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FRENET-SERRET FORMALISM AND
RELATIVISTIC DYNAMICS.

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University of Windsor

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LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L'AVONS RÉCU
FRENÉT-SERRET FORMALISM
AND RELATIVISTIC DYNAMICS

by

Robert Douglas Kent

A Dissertation
Submitted to the Faculty of Graduate Studies
through the Department of Physics
in Partial Fulfillment of
the requirements for the Degree
of Doctor of Philosophy at
The University of Windsor

Windsor, Ontario, Canada
1978.
To Slug, Boo, Babe and Otis
without whom nothing would have
made any sense.
ABSTRACT

FRENET-SERRET FORMALISM AND RELATIVISTIC DYNAMICS

by

Robert Douglas Kent

The Frenet-Serret formalism for an arbitrary timelike curve in Riemannian spacetime is expressed in terms of 2- and 4-component spinors. The equations of transport of the spinors along the curve are derived directly from the vector form of the equations using a conveniently chosen null tetrad. The results are a simplified derivation and generalization of results found by Gursey.

In Minkowski space for the case when all Frenet scalars are constant we give the solutions in spinor form. When the curve is a helix, for which the second Frenet torsion scalar vanishes, we find that the spinor solutions can be expressed using a unitary transformation corresponding to a continuous rotation about the helix axis. This transformation formally resembles the Foldy-Wouthuysen transformation.

We apply the vector and spinor analysis to the motion of a point charge in an electromagnetic field consisting of a constant tensor multiplied by a scalar function. These solutions include the motion of a point charge in a plane polarized radiation field.

We apply the Frenet-Serret formalism to the Frenkel-Thomas equations describing a free spinning point particle. We derive the well-known oscillatory solutions. In spinor form the solutions are
formally identical to the momentum-representation form of solutions
to the one-particle Dirac equation. When the particle energy vanishes
then the solutions allow the 4-velocity to remain timelike while the
momentum 4-vector becomes null. The spinor solutions have only one
helicity component corresponding to the same formal behaviour
exhibited by the neutrino spinor solutions. Further, the energy and
frequency of the particle are related by a classical analogue of the
Planck-Einstein relation.

The motion of a charged, magnetic pole-dipole in a homogeneous
electromagnetic field is studied for the case in which the magnetic
moment is parallel to the spin and the charge-to-mass ratio is equal
to the gyromagnetic ratio. The characteristics of the trajectory are
discussed in terms of the interaction energy as a function of distance
along the curve. The solution process is made simple by the intro-
duction of a convenient dynamical tetrad which can be transformed into
the Frenet-Serret equations for a helix with variable torsion. We
show that there exist five different solutions for the interaction.
The choice of correct solution for a given problem is dependent on
initial conditions only.
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To my supervisor, Dr. G. Szamosi, I owe much for his patient guidance, careful application of physical insight and trusting friendship over these last several years.

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INTRODUCTION

The purpose of this investigation is three fold:

1) to present the Frenet-Serret formalism describing a timelike trajectory in terms of orthonormal and null tetrads and also two- and four-component spinors,

2) to apply the formalisms to the relativistic dynamics of a charged point particle moving in a constant electromagnetic field, and,

3) to apply the formalisms to the relativistic dynamics of a spinning point particle, or magnetic dipole, moving freely and in the presence of an homogeneous electromagnetic field.

The Frenet-Serret description of curves in space-time can be described as the local theory of curves parameterized by arc-length. This is sometimes referred to as the intrinsic theory of curves. The use of the formalism in Physics and Relativity, in particular, has not been frequent. The mathematical description, however, may be found in almost any book on Differential Geometry (e.g. Eisenhart 1949, Kreysig 1968).

The first use of the Frenet-Serret method in Relativistic Physics was made by Synge (1937) in Relativistic Hydrodynamics. Synge described the properties of a fluid by considering the motion of a point-like fluid element as defining a frame. For an arbitrary timelike curve, in the sense that the tangent vector is timelike
everywhere on the curve, this requires that one timelike and three space-like vectors be defined corresponding to the dimensionality and metric structure of Riemannian space-time. By appropriately orienting the tetrad along the curve, Synge showed that the geometry of the curve is completely specified by three invariantly defined scalars which are functions of the distance parameter along the curve. These scalars generalize the classical notions of radius of curvature, precession and nutation of the frame. The generalization to n-dimensional spaces has been studied by Blaschke (1920) and Coburn (1942) (also, Eisenhart 1949).

The choice of the Frenet-Serret description of timelike trajectories is important because the tetrad, or frame, is unique among all possible tetrads. In physical terms the description of the motion is purely local. Thus we can unambiguously describe events in terms of a well-defined, local laboratory frame (eg. Synge 1960).

Following Synge (1937) and results of Bohm, Schiller and Tionno (1955a, b) a spinor approach to the geometry of timelike trajectories was devised by Gursey (1957). This method of transforming the space-time Frenet-Serret equations into 4-component spinor form was accomplished using the properties of the unimodular $2 \times 2$ rotation matrices, the so-called Cayley-Klein parameters (eg. Corben and Stehle 1950). Gursey applied the formalism to the motion of a point charge in a constant electromagnetic field and to a free, spinning point particle. The technique and applications developed by Gursey are, however, cumbersome, often requiring lengthy algebra
and it is not always easy to see their physical interpretation.

Pläss (1961) investigated the Lorentz-Dirac equations describing the radiation reaction from an accelerating point charge. He was able to show, using the Frenet-Serret equations, that run-away solutions were inevitable in 3-dimensions. He also used this method to demonstrate the existence and uniqueness of physical solutions for a broad class of applied electromagnetic fields.

Rohrlich (1965) considered a few special cases of motion of a point charge in an electromagnetic field using the 3-dimensional Frenet-Serret description. More recently, Honig (1973) and Honig, Schucking and Vishveshwara (1974) have investigated the 4-dimensional Lorentz equation for a covariantly constant, or homogeneous, electromagnetic field. Their treatment is extensive and rigorous and provided the basis for much of this work and especially Chapter III. In a recent paper by Lapkoskii and Laptinskii (1976) a number of Honig's, et al., results have been expressed in spinor form.

Hestenes (1974) formulated a relativistic classical mechanics using a real spinor calculus. The Frenet tetrad is used to define a rigid point particle frame comoving with the particle. This method allows the spinors to be unambiguously described in terms of spacetime dynamical quantities.

Finally, Ellis (1975) used the Frenet-Serret formalism to solve the Frenkel-Thomas equations describing a free spinning particle. This method is similar to the method we use in Chapter IV whereby an orthonormal tetrad can be defined using the dynamical vectors associated with a spinning particle. This tetrad can be easily
transformed into the Frenet tetrad and the motion is very simply described geometrically.

Apart from the above references there does not seem to exist further evidence of the use of Frenet-Serret formalism in relativistic dynamics in the available literature. It seems that an important part of physics is concerned with describing the motion of point particles following timelike trajectories, however. Thus, the uniqueness of the Frenet tetrad gives this formalism a special importance.

In this paper we shall use the Frenet-Serret formalism to describe some physical problems from classical relativistic dynamics using both the vector, or space-time, approach and 2- and 4-component spinors.

Chapter I defines the notations, conventions and relevant results relating spinors, tensors and tetrads. These results are usually well-known and well-documented so only the most relevant features of tensor and spinor analysis are given.

We begin Chapter II by writing the Frenet-Serret equations for a timelike curve using an orthonormal tetrad. With a conveniently chosen null tetrad constructed from the original tetrad we derive directly the spinor form of these equations valid in Riemannian space. They are generalizations of Cursey's equations.

We also discuss two alternate methods of describing the curves using the Darboux spinor, which describes the generalized rotation plane of the spinors, and also an axial spinor corresponding to the generalized rotation axis of the tetrad. In the case when the curves
are space-time helices this latter spinor relates spinors in different frames and is formally the same as the Foldy-Wouthysen transformation.

Finally, we completely solve the equations in Minkowski space when all the Frenet scalars are constants along the curve.

In Chapter III we examine the motion of a point charge in an electromagnetic field which consists of a constant tensor multiplied by a scalar function. This is a generalization of some results of Honig, Schucking and Vishveshwara (1974). The spinor equations of motion are derived in a form found by Plebanski (1966) and solved in various fields.

In Chapter IV we use the Frenet-Serret formalism to solve the Frenkel-Thomas equations for a free spinning particle. We repeat some of the analysis of Ellis (1971) and then extend these results to include the spinor description of the motion. We find that there are a number of formal similarities between the spinors used in the quantum mechanical description of spin. This includes classical analogues of the quantum mechanical 'zitterbewegung', Foldy-Wouthysen transformation and the Planck-Einstein relation for a massless particle. Finally, we discuss at length the motion of a charged magnetic dipole in a homogeneous electromagnetic field when the charge-to-mass ratio is equal to the gyromagnetic ratio. This motion will be solved covariantly.

It should be noted that at the close of Chapter III the page numbers 44 and 45 are omitted.
CHAPTER I
NOTATIONS, CONVENTIONS AND DEFINITIONS

We shall consider a Riemannian space-time which we shall refer to as 'world'-space and quantities defined in the space-time called 'world'-quantities.

The invariant measure of distance between the space-time points is given by the line element

\[ d\lambda^2 = g_{\mu\nu} dx^\mu dx^\nu \]  \hspace{1cm} (1.1)

where \( g_{\mu\nu} \) is the world metric and \( \lambda \) is the proper distance measuring the length of \( x^\mu \). We shall take \( g_{\mu\nu} \) to be of -2 signature so that in the limit of Minkowski space

\[ g_{\mu\nu} \rightarrow \eta_{\mu\nu} = \text{Diag} (1, -1, -1, -1) \]  \hspace{1cm} (1.2)

in a Cartesian frame. Greek indices have the values \( \mu = 0, 1, 2, 3 \) and repeated indices imply the summation convention.

Vectors and tensors are expressed by quantities \( K^\mu \) and \( K^{\mu\nu} \). We define the character of these quantities by

\[ K_{\mu}K^{\mu} \begin{cases} > 0 & \text{Timelike} \\ = 0 & \text{Null} \\ < 0 & \text{Spacelike} \end{cases} \]  \hspace{1cm} (1.3)

for vectors, and, for \( K^{\mu\nu} = -K_{\mu\nu} \)

\[ K^{\mu\nu}K_{\mu\nu} \begin{cases} > 0 & \text{Timelike} \\ = 0 & \text{Null} \\ < 0 & \text{Spacelike} \end{cases} \]  \hspace{1cm} (1.4)
\[ K_{\nu}^{\alpha} \quad K_{\mu}^{\gamma} \left\{ \begin{array}{ll} = 0 & \text{Simple} \\ \neq 0 & \text{Non-simple} \end{array} \right. \] (1.5)

where \( K_{\nu}^{\alpha} \) is referred to as a bivector, or anti-symmetric tensor, and \( K_{\mu}^{\gamma} \) is the dual bivector to \( K_{\nu}^{\alpha} \). We define the dual as

\[ \hat{K}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\kappa\rho} K_{\kappa\rho} \] (1.6)

where \( \varepsilon^{\mu\nu\kappa\rho} \) is the alternating tensor

\[ \varepsilon^{\mu\nu\kappa\rho} = \frac{1}{\sqrt{-g}} \gamma^{\mu\nu\kappa\rho} \quad \varepsilon_{\mu
u\kappa\rho} = -\sqrt{-g} \eta_{\mu
u\kappa\rho} \] (1.7)

\[ \eta^{\mu\nu\kappa\rho} = \begin{cases} -1 & \text{cyclic} \\ 1 & \text{anti-cyclic} \\ 0 & \text{two indices equal} \end{cases} = \eta_{\mu\nu\kappa\rho} \] (1.8)

and \( g \) is the determinant of \( g_{\mu\nu} \).

Covariant differentiation is defined as

\[ \nabla_{\mu} K_{\nu}^{\rho} = \partial_{\mu} K_{\nu}^{\rho} + \Gamma_{\mu \nu}^{\gamma} K_{\gamma}^{\rho} - \Gamma_{\nu \rho}^{\gamma} K_{\mu}^{\gamma} \] (1.9)

where \( \partial_{\mu} \) is the operator of partial differentiation and \( \Gamma_{\mu \nu}^{\rho} \) are the connection coefficients, or Christoffel symbols, defined by,

\[ \nabla_{\mu} g_{\alpha\beta} = 0 \] (1.10)
We shall assume that at each point in the space-time we can define a frame of four linearly independent vectors which we call a tetrad, denoted by $e_{(a)}^\mu$ where $a = 0, 1, 2, 3$, denote the particular tetrad member, or 'leg'.

We can use the tetrad to define quantities in that particular frame. Hence, for the metric

$$g_{\mu\nu} e_{(a)}^\mu e_{(b)}^\nu = \eta_{(a)(b)}$$

(1.11)

where $\eta_{(a)(b)}$ is in general a set of ten independent scalars. We assume $e_{(a)}^\mu$ has the inverses $e^{\mu(a)}$ and $e_{(a)}^\mu$ satisfying

$$e_{(a)}^\mu e_{(a)}^{(b)} = \eta_{(a)(b)} e_{(a)}^{(b)} = \delta_{(a)}^{(b)} = \delta_{(a)}^{(a)}$$

(1.12)

where $\delta_{(a)}^{(a)}$ is the tetrad projection of the Kronecker symbol and

$$e_{(a)}^\mu e_{(a)}^{(b)} = \eta_{(a)(b)} e_{(a)}^{(b)} = \eta_{(a)}^{(b)} = \delta_{(a)}^{(a)}$$

(1.13)

where $\delta_{(a)}^{(b)}$ is the usual 4-dimensional Kronecker symbol

$$\delta_{(a)}^{(a)} = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

If we choose for $\eta_{(a)(b)}$ the particular form

$$\eta_{(a)(b)} = \eta^{(a)(b)} = \eta_{(a)(b)} = \text{Diag}(1, -1, -1, -1)$$

(1.14)

then we impose on $e_{(a)}^\mu$ that it be an orthonormal tetrad and we can write $g_{\mu\nu}$ in the Lorentzian form

$$g_{\mu\nu} = e_{\mu(a)} e_{\nu(b)} - e_{\mu(a)} e_{\nu(b)} - e_{\mu(a)} e_{\nu(b)} - e_{\mu(a)} e_{\nu(b)}$$

(1.15)
where it is seen that $e^\mu_{(\alpha)}$ is a timelike unit vector and $e^\mu_{(\alpha)}$ ($\alpha = 1, 2, 3$) are unit spacelike vectors, corresponding to a choice of Cartesian coordinates at each space-time point.

Similarly we can project vectors and tensors onto the tetrad by

$$K^{(\mu)} = K^\mu e^{(\mu)}_{(\alpha)} \quad K^\mu = K^{(\mu)} e_{(\alpha)}^\mu$$

(1.16a)

$$K_{(\alpha\beta)} = K_{\mu\nu} E^{\mu\nu}_{(\alpha\beta)} \quad K^{\mu\nu} = K^{(\alpha\beta)} E_{(\alpha\beta)}^{\mu\nu}$$

(1.16b)

where

$$E_{(\alpha\beta)}^{\mu\nu} = \frac{1}{2} \left( e_{(\alpha)}^\mu e_{(\beta)}^\nu - e_{(\beta)}^\mu e_{(\alpha)}^\nu \right) = - E_{(\alpha\beta)}^{\nu\mu} = - E_{(\beta\alpha)}^{\mu\nu}$$

(1.17)

From the covariant derivative we form the projected derivative and differential

$$\frac{D}{dx^{(\alpha)}} = e^\mu_{(\alpha)} \nabla_\mu \quad d\lambda^{(\alpha)} = e_\mu^{(\alpha)} dx^\mu$$

(1.18)

where the projected 'coordinates' $\lambda^{(\alpha)}$ are defined by the second of (1.18). This derivative is directional along the various legs of the tetrad. We shall specialize in this paper to the case of timelike curves where the relevant derivatives are taken along the curve. In this case we shall write $\lambda^{(a)} = s$ where $s$ is the length along the curve and $e^\mu_{(\alpha)}$ is the tangent vector.

We shall define our tetrad to be 'right-handed' in the sense that the spatial triad $e^\mu_{(\alpha)}$ ($\alpha = 1, 2, 3$) is oriented according to a right-handed screw. We express this invariantly by demanding, with
\begin{align}
\varepsilon^{\mu
u\kappa\lambda} e_{\mu\alpha} e_{\nu\beta} e_{\kappa\gamma} e_{\lambda\delta} &= -I \\
\varepsilon^{\mu
u} e_{\mu\alpha} e_{\nu\beta} &= -I
\end{align}

Thus we find

\begin{align}
\dot{E}^{\mu\nu}_{(\alpha\beta)} &= -E^{\mu\nu}_{(\beta\alpha)} \\
\dot{E}^{\mu\nu}_{(\alpha\beta)} &= -E^{\mu\nu}_{(\beta\alpha)} \\
\dot{E}^{\mu\nu}_{(\alpha\beta\gamma)} &= -E^{\mu\nu}_{(\beta\alpha\gamma)}
\end{align}

which can be inverted simply using the property

\begin{align}
\hat{K}^{\mu\nu} = -K^{\mu\nu}
\end{align}

We shall have occasion to use the exponential tensor

\begin{align}
\exp\{\Theta K^{\mu\nu}\} = \sum_{n=0}^{\infty} \frac{\Theta^n}{n!} (K^{\mu\nu})^n
\end{align}

where \((K^{\mu\nu})^n\) means \(K^{\mu\nu}\) multiplied by itself \(n\) times. This tensor has
the properties, paying special attention to the indices,

\begin{align}
\exp\{\Theta K^{\mu\nu}\} \exp\{-\Theta K^{\mu\nu}\} = \delta^{\mu\nu}
\end{align}

\begin{align}
K^{\mu\nu} \exp\{\Theta K^{\mu\nu}\} = \exp\{\Theta K^{\mu\nu}\} K^{\mu\nu}
\end{align}
where $\delta^\mu_\nu$ is the Kronecker tensor

$$ \delta^\mu_\nu = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases} \quad (1.25) $$

According to a result of Ruse (1935) any two arbitrary bivectors $A^\mu_\nu$ and $B^\mu_\nu$ obey the relation

$$ A^\mu_\nu B^\nu_\alpha - B^\mu_\alpha A^\nu_\nu = -\frac{1}{2} A^\alpha_\rho B^\rho_\nu \delta^\mu_\nu \quad (1.26) $$

Thus, for arbitrary $K^\mu_\nu$ we find

$$ K^\mu_\alpha K^\alpha_\nu = \hat{K}^\mu_\alpha K^\alpha_\nu = -\frac{1}{4} K^{\alpha \beta} \hat{K}^{\nu \rho} \delta^\mu_\nu \quad (1.27) $$

$$ \left( K^\mu_\nu \right)^3 + \frac{1}{2} K^{\alpha_1 \rho} K^{\rho \sigma} K^\nu_\sigma + \frac{1}{4} K^{\alpha_1 \alpha_2} \hat{K}^{\nu \rho} \hat{K}^\mu_\rho = 0 \quad (1.28) $$

In denoting spinors we shall use an abstract index notation. This enables us to avoid using matrix representations of the spinors and simplifies the spinor algebra considerably. When displaying components we shall resort to the use of matrices, however. Much of the notation is essentially the same as that used by Bade and Jehle (1953), Penrose (1960) and Pirani (1964) to name but a few authors. Any deviations are hopefully clearly indicated.

We define the spinor $\xi^A_\mu$ as an ordered pair of complex scalars with the index $A = 1, 2$. The components of $\xi^A_\mu$ are
\[ \{ \zeta^A \} = \{ \zeta^1, \zeta^2 \} \]

The complex conjugate is denoted by \( \zeta^\dagger \) so that \( \zeta^\dagger = \zeta^\dagger \).

We define spin-space as the space spanned by spinors \( \zeta^A, \eta^A \) and the complex conjugates satisfying in general

\[ \zeta^A \eta^B \epsilon_{AB} = \rho e^{-i\theta} \quad ; \quad \rho > 0 \]

where \( \epsilon_{AB} \) is the spinor metric which can be written directly in terms of the spinor dyad

\[ \epsilon_{AB} = \frac{1}{\rho} \left\{ \zeta_A \eta_B - \eta_A \zeta_B \right\} \]

The inverse of \( \epsilon_{AB} \) is \( \epsilon^{AB} \) satisfying

\[ \epsilon^{AC} \epsilon_{CB} = \epsilon^A_B = \epsilon_{BA} \epsilon^{CA} = -\epsilon^A_B = -\delta^A_B \]

where \( \delta^A_B \) is the identity spinor with components

\[ \left\{ \delta^A_B \right\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

Note that the indices on \( \delta^A_B \) are placed one atop the other so as to avoid confusion with the \( \epsilon^A_B \) spinor which is antisymmetric in its indices. We shall specialize to the case where \( \epsilon^A_B \) can be written

\[ \left\{ \epsilon^A_B \right\} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \left\{ \epsilon_{AB} \right\} \]
Raising and lowering of indices is defined according to the rules

\[ \xi^A = \varepsilon^{AB} \xi_B \quad \xi_A = \varepsilon_{BA} \xi^B \]  

(1.35)

Thus we find for the components of \( \xi_A \) and \( \xi^A \)

\[ \xi_1 = -\xi^2 \quad \xi_2 = \xi^1 \]  

(1.36)

and for the normalization (1.30)

\[ \xi_A \eta^A = -\eta_A \xi^A = \xi_A \eta + \xi_A \eta^2 \]  

(1.37)

We shall refer to spinors satisfying

\[ \xi_A \eta^A = 1 \]  

(1.38)

as the unit basis spinors.

We define 'outer' multiplication of spinors \( \xi^A, \eta^A \) by

\[ \begin{pmatrix} \xi^A \eta^B \\ \xi^A \eta^B \end{pmatrix} = \begin{bmatrix} \xi' \eta' & \xi' \eta^2 \\ \xi^2 \eta' & \xi^2 \eta^2 \end{bmatrix} \]  

(1.39)

We define spinor covariant differentiation according to

\[ \nabla_{\mu} \phi_A \phi^A = \partial_{\mu} \phi_A \phi^A + \phi_A \gamma^\mu \phi^c - \gamma_{\mu A} \phi_c \phi^A \]  

(1.40)
where \( \mathcal{A} \) is the spinor connection, sometimes referred to as the Fock-Ivanenko coefficient. In Cartesian coordinates in Minkowski space \( \mathbb{R}^{1+3} \) (e.g., Chapman 1977). We shall assume the strong condition

\[
\nabla_{\mu} e_{\mu}^A = 0
\]  

(1.41)

which allows the metric to be used to raise or lower indices from in or outside of the differential operator. This condition, it should be noted, does not place very severe constraints on the space-time but precludes using the phase scalars \( \rho \) and \( \Theta \) in (1.30) as gauge transformations such as are used by Weyl. Relaxation of (1.41) allows \( \rho \) and \( \Theta \) to generate the so-called extended conformal transformations which were studied extensively by Rohrlich, Fulton and Witten (1961) (also, Penrose 1960).

Let us now define a null tetrad of vectors \( l^\mu, n^\mu, m^\mu \) and \( \bar{m}^\nu = (m^\nu)^* \) satisfying

\[
l_{\mu} n^\mu = -m_{\mu} \bar{m}^\mu = 1
\]  

(1.42)

with all other products vanishing. The metric tensor is then expressed by

\[
\eta_{\mu\nu} = l_{\mu} n_{\nu} + n_{\mu} l_{\nu} - m_{\mu} \bar{m}_{\nu} - \bar{m}_{\mu} m_{\nu}
\]  

(1.43)

From the null tetrad and the unit spinors we define the Hermitian mapping functions

\[
\sigma^{\mu\nu} = \frac{1}{2} \left[ l^{\mu} \bar{\eta} \bar{\gamma}^{\nu} \bar{\gamma}^{\delta} + n^{\mu} \gamma^{\nu} \gamma^{\delta} - m^{\mu} \bar{\eta} \gamma^{\nu} \gamma^{\delta} - \bar{m}^{\mu} \gamma^{\nu} \bar{\eta} \gamma^{\delta} \right]
\]  

(1.44)

We shall assume in what immediately follows that the phase scalars are given by \( \rho = 1 \) and \( \Theta = 0 \).
Projecting the null tetrad onto $\sigma^{\mu\lambda\delta}$ we find

$$\frac{1}{\sqrt{2}} \sigma_{\mu}^{\lambda\delta} = \gamma^{\lambda} \gamma^{\delta}$$

and

$$\frac{1}{\sqrt{2}} \sigma_{\mu}^{\lambda\delta} \eta_{\lambda} = \gamma^{\lambda} \gamma^{\delta}$$

$$\frac{1}{\sqrt{2}} m_{\mu} \sigma_{\lambda\delta} = \gamma^{\lambda} \gamma^{\delta}$$

(1.45a)

$$\frac{1}{\sqrt{2}} \sigma_{\lambda\delta} = \gamma^{\lambda} \gamma^{\delta}$$

(1.45b)

We see that the $\sigma^{\mu\lambda\delta}$ are the spinor representation of the tetrad and that the spinors $\gamma^{\lambda}$ and $\gamma^{\delta}$ are the spin-space images of the vectors $\lambda^{\mu}$ and $\eta^{\lambda}$ respectively. The $\sigma^{\mu\lambda\delta}$ satisfy, using (1.43) - (1.44)

$$\sigma^{\mu\lambda\delta} \sigma_{\nu\delta c} + \sigma_{\nu\lambda\delta} \sigma^{\mu\delta c} = 2 g_{\mu\lambda} \delta^{A} c_{c}$$

(1.46)

$$\sigma^{\mu\lambda\delta} \sigma_{\delta\alpha} = 2 g_{\mu\nu}$$

$$\sigma^{\mu\lambda\delta} \sigma_{\mu}^{\lambda\delta} = 2 \epsilon_{\mu\nu} \epsilon^{\delta\delta}$$

(1.47)

In general, it is always possible to choose the tetrad and spinor dyad such that

$$\nabla_{\mu} \sigma^{\mu\lambda\delta} = 0$$

(1.48)

In Minkowski space we are free to choose a particular representation for the $\sigma^{\mu\lambda\delta}$ and we shall use the Pauli representation
\[
\begin{align*}
\{ \sigma^{0 \alpha \bar{\alpha}} \} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \{ \sigma^{i \alpha \bar{\alpha}} \} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
\{ \sigma^{1 \alpha \bar{\alpha}} \} &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & \{ \sigma^{2 \alpha \bar{\alpha}} \} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{align*}
\] (1.49a)

where \( i \) is the usual root of minus one.

In general the \( \sigma^{-\alpha \beta} \) satisfy, as shown by direct substitution of (1.49),

\[
\frac{1}{4!} \epsilon^{\mu \nu \kappa \beta} \sigma^{-\mu \nu \kappa} \sigma^{-\beta} = -i \delta^\alpha_E
\] (1.50)

\[
\frac{1}{2} \epsilon^{\mu \nu \kappa \beta} \sigma^{-\mu \nu \kappa} \delta^{\beta} c = -i \sigma^{\epsilon \alpha \beta} \delta^{\epsilon} c
\] (1.51)

where

\[
A^{[\mu \nu]} = \frac{1}{2} \left\{ A^\mu B^\nu - A^\nu B^\mu \right\}
\] (1.52)

Using the \( \sigma^{-\alpha \beta} \) we can project vectors and tensors into spin-space by

\[
K^\alpha \beta = \frac{1}{\sqrt{2}} K^\mu \sigma^{-\mu \alpha \beta}
\] (1.53)
\[-i K^{AC} \epsilon^{\hat{A}\hat{B}} + i K^{i\hat{i}} \epsilon^{AC} = \frac{1}{2} K_{\mu\nu} \sigma^\mu \sigma^{\nu} \sigma^c \sigma^b. \quad (1.54)\]

where \( K^{\alpha\nu} = -K^{\nu\alpha} \). The spinor \( K^{AC} \) in (1.54) is symmetric and from (1.51) we see that

\[ K^{AC} = -\frac{i}{2} (K_{\mu\nu} + i K^A_{\mu\nu}) \sigma^\mu \sigma^\nu \sigma^B \sigma^C. \quad (1.55) \]

Thus \( K^{AC} \) is the image of the anti-self-dual bivector \( \hat{K}_{\alpha\nu} = \frac{i}{2} (K^{\alpha\nu} + i K^A_{\alpha\nu}) \), \( \hat{K}_{\alpha\nu} = -i K^{\alpha\nu} \).

If we define the orthonormal tetrad

\[ \sqrt{2} e^{\kappa}_{\alpha\gamma} = l^\kappa + m^\kappa \quad \sqrt{2} e^{\kappa}_{\alpha\gamma} = m^\kappa + \bar{m}^\kappa \quad (1.56a) \]

\[ i\sqrt{2} e^{\kappa}_{\alpha\gamma} = m^\kappa - \bar{m}^\kappa \quad i\sqrt{2} e^{\kappa}_{\alpha\gamma} = l^\kappa - \eta^\kappa \quad (1.56b) \]

then from (1.38), (1.42) and (1.44) we find

\[ e^{\kappa}_{\alpha\gamma} = \frac{1}{2} \sigma^\alpha \partial^\beta (x^\beta \tilde{x}^\gamma + \eta^\beta \tilde{\eta}^\gamma) \quad (1.57a) \]

\[ e^{\kappa}_{\alpha\gamma} = \frac{1}{2} \sigma^\alpha \partial^\beta (\tilde{x}^\gamma \eta^\beta + \eta^\gamma \tilde{\eta}^\beta) \quad (1.57b) \]

\[ e^{\kappa}_{\alpha\gamma} = -\frac{i}{2} \sigma^\mu \partial^\alpha (x^\mu \tilde{x}^\gamma - \eta^\mu \tilde{\eta}^\gamma) \quad (1.57c) \]
\[ \mathcal{E}_{(a)} = \frac{1}{2} \sigma_{\dot{a}}^{\lambda} \left( \gamma^\lambda \gamma^\delta - \gamma^\delta \gamma^\lambda \right) \]  \hfill (1.57d)

We shall find it convenient sometimes to express two 2-component spinors simultaneously. Such constructions are called bispinors, or 4-component spinors. The 2- and 4-component spinor formalisms are equivalent (e.g. Chapman 1977), a fact which we shall display explicitly.

We define the mapping functions

\[ \left\{ \gamma_{\dot{a}}^\alpha \right\} = \begin{bmatrix} 0 & \sigma_{\dot{a}}^{\alpha \dot{b}} \\
\sigma_{\dot{b}}^{\alpha \dot{a}} & 0 \end{bmatrix} \]  \hfill (1.58)

where \( a = 1, 2, 3, 4 \). There should be no confusion here between the Latin indices on the bispinor and the bracketed Latin indices used with the tetrads. To further differentiate between these quantities we shall use a bold script to denote bispinor quantities projected from world space. The components of \( \gamma_{\dot{a}}^\alpha \) may be found from (1.58) by substituting the 2 X 2 zero matrix and the \( \sigma_{\dot{a}}^{\alpha \dot{b}} \) 2 X 2 matrix representations, and then taking the indices \( a \) and \( b \) to denote row and column position respectively.

The bispinor metric is defined by

\[ \left\{ I_{\dot{a} \dot{b}} \right\} = \begin{bmatrix} \epsilon_{\dot{b} \dot{a}} & 0 \\
0 & \epsilon_{\dot{a} \dot{b}} \end{bmatrix} \quad \left\{ I_{\dot{a} \dot{b}} \right\} = \begin{bmatrix} \epsilon_{\dot{a} \dot{b}} & 0 \\
0 & \epsilon_{\dot{b} \dot{a}} \end{bmatrix} \]  \hfill (1.59)
and the identity bispinor by

\[
\{ I_{a^b} \} = \begin{bmatrix} S_B & 0 \\ 0 & S_A \end{bmatrix}
\]

(1.60)

where \( I_{a^c} I^{c^b} = -I_{a^b} \).

Using (1.50) we define the bispinor

\[
\frac{i}{4!} \epsilon_{\mu \nu \kappa \rho} \gamma_{\mu^a_c} \gamma_{\nu^a_d} \gamma_{\kappa^a_e} \gamma_{\rho^a_b} = \gamma^{(s)}_{a^b}
\]

(1.61)

where we note that

\[
\gamma^{(s)}_{a^b} = \begin{bmatrix} S_B & 0 \\ 0 & -S_A \end{bmatrix}
\]

(1.62)

and \( \gamma^{(s)}_{a^c} \gamma^{(s)}_{c^b} = I_{a^b} \).

We define the bispinor projections using (1.53) - (1.54)

\[
\mathcal{K}_{a^b} = \frac{i}{\sqrt{2}} K_{\mu} \gamma^{\mu}_{a^b}
\]

\[
\mathcal{J}_{a^b} = \frac{-i}{\sqrt{4}} J_{\mu \nu} \gamma^{\mu}_{a^c} \gamma^{\nu}_{c^b}
\]

(1.63)

\[
\begin{bmatrix} 0 & K_{A^B} \\ K_{B^A} & 0 \end{bmatrix}
\]

(1.64)
where \( J^{\mu \nu} = - J^{\nu \mu} \).

If we define bispinors

\[
\psi_a = \frac{1}{\sqrt{2}} \left\{ \gamma^a, \xi^a \right\} \quad \chi^a = \frac{1}{\sqrt{2}} \left\{ \xi^a, \gamma^a \right\}
\]  

(1.65)

where \( \xi^a \) and \( \gamma^a \) satisfy (1.30) in general and where \( \psi = \frac{\gamma}{\sqrt{\kappa}}, \psi_2 = \frac{\gamma_2}{\sqrt{\kappa}} \), \( \psi_3 = \frac{\gamma_3}{\sqrt{\kappa}} \), \( \psi_4 = \frac{\gamma_4}{\sqrt{\kappa}} \) and so on then it follows from (1.57) that

\[
\chi^a \psi_a = \rho \cos \theta
\]  

(1.66a)

\[
\chi^a \gamma^c \psi_b = i \rho \sin \theta
\]  

(1.66b)

\[
\chi^a \gamma^c \psi_b = \rho \sigma^a_c
\]  

(1.66c)

\[
\chi^a \gamma^c \psi_b \psi_c = \rho \sigma^a_c
\]  

(1.66d)

\[
2 \chi^a \gamma^c \psi_b \psi_c = i \rho \tilde{\varepsilon}^{a \mu \nu}
\]  

(1.66e)

The expressions (1.66) are usually called bilinear covariants. These expressions arise purely formally from the definitions of the spinors. Note that when \( \rho = 1, \theta = 0 \) then (1.66b) vanishes and
\[ \chi^q = 1 \tag{1.67} \]

Whenever \( \chi^q \) satisfy (1.67) we refer to them as unit bispinors.

We shall have occasion to use exponential spinors, analogous to (1.22). We note the general results

\[ \exp \{ \theta \delta^a_b \} = e^{\theta} \delta^a_b \tag{1.68} \]

\[ \exp \{ \theta I_a^b \} = e^{\theta} I_a^b \tag{1.69} \]

\[ \exp \{ \theta \gamma^{(5)}_a^b \} = \cosh \theta I_a^b + \sinh \theta \gamma^{(5)}_a^b \tag{1.70} \]

\[ \exp \{ \theta K^A\delta^A_b \} = \cos \left( \frac{k \theta}{2} \right) \delta^A_b + \frac{2}{K} \sin \left( \frac{k \theta}{2} \right) K^A \delta^A_b \tag{1.71} \]

where

\[ K^A_{
abla} K^C_{\delta} = - K^2 \delta^A_{\delta} \]

\[ K^2 = - K^A_{\nabla} K^C_{A} \tag{1.72} \]

and so on.

Finally, we shall be concerned with describing motion along timelike trajectories using both dynamical and kinematical language.
We shall endeavour, as much as is possible, to restrict certain symbols for dynamical use only. For example, $P$ is used for momentum, $M$ and $m$ for mass or energy, $S$, $L$ and $J$ for angular momenta, $F$ for the electromagnetic field and a number of other common symbols. We use units where $c = 1$. Any deviation from the conventions stated above will hopefully be clearly noted and explained.
CHAPTER II
FRENET-SERRET FORMALISM

We consider an arbitrary timelike curve, \( C(s) \), in Riemannian space-time. We shall assume that the space-time is orientable at each point and therefore, following Geroch (1968), admits a tetrad structure of one timelike and three spacelike vectors. We denote the tetrad on \( C \) by \( e^\mu_{(\alpha)} \) which satisfies (1.11) - (1.15).

We shall denote points on \( C \) by \( X^\mu(s) \) where the parameter \( s \) measures the invariant 'length' along \( C \). The tetrad is locked onto the curve by demanding that the tangent vector

\[
e^\mu_{(\alpha)} = \frac{dX^\mu(s)}{ds} \quad e^\mu_{(\alpha)} \cdot e^\mu_{(\alpha)} = 1 \quad (2.1)
\]

which is timelike everywhere on \( C \).

The remainder of the tetrad is fixed by the Frenet-Serret equations

\[
\frac{d}{ds} e^\mu_{(\alpha)} = \kappa e^\mu_{(\alpha)} \quad (2.2a)
\]

\[
\frac{d}{ds} e^\mu_{(1)} = \kappa e^\mu_{(0)} + \tau_1 e^\mu_{(2)} \quad (2.2b)
\]

\[
\frac{d}{ds} e^\mu_{(2)} = -\tau_1 e^\mu_{(1)} + \tau_2 e^\mu_{(3)} \quad (2.2c)
\]
\[ \frac{D}{ds} e^\mu_{(s)} = -\tau_1 e^\mu_{(s)} \]  

where the derivative \( \frac{D}{ds} \) was defined by (1.18).

We shall refer to the triad vectors \( e^\mu_{(s)} \) \( (s=1, 2, 3) \) as the normal, binormal and trinormal respectively. The scalars \( \kappa, \tau_1 \) and \( \tau_2 \) are called the curvature and first and second torsion. Both \( \kappa \) and \( \tau_1 \) are taken to be positive and the sign of \( \tau_2 \) is fixed by the requirement that \( e^\mu_{(s)} \) be a right-handed triad, in agreement with the convention of Honig, et al., (1974).

Equation (2.2) is written sometimes in the shorthand form

\[ \frac{D}{ds} e^\mu_{(a)} = D^{\nu\lambda} e^\nu_{(a)} \]  

where the Darboux bivector \( D^{\nu\lambda} \) is defined as

\[ D^{\nu\lambda} = -2\kappa E^{\nu\lambda}_{(\omega\mu)} + 2\tau_1 E^{\nu\lambda}_{(\mu\lambda)} + 2\tau_2 E^{\nu\lambda}_{(\lambda\mu)} \]  

using (1.17). The Darboux bivector defines the generalized rotation plane of \( e^\mu_{(a)} \) along \( C \).

We shall demand that the tetrad be right-handed, hence \( e^\mu_{(a)} \) satisfies (1.19) and from (1.20) we find for the dual bivector

\[ D^{\nu\lambda} = 2\tau_2 E^{\nu\lambda}_{(\omega\mu)} + 2\tau_1 E^{\nu\lambda}_{(\mu\lambda)} + 2\kappa E^{\nu\lambda}_{(\lambda\mu)} \]  

The acceleration and twist, or angular velocity, of \( e^\mu_{(a)} \) relative to a Fermi-propagated frame are defined by the vector
projections
\[ a^\mu = D^\rho \nu \ e^\nu_{\omega_1} = \kappa e^{\nu}_{\omega_1} \]  \hspace{1cm} (2.6)

\[ \Omega^\mu = -\hat{D}^\rho \nu \ e^\nu_{\omega_1} = \tau_2 e^{\rho}_{\omega_1} + \tau_1 e^{\omega_1}_{\omega_1} \]  \hspace{1cm} (2.7)

From (2.6) - (2.7) it follows that
\[ \sqrt{a^2} = -a^\rho a^\rho = \kappa^2 \]  \hspace{1cm} (2.8)

\[ \Omega^2 = -\Omega^\rho \Omega^\rho = \tau_1^2 + \tau_2^2 \]  \hspace{1cm} (2.9)

From \( a \) and \( \Omega \) we define the pitch of the curve
\[ \beta = \frac{a}{\Omega} = \frac{\kappa}{\sqrt{\tau_1^2 + \tau_2^2}} \]  \hspace{1cm} (2.10)

The quantity \( \beta \) can be thought of as an invariant 'velocity' of the tetrad along \( \mathcal{C} \).

From \( D^\mu \nu \) and its dual we can form two invariants, namely
\[ \frac{1}{2} D^\rho \nu D^\nu \mu = \Omega^\rho \Omega^\mu - a^\rho a^\mu = \kappa^2 - \tau_1^2 - \tau_2^2 \]  \hspace{1cm} (2.11)
\[
\frac{1}{2} D^\mu \hat{D}^\nu \gamma = 2 \alpha_\mu \Omega^\mu = -2 \kappa \tau_2. \tag{2.12}
\]

We may classify the Darboux bivector using (1.4) - (1.5) and (2.10) - (2.11) by the properties:

\[
\frac{1}{2} D^\mu \hat{D}^\nu \gamma \left\{ \begin{array}{ll}
> 0 & \text{Timelike} \\
= 0 & \text{Null} \\
< 0 & \text{Spacelike}
\end{array} \right. \quad (2.13a)
\]

\[
\frac{1}{2} D^\mu \hat{D}^\nu \gamma \left\{ \begin{array}{ll}
\neq 0 & \text{Non-simple} \quad \kappa, \tau_2 \neq 0 \\
= 0 & \text{Simple} \quad \tau = 0
\end{array} \right. \quad (2.13b)
\]

We shall now derive the Frenet-Serret equations in spinor form. First we define the null-tetrad

\[
\sqrt{2} \ell^\mu = e^{\mu}_{(\text{co})} + e^{\mu}_{(\text{cs})}. \tag{2.14a}
\]

\[
\sqrt{2} m^\mu = e^{\mu}_{(1)} + i e^{\mu}_{(2)} \tag{2.14b}
\]

\[
\sqrt{2} \overline{m}^\mu = e^{\mu}_{(1)} - i e^{\mu}_{(2)} \tag{2.14c}
\]

\[
\sqrt{2} \eta^\mu = e^{\mu}_{(\text{co})} - e^{\mu}_{(\text{cs})} \tag{2.14d}
\]

Substituting (2.14) in (2.2) we find
\[
\frac{\mathcal{D}}{ds} \bar{\ell}^\mu = \frac{1}{2} (\kappa + i\tau_3) \bar{\ell}^\mu + \frac{1}{2} (\kappa - i\tau_3) \bar{\eta}^\mu
\] (2.15a)

\[
\frac{\mathcal{D}}{ds} \bar{\eta}^\mu = \frac{1}{2} (\kappa - i\tau_3) \bar{\eta}^\mu + \frac{1}{2} (\kappa + i\tau_3) \bar{\ell}^\mu
\] (2.15b)

\[
\frac{\mathcal{D}}{ds} \bar{\eta}^\mu = \frac{1}{2} (\kappa - i\tau_3) \bar{\eta}^\mu + \frac{1}{2} (\kappa + i\tau_3) \bar{\eta}^\mu
\] (2.15c)

where we have suppressed the equation for \(\bar{m}^\mu\) since it is simply the complex conjugate of (2.15b). The derivative is still along \(\mathcal{C}^\mu\), or \(\mathcal{K}^\mu + \eta^\mu\).

The choice of (2.14) is a matter of convenience. We use the null tetrad to facilitate the transformation of (2.2) into spinor form. The choice of (2.14) was made originally to reflect the symmetry between \(\mathcal{C}^\mu_{(e)}\) and \(\mathcal{C}^\mu_{(3)}\) and between \(\mathcal{C}^\mu_{(e)}\) and \(\mathcal{C}^\mu_{(3)}\) in (2.2).

We found henceforth that this choice also gives the simplest expression of results.

We now introduce unit spinors \(\xi^A, \eta^A\) satisfying

\[
\xi_A^A \eta_A^A = 1
\] (2.16)

and the mapping functions \(\sigma^{A\bar{A}}\) defined by (1.44). Multiplying (2.15) by \(\sigma^{A\bar{A}}\) and using (1.45) we find...
\[
\frac{D \xi^A \xi^B}{ds} + \frac{\xi^A D \xi^B}{ds} = \left( \frac{\kappa + i \tau_2}{2} \right) \xi^A \xi^B + \left( \frac{\kappa - i \tau_2}{2} \right) \eta^A \xi^B
\]  
(2.17a)

\[
\frac{D \xi^A \eta^B}{ds} + \frac{\xi^A D \eta^B}{ds} = \left( \frac{\kappa + i \tau_2}{2} \right) \xi^A \eta^B + \left( \frac{\kappa - i \tau_2}{2} \right) \eta^A \eta^B - i \tau_2 \xi^A \eta^B
\]  
(2.17b)

\[
\frac{D \eta^A \xi^B}{ds} + \frac{\eta^A D \xi^B}{ds} = \left( \frac{\kappa - i \tau_2}{2} \right) \eta^A \xi^B + \left( \frac{\kappa + i \tau_2}{2} \right) \eta^A \eta^B + i \tau_2 \eta^A \xi^B
\]  
(2.17c)

\[
\frac{D \eta^A \eta^B}{ds} + \frac{\eta^A D \eta^B}{ds} = \left( \frac{\kappa - i \tau_2}{2} \right) \eta^A \eta^B + \left( \frac{\kappa + i \tau_2}{2} \right) \eta^A \eta^B
\]  
(2.17d)

Multiplication of (2.17a) by \( \eta^B \) and (2.17b) by \( \xi^B \) and adding the

two results gives

\[
\frac{D \xi^A}{ds} = -i \tau_2 \xi^A + \frac{(\kappa - i \tau_2)}{2} \eta^A
\]  
(2.18)

Proceeding similarly with (2.17c) and (2.17d) we find for \( \eta^A \)

\[
\frac{D \eta^A}{ds} = i \tau_2 \eta^A + \frac{(\kappa - i \tau_2)}{2} \xi^A
\]  
(2.19)
Equations (2.18) and (2.19) with the complex conjugate equations are the 2-component spinor Frenet-Serret equations.

Regarding the derivative \( \frac{D}{ds} \) we find that

\[
\frac{D}{ds} = \frac{i}{\sqrt{2}} \left( \xi^A \dot{\eta}^A + \eta^A \dot{\xi}^A \right) \nabla_{\dot{\theta}A}
\]  

(2.20)

where the spinor planar derivative \( \nabla_{\dot{\theta}A} \) is related to \( \nabla_\mu \) defined by (1.40) (eg. Rzewuski 1958)

\[
\nabla_{\dot{\theta}A} = \frac{i}{\sqrt{2}} \sigma^\mu \nabla_\mu
\]  

(2.21)

We can write (2.18) - (2.19) in bispinor form using (1.58) - (1.62) and (1.65)

\[
\frac{D}{ds} \chi^a = -\frac{i}{2} \chi^a + \frac{1}{2} \left\{ K I_{ba}^{ab} - i \tau_2 \chi^{(s)ab} \right\} \psi^b 
\]  

(2.22a)

\[
\frac{D}{ds} \psi^a = -\frac{i}{2} \chi^a + \frac{1}{2} \left\{ K I_{ba}^{ab} - i \tau_2 \chi^{(s)ab} \right\} \psi_b
\]  

(2.22b)

where

\[
\chi_a = \chi^b I_{ba} \quad \psi^a = I^{ab} \psi_b
\]  

(2.23)
The ordering of the bispinors in (2.22) and (2.23) is not important because of the summation convention. However, it is useful to place summed indices adjacent to each other in the configuration shown for calculation purposes.

In Minkowski space (2.22) are identical to equations derived by Gursey (1957). Therefore, (2.22) are the generalization of those equations to Riemann space. The method we have used to derive (2.22) is much simpler than that used by Gursey which necessitated lengthy algebra and is much less transparent than our derivation.

The spinor forms of the Frenet-Serret equations are formally simpler than the vector form. On occasion they may be much easier to solve.

It is often convenient to express (2.18) - (2.19), or (2.22), in a shorthand form involving the Darboux bivector. It is easy to show, by substitution, that

\[ \frac{d}{ds} \xi^A = i D^A \epsilon \xi^\epsilon \]
\[ \frac{d}{ds} \eta^A = i D^A \epsilon \eta^\epsilon \]

(2.24)

where, from (1.54) - (1.55)

\[ \frac{1}{2} \frac{d}{d\nu} \sigma^{\epsilon \epsilon_A \delta} \sigma^{\nu \nu \epsilon} = i \epsilon^{AC} D^\epsilon - i D^A \epsilon \delta \]

(2.25)

\[ D^{AC} = \frac{1}{2}(\xi^A \eta^C + \eta^A \xi^C) - i (\kappa - i \tau_2)(\xi^A \eta^C - \eta^A \xi^C) \]

(2.26)
We see that $D^{AC}$ is just the spin-space image of $D^\mu\nu$ hence we shall call $D^{AC}$ the Darboux spinor.

Similarly, using the bispinors, we find

$$\frac{d\psi_a}{ds} = i D_a^b \psi_b, \quad \frac{d\chi^a}{ds} = -i \chi^b D_b^a$$ \hspace{1cm} (2.27)

where

$$D_a^b = -\frac{i}{4} D_{\mu\nu} \gamma^\mu_{a c} \gamma^\nu_{c b} = \begin{bmatrix} D_{ab} & 0 \\ 0 & D_{ba} \end{bmatrix}$$ \hspace{1cm} (2.28)

is the Darboux bispinor.

We note the results, using (1.70) and (1.72),

$$D^c \psi^b = \frac{\omega^2}{4} \exp \left[ -2i \rho \sigma^{ab} \right]$$ \hspace{1cm} (2.29a)

$$D^c \psi^b = \frac{\omega^2}{4} \exp \left[ -2i \rho \sigma^{ab} \right]$$ \hspace{1cm} (2.29b)

where

$$\omega^2 \cos 2\rho = \tau_1^2 + \tau_2^2 - \kappa^2 = \frac{1}{2} D_{\mu\nu} D^{\mu\nu}$$ \hspace{1cm} (2.30a)
\[ \omega^2 \sin 2\rho = 2\kappa \tau_2^1 = \frac{i}{2} D_{\nu \mu} \nabla^{\mu} \] (2.30b)

From (2.30) it follows that

\[ \omega^4 = \left\{ \left( \tau_1^2 + \tau_2^1 - \kappa^2 \right)^2 + 4\kappa^2 \tau_1^2 \right\} \] (2.31a)

\[ \tan 2\rho = \frac{2\kappa \tau_1^2}{\tau_1^2 + \tau_2^1 - \kappa^2} \] (2.31b)

Now we shall examine the spinors in Minkowski space when the Darboux bivector is constant along the curve C. This represents the simplest non-trivial case which can be completely solved. From the constancy of \( D^{\mu \nu} \), namely

\[ \frac{1}{2} \frac{d}{ds} D^{\mu \nu} = - \frac{dx}{ds} \varepsilon^{\mu \nu \alpha \beta} + \frac{dx_1}{ds} E^{\mu \nu}_{\alpha \beta} + \frac{dx_2}{ds} E^{\mu \nu}_{\alpha \beta} = 0 \] (2.32)

it follows that each Frenet scalar is constant. It should be noted that as a consequence of using the Frenet-tetrad the first and second terms of (2.32) are equal in general. Also, with (1.48), it follows that the Darboux spinors are constant. Thus, by repeating the differentiation of (2.24) we find the second order linear equations

\[ \frac{D^2}{ds^2} \psi^A = -\frac{\nu^2}{4} \psi^A \] \[ \frac{D^2}{ds^2} \eta^A = -\frac{\nu^2}{4} \eta^A. \] (2.33)
where the complex frequency scalar $\nu$ is simply

$$\nu = \omega e^{-\beta}$$

This yields the well-known helical solutions in a simple way.

We write solutions to (2.33) in the form

$$\xi^A(s) = \exp\left\{\frac{i\omega s}{2} e^{i\beta}\right\} \xi^A + \exp\left\{-\frac{i\omega s}{2} e^{i\beta}\right\} \overline{\xi}^A$$

(2.35a)

$$\eta^A(s) = \exp\left\{\frac{i\omega s}{2} e^{i\beta}\right\} \eta^A + \exp\left\{-\frac{i\omega s}{2} e^{i\beta}\right\} \overline{\eta}^A$$

(2.35b)

where $\xi^A$ and $\eta^A$ are constant spinors which are related to the spinors $\xi^A(o)$ and $\eta^A(o)$ by, setting $D^A_B(o) = D^A_B$,

$$\xi^A = \frac{1}{2} \left\{ \xi^A_B + \frac{2}{\nu} D^A_B \right\} \xi^B(o)$$

(2.36a)

$$\eta^A = \frac{1}{2} \left\{ \xi^A_B - \frac{2}{\nu} D^A_B \right\} \eta^B(o)$$

(2.36b)

$$\overline{\xi}^A = \frac{1}{2} \left\{ \xi^A_B + \frac{2}{\nu} D^A_B \right\} \overline{\xi}^B(o)$$

(2.36c)

$$\overline{\eta}^A = \frac{1}{2} \left\{ \xi^A_B - \frac{2}{\nu} D^A_B \right\} \overline{\eta}^B(o)$$

(2.36d)
The $s=0$ spinors are related by

$$\gamma^A(0) = \frac{i}{\kappa - i \tau_2} \left\{ \gamma^A + \frac{2}{\tau_1} D^A B \right\} \gamma^B(0) \tag{2.37}$$

To clarify what we mean by the spinors $\gamma^A(0)$ and $\gamma^B(0)$ we shall assume that for $s<0$ the curve is geodesic, that is $\kappa = \tau_1 = \tau_2 = 0$. We then map the geodesic spinors into the initial ($s=0$) spinors by the requirement that $\gamma^A(s=0; 0, 0) = \gamma^A(s=0; \kappa, \tau_1, \tau_2)$. Thus the $\gamma^A(0)$ and $\gamma^B(0)$ spinors serve as constant reference spinors along the curve.

We can also write the solutions directly from (2.24) in the form, for $\gamma^A(0)$ only,

$$\gamma^A(s) = \exp\left\{ is D^A B \right\} \gamma^A(0) \tag{2.38}$$

where

$$\exp\left\{ \pm is D^A B \right\} = \sum_{n=0}^{\infty} \frac{(\pm is)^n}{n!} (D^A B)^n$$

$$= \cos(\frac{vs}{2}) \delta^A_B \mp \frac{i}{v} \sin(\frac{vs}{2}) D^A B \tag{2.39}$$

In the form (2.38) we see that the spinor solution of $\gamma^A$ when all Frenet scalars are zero, that is, for geodesics, is just the 'initial' spinor $\gamma^A(0)$. The value of $\gamma^A(0)$ is just a unitary transformation, corresponding to the Darboux bivector, multiplying the 'geodesic' spinor $\gamma^A(0)$. From (2.39) we see that this is equivalent to a 'boost' along $\gamma^A(0)$ and a rotation in the spin plane $D^A B$.

We can express solutions for the bispinors in a similar fashion although the reduction of the analogous exponential bispinor is quite complicated in general compared to the simple result (2.39).
There is another shorthand method of writing the Frenet-Serret equations (2.18) - (2.19), namely

\[ -i \frac{D}{ds} \xi^A = \Pi^{\dot{A}B} \eta^B - i \frac{D}{ds} \eta^A = \Pi^{AB} \xi^B \tag{2.40} \]

By expanding \( \Pi^{AB} \) in its most general form and substituting in (2.18) - (2.19) we find

\[ \Pi^{\dot{A}B} = \frac{c}{\lambda} (\xi^\dot{A} \xi^B + \eta^A \eta^B) - \frac{i}{\lambda} \eta^C (\xi^A \eta^\dot{B} - \eta^A \xi^\dot{B}) \tag{2.41} \]

Multiplying (2.41) by \( \sigma^\mu_{\delta A} \) we find that \( \Pi^{\dot{A}B} \) corresponds to the vector, using (1.57)

\[ \Pi^\mu = \frac{i}{\sqrt{2}} \sigma^\mu_{\delta A} \Pi^{\dot{A}B} = \frac{c}{\lambda} e^\mu_{(e)} + \frac{i}{\lambda} \eta^C (\xi^A \eta^\dot{B} - \eta^A \xi^\dot{B}) \tag{2.42} \]

This vector is further found to be simply the projection of the trinormal vector \( e^\mu_{(e)} \) onto the anti-self-dual Darboux bivector, as seen from

\[ \frac{i}{2} \left( \mathcal{D}_\mu + i \mathcal{D}^\mu \right) e^\nu_{(e)} = -i \Pi^\mu \tag{2.43} \]

The bispinor forms of (2.40) and (2.41) are
\[-i \frac{d}{ds} \Psi_a = \prod_a^b \Psi_b \quad i \frac{d}{ds} \chi^a = \chi^b \prod_b^a\]  

(2.44)

where

\[\prod_a^b = \begin{bmatrix} \hat{0} & \Pi^{A_b} \\ \Pi^A_{\delta A} & 0 \end{bmatrix}\]  

(2.45)

The \(\prod^{A_b}\), and therefore the \(\prod_a^b\), are not Hermitian in general which we 'see' immediately from the expression

\[-\prod_a^b = \frac{1}{2 \sqrt{2}} \left\{ \tau_1 e^{\mu(\delta)} + \chi e^{\mu(\gamma)} \right\} \gamma_{\delta B} - \frac{i \tau_2}{2 \sqrt{2}} e^{\mu(\delta)} \gamma_{a^{(5)}} \gamma_{\delta c} \gamma^\mu b\]  

(2.46)

A bar over \(\prod_a^b\) denotes changing \(\alpha \leftrightarrow \beta\) in (2.46).

The \(\prod^{A_b}\) and \(\prod_a^b\) satisfy

\[-\prod^{A_b} \prod^{A_{\delta}} = \frac{1}{4} \delta^A_{\delta}\]  

(2.47a)

\[-\prod_a^b \prod_b^c = \frac{1}{4} \exp \left\{ -2i \rho \delta_{a^{(c)}} \right\}\]  

(2.47b)

Thus, the squares of the \(\prod\) spinors are seen from (2.29) to be just the
same as the squares of the Darboux spinors. In general \( \Pi^{ab} \) is not constant when all Frenet scalars are constant. In fact, we find on differentiating (2.46), with \( \kappa, \tau, \) and \( \tau_2 \) constant, that

\[
\frac{d}{ds} \Pi_a^b = -\tau_2 \left\{ \gamma_{a}^{\mu} \gamma_{b}^{\nu} + i \sigma_{a}^{\mu} \gamma^{(s)}_{a} \gamma_{c}^{\nu} \right\} \tag{2.48}
\]

where \( \gamma^\mu \) and \( \sigma^a \) were defined in (2.6) and (2.7). As we see from (2.48) the spinor \( \Pi_{a}^b \), and also \( \Pi^{ab} \), is constant only if \( \tau_2 = 0 \). In this case it follows from (2.46), or (2.41), that \( \Pi_{a}^b \) is Hermitean.

Just as the Darboux bivector represents the generalized rotation plane of the Frenet tetrad on the curve \( C(s) \), the vector \( \Pi^a \) describes the generalized rotation axis of the tetrad. This is best illustrated when \( \tau_2 = 0 \), or, as noted above, when \( \Pi^a \) is a constant vector. In this case the trinormal is also constant, as seen from (2.2d). The curve is restricted to a 3-dimensional subspace orthogonal to \( \sigma^a \), and has \( \Pi^a \) as a constant axial vector. Thus the curve is a space-time helix with pitch \( \beta = \frac{\kappa}{\tau} \) from (2.10).

In Minkowski space, when \( \kappa \) and \( \tau \) are constant, we can write bispinor solutions in the form, using \( \psi_a \) only,

\[
\psi_a(s) = \exp \left\{ is \Pi_a^b \right\} \psi_b(0)
\]

\[
= \left\{ \cos \frac{\omega s}{2} \Pi_a^b + \frac{2i}{\omega} \sin \frac{\omega s}{2} \Pi_a^b \right\} \psi_b(0) \tag{2.49}
\]
where

\[ \mathcal{I}^a \cdot \mathcal{I}^b = \frac{1}{\sqrt{2}} \mathcal{I}^c \mathcal{I}^d \mathcal{G}^{\mu \nu} \mathcal{I}^a \cdot \mathcal{I}^b \]

\[ \mathcal{I}^{\mu \nu} = \frac{\tau_\gamma}{\kappa} e^\gamma_{\nu} + \kappa e^\mu_{\mu} \]  \hspace{1cm} (2.50a)

\[ \omega^2 = \tau_\gamma^2 - \kappa^2 \]  \hspace{1cm} (2.50b)

The bispinor \( \psi_{\alpha}^{(s)} \) is the value of the bispinor \( \psi_{\alpha}^{(s)} \) for the case \( \kappa = \tau_\gamma = 0 \) corresponding to a geodesic curve. The unitary transformation (2.49) thus represents a boost and a rotation about the axis \( \mathcal{I}^{\mu \nu} \). Furthermore, (2.49) formally resembles the Foldy-Wouthysen transformation used in the one particle Dirac equation.

The curve types described by (2.49) - (2.50) can be categorized as follows (e.g. Synge 1937):

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \mathcal{I}^{\mu \nu} )</th>
<th>( \mathcal{D}^{\mu \nu} ) (Simple)</th>
<th>Helix Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega =</td>
<td>\omega</td>
<td>)</td>
<td>Timelike</td>
</tr>
<tr>
<td>( \omega = 0 )</td>
<td>Null</td>
<td>Null</td>
<td>Parabolic</td>
</tr>
<tr>
<td>( \omega = i</td>
<td>\omega</td>
<td>)</td>
<td>Spacelike</td>
</tr>
</tbody>
</table>
CHAPTER III

RELATIVISTIC DYNAMICS OF A POINT CHARGE

We describe the motion of a point charge, \( e \), of mass \( m \) in an electromagnetic field \( F^{\mu\nu} \) by the Lorentz equation

\[
\frac{du^\mu}{ds} = \frac{e}{m} F^{\mu\nu} u^\nu ; \quad u_\mu u^\mu = 1
\]  

(3.1)

where \( u^\mu \) is the 4-velocity and unit tangent vector of the charge trajectory.

We shall consider a field of the form

\[
F^{\mu\nu} = f(s) F^{\mu\nu}_H , \quad \frac{df}{ds} F^{\mu\nu}_H = 0 , \quad f(0) = 1
\]  

(3.2)

where \( f(s) \) is a scalar function of the proper distance and \( F^{\mu\nu}_H \) is a static tensor field.

Setting \( u^\mu = e^\mu_\omega \) and using the Frenet-Serret formalism we find from (2.2)

\[
\frac{de^\mu_\omega}{ds} = \kappa e^\mu_\omega = \frac{e}{m} f F^{\mu\nu}_H e^\nu_\omega
\]  

(3.3)

Differentiating once more with (3.2) we find, assuming \( \kappa \neq 0 \),

\[
\frac{de^\mu_\omega}{ds} - \frac{e}{m} F^{\mu\nu}_H e^\nu_\omega = - \left\{ \frac{1}{\kappa} \frac{df}{ds} + \frac{f}{s} \frac{df}{ds} \right\} e^\mu_\omega
\]  

(3.4)

Since the two sides of (3.4) are linearly independent it follows that

\[
\frac{de^\mu_\omega}{ds} = \frac{e}{m} F^{\mu\nu}_H e^\nu_\omega
\]  

(3.5)

where

\[
k^2 = - \left( \frac{e}{m} \right)^2 E^\mu E_\mu , \quad k(s) = f(s) k_0
\]  

(3.6)
and

\[ E^\mu = F^\mu_\nu \epsilon^\nu = \tilde{F}^\mu_\nu \epsilon^\nu_{(\alpha)} \]  

(3.7)

is the local electric projection of the field.

Proceeding similarly we find

\[ \frac{D}{ds} e^\mu_{(\alpha)} = e^{\mu}_{(a)} F^\mu_\nu \epsilon^\nu_{(a)} \]  

(3.8)

for all Frenet vectors. Thus the tetrad is also a Michel tetrad (Fierz and Telegdi 1970). The Frenet scalars are given by

\[ \tau_1^2 = \left( \frac{e}{m} \right)^2 \frac{R^\mu_\kappa R_\mu^\kappa}{E^\mu E_\kappa} \quad \tau_2^2 = -\left( \frac{e}{m} \right)^2 \frac{E^\mu H_\mu}{E^\kappa E_\kappa} \]  

(3.9)

where

\[ H^\mu = \hat{F}^\mu_\nu \epsilon^\nu_{(\alpha)} \]  

(3.10)

\[ R^\mu = \epsilon^{\mu_\nu_\rho_\kappa} E_\nu H_\rho \epsilon^\kappa_{(\alpha)} \]  

(3.11)

We refer to the vectors \( \hat{H}^\mu \) and \( R^\mu \) as the local magnetic and 'Poynting' projections. In the instantaneous rest frame of the charge, \( U^\mu = \delta^\mu_0 \), the expressions for the Frenet scalars are given by the three-dimensional expressions, following Honig, et al., (1974)

\[ \kappa = \left| \frac{e}{m} \right| \left| \vec{E} \right| \]  

(3.12a)

\[ \tau_1 = \left| \frac{e}{m} \right| \frac{\left| \vec{E} \times \vec{H} \right|}{\left| \vec{E} \right|} \]  

(3.12b)
\[ \xi_\mu = - \frac{|e|}{m} \cdot \frac{\vec{E} \cdot \vec{H}}{|E|} \quad (3.12c) \]

where the minus sign in (3.12c) arises from the requirement that the tetrad be right-handed.

Comparing the expressions (3.8) and (2.3) we see that the Darboux bivector is related to the field by

\[ D^{\mu \nu} = \frac{e}{m} F^{\mu \nu} = \frac{e}{m} f(s) F^{\mu \nu}_H \quad (3.13) \]

Thus, from (2.4), (2.6) and (2.7) we see that

\[ \frac{e}{m} E^\mu = a^\mu = \kappa e^\mu_{(1)} \quad (3.14a) \]

\[ \frac{e}{m} H^\mu = - \Omega^\mu = - \tau_2 e^\mu_{(1)} - \tau_1 e^\mu_{(2)} \quad (3.14b) \]

\[ (\frac{e}{m})^2 R^\mu = - \kappa \tau_1 e^\mu_{(2)} \quad (3.14c) \]

The two field invariants are expressed by

\[ \frac{1}{2} \left( \frac{e}{m} \right)^2 F^{\mu \nu} F_{\mu \nu} = \left( \frac{e}{m} \right)^2 \left\{ E^\mu E_{\mu} - H^\mu H_{\mu} \right\} = \tau_1^2 + \tau_2^2 - \kappa^2 \quad (3.15a) \]

\[ \frac{1}{2} \left( \frac{e}{m} \right)^2 F^{\mu \nu} F_{\mu \nu} = 2 \left( \frac{e}{m} \right)^2 E_{\mu} H^\mu = 2 \kappa \tau_2 \quad (3.15b) \]

From (3.8) and (2.24) we can define spinors \( \xi^A \) and \( \eta^A \) as
in the second chapter, satisfying

\[
\frac{D}{ds} \xi^A = \frac{i e}{m} F^A_{\ e} \xi^e \quad \frac{D}{ds} \eta^A = \frac{i e}{m} F^A_{\ e} \eta^e
\]  

(3.16)

These equations are a special case of those derived by Plebanski (1966). We show them here to demonstrate the ease by which they may be obtained using the Frenet-Serret formalism.

If we rescale the distance parameter

\[
dt = f(s) \ ds
\]  

(3.17)

then, differentiating (3.16) with (3.17), we find

\[
\frac{d^2}{dt^2} \xi^A = -\frac{\nu^A}{4} \xi^A \quad \frac{d^2}{dt^2} \eta^A = -\frac{\nu^A}{4} \eta^A
\]  

(3.18)

where, from (2.29) - (2.31) and (2.33) - (2.34),

\[
\nu = \omega e^{-\lambda t}
\]  

(3.19a)

\[
\omega^4 = \frac{f_{\ x}^{-1}}{4} \left( \frac{e}{m} \right)^2 \left\{ \left( F_{\ mu} F^{\ mu} \right)^2 + \left( F_{\ mu} \hat{F}^{\ mu} \right)^2 \right\}
\]  

(3.19b)

\[
\tan 2\rho = \frac{F_{\ \mu \nu} \hat{F}^{\ \mu \nu}}{2 F_{\ \mu \nu} F^{\ \mu \nu}}
\]  

(3.19c)

We can then proceed to write solutions in Minkowski space in terms of the rescaled parameter \( t \) just as we did for (2.35) - (2.39).

We shall not repeat these details here, however.
It is to be noted that when the scalar function $f$ is constant ($f(x) = 1$) then all our results go over to results of Honig, Schucking and Vishveshwaro (1974).

If the field (3.2) is a plane-wave radiation field (linear polarized) characterized by the field plane $\hat{F}_{\lambda}^{\mu\nu}$ and $f(x) = f(k_x)$ where the propagation vector $k_\mu$ satisfies

$$k_\mu k_\mu = 0, \quad k_\mu F^{\mu\nu} = 0, \quad k_\nu \nabla_\nu k_\mu = 0 \quad (3.20)$$

from Maxwell's equations, then the motion is particularly simple. By definition the field invariants for a radiation field are zero

$$F^{\mu\nu} F_{\mu\nu} = F^{\mu\nu} \hat{F}_{\mu\nu} = 0 \quad (3.21)$$

hence, from (2.11) and (2.12) it follows that

$$\tau_2 = 0, \quad \kappa = \tau_1 \quad (3.22)$$

The motion is therefore a parabolic helix.

Multiplying (3.8) by $k_\mu$ and using (3.20) and (3.22) it follows that $k_\mu$ can be written as

$$k_\mu = k_o \left( e_\mu^{(n)} + e_\mu^{(e)} \right) \quad (3.23)$$

Comparing (3.23) with (2.50) we see that $k_\mu$ is collinear with the helix axis $\Pi_\mu$ defined previously.
CHAPTER IV

RELATIVISTIC DYNAMICS OF A POINT DIPOLE

We describe the motion of a 'freely-moving' spinning point particle having spin \( S \) and total energy by the Frenkel (1926) - Thomas (1926, 1927) equations

\[
\frac{d}{ds} P^\mu = 0 \tag{4.1}
\]

\[
\frac{d}{ds} S^{\mu\nu} = P^\mu u^\nu - u^\mu P^\nu \tag{4.2}
\]

\[
S^\mu_\nu u^\nu = 0 \quad u_\mu u^\mu = 1 \tag{4.3}
\]

where \( P^\mu \) is the energy-momentum 4-vector satisfying

\[
m^2 = P_\mu P^\mu \tag{4.4}
\]

\( S^{\mu\nu} \) is the spin tensor, \( u^\mu \) the 4-velocity and tangent to the trajectory.

The 'helicity', or spin scalar, \( S_0 \) is defined as

\[
S_0^2 = \frac{1}{2} S_\mu S^{\mu\nu} > 0 \tag{4.5}
\]
and, from (4.2) - (4.3), is conserved. The local projection of $P^\alpha$ in the velocity frame is also a mass:

$$M = u^\nu P_\nu$$  \hspace{1cm} (4.6)

which, in general, need not be equal to $m$. Hence, we assume, in general, that $P^\alpha$ and $u^\nu$ are non-collinear. In the velocity rest frame, $u^\mu = S^\mu$, $M$ is the proper energy of the particle.

In the case where $P^\alpha = M u^\alpha = m u^\alpha$, it follows that $S^\mu$ is constant hence the particle follows a geodesic and the intrinsic frame of the particle undergoes constant rotation in the plane orthogonal to the trajectory and to the spin axis defined by

$$S^\mu = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} S_{\nu\alpha} u_\beta$$

$$S_{\mu} S^\mu = - S^2$$  \hspace{1cm} (4.7)

which satisfies, from (4.1) - (4.3)

$$\frac{D}{ds} S^\mu = 0$$  \hspace{1cm} (4.8)

The equations (4.1) - (4.3) and (4.8) are rather simple to solve and yield oscillatory solutions when $P^\alpha \neq M u^\alpha$. We shall use the Frenet-Serret formalism to demonstrate the ease with which solutions can be obtained. This method is similar to one used by Ellis (1975) who solved the same equations.

We consider the tetrad projections setting $u^\alpha = e^\alpha_{(\omega)}$ and,
taking into account (4.3) and (4.6)

\[ P^\mu = M e^{\lambda}_{(\lambda)} - P_{(1)} e^{(1)}_{(1)} - P_{(2)} e^{(2)}_{(2)} - P_{(3)} e^{(3)}_{(3)} \]  (4.9a)

\[ S^{\mu\nu} = S_{(12)} E^{(12)}_{(12)} - S_{(23)} E^{(23)}_{(23)} + S_{(13)} E^{(13)}_{(13)} \]  (4.9b)

where we have defined \( S_{(12)} = S_{(1212)} \), \( S_{(23)} = -S_{(2323)} \) and \( S_{(13)} = S_{(1313)} \).

Substituting (4.9) in (4.1) and (4.2) we find the ten scalar equations:

\[ \frac{d}{ds} M = \kappa P_{(1)} \]  (4.10a)

\[ \frac{d}{ds} P_{(1)} = \kappa M + \tau_1 P_{(23)} \]  (4.10b)

\[ \frac{d}{ds} P_{(2)} = -\tau_1 P_{(1)} + \tau_2 P_{(3)} \]  (4.10c)

\[ \frac{d}{ds} P_{(3)} = -\tau_2 P_{(2)} \]  (4.10d)

\[ \frac{d}{ds} S_{(3)} = -\tau_2 S_{(23)} \]  (4.10e)
\[
\frac{d}{ds} S_{(\tau)} = -\tau_1 S_{(\kappa)} + \tau_2 S_{(\tau)} \quad (4.10f)
\]

\[
\frac{d}{ds} S_{(\kappa)} = \tau_1 S_{(\tau)} \quad (4.10g)
\]

\[
P_{(\tau)} \equiv 0 \quad , \quad P_{(\kappa)} = \kappa S_{(\tau)} \quad , \quad P_{(\tau)} = -\kappa S_{(\tau)} \quad (4.10h)
\]

It should be noted that any tensor equation can be resolved into scalar components by choosing a particular tetrad. The beauty of the Frenet tetrad is that the number of arbitrary curvature scalars is minimized.

In addition to the trivial solutions in which \( \kappa = \tau_1 = \tau_2 = 0 \) which imply in turn \( \omega = M \kappa \) (M) we also have solutions corresponding to the case

\[
\tau_2 = 0 \quad , \quad \kappa, \tau_1 \neq 0 \quad , \quad \frac{d}{ds} \kappa = \frac{d}{ds} \tau_1 = 0 \quad (4.11)
\]

It may be noted that these are the only solutions which make any sense in the context of the Frenet-Serret formalism. From (4.10) it follows that

\[
P_{(\kappa)} = P_{(\tau)} = 0 \quad , \quad S_{(\kappa)} = S_{(\tau)} = 0 \quad (4.12)
\]

and, therefore \( \omega^\mu \) and \( S^{\mu\nu} \) have the forms
\[ p^\mu = m e^{\mu}_{\alpha_3} + m e^{\mu}_{\epsilon_1} \frac{\tau_1}{\epsilon_1} \] (4.13)

\[ s^\mu = -s_0 e^{\mu}_{\alpha_3} \quad \quad s^\mu = s_0 e^{\mu}_{\epsilon_1} \] (4.14)

Comparing (4.13) with (2.42) we see that \( p^\mu \) is collinear with \( \Pi^\mu \), while from Table I we see that, since, by (4.13),

\[ \frac{M}{S_0} = \tau_1 > \kappa \] (4.15)

the trajectory is a circular helix of pitch

\[ \beta = \frac{\kappa}{\epsilon_1} < 1 \] (4.16)

and axis \( p^\mu \) and spin axis \( s^\mu \).

From (4.4) and (4.13) we find

\[ m^2 = M^2 (1 - \beta^2) \] (4.17)

Defining the frequency

\[ \omega = (\tau_1^2 - \kappa^2)^{1/2} \] (4.18)
we find from (4.15) and (4.17)

$$\omega = \frac{\mathbf{m}}{\mathbf{S}_e} = \frac{\mathbf{m}}{\mathbf{M}} \tau, \quad \tau = \frac{\mathbf{m}}{\mathbf{M}} \tau. \quad (4.19)$$

With (4.19) in (4.13) then

$$P^\mu = \frac{\mathbf{m}}{\omega} \left\{ \tau, e^\mu_{(\nu)} + \nu e^\mu_{(\nu)} \right\} = 2S_e \Pi^\mu. \quad (4.20)$$

The motion described above has been suggested by several authors (eg. Corben 1968) to be a classical analogue of the quantum mechanical 'Zitterbewegung'.

This motion is interesting in spinor description. First, we define new coordinates by

$$\rho^\mu = -\frac{1}{\mathbf{m}^2} S^\mu \sqrt{\mathbf{P}^\nu = -\nu e^\mu_{(\nu)}} \quad (4.21)$$

which describe the deviation of the trajectory from a trajectory parallel to $P^\mu$, that is for $\nu = 0$. We find from (4.2)

$$\frac{d\rho^\mu}{d\delta} = -\frac{\mathbf{m}}{\mathbf{m}^2} P^\mu + \upsilon^\mu. \quad (4.22)$$

hence the position vector of the particle is given in Minkowski space, up to constant translations, by
\[ X^\kappa (s) = \frac{m}{\kappa} P^\kappa + \gamma^\kappa (s) \] (4.23)

We find for the orbital and total angular momenta

\[ L^{\mu \nu} = X^\mu P^\nu - P^\mu X^\nu \]
\[ = \kappa \frac{m}{\omega} \left\{ \tau_i E^{\mu \nu}_{(e)} - k E^{\mu \nu}_{(s)} \right\} \] (4.24)

\[ M^{\mu \nu} = L^{\mu \nu} + S^{\mu \nu} = -S^i \tau_i D^{\mu \nu} \] (4.25)

where we see that the rotation plane of the Frenet tetrad is proportional to the total angular momentum. It also follows that

\[ M^\mu \cdot P^\nu = 0, \quad M^\mu \cdot S^\nu = 0 \] (4.26)

or, simply

\[ M^{\mu \nu} = -\frac{\tau_i}{m \omega} \epsilon^{\mu \nu \kappa \rho} S_\kappa P_\rho \] (4.27)

Now, using (2.44) - (2.50) and our \( P^\mu \) vector, we write for the bispinors
\[-iS_a \frac{D\psi_b}{ds} = \mathcal{P}_a \psi_b, \quad iS_a \frac{DX^a}{ds} = \chi^b \mathcal{P}_b \chi^a \quad (4.28)\]

These equations are formally similar to the quantum mechanical Dirac equations for a 'free' particle.

In Minkowski space we can express the solutions of (4.28) in the form

\[
\psi_a(x^s(s)) = e^{-i\frac{P_a \chi^s}{2S_a} P_a} + e^{-i\frac{P_a \chi^s}{2S_a} P_a} \quad (4.29a)
\]

\[
\chi^a(x^s(s)) = e^{-i\frac{P_a \chi^s}{2S_a} \Lambda_{\psi}^a} + e^{i\frac{P_a \chi^s}{2S_a} \Lambda_{\psi}^a} \quad (4.29b)
\]

where, from (4.23)

\[
\frac{1}{2S_a} P_a \chi^s(s) = \frac{1}{2S_a} \left( {m \over M} \right) P_a \chi^s(s) = \frac{\omega s}{2} \quad (4.30)
\]

and the constant bispinors

\[
\mathcal{P}_a^{(\pm)} = \frac{1}{\sqrt{2}} \left( \mathcal{I}_a \right)^b \pm \frac{2}{m} \mathcal{P}_a \mathcal{P}_b \psi_b^{(0)} \quad (4.31a)
\]
\[ \Lambda_{(\pm)} = \frac{1}{\sqrt{2}} \chi^{(0)} \left\{ \mathcal{I}_b \mathcal{I}^c + \frac{2}{m} \mathcal{P}_b \mathcal{P}^c \right\} \]  

(4.31b)

where

\[ \Lambda_{(+)} \Phi^{(\pm)} = -\frac{1}{2} \left( \frac{\hbar}{m} - 1 \right) \chi^{(0)} \]  

(4.32a)

\[ \Lambda_{(-)} \Phi^{(\pm)} = \frac{1}{2} \left( \frac{\hbar}{m} + 1 \right) > 0 \]  

(4.32b)

with all other products vanishing. From (4.31) we note that \( \Lambda_{(\pm)} \) and \( \Phi^{(\pm)} \) depend on \( \mathcal{P}^n \) and the initial conditions explicitly. The solutions (4.29) are formally similar to the 'momentum representation' form of the solutions to the free-particle Dirac equation.

From (2.49) we can also express the solutions (4.29) in the form, using \( \psi \) only,

\[ \psi_a(s) = \exp \left\{ \frac{i s}{S_a} \mathcal{P}_a \right\} \psi_b(0) \]

\[ = \left\{ \cos \frac{\omega s}{2} \mathcal{I}_a \mathcal{I}^b + \frac{2i}{m} \sin \frac{\omega s}{2} \mathcal{P}_a \mathcal{P}^b \right\} \psi_b(0) \]  

(4.33)
Thus the spinor \( \psi_\alpha(0) \) is simply a unitary transformation of the spinor \( \psi_\alpha(0) \) along the trajectory. Now the spinor \( \psi_\alpha(0) \) corresponds
to the case \( \kappa^\mu \tau_\nu = \omega \phi \). In this case \( \frac{p^\nu}{\mu} \) and \( u^\lambda \) coincide. We shall
refer to \( \psi_\alpha(0) \) therefore as the 'momentum' spinor since it does
not change when \( u^\lambda \) and \( p^\nu \) are non-collinear. Then \( \psi_\alpha(0) \) can be termed
the 'velocity' spinor. The transformation (4.33) relates the
bispinors in the different frames, hence it is formally the same as
the Foldy-Wouthysen transformation (Foldy and Wouthysen 1958; see
also Messiah 1961).

From (1.66c) we find

\[
\psi_{\alpha(1)} = \chi^{\alpha(0)} \gamma^\lambda_{a \ b} \psi_b(0)
\]

\[
= \chi^{\alpha(0)} \gamma^\lambda_{a \ b(0)} \psi_b(0)
\]

(4.34)

where

\[
\gamma^\lambda_{a \ b}(0) = \exp \left\{ \frac{is}{\hbar} \gamma^\lambda_{d \ c} \right\} \gamma^\mu_{c \ d} \exp \left\{ \frac{-is}{\hbar} \gamma^\nu_{b \ c} \right\}
\]

(4.35)

Thus the \( \gamma^\lambda_{a \ b} \) are transformed by a continuous time dependent
similarity transformation which keeps the form of (4.34) independent
of spin transformations.

It is of interest to consider the solutions to (4.1) - (4.3)
in the limit as \( m \) vanishes, or \( p^\mu \) becomes a null vector, representing
a classical description of a spinning massless dipole. This problem
has been considered by Bailyn and Ragusa (1977) in General Relativity. When $p^\mu \neq 0$ it follows that $\kappa = \tau_1$ and thus $\omega = 0$. Then we write for $p^\mu$

$$p^\mu = M (e^{\mu}_{(\alpha)} + \delta^\mu_{(\alpha)}) - \kappa^\nu$$  \hspace{1cm} (4.36)

where $M \delta^\alpha_{(\alpha)}$ expresses the classical analogue of the Planck-Einstein relation, $E = h\nu$, relating energy and frequency of a massless particle. This result states that although the motion of the particle in the circle is timelike the plane of the orbit moves along $p^\mu$ with the velocity of light.

From (4.33) we write the solutions in the form

$$\psi_a(s) = \exp \left\{ \frac{iMs}{S_0} \Gamma_{\alpha b}^{\mu} \right\} \psi_b(0)$$

$$= \psi_a(0) + \frac{iMs}{S_0} \Gamma_{\alpha b}^{\mu} \psi_b(0) \hspace{1cm} (4.37)$$

Thus the velocity bispinor $\psi_a(s)$ is related to the momentum, or geodesic (since $\psi_a(0)$ corresponds to the case $\kappa = \tau_1 = 0$), bispinor $\psi_a(0)$ by a linear transformation along $\Gamma_{\alpha b}^{\mu} \psi_b(0)$. Further, $\chi^a(0)$ and $\psi_a(0)$ corresponds to a null vector, that is

$$\chi^a(0) \Gamma_{\alpha b}^{\mu} \psi_b(0) = \kappa^\mu, \hspace{0.5cm} \kappa^\mu \kappa_\mu = 0 \hspace{1cm} (4.38)$$

Comparing (4.37) and (4.29a) we see that the velocity bispinor $\psi_a(s)$ defined by (4.37) contains only positive ($\tau_1 > 0$, $S_0 > 0$) or negative
(κ < 0, S < 0) frequency parts whereas from (4.29a) and (4.31a) we see that both positive and negative frequency parts are included in the solution when κ ≠ κ. This property of the m = 0 solutions is formally the same behaviour that is seen in the quantum mechanical description of the neutrino (e.g. Ziman 1969). The solutions for the spinors are not invariant under reversal of the sign of the spin corresponding to parity reversal.

Before proceeding we shall briefly discuss the relevance of our results to the question of whether a spinning pole-dipole is better described by the so-called Pirani (1956) constraint (4.3) or by the Tulczyjew (1959) constraint replacing U by ρ in (4.3). The controversy over which constraint to use has a lengthy history which we shall not repeat here. It seems, however, that each school of thought is justified according to certain arguments.

For example, Mashoon (1971) has found that for a spinning test particle in the gravitational field of the earth, (represented by the Kerr metric, in the limit of small test-mass to separation (m/l) ratio) Schiff's (1960) mass-current effect follows only if the Pirani constraint is used. For any other constraint it follows that the magnitude of the spin is not a constant of the motion.

Following Hanson and Regge (1974) Hojman and Regge (1976) examined the equations of motion for a rigid top based on a rigorous Lagrangian method. They found that self-consistency demanded that the Tulczyjew constraint apply when describing spinning particles.

Finally, it should be noted that experiments have begun to test whether the momentum and velocity vectors of a spinning particle
are non-collinear. Results have been achieved by Imbert (1970) and discussed by de Beauregard (1972) in which it is found that for a light beam the two vectors are indeed non-collinear. A similar result for Fermions would clarify the controversy, therefore.

Our results are not of such a nature that they can be used to distinguish between the apparent correctness of either constraint. Our results do indicate, however, that the choice may be considered either on the basis of physical and mathematical self-consistency within a classical model or on the basis that the classical equations of motion should reflect some limit of the quantum mechanical description of spin afforded by the Dirac equation. Since we have shown that our results closely resemble the quantum mechanical free-particle solutions using the Pirani constraint, but not for the Tulczyjew constraint, it seems that (4.3) is the appropriate constraint when used in the context of a classical description of a quantum-mechanical particle.

We shall now consider the motion of a pole-dipole particle with charge $\mathcal{E}$ and magnetic moment $\mu$ in an electromagnetic field $F^{\mu \nu}$. The motion is described by the equations

$$\frac{d}{ds} P^\mu = e F^\mu \nu u^\nu + \frac{i}{2} \mu^{\nu} \nabla^\mu F^\nu \rho$$ \hspace{1cm} (4.39)

$$\frac{d}{ds} S^\mu \nu = P^\mu u_\nu - u_\nu P^\mu + F^\mu \nu \mu^{\sigma} \rho - \mu \sigma F^\nu \rho$$ \hspace{1cm} (4.40)
where $\mu^\Phi$ is the magnetic moment and the usual definitions apply to the other variables.

Equations (4.39) - (4.41) have been discussed by a number of authors including Corben (1968), Dixon (1965) and Ellis (1970). However, we have not been able to find any examples of covariant solutions for the full set of equations. Corben (1968) considered the approximate case when $P^\mu \tilde{u}^\mu$ and found solutions when $\nabla \cdot E = 0$ or for a homogeneous field.

We shall specialize to the case when

$$\mu^\Phi = \frac{\mu}{S_0} S^\Phi$$  \hspace{1cm} (4.42)

where $S_0$ is the constant spin scalar (4.5) and $\mu$ is in general (Dixon 1965) a function of the trajectory parameter $s$. Equation (4.42) with (4.41) states that the particle is a pure magnetic dipole in the velocity rest frame.

We define the quantities, dot above a quantity noting derivative with respect to $s$,

$$\dot{u}^\mu = a^\mu \hspace{1cm} F^\mu \cdot \dot{u}^\mu = E^\mu$$  \hspace{1cm} (4.43a)

$$P^\mu - M u^\mu = B^\mu$$  \hspace{1cm} (4.43b)
Recalling (4.4), (4.6), (4.7) we find from (4.39) - (4.41)

\[ \frac{d}{ds} M = \frac{\mu}{2S} \frac{d}{ds} (F_{\mu\nu} S_{\nu\rho}) \]  
(4.44a)

\[ \frac{d}{ds} \left( m - e F_{\mu\nu} S_{\nu\rho} \right) = -e S_{\alpha\rho} \frac{d F_{\alpha\rho}}{ds} + \frac{\mu}{S} \rho \left( S_{\sigma\nu} \nabla_\rho F^{\sigma\nu} \right) \]  
(4.44b)

\[ \frac{d}{ds} S^\alpha = \frac{\mu}{S} F_{\alpha\nu} S^\nu - \left( a_\alpha - \frac{\mu}{S} E_\alpha \right) S^\nu u^\nu \]  
(4.45a)

\[ \frac{d}{ds} u^\nu = \frac{\mu}{S} F_{\alpha\nu} u^\alpha + \frac{1}{S} S^\alpha \nu B^\alpha - \left( a_\nu - \frac{\mu}{S} E_\nu \right) \frac{S^\nu}{S^2} S^\alpha \]  
(4.45b)

\[ B^\nu = -S^\alpha \nu \left( a_\nu - \frac{\mu}{S} E_\nu \right) \]  
(4.46)

Note that (4.45a) is just the Bargmann, Michel and Telegdi equation

(Bargmann, et al. 1959). Defining the vector

\[ R^\alpha = \frac{1}{S} S^\alpha \nu B^\nu \]  
(4.47)

it follows from (4.45b) that

\[ R^\alpha = S \left[ a^\alpha - \frac{\mu}{S} E^\alpha + \left( a_\alpha - \frac{\mu}{S} E_\alpha \right) \frac{S^\nu}{S^2} S^\alpha \right] \]  
(4.48)

Thus, we find that

\[ u^\nu R_\nu = \rho^\nu R_\nu = B^\nu R_\nu = S^\nu R_\nu = 0 \]

From (4.46) we see that
\[ B^\mu = -\frac{i}{S_o} S^\mu_\nu R^\nu = -S^\mu_\nu (a^\nu - \frac{m}{S_o} E^\nu) \]  

(4.49)

From (4.47) and (4.49) it follows that

\[ \left( S^\mu_\nu + \frac{1}{5} S^\mu_\sigma S^\sigma_\nu \right) \begin{bmatrix} B^\nu \\ R^\nu \end{bmatrix} = 0 \]  

(4.50)

where we have used the fact that, from (1.28)

\[ (S^\mu_\nu)^3 = -S^2 S^\mu_\nu \]  

(4.51)

Equation (4.50) states that \( B^\mu \) and \( R^\nu \) span the spin plane \( S^{\mu\nu} \).

From (4.43) and (4.49) it follows that

\[ B^\mu B^\nu = R^\mu R^\nu = - (M^2 - m^2) \]  

(4.52)

hence, we may write

\[ S^{\mu\nu} = \frac{-2S_o}{(M^2 - m^2)} \begin{bmatrix} R^\mu \, B^\nu \\
\end{bmatrix} \]  

(4.53)

Using the results

\[ S^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\kappa\beta} S_{\kappa\beta} \]  

(4.54)

and also

\[ \hat{S}^\mu_\nu B^\nu = \hat{S}^\mu_\nu R^\nu = 0 \]  

(4.55)
we can show that $F_\alpha S^\alpha = 0$. Without any loss of generality we may further assume that

$$ (q_\alpha - \frac{m}{S_0} F_\alpha) S^\alpha = 0 \quad (4.56) $$

This assumption is consistent with $S_\alpha$ being Fermi propagated along the trajectory. It also states that the vector $R^\alpha$, which gives the deviation of the particle position from its position when we turn off the spin, is given in the particle rest frame by

$$ R = \frac{1}{S_0} S \times \vec{P}. $$

in agreement with Corben (1968).

Substituting (4.47) and (4.49) in (4.39) - (4.40) and (4.56) in (4.45) we find the set of equations

$$ \frac{D}{ds} U^\alpha = \frac{i}{S_0} R^\alpha + \frac{m}{S_0} F_\alpha \nu U \quad (4.57a) $$

$$ \frac{D}{ds} R^\alpha = \left( \frac{m^2 - m}{S_0^2} \right) U^\alpha + \frac{1}{S_0} \left( M + \frac{m}{S_0} F_\alpha S_\alpha \right) B^\alpha + \frac{m}{S_0} F_\alpha \nu R^\alpha $$

$$ + \frac{m}{S_0^2} \left( \frac{e}{m} - \frac{m}{S_0} \right) S^\alpha \nu E^\gamma + \frac{m}{2S_0} S_\alpha \beta (\vec{\nabla}_\nu F_\nu^\alpha) S^\beta \quad (4.57b) $$

$$ \frac{D}{ds} B^\alpha = -\frac{1}{S_0} \left( M + \frac{m}{S_0} F_\alpha S_\alpha \right) R^\alpha + \frac{m}{S_0} F_\alpha \nu B $$

$$ - \frac{m}{S_0^3} \left( \frac{e}{m} - \frac{m}{S_0} \right) S^\alpha \nu S^\beta \nu E^\gamma + \frac{m}{2S_0^3} S_\alpha \beta (\vec{\nabla}_\nu F_\nu^\alpha) S^\gamma S^\beta \nu \quad (4.57c) $$
\[
\frac{dS^\gamma}{ds} = \frac{\lambda}{S_0} S^\gamma 
\] (4.57d)

\[
\frac{dM}{ds} = \frac{\lambda}{2S_0} \frac{d}{ds} (F^{\alpha} S_{\alpha}) 
\] (4.57e)

\[
\frac{d}{ds} (M_i^2 - e F^{\alpha} S_{\alpha}) = -M \left( \frac{e}{M} - \frac{M}{S_0} \right) S_{\alpha} \frac{dF^{\alpha}}{ds} + \frac{\lambda}{S_0} S_{\alpha} \nabla_\alpha F^{\beta} B^\beta 
\] (4.52f)

Now when \( F^{\alpha} = 0 \) these equations go over to the Frenet-Serret equations with

\[
\psi B^\alpha = (M_i^2 - M^2)^{1/2} e^{\alpha}_{(2)} \quad R^\alpha = (M_i^2 - M^2)^{1/2} e^{\alpha}_{(3)} 
\] (4.58a)

\[
U^\alpha = e^{\alpha}_{(3)} \quad S^\alpha = S_0 e^{\alpha}_{(3)} 
\] (4.58b)

\[
\tau_1 = \frac{M}{S_0} \quad \kappa = \frac{(M_i^2 - M^2)^{1/2}}{S_0} 
\] (4.58c)

For general fields equations (4.57) are still quite difficult to solve, but for a wide variety of problems they are more tractable than the set (4.39) – (4.41).

The simplest non-trivial case for which solutions can be obtained is for homogeneous fields

\[
\nabla_\mu F^{\mu} = 0 
\] (4.59)
with the magnetic moment given by

\[ \mu = \frac{\varepsilon}{M} S \]

(4.60)

Substituting (4.59) and (4.60) in (4.57) the equations reduce to

\[ \frac{d}{ds} U^\alpha = K_\alpha R^\alpha + \frac{\varepsilon}{M} F^\alpha_\beta U^\beta \]

(4.61a)

\[ \frac{d}{ds} \hat{R}^\alpha = K_\alpha U^\alpha + \frac{1}{S_0} \left( M + \frac{\varepsilon}{2M} F^\alpha_\beta S^\beta \right) \hat{B}^\alpha + \frac{\varepsilon}{M} F^\alpha_\beta \hat{R}^\beta \]

(4.61b)

\[ \frac{d}{ds} \hat{B}^\alpha = -\frac{1}{S_0} \left( M + \frac{\varepsilon}{2M} F^\alpha_\beta S^\beta \right) \hat{R}^\alpha + \frac{\varepsilon}{M} F^\alpha_\beta \hat{B}^\beta \]

(4.61c)

\[ \frac{d}{ds} \hat{S}^\alpha = \frac{\varepsilon}{M} F^\alpha_\beta \hat{S}^\beta \]

(4.61d)

where we have defined unit vectors

\[ \hat{R}^\alpha = (M^2 - m^2)^{1/2} R^\alpha \quad \hat{B}^\alpha = (M^2 - m^2)^{-1/2} B^\alpha \quad \hat{S}^\alpha = \frac{1}{S_0} S^\alpha \]

(4.62)

and

\[ K_\alpha = \frac{(M^2 - m^2)^{1/2}}{S_0} \]

(4.63)
From (4.57e) and (4.57f) we find

\[ M^2 = M_0^2 + e F^{\alpha \beta} S_{\alpha \beta}, \quad m^2 = m_0^2 + e F^{\alpha \beta} S_{\alpha \beta} \]  

(4.64)

Thus \( M^2 - m^2 = M_0^2 - m_0^2 \) and \( \kappa \) is constant and corresponds to the field-free value of \( \kappa \) from (4.16) - (4.18). Also the magnitudes of \( \kappa^\alpha \) and \( \theta^\alpha \) are conserved.

We find for the magnetic moment \( \mu \) from (4.60),

\[ \mu = \mu_0 \left( 1 + \frac{e}{M_0^2} F^{\alpha \beta} S_{\alpha \beta} \right)^{-\frac{1}{2}} \]  

(4.65)

where the free particle moment \( \mu_0 = eS_0 \). Thus, the 'interaction' of the field and magnetic moment of the particle produces a change in the value of the magnetic moment.

Using (1.22) we define the exponential tensor

\[ \exp \left\{ \alpha F_{\alpha \beta} \right\} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \left( F_{\alpha \beta} \right)^n \]  

(4.66)

where

\[ \alpha(s) = \int_{S_{\text{initial}}}^{S_s} \frac{e}{M(s')} ds' \]  

(4.67)

is assumed to exist.

We define the transformed tetrad
\[
\begin{bmatrix}
    \dot{h}_1^N \\
    \dot{h}_2^N \\
    \dot{h}_3^N \\
\end{bmatrix} = \exp\left\{ -\alpha F^N \right\}
\begin{bmatrix}
    u^N \\
    \hat{R}^N \\
    \hat{B}^N \\
\end{bmatrix}
\tag{4.68}
\]

and from (4.61) we find that \( h_1^N \) satisfies

\[
\frac{D}{ds} h_1^N = \kappa_1 h_1^N
\tag{4.69a}
\]

\[
\frac{D}{ds} h_2^N = \kappa_2 h_2^N + \tau h_1^N
\tag{4.69b}
\]

\[
\frac{D}{ds} h_3^N = -\tau h_1^N
\tag{4.69c}
\]

\[
\frac{D}{ds} h_4^N = 0
\tag{4.69d}
\]

where, setting \( \tau = \frac{M_0}{S_0} \)

\[
\tau = \tau_0 \left( 1 + \frac{3}{2} \frac{\varepsilon}{M_0^2} F^N \right) \left( 1 + \frac{\varepsilon}{M_0^2} F^N \right)^{-\frac{1}{2}}
\tag{4.70}
\]

Equations (4.69) are simply the Frenet-Serret equations for the transformed trajectory and are identical to the free particle
equations except for the fact that $\tau$, in general, is a function of the distance $s$ through the expression $F^\mu \Sigma^\nu_{\mu}$. The acceleration $\Sigma_{\nu}$ is constant, however. Even so, we should expect the same general oscillatory behaviour as for the free particle case.

We can determine the functional dependence of $\tau(s)$ by noting that we can define the transformed spin tensor

$$\Sigma^\mu \Sigma^\nu_{\mu} = \exp \left\{ -\alpha F^\mu \Sigma^\nu_{\mu} \right\} \Sigma^\nu_{\mu} \exp \left\{ \alpha F^\mu \right\}$$

$$= -2S \Sigma^\nu_{\mu} \left( \Sigma^\mu_{\nu} \right)$$

(4.71)

Further, from (1.24) we see that

$$F^\nu \Sigma^\mu_{\nu} = F^\nu \Sigma^\mu_{\nu} = -2S \Sigma^\nu_{\mu} \Sigma^\nu_{\mu}$$

(4.72)

We define the dimensionless interaction scalar

$$\rho = \frac{e}{M^2} F^\nu \Sigma^\mu_{\nu} = -\frac{2eS}{M^2} F^\nu \Sigma^\mu_{\nu}$$

(4.73)

and by repeated differentiation, using (4.69) and (4.70) we find the non-linear third-order equation

$$(1+\rho)(1+\frac{3}{4} \rho) \frac{D^3}{ds} \rho - (1+\frac{3}{4} \rho) \frac{D^2}{ds} \frac{D}{ds} \rho + f(\rho) \frac{D}{ds} \rho = 0$$

(4.74)

where
\[
\mathcal{L}(\rho) = \zeta^2 \left(1 + \frac{3}{2} \rho \right)^3 - K_o \left(1 + \frac{3}{2} \rho + \frac{3}{4} \rho^2 \right) \quad (4.75)
\]

To find solutions we first substitute

\[
\frac{D_p}{ds} = \eta(\rho) \quad ; \quad \frac{D^2_p}{ds^2} = \frac{D\eta}{d\rho} \quad ; \quad \frac{D^3_p}{ds^3} = \frac{D^2\eta}{d\rho^2} \eta^2 + \left(\frac{D\eta}{d\rho}\right)^2 \eta \quad (4.76)
\]

and find

\[
\eta \left[ (1+\rho)(1+\frac{3}{2}\rho) \left\{ \eta \frac{D\eta}{d\rho} + \left(\frac{D\eta}{d\rho}\right)^2 \right\} - (1+\frac{3}{2}\rho) \eta \frac{D\eta}{d\rho} + f(\rho) \right] = 0 \quad (4.77)
\]

Equation (4.77) has two solution sets, namely when \( \eta = 0 \), or

when \( \eta \neq 0 \) in general and

\[
(1+\rho)(1+\frac{3}{2}\rho) \left\{ \eta \eta'' + \eta' \right\} - (1+\frac{3}{2}\rho) \eta \eta' + f(\rho) = 0 \quad (4.78)
\]

where \( \eta' = \frac{d\eta}{d\rho} \).

In the case when \( \eta = \frac{D\eta}{ds} = 0 \) it follows that the field component in the spin plane expressed by (4.72) is constant. It follows further that the solutions to the motion are renormalizable to the free-particle solutions. By this we mean that the results are the same for the 'free'-particle example replacing the masses by \( M = M_o \left(1+\rho_0 \right)^{1/2} \)

and \( m = m_o \left(1+\rho_0 \right)^{1/2} \) where \( \rho_0 \) is constant. These fields were considered by Corben (1968) and used to explain the classical analogue of the anomalous magnetic moment correction to the motion of a free particle.

The motion described by (4.78) is considerably more difficult and is non-renormalizable in general. The presence of the \( \eta'^2 \) term implies that \( \eta \) is a multiple-valued function of \( \rho \). We shall assume
that \( \eta \) is finite for finite \( \rho \).

With this assumption on \( \eta \), it follows that the solution of (4.78) is

\[
\eta(\rho) = \frac{D\rho}{ds} = i\eta_0 \omega_0 \rho (1 + \gamma_0^2 \rho)^{\gamma_0^2}/2 \tag{4.79}
\]

where \( \eta_0 = 0, \pm i, \pm i \) depending on the initial value of \( \rho \) and

\[
\omega_0 = (\omega_0^2 - K_0^2)^{\gamma_0^2}/2 \quad \gamma_0 = \frac{\omega_0}{\omega_0^2} > 1 \tag{4.80}
\]

We can integrate (4.79) and write

\[
i\eta_0 \omega_0 s = C \ln \left\{ \frac{(1 + \gamma_0^2 \rho)^{\gamma_0^2}/2 - 1}{(1 + \gamma_0^2 \rho)^{\gamma_0^2}/2 + 1} \right\} \tag{4.81}
\]

where we have chosen initial conditions such that at \( s = 0 \) the argument of the right side equals one.

Inverting (4.81) we find

\[
\rho(s) = -\frac{2}{\gamma_0^2} \left\{ (1 + \cot^2 \omega_0 s) \mp \left[ (1 + \cot^2 \omega_0 s)^2 - (1 + \cot^2 \omega_0 s)^{\gamma_0^2}/2 \right]^{1/2} \right\} \tag{4.82}
\]

Since the solutions are symmetric about \( \pm \omega_0 s \), we need only consider \( \eta_0 = 0, 1, i \). There are four solutions in addition to the case for which \( \eta_0 = 0 \) is constant. We list them with their limits below:

A) \( \eta_0 = i \quad 0 \leq \rho_i < \infty \)

\[
\rho_i(s) = \frac{2}{\gamma_0^2} \left\{ (\coth^2 \omega_0 s - 1) \mp \left[ (\coth \omega_0 s - 1)^2 + (\coth^2 \omega_0 s - 1)^{\gamma_0^2}/2 \right]^{1/2} \right\} \tag{4.83}
\]
B) \( \eta_0 = 1 \quad 0 \leq p^2 < p^2_{(\text{max})} \)

\[
p^2_{(s)} = \frac{2}{\delta^2} \left\{ \cosh^2 \omega s - 1 \right\} - \left[ (\cosh^2 \omega s - 1)^2 + (\cosh^2 \omega s - 1) \right]^{1/2} \tag{4.84}
\]

C) \( \eta_0 = 1 \quad -\frac{2}{\delta^2} \leq p^3 \leq 0 \)

\[
p^3_{(s)} = -\frac{2}{\delta^2} \left\{ (1 + \cosh^2 \omega s) - (1 + \cosh^2 \omega s)^2 - (1 + \cosh^2 \omega s) \right\}^{1/2} \tag{4.85}
\]

D) \( \eta_0 = 1 \quad -\infty < p^4 \leq -\frac{2}{\delta^2} \)

\[
p^4_{(s)} = -\frac{2}{\delta^2} \left\{ (1 + \cosh^2 \omega s) + (1 + \cosh^2 \omega s)^2 - (1 + \cosh^2 \omega s) \right\}^{1/2} \tag{4.86}
\]

These solutions have been graphed in Figure I. We note the symmetry about \( \omega s = \delta \). Also, we see that if the initial value of \( \rho(s) \) places it in one particular solution category then it remains in that solution set for all time.

From the expression (4.70) we see that the torsion becomes infinite as \( \rho \) tends to the value \(-1\) and changes sign for \( \rho \to -\frac{1}{3} \).

These values of \( \rho \) correspond to field strengths of the order of the particle rest energy \( M_0 \) but opposite in sign. The changing of the sign of \( \tau \) corresponds to spin-flipping, which, for either \( p^3 \) or \( p^4 \) (depending on the magnitude of \( \eta_0 \) ) occurs once every period of \( \omega s \).

When \( \rho \to -1 \) \( \tau \) becomes complex and the Frenet tetrad (4.68) also becomes
complex. Although this motion is an allowable solution mathematically, it corresponds to no physical motion that we know of.

In general the solutions to (4.69) are very complicated but can be expressed analytically. We have not found any solutions to this problem in the available literature and believe these results to be original. The beauty of the tetrad approach as we have formulated it is in the ease with which it allows a very complicated motion to be solved.
a) Interaction energy $\rho$ as a function of $\omega s$

b) Torsion $\tau$ as a function of $\omega s$: $\tau = \tau_0 (1 + \frac{3}{2} \rho)(1 + \rho)^{-\frac{1}{2}}$

$\tau_3$ is shown for $\gamma_0 = \sqrt{2}$.
CONCLUSION

We have presented a novel way of deriving and using the Frenet-Serret equations using 2- and 4-component spinors. The equations of motion in 2-component spinor form are for $\xi^a, \eta^a = 1$

$$\frac{D}{ds} \xi^a = -i \tau_1 \xi^a + \frac{(\kappa - i\tau_2)}{2} \gamma^a$$

$$\frac{D}{ds} \eta^a = i \tau_1 \eta^a + \frac{(\kappa - i\tau_2)}{2} \xi^a$$

where $\kappa$, $\tau_1$, and $\tau_2$ are the curvature and first and second torsion. In Minkowski space these equations are identical to those found by Gursey.

We can alternately write the equations of motion in the forms

$$\frac{D}{ds} \xi^a = i D^a \xi^c$$

$$\frac{D}{ds} \eta^a = i D^a \eta^c$$

where $D^a$ is the Darboux spinor, or

$$-i \frac{D}{ds} \xi^a = \Pi^{a\dot{a}} \gamma_{\dot{a}}$$

$$-i \frac{D}{ds} \eta^a = \Pi^{a\dot{a}} \xi_{\dot{a}}$$

where $\Pi^{a\dot{a}}$ corresponds to the vector

$$2 \Pi^a = \tau_1 e^a_{(a)} + \kappa \dot{e}^a_{(a)} - i \tau_2 e^a_{(a)}$$

$$= \dot{\gamma}^a + i \dot{\xi}^a$$

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These latter equations are formally similar to the Dirac equations.

When the Frenet scalars are constant along the curve then in Minkowski space we can write the solutions as

\[ \frac{d}{ds} \phi^A = \cos \frac{\nu}{2} \phi^A - \frac{\nu}{2} \sin \frac{\nu}{2} \Pi^A \phi^B \]

where \( \phi^A = \phi^A(\sigma) \) and \( \nu = \omega e^{-\sigma} \) with

\[ \omega^+ = \sqrt{\tau_1^2 + \tau_2^2 - \kappa^2} + \frac{4 \kappa^2 \tau_2}{\sqrt{\tau_1^2 + \tau_2^2 - \kappa^2}} \]

\[ \tan 2\psi = \frac{2 \kappa \tau_2}{\tau_1^2 + \tau_2^2 - \kappa^2} \]

The different types of curves can then be described by the relative values of the Frenet scalars. In particular, whenever \( \tau_2 = 0 \) the curves are space-time helices. The bispinor solutions are then written

\[ \psi^A(\sigma) = \cos \frac{\omega}{2} \psi^A(\sigma) + \frac{\nu}{\omega} \sin \frac{\omega}{2} \Pi^A \psi^B(\sigma) \]

where \( \psi^A(\sigma) \) is the geodesic spinor, \( \omega = (\tau_1^2 - \kappa^2)^{1/2} \) and \( \Pi^A \) can be time-like \((\tau_1 > \kappa)\), spacelike \((\tau_1 < \kappa)\) or null \((\tau_1 = \kappa)\). This unitary transformation formally resembles the Foldy-Wouthuysen transformation.

We next considered the motion of a point charge in a constant electromagnetic field. The spinors obey

\[ \frac{D}{ds} \xi^A = \frac{ie}{m} F^A \xi^c \]

\[ \frac{D}{ds} \eta^A = \frac{ie}{m} F^A \eta^c \]
in agreement with Plebanski (1966). We defined the local field projections

\[ \frac{e}{m} E^\mu = \frac{e}{m} F^\mu\nu \gamma^\nu = i \epsilon_{\mu} \rho_{0} \]

**ELECTRIC**

\[ \frac{e}{m} H^\mu = -\frac{e}{m} F^\mu\nu \gamma^\nu = \tau_{+} e^{\mu} \rho_{0} + \tau_{-} e^{\mu} \rho_{1} \]

**MAGNETIC**

\[ \left( \frac{e}{m} \right)^{2} R^\mu = \left( \frac{e}{m} \right)^{2} E^\mu \gamma^\nu \gamma^\rho H_{\nu\rho} = -\kappa \tau_{1} e^{\mu} \rho_{1} \]

**POYNTING**

and therefore related the Frenet scalars to the field by

\[ \kappa = -\left( \frac{e}{m} \right)^{2} E_{\mu} E^{\mu} \quad \tau_{1} = \left( \frac{e}{m} \right)^{2} R_{\mu} R^{\mu} \quad \tau_{2} = -\left( \frac{e}{m} \right)^{2} \frac{E_{\mu} H^{\mu}}{E_{\mu} E^{\mu}} \]

We showed how the fields include the plane polarized radiation field as one example of a non-trivial field.

We then turned to the problem of a spinning point particle. Using a method of tetrad projections we were able to easily solve the Frenkel-Thomas equations and determine the quantities, for a free particle,

\[ \kappa = \frac{(m^{2} - \omega^{2})^{1/2}}{s_{0}} \quad \tau_{1} = \frac{M}{s_{0}} \quad \omega = \frac{m}{s_{0}} = \left( \tau_{1}^{2} - \kappa^{2} \right)^{1/2} \]

\[ p^{\mu} = M \left( e_{\mu 0} + \frac{\kappa}{\tau_{1}} e_{\mu 1} \right) \quad p_{\mu} p^{\mu} = \omega^{2} \]

\[ S^{\mu\nu} = -s_{0} E_{\mu}^{\rho} \gamma_{\rho} \gamma_{\nu} \quad S^{\mu} = s_{0} e_{\mu 1} \quad M^{\mu\nu} = -s_{0} \frac{\tau_{1}}{\omega^{2}} D^{\mu\nu} \]
where $M$ is the projected particle energy, $E$ the total energy, $S$ the spin, $J$ the total angular momentum and $P^\mu$ is the Darboux bivector, or generalized rotation plane of the tetrad. The motion is a circular helix with spin axis $S^\mu$ and rotation axis $P^\mu$.

From the solution of the trajectory we were able to write solutions in the form

$$
\psi(x^{(s)}) = \frac{i}{\sqrt{2}} \left( \phi_a + \frac{2}{m} P_a \phi_b \right) e^{iP^\mu x^\mu \frac{m}{2}} + \frac{i}{\sqrt{2}} \left( \phi_a - \frac{2}{m} P_a \phi_b \right) e^{-iP^\mu x^\mu \frac{m}{2}}
$$

These solutions are identical to the momentum representation form of solutions to the one-particle Dirac equation. This formal similarity supports Corben's (1968) contention that the motion is a classical analogue of 'zitterbewegung'. In the form

$$
\psi(s) = \exp \left\{ \frac{i\alpha s}{2m} P_a \right\} \psi(0)
$$

the unitary transformation connects the momentum, or geodesic, spinor $\psi(s)$ with the velocity spinor $\psi(0)$ hence it is formally the same as the Foldy-Wouthysen transformation.

For consistency in the limit $m \to 0$, we showed that $m = 0$. This case describes a classical analogue of a spinning massless particle.

In agreement with Bailyn and Ragusa (1977) we found that solutions are defined where $e^{\mathbf{P}_\nu}$ is timelike but $\mathbf{P}^\mu$ is null. In this case we find
that the solutions have only one helicity hence can be considered as classical analogues of neutrino solutions.

In the limit of $\kappa = 0$ it is impossible to uniquely define the solutions. This is a direct result of the Frenet-Serret formalism. The motion corresponds to $\mathbf{P}^\mu$ and $\mathbf{u}^\mu$ collinear, but all resemblance between the quantum mechanical and classical spinors vanishes.

Finally we considered in detail the motion of a charged spinning dipole with magnetic moment parallel to the spin and

$$\frac{e}{M} = \frac{\mu}{S_0}$$

We introduced a tetrad consisting of the 4-velocity $\mathbf{u}^\mu$, spin vector $\mathbf{S}^\mu$, deviation vector $\mathbf{R}^\mu$ and $\mathbf{S}^\mu$ which measures the difference between the 4-momentum and local energy vector $\mathbf{M}u^\mu$. In a homogeneous field we found that the tetrad satisfies the Frenet-Serret equations for a helix of variable torsion

$$\begin{align*}
\frac{D}{ds} h^\mu_{(1)} &= \kappa h^\mu_{(0)} \\
\frac{D}{ds} h^\mu_{(0)} &= \kappa h^\mu_{(1)} + \tau h^\mu_{(2)} \\
\frac{D}{ds} h^\mu_{(2)} &= -\tau h^\mu_{(1)} \\
\frac{D}{ds} h^\mu_{(3)} &= 0
\end{align*}$$
where the tetrad \( h^\mu_{(a)} \) is related to the original tetrad by the transformation

\[
\begin{bmatrix}
U^a \\
R^a \\
B^a \\
S^a
\end{bmatrix} = \exp \left[ \alpha(s) F^a \right] \begin{bmatrix}
h^\gamma_{(a)} \\
(m^a - m^a)^{1/2} h^\gamma_{(a)} \\
(m^a - m^a)^{1/2} h^\gamma_{(a)} \\
S, h^\gamma_{(a)}
\end{bmatrix}
\]

The Frenet scalars are given by

\[
\kappa = \frac{(m^a - m^a)^{1/2}}{S} \\
\tau = \tau_0 \frac{(1 + \frac{3}{2} \rho)}{(1 + \rho)^{1/2}} \\
\tau_0 = \frac{M_0}{S_0}
\]

while for the masses, magnetic moment and \( \rho \) we find

\[
M^2 = m^2 + eF^\alpha \cdot S_{\alpha \beta} \\
M^2 = M^2 + eF^\alpha \cdot S_{\alpha \beta} \\
M = \mu_0 \frac{eS_0}{M_0} \\
\alpha(s) = \int \frac{\mu(s') ds'}{S}
\]

\[
\rho = \frac{eF^\alpha S_{\alpha \beta}}{M^2} = -\frac{2eS}{M^2} F^\gamma \cdot h_{\mu(0), h^\gamma_{(a)}}
\]

Using the Frenet-Serret equations we found that \( \rho \) satisfies the third-order non-linear equation in the distance parameter \( S \), denoting derivative,
\[
(1+\rho)(1+\frac{3}{2}\rho)\ddot{\rho} - (1+\frac{3}{2}\rho)^2 \dot{\rho} + f(\rho) \dot{\rho} = 0
\]

where

\[
f(\rho) = (1+\frac{3}{2}\rho)^3 - \kappa_0^2 (1+\frac{3}{2}\rho + \frac{3}{4}\rho^2)
\]

In addition to the solution \(\rho = \text{constant}\), corresponding to a renormalization of the free particle solutions we found that \(\rho\) satisfied the equation

\[
\dot{\rho} = i\eta_0 \omega_0 \rho (1+\chi^2 \rho)^{1/2}
\]

where \(\eta_0 = 1\), \(i\) and \(\chi = \frac{\kappa_0}{\omega_0}\) depending on the initial value of \(\rho\). The general solution of \(\rho\) is found to be

\[
i\eta_0 \omega_0 s = \ln \left\{ \frac{\sqrt{1+\chi^2 \rho} - 1}{\sqrt{1+\chi^2 \rho} + 1} \right\}
\]

Depending on the initial conditions, then \(\rho\) can be steadily decreasing, it can first increase and then decrease, it can oscillate between zero and some finite negative value, or it can oscillate between zero and negative infinity.

For the value \(\rho = -\frac{2}{3}\), the torsion changes sign corresponding to a spin-flipping since in order to preserve the handedness of the tetrad it is necessary to change the sign of the spin scalar \(s\).

This corresponds to parity reversal in a classical sense. For the value \(\rho \leq -1\) the torsion becomes complex and the Frenet tetrad is also
complex. This motion is non-physical and corresponds to the extremely relativistic case where the interaction energy is opposite in sign but of equal magnitude as the particle energy $M$.

In all the cases considered we have utilized the Frenet-Serret formalism with particular ease to describe the motion. Thus, it is apparent that results can be obtained with elegance and simplicity using this formalism.
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Robert (Bob) Douglas Kent was born on September 14, 1950 at Campbell River, B.C., Canada. After attending several elementary and secondary schools throughout the province of B.C. he graduated from Chilliwack Senior Secondary School in 1968 and entered the University of British Columbia. In 1972 he graduated with Honours with the degree of B.Sc. in Physics and Astronomy. He then entered the University of Windsor where he received first the M.Sc. degree in 1973 and then graduated with the Ph.D. degree in 1979 specializing in Relativistic Physics.

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