to identify the uncertain parameter, and the error in the identified parameter. For the majority of experiments performed, the values used were $P_{21}(0) = 4.0$ and $P_{22}(0) = 2.0$. The effects of different values for $P_{21}(0)$ and $P_{22}(0)$ on the control system performance will be outlined in a later section of this chapter.

The initial conditions on the state vector and control vector were chosen as,

$$
\dot{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad u_0 = \begin{bmatrix} 0.66 \\ 0.7 \end{bmatrix}
$$
This choice of initial conditions implies that the only initial information available to the control algorithm is that the control vector $\hat{u}_o$ has been applied to the system, (i.e. owing to the parameter uncertainty and state inaccessibility, no prior computation or measurements have been performed). The initial nominal trajectory point is obtained from $\pi_{-1} = \pi_o = \{\hat{\pi}_o, \hat{\pi}_o | \hat{\phi}_o\}$, where $\hat{\phi}_o$ is the initial estimate of the parameter $a_{11}(0)$.

4.4 Experimental Results

4.4a Adaptive regulation

Temperature regulation of the furnace model was considered with a normalized set-point $\rho = 0.5$ being the desired temperature. A series of runs were generated for both the conditional optimal control policy (COCP) and the nominal optimal control policy (NOCP), using a number of different initial parameter values, $\hat{\phi}_{11}(0)$. The true system parameter was $a_{11} = -0.48$, and the set of initial parameter values used was $\hat{\phi}_{11}(0) \in \{-0.2, -0.4, -0.48, -0.60, -0.70\}$. The response of the controlled state (the temperature $x_1$), for the different initial parameter estimates, is plotted in Figure 4.4. In all the plots, the response obtained using the COCP (solid line) is shown together with the response obtained using the NOCP (dotted line). The percentage values associated with each plot represent the percentage error between
FIG. 4.4 REGULATOR CONTROL RESPONSES WITH THE COCP & NOCP ALGORITHMS
the true parameter value and the chosen initial value, i.e.

\[ \Delta a_{11} (\%) = \frac{[a_{11} - \hat{a}_{11}(0)]}{|a_{11}^{\text{max}} - a_{11}^{\text{min}}|} \times 100 \]

where \( a_{11}^{\text{max}} \) and \( a_{11}^{\text{min}} \) are the maximum and minimum values of \( a_{11} \), respectively, and the denominator of the above equation represents the modulus of the maximum change in \( a_{11} \) due to variations in the gas flow (see Appendix I).

It is evident from Figure 4.4 that, without parameter identification, extremely poor regulation of the furnace temperature results when errors exist between the actual and modelled parameters. Whereas, with the adaptive COCP, good temperature regulation is attained for all the initial parameter estimates considered. There is an initial transient period of poor regulation, during which the parameter is being identified – the transients being more pronounced the greater the initial error. However, in all cases the COCP assured attainment of the desired temperature set-point.

Utilizing the results obtained from this series of tests, comparisons between the COCP and the NOCP may be made. Quantities characterizing various aspects of regulator performance are the control performance criterion \( J(\hat{u}_{k}) \), the mean steady state regulation error \( \| u - \hat{u} \| \) and the mean steady-state modelling error norm \( ||x - \hat{x}||^2 \). These quantities are shown in Figures 4.5, 4.6 and 4.7 as functions of the percentage initial parameter error \( \Delta a_{11} \).

It may be seen that, with the exception of a region of small parameter error \( |\Delta a_{11}| \lesssim 2\% \), the COCP provides better performance than the NOCP with respect to all three performance indicators. The significance of the region \( |\Delta a_{11}| \lesssim 2\% \) will be discussed in the next chapter.
FIG. 4.5 CONTROL COST AS A FUNCTION OF $\Delta a_{11}$
FIG. 4.6 STEADY STATE REGULATION ERROR AS A FUNCTION OF $\Delta a_{11}$

Steady state control offset error

$\frac{(p - x_1)}{}$

- non-adaptive (NOCP)
- adaptive (COCP)
\( \{ \| x - \hat{x} \|_2^2 \} \)

Percent error in initial parameter estimate

\( \Delta \quad \text{adaptive system (COCP)} \)
\( \circ \quad \text{non-adaptive (NOCP)} \)

**Fig. 4.7** Steady state error in modelled states as a function of \( \Delta a_{11} \).
\[ \| x_1 - \hat{x}_1 \|^2 \]
\[ |a_n - \hat{a}_n| \]
\[ \hat{a}_n(0) = -0.2 \]
\[ \hat{a}_n(0) = -0.4 \]

**FIG. 4.8** CONVERGENCE OF THE MODELLED PARAMETER $\hat{a}_{11}$ AND STATE $\hat{x}_1$
It is interesting to note that, for all initial parameter errors, the COCP reduces the mean steady-state regulation error to zero; which is definitely not the case with the NOCP.

Typical convergence characteristics of the errors in modelling the furnace temperature $\hat{x}_1$ and parameter $\hat{a}_{11}$, are shown in Figure 4.8 for two different initial parameter errors.

4.4b Regulation with parameter tracking

The recorded responses of the controlled furnace temperature, when parameter variations occur during operation, are shown in Figures 4.9a and 4.9b. As before, the control system is required to regulate the temperature at a set-point $= 0.5$. In the first case, Figure 4.9a, a step change in $a_{11}$ occurs during operation. A transient in the controlled temperature appears during the period in which the new parameter value is being identified. Following successful identification, the desired regulation is once again achieved. In Figure 4.9b regulation of the furnace temperature is shown for the case of a continuous triangular variation in $a_{11}$. The variation in $a_{11}$ represents a 40% change in the parameter value, relative to the maximum possible change in the parameter. The response denoted by the dotted trajectory is the response of the NOCP, evaluated with a nominal parameter estimate $\hat{a}_{11} = -0.5$. It is evident that the COCP is capable of accommodating the parameter variations whilst maintaining relatively good temperature regulation; whereas the NOCP merely reflects the parameter variations as if they were unknown (i.e. unmodelled), slowly varying system disturbance inputs.

4.4c Trajectory tracking control

Consideration was given to situations in which the desired furnace
FIG. 4.9 REGULATOR CONTROL WITH PARAMETER TRACKING

a. Step change in $a_{11}$

b. Continuously varying $a_{11}$
temperature is not a fixed value, but varies in some manner. Such variations in the desired temperature might arise due to either a change in the set point, introduced by the process control operator or, a demand generated by another system within the overall process. In such situations the control system is required to operate as an 'adaptive servomechanism', as opposed to the 'adaptive regulator' considered previously.

The response of the furnace temperature to step changes in the desired temperature set-point is shown in Figure 4.10a. Also shown in this plot is a recorded trace of the noisy temperature observation being monitored by the control computer. The purpose of including this trace is merely to illustrate the level of noise in the observed signal.

The trajectory-tracking response, with a continuous sinusoidal variation of the desired furnace temperature, is shown in Figure 4.10b. The peak-to-peak variation of the desired temperature represents 55% of the range of furnace temperature ($x_1 = 0$ to $x_1 = 1.0$ in normalized units). As the experimental results indicate, the COCP algorithm operates extremely well in the trajectory-tracking mode.

The weighting factors of the control performance index $J(\delta u_k)$ play a dominant role in determining the response characteristics of the control system. In particular, the control weighting matrix $R_1$ may be chosen to yield a particular response characteristic, e.g. damped, underdamped, etc., since the elements of this matrix attach penalties to excursions of the corresponding control vectors — with the penalties assessed proportional to the relative magnitudes of the matrix elements. This is demonstrated in Figure 4.11, where the step responses of the control system, with various $R_1$ weighting matrices, are plotted. The
FIG. 4.10 TRAJECTORY - TRACKING ADAPTIVE CONTROL RESPONSE

a. Step response
b. Sine wave following
FIG. 4.11  STEP RESPONSE OF THE COCP ALGORITHM WITH DIFFERENT $R_1$ MATRICES

(1) $R_1 = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$  
(2) $R_1 = \begin{bmatrix} 0.05 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$

(3) $R_1 = \begin{bmatrix} 0.02 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$  
(4) $R_1 = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.1 \end{bmatrix}$
effect of increasing the weighting factor $q_1$ was to speed up the step response of the system. Furthermore, the control gain Ricatti equation is more sensitive to variations in $q_1$ than it is to variations in $R_1$. It was found that if $q_1$ was made too large, numerical divergence of the Ricatti equations occurred. For the system considered, divergence was encountered for all $q_1 > 0.02$ (i.e. a relative weighting factor $q_1 \approx 50.0$).

4.4d Factors influencing the identification process

The primary factors influencing parameter identification are the initial values $p_{21}(0)$ and $p_{22}(0)$, since they effectively control the identification gain (in this case the gain is element $k_{61_k}$ of the modelling gain matrix $K$). The manner in which these initial values may be chosen is not obvious, since the statistics $\text{cov}(X_{10}, a_{11_0})$ and $\text{var}(a_{11_0})$ do not exist. The values of $p_{21}(0)$ and $p_{22}(0)$ may be interpreted as reflecting the level of initial uncertainty in the initial parameter estimate. However, the problem of rationally quantifying this uncertainty still remains. As a result, numerical experimentation was resorted to in order to select suitable initial values.

A series of regulator control runs were performed using different combinations of the initial values $p_{21}(0)$ and $p_{22}(0)$. A sample of the parameter identification trajectories obtained are shown in Figure 4.12. It was found that, the larger the values of $p_{21}(0)$ and $p_{22}(0)$ were, the more rapidly the parameter error decreased. This corresponds to the case of higher levels of initial uncertainty in the parameter estimate (as reflected by the larger values of $p_{21}(0)$ and $p_{22}(0)$), which result in a larger identification gain and, thereby, act to produce a
FIG. 4.12 EFFECT OF \( p_{21}(0) \) & \( p_{22}(0) \) ON THE IDENTIFICATION PROCESS

(1) \( p_{21}(0) = 1.0 \) ; \( p_{22}(0) = 1.0 \)
(2) \( p_{21}(0) = 2.0 \) ; \( p_{22}(0) = 2.0 \)
(3) \( p_{21}(0) = 3.0 \) ; \( p_{22}(0) = 3.0 \)
(4) \( p_{21}(0) = 4.0 \) ; \( p_{22}(0) = 4.0 \)
(1) $p_{22}(0) = 1.0$
(2) $p_{22}(0) = 2.0$
(3) $p_{22}(0) = 3.0$
(4) $p_{22}(0) = 4.0$

**FIG. 4.13** IDENTIFICATION CHARACTERISTICS AS FUNCTIONS OF $p_{21}(0)$ & $p_{22}(0)$

--- Steady-state identification error (%)
--- Δ --- Relative identification time
more rapid response in the identification process. However, as might be expected, the larger identification gains result in larger steady state errors in the identified parameter.

The results of these series of tests are summarized in Figure 4.13, where the mean steady state error in the identified parameter and the relative identification time are plotted as functions of $p_{21}(0)$ and $p_{22}(0)$. Note that the identification time for $p_{21}(0) = p_{22}(0) = 2.0$ is used as the unit identification time. From these results it may be seen that more rapid parameter identification is obtained at the expense of greater steady state parameter errors.

4.5 Sampling Interval Sensitivity

From the point of view of real-time control, an important consideration is the sensitivity of the control algorithm to variations in the sampling interval. Typically, a design procedure for a particular application involves selecting suitable control performance criterion weight functions based upon, (i) physical considerations, i.e. dimensional normalization of quantities appearing in the performance index (see Appendix I), (ii) the relative weightings reflecting operating cost and, (iii) the desired response characteristics. Having chosen a set of weighting coefficients for a particular sampling interval, there is no guarantee that the same control performance will be achieved for a different sampling interval. It is therefore desirable to obtain both qualitative and quantitative information regarding the effects of sampling interval variations upon the control system performance.

Owing to the lower limit imposed upon the sampling interval by the time required to complete one iteration of the algorithm, only relatively small variations in the sampling interval could be
accommodated in the real-time experiments. It was found that in the range \( \Delta = 1.3 - 1.6 \) sec. very slight changes in the control performance occurred. The weighting coefficients of \( J(\delta u_k) \) had been selected for \( \Delta = 1.36 \) and, as \( \Delta \) increased, there was a small increase in the control cost. An identification error function, computed from

\[
J(\hat{\Delta}) = \sum_{k=1}^{256} (a_{11} - \hat{a}_{11})^2
\]

was found to decrease slightly as \( \Delta \) increased from 1.36 sec. to 1.6 sec.

Since the effects of large scale variations in \( \Delta \) are of major interest, recourse was made to a digital simulation of the real-time control system. Different sampling intervals in the range \( \Delta = 0.5 - 1.6 \) sec. were considered and, in this case, the results obtained were much more spectacular. As was the case in the real-time control experiments, there were slight increases in control cost for \( \Delta > 1.36 \). However, for \( \Delta < 1.36 \) the performance degraded rapidly with decrease in the sampling interval. The regulation response became much more oscillatory as shown in Figure 4.14, and the control performance criterion increased rapidly; see Figure 4.15. It is interesting to note that the increased control costs were accompanied by more rapid parameter identification, Figure 4.14, and lower values of the identification error function \( J(\hat{\Delta}) \), Figure 4.15.

In view of the sensitivity of the control algorithm to variations in the sampling interval, it was decided to desensitize the algorithm. This may be readily achieved by adjusting the values of the control performance criterion weighting factors dependent upon the sampling interval used, thereby making the control gains effectively invariant of the sampling interval. Considering the equations defining the control vector, i.e. from Table 4.1,
FIG. 4.14  ADAPTIVE REGULATION WITH DIFFERENT SAMPLING INTERVALS
FIG. 4.15 CONTROL AND IDENTIFICATION COSTS AS FUNCTIONS OF $\Delta$

- Sampling interval sensitive
- --- desensitized

$x = \overline{J(\delta u)} \times 10^{-2}$

$\circ \overline{J(\hat{\alpha})}$
\[ \delta \hat{u}_k = r^{-1} q_k \psi^T Q_1 (p_k - \psi \hat{u}_k) \]

\[ N_k = \delta_k \left( M_k - M_k \psi^T I_{Q_1}^{-1} \psi M_k \psi^T I_{Q_1}^{-1} \psi \right) \]

\[ N_{k+1} = N_k \delta_k + \Gamma_k R_1^{-1} \Gamma_k^T \]

it may be seen that \( \delta \hat{u}_k \) is directly proportional to \( Q_1 \) and, through dependence on \( N_k \), inversely proportional to \( R_1 \). Therefore, by selecting the weighting factors for a sampling interval \( \Delta_2 \) as,

\[ Q_{12} = \left( \frac{\Delta_2}{\Delta_1} \right) Q_1 \]

\[ R_{12} = \left( \frac{\Delta_1}{\Delta_2} \right) R_{11} \]

where \( Q_{11} \) and \( R_{11} \) are the weighting factors for a sampling interval \( \Delta_1 \), the effective control gain, \( r^{-1} q_k \psi^T Q_1 \), will become invariant of the sampling interval. The results obtained from the simulation, shown by the dotted line in Figure 4.15, indicate that the desired sampling interval insensitivity is achieved. Both the control cost \( J(\delta \hat{u}_k) \) and identification error function \( J(\hat{u}) \) remain nearly constant over a range of sampling intervals from 0.1 sec. to 1.6 sec.

The results obtained from the study of sampling interval sensitivity have indirect, and extremely important, bearing upon the more general questions of control and identification interaction in adaptive systems, and upon the overall performance of adaptive control systems.

These questions will be considered in the next chapter, where the implications of the results documented in this chapter will be discussed.
CHAPTER 5

DISCUSSION OF RESULTS

The experimental results presented in the previous chapter have demonstrated the effectiveness of the proposed adaptive control policy in achieving desired control objectives, despite wide variations in the uncertain system parameter. The control algorithm was shown to perform well in both the adaptive regulating and trajectory tracking control modes. The results pertaining to the adaptive COCP and non-adaptive NOCP are of particular interest since they illustrate a number of inherent features of adaptive control systems.

The performance of the NOCP is extremely sensitive to errors in the assumed value of the uncertain system parameter $\hat{A}_{11}$. With increase in the error between the actual and assumed parameter values, the overall performance deteriorates very rapidly; this performance degradation being characterized by increased control costs (Figure 4.5), increased offset error in the regulated state variable (Figure 4.6) and increased error in the modelled state variables (Figure 4.7). On the other hand, with the COCP the sensitivity of the overall performance to initial parameter errors is considerably reduced. Not only are the control costs significantly decreased for large initial parameter errors, but the steady state offset in the regulated state is reduced to zero and the errors in the modelled states are reduced to very small levels. It is evident that, with increase in the initial parameter error, there is some increase in the control cost; however, in comparison to the increase associated with the NOCP, it is much less significant. This slight increase in control cost of the COCP may be accounted for in terms of the additional cost incurred during the transient.
Fig. 5.1  Control Cost as a Function of $\Delta a_{11}$

\[ J(COCP) \geq J(NOCP) \]
parameter identification period.

When the parameter errors are small it is possible for the NOCP to yield lower control cost than the COCP, as shown in Figure 5.1 where a portion of the curves of $J(\Delta u)$, given in Figure 4.5, are reproduced. The shaded region in Figure 5.1 indicates the range of parameter errors over which the NOCP yields lower control cost than the COCP. It may be seen that this range of percentage parameter errors is approximately defined by $|\Delta a_{11}| \approx 1.5\%$. It is interesting to note that this value corresponds to the bound on the accuracy of the parameter estimate, obtained from the identification process with the initial conditions $P_{21}(0) = 4.0$ & $P_{22}(0) = 2.0$. It may therefore be concluded that there is a lower bound on the performance of the adaptive control policy, which is imposed by the identification process — this bound being determined by the accuracy with which the uncertain system parameter can be identified.

In addition, these results provide empirical validation for the ordering of the control policies proposed in Chapter 2. It was stated in the aforementioned chapter that, providing there is a finite integer $M$, such that $|| a_k - \hat{a}_k || < || a_k - a_n ||$, $\forall \, k > M$, then,$$ J(U^*_k) \leq J(\hat{U}_k) \leq J(U^n_k) $$where, $U^*_k$, $\hat{U}_k$ and $U^n_k$ represent the strictly, conditional and nominal optimal control policies, respectively and $a_k$, $\hat{a}_k$ and $a_n$ are the actual, identified and nominal values of the parameter vector, respectively. From Figure 4.5 it may be seen that for all percentage parameter errors given by $|\Delta a_{11}| > 1.5\%$ the conditional optimal control policy does yield lower values of J than the nominal optimal control policy. Furthermore, the control costs are always greater than the cost.
associated with the NOCP evaluated with a nominal parameter value equal to the true value of parameter, i.e. $a_{11n} = a_{11}$. This local minimum of $J(\delta u)$ represents an approximation to the cost associated with the strictly optimal control policy. Further evidence of the ordering of control policies may be ascertained from the plots of control cost as a function of the number of iterations performed (Figure 5.2). It may be seen that the control cost for the COCP becomes constant after a certain number of iterations of the algorithm have elapsed, while the cost for the NOCP continues to increase due to the presence of non-zero offset error in the regulated state variable. It may, therefore, be concluded that, if an integer $M_1$ exists such that, 

$$||a_k - \hat{a}_k|| < ||a_k - a_n||, \forall_k > M_1,$$

then an integer $M_2 > M_1$ exists such that, 

$$J(U_k^*) < J(\hat{U}_k) < J(U_k^n), \forall_k > M_2.$$

An inherent characteristic of adaptive control systems is the interaction between the control and identification processes. Some quantitative information regarding this interaction is provided by the experimental results. An important observation which can be made, based upon the overall adaptive system performance, is that the requirements for 'good' control tend to be in conflict with the requirements for 'good' parameter identification. From a control point of view, it is desirable that the controlled states do not undergo large excursions from their desired values, whilst from the point of view of parameter identification it is desirable that the uncertain system parameters be identified accurately. The experimental results obtained from the study of the identification process (summarized in Figure 4.13) indicate that more accurate parameter identification may be achieved, providing
Fig. 5.2 Control cost as a function of number of iterations
the convergence of the parameter is concomitantly, less rapid. However, a longer period of identification will result in increased control cost, since the identification transient in the controlled state variable will last for a greater length of time. Furthermore, the identification process is dependent upon the control inputs. When large or numerous control excursions occur, as in the case of an oscillatory adaptive control response, the convergence of the identified parameter tends to be more rapid than when the control response is sluggish (see the response in Figure 4.4).

These observations highlight the dual purpose of control inputs in adaptive systems; i.e. not only do the control inputs serve the purpose of attaining control objectives, but they also assist in identifying the uncertain system parameters (c.f. Fel'dbaum's 'dual control theory' [30]). The results obtained from the digital simulation study also provide evidence of the dual nature of the control inputs. It can be seen, from Figures 4.14 and 4.15, that 'good' identification, as measured by \( J(\hat{\Theta}) \), is obtained at the expense of 'poor' control, as measured by \( J(\delta u) \); i.e. as \( J(\hat{\Theta}) \) decreases, \( J(\delta u) \) increases, and vice-versa.

In view of this behaviour in adaptive control systems, it was decided to define an index of 'overall adaptive performance' as,

\[ z = J(\delta u_N) J(\hat{\Theta}_N) \]  
(5.1)

where, \( J(\delta u_N) \) is the control performance index evaluated over \( N \) stages and \( J(\hat{\Theta}_N) \) is the identification cost function, defined by \( J(\hat{\Theta}_N) = \sum_{j=1}^{N} ||a_j - \hat{a}_j||^2 \). Based upon this index of overall adaptive performance, a 'principle of indeterminacy' is proposed for adaptive systems,
i.e.

$$
\mathcal{I} = \mathcal{J}(\delta \mathcal{U}) \mathcal{J}(\mathcal{U}) \geq \varepsilon
$$

(5.2)

where \( \varepsilon \) is some finite, scalar real number. The analogy between the relationship of (5.2) and the uncertainty principle in quantum physics [67] should be noted. The application of this indeterminacy principle is illustrated using the results of the digital simulation study (Figure 4.15).

In Figure 5.3, the index \( \mathcal{I} \) is plotted for the cases of sampling interval sensitive control and sampling interval desensitized control. It is evident that for both cases \( \mathcal{I} \geq 0.6 \times 10^{-2} \) and, specifically for the desensitized control algorithm, \( \mathcal{I} \geq 0.6 \times 10^{-2} \). When the control algorithm was not made insensitive to the sampling interval it was found that the response became more oscillatory as the sampling interval decreased. These oscillations resulted in greater control cost and tended to improve the identification (i.e. \( \mathcal{J}(\mathcal{U}) \) decreased). The fact that the product index \( \mathcal{I} \) in Figure 5.3 increases indicates that, for the level of improved identification, excessive control costs are being incurred, over and above the increase in control cost required to attain the specific level of improvement in parameter identification. In other words, the response could be made less oscillatory, and the control cost thereby decreased, without causing a change in the identification response. The control cost could be decreased up to the point at which \( \mathcal{I} \leq 0.6 \times 10^{-2} \) — any further decrease in control cost would be offset by an increase in parameter identification cost. It is evident, therefore, that the overall performance index \( \mathcal{I} \) also serves as an indication of the efficiency of operation of an adaptive control system and can prove useful in future comparative studies of possible adaptive
FIG. 5.3 OVERALL ADAPTIVE CONTROL PERFORMANCE INDICATOR AS A FUNCTION OF $\Delta$

- $\chi = \bar{J}(\delta u), \bar{J}(\tilde{u})$

- $\Delta$ (sec.)

- Sampling interval sensitive

- Desensitized
control algorithms. An efficiency figure may be defined as,

$$\eta = \frac{z_{\min}}{z} \times 100 \%$$

(5.3)

where, $z_{\min} = \min \{J(\delta u_N)J(\delta N)\}$. In most cases the value of $z_{\min}$ would have to be determined experimentally.

An interesting conclusion which may be drawn is that adaptive control systems need not be perfect parameter identifiers. This conclusion is arrived at as a result of the conflicting requirements for low control cost and low parameter identification errors, and is embodied in the indeterminancy relationship. Perfect identification necessarily implies, (i) the application of forcing inputs to the system in order to separate the effects of the states and parameters in the dynamic response and, (ii) an identification gain which approaches zero with increasing time, in order to prevent variations of the states from affecting the identified parameter value. However, the first requirement for a perfect identifier results in increased control costs due to the additional perturbation of the system, while the second requirement leads to decreasing ability of the identification scheme to adapt to any future changes which may occur in the parameters.

In light of these results, one avenue for future research might be the application of decision theory to the parameter adaptive problem, with a view toward generating a multi-modal adaptive scheme. The purpose of the mode controller would be to make decisions as to whether rapid parameter identification, or accurate parameter identification is required, and then to adjust the identification gains accordingly. Major problems encountered would be, (i) the development of a decision
strategy capable of discriminating between changes in parameters and changes in other system variables and, (ii) the selection of suitable gain levels for the rapid and slow identification phases.
CHAPTER 6

SUMMARY

The major results, conclusions and contributions of the research described in this dissertation may be summarized as follows:

(i) A suboptimal adaptive control algorithm for a class of nonlinear systems has been derived. The control algorithm has particular applicability to continuously operating systems (i.e. where control over an 'infinite interval' is required). Important features of the algorithm are, (a) a precomputed and stored nominal system trajectory is not required for generating the control vector sequence, (b) prediction of future values of states, parameters or reference inputs is not required for generation of the present control vector, (c) the nominal trajectory is based solely upon the past operating history of the system and, (d) the feedback control gains are generated in a forward recursive manner, with only one iteration required for the incorporation of new information regarding the uncertain system parameters.

(ii) The features outlined in (a) - (d) above make the algorithm particularly suitable for real-time control applications.

(iii) Conditions for uniqueness of the control vectors, generated by the forward recursive solution to the control optimization problem, were derived, and proof was given that these conditions are upheld by the control feedback gains of the algorithm.

(iv) The algorithm is capable of performing in both the regulator and trajectory control modes without any additional computational complexity. In addition, the formulation of the control problem is such that control of inaccessible system states can be directly accommodated by the algorithm.
(v) A software package was developed for implementation of the adaptive control algorithm on a PDP 8/I minicomputer and real-time control of a model of a plasma torch furnace was studied. The ability of the adaptive control strategy to effectively handle large initial errors in parameter values was demonstrated and the strategy was shown to possess the desirable property of low performance index sensitivity to initial parameter errors. Comparison of control performance with a non-adaptive strategy, the NOCP, was made and the improvement in performance, obtained by using the adaptive strategy, was demonstrated in a conclusive manner.

(vi) The dual purpose served by the control inputs, vis-a-vis the conflicting requirements of control and parameter identification, was investigated during the experimental study. As a result of this investigation an 'indeterminacy relationship' for parameter adaptive control systems was proposed. This relationship expresses the limitation on overall adaptive performance arising from the interaction of the control and identification processes.

(vii) The adaptive control algorithm meets the stringent application requirements imposed at the outset; viz., (a) less than 4K computer core memory is required for storage and, (b) real-time control implementation is possible for non-trivial systems described by nonlinear dynamic equations.

(viii) A mathematical model representing the electrothermal behaviour of an a.c. plasma torch has been derived. The model is not specific to one particular torch design but has general applicability. The model utilizes the physical and structural parameters pertaining to a plasma torch and it may, therefore, be applied in design optimization studies
as well as in control studies. The results obtained from a simulation
study of a particular a.c. plasma torch were shown to be in good agree-
ment with the experimental results obtained by other researchers.

In conclusion the major contributions of this research are:
(a) The development of an approach to the optimal adaptive control
problem which overcomes the excessive computational burdens and storage
requirements normally associated with approaches based upon optimal
control theory. This approach is both novel and significant in that
the form of the solution is a direct analog of the 'optimal filtering'
problem, as opposed to being the dual of the 'optimal filtering'
problem.
(b) The demonstration of practical, real-time adaptive control
capability and, the empirical derivation of certain inherent limitations
of the optimal-adaptive approach to controlling systems with uncertain
parameters.
(c) The derivation of a general mathematical model for an a.c. plasma
torch, which is the first complete, deductive mathematical model for
such a device available in the literature.
APPENDIX I

MATHEMATICAL MODELLING OF AN A.C. PLASMA TORCH HEATER [78, 79]

A plasma torch is, essentially, an arc-based device in which an electric arc discharge is utilized as a high temperature, high enthalpy thermal source. Such devices may be classified into two basic types, dependent upon the mode of operation; viz., transferred devices, in which the workpiece being processed serves as one electrode for the arc discharge, and non-transferred devices, in which the electrodes are isolated from the workpiece. Additional classification can be made in terms of the type of arc power supply; i.e. a.c. or d.c. supply. Some typical present and future applications of plasma torches are: (i) spray coating and forming of metallic and non-metallic materials, (ii) cutting and welding [68], (iii) surface cleaning (e.g. improving rail-wheel adhesion of locomotives), (iv) chemical synthesis [69,70], (v) Melting furnaces [71], (vi) metallurgical furnaces [72] and, (vii) re-entry simulation for aerospace studies [73].

The particular plasma torch studied is an a.c. non-transferred torch developed by Harzy [74-76] at the University of Aston in Birmingham, England. The torch consists of concentric cylindrical, water-cooled, copper electrodes between which an a.c. arc is struck. A gas (air in this case) is forced between the annular electrode gap and is heated by the arc discharge. An external, transverse magnetic field is applied to the electrode gap causing the arc to rotate, thereby ensuring a more uniform heat transfer to the gas. A separate high-frequency, high-voltage supply ensures reignition of the arc following each current zero. A simplified diagram, showing the basic construction of the torch, is given in Figure A.1. The high temperature
gas exiting from the torch nozzle may be utilized in a large number of
electroheat process applications. Concurrent with the application of
plasma torches in electroheat processes will be the necessity to control
their operation. In order that rational control strategies may be
evolved, valid mathematical models describing the behaviour of such
devices are required. For this reason, the modelling study, described
herein, was undertaken.

AI-1. A brief review of pertinent studies on arc processes

In the plasma torch under consideration the arc discharge occurs
within both a transverse magnetic field and a transverse aerodynamic
field. A number of experimental investigations of arc processes,
occuring under similar conditions, have been conducted and documented
in the literature. In this section the results of investigations
pertinent to this study will be summarized. No attempt will be made
to provide a complete review of all facets of arc processes, since
such information is available in the literature; e.g. [77].

(a) Arcs in Transverse Magnetic Fields

The electric arc discharge constitutes a highly ionized plasma
characterized by a low voltage drop across its length and a high
current-carrying capability. The arc discharge may be divided into
three regions, the cathode root region, anode root region and arc
column. When an external magnetic field is applied to an arc discharge
the average force/unit volume on the arc may be expressed as,

$$
\mathbf{F} = \mathbf{J} \times \mathbf{B} - \nabla p
$$

(AI-1)

where \( \mathbf{J} \) is the current density, \( \mathbf{B} \) the magnetic flux density and \( \nabla p \) is
the pressure gradient. The force on the arc causes the arc to move and,
if the arc is struck between concentric cylindrical electrodes with a
transversely applied magnetic field, the movement is one of rotation in the annular gap (see Figure A1-2). Since the current density is greatest in the cathode root region, the arc movement is normally 'cathode dominated'.

Assuming the magnetic field to be uniform in the annular region of the arc gap, the velocity of arc propagation will, in the average, be uniform and in a direction everywhere perpendicular to the arc column. Under such conditions it has been shown that the arc column will assume an involute shape of fixed length given by [80],

\[
e = \frac{r_1}{2} \left[ \left( \frac{r_o}{r_1} \right)^2 - 1 \right]
\]  

(AI-2)

where \( r_1 \) is the radius of the inner electrode and \( r_o \) the radius of the outer electrode.

---

**FIG. A2**  
ROTATION OF THE ARC IN THE ELECTRODE GAP
No complete theoretical model exists to describe the effects of an applied magnetic field on the variables characterizing the arc, e.g. the voltage gradient in the arc column, the velocity of arc movement etc. However, a number of researchers [81-85] have applied dimensional analysis techniques coupled with correlation of experimental data to obtain characteristic functional relationships for arcs in magnetic fields. It has been found that for short to medium length arcs, typical of those found in plasma torches, the following functional relationships provide the best fit to available data,

\[
\left( \frac{E}{I} \right) = 10^{3} \left( \frac{U}{I} \right)^{2/3} \tag{AI-3}
\]

\[
\left( \frac{U}{I} \right) = 4.6 \left( \frac{B}{I} \right)^{0.6} \tag{AI-4}
\]

where, \( E \) = voltage gradient in the arc column (V/m)
\( U \) = arc velocity (m/sec)
\( d \) = electrode separation (m)
\( B \) = flux density in the electrode gap (Tesla)

The arc velocity is normally taken as the velocity of the cathode spot for the reason alluded to above.

(b) Arcs in Transverse Aerodynamic Fields

The gas flow transverse to the arc, results in constriction of the arc column due to forced convection. The net effect of this constriction is to increase the voltage gradient in the arc column. In addition, aerodynamic drag forces on the arc column result in a lengthening of the arc which also causes an increase in the voltage drop across the arc column. As before, no satisfactory theoretical models describing the effects of aerodynamic fields upon arc behaviour exist. Experimental
approaches have concentrated upon studying the effects of transverse gas flows upon an arc kept stationary by an applied 'balancing' magnetic field [86-89]. Theoretical approaches have concentrated upon obtaining expressions for the drag force on the arc column from fluid dynamic considerations [90]. Experimental data appears to indicate that the arc may be represented aerodynamically as a solid sphere, characterized by an effective drag width defined as [89],

$$C_D^D = \frac{B_1 I}{1/2 \rho_w U_w^2}$$  \hspace{1cm} (AI-5)

where

- $C_D$ = drag coefficient of arc
- $D$ = effective diameter of arc (cm)
- $\rho_w$ = free stream gas density (gm/c.c)
- $U_w$ = free stream gas velocity (m/s)
- $B_1$ = magnetic field flux density required to balance arc (Tesla)
- $I$ = arc current (amps)

When the arc is rotated by a transverse magnetic field the emerging gas column has an induced rotational velocity component due to the applied electromagnetic torque. An expression for the induced rotation frequency has been obtained as [91, 92],

$$\omega_g = \frac{IB}{2 \rho_w U_w \pi^2 (r_1^2 + r_o^2)} \hspace{1cm} \text{revs/sec.}$$ \hspace{1cm} (AI-6)

where, $I$, $B$, $\rho_w$, $U_w$, $r_1$ and $r_o$ are as previously defined.

**AI-2. Modelling the Plasma Torch**

(a) **Average arc current and power relationships**

Initially, attention will be focussed on obtaining general expressions for the arc current and the average power dissipated over one-half cycle of the arc current. Consider the typical arc voltage-current
characteristic shown in Figure A3. There is a minimum supply voltage, \( V_{ig} \), which must be attained before arc ignition (i.e. a sustained discharge) will occur. This ignition voltage is dependent upon both the arc characteristic and the associated current limiting resistor \( R \) and stabilizing inductance \( L \) in the supply circuit. The conditions for arc ignition may be stated in terms of an impedance criterion and a voltage amplitude criterion, i.e.

\[
\frac{d V_a}{d I_a} \leq -z = - \left( R^2 + \omega^2 L^2 \right)^{1/2} \quad \text{(AI-7a)}
\]

\[
v(t) \geq V_{ig} = V_A + I_A Z \quad \text{(AI-7b)}
\]

where \( v(t) \) is the supply voltage and \( V_A \) and \( I_A \) are the points on the arc characteristic at which \( \frac{d V_a}{d I_a} = -z \).

**FIG. A3** CONDITIONS FOR ARC IGNITION.
For high current arcs typically encountered in plasma torches, the arc voltage following ignition drops to an approximately constant average value (the arc burning voltage), which varies inappreciably over a wide range of current. Hence, the arc burning voltage will be approximated by \( V_A \). Over one-half cycle the arc current is defined by the equations,

\[
i_a(t) = 0 \quad 0 < \omega t \leq \Theta \quad (AI-8a)
\]

\[
L \frac{di_a}{dt} + R_i a = V_m \sin(\omega t - \Theta) - V_A \quad \Theta + \gamma > \omega t \geq \Theta \quad (AI-8b)
\]

where \( \Theta \) = ignition angle with reference to supply voltage zero crossing \( \gamma = \arcsin\left(\frac{V_{ig}}{V_m}\right) \)

\( \gamma \) = arc conduction angle

Making the simplifying assumption that the arc burning voltage remains constant and equal to \( V_A \) over the interval \( \gamma + \Theta \geq \omega t \geq \Theta \), the solution for the arc current \( i_a(t) \) is obtained from (AI-8b) as,

\[
i_a(t) = I_m \left[ \sin(\omega t - \phi) - \sin(\Theta - \phi) \exp\left\{ -\frac{R(\omega t - \Theta)}{\omega L} \right\} \bigg] - a\phi(1 - \exp\left\{ -\frac{R(\omega t - \Theta)}{\omega L} \right\}) \quad (AI-9)
\]

where, \( I_m = V_m / Z \)

\( a = \frac{V_A}{V_m} \)

\( \phi = Z/R \)

\( \phi = \arctan\left(\frac{\omega L}{R}\right) \)

The point of arc extinction is defined by \( i_a(t) = 0 \) and \( \omega t = \gamma + \Theta \). From (AI-9) the arc conduction angle is defined by the functional relationship,

\[
\sin(\gamma + \Theta - \phi) - \sin(\Theta - \phi) \exp\left\{ -\frac{R\gamma}{\omega L} \right\} - a\phi(1 - \exp\left\{ -\frac{R\gamma}{\omega L} \right\}) = 0 \quad (AI-10)
\]
Typical waveforms of arc current, for three distinct conduction angle regions, viz., $\gamma < \pi$, $\gamma = \pi$, and $\gamma > \pi$, are shown in Figure A4. The relevance of the condition $\gamma > \pi$ to macroscopic arc instability has been reported in the literature by Luxat and Lees [93].

**FIG. A4 TYPICAL ARC CURRENT WAVEFORMS**

The average arc power over one complete cycle of arc conduction is given by

$$
\overline{P}_a = \frac{1}{2\pi} \left[ \int_{\theta_1}^{\gamma_1 + \theta_1} V_a i_a (\omega t) \, d\omega t + \int_{\theta_2}^{\gamma_2 + \theta_2} V_a i_a (\omega t) \, d\omega t \right] 
$$

(AI-11)

where
- $\gamma_1$ = conduction angle for 1st half cycle
- $\gamma_2$ = conduction angle for 2nd half cycle
- $\theta_1$ = ignition angle for 1st half cycle
  - $= \arcsin \left( \frac{V_i}{V_m} \right)$
- $\theta_2$ = ignition angle for 2nd half cycle
  - $= \arcsin \left( \frac{V_{i_2}}{V_m} \right)$
with,
\[ V_{\text{dig}} = \begin{cases} V_m \sin(\gamma_1 + \theta_1 - \pi) & \gamma_1 > \pi \\ V_m \sin(\theta_1) & \gamma_1 \leq \pi \end{cases} \]

When \( \gamma_1 \leq \pi \) the arc current is symmetrical for both half-cycles and \( \gamma_1 = \gamma_2 \). This will be the normal case since the design of the arc circuit should be such that \( \gamma_1 = \gamma_2 = \pi \), i.e. for maximum utilization. With \( \gamma_1 = \gamma_2 = \gamma \), the average arc power is equal to the half cycle average value, i.e.

\[ \overline{P_a} = \frac{1}{\pi} \int_0^{\pi} V_A \cdot i_a(\omega t) \, dt \quad \text{(AI-12)} \]

Using equation (AI-9) the average arc power may be obtained as,

\[ \overline{P_a} = \frac{V_m}{\pi} \left[ \cos(\theta - \phi) - \cos(\gamma + \theta - \phi) + \frac{\omega L}{R} [a\beta - \sin(\theta - \phi)] \\ \left[1 - \exp(-\gamma R/\omega L)\right] - a\beta \gamma \right] \quad \text{(AI-13)} \]

The power dissipated in the arc discharge may be divided into two components, (i) the power dissipated in the arc column, \( P_c \), and, (ii) the power dissipated in the anode and cathode root regions, \( P_r \), with,

\[ \overline{P_a} = \overline{P_c} + \overline{P_r} \quad \text{(AI-14)} \]

Furthermore, since the inner and outer electrodes alternately act as anode and cathode, the average power, dissipated over one cycle, at either electrode due to the arc root regions is,

\[ \overline{P_r} = \frac{1}{2} \overline{P_c} \]

(b) The effects of transverse magnetic and aerodynamic fields

The arc current and the average power dissipated in the arc arc, to a large extent, governed by the arc burning voltage \( V_A \) (refer to equations (AI-9) and (AI-13)). Apart from appearing directly in these
equations, the arc burning voltage also influences the arc ignition angle $\theta$ and conduction angle $\gamma$. The burning voltage may be expressed as,

\[ V_A = E\theta + V_r \]  \hspace{1cm} (AI-15)

where $E$ is the electric field in the arc column, $\theta$ is the arc column length and $V_r$ is the sum of the voltage drops occurring across the anode and cathode root regions. For specific electrode materials and gas medium, the voltage $V_r$ is an approximately constant value determined primarily by the physical processes occurring in the root regions.

The arc length, for an arc rotating in a cylindrical annular region, may be expressed as a function of the electrode dimensions (equation AI-2). Consequently, the arc voltage is governed primarily by the interrelationship between the column electric field and the magnetic and aerodynamic fields.

Considering the forces produced upon the arc column by the magnetic and aerodynamic fields, as shown in Figure A5, $F_B$ is the force on the arc column due to the applied magnetic field and $F_D$ the force due to aerodynamic drag on the column. The drag force on the column is given by,

\[ F_D = F_D' \rho \]  \hspace{1cm} \text{where } F_D' \text{ is the drag force per unit length.}

From (AI-4) the drag force per unit length is,

\[ F_D' = C_D \frac{1}{2} \rho_w U_m^2 \] \hspace{1cm} (AI-16)

The drag force may be modelled as a force due to an 'equivalent magnetic field', $B_{eff}$ such that,

\[ F_D' = I_a B_{eff} \] \hspace{1cm} (AI-17)
From (AI-16) and (AI-17), the equivalent magnetic field $B_{\text{eff}}$ is obtained as,

$$B_{\text{eff}} = \frac{1}{2} \frac{C_D \rho_m U_w^2}{\bar{I}_a}$$  \hspace{1cm} (AI-18)

where $\bar{I}_a$ is the average arc current. $\bar{I}_a$ may be obtained from (AI-13) as,

$$\bar{I}_a = \frac{\bar{I}_A}{V_A}$$

$$= \frac{I_m}{\pi} \left\{ \cos(\theta - \phi) - \cos(\gamma + \theta - \phi) + \frac{\omega L}{R} [\alpha \beta - \sin(\theta - \phi)] \right\} \left[ 1 - \exp(-R/L) \right] - \alpha \beta \gamma$$  \hspace{1cm} (AI-19)

The direction of $B_{\text{eff}}$ is perpendicular to $F_D$, consistent with the vector relationship, $F_D^2 = \bar{I}_a \times B_{\text{eff}}$. Consequently, for modelling
purposes the arc column may be considered as interacting with two crossed magnetic fields, B and \( B_{\text{eff}} \) as depicted in Figure A.5. The resultant 'magnetic field' is \( B_x = \left[ B^2 + B_{\text{eff}}^2 \right]^{\frac{1}{2}} \), or more generally, when the flow is at an angle \( \psi \) to the magnetic field \( B \),

\[
B_x = \left[ (B \pm B_{\text{eff}} \sin \psi)^2 + (B_{\text{eff}} \cos \psi)^2 \right]^{\frac{1}{2}} \tag{AI-20}
\]

Furthermore, in terms of the mass flow rate of the gas, \( f(\text{kgm/sec.}) \), the equivalent magnetic field may be expressed as,

\[
B_{\text{eff}} = \frac{\frac{1}{2} C_D \rho_\infty}{\bar{I}_a} \left( \frac{f^2}{\rho_\infty \pi^2 (x_0^2 - x_1^2)^2} \right)
= \frac{C_D}{2 \bar{I}_a \rho_\infty \pi^2} \frac{f^2}{(x_0^2 - x_1^2)^2} \tag{AI-21}
\]

Utilizing the 'resultant' magnetic field of equation (AI-20) and the relationships given in (AI-3) and (AI-4), the column electric field may be expressed as,

\[
E = \frac{(2.76 \times 10^3) B_{\text{eff}}^{0.4}}{d^{0.2667} \bar{I}_a^{0.066}} \tag{AI-22}
\]

with \( d = (x_0 - x_1) \). Thus, from equations (AI-20), (AI-21) and (AI-22) a functional relationship may be obtained, relating \( E \) and the variables \( (f, B, \bar{I}_a) \) and physical parameters \( (x_0, x_1, \rho_\infty, C_D) \).

(c) Gas heating at the arc column

Heat transfer occurs from the arc column to the gas flowing in the immediate vicinity of the column. The gas flow past the arc column consists of two flow components, viz. (i) the direct flow due to the gas forced through the annular gap (with velocity \( U_\theta \)) and, (ii) the relative
flow due to the rotation of the arc in the annular gap. The relative
gas flow past the arc column has the velocity \( U' = (U - U_g) \), where \( U \) is
defined by (AI-4) and \( U_g \) is the induced rotation of the gas, which
may be obtained from (AI-5) as,

\[
U_g = \frac{1}{2} \cdot 2\pi (r_1 + r_0) \nu \gamma \\
= \frac{\frac{I}{a} B \cdot \pi (r_1 + r_0)}{2\rho_0 \nu \pi^2 (r_1^2 + r_0^2)}
\]

(AI-23)

The enthalpy transfer to the gas is proportional to the mass flow rate
of the gas passing the arc. Hence, if the direct mass flow rate is \( f \)
and the relative mass flow rate due to rotation is \( f_r \), where

\[
f = U_\infty \rho_\infty (r_0^2 - r_1^2) \tag{AI-24a}
\]

and

\[
f_g = U' \rho_\infty (r_0 - r_1)^2 \tag{AI-24a}
\]
a proportion \( \chi \) of the heat release will be distributed circumferentially
in the wake of the rotating arc. The proportion factor will be
defined as,

\[
\chi = \frac{f_g}{(f^2 + f_g^2)^{\frac{1}{2}}}
\]

(AI-25)

The component of the heat release due to arc rotation, contributes to
a more even thermal distribution within the heated gas and, therefore,
\( \chi \) will be termed the 'uniform heating factor'.

Uniform heating of the gas may be interpreted in terms of \( \chi \) as
follows: when \( f \ll f_g \), \( \chi \rightarrow 1 \) (the condition for completely uniform
heating of the gas). In this case the arc heat release is distributed
uniformly to the gas flowing through the annular region, i.e. a
cylindrical shell of uniformly heated gas emerges from the torch nozzle. Conversely, when \( f \gg f_g \), \( x \ll 1 \), and the gas emerging from the nozzle will consist of helical bands of locally heated gas. These two situations are depicted in Fig. A6.

\[ f \ll f_g \quad x \rightarrow 1 \]
\[ f \gg f_g \quad x \ll 1 \]

Heated gas

**FIG. A6** UNIFORM \\& NON-UNIFORM GAS HEATING

(d) Dynamic heat balance within the electrode assembly.

Heat losses occur at the electrodes due to, (i) the heat dissipated in the arc root regions, (ii) conduction and convection at the inner surface of the outer electrode and, (iii) conduction at the face of the inner electrode. The heat transfer components (ii) and (iii) above, originate from the containment of the heated gas within the electrode assembly. Considering these heat loss components, as shown in Figure A7, they may be represented as,

**Conduction loss (inner electrode)**

\[ q_{cl} = h_{cl} (\theta_s - \theta_w) \]

kg-cal/sec.
**Conduction loss (outer electrode)**

\[ q_{c2} = h_{c2} (\theta_s - \theta_2) \text{ kg.-cal/sec.} \]

**Convection loss (outer electrode)**

\[ q_{cv} = h_{cv} L(\theta_i - \theta_w) \text{ kg.-cal/sec.} \]

**Heat loss from arc roots (average per electrode)**

\[ q_e = k \frac{1}{2} \frac{r^2}{r} \text{ kg.-cal/sec.} \]

where, \( h_{cj} = \text{conductive heat transfer coefficient (kg.-cal/sec.} \cdot \text{C)} \)

\[ = K_j \frac{A_j}{\delta_j} \]

\( A_j = \text{electrode area over which conduction losses occur.} \)

\( \delta_j = \text{thickness of electrode wall} \)
\( \theta_{ej} \) = electrode coolant temperature

\( K_j \) = thermal conductivity of electrode material

\( j \) = index subscript (1 for inner electrode and 2 for outer electrode).

\( \theta_s \) = effective surface temperature at electrode walls

\( h_{cv} \) = convective heat transfer coefficient of outer electrode

\( L \) = distance inner electrode is inset with respect to outer electrode (Figure A7)

\( \theta_i \) = bulk temperature of gas emerging from the electrode assembly.

\( k \) = 0.238 kg·cal/kW·sec.

The effective surface temperature at the electrode walls will be dependent upon the degree to which the gas is uniformly heated. If the gas is uniformly heated it can be expected that the effective surface temperature will approach the bulk temperature of the heated gas. When the gas is non-uniformly heated, however, the effective surface temperature will be some lesser value owing to the non-homogeneous mixing of the heated gas within the electrode assembly. Hence, the surface temperature will be defined as the product of the uniform heating factor and the bulk gas temperature \( \theta_i \), i.e.

\[ \theta_s = x \theta_i \]

The dynamic heat balances to be considered may be expressed as,

(i) Gas heating

\[ C_i \dot{\theta}_i = q_a - q_{c1} - q_{c2} - q_{cv} - f \frac{dA}{dp} \theta_i \]

with \( q_a = k \bar{F}_c = k (V_A - V_x) \bar{I}_a \). Utilizing the expressions for the heat loss terms defined above, the dynamic equation governing the temperature of the exit gas is,
\[ \dot{e}_1 = \frac{1}{C_i} \left\{ k (v_A - v_x) \overline{I}_a - \left( \frac{v h_{cl} + v h_{c2} + h_{cv} L + f c_p}{c_p} \right) \dot{e}_1 + h_{cl} \dot{e}_{wl} + \left( h_{c2} + h_{cv} L \right) \dot{e}_{u2} \right\} \] (AI-26)

where \( C_i \) is the thermal capacitance of the heated gas, and \( c_p \) is the specific heat of the gas.

(ii) Outer electrode heat losses

The dynamic heat balance equation describing the heat loss to the outer electrode is,

\[ C_{u2} \dot{e}_{u2} = \frac{k V_x \overline{I}_a}{2} + q_{c2} + q_{cv} - f_{u} c_{u} \dot{e}_{u2} \]

where \( f_{u} \) is the mass flow rate of the electrode coolant and \( c_{u} \) its specific heat. \( C_{u2} \) is the thermal capacitance of the electrode coolant within the electrode. Substituting for \( q_{c2} \) and \( q_{cv} \), the dynamic equation for the outer electrode coolant temperature are obtained as,

\[ \dot{e}_{u2} = \frac{1}{C_{u2}} \left\{ \frac{k V_x \overline{I}_a}{2} + \left( h_{c2} + h_{cv} L \right) \dot{e}_1 - \left( f_{u} c_{u} + h_{c2} + h_{cv} L \right) \dot{e}_{u2} \right\} \] (AI-27)

(iii) Inner electrode losses

In a similar fashion, the dynamic equation for the coolant temperature exiting from the inner electrode may be obtained as,

\[ \dot{e}_{wl} = \frac{1}{C_{wl}} \left\{ \frac{k V_x \overline{I}_a}{2} + h_{cl} \dot{e}_i - \left( f_{w} c_{w} + h_{cl} \right) \dot{e}_{wl} \right\} \] (AI-28)

An overall block diagram representation of the interactions occurring within the electrode assembly is shown in Figure A8.

AI-3 Simulation Study

A simulation study of the steady state operating characteristics
FIG. A8  INTERACTIONS IN THE PLASMA TORCH ASSEMBLY
of the plasma torch was conducted. The purpose of this study was to verify the model against experimental data obtained from the Aston torch by other researchers [74, 94-97].

Central to the problem of obtaining the steady state operating conditions for the plasma torch model, is the solution for the arc conduction angle and the average arc current. An analytic expression for these quantities cannot be obtained directly, due to the complexity of the functional relationship existing between the conduction angle and average arc current. Furthermore, both quantities are dependent upon the arc burning voltage $V_A$, which is itself a function of the arc current. The approach adopted for evaluating $\gamma$ and $I_a$ was therefore, to compute these values via a numerical procedure.

Initially, the arc ignition conditions are obtained from the requirement

$$ - \frac{3V_A}{9I_a} = \frac{Z}{I_a} $$

with $V_a = ES + V_r$

and

$$ E = \frac{(2.76 \times 10^3) [B^2 + E^2 \text{eff}]^{0.2}}{d^{0.2667} I_a^{0.066}} \quad \text{from (AI-22).} $$

Substituting for $B_{\text{eff}}$ from (AI-22), the electric field may be expressed as,

$$ E = \frac{C_o [B^2 x^2 + C^2 x^4]^{0.2}}{I_a^{0.466}} \quad \text{(AI-29)} $$

where, $C_o = \frac{2.76 \times 10^3}{(r_0 - r_1)^{0.2667}}$

$$ C_1 = \frac{C_d b_d}{2\rho \pi^2 (r_0^2 - r_1^2)^2} $$
Evaluating the impedance criterion for arc ignition, using $E(I_a)$ defined in (AI-29), the ignition point arc current $I_A$ (see Figure A3) is defined by,

$$C_0 \theta \left( 0.066 \frac{B^2}{I_A} 0.534 + \frac{0.466 C_1^2 f^4}{I_A 1.466} \right) - \frac{Z}{Z} \left( B^2 I_A^2 + C_1^2 f^4 \right) = 0$$

(AI-30)

The solution for $I_A$, obtained from (AI-30), enables the approximation to the arc burning voltage $V_A$ to be obtained. Furthermore, the ignition voltage may be obtained from,

$$V_{ig} = V_A + I_A Z$$

and, consequently, the arc ignition angle $\theta$ may be evaluated.

Utilizing the values of $V_A$ and $\theta$ thus obtained, the arc conduction angle and average arc current may be evaluated from (AI-10) and (AI-19). A flowchart of the numerical procedure used to obtain the solutions is shown in Figure A9. The values of parameters used in the simulation study are tabulated in Table A1.

Thermal variables of interest are the bulk temperature of the gas emerging from the torch $T_i$, the electrothermal efficiency of the torch $\eta$ and the electrode losses $q_L$, where,

$$\eta = \frac{k \frac{P_a - q_L}{P_a}} \times 100 \%$$

$$q_L = 2q_e + q_{cl} + q_{c2} + q_{cv}$$

These variables are shown as functions of gas flow rate in Figures A10 and A11 together with pertinent experimental data obtained from a similar torch by other workers [74, 94-87]. It is evident that there is
Initial current estimate

\[ I_{a_0} \approx \left[ \frac{0.066 \cdot c_0 \cdot e \cdot B^{0.44}}{Z} \right]^{1/4.066} \]

\[ \Delta Z_n = Z - \frac{c_0 \cdot e}{(B^2 \cdot I_{a_n}^2 + c_i^2 \cdot f^4)^{0.68}} \left[ 0.066 \cdot B^4 \cdot I_{a_{n-1}}^{0.536} + \frac{0.466 \cdot c_i^2 \cdot f^4}{I_{a_{n-1}}} \right] \]

YES

\[ \Delta Z_n < \epsilon ? \]

NO

\[ |\Delta I_{a_n}| = \left\{ \frac{|\Delta Z_n|}{0.066 \cdot c_0 \cdot e \cdot B^{0.44}} \right\}^{1/4.066} \cdot I_{a_{n-1}} \]

\[ I_{a_{n+1}} = I_{a_{n}} - K \cdot \text{sgn}(\Delta Z_n) \cdot |I_{a_n}| \cdot \epsilon ; K < 1 \]

\[ V_A = \frac{c_0 \cdot e \cdot [B^4 \cdot I_{a}^2 + c_i^2 \cdot f^4]^{0.2}}{I_{a_{0.466}}} + V_r \]

\[ I_A = I_2 ; \quad V_{ig} = V_A + I_A \cdot Z \]

\[ \theta = \arcsin \left( V_{ig} / V_m \right) \]

Obtain initial estimate of \( \delta \) from eqn. (AI-10) using half interval method

Improve solution for \( \delta \) using Newton Raphson method

Compute average arc current:

\[ I_{\overline{a}} = \frac{1}{n} \left\{ \cos(\theta - \phi) - \cos(\delta + \theta - \phi) + \cos L[Q_{p} - \sin(\theta - \phi)] \right\} \]

\[ \times \left[ 1 - \exp \left( -R \delta / \omega_0 L \right) \right] - \alpha \delta \]

Fig. A9 Flowchart of computations for arc conditions
Inner electrode radius, \( r_1 = 0.0125 \text{ m} \)

Outer electrode radius, \( r_o = 0.0205 \text{ m} \)

Effective drag width of arc, \( (C_D) = 0.01 \text{ m} \) \[85, 88\]

Electrode thickness \( \delta = 0.0025 \text{ m} \)

Thermal conductivity of electrode material (copper) \( k = 0.0918 \text{ kg.cal/m.sec}^{\circ}\text{C} \)

Inset distance of inner electrode \( L = 0.0051 \text{ m} \)

Specific heat of gas (air) at 1000°\text{C}, \( C_p = 0.25 \text{ kg.cal/kg}^{\circ}\text{C} \)

Specific heat of coolant (water), \( C_\omega = 1.002 \text{ kg.cal/kg}^{\circ}\text{C} \)

Coolant mass flow rate, \( f_\omega = 0.5 \text{ kg/sec} \)

Sum of electrode root voltages, \( V_x = 25 \text{ V} \)

Arc circuit resistance, \( R = 0.2 \text{ ohms} \)

Arc circuit inductance, \( L = 2.8 \text{ mH} \)

Supply voltage \( (50 \text{ Hz}) = 415 \text{ V} \)

Freestream gas density, \( \rho_\infty = 0.035 \times 16.02 \text{ kg/m}^3 \)

---

**TABLE A1. PARAMETERS RELATING TO PLASMA TORCH MODEL**
FIG. A10  EXIT GAS TEMPERATURE AS A FUNCTION OF GAS FLOW
FIG. A11  ELECTROTHERMAL EFFICIENCY & ELECTRODE LOSSES vs. GAS FLOW
FIG. A12  ARC BURNING VOLTAGE AS A FUNCTION OF
GAS FLOW
a.  B = 0.4  Tesla
b.  B = 0.2  "
    V  Harry (74), B = 0.4
good correspondence between the electrothermal behaviour of the plasma torch model and the empirical results obtained from the Aston plasma torch. This would indicate validity of the proposed model. The variations between the model and experimental data arise from a number of possible factors. Firstly, the results of Kadhim [95] were obtained with the torch operating in a different mode to that studied by Harry [74]. Harry operated the torch such that the arc electrical characteristics were relatively independent of the gas flow, whilst in Kadhim's experiments the arc characteristics showed significant variations with gas flow. The electrical characteristics of the model torch are relatively independent of gas flow, as shown in Figure A12, hence it can be expected that the performance of the model will correspond more closely to the results of Harry.

The variation of electrothermal performance of the torch with respect to the magnetic flux density is of interest, since the magnetic flux density represents a possible control variable. In Figures A13, A14 and A15 the variables $\theta_1$, $\eta$ and $q_L$ are plotted as functions of the magnetic flux density. The following points are to be noted, (i) for a given gas flow greater than 0.01 kgm/sec the electrothermal efficiency tends to an approximately constant value for all flux densities in excess of 0.005 tesla, (Figure A14).

(ii) At low values of gas flow there is an optimum flux density at which the electrode losses attain a minimum with respect to $B$, (Figure A15). Making the assumption that evaporation of material from the electrodes is directly proportional to the heat losses at the electrodes, such behaviour would indicate that there is an 'optimum' value of flux density, for low gas flow rates, at which the material evaporation is
FIG. A13  EXIT GAS TEMPERATURE vs. MAGNETIC FLUX DENSITY
FIG. A14  ELECTROTHERMAL EFFICIENCY vs. MAGNETIC FLUX DENSITY
FIG. A15  ELECTRODE LOSSES vs. MAGNETIC FLUX DENSITY
a minimum. Experimental investigation [97] has shown that this does indeed occur; thus giving further indication the validity of the model.

(iii) At low gas flow rates the gas temperature is sensitive to changes in the magnetic flux density and, as the gas flow rate increases the sensitivity decreases, as does the maximum attainable temperature.

AI-4. Further Generalizations of the Plasma Torch Model

The generalization of the model to account for different gas mediums and to improve the model in the region of low gas flow rates is desirable. Although such generalizations were not considered quantitatively in this research, the ability to account for a large range of physical parameters would greatly increase the utility of the model. Firstly, the factors to be considered in modelling a torch operating with a different gas medium will be detailed (argon and nitrogen are typical gases used in some torches).

(a) Different gas medium

The use of a different gas medium has an effect on the voltage gradient in the arc column. Correlations of empirical data [84] have shown that a more general expression relating the arc velocity to arc current, magnetic field and electrode separation is, (c.f. equation AI-4).

\[
\left( \frac{U_d}{I} \right) = \frac{4.6}{pg^{0.5}} \left( \frac{B_d}{I} \right)^{0.6}
\]

\[(AI-4a)\]

where \(pg\) = \(density \ of \ gas \ at \ given \ pressure \)

\(\frac{density \ of \ air \ at \ 1 \ atmosphere}{\}

Hence, (AI-4a) in conjunction with (AI-3) yields the general expression for arc column voltage gradient as,

\[
E = \frac{2.76 \times 10^3}{pg^{0.333} d^{0.2667} i^{0.066}} \]

\[E^{0.4}\]
The other physical parameters appearing in the model which have to be considered are,

(i) the voltage drop across the anode and cathode root regions, $V_x$.

This voltage varies with both the electrode material used and the ionization potential of the gas medium in which the arc discharge occurs.

(ii) the specific heat $c_p$ and freestream density of the gas $\rho_w$.

(b) Operation at low gas flow rates

It is unlikely that the plasma torch would be operated continuously at low gas flow rates, owing to high electrode losses which result in, (i) low electrothermal efficiencies and (ii) greater likelihood of electrode failure due to excessive evaporation. However, this in itself, is insufficient justification for not considering the peculiarities of the low gas flow regime.

Due to the higher temperatures encountered at the low gas flow rates, the dissociation of the gas molecules becomes a significant factor. The following simplified analysis is proposed to account for gas dissociation. The heat exchanges are considered in two distinct regions; viz. the region within the electrode assembly and the region external to the electrode assembly (Figure A16). Region 1 is the region in which gas dissociation takes place while region 2 is the region in which it is assumed the dissociated gas molecules recombine (due to the high temperature gradients encountered as the emerging gas comes into contact with cooler gas). In region 1 not all the arc column energy is converted to joule heating of the gas since some energy will be used in dissociating the gas molecules. Considering the heat transfer in the region of the arc column and, defining a bulk temperature of the arc column as $\theta_a$, the heat balance is governed by,
FIG. A16  Gas dissociation & recombination regions (low flow rates)

Exit gas temperature $\Theta_2 (^\circ \text{C})$

FIG. A17  Exit bulk gas temperature vs. gas flow rate (low flow rate region)
\[ k(v_e - v_r) \bar{I}_a - kP_D(\theta_a) - \frac{f'}{cp} \theta_a = 0 \]

where \( P_D(\theta_a) \) is the power absorbed by the dissociating gas molecules (a function of \( \theta_a \)) and \( f' \) is the effective mass flow rate of gas passing the arc column. This flow rate is defined as before, by

\[ f' = \left[ f_1^2 + f_2^2 \right]^{1/2} \]

with \( f_g = \rho_w (U - U_g) (r_o - r_1) D \).

\( U \) is the arc rotational velocity, \( U_g \) is the induced rotational velocity of the gas in the annular gap, and \( D \) is the effective arc diameter.

The dissociation power may be expressed as,

\[ P_D = \lambda(\theta_a) v_p \bar{I}_a \]

where \( V_p = \) dissociation potential of the gas,

\[ \lambda(\theta_a) = \text{partial pressure of dissociated gas at a temperature } \theta_a \].

Consequently, by solving for \( \theta_a \), the enthalpy entering region 1 from the arc column is obtained simply as \( c_p \theta_a \). From equation (AI-26) the steady state bulk temperature in region 1 may be obtained from the heat balance equation,

\[ \frac{f}{c_p} \theta_a - \left[ x(h_{c1} + h_{c2}) + h_{cvL} + f c_p \theta_1 + h_{c1} \theta_1 \right] + \left[ h_{c2} + h_{cvL} \right] \theta_{o2} = 0 \]

Similarly, the enthalpy entering region 2 is that carried by the gas leaving region 1, i.e. \( c_p \theta_1 \). In addition there is the heat source created by recombination of the dissociated gas molecules.

Hence, the bulk temperature in region 2 is given by,

\[ \theta_2 = \theta_1 + \frac{k \lambda(\theta_a) v_p \bar{I}_a}{f c_p} \]
The effect of gas dissociation is therefore, to effectively decrease the electrode losses to a certain extent. As a result, the bulk gas temperature obtained in the low gas flow region will be higher than would be the case if dissociation were neglected. The variation of gas temperature with gas flow, for low gas flow rates, would typically take the form shown in Figure A17.

AI-5. An Illustrative Application for Control Studies

The application of the plasma torch in a gas heating furnace is briefly presented in this section. The primary purpose of presenting this possible application is to obtain a dynamic model of a nonlinear multivariable plant, to which the adaptive control algorithm developed in the main body of this dissertation may be applied and tested.

The furnace, shown in Figure A18, consists of a refractory lined, and thermally lagged container with the plasma torch mounted at the top of the container. Heated gas exits from outlets situated at the bottom (or top) of the furnace. The plasma torch variables being controlled are the arc current and the arc gap flux density. Control is achieved via separate a.c. and d.c. generators which supply the arc and field coil power, respectively.

Such a furnace might possibly be used in reaction processes such as production of titanium oxide, or for melting purposes. It may also serve as a mixing chamber in which the heated gas from the plasma torch, which will not be at a completely uniform temperature is mixed so as to attain homogeneous temperature in the body of gas. The heated gas exiting from the furnace would be used in some process requiring gases at elevated temperature, e.g. flash dryers, chemical reactors, cement kilns etc.
DIGITAL COMPUTER

PLASMA TORCH

FIELD COIL

A.C. GEN.

D.C. GEN.

HEATED GAS

LINED & THERMALLY LAGGED FURNACE BODY

FIG. 418 SIMPLIFIED DIAGRAM OF PLASMA TORCH FURNACE
The average arc current, defined by equation (AI-19), may be expressed in the form,

\[ \bar{I}_a = a_1 V_m - a_2 V_A \]  

(AI-30)

where,  

\[ a_1 = \frac{1}{\pi Z} \left[ \cos(\theta - \phi) - \cos(\gamma + \theta - \phi) - \frac{\omega L}{R} \sin(\theta - \phi) \right] \]

\[ a_2 = \frac{1}{\pi R} \left[ \gamma - \frac{\omega L}{R} \left( 1 - \exp \left( -\frac{R_y}{\omega L} \right) \right) \right] \]

It is not possible to obtain an expression for \( V_A \) in terms of \( B \) only. However, the arc voltage may be represented by a linear function of \( B \) over the range \( 0.1 < B < 0.4 \), as shown in Figure AI9. This is the normal operating range of the magnetic flux density in the torch — at lower flux densities the electrode erosion rate increases rapidly and can lead to failure of the electrode. For these reasons the arc voltage will be taken as a linear function of the magnetic flux density, i.e.,

\[ V_A = \xi_0 + \xi_1 B \]  

(AI-31)

The magnetic flux density is a function of the current in the coil, \( I_f \) and can be expressed as,

\[ B = 2 \times 10^{-7} \frac{N I_f}{r_f} \]  

Tesla.

where \( N \) is the number of turns on the coil and \( r_f \) is the effective radius of the coil. In terms of the applied field coil voltage \( V_f \) the flux density is given by,

\[ B = 2 \times 10^{-7} \frac{N V_f}{r_f R_f} \]

where \( R_f \) is the resistance of the coil. The time constants of the arc
FIG. A19  ARC VOLTAGE AS A FUNCTION OF MAGNETIC FLUX DENSITY

a. Simulation results
b. Guile et al. (Ref. 85), $I_a = 800 \pm 400$ A; $d = 1.27$ cm.
c. Results of b renormalized for $d = 0.8$ cm.
circuit and field coil are typically very small (<10^{-2} sec.), hence they can effectively be neglected. The dynamics of the arc and magnetic field circuits will therefore be governed by the response of the generators supplying the power to these circuits. Normalized equations for current and flux density may be obtained by substituting for \( V_A \) in (AI-30) from equation (AI-31), i.e.

\[
\begin{align*}
I_a &= \beta_0 V_m - \beta_1 B - \beta_2 \\
B &= \beta_3 V_f
\end{align*}
\]

\begin{align*}
0 < I_a < 1 \\
0 < B < 1 \\
0 < V_m < 1 \\
0 < V_f < 1
\end{align*}

(AI-32)

where \( \beta_0 = \left( \frac{a_1 V_{mm}}{I_{am}} \right) \)

\( \beta_1 = \left( \frac{a_2 t_1 B_m}{I_{am}} \right) \)

\( \beta_2 = \left( \frac{a_2 t_0}{I_{am}} \right) \)

\( \beta_3 = \left( \frac{2 \times 10^{-7} N V_{fm}}{x_f R_f B_m} \right) \)

and \( I_{am}, B_m, V_{mm} \) and \( V_{fm} \) are the nominal maximum values of arc current, magnetic flux density, arc supply voltage and field supply voltage, respectively.

Furthermore, it is assumed that the generators may be represented by the equivalent, normalized first order dynamic equations (i.e. their response is damped),

\[
\begin{align*}
\dot{V}_m &= \beta_4 (V_{mc} - V_m) \\
\dot{V}_f &= \beta_5 (V_{fc} - V_f)
\end{align*}
\]

(AI-33)
where $V_{mc}$ and $V_{fc}$ are the excitation voltages controlling the output voltages.

Considering the state variables of the furnace to be the gas temperature $\theta_1$, the inner and outer electrode coolant exit temperatures $\theta_{o1}$ and $\theta_{o2}$, the arc current $I_A$ and the flux density $B$, the normalized state equations may be obtained from (AI-26), (AI-27), (AI-28), (AI-32) and (AI-33) in the form,

\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_{o1} \\
\dot{\theta}_{o2} \\
\dot{I}_A \\
\dot{B}
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & 0 \\
0 & a_{22} & a_{23} & a_{24} & 0 \\
0 & 0 & a_{33} & a_{34} & 0 \\
0 & 0 & 0 & a_{44} & a_{45} \\
0 & 0 & 0 & 0 & a_{55}
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_{o1} \\
\theta_{o2} \\
I_A \\
B
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
V_{fc} \\
V_{mc}
\end{bmatrix}
\begin{bmatrix}
b_{41} \\
b_{42} \\
b_{51}
\end{bmatrix}
\begin{bmatrix}
I_A \\
B
\end{bmatrix}
\]

The parameters $a_{ij}$, $b_{ij}$, and $f_i$ are defined in Table A2. Note that in Table A2 $V_g$ is the volume of the furnace, $V_{el}$ is the volume of coolant in the inner electrode and $V_{e2}$ is the volume of coolant in the outer electrode.
\[ a_{11} = - \frac{(x h_{c1} + x h_{c2} + h_{dv} L + f_{cp})}{\rho \omega C_p V g} \]

\[ a_{12} = \frac{h_{c1}}{\rho \omega C_p V g} \left( \frac{\theta_{1lm}}{\eta_{1m}} \right) \]

\[ a_{13} = \frac{h_{c2} + h_{cv} L}{\rho \omega C_p V g} \left( \theta_{2m} \right) \]

\[ a_{14} = \frac{k \left( \frac{f}{\omega} - \frac{V}{V} \right)}{\rho \omega C_p V g} \left( I_{Am} \right) \]

\[ a_{15} = \frac{h_{c1}}{\rho \omega C_p V e_1} \left( \frac{\theta_{1m}}{\eta_{1m}} \right) \]

\[ a_{16} = \frac{h_{c2} + h_{cv} L + f_{w} C_w}{\rho \omega C_p V e_2} \left( \frac{\theta_{1m}}{\eta_{1m}} \right) \]

\[ a_{22} = - \frac{(h_{c1} + f_{w} C_w)}{\rho \omega C_w V e_1} \]

\[ a_{23} = \frac{k V_r}{2 \rho \omega C_w V e_1} \left( I_{Am} \right) \]

\[ a_{24} = \frac{(h_{c2} + h_{cv} L + f_{w} C_w)}{\rho \omega C_w V e_2} \left( \frac{\theta_{1m}}{\eta_{1m}} \right) \]

\[ a_{34} = \frac{k V_r}{2 \rho \omega C_w V e_2} \left( I_{Am} \right) \]

\[ a_{44} = - \beta_4 \]

\[ a_{45} = - \left( \frac{\alpha_2 \xi_1 B_m}{I_{Am}} \right) \left( \beta_4 - \beta_5 \right) \]

\[ a_{55} = - \beta_5 \]

\[ b_{41} = - \left( \frac{\alpha_2 \xi_1 B_m}{I_{Am}} \right) \beta_5 \]

\[ b_{42} = \left( \frac{\alpha_1 V_{mm}}{I_{Am}} \right) \beta_4 \]

\[ b_{51} = \beta_5 \]

\[ f_1(B, I_A) = \left( \frac{k \xi_1}{\rho \omega C_p V g} \right) \left( \frac{B_m I_{Am}}{\theta_{1m}} \right) B B_a \]

\[ \delta_4 = \left( \frac{\alpha_2 \xi_0}{I_{Am}} \right) \beta_4 \]

\[ \theta_{1m}, \theta_{1lm}, \theta_{2m}, I_{Am}, B_m \] are the nominal maximum values of the variables

\[ \theta_1, \theta_{1m}, \theta_{2m}, I_A \] and \( B \), respectively.

**TABLE A2. NORMALIZED STATE EQUATION PARAMETER DEFINITIONS**
The exit gas temperature from the furnace is required to be regulated at a particular set-point value, irrespective of the mass flow rate. The 'desired' gas temperature may vary dependently with another process variable (e.g. the solution concentration entering a flash drier), while the gas flow rate may be governed by another variable in the process (e.g. the solution flow rate entering a flash drier). The control objective for the furnace is to keep the gas temperature at the desired value and to minimize the cost of operating the furnace. Hence, a performance criterion for control may be expressed as,

\[ J = \sum_{k=1}^{K} \left[ q'_{1} \ a_{1} \ (\theta_{d,k} - \theta_{i,k})^2 + r'_{1} \ a_{2} \ (v_{f,k})^2 + r'_{2} \ a_{3} (v_{m,k})^2 \right] \]

where \( q'_{1} \), \( r'_{1} \) and \( r'_{2} \) are weighting factors and \( a_{1}, a_{2} \) and \( a_{3} \) are 'rationalizing' coefficients selected in such a manner that the dimensions of the terms in \( J \) are consistent. The dimension normalizing coefficients are obtained as follows. Utilizing the nominal maximum values of relevant variables and parameters of the plasma torch, the terms appearing in the performance index can be given the dimensions of power by premultiplication with appropriate factors, i.e.

\[
\text{Thermal demand power} = \left( \frac{f_{m} c_{p}}{k} \right) (\theta_{d} - \theta_{i}) \quad [\text{kw}]
\]

\[
\text{Arc circuit losses} = \left( \frac{v_{nm}}{R} \right) v_{m} \quad [\text{kw}]
\]

\[
\text{Field coil losses} = \left( \frac{v_{fm}}{R_{f}} \right) v_{f} \quad [\text{kw}]
\]

Consequently, the performance index may be represented in rationalized form as,
\[ J = \frac{1}{\Lambda^2} \sum_{k=1}^{k_f} \left\{ q'_1 \left( \frac{q_m}{k} \right)^2 \left( \theta_d - \theta_i \right)^2 + r'_1 \left( \frac{v_{fm}}{R_f} \right)^2 v^2_{fk} \right. \]
\[ + \left. r'_2 \left[ \frac{v_{nm}}{R} \right]^2 \right\} [kW^2] \]

with \( \Lambda = \max \left\{ \left( \frac{q_m}{k} \right), \left( \frac{v_{fm}}{R_f} \right), \left( \frac{v_{nm}}{R} \right) \right\} \).

Using the values of \( C_p, k \) and \( R \) given in Table A1, with \( V_{nm} = 0.450 \text{ kv}, \)
\( V_{fm} = 0.125 \text{ kv}, \)
\( f_m = 0.04 \text{ kgm/sec} \)
and \( R_f = 0.43 \Omega \) [74], the performance index is obtained as,
\[ J = \sum_{k=1}^{k_f} \left\{ q'_1 \times 4.0 \times 10^{-4} \left( \theta_{dk} - \theta_{ik} \right)^2 + r'_1 \times 2.0 \times 10^{-2} v^2_{fk} \right. \]
\[ + \left. r'_2 \frac{v^2_{mk}}{10} \right\} \]

Note that the first term of \( J \) represents the squared error between the desired and actual gas temperature, while the remaining two terms are the power losses occurring in the arc circuit and field coil.

Since the gas flow through the plasma torch acts as a variable process parameter of the furnace, it is necessary to consider the variations with flow of the parameters of the dynamic mode. The parameters dependent upon gas flow are those in which \( f, \chi \) or \( h_{cv} \) appear; viz., \( a_{11}, a_{13}, a_{21}, a_{31}, a_{33} \). These parameters are shown as functions of gas flow rate in Figure A20.
FIG. A20  NORMALIZED PLASMA TORCH PARAMETERS vs. GAS FLOW.
APPENDIX II

THE ANALOG COMPUTER SIMULATION

A simulation of the model of the plasma torch furnace, for use in real-time control studies, was undertaken using an EAI 580 analog computer. Values for the parameters of the dynamic equations were obtained from the relationships defined in Table A2 of Appendix I. As may be seen from Figure A20, only the parameter $a_{11}$ varies significantly with gas flow rate in the normal operating region of the plasma torch. Consequently, the system parameters other than $a_{11}$ are regarded as constant over the operating range of gas flow rate. The system parameters are tabulated in Table A3, and the analog computer patching diagram is given in Figure A21.

The variation of the gas temperature with magnetic flux density $B$, as obtained from the analog computer model, is plotted in Figure A22 together with the results obtained from the more complete digital simulation described in Appendix I. The variable $U_2$ characterizing the curves shown in Figure A22 represents the arc circuit supply voltage; i.e. $U_2 = [V_e/450]$. 
Dynamic equation:
\[
\dot{x} = Ax + f(x) + Bu + d
\]
\[
A = \begin{bmatrix}
a_{\eta} & 0.0173 & 0.002 & 0.933 & 0 \\
0.0156 & -0.028 & 0 & 0.018 & 0 \\
0.0093 & 0 & -0.057 & 0.021 & 0 \\
0 & 0 & 0 & -0.4 & 0.054 \\
0 & 0 & 0 & 0 & -0.5
\end{bmatrix}
\]
\[
a_{\eta} \in (-0.2 \rightarrow -0.7)
\]
\[
B = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
-0.27 & 0.485 \\
0.5 & 0
\end{bmatrix}
\]
\[
d = \begin{bmatrix}
0 \\
0 \\
0 \\
-0.143 \\
0
\end{bmatrix}
\]
\[
f(x) = \begin{bmatrix}
0.9 x_4 x_5 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

**TABLE A3.** PARAMETER VALUES FOR MODEL SIMULATION
FigA21 Analog computer diagram for plasma torch simulation
FIG. A22  EXIT GAS TEMPERATURE vs. MAGNETIC FLUX DENSITY

--- Analog simulation  ---o--- Digital simulation
APPENDIX III
SOFTWARE PACKAGE FOR ADAPTIVE CONTROL

III.1 The Computational Algorithm

The adaptive control algorithm developed in Chapter 3 is composed of two sets of 'filter-type' vector equations with associated matrix equations which generate the 'filter gains'. Firstly, there is the 'modelling filter' which generates sequences of one-step-ahead estimates of the state vector and uncertain parameter vector and, secondly, a 'feedback control filter' which generates sequences of incremental control vectors from the model feedback error \( (p_k - \hat{x}_k) \). The modelling and control update equations are,

\[
\begin{align*}
\dot{x}_{k+1} &= \phi_k \hat{x}_k + \phi_{ak} \hat{a}_k + \tau_k \hat{u}_k + \gamma_k \\
\hat{x}_{k+1} &= \hat{x}_{k+1} + K_{1k}H_T (y_k - \hat{H}_k) \\
\hat{a}_{k+1} &= \hat{a}_k + K_{2k}H_T (y_k - \hat{H}_k) \\
\delta U_{k+1} &= M_k \psi_T (p_{k+1} - \psi \hat{x}_{k+1}) \\
\hat{u}_{k+1} &= \hat{u}_k + \delta U_{k+1}
\end{align*}
\]

(A3.1)

Recognizing the fact that the matrix equations for updating the gain matrices \( K_k \) and \( N_k \) are of identical structural form, a common subroutine was written to calculate the gain matrices and to update the Ricatti matrices upon which they are dependent. Utilizing the matrix inversion lemma and the fact that the Ricatti matrices for modelling and control, \( P_k \) and \( M_k \) respectively, are symmetric, the following computational scheme was developed,

(a) Intermediate gain matrix
\[ V^*_k = W_k - c(W_k) \lambda_k c(W_k)^T \]
\[ K_{21k} = P_{21k} - P_{21k} H_k^T \lambda_k c(W_k)^T \]
\[ K_{22k} = P_{22k} - P_{21k} H_k^T \lambda_k H_k P_{21k} \]

(b) Gain matrix update
\[ V_k = \phi_k V^*_k + \phi_{ok} K_{21k} \]
\[ K_{21k} = K_{21k} \]

(c) Ricatti matrix update
\[ W_{k+1} = V_k \phi_k^T + (\phi_k P_{21k})^T + SR^{-1} S^T \]
\[ P_{21k+1} = K_{21k} \phi_k^T + P_{22k} \phi_{ok} \]
\[ P_{22k+1} = K_{22k} \]

The operations involving the general matrix arguments \( W_k \), \( V_k \), \( S \), \( R \), \( \lambda_k \) and \( c(W_k) \) define the computations common to both modelling and control, i.e. the dimensions conform to the dimensions of the state vector. The operations involving \( K_{21k} \), \( K_{22k} \), \( P_{21} \) and \( P_{22} \) are the additional computations required for parameter modelling (identification). The general matrix arguments have the following identities for the two different modes of operation,

(i) Modelling
\[ W_k = P_{11k} ; \quad V_k = K_{11k} ; \quad R = R_2 ; \quad S = A \]
\[ \lambda_k = \left[ Q_2^{-1} + H_k P_{11k} H_k^T \right]^{-1} ; \quad c(W_k) = P_{11k} H_k^T \]

(ii) Control
\[ W_k = M_k ; \quad V_k = H_k ; \quad S = R_k ; \quad R = R_1 \]
\[ \lambda_k = \left[ Q_1^{-1} + \psi M_k \psi^T \right]^{-1} ; \quad c(W_k) = M_k \psi^T \]
The system matrices are updated using the incremental values for the nominal trajectory, i.e.

$$
\delta \pi_k = \{(x_{k-1} - x_{k-2}) , \delta U_{k-1} | (a_{k-1} - a_{k-2}) \} , \text{ with,}
$$

$$
\phi_k = \phi_{k-1} + \Delta \phi(\delta \pi_k)
$$

$$
\phi_a = \phi_{a,k-1} + \Delta \phi_a(\delta \pi_k)
$$

$$
\Gamma_k = \Gamma_{k-1} + \Delta \Gamma(\delta \pi_k)
$$

(3.6)

where $\Delta$ is the sampling interval and the incremental matrices are given by,

$$
\Delta \phi(\delta \pi_k) = \frac{\partial f}{\partial x} |_{x = x_{k-1} - x_{k-2}}
$$

$$
\Delta \phi_a(\delta \pi_k) = \frac{\partial f}{\partial a} |_{a = a_{k-1} - a_{k-2}}
$$

$$
\Delta \Gamma(\delta \pi_k) = \frac{\partial f}{\partial u} |_{u = \delta U_{k-1}}
$$

Initial values $\phi_o$, $\phi_{ao}$ and $\Gamma_o$ are evaluated prior to initiating on-line control operation using the initial nominal trajectory points $\pi_o = (x_o , u_o | a_o)$.

III.2: The Off-Line Package

This is a set of routines which accept from either the teletype (TTY) keyboard or tape reader, data regarding the dimensions of matrices and vectors and the values of their elements. The data is stored in core memory locations specified prior to inputting the data. The stored data, which are either binary fixed point or floating point numbers, can be subsequently punched on paper tape in binary format, using either the high speed or low speed punch. This binary tape serves as initial data input to the on-line package.

Features of the off-line package are:
(1) Data input is specified by three formats,

   (a) Mxxxx - memory address; denoted by M followed by the octal value of the memory location.

   (b) Ixxxx - an integer number; denoted by I followed by the octal value of the number.

   (c) Fxxx.xxxc - a decimal floating point number; denoted by F followed by the decimal value and any delimiting character.

(Note: the DEC floating point package is used for floating point input, hence the rules pertaining to selection of a suitable delimiting character must be adhered to (see [98])).

Details of the data input and storage routines are shown in the flowchart of Figure A23.

(2) Selection of the paper tape punch used to output the assembled data is made from the keyboard. The data may be relocated in different areas of core at this stage by specifying values for the address pointers SA, FA and SDA; where SA is the starting address in core of a block of assembled data, FA is the end address of the data block and SDA is the relocated starting address of the data block. The starting address SDA is punched on the paper tape prior to punching the data block. A flow chart of the output punching routine is shown in Figure A23.

III.3 The On-Line Package

This package contains routines which perform the following functions.

(1) Selection of the sampling interval. This is done by calibrating the RC clock against the crystal clock.

(2) Computation of the initial system matrices using the initial nominal trajectory point $\pi_0$ and the previously obtained floating point value for the RC clock period (i.e. the sampling interval).
START
IC printed
Input charac.
from TTY

2
YES
charac. = $?
NO

DATA INPUT
SUBROUTINE

1

Input charac.
from TTY

3
YES
=$?
NO

YES
Read octal
memory address
Store in BUFF

NO

NO

YES
Read octal integer
Store in location
contents (BUFF)

NO

YES
Read spacing pt.
number. Store
in location C(Buf)

NO

BUFF + = 1
BUFF = BUFF - 1
BUFF = BUFF + 1
BUFF = BUFF - 1

2

Input charac.
from TTY

4
YES
Set PFLAG = 1
(HSP select)

NO

YES
Set PFLAG = 0
(LSP select)

NO

FIG. A23
FLOWCHART OF OFF-LINE PACKAGE

(Cont.)
FIG. A23 (Cont.)
(3) Sampling of the observations $y_k$ and reference trajectory $\rho_k$ and output of the control signals $u_k$, following an RC clock interrupt.

(4) Performance of the computations specified by the adaptive control algorithm, equation (A3.1) - (A3.6). A check is made to ascertain whether the computations have been completed following each RC clock interrupt.

The mode of operation of the algorithm may be selected by changing the contents of two memory locations EST and CTFLAG as shown in Table A3.1.

<table>
<thead>
<tr>
<th>LOCATION</th>
<th>MNEMONIC</th>
<th>CONTENTS</th>
<th>FUNCTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$22\emptyset\emptyset$</td>
<td>EST</td>
<td>$73\emptyset\emptyset$</td>
<td>State modelling</td>
</tr>
<tr>
<td>$\emptyset\emptyset33$</td>
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TABLE A3.1 - Selection of Operating Mode
Details of the on-line package are shown in the flowcharts given in Figure A.24.

Note that the on-line package also contains a matrix package to perform the following matrix operations: (a) matrix addition $A + B + C$, (b) matrix transposition $A^T + B$, (c) multiplication by a scalar $\beta A + B$, (d) matrix multiplication $AB + C$ and, (e) inversion of a $2 \times 2$ matrix.

For complete generalization of the matrix package a general matrix inversion routine would be required. Considering the common core area available for storage of intermediate matrices, the escalator method [99] appears to be the most suitable technique.
FIG. A24  FLOWCHART OF ON-LINE PACKAGE
FIG. A24 (Cont.)
FIG. A24 (Cont.)
APPENDIX IV

A COMPARISON OF THE SUBOPTIMAL AND OPTIMAL ADAPTIVE APPROACHES

A comparison of the suboptimal adaptive control algorithm and the corresponding optimal adaptive algorithm will be made in terms of the difficulties of implementation and relative computation times. Firstly, the optimal formulation for the servomechanism problem will be derived.

Based on the modelled linearized system equations,

\[ \hat{x}_{k+1} = \Phi_k \hat{x}_k + \Phi_{d_k} \hat{a}_k + \Gamma \delta u_k + \lambda_k + \hat{w}_k \]

with the control performance index,

\[ J(\delta u_k) = \frac{1}{2} \sum_{k=1}^{k_f} \{ || \rho_k - \psi \hat{a}_k ||^2 Q_1 + || \delta u_k ||^2 R_1 \} \]

the two point boundary value control problem is characterized by the set of equations (see Chapter 2),

\[ \begin{align*}
  x_{k+1} &= \Phi_k x_k + \Phi_{d_k} a_k + \lambda_k + \hat{w}_k - \Gamma R^{-1} \Gamma^T \lambda_{k+1} \\
  \lambda_k &= \Phi_k^T \lambda_{k+1} - \psi^T Q (\rho_k - \psi x_k) \\
  x_o &= x_o, \quad \lambda_{k_f} = 0
\end{align*} \]  \hspace{1cm} (IV.1)

Assuming a linear feedback solution for \( \lambda_k \) with an optimal driving vector \( \hat{n}_k \); i.e.,

\[ \lambda_k = P_k x_k + \eta_k \]

the set of equations (IV.1) may be reduced to,

\[ \begin{align*}
  x_{k+1} &= \Phi_k x_k + \Phi_{d_k} a_k + \lambda_k + \hat{w}_k - \Gamma R^{-1} \Gamma^T \left( P_{k+1} x_{k+1} + \eta_{k+1} \right) \\
  P_k x_k + \eta_k &= \Phi^T \left( P_{k+1} x_{k+1} + \eta_{k+1} \right) - \psi^T Q (\rho_k - \psi x_k) \hspace{1cm} (IV.2)
\end{align*} \]

Solving for \( x_{k+1} \) and substituting out in (IV.2), the following
relationship is obtained,

\[
(p_k - \Psi^T \Omega \Psi) \dot{x}_k + \eta_k = \dot{\rho}_k^T \left[ P_{k+1}^{-1} + \Gamma R_k^{-1} \Gamma^T \right]^{-1} \quad \eta_{k+1} \quad \rho_k \\
\left\{ \dot{\rho}_k \dot{x}_k + \dot{\rho}_k^T \dot{\rho}_k + \lambda_k + \hat{\omega}_k - \Gamma R_k^{-1} \Gamma^T \eta_{k+1} \right\} + \eta_k^T \rho_k + \Psi^T \Omega \rho_k
\]

Equating coefficients of the powers of \( \dot{x}_k \) (i.e. \( \dot{x}_k^0, \dot{x}_k^1 \)) the control algorithm may be obtained as,

\[
P_k = \dot{\rho}_k^T \left[ P_{k+1}^{-1} + \Gamma R_k^{-1} \Gamma^T \right]^{-1} \dot{\rho}_k + \Psi^T \Omega \Psi
\]

\[
\eta_k = \dot{\rho}_k^T \left\{ I - \left[ P_{k+1}^{-1} + \Gamma R_k^{-1} \Gamma^T \right]^{-1} \Gamma R_k^{-1} \Gamma^T \right\} \eta_{k+1} \\
+ \dot{\rho}_k^T \left[ P_{k+1}^{-1} + \Gamma R_k^{-1} \Gamma^T \right]^{-1} \{ \dot{\rho}_k a_k + \lambda_k + \hat{\omega}_k \} - \Psi^T \Omega \rho_k
\]

\[
\delta u_k = -R_k \Gamma^T \left\{ K_k \dot{x}_k + \dot{\rho}_k^T \left[ \eta_k + \Psi^T \Omega \rho_k \right] \right\}
\]

with, \( K_k = \left[ P_{k+1}^{-1} + \Gamma R_k^{-1} \Gamma^T \right]^{-1} \dot{\rho}_k \)

and \( P_{kf} = 0, \quad \eta_{kf} = 0 \)

From the set of equations (IV.3) it may be seen that the optimal adaptive approach requires the backward recursive solution of, (i) a matrix Riccati equation which generates the feedback gains and (ii) a vector equation which generates the driving vector \( \eta_k \). It is evident furthermore, that the solution for \( \eta_k \) is directly dependent upon the future values of the vectors \( \dot{x}_k, \rho_k \) and \( \hat{\omega}_k \). For the sequence of \( \eta_k \) vectors to be the optimal driving input, \( \dot{\rho}_k, \rho_k \) and \( \hat{\omega}_k \) must be known for all future stages - in most practical systems the future values of \( \rho_k \) and \( \hat{\omega}_k \) will not be known, and in adaptive systems the future values of \( \dot{\rho}_k \) will in addition be unknown. Similarly, for the control gains to be optimum the future values of the system transition matrix \( \dot{\rho}_k \) must be known - which is not the case in adaptive systems.
In view of these difficulties the computational approach will require repeating the solution for \( \hat{P}_k \) and \( \hat{n}_k \) from the terminal time to the present time following any change in either \( \rho_k \) or \( \hat{\gamma}_k \). In addition, the approximation that \( \rho_k \), \( \hat{\gamma}_k \), and \( \hat{\alpha}_k \) remain constant for all future stages will have to be made at each solution stage.

Comparison of Computing Times

An approximate comparison of the computing times required to update the control vector using the suboptimal and optimal adaptive algorithms will be made. The computing times will be evaluated from the number of basic arithmetic operations to be performed in order that a new control vector may be computed. For the optimal adaptive approach it will be assumed that a total of \( N \) iterations of equations (IV.3) from the terminal conditions are required before steady state values of \( P_k \) and \( n_k \) are obtained, i.e. \( P_k = P_{k+1} \) and \( n_k = n_{k+1} \).

The following pertinent dimensions may be defined:

- \( n \) = order of the system
- \( n_c \) = number of control inputs
- \( n_x \) = number of controlled states
- \( n_o \) = number of observed states
- \( n_p \) = number of uncertain parameters.

The number of basic arithmetic operations required by the two algorithms may be obtained as:

(a) Optimal algorithm

Gain update

Number of multiplications - \( N_{GOM} = n \left( 4n^2 + 2nn_c + 2n^2_c \right) \)

\[ + n_c^2 + \left( n_c! \right) \left( n^2 - n_c - 1 \right) \]
No. of additions $N_{GOA} = (n-1)(4n^2 + 2mn_c + n_c^2) + nn_c \left( n_c - 1 \right) + n_c^2 + n^2 + (n_c-1)(n_c - 1) \left( n_c - 1 \right)$

Driving input
No. of multiplications $N_{IOM} = n(3n^2 + 4n + n_x)$
No. of additions $N_{IOA} = (n-1) \left( 3n^2 + 4n + n_x \right)$

Total Operations per iteration:
Multiplication $N_{OM} = N_{GOM} + N_{IOM} = n \left( 7n^2 + 4n + 2mn_c + 2n_c^2 + n_x \right) + n_c^2 + (n_c-1)(n_c^2 - n_c - 1)$
Addition $N_{OA} = N_{GOA} + N_{IOA} = (n-1) \left( 7n^2 + 2mn_c + n_c^2 + 4n + n_x \right) + n n_c \left( n_c - 1 \right) + n_c^2 + n^2 + (n_c-1)(n_c - 1)$.

Number of Operations Required for Complete Control Update
Multiplication $N_{OMT} = N_x N_{OM} + N_{OCM}$
Additions $N_{OAT} = N_x N_{OA} + N_{OCA}$
where $N_{OCM} = n \left( n^2 + 2n + n_c \right)$
$N_{OCA} = (n-1) \left( n^2 + 2n + n_c \right) + 2n$ control vector:
update operations

(b) Suboptimal Algorithm
Gain update:
Multiplications $N_{GSM} = n \left( 4n^2 + 2mn_x + 2n_x^2 + n_x^2 \right) + (n_x-1)(n_x^2 - n_x - 1)$
Additions: $N_{GSA} = (n-1) \left( 4n^2 + 2mn_x + 2n_x^2 \right) + nn_x \left( n_x^2 - 1 \right) + n x^2 + n^2 + (n_x-1)(n_x - 1)$

Control update:
Multiplications $N_{SCM} = n(n + n_x^2)$
Additions $N_{SCA} = n[n \times (n_x - 1) + N-1] + n$
Let $\tau_m$ be the time required for a multiplication and $\tau_a$ be the time required for an addition and let $\tau_m/\tau_a = \sigma$, where $\sigma > 1$.

The total computing time required for complete update of the control vector using the two approaches may therefore be expressed as,

(a) Optimal algorithm

$$T_o = \left\{ \left[ (N \cdot N_{OM} + N_{OCH}) + (N \cdot N_{OCN} + N_{OCA}) \right] \right\} \tau_a$$

(b) Suboptimal algorithm

$$T_s = \left\{ \left[ (N_{GSM} + N_{SCM}) + (N_{GCA} + N_{SCA}) \right] \right\} \tau_a$$

Example:

For the plasma torch furnace control problem the relevant dimensions are,

$$n = 5 ; \quad n_c = 2 ; \quad n_x = 1$$

Hence the ratio of computing times is obtained as,

$$\lambda = \frac{T_o}{T_s} = \left\{ \frac{(1126N + 185)\sigma + (921N + 148)}{592 \sigma + 494} \right\}$$

A plot of $\lambda$ against the number of required iterations of the optimal algorithm $N$ is shown in Figure A.25. As might be expected the computation time required for updating the control vector using the optimal adaptive approach is directly proportional to the number of iterations of the algorithm required before steady state values of $p_k$ and $n_k$ are obtained.

In Figure A.26 the variation of $\lambda$ with the dimension of the system equations $n$ has been plotted for different values of the number of iterations of the optimal algorithm.
\[ \lambda = \frac{b}{T_s} \]

**FIG. A25** VARIATION OF \( \lambda \) WITH \( N \)
\[ \lambda = \frac{T_0}{T_s} \]

\( \sigma \geq 6 \)

\( N = 20 \)

\( 5 \)

\( 10 \)

\( 20 \)

\( 40 \)

\( 60 \)

\( n \)

\( 0 \)

\( 1 \)

\( 2 \)

\( 3 \)

\( 4 \)

\( 5 \)

\( 6 \)

**FIG. A26** VARIATION OF \( \lambda \) WITH \( n \)
APPENDIX V

BAYESIAN ESTIMATION AND THE MODELLING SOLUTION

Consider the problem of maximizing the conditional probability density function \( p(z_{k+1} | y_k) \) for the system described by the set of equations,

\[
\begin{align*}
\dot{x}_{k+1} &= \dot{x}_k + \gamma_{k+1} + G w_k \\
y_k &= H_k x_k + \beta_k + v_k
\end{align*}
\]

where \( y_k \) is the sequence of observation vectors \( y_{k0}, y_{k1}, \ldots, y_k \).

and \( v_k \) and \( w_k \) are the Gauss-Markov white noise sequences with statistics,

\[
\begin{align*}
E \{ w_k \} &= 0 & E \{ w_k w_j^T \} &= R_k \delta_{jk} \\
E \{ v_k \} &= 0 & E \{ v_k v_j^T \} &= Q_k \delta_{jk}
\end{align*}
\]

\( \delta_{jk} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases} \) \hspace{1cm} (V.1)

Application of Bayes rule yields the relationship

\[
p(z_{k+1} | y_k) = \frac{p(y_k | z_{k+1}) p(z_{k+1})}{p(y_k)}
\]

Since \( y_k \) is independent of \( z_{k+1} \), the likelihood function \( p(y_k | z_{k+1}) \) may be reduced to \( p(y_k | z_k) \) and, therefore,

\[
p(z_{k+1} | y_k) = \frac{p(y_k | z_k) p(z_{k+1})}{p(y_k)}
\]

It may be shown [24, 101, 102] that, for Gauss-Markov noise sequences,
\[
p(z_{k+1}|y_k) = \frac{\prod_{n=k_0}^k p(y_n - H_n z_n - \beta_n) \prod_{n=k_0}^k p(z_{k+1}|z_n) p(z_{k_0})}{p(z_k)}
\]
\[
= c(y_k) \exp \left\{ \frac{1}{2} \left\| z_{k_0} - \mu_{k_0} \right\|^2 v_0^{-1} \right. 
+ \frac{1}{2} \sum_{n=k_0}^k \left\| y_n - H_n z_n - \beta_n \right\|^2 q_n^{-1} 
+ \left. \frac{1}{2} \sum_{n=k_0}^k \left\| z_{k+1} - g'_k z_k - \gamma_k' \right\|^2 \left[ G R_n G_n^T \right]^{-1} \right\} \quad \text{(V.2)}
\]

Maximization of \( p(z_{k+1}|y_k) \) is equivalent to minimization of a quadratic function,
\[
I = \frac{1}{2} \left\| z_{k_0} - \mu_{k_0} \right\|^2 v_0^{-1} + \frac{1}{2} \sum_{n=k_0}^k \left\| y_n - H_n z_n - \beta_n \right\|^2 q_n^{-1} 
+ \left\| z_{k+1} - g'_k z_k - \gamma_k' \right\|^2 \left[ G R_n G_n^T \right]^{-1}
\]
with respect to \( z_k \), \( v_0 \) and \( \mu_{k_0} \) are the a-priori statistics,
\[
E\{z_{k_0}\} = \mu_{k_0} \quad ; \quad \text{var}\{z_{k_0}\} = v_0
\]

Furthermore, minimization of \( I \) with respect to \( z_k \) is equivalent to minimizing,
\[
J = \frac{1}{2} \left\| z_{k_0} - \mu_{k_0} \right\|^2 v_0^{-1} + \frac{1}{2} \sum_{n=k_0}^k \left\| y_n - H_n z_n - \beta_n \right\|^2 q_n^{-1} 
+ \left\| w_n \right\|^2 r_n^{-1}
\]
with respect to \( z_k \), subject to the equality constraint
\[
z_{k+1} = g'_k z_k + \gamma_k' + G w_k \quad \text{(V.4)}
\]
Equations (V.3) and (V.4) constitute a classical optimization problem which may be resolved by, (i) application of the discrete maximum principle and (ii) assuming a Riccati transformation for the unknown boundary conditions in terms of the known boundary conditions (see Appendix V). The solution to the optimization problem may be obtained as,

\[
\hat{z}_{k+1} = \phi_k \hat{z}_k + \gamma_k + \kappa_k \left[ y_k - H_k \hat{z}_k - \beta_k \right]
\]

\[
K_k = \phi_k \hat{S}_k H_k^T \left[ Q_k + H_k S_k H_k^T \right]^{-1} H_k S_k \phi_k^T + GR_k^{-1} G^T
\]

\[
S_{k+1} = \phi_k \left[ S_k - S_k H_k^T \left[ Q_k + H_k S_k H_k^T \right]^{-1} H_k S_k \right] \phi_k^T + GR_k^{-1} G^T
\]

where \( z_{k+1} = E \{ z_{k+1} \mid y_k \} \)

and \( S_k = E \{ (z_{k+1} - \hat{z}_{k+1}) (z_{k+1} - \hat{z}_{k+1})^T \} \)

(V.5)

with initial conditions \( z_{k0} = \mu_{k0} \) and \( S_{k0} = V_0 \). The set of equations above corresponds to the Kalman one-step-ahead predictor [100].

When the system dynamics are nonlinear and equations (V.1) represent the dynamics linearized about the previous estimate \( \hat{z}_{k-1} \), equations (V.5) correspond to the solution for the 'extended' Kalman one-step predictor [24, 101].

If the original nonlinear dynamics are of the form,

\[
z_{k+1} = f(z_k) + Gw_k
\]

\[
y_k = h(z_k) + v_k
\]

the estimate \( z_{k+1} \) obtained from (V.5) represents a first order approximation to \( E\{ z_{k+1} \mid y_k \} \). However, if the original nonlinear equations are of the form,
\[ z_{k+1} = f(z_k, w_k) \]
\[ y_k = h(z_k, v_k) \]

then \( \hat{z}_{k+1} \) is a first order estimate of the mode of the distribution function \( p(z_{k+1} | y_k) \).

A comparison of the modelling equations derived by application of invariant imbedding in Chapter 3 with the equations (V.5) indicates that they are of analogous form. Therefore, if the noise terms \( V_k \) and \( W_k \) are Gauss-Markov white noise sequences with known covariances \( Q_k \) and \( R_k \) respectively, the modelling solution will yield approximations to \( E \{ z_{k+1} | y_k \} \) providing,

\[ Q_k = Q_k^{-1} \; ; \; R_k = R_k^{-1} \]

and the initial conditions are selected from the a-priori statistics, i.e.,

\[ \hat{z}_0 = \mu_0 \; ; \; P_0 = V_0 \]
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