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NUMERICAL SIMULATION OF FLOW THROUGH POROUS MEDIA

BY

MOHAMMAD HAFIZ HAMDAN

A Dissertation submitted to the Faculty of Graduate Studies and Research through the Department of Mathematics and Statistics in Partial Fulfillment of the requirements for the Degree of Doctor of Philosophy at the University of Windsor

Windsor, Ontario, Canada
1989
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ABSTRACT

NUMERICAL SIMULATION OF
FLOW THROUGH POROUS MEDIA

by

Mohammad Hafiz Hamdan

This work is devoted to the numerical treatment of single-phase fluid flow through porous media and the development and testing of new mathematical models describing the flow of a dusty fluid through porous media. The work is subdivided into two parts. The first part discusses the single-phase fluid flow through a porous domain with curved boundaries and the flow into a line sink. The second part discusses the development of models describing dusty fluid flow through porous media and considers the flow governed by these new models over curved boundaries and the flow into a line sink, in a manner that parallels the single-phase fluid flow, considered in the first part.

When the flow is considered over curved boundaries,
the von Mises coordinate system is extended to a double transformation which provides a new method of numerically analysing multi-phase flow over curved boundaries.

Single-phase and dusty fluid flows through porous media into a line sink are studied in this work to shed some light and to offer further insight into the structure of separated eddies and the effect of permeability on viscous separation. The effect of dust parameters and the flow Reynolds number on viscous separation is also studied. Comparisons are made between various single-phase flow models, and between the different dusty fluid flow models that arise due to the subdivision of the newly proposed models.

In the process, a modification of one of the existing single-phase flow models has been proposed. This modification is based on an "artificial vorticity" method.
Respectfully Dedicated to

my daughters Nadia and Nadia
my wife Carol
my parents Safiz and Nadia
my brothers and sisters
ACKNOWLEDGEMENT

I would like to express my heartfelt gratitude and appreciation to Professor Ronald M. Barron, whom I shall always be indebted to, for the encouragement and support he has given me throughout this work. His capable supervision, guidance and his continuous strive for high quality work has been a great inspiration to me.

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To members of my dissertation committee I offer my sincere thanks for their constructive comments and advice. To this extent I, once again, would like to thank Professor Barron for arranging for one of the pioneers in the science of porous media to be my external examiner.

To Arab Student Aid International I would like to offer my appreciation for the scholarship loan they have supported me with and for their continuous encouragement. I would also like to say to them it is always nice to know that somebody really cares.

To my family I extend my gratitude and love for their unfailing support and patience. To my parents, Hafiz and Nadia, and to my brothers and sisters I offer my heartfelt appreciation for being there for me. To my loving wife Carol, and to my daughters Nadia and Fadia I would like to renew my love to them and appreciation for their waiting for me during the lonely nights I spent away from them. Finally, to my wife I would like to say this one is for us.
PREFACE

In the past two decades there has been a noticeable trend towards treating the study of fluid flow through porous media as a separate entity from the general study of fluid dynamics. This stems from the many applications that this kind of study entails, together with the fact that not only fluid dynamicists have shown great interest in this challenging phenomenon but also scientists from all branches of the physical sciences.

The dynamics of fluid flow through porous media deal with the study of flow through a complicated, disordered structure of capillaries or pores existing within a solid. This structure is referred to as a porous medium through which a fluid can only flow provided that some of these pores are interconnected.

The study of fluid flow through these media may be subdivided into two inter-related fields: the study of single-phase and the study of multiphase fluid flow. This classification stems from the diversity of possible applications and the methods of treatment associated with each field.
An enormous amount of literature is available dealing with the study of single and multiphase flow through porous media, as outlined in the bibliography, and treating diverse problems such as the study of miscible and immiscible displacement, percolation theory, dispersion through porous media, reservoir simulation, groundwater flow problems, fluid flow past porous-layers and surfaces, colmatage and infiltration processes and heat and mass transfer through porous media, to name only a few.

The first empirical model describing fluid flow through porous media, Darcy's law, came into existence over thirteen decades ago. The up-to-date available literature witnesses the modelling of flow through porous media in such a way as to account for the type of flow and porous medium considered, together with the type of application and the range of applicability that is being sought.

This treatise is devoted to the comprehensive treatment of five leading models of single-phase fluid flow through porous media and their applicability in the numerical study of flow through porous media overlaying curved boundaries. This is accomplished by treating the flow problems through the well-known von Mises
transformation, and the transformed governing equations are solved numerically using the classical finite differences procedure. Parallel to the above models, mathematical models describing the flow of dusty gases through porous media, in the absence of colmatage, have been developed in this dissertation. The von Mises transformation is extended to handle this type of two-phase flow. Numerical solutions to the flow of dusty fluids through porous media overlaying curved boundaries are also obtained.

The problem of viscous separation of single-phase fluid flow through porous media is studied by considering the steady flow into a line sink. Dusty fluid flow through a porous medium into a line sink based on the proposed dusty fluid models, is also investigated.

In accomplishing a treatise of this type, there arises the inevitable problem of completeness. Although the topics selected are thoroughly treated in their own rights, they are not exhausted in the sense that the various ideas presented are easily extendable to the general multiphase flow through porous media. It should be pointed out that for the flow of multi-phase fluids through porous media only the two-phase dusty gas flow is considered. Furthermore, other aspects of the problems
discussed, e.g. connection with heat and mass transfer, dispersion and hydrodynamic stability considerations, are not touched upon.

Finally, the author wishes to extend his gratitude to Professor Ronald M. Barron for initiating the author's interest in the challenging field of fluid dynamics in porous media.
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NOMENCLATURE

\( \rho \) fluid density
\( \rho_1 \) fluid-phase density
\( \rho_2 \) dust-phase density
\( \tilde{\rho}_j \) intrinsic volume-averaged density of phase \( j \)
\( \hat{\rho}_j \) volume-averaged density of phase \( j \)
\( \mu \) fluid viscosity, kinematic viscosity
\( \mu_{ef} \) effective viscosity of the fluid saturated porous medium
\( \mu_j \) viscosity of phase \( j \)
\( \mu_1 \) fluid-phase viscosity
\( \psi \) streamfunction
\( \psi_1 \) fluid-phase streamfunction
\( \psi_2 \) dust-phase streamfunction
\( \Psi \) dimensionless streamfunction
\( \Psi_1 \) dimensionless fluid-phase streamfunction
\( \Psi_2 \) dimensionless dust-phase streamfunction
\( \sigma \) typical streamline (constant streamfunction)
\( \omega \) vorticity; relaxation parameter
\( \xi_1 \) fluid-phase vorticity
\( \xi_2 \) dust-phase vorticity
Ω  dimensionless vorticity
Ω₁ dimensionless fluid-phase vorticity
Ω₂ dimensionless dust-phase vorticity
Ω̃ vorticity solution vector
φ velocity potential
φ̃ perturbation velocity potential
ξ, ξ₁, ξ₂ curvilinear coordinates
τ, τ₁, τ₂ time variables in the von Mises and the extended von Mises transformations
ε aspect ratio
δ error tolerance
η porosity
β volume of the solid matrix (1−η)
θ ratio of effective viscosity to viscosity
χₚ binary parameter
ζ dimension of a unit comprizing the solid matrix
γᵢ volume fraction of jᵗʰ phase
F̂ₗ intrinsic volume-averaged quantity
̂Γₗ volume-averaged quantity
i grid line in the horizontal direction
j grid line in the vertical direction
A tridiagonal matrix
αᵢ,j upper diagonal elements of A
ΔX step size in i-direction
\( \Delta \Psi \) step size in \( j \)-direction

\( h \) typical step size

\( \text{Imax} \) maximum number of grid points in \( i \)-direction

\( \text{Jmax} \) maximum number of grid points in the \( j \)-direction

\( A \) cross-sectional area

\( A_{i,j}, B_{i,j}, C_{i,j}, D_{i,j} \) matrix coefficients

\( F_{i,j} \) matrix vector of unknowns

\( c_1 \) constant pressure gradient

\( C_d \) drag constant

\( C_r \) coefficient of resistance in the porous medium

\( f, f_1, f_2, f_3 \) functions of \( X \)

\( f' \) derivative of \( f \) with respect to \( X \)

\( F' \) friction force exerted on a unit volume of fluid

\( F'_{1,2} \) sum of forces exerted on a unit volume of the fluid-phase

\( F''_{1,2} \) sum of forces exerted on a unit volume of the dust-phase

\( F_{j,1} \) sum of external forces exerted on a unit volume of phase \( j \)

\( f_{j,1} \) sum of external forces exerted on phase \( j \)

\( F_{1,1} \) friction force on a unit volume of the fluid-phase due to the solid matrix

\( F_{1,2} \) force exerted by the dust-phase on a unit volume of the fluid-phase
n iteration level
N particle number density
\( m \) mass of a single particle
\( \delta p \) pressure drop
\( p \) pressure, interstitial pressure
\( P \) dimensionless pressure
\( p_f \) fluid-phase partial pressure
\( p_d \) dust-phase partial pressure
\( \hat{p}_j \) intrinsic volume-averaged pressure of phase \( j \)
\( \hat{p}_j \) volume-averaged pressure of phase \( j \)
\( x, y \) cartesian coordinates
\( X, Y, X'_1, Y'_1, X'_2, Y'_2 \) dimensionless cartesian coordinates
\( t \) time variable
\( T \) dimensionless time variable
\( D \) reference depth
\( D_1, D_2 \) dimensionless dust parameters
\( d_p \) average pore diameter
\( L \) length, reference length
\( k \) dimensional permeability
\( K \) dimensionless permeability
\( q \) volumetric flow rate
\( q_r \) relative velocity vector
\( g \) gravitational acceleration vector
Re  Reynolds number
Re* effective Reynolds number
Rec dimensionless parameter (coefficient of artificial vorticity)
Re  pore Reynolds number
J_1', J_{1S}', J_{2S}' spatial Jacobians of transformation
J_2', J_{1T}', J_{2T}' temporal Jacobians of transformation
v  velocity vector
\tilde{v}  Darcy velocity vector
\tilde{v}_p  average pore velocity vector
\tilde{v}  dimensionless velocity vector
\tilde{V}  dimensionless velocity components
U, V  dimensionless velocity components
U_0  characteristic velocity
\tilde{u}_1', \tilde{v}_1' fluid-phase velocity vectors
\tilde{u}_2', \tilde{v}_2' dust-phase velocity vectors
\tilde{U}_1', \tilde{V}_1' fluid-phase velocity components
\tilde{U}_2', \tilde{V}_2' dust-phase velocity components
\tilde{u}_j' intrinsic volume-averaged phase j velocity vector
\tilde{u}_j  volume-averaged phase j velocity vector
V_j  volume occupied by phase j
CHAPTER 1

INTRODUCTION

1.1. OVERVIEW

The study of fluid flow through porous media has received considerable attention due to its many practical applications and implications in the physical sciences and in the applied sciences.

On the one hand, the movement of groundwater is represented by flow through porous media, the study of which becomes essential in the recovery of fresh water [1-4]. In addition, the increasing demand for energy necessitates the study of oil and gas movement through the porous earth layers [5]. Interaction of oil, gas and water, their movement and the displacement processes that occur within oil reservoirs is a study of fundamental importance in this field [8-12].

In agriculture, the importance of flow through porous media is witnessed in irrigation processes [13] and the movement of nutrients, fertilizers and pollutants into plants [14].
The study of convection processes in porous media has also received considerable attention in the literature due to the importance of these processes in geothermal, geophysical and industrial environments [15-20], where analyses of thermal energy storage systems, thermal insulation and solar collectors with porous absorbers are vital [15].

The importance of porous media is also witnessed in the theory of filtration due to its direct applicability to the design of deep filters, industrial filters and water purification plants [21-23]. Mass transport in composite materials, which has many applications ranging from environmental studies and pollutant disposals to energy storage systems [15,24] is also an important aspect of the study of flow through porous media.

The importance of studies of flow through porous media is also exhibited in its various applications in the biophysical sciences and in biomechanics where the human lungs and tissues, for instance, are idealized into layers of flocs and other kinds of porous media [25]. Applications of flow through porous media in biological studies and the study of macromolecular systems have been reported by various authors [26,27].

Another important aspect is the study of flow over porous layers. This has been reported to have many
applications in biomedical engineering [28-32], in addition to other applications. In these studies, the crux of the problem is the derivation of the boundary conditions along the interface between the porous medium and free space. Work in this field includes that of Beavers and Joseph [33] and of Rudraiah [31,32], among others. The bibliography includes a survey of the up-to-date work that has contributed to the advances of fluid flow over porous layers together with a section on some of the work devoted to employing these concepts to solve some physical problems.

Although the above discussion does not cover all of the different areas, and topics of research, of applications of flow through porous media and over porous surfaces, it nevertheless stresses the fundamental importance and the many applications of this type of flow. This review emphasizes the importance of seeking solutions to boundary value and initial value problems in this field.

Although scientific treatment of fluid flow through porous media started out, about a century and a half ago, as an experimental study, some of the up-to-date theoretical investigations in certain branches of porous media studies rely on empirical laws. In addition, the treatment of fluid flow problems in either finite or
infinite domains may also be based on sets of
differential equations governing the flow phenomena.

The solution methodology to a given boundary value
problem has ranged, depending on the type of the
governing equations and on the type of flow and the flow
domain considered, from analytical methods or
semi-analytical to numerical solutions that became
possible with the advent of high speed digital machines.

Although one might seek numerical solutions due to
the difficulty of obtaining analytical solutions to the
complicated flow problems through porous media, obtaining
numerical solutions might be hindered at times due to the
shape of the flow domain. This is witnessed by the
impracticality of using the finite differences approach
in solving a given boundary value problem possessing
arbitrarily curved boundaries. In this case one might
resort to techniques such as the finite element technique
which has recently gained popularity in the treatment of
flow problems in porous media [34,35,38].

An overall objective of the current work is the
development and testing of methods that are applicable to
the study of flow through porous media and are capable of
handling the inevitable problem of curved boundaries in
porous domains. The methods are based on the von Mises
transformation which was implemented in the numerical
study of flow over arbitrary airfoils by Barron [37]. In
the current work extensions of this streamfunction
coordinate system method to handle the general flow of a
viscous fluid over bodies, and a generalization of the
method to treat unsteady flows, have been carried out.
Although a great amount of work has been devoted to
the study of deep filtration processes and the study of
suspension in porous media (cf. Herzig et al [21] and the
references therein), most of the available work
approaches the problem empirically without much attention
being given to the phenomenological approach or to the
possibility of treating the flow of a dusty fluid through
porous media via a set of partial differential equations.
This lack of models leaves a gap that has been addressed
in the current work. The proposed models are expected to
serve as an initial step in advancing the study of dusty
fluid flow through porous media in a manner that
parallels the well-established dusty fluid flow in free
space [38,39,40]. Since no treatment is complete without
devising the methodology to solve the proposed models,
this work introduces a method that is expected to treat
the flow of a dusty fluid in porous media.
Another important aspect of flow through porous
media is that of viscous separation. As will be discussed
in a later section, in this study fluid flow through
porous media into a line sink is considered in an attempt to offer some contribution in the field of viscous separation in porous media.

To parallel the ideas on viscous separation for the flow of a single-phase fluid, mentioned above, the proposed dusty fluid models of flow through porous media are used to study the structure of separated eddies when the flow considered is through a porous channel into a line sink.

In this introduction, a review of the leading models of flow through porous media is presented together with a brief introduction to the range of validity of each of the discussed models. This is followed by a discussion of the scope of the current work.

Due to the variety of topics discussed in this work, a literature review is included in the discussion on each of the topics. The relationship of each of the topics with the existing literature is also discussed.

1.2. FUNDAMENTALS OF FLOW THROUGH POROUS MEDIA

The flow of a viscous fluid is described mathematically by a set of equations representing two conservation principles, namely conservation of mass and
conservation of momentum. When the fluid is Newtonian the momentum equations are given by Navier-Stokes equations which, when the fluid is incompressible and body forces are neglected, are expressed as

\[
\rho \frac{Dv}{Dt} = -\nabla p + \mu \nabla^2 v.
\]  \hspace{1cm} \ldots (1.1)

The conservation of mass, on the other hand, is given as an equation of continuity which, in the absence of sources and sinks, takes the following form

\[
\nabla \cdot v = 0.
\]  \hspace{1cm} \ldots (1.2)

In equations (1.1) and (1.2), \( t \) is the time variable, \( \rho \) is the fluid density, \( p \) is the pressure, \( \mu \) is the viscosity coefficient, \( v \) is the velocity vector, \( \nabla \) is the gradient operator, \( \nabla^2 \) is the laplacian operator and \( \frac{D}{Dt} \) is the material derivative given by

\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (v \cdot \nabla).
\]  \hspace{1cm} \ldots (1.3)

7
The application of the above mathematical system of equations to a particular flow problem requires the mathematical description of the boundaries of the system in question.

When the system is taken to be a porous medium, a well defined description of the porous matrix boundaries is a formidable task and perhaps impossible owing to the great complexity of the pore and solid matrix structures. This has given rise to the idea of studying a particular flow problem based upon the mathematical description of the macroscopic features of the fluid flow and upon the statistical averaging of the porous media quantities [5].

This approach requires that the conservation of mass principle be expressed as a macroscopic continuity equation while the macroscopic linear momentum equations, as will be seen in the following section, take into account the type of flow considered together with the type of the porous matrix structure. The macroscopic equation of continuity for all of the models to be discussed takes a form similar to that of equation (1.2).

Depending on the type of the porous medium and the flow under consideration the differential equations governing fluid flow through porous media are the macroscopic equation of continuity and the macroscopic momentum equations which are either given by Darcy’s law
or by other models which, depending on the porous medium structure and the flow considered, incorporate the viscous shear and inertial effects of the flow.

In the following, Darcy's law and the other existing models of flow through porous media are briefly presented and the range of validity and applicability are discussed, based on what has been reported in the literature.

1.2.1. Darcy's Law

In 1856 Darcy conducted an experiment which resulted in what is now known as Darcy's law which states that the volumetric rate of flow, \( q \), through a horizontal sample of porous material of length \( L \) and of cross-sectional area \( A \), is given by

\[
q = \frac{k A \delta P}{\mu L} \quad \text{... (1.4)}
\]

where \( \delta P \) is the pressure drop across the sample, \( k \) is the medium permeability and \( \mu \) is the fluid viscosity.

Equation (1.4) was generalized for three dimensional flow and cast in the following differential form by Muskat [5]
\[ \gamma = - \frac{k}{\mu} (\nabla P + \rho g) \quad \cdots (1.5) \]

where \( P \) is the interstitial pressure, \( g \) is the body force (taken here as that due to gravity), \( \rho \) is the fluid density and \( \gamma \) is the mean filter velocity vector. It should be noted that the mean filter velocity, denoted \( \gamma_d \) and called the Darcy velocity, is related to the average pore velocity, \( \gamma_p \), by the following relation [15]

\[ \gamma_d = \eta \gamma_p \quad \cdots (1.6) \]

where \( \eta \) is the medium porosity, defined as the ratio of the volume of the pores in the medium to the total volume of the medium.

Darcy's law as described by equation (1.5) is postulated to have some limitations on its validity, [5,14,41,42], the most important of which is its possible validity only when the velocity domain is of the seepage type and the fluid is homogeneous. Thus Darcy's law can be considered valid in situations where the flow is of the creeping type [14] or when the porous medium is densely packed with small enough permeability [41], \( k^{1/2} \ll 1 \), such that the pore Reynolds number, \( \text{Re}_p \), based on the local volume averaged speed, is less than unity.
Discussions on the pore Reynolds number and its relation to the medium Reynolds number, Re, are given by Rudraiah [18,41] and by Greenkorn [14]. The transition of flow from laminar to turbulent is reported to occur around Reynolds number, Re = 10 [42].

It is clear that Darcy's law neglects the boundary and inertia effects of the fluid flow [15] due to the small porosity associated with it. In certain porous media where the porosity is close to unity and the flow is of high enough speed such that the pore Reynolds number is of order unity or greater the inertia effects, which arise since the flow is curvilinear, have to be taken into account. Furthermore, Darcy's law does not account for the high velocity gradients that arise when the viscous effects are accounted for on a solid boundary due to the low order of this law. In these cases Darcy's law is no longer valid and therefore a non-Darcy equation that incorporates inertia and viscous shear effects has to be used. This has given rise to the different models that are discussed in the sub-sections to follow.

In spite of the absence of inertia terms in equation (1.5), Yih [13] offered the following generalization of Darcy's law when the inertia effects are not negligible:

\[ \frac{\rho}{\eta} \left( \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\eta} ( \mathbf{v} \cdot \nabla ) \mathbf{v} \right) = -\nabla p - \rho g - \frac{\mu}{k} \mathbf{v}. \quad \ldots(1.7) \]
Yih [13] indicated that the temporal acceleration should be taken into account when dealing with unsteady problems, while the convective acceleration is too small.

It should be noted that the flow of an incompressible fluid through a porous medium in which Darcy's law is applicable is necessarily steady, and the continuity equation is given by (1.2). In the event that the fluid is compressible the continuity equation, in the absence of sources and sinks and for a medium with constant porosity \( \eta \), takes the form

\[
\eta \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \nu = 0 \\
\eta \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nu) = 0 \quad \ldots (1.8)
\]

and thus the equations governing the flow in the medium in question are obtained by substituting the momentum equation (1.5) into the continuity equation (1.8). In the absence of body forces, the resulting equation can be expressed, in terms of the pressure gradient, to yield

\[
\eta \frac{\partial \rho}{\partial t} = \nabla \cdot \{ \frac{k \rho}{\mu} (\nabla P) \} \quad \ldots (1.8)
\]

For further insight into the validity and
limitations of Darcy's law one is referred to Scheidegger's The Physics of Flow Through Porous Media [42], Rudraiah's Some Flow Problems in Porous Media [41] or to the work of Greenkorn [14].

1.2.2. Brinkman's Equation

Brinkman [43] offered a modification of Darcy's law by examining the flow past a spherical particle, in order to account for the viscous shear stresses which act on the fluid elements. Viscous shearing action is neglected in Darcy's law since it only retains the viscous damping effects due to the porous matrix.

Brinkman [43] has considered steady-state fluid flow for which the governing equation is taken in the following form

\[ \nabla \cdot \left( \frac{\mu}{k} \nabla v \right) + \frac{\mu}{k} v = \mu_{ef} \nabla^2 v \]  

... (1.10)

where \( v \) is the ensemble-averaged velocity within the porous medium, \( \mu_{ef} \) is the effective viscosity of the fluid saturated porous medium, and \( \mu \) is the viscosity of the fluid. It has been argued that \( \mu_{ef} \) is, in general, different than \( \mu \) [25, 32, 43, 44, 45, 48], and an expression
relating the two viscosities has been derived by Rudraiah et al. [52], having the form

\[ \mu_{eff} = \mu \left\{ 1 + \frac{5}{2} \beta + \left[ \frac{81}{32} \log \beta + 19.68 \right] \beta^2 \right. \\
\left. \quad + 6.59 \beta^{5/2} \log \beta \right\} \quad \text{(1.11)} \]

where \( \beta = 1 - \eta = \frac{4}{3} \pi \frac{d_p^3}{\rho} \) and \( d_p \) is the average pore diameter.

It has been reported that equation (1.10) is also associated with the name of Debye [47], provided that \( \mu = \mu_{eff} \), and this equation has been shown to have many applications in the biophysical sciences and the study of macromolecular systems [26,27]. Since Debye [47] never offered any theoretical justification, equation (1.10) is currently better known as Brinkman's equation.

In equation (1.10) the inertial terms are neglected, based on the assumption that effects of Reynolds number are minimal compared to the dominant viscous shear terms. Although Brinkman's formulation of equation (1.10) is heuristic, other investigators have attempted to give a theoretical justification of the model equation [45,48,49,50] and to emphasize that high porosity is associated with the
medium where this equation is valid [32,18].

It should be noted that equations (1.10) reduce to the steady inertia-free Navier-Stokes equations for large values of permeability, $k$, and reduce to Darcy's law for small values of $k$.

An alternative form of equation (1.10) was offered by Mandl (see [27] for reference) and takes the form

$$\nabla P + \frac{\mu}{k} \nabla \nu = \frac{\mu}{\eta} \nabla^2 \nu.$$  \hspace{1cm} (1.12)

With $\mu = \mu_{ef}$ and $\eta = 1$, equations (1.10) and (1.12) are equivalent.

In equations (1.5) and (1.10) the inertia effects are neglected when describing fluid flow through porous media and thus they are classified as Darcian models. In some types of porous media it has been postulated that the medium porosity is close to unity and one is compelled to use a non-Darcian model that incorporates the inertia effects which arise due to the rather high flow speed [32], and which are neglected in Brinkman's model. These non-Darcian models incorporate the viscous shearing action which arises due to the fluid velocity distortion, and is neglected in Darcy's law. The non-Darcian models are discussed in the following
sections.

1.2.3. Darcy-Lapwood Equation

This model was obtained by Lapwood [51] by replacing the viscous term, \( \mu \nabla^2 v \), in the Navier-Stokes equations \( \nabla \cdot v = 0 \) by the viscous damping term in Darcy’s equation \( \frac{\mu}{k} v \), to obtain

\[
\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{\nabla p}{\rho} - \frac{\mu}{\rho k} v + g . \tag{1.13}
\]

The Darcy-Lapwood equation is postulated to be valid for flow through a sparse distribution of particles fixed in space [32].

Arguing from a convection analysis point of view, some authors have stressed the errata in the inclusion of the inertia term in equation (1.13), as Lapwood [51] has done. They argue that the inclusion of a such term in Darcy’s law raises the order of the equation, when the base flow is not quiescent, which in turn renders a given boundary value problem indeterminate due to the unavailability of additional boundary conditions to accompany this increase in the order of the governing equations [16,19]. Beck [16] and Rudraiah [52] indicated
that an under-determined system results, when using Lapwood's equation, if the normal component of velocity is the only condition imposed on the boundary and an over-determined system results if the velocity distribution is given on the boundary. The Darcy-Lapwood equation (1.13) has, nevertheless, attained a certain popularity in the study of nonlinear convection [53-56].

1.2.4. Darcy-Forchheimer Equation

The Darcy-Lapwood model treats the inertial effects in the porous medium by incorporating inertia terms like those appearing in the Navier-Stokes equations. The Darcy-Forchheimer model, on the other hand, replaces the aforementioned inertia terms by nonlinear terms of the form

\[
\frac{C_d}{\sqrt{k}} \frac{|v|}{\sqrt{v}} 
\]

... (1.14)

Equation (1.13) is thus replaced by

\[
\frac{\partial v}{\partial t} = - \frac{\nabla p}{\rho} + g - \frac{\mu}{\rho k} \frac{|v|}{\sqrt{v}} - \frac{C_d}{\sqrt{k}} \frac{|v|}{\sqrt{v}} 
\]

... (1.15)
where \( C_d \) is the form drag constant \([15,41,57,58]\).

Equation (1.15) has received a certain popularity due to the belief that it describes flow phenomena through naturally occurring porous media \([57]\). Experiments in support of representing the inertial effects by a drag term that is quadratic in the velocity have been given by Beavers et.al.\([59]\), Ward \([58]\) and Macdonald et.al. \([60]\).

The value of the drag constant \( C_d \) has been reported to approach asymptotically the value of 0.55, when the pore Reynolds number \( Re_p > 10 \), \([15,58]\).

Derivation of the Darcy-Forchheimer model through the Slattery-Whittaker averaging theorem is given by Rudraiah et.al. \([41]\). This equation has also been employed in the study of convection \([16,57,61,62]\).

1.2.5. Darcy-Lapwood-Brinkman's Equation

Although Darcy-Lapwood and Darcy-Forchheimer models take into account the fluid inertia they, nevertheless, neglect the viscous shearing effect. Incorporating the viscous term \( \mu_e \frac{\hat{V}^2}{
abla} \) in the Darcy-Lapwood equation \( (1.13) \) results in the following equation, referred to as the Darcy-Lapwood-Brinkman's equation.
\[
\frac{\partial v}{\partial \tau} + (v \cdot \nabla) v = -\frac{\nabla p}{\rho} - \frac{\mu}{\rho k} v + g + \frac{\mu_{ef}}{\rho} \nabla^2 v. \quad \ldots (1.16)
\]

Equation (1.16) is postulated to govern the flow in a slurry or in a mushy region in a medium undergoing rapid freezing [32]. It should be noted that equation (1.16) is similar in nature to the Darcy-Lapwood equation (1.13) except for the inclusion of the Laplacian term. This renders a given boundary value problem determinate [57].

Equation (1.16) was derived by Tam [49] using the ensemble averaging technique, and by Rudraiah et al. [41] using a non-equilibrium thermodynamics method.

The effect of the boundary layer that arises near a no-slip boundary [19] has been studied by many authors [55,63,64,65,66] using equation (1.16) and this equation has also received certain popularity in the study of convection [67].

It should be noted that when the inertial terms are neglected in equation (1.16), the unsteady form of Brinkman’s equation results.

1.2.6. Darcy-Forchheimer-Brinkman Model

When the inertial effects in equation (1.16) are
replaced by the inertia effects proposed by Forchheimer, given by equation (1.14), the following equation results:

\[
\frac{\partial \bar{v}}{\partial t} = -\frac{\nabla p}{\rho} - \frac{\mu}{\rho k} \bar{v} + g + \frac{\mu_{ef}}{\rho} \nabla^2 \bar{v} - \frac{C_d}{\sqrt{k}} \bar{v} | \bar{v} | \bar{v} \ldots (1.17)
\]

Equation (1.17) is termed the Darcy-Forchheimer-Brinkman equation and was derived by Vafai and Tien [66] using the local averaging technique. This equation has also been employed by many authors in convection problems [19,69,70].

1.2.7. Rudraiah's Binary Equation

In terms of accounting for the inertia and/or the viscous shear, the models presented thus far may be classified in the following four groups:

1. When the boundary and inertial effects are neglected, the flow is governed by Darcy’s law.

2. When only the inertia is accounted for, the flow is governed by the Darcy-Lapwood or the Darcy-Forchheimer models.
3. When the boundary effects are accounted for, through the viscous shear term, and the inertia is neglected, the flow is governed by the Brinkman equation.

4. When both of the inertia and the boundary effects are accounted for, the flow is governed by the Darcy-Lapwood-Brinkman or by the Darcy-Forchheimer-Brinkman models.

In group 4, it is clear that when the inertia is accounted for by a form similar to that given by Lapwood, the resulting model is termed the Darcy-Lapwood-Brinkman model, and when the interia is accounted for by the Forchheimer's suggested drag form that is quadratic in the velocity, the resulting model is termed the Darcy-Forchheimer-Brinkman model. This of course leaves out the more general case in which the boundary effects are accounted for, through the viscous shear effects, and the inertia is accounted for through the most general quadratic velocity drag and the convective terms. This case has been discussed by Rudraiah [18] and the resulting equation, which is termed here the Rudraiah binary equation, takes the following form:
\[
\rho \left\{ \chi_p \left[ \eta^{-1} - 1 \right] + 1 \right\} \frac{\partial \tilde{v}}{\partial t} \\
+ \rho \left\{ \chi_p \left[ \eta^{-2} - 1 \right] + 1 \right\} \left( \tilde{v} \cdot \nabla \right) \tilde{v} = -\nabla \tilde{p} + \rho \tilde{g} \\
+ \mu \left\{ \chi_p \left[ \Theta - 1 \right] + 1 \right\} \nabla^2 \tilde{v} - \chi_p \left( \frac{\mu}{k} \tilde{v} + \frac{\rho \, C_d}{\sqrt{k}} |\tilde{v}| \tilde{v} \right).
\]

\[\cdots (1.18)\]

In equation (1.18), \( \Theta = \frac{\mu_{ef}}{\mu} \), \( \tilde{v} \) is the Darcian velocity, \( \tilde{v}_d \), in the porous medium and the true velocity \( \tilde{v} \) in the fluid. \( \chi_p \) is the binary parameter given by [15]

\[
\chi_p (x, y) = \begin{cases} 
1 & \text{if } (x, y) \text{ is in the porous medium} \\
0 & \text{if } (x, y) \text{ is in the fluid.} 
\end{cases} \quad \cdots (1.19)
\]

From (1.18) and (1.19), it is clear that when \( \chi_p = 0 \), equation (1.18) reduces to the usual Navier-Stokes equations. If \( \chi_p = 1 \), then all of the models of flow through porous media may be obtained by assigning
appropriate values to the parameters $\theta$, $\eta$ and $C_d$.

Finally, the most general momentum equation governing the single-phase fluid flow through porous media is obtained from equation (1.18) by taking $\chi_p = 1$, $\eta \to 1$ and $\theta \to 1$. The following equation is, therefore, obtained

$$\frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \rho \mathbf{g}$$

$$+ \mu \nabla^2 \mathbf{v} - \left( \frac{\mu}{k} \nabla \frac{\rho C_d}{\sqrt{k}} |\nabla| \nabla \right). \quad \ldots (1.20)$$

Equation (1.20) is recognized as the Darcy-Lapwood-Forchheimer-Brinkman equation.

1.3. SCOPE OF THE CURRENT WORK

The present work may be subdivided into two parts, the first of which is concerned with the study of single-phase fluid flow through porous media. This topic is covered in chapters 2, 3 and 4. The second part of this work deals with the dusty fluid flow through porous media, and is covered in chapters 5, 6, 7 and 8.
Although this chapter includes the problems that will be considered in both parts, and the literature review relates to both parts, further discussion and suggested applications of the dusty fluid flow through porous media will be given in the second part of this work.

1.3.1. The Flow of a Single-phase Fluid through Porous Media

1.3.1.1. Flow Over Curved Bodies

The literature cites numerous studies of flow through porous media where the models discussed in section 1.2 have been used to study certain flow situations, for example groundwater flow problems, irrigation problems, heat transfer problems, or problems relating to biomedical and biological studies.

Although existing analytical solutions to flow problems through porous media are numerous they are, nevertheless, obtainable for cases where the geometry of the flow domain is simplified or the flow situation is highly idealized. Some of the existing analytical solutions can be found in the study of groundwater flow problems [1,2] and solutions for the flow of gases have
been treated by Carman [71] for certain flow situations.

Several other methods of solution have also been used in studies of heat transfer problems by Rudraiah et al. (cf. [15, 19, 41] and the references therein). Other situations where analytical solutions are possible are given by Scheidegger [42], Muskat [5] and Bear [72].

It is important to note that in many of the practical problems encountered in the study of flow through porous media the geometries of the flow domains considered are of a complex irregular type for which analytical solutions are either hard to find or impossible. This has given rise to a variety of ways to approach this type of problem. The main methods employed are conformal mapping and complex variable techniques [1, 4, 5, 9] in addition to the hodograph transformation approach and the use of flow nets [72]. In all of these semi-analytical methods, the element of generality of a given method is lost in the sense that a different geometry of the flow domain and different boundary conditions associated with a given problem require different transformations.

An alternative to the above techniques is the numerical approach which has gained popularity in the study of single and multiphase flows through porous media. Applications of the finite difference technique in
the study of reservoir engineering has been offered, among others, by Peaceman [73] and Aziz and Settari [74]. Numerical study of subsurface flow problems incorporating the finite element and boundary element techniques has also been reported [34,35,36].

Although numerical studies offer an attractive alternative approach in the study of flow through porous media, they too have their own drawbacks, ranging from the impracticality of the finite differencing technique in handling cases where the boundaries are arbitrarily curved, to the impracticality of the finite element method in dealing with partial differential equations that do not possess a functional. Of course, the difficulty of handling curved boundaries in finite difference calculations can be overcome using numerical grid generation. However, grid generation techniques also suffer from their drawbacks [37] and have not yet received popularity in the study of flow through porous media.

In this study an alternative approach is offered to better handle some of the problems arising in the numerical approach to flow through porous media. The approach is based on the transformation of the governing equations using the von Mises transformation and the numerical solution is obtained in the computational
domain through the use of the finite difference technique. This approach was first introduced by Barron [37] in his study of flow over airfoils of arbitrary shape and in the design problem of airfoils. Some of the advantages of the approach were discussed [37] and include the transformation of any smooth streamline boundary, although curved, into a straight line, which does away with the disadvantage of implementing finite differences in cases of curved boundaries.

The easy implementation of this method makes it more favourable than grid generation. Furthermore, since the occurrence of natural porous media, or the earth strata, is arbitrary in shape and a given flow domain is, more than likely, of the type that has curved boundaries, this method of approach seems to be realistic in dealing with flow problems in porous media.

As is well-known, the von Mises transformation exists in the time independent form. In this work, and for the sake of completeness, the von Mises transformation is extended to include the unsteady state case in an attempt to offer more generality in the study of different physical problems which are time dependent.

In chapter 2, the single-phase flow equations of section 1.2 are cast in vorticity-streamfunction form and then transformed using the derived unsteady von Mises
transformation. The vorticity-streamfunction form of the governing equations will be implemented in chapter 4, while the governing equations in terms of the von Mises variables will be implemented in chapter 3.

The von Mises transformation is applied to the numerical study of single-phase fluid flow through an infinite porous domain in order to illustrate the applicability of the method in this area. The flow domain is assumed to be structured in such a way that it is physically feasible to consider some of the flow models discussed in section 1.2.

To accomplish this study, five of the models discussed in section 1.2 are employed in an attempt to offer a comparison between the different flow regimes for different permeabilities and different Reynolds numbers, where applicable. The flow models considered are Darcy's law, the Darcy-Lapwood model, Brinkman's equation, the Darcy-Forchheimer model and the Darcy-Lapwood-Brinkman model. These models are considered as representative of the different types of models discussed in section 1.2 in the sense that they include the seepage-flow model of Darcy, the inertia-free model of Brinkman, the purely inertial model of Lapwood, the quadratic-velocity drag model of Forchheimer and the mixed inertial-viscous shear Darcy-Lapwood-Brinkman model. The two other models, the
Darcy-Forchheimer-Brinkman and the Darcy-Lapwood-Forchheimer-Brinkman, are left for future consideration.

The problems arising due to the use of the Darcy-Lapwood model are discussed in chapter 3, and a modification of the model by a proposed "artificial vorticity" method is also discussed. The proposed model is termed, in the current study, the Darcy-Lapwood (DL) modified model.

Comparison of the DL modified model with the Darcy-Lapwood-Brinkman model demonstrate that the effects of the viscous term $\mu \nabla^2 \nu$ are minimal when the flow is an external flow over curved boundaries.

1.3.1.2. Flow into a Line Sink

The general area of fluid flow into a line sink has been reported by several authors to be of practical importance in many physical aspects. For example, stratified fluid flow into a line sink is useful in the cooling of thermo-electric generating stations [73,75]. In the study of fluid flow through porous media, the sink model is of importance as being representative of a production well, in the case of crude oil drawing from the reservoir, or of an artesian well, in the case of
groundwater recovery. On the agricultural scene, the sink-source model is of importance in some irrigation problems [13].

Although the study of flow through porous media into a two-dimensional sink has received a considerable amount of attention [77-81] due to its many applications, most of the cited literature treats the stratified fluid flow into the sink, whether the stratification is in density [77, 78, 79] or in viscosity [80]. The approaches taken in these studies were both experimental and theoretical, where exact solutions have been obtained. The cases considered thus far in the literature assume the flow through the porous medium to be governed by Darcy's law, and exact solutions were sought. The existence of a boundary-layer-type solution was suggested by Koh [81] and has been employed by List [78] in his study of density-stratified fluid flow in porous media.

In different flow situations, when the flow is governed by a model that accounts for the boundary effects, the subject matter of fluid flow into a sink is hardly reported in the existing literature. For this reason, the current study considers the flow through a porous medium into a line sink when the flow is governed by other flow models discussed in section 1.2. In particular, to account for the effect of the boundary.
i.e. the effect of the no-slip conditions, a finite porous domain is considered and the flow is assumed to be governed by the Darcy-Lapwood-Brinkman model. As will be seen in chapter 4, when the inertia effects are ignored in this model, Brinkman's model is recovered.

An important aspect in the study of fluid flow through a finite dimensional porous medium into a line sink is viscous separation. Although this phenomenon is not encountered when the flow is of the seepage type, as governed by Darcy's law, the models employed in this study demonstrate the regions of viscous separation and the effects of the permeability and the flow Reynolds number on these regions. An unfavourable aspect of viscous separation is its occurrence in the reservoir when the crude oil is being drawn into the production well. Viscous separation in this case results in an unrecoverable portion of the entrapped oil, termed the residual oil. Although no claim is made in the current work about the direct applicability of the conclusions about viscous separation, the general area of flow separation is of interest in the field of reservoir engineering and the current work has provided an initial contribution to this area. The ideas presented in the current work serve as an illustration of the phenomena arising under different flow situations and for different
flow models. They represent an attempt to offer a further understanding of the occurrence of separated corner eddies and their dependence on the permeability.

When the flow is governed by Brinkman’s model, the effect of the sink location on the flow patterns, the velocity profiles and on the regions of viscous separation is studied for different permeabilities. The breakdown of the Darcy-Lapwood model is also discussed together with the DL modified model. Unlike in the case of flow over curved boundaries, the DL modified model provides less accurate results when the flow considered is that of a viscous fluid in a closed, or semi-closed, domain.

1.3.2. The Flow of Dusty Fluids through Porous Media

1.3.2.1. Models Describing Dusty Fluid Flow through Porous Media

The study of dusty gases has received considerable attention when the dusty fluid flow is considered in free space [38-40]. The equations governing the flow phenomena, as based on Saffman’s work [38], Marble’s work [39] or Soo’s work [40], are well established in the
literature on dusty gases.

Dusty fluid flow in porous media, on the other hand, has received little or no attention from the point of view of paralleling the dusty fluid theories and equations of flow in free space. The majority of the work of an enormous number of researchers approaches the subject matter from an empirical point of view, as is witnessed by the work of Herzig et al. [21], Soo and Radke [22], Payatakes [23] and Tien and Payatakes [82]. This might be ascribed to one major objective of the study of dusty fluids in porous media, namely the process of filtration, where sand filters have received certain popularity. In one aspect, it has always been assumed that the flow is governed by Darcy’s law and, therefore, a phenomenological approach to the subject matter is thought of as possible from the point of view of considering single-phase fluid flow through a porous medium with variable porosity. Details of this approach and an extensive literature review is given by Tien and Payatakes [82].

Filtration, or better known deep-bed filtration, is an old process that received considerable attention during the days of Darcy as a means of water purification. This long-known idea of filtration is currently used in the design of industrial filters, in
water purification plants and in the study of ultrafiltration which is important in the medical treatment of kidney problems [24]. Aerosol filters have also been reported [23] in addition to the most up-to-date self-cleaning filters [24].

Another aspect of filtration is when the fluid is composed of dilute emulsions flowing in a porous medium. This matter has been discussed in detail by Soo and Radke [22], Alvarado and Marsden [83] and Devereux [84,85], and it differs from the flow of suspension through porous media in that the emulsion particles squeeze into the pores resulting in flow re-distribution.

In spite of the great amount of work devoted to the different aspects of the flow of a dusty fluid through porous media, the up-to-date literature reports only on the flow through media composed of either sand or granular materials, the porosity and the permeability of which might be thought of as small. The type of flow is always considered as governed by Darcy's law, and the subject matter is studied to determine retention rates and probability of capture, in addition to other pertinent quantities that are important in the determination of the filter efficiency [21,23,82].

In cases where the porous medium is of the type suitable for the flow to be governed by one of the
non-Darcian models discussed in section 1.2, or of the type that is composed of a sparse distribution of particles fixed in space, the current work proposes to approach the study of dusty fluid through porous media from the continuum point of view. The continuum approach is applied to each of the phases present and, via intrinsic volume averaging, a set of partial differential equations is developed to govern each of the phases involved.

The governing sets of equations are coupled together through appropriate forcing terms which take into account the interaction between the phases involved and the effect of the porous matrix on the flowing fluid. To accomplish this, Darcy's law is modified to take into account the relative seepage velocity of the phases involved.

Details of the methodology used and the models developed are given in chapter 5. For the sake of completeness, this chapter includes a discussion of the relationship between the proposed models and the existing ideas about the flow of a dusty fluid in porous media.
1.3.2.2. Numerical Study of Dusty Fluid Flow through Porous Media

1.3.2.2.1. Dusty Fluid Flow into a Line Sink

In an attempt to offer comparisons between the different dusty fluid flow models in porous media, developed in chapter 5, the flow of a dusty fluid in a porous medium into a line sink is considered when the flow is governed by the different dusty fluid flow models. The flow domain is the same as the one considered in the study of single-phase fluid flow into a line sink, discussed in section 1.3.1.2.

In order to accomplish the study, the governing equations for each of the developed models are cast into vorticity-streamfunction form in chapter 6. The validity of the different models is discussed after the numerical solution to each of the models is obtained in chapter 7. The effects of the presence of dust on the flow patterns, velocity profiles and on the development of corner eddies are discussed in details for each of the models. The effects of the dust parameters, permeability and Re are also discussed in chapter 7.
1.3.2.2. Dusty Fluid Flow over Curved Boundaries

Due to the complexity of the governing equations and the coupling between them, solutions of dusty gas flows in closed domains are, in general, scarce. This lack of solutions is not only noticed in analytical solutions, but also in numerical solutions.

Although there exists a number of numerical methods that are applicable to a variety of flow situations [86–92], dusty fluid flow separation in closed domains is an even more challenging problem, especially in cases where the particle number density is a quantity to be determined. When the number density is assumed constant, Barron and Hamdan [86] have considered the structure of separated dusty gas eddies in a closed domain. In cases where the flow of the dusty fluid is taken to be in an irregular domain, it has been customary to map the domain using coordinate transformations [87] and then employ the well-known trajectory model [87, 91, 92].

In the current work, the flow of a dusty fluid through a porous medium overlaying a curved boundary is considered. The study is undertaken to illustrate the determinate nature of the proposed dusty fluid flow models through porous media, discussed in chapter 5. The flow domain is similar to the one considered in the study
of single-phase flow through porous media overlaying curved boundaries.

In the case of single-phase fluid flow, the von Mises transformation was adopted, as discussed in chapters 2 and 3. In an attempt to parallel that method, a new approach to the numerical study of dusty fluids is proposed here. A complete discussion of the proposed method is given in chapter 6, and is an extension of the von Mises coordinate transformation in the sense that a double transformation is introduced such that the two sets of equations governing the two phases in the same physical domain are transformed into two sets of equations governing the two phases in the same computational domain. This approach, as shall be seen in chapters 6 and 8, might very well serve as a replacement to the trajectory models and the particle tracking techniques which have received popularity in the dusty gas literature. With this new approach, the fluid-phase streamlines may be predicted. The dust-phase streamlines can also be predicted and since they represent the movement of the dust particles, this method eliminates the need for particle tracking method. The particle number density may also be determined using this approach.

In chapter 6, the details of the new method are
given and the governing equations are cast in terms of the extended von Mises coordinates. In chapter 8, the results of the implementation of the method applied to the study of dusty fluid flow through porous media over curved boundaries are discussed.

It is worthwhile noting that in a manner similar to this approach, Soo [40] considered the flow of a conducting dusty gas between parallel plates under the assumption of a constant fluid-phase velocity. In his formulation [40], the dust-phase equations were transformed in terms of the dust-phase streamfunction and, thus, the regions of dust accumulation were determined by plotting the dust-phase streamlines. The current method is more general, offering more flexibility and ability to handle a variety of flow situations.

1.4. SUMMARY

A review of the leading models governing the flow of a single-phase fluid through porous media has been made. This has been undertaken to accommodate the testing of the von Mises transformation in the numerical study of flow through porous media. In addition, this review has furnished the grounds for gaining a deeper insight into
the occurrence of flow separation as the permeability increases. This covers the discussion of the first part of this work.

Another topic that has been reviewed in this chapter is the dusty fluid flow through porous media. The lack of mathematical models describing this type of two-phase fluid flow gave rise to the new flow models which are derived in the second part of this work. In testing these models, in a parallel way to the first part of this work, a new method evolved. The method may be considered as an extended form, or double transformation, of the von Mises transformation.

The optimism of writing a piece of work of the current size gives rise to the inevitable problem of selection. As in every one's work, the first impulse is usually to cover every topic. Selection then starts by narrowing down the scope of the work. For this reason, the current work may by no means be classified as complete. Different aspects of the flows considered are hardly touched or analysed. The number of parameters involved, together with the variety of flow situations, may take a number of years to be fully analysed. On the other hand, the proposed dusty fluid flow models represent an initial step for further studies. More general studies and studies that incorporate the
different aspects of swelling, compressibility and the unsteady-state form of the equations are needed. Dispersion, diffusion, convection and stability analyses of the developed models are also needed to verify the applicability of the models in the study of atmospheric processes, filtration processes and in biomedical research.
CHAPTER 2

THE VON MISES TRANSFORMATION

2.1. INTRODUCTION

In 1927 von Mises [93] introduced a coordinate transformation, now recognized as the von Mises transformation, to transform the two dimensional boundary layer equations into a form in which the independent variables \( x, y \) are replaced by \( x, \psi \), respectively, where \( \psi \) is the streamfunction.

Although the von Mises transformation has been known for over half a century it has mainly been of considerable importance in theoretical boundary layer investigations [94]. In a deviation from boundary layer analysis, Benjamin [95] employed the transformation in his study of solitary waves with arbitrary vorticity distribution.

Numerical implementation of the von Mises transformation came about in 1986 when Barron [37] presented a formulation and computations involving von
Mises variables, in his study of flow over airfoils of arbitrary shapes, as a substitute to grid generation. His analysis showed that the von Mises transformation can be arrived at through Martin's approach [96], of which the von Mises transformation is a special case, obtained by a 'judicious choice of the coordinate curves'.

The success of the transformation in the numerical study of a particular flow problem relies heavily on the requirement that each of the boundaries of the flow domain remains a streamline, or part of a streamline. The streamlines $\psi = \text{constant}$, which may not be straight lines in the physical plane, are mapped into horizontal straight lines in the transformed plane and $\psi$ replaces the original independent variable.

The problem of determining $\psi$ in the physical plane is replaced by the problem of determining $y$ in the rectangular computational plane. Other flow variables of interest (velocity, pressure and vorticity) become, in the computational plane, functions of $x$ and $\psi$.

In this work we consider the two dimensional flow of a viscous, incompressible and homogeneous fluid through a homogeneous porous medium of constant permeability. The porous domains considered in the current analyses are assumed to have curved boundaries. The method of solution for a particular problem is to
transform the governing equations into the new coordinate system so that the curved boundaries of the porous domains are transformed into straight lines, with the whole of the domain being thus transformed into a rectangular region, designated the computational domain.

Two methods are possible to transform the governing equations into the new coordinates: the first relies on the direct transformation of the derivatives and the primitive variables involved by using the transformation operators which will be discussed in section 2.4. In this case the pressure terms are treated through the introduction of an energy function and the resulting equations are cast in the vorticity-Y form. Details of this method are explained in the work of Barron [37]. The second approach is to cast the governing equations in vorticity-streamfunction form and then to transform these equations in terms of the new coordinate transformation. This second approach is implemented here since the vorticity-streamfunction formulation is also used in this work to study the flow through porous media into a line sink, as will be discussed in chapter 4.

In section 2.2, the equations governing the flow in each of the porous media models of interest and the Navier-Stokes equations are cast in vorticity-streamfunction form. In section 2.3, the
pressure equations pertaining to each of the flow models are derived, for the sake of completeness. In section 2.4, the differential operators stemming from the von Mises transformation are derived and the governing equations in vorticity-streamfunction form and the pressure equations are transformed in terms of the von Mises coordinates.

2.2. VORTICITY-STREAMFUNCTION FORMULATION

After expressing each of the momentum equations in terms of its components in the cartesian plane, the pressure terms can be eliminated by cross differentiation. Letting Ψ represent the dimensionless streamfunction for the Navier-Stokes equations and the macroscopic streamfunction for each of the models, defined in terms of the dimensionless velocity components by

\[
U = \frac{\partial \Psi}{\partial Y} \quad \text{and} \quad V = -\frac{\partial \Psi}{\partial X},
\]

... (2.1)

and letting Ω represent the dimensionless macroscopic vorticity, given in terms of the velocity components by
\[ \Omega = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}, \] 

... (2.2)

the following dimensionless vorticity-streamfunction representations are obtained. For the porous media flow models, the streamfunction equation is of the same form (2.3) as for the Navier-Stokes equation and thus only the vorticity equation is indicated for each of these models.

2.2.1. Navier-Stokes Equations

**Streamfunction equation**

\[ \Omega = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}, \] 

... (2.3)

**Vorticity equation**

\[ \frac{\partial \Omega}{\partial t} + \frac{\partial \psi}{\partial x} \frac{\partial \Omega}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial \Omega}{\partial y} = \frac{1}{Re} \left( \frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} \right), \] 

... (2.4)
2.2.2. Darcy’s Law

\[ \Omega = 0 \quad \ldots (2.5) \]

2.2.3. Brinkman’s Equation

\[ \Omega = \theta K \left( \frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} \right) \quad \ldots (2.6) \]

2.2.4. Darcy-Lapwood Model

\[ \frac{\partial \Omega}{\partial T} + \frac{\partial \psi}{\partial y} \frac{\partial \Omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \Omega}{\partial y} = - \frac{1}{K \text{Re}} \Omega \quad \ldots (2.7) \]

2.2.5. Darcy-Lapwood-Brinkman’s Model

\[ \frac{\partial \Omega}{\partial T} + \frac{\partial \psi}{\partial y} \frac{\partial \Omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \Omega}{\partial y} = \frac{1}{\text{Re}} \left( \frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} - \frac{\Omega}{K} \right) \quad \ldots (2.8) \]
2.2.6. Darcy-Forchheimer's Model

\[
\frac{\partial \Omega}{\partial T} + \Omega \left( \frac{1}{\text{Re} \ K} + \frac{C_d}{\sqrt{K}} \sqrt{\left( \frac{\partial \Psi}{\partial x} \right)^2 + \left( \frac{\partial \Psi}{\partial y} \right)^2} \right)
\]

\[
= \frac{C_d}{\sqrt{K}} \left( \frac{\left( \frac{\partial \Psi}{\partial x} \right)^2 \frac{\partial^2 \Psi}{\partial x^2} + 2 \frac{\partial \Psi}{\partial x} \frac{\partial \Psi}{\partial y} \frac{\partial^2 \Psi}{\partial x \partial y} + \left( \frac{\partial \Psi}{\partial y} \right)^2 \frac{\partial^2 \Psi}{\partial y^2}}{\left( \frac{\partial \Psi}{\partial x} \right)^2 + \left( \frac{\partial \Psi}{\partial y} \right)^2} \right)
\]

\[\ldots (2.9)\]

The above equations were rendered dimensionless with respect to a characteristic length, L, and a characteristic velocity, \( U_o \), using the following dimensionless quantities:

\[ X = x/L, \ Y = y/L, \ V = v/U_o, \ T = t U_o / L, \ \Omega = \omega L/U_o \]

\[ K = k/L^2, \ \Psi = \psi/L U_o \] and the Reynolds number is defined as \( \text{Re} = \rho U_o L / \mu \). Variables represented by lower-case Greek and Roman letters represent the dimensional quantities, and those represented by upper-case letters are the dimensionless variables.
2.3. PRESSURE EQUATIONS

To cast the governing equations in vorticity-streamfunction form the pressure terms were eliminated from the governing equations. Once $\Psi$ and $\Omega$, and hence $U$ and $V$ are determined, the pressure can then be determined. The pressure equation is derived by taking the divergence of the momentum equations in vector form and thus the following dimensionless equations are obtained for each of the models discussed above, where the nondimensional pressure is defined by

$$P = \frac{p}{\rho U_0^2}.$$ 

2.3.1. Navier-Stokes Pressure Equation

For unsteady, two-dimensional flow of an incompressible, viscous fluid, equation (1.1) can be expressed in dimensionless component form, in $X$- and $Y$-directions respectively, as

$$\frac{\partial P}{\partial x} = \frac{1}{Re} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} \frac{\partial U}{\partial t}$$

... (2.10)
\[
\frac{\partial P}{\partial Y} = \frac{1}{Re} \left( \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) - U \frac{\partial V}{\partial X} - V \frac{\partial V}{\partial Y} - \frac{\partial V}{\partial T} \quad \ldots \quad (2.11)
\]

and the pressure equation takes the form

\[
\frac{\partial^2 P}{\partial X^2} + \frac{\partial^2 P}{\partial Y^2} = - \left\{ \left( \frac{\partial U}{\partial X} \right)^2 + 2 \frac{\partial V}{\partial X} \frac{\partial U}{\partial Y} + \left( \frac{\partial V}{\partial Y} \right)^2 \right\} \quad \ldots \quad (2.12)
\]

The pressure equation (2.12) shows that the effect of Reynolds number, Re, does not enter through this equation but rather through the expressions for the pressure gradients, given by equations (2.10) and (2.11). In order to find the pressure, P, in a given flow problem, for a given Re, it is necessary to incorporate equations (2.10) and (2.11) through specification of the boundary conditions. The effect of Re at the boundary is then transmitted into the flow domain, since the boundary conditions influence the solution of equation (2.12).

2.3.2. The Darcy's Pressure Equation

In the absence of body forces, equation (1.5) can
be expressed in the following, dimensionless, component form in the \( X \)- and \( Y \)-directions, respectively.

\[
\frac{\partial P}{\partial x} = -\frac{U}{Re \, K} \quad \ldots \text{(2.13)}
\]

\[
\frac{\partial P}{\partial y} = -\frac{V}{Re \, K} \quad \ldots \text{(2.14)}
\]

and, therefore, the Darcy's pressure equation takes the form

\[
\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0 \quad \ldots \text{(2.15)}
\]

An alternative approach to determining the pressure in Darcy's systems is to define the dimensionless velocity potential by

\[
\phi = Re \, K \, P \quad \ldots \text{(2.16)}
\]

and thus equations (2.13) and (2.14) are expressed in
terms of $\phi$ as

$$U = - \frac{\partial \phi}{\partial x} \quad \ldots \ (2.17)$$

and

$$V = - \frac{\partial \phi}{\partial y} \quad \ldots \ (2.18)$$

where $\phi$ is seen to satisfy

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \ldots \ (2.19)$$

Once $\phi$ is determined, the pressure $P$, for a given permeability, can then be determined through equation (2.16).

It should be noted that in the definition of $Re$ the length $L$ in the context of Darcy's systems refers to a characteristic pore diameter, and thus for many practical situations the choice of $Re \leq 1$ is quite acceptable [1].
2.3.3. Brinkman's Pressure Equation

The pressure gradients in this case are obtained from the dimensionless form of equation (1.10) and take the following form

\[
\frac{\partial P}{\partial x} = \frac{1}{Re} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) - \frac{1}{Re} \frac{U}{K} \quad \ldots \ (2.20)
\]

\[
\frac{\partial P}{\partial y} = \frac{1}{Re} \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) - \frac{1}{Re} \frac{V}{K} \quad \ldots \ (2.21)
\]

and the pressure equation takes the form

\[
\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0 \quad \ldots \ (2.22)
\]

In equations (2.20) and (2.21), \( Re^* \) is the effective Reynolds number defined by \( Re^* = \frac{p L U_0}{\mu_{ef}} \).

Although Brinkman's equation (1.10) is valid under
the assumption that the effects of inertial terms are negligible as compared to the dominant viscous terms, the appearance of Reynolds numbers in equations (2.20) and (2.21) does not violate the above assumption. These Reynolds numbers arise in the nondimensionalization process and emphasize the concept of different viscosities associated with equation (1.10). It can also be seen that the ratio $\theta$, appearing in equations (1.18) and (2.8), also expresses the ratio of the Reynolds numbers appearing in equations (2.20) and (2.22), namely

$$\theta = \frac{Re_\text{e}}{Re} = \frac{\mu_\text{ef}}{\mu}. \quad \ldots \ (2.23)$$

It should be noted that the form of the pressure equation (2.28) is in agreement with the results of Ooms et al. [27].

For the sake of simplicity, the current work considers the viscosities, $\mu$ and $\mu_\text{ef}$, equal, and therefore $\theta = 1$.

2.3.4. Darcy-Lapwood Pressure Equation

The dimensionless component form of the momentum equations (1.13), in the $X$- and $Y$-directions,
respectively, are given by

\[
\frac{\partial P}{\partial x} = -\frac{U}{Re\ K} - \left( U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + \frac{\partial U}{\partial t} \right) \quad \ldots (2.24)
\]

\[
\frac{\partial P}{\partial y} = -\frac{V}{Re\ K} - \left( U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + \frac{\partial V}{\partial t} \right) \quad \ldots (2.25)
\]

while the pressure equation takes the form

\[
\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = - \left( \left( \frac{\partial U}{\partial x} \right)^2 + 2 \frac{\partial V}{\partial x} \frac{\partial U}{\partial y} + \left( \frac{\partial V}{\partial y} \right)^2 \right) \quad \ldots (2.26)
\]

2.3.5. Darcy-Lapwood-Brinkman Pressure Equation

Equation (1.16) can be expressed in dimensionless component form to yield the pressure gradients in \( x \)- and \( y \)-directions, given respectively by

\[
\frac{\partial P}{\partial x} = \frac{1}{Re} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \frac{U}{K} \right) - \left( U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + \frac{\partial U}{\partial t} \right) \quad \ldots (2.27)
\]
\[
\frac{\partial P}{\partial Y} = \frac{1}{Re} \left( \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} - \frac{V}{K} \right) - \left( U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} + \frac{\partial V}{\partial T} \right) \ldots (2.28)
\]

and the pressure equation takes the form

\[
\frac{\partial^2 P}{\partial X^2} + \frac{\partial^2 P}{\partial Y^2} = -\left( \left( \frac{\partial U}{\partial X} \right)^2 + 2 \frac{\partial V}{\partial X} \frac{\partial U}{\partial Y} + \left( \frac{\partial V}{\partial Y} \right)^2 \right) \ldots (2.29)
\]

2.3.6. Darcy-Forchheimer Pressure Equation

Equation (1.15) can be written in dimensionless component form, in X- and Y-directions respectively, as

\[
\frac{\partial P}{\partial X} = -\frac{U}{Re K \sqrt{U^2 + V^2}} - \frac{\partial U}{\partial T} \ldots (2.30)
\]

\[
\frac{\partial P}{\partial Y} = -\frac{V}{Re K \sqrt{U^2 + V^2}} - \frac{\partial V}{\partial T} \ldots (2.31)
\]

while the pressure equation takes the form

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\[
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = - \frac{C_d}{\sqrt{K}} \left\{ \frac{UV \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) + \frac{\partial V}{\partial y} \left( V^2 - U^2 \right)}{\sqrt{U^2 + V^2}} \right\}. \]
\[
\ldots (2.32)
\]

2.4 The von Mises Form of the Governing Equations

2.4.1. The Transformation

Consider the two dimensional transformation of coordinates between the dimensionless cartesian coordinates \((X,Y)\), and dimensionless time \(T\), and the curvilinear coordinates \((\Phi, \Psi)\), and time \(\tau\), defined by

\[
\Phi = \Phi (X) \quad \ldots (2.33)
\]

\[
\Psi = \Psi (X,Y,T) \quad \ldots (2.34)
\]

\[
\tau = \tau (T) \quad \ldots (2.35)
\]

Expanding (2.33), (2.34) and (2.35) by the chain rule, the following expressions are obtained:
\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial x} \frac{\partial}{\partial \psi} + \frac{\partial}{\partial x} \frac{\partial}{\partial \psi} + \frac{\partial}{\partial x} \frac{\partial}{\partial t} \quad \ldots \quad (2.36)
\]

\[
\frac{\partial}{\partial y} = \frac{\partial}{\partial y} \frac{\partial}{\partial \psi} + \frac{\partial}{\partial y} \frac{\partial}{\partial \psi} + \frac{\partial}{\partial y} \frac{\partial}{\partial t} \quad \ldots \quad (2.37)
\]

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t} \frac{\partial}{\partial \psi} + \frac{\partial}{\partial t} \frac{\partial}{\partial \psi} + \frac{\partial}{\partial t} \frac{\partial}{\partial \psi} \quad \ldots \quad (2.38)
\]

Taking \( x = \xi \) and \( t = \tau \), equations (2.36), (2.37) and (2.38), take the following form, respectively,

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \psi} \quad \ldots \quad (2.39)
\]

\[
\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} \quad \ldots \quad (2.40)
\]

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \psi} \quad \ldots \quad (2.41)
\]
The spatial Jacobian of the above transformation is given by

\[ J_1 = \left| \begin{array}{c} \frac{\partial (X,Y)}{\partial (\xi,\eta)} \\ \frac{\partial (\xi,\eta)}{\partial (\psi,\tau)} \end{array} \right| = \frac{\partial Y}{\partial \psi} \]

... (2.42)

while the temporal Jacobian of the transformation is given by

\[ J_2 = \left| \begin{array}{c} \frac{\partial (X,Y)}{\partial (\xi,\tau)} \\ \frac{\partial (\xi,\tau)}{\partial (\psi,\tau)} \end{array} \right| = \frac{\partial Y}{\partial \tau} \]

... (2.43)

It should be noted at this point that if \( J_1 = 0 \), or is infinite, or equivalently \[ \frac{\partial Y}{\partial \psi} = 0 \], or is infinite, then the transformation is singular.

If \( J_2 \equiv 0 \), or equivalently \[ \frac{\partial Y}{\partial \tau} \equiv 0 \], then the transformation reduces to the steady transformation, that is the usual von Mises transformation. If, however, \( 0 < |J_1| < \infty \) and \( 0 < |J_2| < \infty \) then equations (2.33), (2.34) and (2.35), with \( X = \xi \) and \( T = \tau \), can be solved for \[ \frac{\partial \psi}{\partial X}, \frac{\partial \psi}{\partial Y} \] and \[ \frac{\partial \psi}{\partial T} \] in terms of the first
derivatives of $Y$, and thus the following expressions are obtained:

$$\Psi_T = -\frac{Y_T}{Y_{\Psi}} = -\frac{J_2}{J_1} \quad \ldots \ (2.44)$$

$$\Psi_X = -\frac{Y_{\phi}}{Y_{\Psi}} = -\frac{Y_{\phi}}{J_1} \quad \ldots \ (2.45)$$

$$\Psi_Y = \frac{1}{Y_{\Psi}} = \frac{1}{J_1} \quad \ldots \ (2.46)$$

where subscript notation denotes partial differentiation.

Substituting (2.44), (2.45) and (2.46) into (2.39), (2.40) and (2.41), the following first derivative expressions in the new coordinate system are obtained, where the independent variable $\tau$ is replaced by its equivalent $X$, and $\tau$ is replaced by $T$:

$$\partial_X = \partial_X - \frac{Y_X}{Y_{\Psi}} \partial_{\Psi} \quad \ldots \ (2.47)$$

$$\partial_Y = \frac{1}{Y_{\Psi}} \partial_{\Psi} \quad \ldots \ (2.48)$$
\[ \partial_T = \partial_T - \frac{y_T}{y_\psi} \partial_{y_\psi}. \quad \cdots (2.49) \]

Expressions for the second order spatial derivatives can also be obtained in the new coordinate system by applying these operators on themselves:

\[ \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x^2} - 2 \left( \frac{y_x}{y_\psi} \right) \frac{\partial^2}{\partial x \partial y} + \left( \frac{y_x}{y_\psi} \right)^2 \frac{\partial^2}{\partial y^2} \]

\[ + \left( \frac{2 y_x y_{xx}}{y_\psi^2} - \frac{y_{xx}}{y_\psi} \right) \frac{\partial}{\partial y} \quad \cdots (2.50) \]

\[ \frac{\partial^2}{\partial y^2} = \frac{1}{y_\psi^2} \frac{\partial^2}{\partial y^2} \quad \frac{\partial^2}{\partial y_\psi^2} - \frac{y_{y_\psi}}{y_\psi^3} \frac{\partial}{\partial y_\psi} \quad \cdots (2.51) \]

The cross derivative takes the form:

\[ \frac{\partial^2}{\partial x \partial y} = \frac{y_x}{y_\psi^3} \frac{\partial^2}{\partial y^2} + \frac{y_x y_{y_\psi} - y_{y_\psi} y_x y_\psi}{y_\psi^3} \frac{\partial}{\partial y_\psi}. \quad \cdots (2.52) \]
2.4.2. Flow Equations in the von Mises Variables

As indicated in section 2.2, the equations governing single-phase fluid flow in porous media and in free space were expressed in terms of the two-dimensional streamfunction and vorticity. In this section the roles of the dimensionless streamfunction and the independent variable $Y$ have been interchanged to yield new independent variables, $X$, $\Psi$ and $\tau$, in terms of which the flow variables are to be expressed. The governing equation that the streamfunction $\Psi$ satisfies will be transformed into an equation that the new dependent variable $Y$ satisfies, where $Y = Y(X,\Psi,\tau)$. Furthermore, other flow variables that are originally functions of $X$, $Y$ and $T$ and satisfy the governing equations in $\Omega - \Psi$ form, as discussed in section 2.2, become functions of the new independent variables $X$, $\Psi$ and $T$ and thus satisfy the governing equations in the transformed $\Omega - Y$ form.

To this end, the two-dimensional velocity components $U$ and $V$, defined by equations (2.1), are transformed using equations (2.33) and (2.34) to yield:

$$U = \frac{1}{Y\Psi} \quad \ldots \quad (2.33)$$
\[ v = \frac{y_x}{y_\psi} = u \frac{y_x}{y_\psi} \quad \ldots \quad (2.54) \]

The governing equations in \( \Omega - \Psi \) form, discussed in section 2.2, are now expressed in terms of the new variables for general unsteady flow. The steady states of these governing equations in \( \Omega - Y \) form can be deduced. The pressure equations are also transformed into the new coordinates.

2.4.2.1. Navier-Stokes Equations in von Mises Coordinates

The vorticity and the streamfunction equations, (2.3) and (2.4), are transformed into \( \Omega - Y \) equations using the transformation equations (2.47) through (2.52) to yield the \( Y \)-equation of the form

\[ \frac{y_\psi}{y_x} y_{xx} - 2 \frac{y_x}{y_\psi} y_\psi y_{x\psi} + \left( 1 + \frac{y_x}{y_\psi} \right) y_{\psi\psi} = \frac{y_\psi}{y_x} \Omega. \]

\[ \ldots \quad (2.55) \]

The \( \Omega \) - equation takes the following form when the flow is unsteady.
\[ \Omega_T = \frac{Y_T \Omega_\Psi - \Omega_X}{Y_\Psi} - \frac{1}{Re} \Omega_\Psi \]

\[ + \frac{1}{Re} \left\{ \frac{\Omega_{XX} - 2 \frac{Y_X}{Y_\Psi} \Omega_X \Psi + \left[ \frac{1 + Y_X^2}{Y_\Psi^2} \right] \Omega_\Psi \Psi}{Y_\Psi} \right\} \]. \quad \ldots (2.56) \]

The pressure equation (2.12) associated with the Navier-Stokes equations is also transformed using the above mentioned transformation to yield the following equation in terms of the von Mises variables,

\[ Y_\Psi^2 P_{XX} - 2 Y_X Y_\Psi P_X \Psi + \left[ 1 + Y_X^2 \right] P_\Psi \Psi \]

\[ - \Omega Y_\Psi^2 P_\Psi = \frac{2 Y_{XX} Y_\Psi \Psi - 2 Y_X \Psi}{Y_\Psi^2} \]. \quad \ldots (2.57) \]

while the pressure gradients, equations (2.10) and (2.11), are also transformed using (2.47) through (2.49) to yield the following gradients in terms of the von Mises variables, respectively.
\[ y_\psi p_x - y_x p_\psi = \frac{y_{x\psi}^2}{y_\psi^2} - u_t + \frac{y_t}{y_\psi} u_\psi \]

\[ + \frac{1}{\text{Re} \ y_\psi} \left[ y_\psi^2 u_{xx} - 2 y_x y_\psi u_{x\psi} + \left( 1 + y_x^2 \right) u_{\psi\psi} \right] + \Omega y_{\psi\psi} \] ...

(2.58)

\[ p_\psi = -\frac{y_\psi y_{\psi\psi} - y_x y_{x\psi}}{y_\psi^2} - v_t + \frac{y_t}{y_\psi} v_\psi \]

\[ + \frac{1}{\text{Re} \ y_\psi} \left[ y_\psi^2 v_{xx} - 2 y_x y_\psi v_{x\psi} + \left( 1 + y_x^2 \right) v_{\psi\psi} \right] - \Omega \left( y_\psi y_{x\psi} - y_x y_{\psi\psi} \right) \] ...

(2.59)

### 2.4.2.2. Darcy's Law in von Mises Coordinates

The streamfunction equation for Darcy's regime, in terms of von Mises coordinates, takes a similar form to equation (2.55) with \( \Omega = 0 \). The governing equation in
this case is the \( Y \)-equation which takes the form

\[
Y_{\psi}^2 \psi_{XX} - 2 Y_{\psi} \psi_{X} \psi_{XX} + \left( 1 + Y_{\psi}^2 \right) \psi_{X\psi} = 0 \quad \ldots (2.60)
\]

and expresses the irrotationality of the flow.

The pressure equation (2.18) associated with Darcy's law takes the following form in \( X-\psi \) plane

\[
Y_{\psi}^2 \psi_{XX} - 2 Y_{\psi} \psi_{X} \psi_{XX} + \left( 1 + Y_{\psi}^2 \right) \psi_{X\psi} = 0 \quad \ldots (2.61)
\]

with the pressure gradients, equations (2.18) and (2.17), taking the following forms, respectively,

\[
Y_{\psi} \psi_{X} - Y_{\psi} \psi_{X} = - \frac{1}{Re \ K} \quad \ldots (2.62)
\]

\[
\psi_{X} = - \frac{Y_{\psi}}{Re \ K} \quad \ldots (2.63)
\]

2.4.2.3. Brinkman's Equation in von Mises Coordinates

The vorticity equation (2.8) takes the following
transformed form in the $X-\Psi$ plane

$$
\left\{ \frac{\Omega_{xx}}{Y_\Psi} - 2 \frac{Y_X Y_\Psi}{Y_\Psi} \frac{\Omega_{x\Psi}}{Y_\Psi} + \left[ \frac{1 + Y_X^2}{Y_\Psi^2} \Omega_{\Psi\Psi} - \Omega_X \Omega_\Psi \right] \right\} = \frac{\Omega}{K}
$$

...(2.64)

while the streamfunction equation takes the form of equation (2.55). The pressure equation (2.22) takes the form

$$
Y_\Psi^2 P_{xx} - 2 Y_X Y_\Psi P_{x\Psi} + \left[ 1 + Y_X^2 \right] P_{\Psi\Psi}
$$

$$
- \Omega Y_\Psi^2 P_\Psi = 0. \quad ...(2.65)
$$

The pressure gradients, equations (2.20) and (2.21), are also transformed to take the following forms, respectively, in the $X-\Psi$ plane:

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\[ y_\psi P_x - y_x P_\psi = \frac{-1}{\text{Re } K} \]

\[ + \frac{1}{\text{Re } y_\psi} \left[ y_\psi^2 u_{xx} - 2 y_x y_\psi u_{x\psi} + \left( 1 + y_x^2 \right) u_{\psi\psi} \right. \]

\[ + \left. \Omega \frac{\partial}{\partial \psi} \right] \] \hspace{1cm} \ldots (2.66)

\[ P_\psi = -\frac{y_x}{\text{Re } K} \]

\[ + \frac{1}{\text{Re } y_\psi} \left[ y_\psi^2 v_{xx} - 2 y_x y_\psi v_{x\psi} + \left( 1 + y_x^2 \right) v_{\psi\psi} \right. \]

\[ - \left. \Omega \left( y_\psi \frac{\partial}{\partial \psi} - y_x \frac{\partial}{\partial \psi} \right) \right] \] \hspace{1cm} \ldots (2.67)

### 2.4.2.4. Darcy-Lapwood Model in von Mises Coordinates

The vorticity equation (2.7) is transformed into the \( X-\Psi \) plane and takes the following form
\[ \Omega_T = \frac{Y_T \Omega_X + \Omega_X}{Y_\Psi} + \frac{1}{K \text{Re}} \Omega \] ... (2.68)

and the streamfunction equation takes the form of equation (2.55).

The pressure equation (2.26) is also transformed to yield

\[ Y_\Psi^2 P_{XX} - 2 Y_X Y_\Psi P_{X\Psi} + \left(1 + Y_X^2\right) P_{\Psi\Psi} \]

\[- \Omega Y_\Psi^2 P_{\Psi} = \frac{2 Y_{XX} Y_{\Psi\Psi} - 2 Y_{X\Psi}}{Y_\Psi^2} \] ... (2.69)

The pressure gradients associated with the Darcy-Lapwood model, namely equations (2.24) and (2.25), are transformed to the following forms, respectively.

\[ Y_\Psi P_X - Y_X P_\Psi = \frac{-1}{\text{Re} K} + \frac{Y_{X\Psi}}{Y_\Psi^2} - U_T + \frac{Y_T}{Y_\Psi} U_\Psi \] ... (2.70)

\[ P_\Psi = \frac{-Y_X}{\text{Re} K} - \frac{Y_\Psi Y_{XX} - Y_X Y_{X\Psi}}{Y_\Psi^2} - V_T + \frac{Y_T}{Y_\Psi} V_\Psi \] ... (2.71)
In this model the vorticity equation (2.8) is transformed into the $X-\Psi$ plane and takes the following form

$$\Omega_T = \frac{Y_T \Omega_{\Psi} - \Omega_X}{Y_{\Psi}} - \frac{1}{Re} \frac{\Omega \Omega_{\Psi}}{Re K}$$

$$+ \frac{1}{Re} \left[ \Omega_{XX} - 2 \frac{Y_X}{Y_{\Psi}} \Omega_{X \Psi} + \left( 1 + \frac{Y_{X}^2}{Y_{\Psi}^2} \right) \Omega_{\Psi \Psi} \right] \ldots (2.72)$$

while the streamfunction equation takes the same form as equation (2.54). The pressure equation (2.29) associated with this model takes the following transformed form

$$Y_{\Psi}^2 P_{XX} - 2 Y_X Y_{\Psi} P_{X \Psi} + \left[ 1 + Y_X^2 \right] P_{\Psi \Psi}$$

$$- \Omega Y_{\Psi}^2 P_{\Psi} = \frac{2 Y_{XX} Y_{\Psi \Psi} - 2 Y_{X \Psi} Y_{\Psi}^2}{Y_{\Psi}^2} \quad \ldots (2.73)$$
while the pressure gradients, equations (2.27) and (2.28), are also transformed to yield the following gradients, respectively, in terms of the von Mises variables

\[
\begin{align*}
P_{\Psi} X - Y_{\Psi} P_{\Psi} &= \frac{Y_{\Psi} X_{\Psi}}{Y_{\Psi}^2} - \frac{1}{Re \, K} - U_{T} + \frac{Y_{T}}{Y_{\Psi}} U_{\Psi} \\
&+ \frac{1}{Re \, Y_{\Psi}} \left[ Y_{\Psi}^2 U_{XX} - 2 Y_{X} Y_{\Psi} U_{X\Psi} + \left( 1 + Y_{X}^2 \right) U_{\Psi\Psi} \right] + \Omega Y_{\Psi\Psi} \right] \quad \ldots \text{(2.74)}
\end{align*}
\]

\[
\begin{align*}
P_{\Psi} &= -\frac{Y_{\Psi} Y_{\Psi\Psi} - Y_{X} Y_{X\Psi}}{Y_{\Psi}^2} - \frac{Y_{X}}{Re \, K} - V_{T} + \frac{Y_{T}}{Y_{\Psi}} V_{\Psi} \\
&+ \frac{1}{Re \, Y_{\Psi}} \left[ Y_{\Psi}^2 V_{XX} - 2 Y_{X} Y_{\Psi} V_{X\Psi} + \left( 1 + Y_{X}^2 \right) V_{\Psi\Psi} \right] - \Omega \left( Y_{\Psi} Y_{X\Psi} - Y_{X} Y_{\Psi\Psi} \right) \quad \ldots \text{(2.75)}
\end{align*}
\]
2.4.2.6. Darcy-Forchheimer Model in von Mises Coordinates

The transformation equations (2.47) through (2.52) are employed to transform the vorticity equation (2.9), which yields

\[ Y_{\Omega} \Omega_T - Y_T \Omega_{\Psi} = - \frac{\Omega Y_{\Psi}}{Re_K} - \frac{C_d}{\sqrt{K}} \Omega \sqrt{1 + \frac{y^2}{X}} \]

\[ + \frac{C_d}{\sqrt{K} \sqrt{1 + \frac{y^2}{X}}} \left\{ \frac{2 Y_X + 2 Y_X^3}{Y_{\Psi}} \right\} Y_{X\Psi} - \frac{y_X^2 Y_{XX}}{Y_{\Psi}} \]

\[ - \frac{\left(1 + Y_X^4 + 2 Y_X^2\right)}{Y_{\Psi}^3} \right\} \]

... (2.76)

The streamfunction equation for this model is also similar to equation (2.55). For steady state, equations (2.76) and (2.55) can be combined to yield, after simplification, the following single equation for \( Y \):
\[
\left\{ (1 + 2 Y_x^2) Y_{\Psi}^2 \sqrt{1 + Y_x^2} + \frac{(1 + Y_x^2) Y_{\Psi}^3}{\text{Re} \ C_d \sqrt{K}} \right\} Y_{XX} - \\
\left\{ (4 Y_x^2 + 4 Y_x^3) Y_{\Psi} \sqrt{1 + Y_x^2} + 2 Y_x \frac{(1 + Y_x^2) Y_{\Psi}^3}{\text{Re} \ C_d \sqrt{K}} \right\} Y_{X\Psi} \\
+ \left\{ (2 + 4 Y_x^2 + 2 Y_x^4) \sqrt{1 + Y_x^2} + \frac{(1 + Y_x^2) Y_{\Psi}^2}{\text{Re} \ C_d \sqrt{K}} \right\} Y_{\Psi\Psi} \\
= 0. \quad \ldots \text{(2.77)}
\]

The pressure equation (2.32) associated with this model is transformed to yield the following equation in $X-\Psi$ plane:

\[
Y_{\Psi}^2 P_{XX} - 2 Y_x Y_{\Psi} P_{X\Psi} + \left\{ 1 + Y_x^2 \right\} P_{\Psi\Psi} - \Omega Y_{\Psi}^2 P_{\Psi} \\
= \frac{-C_d}{\sqrt{K} Y_{\Psi} \left[ 1 + Y_x^2 \right]^{1/2}} \left\{ Y_x Y_{\Psi} - 1 \right\} Y_{XX} - Y_x^2 Y_{X\Psi} \\
= \ldots \text{(2.78)}
\]

with the pressure gradients, equations (2.30) and (2.31), transformed, respectively, into the forms

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\[ Y_\Psi P_x - Y_X P_\Psi = \frac{-1}{\text{Re} K} - \frac{C_d}{Y_\Psi \sqrt{K}} \left( 1 + Y_X^2 \right)^{1/2} - U_T + \frac{Y_T}{Y_\Psi} U_\Psi \] ... (2.79)

\[ P_\Psi = \frac{-Y_X}{\text{Re} K} - \frac{C_d}{Y_\Psi \sqrt{K}} \left( 1 + Y_X^2 \right)^{1/2} - V_T + \frac{Y_T}{Y_\Psi} V_\Psi . \] ... (2.80)

2.5. SUMMARY

In this chapter the Navier-Stokes equations and the equations governing the two-dimensional flow of a viscous fluid through homogeneous porous media have been cast into vorticity-streamfunction form and then transformed, using unsteady von Mises transformations, into forms which are convenient for a numerical study in a rectangular computational domain. The von Mises form of the governing equations will be used in chapter 3 to obtain the steady state solutions of flow through porous media overlying a static fluid with curved interface. The vorticity-streamfunction form of the governing equations
will be employed in chapter 4 to solve the steady flow through different porous media into a line sink.
3.1 INTRODUCTION

In order to illustrate the numerical implementation of the von Mises approach, discussed in chapter 2, and to show its importance and applicability in the study of fluid flow through porous media, we consider the single-phase fluid flow through a semi-infinite porous medium that is bounded below by a static fluid. The interface between the static fluid and the porous medium is assumed to take the form, in the cartesian plane, $Y = f(X)$, where $f(X)$ is a smooth function of $X$.

The viscous, incompressible single-phase fluid is assumed to flow steadily through the given porous medium, overlying the static fluid, as shown in Fig. 3.1. The boundary conditions are those of uniform flow at infinity and the flow tangency condition along the porous medium-static fluid interface. Since, in the von Mises approach, the roles of $Y$ and $\Psi$ have been interchanged and

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the governing equations have been expressed in terms of the new independent variables $X$, $\Psi$, as explained in chapter 2, the solution to a given set of governing equations is, therefore, sought in the $CX, \Psi$ plane, for the new dependent variable $Y$, and for the vorticity $\Omega$.

Taking the interface between the porous medium and the static fluid to represent the zero streamline of the flow and the farthest $Y$-curve, at infinity, to be a streamline of the flow as well, our physical plane is then transformed into the computational $X-\Psi$ plane, as depicted in Fig. 3.2, which also gives the boundary conditions in the computational plane.

In this work, five models of single-phase flow through porous media are treated in order to illustrate the similarities and differences between these models as they apply to the current physical situation. Solutions are obtained, therefore, for each of the models for different porous media parameters.

In the situation where the flow through the porous medium is of the Darcy type, flow over a static fluid of arbitrary shape have been solved by the finite difference method. We also include the case where the lower wall is impermeable. Although the vanishing velocity on a solid boundary implies an infinite Jacobian of the von Mises transformation, the current work illustrates how this
problem can be overcome for Darcy's flow. The applicability of the von Mises transformation in determining the pressure is illustrated for the Darcy model through the determination of the perturbation potential.

When the porous medium is assumed to be naturally occurring [57], the fluid flow is described by Forchheimer's equation which has proven to be easily transformed into the von Mises coordinates inspite of the complex structure of this equation. Fluid flow described by Brinkman's model has also been studied and the results are compared with those obtained for Darcy's law and for the Forchheimer's equation. For the same flow domain the Darcy-Lapwood-Brinkman and Darcy-Lapwood equations are solved numerically. Numerical solutions to fluid flow governed by the Darcy-Lapwood model is obtained by a method of "artificial vorticity" which is introduced in the model equation through the Darcy-Lapwood-Brinkman's model. The failure of the Darcy-Lapwood model in describing certain flow situations is also illustrated. The failure of the Darcy-Lapwood model in this case has resulted in a modified version of this model based on an artificial vorticity method.
3.2. BOUNDARY CONDITIONS

The physical boundary conditions discussed above, are expressed, for a given interface geometry \( Y = f(X) \), in the following form in the computational plane:

1. Uniform flow at infinity:

\[
Y = \Psi \text{ at } X = \pm \infty \text{ and at } \Psi = \infty \quad \ldots (3.1)
\]

\[
\Omega = 0 \text{ at } X = \pm \infty \text{ and at } \Psi = \infty \quad \ldots (3.2)
\]

2. Flow tangency (along the interface):

\[
Y(X,0) = f(X) \quad -\infty < X < \infty \quad \ldots (3.3)
\]

The vorticity along the interface remains a quantity to be determined.

3.3. FINITE DIFFERENCE APPROXIMATIONS

3.3.1. The Computational Grid

For the purpose of this work, the extent of the computational domain is taken to be four times the length of the major axis of the geometrical shape of \( f(X) \), i.e.
4 times the chord length, in either of the \( \Psi \)-direction and the \( X \)-direction.

In order to obtain numerical solutions to the problems posed in section 3.1, subject to the boundary conditions of section 3.2, the computational \( X-\Psi \) plane is discretized using a uniform rectangular grid. A typical grid point in this computational domain is labelled as grid point \((i,j)\), for \( i = 1, 2, \ldots, I_{\text{max}} \), and \( j = 1, 2, \ldots J_{\text{max}} \), with the grid line \( i = 1 \) corresponding to \( X = -2 \), and \( i = I_{\text{max}} \) corresponding to \( X = 2 \). Similarly, \( j = 1 \) is taken to correspond to \( \Psi = 0 \) and \( j = J_{\text{max}} \) corresponds to \( \Psi = 2 \). For the purpose of this work, the grid size is chosen to be \( I_{\text{max}} \times J_{\text{max}} = 64 \times 22 \). Although it can be argued that the accuracy of the solutions obtained can be improved by choosing a much finer grid, it has been found that the chosen mesh sizes of \( \Delta X = 4/63 \) and \( \Delta \Psi = 2/21 \) are sufficient to illustrate our objectives of the possible implementation of the von Mises transformation in the study of flow through porous media.

It should be noted that solutions for fine grid are also possible when this approach is used and, in fact, have been obtained when the flow is of Darcy's type in order to illustrate that there are no "numerical blow-ups" when the grid is refined. This has been illustrated for a 122x22 grid.
3.3.2. The Finite Differencing Procedure

Following Barron [37], for the incompressible flow considered each of the governing equations is linearized and solved iteratively by considering the second order derivatives at the \( n+1 \)st iteration level while the coefficients are evaluated at the \( n \)th iteration level. Three-point central differencing of all of the derivatives involved has been employed and the resulting difference equations are then expressed in the form

\[
A_{i,j} F_{i,j-1}^n + B_{i,j} F_{i,j}^n + C_{i,j} F_{i,j+1}^n = D_{i,j} \quad \ldots (3.4)
\]

which is suitable for the successive line relaxation procedure with sweep in the \( i \)-direction, i.e. relaxation along the \( j \)-direction. In equation (3.4), \( F_{i,j}^n \) stands for the unknown quantity to be determined, at the \( n+1 \)st iteration level, and \( A_{i,j}^n, B_{i,j}^n, C_{i,j}^n \) and \( D_{i,j}^n \) are coefficients. It should be noted that the matrices resulting from expressing (3.4) at every grid point \((i,j)\) in the flow field, and incorporating the numerical values of the boundary conditions, are tridiagonal. The diagonal elements are \( B_{i,j}^n \) and the super-diagonal and sub-diagonal elements are \( A_{i,j}^n \) and \( C_{i,j}^n \), respectively, for \( i = 2, 3, \ldots, I_{\text{max}}-1 \) and \( j = 2, 3, \ldots, J_{\text{max}}-1 \). The right-hand
side vector of known quantities is $D_{i,j}$, for the same values of $i$ and $j$, above. These differenced forms are incorporated in the STATIC Computer Code in Appendix A.

3.4. SOLUTION PROCEDURE

It is clear that the diagonal matrices resulting from representing all of the governing equations in the form (3.4) are diagonally dominant. These diagonally dominant matrices are inverted using a tridiagonal solver (Thomas' algorithm) with successive line over-relaxation (SLOR) for the streamfunction equations and under-relaxation for the vorticity equations.

For each of the models considered, the solution is obtained by implementing the solution procedure which is summarized as follows:

For a given permeability $K$, and Reynolds number, $Re$ where applicable,

1. The flow domain is initialized by giving $Y$ and $\Omega$ some small starting values.

2. The $Y$-equation is solved for $Y$, using the tridiagonal solver, along each grid line, $i = 2$ to $I_{\text{max}}-1$, and the solution is iterated along each $i$-line using SLOR of the form

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\[ Y_{n+1} = Y^n + \omega Y_{n+1} \quad \ldots (3.5) \]

where \( Y_{n+1} \) is the value obtained from the tridiagonal solver, and \( Y^n \) is the value obtained from the previous iteration. This step is repeated a few times in order to accelerate convergence.

3. \( U_{i,1}, U_{i,2}, V_{i,1}, V_{i,2}, \Omega_{i,1} \) are calculated, along the static interface, using the expressions derived in section section 3.5.

4. The vorticity equation is solved for \( \Omega \) by a similar procedure to that in step 2.

5. Steps 2 to 4 are repeated until the following convergence criterion is met:

\[ | P_{i,j}^{n+1} - P_{i,j}^n | \leq \delta \quad \ldots (3.6) \]

where \( \delta = 5 \times 10^{-3} \).

6. The velocity components \( U \) and \( V \) are calculated along the static interface and in the field using the expressions derived in section 3.5.

It should be noted that when dealing with Darcy's and Forchheimer's equations, the steps in the above algorithm concerning the vorticity are redundant. In Darcy's case, equation (2.80) is solved until
convergence. The velocity components are then calculated in the flowfield and along the interface. The \( Y \)-values are then used as the initialization step for the rest of the models. This practice has proven to result in faster convergence.

In the case of Forchheimer's model, equation (2.77) is solved using the previous algorithm with the steps dealing with vorticity ignored. The vorticity, however, is calculated in the flowfield by an explicit second order accurate finite difference form of equation (2.55). Along the static interface, the vorticity is calculated using the expressions derived in section 3.5.

3.5. VELOCITY EXPRESSIONS

Since the interface is given by \( Y = f(X) \) and, in terms of the von Mises variables, we have

\[
U = \frac{1}{Y_{\Psi}} \quad \ldots \quad (3.7)
\]

and

\[
V = \frac{Y_{X}}{Y_{\Psi}} = U Y_{X} \quad \ldots \quad (3.8)
\]
U can be calculated along the interface using a first order accurate forward differencing scheme of the form

\[ U_{i,1} = \frac{\Delta \Psi}{Y_{i,2} - f(X_i)} \]  \hspace{1cm} \ldots (3.9)

\[ V_{i,1} \] can thus be calculated in terms of \( U_{i,1} \) and \( f'(X_i) \), and takes the form

\[ V_{i,1} = U_{i,1} f'(X_i) \]  \hspace{1cm} \ldots (3.10)

The vorticity along the static interface is determined using a first order accurate scheme of the form

\[ \Omega_{i,1} = \frac{V_{i+1,1} - V_{i,1}}{\Delta X} - V_{i,1} \left( \frac{V_{i,2} - V_{i,1}}{\Delta \Psi} \right) - U_{i,1} \left( \frac{U_{i,2} - U_{i,1}}{\Delta \Psi} \right) \]  \hspace{1cm} \ldots (3.11)

Higher order accurate differencing schemes for the velocity components and the vorticity along the static interface either resulted in delayed convergence (in
Brinkman's model), no convergence (Darcy-Lapwood-
Brinkman's model), or in velocity profiles for the
Darcy's model that were not symmetric. These first order
accurate schemes are justifiable in the sense that they
are only valid at the boundary of the computational
domain, but not used in the computational domain itself.

The velocity components are evaluated at internal
grid points using second order accurate schemes of the
form

$$U_{i,j} = \frac{2 \Delta \Psi}{Y_{i,j+1} - Y_{i,j-1}}$$  \hspace{1cm} ... (3.12)

$$V_{i,j} = \frac{Y_{i+1,j} - Y_{i-1,j}}{Y_{i,j+1} - Y_{i,j-1}} \frac{\Delta \Psi}{\Delta x}$$  \hspace{1cm} ... (3.13)

for $i = 2, 3, \ldots, I_{\text{max}} - 1$ and $j = 2, 3, \ldots, J_{\text{max}} - 2$.

3.8. RESULTS AND ANALYSIS

For the sake of clarity, this section is sub-divided
into three parts, the first of which demonstrates the
results of the calculations involving Darcy's law. The
second part deals with the results of Forchheimer's and Brinkman's equations, and the third part discusses the solutions of the Darcy-Lapwood model, the Darcy-Lapwood modified model and the Darcy-Lapwood-Brinkman equation.

3.6.1. Darcy's Regime

3.6.1.1. Analyses

When Darcy's law is the equation governing the fluid flowing through the porous medium the applicability of the von Mises transformation is illustrated by considering two types of lower boundaries. The first is the case when the porous medium is bounded from below by a static fluid. The interface between the two regions is referred to as the static interface. In the second case, the lower boundary is assumed to be a solid boundary. In both cases the lower boundary is described mathematically by \( Y = f(X) \).

Although the transformation is valid only when the Jacobian of the transformation takes non-zero values and remains finite, it will be shown that the vanishing Jacobian does not arise in the current investigation and
the case of infinite Jacobian, which arises when the lower boundary is a solid wall, can be treated numerically in the type of flow considered. Darcy's flow is analysed in this study by numerically solving for $Y(X, \Psi)$ and then determining the velocity potential, $\phi$, defined by equations (2.16) to (2.19). The potential $\phi$ satisfies the following equation in von Mises variables:

$$Y^2 \frac{\partial^2 \phi}{\partial X^2} - 2 Y \frac{\partial Y}{\partial X} \frac{\partial \phi}{\partial X} + (1 + Y^2) \frac{\partial^2 \phi}{\partial \Psi^2} = 0. \quad \ldots (3.14)$$

In the von Mises coordinate system, the velocity potential and velocity components are related by

$$U = -\phi_X + \frac{Y}{Y_\Psi} \frac{\partial \phi}{\partial \Psi}, \quad \ldots (3.15)$$

$$V = -\frac{1}{Y_\Psi} \phi_\Psi. \quad \ldots (3.16)$$

In order to determine $\phi$, it is convenient to introduce the perturbation potential, $\phi^*$, given by

$$\phi = -X + \phi^*. \quad \ldots (3.17)$$
From (3.14) and (3.17), \( \phi^* \) satisfies

\[
y \frac{\phi^*_{xx}}{\psi} - 2 \frac{\phi^*_x}{\psi} \left( 1 + \frac{\phi^*_x}{\psi} \right) \frac{\phi^*_{\psi}}{\psi} = 0 \quad \cdots (3.18)
\]

in the flowfield and Neumann conditions on the boundary, obtained by applying equation (3.15) at \( X = \pm \infty \) and equation (3.16) along \( \Psi = 0 \) and \( \Psi = \infty \). Thus it is easily seen that \( \phi^* \) satisfies

\[
\phi^*_x = 0 \quad \text{at} \quad X = \pm \infty \quad \cdots (3.19)
\]

\[
\phi^*_\Psi = 0 \quad \text{at} \quad \Psi = \infty \quad \cdots (3.20)
\]

\[
\phi^*_\Psi = -f'(X) \quad \text{at} \quad \Psi = 0. \quad \cdots (3.21)
\]

Equation (3.18) is central differenced, with its coefficients calculated from the converged values of \( Y \), and the solution is obtained for \( \phi^* \) using successive line over-relaxation. The values of \( \phi^* \) at the boundary are calculated after every iteration by defining \( \phi^* \) at the boundary in terms of its values at internal grid points via first order accurate one-sided differencing of
equations (3.19) through (3.21). The iterative procedure is repeated until convergence, where the convergence criterion used is the same as that satisfied by Y.

When the porous medium is bounded below by a solid wall, equations (3.14) and (2.60) are still valid, for \( \phi \) and \( Y \), respectively. Furthermore, the vanishing velocity on the solid wall does not influence the determination of \( Y \) through equation (2.60). This is due to the fact that the solid wall is still a streamline and the boundary conditions on \( Y \) are Dirichlet. Hence the solution obtained is the same as that obtained when the tangency condition is applied. The effect of the vanishing velocity is, nevertheless, transmitted through the determination of the velocity potential. It should be noted at this point that when the flow is governed by Darcy's law the velocity boundary conditions far upstream have to be compatible with Darcy's law and thus, in the case of a solid lower boundary, the only allowable upstream velocity condition is that of a constant velocity at infinity, with the velocity dropping to zero along the solid wall.

Although the spatial Jacobian of the transformation, defined by equation (2.42), remains a finite non-zero value everywhere in the flowfield it becomes infinite on the solid boundary. This fact leads to difficulty in
determining the velocity potential along the solid wall since equation (3.21) does not hold on the solid boundary \( \Psi = 0 \). The velocity potential is nevertheless determined in the flowfield, in this case, by applying the condition

\[
\phi^* \Psi = -Y_x \tag{3.22}
\]

along the first streamline adjacent to \( \Psi = 0 \). Thus equation (3.18) is solved in the region \( 2 \leq i \leq I_{\text{max}} - 1, 3 \leq j \leq J_{\text{max}} - 1 \), subject to the conditions given by equations (3.19), (3.20) applied along \( i = 1 \) and \( i = I_{\text{max}} \), and along \( j = J_{\text{max}} \). Along \( j = 2 \), the condition (3.21) is employed in forward-differenced form, and the solution to equation (3.21) satisfying the above conditions is obtained by a similar procedure to the one used for the case when the tangency condition was employed.

3.8.1.2. Results and Discussion

Numerical solutions have been obtained when the interface between the porous medium and the static fluid is given by
\[
f_1(x) = \begin{cases} 
0.2 \left( 0.25 - x^2 \right)^{1/2}, & -0.5 \leq x \leq 0.5 \\
0, & x < -0.5 \text{ or } x > 0.5
\end{cases}
\]

\(\ldots \ (3.23)\)

\[
f_2(x) = \begin{cases} 
-0.2 \left( 0.25 - x^2 \right)^{1/2}, & -0.5 \leq x \leq 0.5 \\
0, & x < -0.5 \text{ or } x > 0.5
\end{cases}
\]

\(\ldots \ (3.24)\)

\[
f_3(x) = \begin{cases} 
0.1 \left( 1 - 2x \right) \left( 1 - 4x^2 \right)^{1/2}, & -0.5 \leq x \leq 0.5 \\
0, & x < -0.5 \text{ or } x > 0.5
\end{cases}
\]

\(\ldots \ (3.25)\)

where the points \(x = 0.5\) and \(x = -0.5\) are chosen to fall between grid points.

In this analysis the results are based on the computational domain \(0 \leq \psi \leq 2, \ -4 \leq x \leq 4\), although \(-2 \leq x \leq 2\) and \(-6 \leq x \leq 6\), with \(0 \leq \psi \leq 2\), were also tested. The extent of the domain in the \(\psi\)-direction has proven to be satisfactory as illustrated in Figure 3.3,
which demonstrate the streamlines of the flow when the lower boundary is given by \( f_3(X) \) and shows that the streamlines at infinity straighten out.

The above figure also indicates the acceptable extent of the domain in the \( X \)-direction since the streamlines at \( X = -4 \) and at \( X = 4 \) appear to be straight and thus reflect the boundary conditions employed. Although the choice \(-2 \leq X \leq 2\) reflected a similar behaviour, a larger computational domain has proven to be necessary when solving the perturbation potential equation. This is due to the fact that although the \( U \)-velocity component is taken to be unity at \( X = 2 \), the calculated values of this velocity component are less than unity. Part of this discrepancy is ascribed to round-off errors and errors due to the accuracy of the differencing scheme, but also suggests that the extent of the computational domain might not be large enough to correctly impose the boundary conditions on \( \phi^* \). The results obtained using \(-4 \leq X \leq 4\) and \(-8 \leq X \leq 8\) have proven to be close to each other and, to achieve relatively fine grid calculations, the range \(-4 \leq X \leq 4\) is employed in the current calculations. Convergence of the solution, in terms of increasing accuracy with which the continuity equation is satisfied, has been achieved for fine grids. Our results are based on a 122x22 grid
for the flow tangency case and on a 84x22 grid when the lower boundary is a solid wall. In this latter case the extent of the domain in the X-direction was taken to be $-2 \leq x \leq 2$.

Figure 3.4 illustrates the horizontal slip velocity component at the lower boundary along the nonzero parts of $f_1(x)$, $f_2(x)$ and $f_3(x)$ and illustrates the symmetry of this component for $f_1(x)$ and $f_2(x)$. The maximum values of this component occur at around the maxima of the functions defining the interface and the minima occur at the minima of the functions. The U-component of velocity along the vertical lines passing through points around the maxima of the functions are illustrated in Fig. 3.5, which demonstrates the decrease in these components as we move away from the interface. Although the decrease is drastic in the regions closer to the interface, the components are seen to assume more uniform profiles of almost constant values in the upper parts of the domain. This conclusion is consistent with the well known flow structure when Darcy's law is used.

In Fig. 3.6 the vertical slip velocity component at the lower boundary is illustrated over the nonzero part of the above functions and shows that the maximum values of this component occur at the minima of the functions
and the minimum values occur at the maxima of the functions.

The perturbation potential along the interface, over the nonzero parts of the functions, is illustrated in Fig. 3.7 while the velocity potential is illustrated in Fig. 3.8, when the lower boundary is a static fluid. In the case of a solid lower boundary, Fig. 3.9 illustrates the velocity potential distribution along the grid line (streamline) that is closest to the solid wall.

3.6.2. Brinkman's and Forchheimer's Models

3.6.2.1. Analysis

From the previous analysis using Darcy's law, it has become clear that Darcy's law does not distinguish between the slip and the no-slip boundary conditions. This has been witnessed by the fact that the $Y$ values determined for both of the cases involved are identical. Furthermore, compatibility of the boundary conditions with Darcy's law forces the incoming flow profile (boundary condition at $X = -\infty$) to be uniform and thus becomes inappropriate in describing cases of no-slip boundary conditions and simple entry conditions like the
Poiseuille-type flow, even though the idea of plug flow is quite common in porous media [10]. At this point one must reject Darcy’s law and provide an alternative model in describing these flow phenomena. This has given rise to extensions of Darcy’s law, not only extensions to describe three dimensional and unsteady flows but also extensions to better handle the above discrepancies. Two such extensions are given by Brinkman’s and Forchheimer’s equations, whose solutions are compared in this section.

Due to the viscous nature of the fluid, the presence of a solid boundary requires the imposition of a no-slip condition along that boundary. In the von Mises transformation, the no-slip condition implies the transformation is singular along the boundary. For rotational flows this singularity, although it might not influence the Y-equation, does indeed make the determination of the vorticity on the solid wall rather impossible. The determination of \( \Omega \) on the solid boundary might be resolved using some other approach and thus the spirit of using the von Mises transformation to simplify matters might be lost. In as far as this work is concerned, the problem of no-slip along a solid surface, and consequently the singularity of the von Mises transformation, even in fluid flow problems through porous media, remains unresolved.
For the reasons cited above, the Brinkman's and the Forchheimer's equations have been analysed for the flow through a porous medium overlaying a static fluid.

3.6.2.2. Results and Discussion

Numerical solutions have been obtained when the interface between the porous medium and the static fluid is given by $f_4(X)$, in equation (3.23).

Results are obtained in the case of Brinkman's model for $K = 0.001$, $0.01$, $0.1$ and $K \to 1$. In the case of Forchheimer's model results are obtained for $Re \sqrt{K} C_d = 1$, $5$, $9$ and $25$. In the analysis to follow the behaviour of some of the components were very similar to each other for the cases of $Re \sqrt{K} C_d = 5$, $9$ and $25$ and thus only one of these values has been considered, unless a change in the behaviour was noticed. The results are based on the computational domain $0 \leq \psi \leq 2$, $-2 \leq X \leq 2$. The streamline pattern for Forchheimer's model is illustrated in Fig. 3.10 for $Re \sqrt{K} C_d = 1$, and shows that the streamlines at infinity straighten out. This plot also indicates the acceptable extent of the domain in the $X$-direction since the streamlines at $X = -2$ and at $X = 2$ appear to be straight and thus reflect the boundary conditions employed. Although the streamlines for the
different models considered differ slightly in their quantitative behaviour, their qualitative behaviour follows a similar pattern to that shown for the Forchheimer's case. The quantitative behaviour of the various models is illustrated in the velocity and vorticity profiles which are discussed in what follows. The results are based on a 64x22 grid.

Fig. 3.11 illustrates the horizontal slip velocity component at the lower boundary. Fig. 3.11(a) gives the comparison between Darcy's and Forchheimer's solutions and illustrates the increase in this component over the hump with a decrease in the product $Re\sqrt{K} C_d$. In Darcy's regime the slip velocity along the interface is greater than that of Forchheimer's case. For both of these cases, the symmetry of the profiles is clear. In Fig. 3.11(b), the U-velocity component along the interface is illustrated for the case of Brinkman's flow for different permeabilities. A decrease in the permeability results in an increase in this velocity component over the hump. As the permeability becomes small, e.g., $K = 0.001$, the behaviour resembles that of Darcy's and, in fact, at this value of $K$ the numerical results are very close to each other. As the permeability is increased, the loss of symmetry of the velocity profile becomes apparent.
The X-component of velocity along the vertical line passing through points near the maximum of \( f(X) \) are illustrated in Figs. 3.12(a) and 3.12(b). In Fig. 3.12(a), the comparison between the Darcy's case and the Forchheimer's case is given and illustrates the decrease in this component as we move away from the interface with decreasing \( \text{Re} \sqrt{K} C_d \). In this upper region, the velocity is slower for the Darcy's case. In regions close to the interface, the opposite behaviour occurs. In Fig. 3.12(b), the effect of the permeability on the U-velocity component is illustrated to show that in regions close to the interface this velocity component increases with decreasing permeability. Again, the results for Brinkman's model with \( K = 0.001 \) are very close, numerically, to the case of Darcy's model. This behaviour is, of course, expected since Brinkman's model is expected to represent Darcy's law for low permeability.

It should be noted that these profiles use the calculated values of \( U \) at infinity rather than the boundary values of \( U \). This is to be consistent with errors involved in the computations and thus \( U \) was calculated using a first order accurate scheme. This explains the deviation of about 0.01 in the plotted values. Part of this discrepancy is ascribed to round-off errors and errors
due to the accuracy of the differencing scheme, but also suggests that the extent of the computational domain might not be large enough to correctly impose the boundary conditions.

In Fig. 3.13 the vertical slip velocity component at the lower boundary is illustrated over the nonzero part of $f(x)$ and shows that the maximum absolute values of this component occur at the minimum of $f(x)$ and the minimum absolute values occur at the maximum of $f(x)$. The velocity profile in Fig. 3.13 is representative of all of the models considered since the numerical values for this velocity component are very close to each other for the different permeabilities and the different models considered.

In Figs. 3.14(a) and 3.14(b), the vorticity over the hump is illustrated for Forchheimer's model and Brinkman's model, respectively, and demonstrates the slight decrease in vorticity with increasing $Re\sqrt{K}c_d$ in Forchheimer's model, and an increase in the vorticity with decreasing permeability in Brinkman's model. The vorticity along the interface reaches a higher maximum in the case of Brinkman's model. This maximum is, of course, approaching zero with decreasing permeability and as $K$ approaches a very small value the vorticity should approach zero, which will be the point where Darcy's
model provides a good representation of the flow.

The vorticity along the vertical line passing through points near the maximum of \( f(x) \), for the Forchheimer's and Brinkman's models, is illustrated in Figs. 15(a) and 15(b). As \( \text{Re} \sqrt{K} C_d \) increases, the vorticity decreases, i.e. becomes larger in absolute value, for the Forchheimer's case, and as \( K \) decreases, in the Brinkman's case, the vorticity decreases in absolute value and approaches zero which is, once again, the expected behaviour. It should also be noted that a further decrease in the product \( \text{Re} \sqrt{K} C_d \) is expected to produce results that are close to that of Darcy's law.

This conclusion can also be seen from the Forchheimer's model, equation (2.77), as it reduces to Darcy's law, (2.80), if the drag coefficient is taken to be zero, while the permeability is kept constant and \( \text{Re} \) is taken to be small enough to be compatible with Darcy's law.

3.6.3. The Darcy-Lapwood and the Darcy-Lapwood-Brinkman Models

3.6.3.1. Analyses

As mentioned in chapter 1, the Darcy-Lapwood equation is postulated to be valid for flow through a
sparse distribution of particles fixed in space [32].
Arguing from a convection analysis point of view, some authors stressed the errata in the inclusion of the inertia term in Darcy’s law, in the manner that Lapwood has done [16]. This is due to the fact that the inclusion of such a term in Darcy’s law raises the order of the equation which in turn renders a given boundary value problem indeterminate due to the inavailability of additional boundary conditions to accompany this increase in the order of the governing equations [16]. Beck [16] indicated that an under-determined system results, when using Lapwood’s equation, if the normal component of velocity is the only condition imposed on the boundary and an over-determined system results if the velocity distribution is given on the boundary. This implies that the Darcy-Lapwood equation is not a useful model in describing some physical flow phenomena in porous media. It was pointed out by Nield and Joseph [57] that the inclusion of the Laplacian term in the Darcy-Lapwood model should render a given boundary value problem determinate.

In the current work the flow through a porous medium overlying a static fluid is considered. The cases where the flow is assumed to be governed by the Darcy-Lapwood-Brinkman model and by the Darcy-Lapwood
model are considered in an attempt to illustrate how the Darcy-Lapwood equation may be modified, and how to handle the increase in the order of the equation so that it becomes compatible with the boundary conditions. The similarity between the two models, except for the viscous shearing effect which arises in the Darcy-Lapwood-Brinkman model, suggests solving the Darcy-Lapwood model through the introduction of what is termed here as the artificial vorticity.

Although the boundary conditions given by equations (3.1) to (3.3) are compatible with the Darcy-Lapwood-Brinkman model they are, nevertheless, incompatible with the Darcy-Lapwood model as can be seen in the following analysis. The vorticity equation (2.68), associated with the Darcy-Lapwood model, takes the following form when the flow is steady

\[
\frac{\Omega_x}{\nu} + \frac{1}{K \text{Re}} \Omega = 0. \quad \cdots (3.26)
\]

This is a homogeneous first order partial differential equation in \( \Omega \), and possesses homogeneous boundary conditions. The solution to the BVP (3.26) is, therefore, \( \Omega = 0 \). Furthermore, the effect of the lower boundary is
only transmitted to the differential equation (3.26) through the horizontal component of velocity along that boundary, and thus the differential equation is not directly dependent on the vorticity value along the lower boundary due to the presence of a derivative of $\Omega$ with respect to $X$ only. From a numerical viewpoint, central differencing equation (3.26) and expressing it in matrix form results in a system of linear equations of the form

$$A \Omega = 0 \quad \ldots (3.27)$$

when the boundary conditions on $\Omega$ are incorporated, where $A$ is the tridiagonal matrix $TRIDX(-\alpha_{i,j}, 1, \alpha_{i,j})$, where

$$\alpha_{i,j} = \frac{\Delta \Psi K Re}{\Delta X (Y_{i,j+1} - Y_{i,j-1})}, \quad i = 2, 3, \ldots, I_{\text{max}}-1, \quad j = 1, 2, \ldots, J_{\text{max}}-1,$$

and $\Omega$ is the vorticity solution vector.

It is clear that the system given by (3.27) has a non-trivial solution iff the determinant of $A \equiv 0$, which is not the case. We conclude, therefore, that the Darcy-Lapwood model is not compatible with the boundary conditions associated with the problem at hand. This
problem, however, can be overcome by modifying the Darcy-Lapwood model based on the Darcy-Lapwood-Brinkman model by introducing viscous shear effects into the Darcy-Lapwood equation (1.13), or equivalently into equation (2.7), and therefore equation (2.7) is replaced by

\[
\frac{\partial \Omega}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \Omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \Omega}{\partial y} = - \frac{1}{\text{Re}} \frac{\partial \Omega}{\partial y} - \frac{1}{\text{Re} \text{K}} \left( \frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} \right) \quad (3.28)
\]

where \( \text{Rec} \) is a dimensionless number. Equation (3.28) is referred to here as the modified form of the Darcy-Lapwood model. This equation includes an "artificial vorticity" term and thus it is compatible with the boundary conditions imposed in the flow problem discussed above. It should be noted that if \( \text{Rec} = \text{Re} \), then equation (3.28) is the same as the Darcy-Lapwood-Brinkman equation (2.8). As \( \text{Rec} \) becomes large, \( \text{Rec} \to \infty \), then equation (3.28) reduces to the Darcy-Lapwood equation (2.7).

Equation (3.28) takes the following form in terms of
the von Mises variables:

$$\Omega_T = \frac{Y_T \Omega_{\psi} - \Omega_X}{y_\psi} - \frac{\Omega}{\text{Re } K} - \frac{\Omega}{\text{Rec}}$$

$$+ \frac{1}{\text{Re}} \left[ \Omega_{XX} - 2 \left( \frac{Y_X}{y_\psi} \right) \Omega_{X\psi} + \frac{1}{y_\psi^2} \Omega_{\psi\psi} \right]$$

$$\ldots \text{(3.29)}$$

3.6.3.2. Results and Discussion

The Darcy-Lapwood-Brinkman model will be referred to as the DLB model in this discussion. The Darcy-Lapwood model will be referred to as the DL model, and the modified Darcy-Lapwood model will be referred to as the DL modified model.

Numerical solutions have been obtained when the static interface is given by $f_i(X)$, as defined in equation (3.23). Results are obtained for $K = 0.0001$, $0.001$, $0.01$, $0.1$ and $K \rightarrow 1$, for the DLB model and for the DL modified model. The range of Re considered for the DLB model is 10 to 500, while for the DL modified model the Re range 100, 200, 300, 400 and 500 has been considered. The Rec range in the case of DL modified
model is 500, 1000, 10,000 and 100,000. The extent of the computational domain is the same as that discussed in section 3.6.2.2.

The behaviour of the DLB model for a given Re and different permeabilities, and for a given permeability and different Re, is demonstrated in the velocity and vorticity profiles which are discussed in the following.

Fig. 3.16 illustrates the horizontal slip velocity component at the lower boundary. Fig. 3.16(a) gives the comparison of this velocity component for Re = 10 and different permeability K, and indicates the increase in this component over the hump with a decrease in the permeability. Over the zero portion of the interface, an increase in this component is noticed with increasing permeability, especially downstream of the hump. The symmetry of the profiles is lost with the increase in permeability.

The effect of Re, for a given permeability, on this component of velocity is illustrated in Fig. 3.16(b), where the velocity increases with increasing Re over all sections of the interface. The X-component of velocity along the vertical line passing through points near the maximum of f(X) are illustrated in Fig. 3.17, where the effect of decreasing permeability, for a given Re, is seen to decrease this component as we move away from the
interface. A similar pattern of behaviour was noticed for the case of Brinkman’s and Forchheimer’s models.

In Figs. 3.18(a) and 3.18(b), the vorticity over the hump is illustrated for different permeabilities, and a given Re, and for different Re, and a given permeability, respectively. It is clear that decreasing K results in an increase in the vorticity, as indicated in Fig. 3.18(a), while increasing Re from 10 to 500 results only in a slight increase in the vorticity, for a given permeability, as indicated in Fig. 3.18(b).

This behaviour for large Re, and low permeability, as illustrated in Figures 3.18(b) and 3.18(b), indicates that for this type of flow the Reynolds number has to be small, and it might imply that after a certain value of Re any further increase does not have a great influence on the results. This pattern of behaviour is not what happens in the usual flow of a viscous fluid in free space, where one expects a noticeable increase in the vorticity and velocity components when Re is increased from 10 to 500. For the D LB model, the results indicate that the most important parameter of the flow is the permeability. This is also clear in Fig. 3.19 which illustrates the noticeable change in the vorticity along the vertical line passing through a point near the maximum of $f_1(X)$. 

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For the DL modified model, Fig. 3.20 illustrates the streamline pattern for the case of \( \text{Re} = 200, \ K = 0.001 \) and \( \text{Rec} = 1000 \). The flow pattern follows a similar qualitative behaviour to that of Brinkman's.

As indicated in the analyses of section 3.8.3.1., as the parameter Rec approaches infinity, the DL modified model reduces to the DL model. This is illustrated in Fig. 3.21 which shows the unexpected behaviour of the slip, that is the U-component of velocity, along the static interface. This profile is obtained for \( \text{Re} = 100, \ K = 0.001 \) and for \( \text{Rec} = 10,000 \). This idea is further illustrated in Fig. 3.24, which shows the vorticity along the hump of the static interface, and indicates an unexpected behaviour when \( \text{Rec}=10,000 \), and compares it with the results obtained when \( \text{Rec} = 1000 \). In Fig. 3.25, the vorticity along the vertical line passing near the maximum of \( f(x) \) is illustrated for \( \text{Re} = 100, \ K = 0.01 \), and for \( \text{Rec} = 1000 \) and 100,000. The closeness of the vorticity to zero when \( \text{Re} = 100,000 \) is not surprising.

For other values of Rec, Fig. 3.22 illustrates this velocity component for \( \text{Re} = 100, \ K = 0.01 \) and \( \text{Rec} = 500 \) and 1000. Comparison is also made with the results from the DLB model for the same permeability and Reynolds number. The closeness of these profiles is surprising enough to suggest that the term \( \nabla^2 \Omega \) that appears in the
DLB model, although essential, might not be very significant in the sense that even though one-fifth of this term was accounted for, in the DL modified model with \( \text{Re}_{c} = 500 \), the indicated velocity profiles are very close, numerically, to each other. The behaviour is also illustrated in Fig. 3.23, where the U-component profile along a vertical line passing near the maximum of \( f_{u}^{*}(x) \) is plotted for different \( K, \text{Re}, \text{Rec} \) and compared with the results of the DLB model.

3.7. CONCLUSIONS

The applicability of the von Mises coordinates in the study of fluid flow through porous media, governed by different flow models, has been tested.

It has been illustrated that Darcy's law does not differentiate between slip or no-slip conditions, in terms of the von Mises variables. Departure from the Darcy regime has, therefore, to be taken into account in order to account for the existence of a boundary layer, which arises when solid boundaries are implemented \([31,32]\). This departure leads to using one of the models that account for the existence of the boundary layer, which in turn implies the singularity of the von Mises transformation.
CHAPTER 4

SINGLE-PHASE FLUID FLOW INTO A TWO-DIMENSIONAL SINK

4.1. INTRODUCTION

Fluid flow through porous media into a two-dimensional sink is considered to illustrate the similarities and differences when the flow through the porous media is governed by the different models discussed in chapter 1. Detailed comparison is also made between the flow governed by the Navier-Stokes equations at \( \text{Re} = 0 \) and by Brinkman's model with \( \mu = \mu_{\text{ef}} \).

Extensive analysis is made for the case of Brinkman’s model, including the effect of the sink location on the secondary eddies and on the flow pattern. Other models considered for this study include the Darcy-Lapwood-Brinkman model and the Darcy-Lapwood modified model, which was suggested in chapter 3 by equation (3.28). The Darcy-Lapwood model, which was abandoned in the study of chapter 3 is also rejected here.
due to its incompatibility with the boundary conditions and with the type of flow considered in the current problem.

The porous medium is assumed to be bounded by three impermeable walls, ab, bc, cd, as shown in Fig. 4.1, and the flow is generated by the line sink at location "e" situated perpendicular to the wall cd and parallel to the z-axis. The fluid is assumed to enter the flow domain through the section "ad". For the sake of simplicity, the effect of gravity is ignored in the current problem. The bounded porous medium is assumed to be finite in length L, and depth D, where L and D are chosen so that LD \gg \xi, where \xi is the dimension of each unit comprizing the solid matrix of the porous medium.

4.2. GOVERNING EQUATIONS

Using the vorticity-streamfunction formulation, we consider the flow through the porous channel into the two-dimensional sink to be governed by the Darcy-Lapwood-Brinkman model, given by equations (2.3) and (2.8), and by the Darcy-Lapwood modified model, given by (2.3) and (3.28). The flow is also considered when it is governed by the Brinkman model given by equations (2.3) and (2.8), with \theta = 1. The problem can also be
examined through the Darcy-Lapwood-Brinkman model with \( \text{Re} = 0 \), and \( \vartheta = 1 \). It should be noted that although the choice of \( \vartheta = 1 \) is restrictive for the case of Brinkman's model, the numerical procedure employed is valid for any constant value of \( \vartheta \).

To cast the flow equations into dimensionless form the flow variables have been nondimensionalized with respect to a characteristic velocity \( U_0 \), length \( L \), of the channel and depth \( D \). This gives rise to the aspect ratio of the channel defined as \( \varepsilon = D/L \). The governing equations can thus be expressed in terms of the aspect ratio, \( \varepsilon \), and for the sake of illustration, Brinkman's model given by (2.3) and (2.6), takes the form

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{\varepsilon^2} \frac{\partial^2 \psi}{\partial y^2} = -\Omega 
\]  
\ldots(4.1)

\[
\varepsilon \frac{\partial^2 \Omega}{\partial x^2} + \frac{1}{\varepsilon^2} \frac{\partial^2 \Omega}{\partial y^2} = \frac{1}{K} \Omega \n\]  
\ldots(4.2)

\[\text{streamfunction equation}\]

\[\text{vorticity equation}\]
and the equation of continuity takes the form

\[
\varepsilon \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]

...(4.3)

where

\[
U = \frac{1}{\varepsilon} \frac{\partial \psi}{\partial y}, \quad V = -\frac{\partial \psi}{\partial x}
\]

...(4.4)

In the current analysis we only consider the case of \( \varepsilon = 1 \), and thus the flow domain becomes the region bounded by \( 0 < X < 1 \), and \( 0 < Y < 1 \). The analyses and method of solution can easily be extended to treat the full range of aspect ratio \( \varepsilon \).

4.3. FINITE DIFFERENCE APPROXIMATIONS

In order to carry out the numerical integration procedure, the governing differential equations are approximated using a second order accurate central differencing scheme for all of the derivatives involved. A square grid of uniform mesh size is used and the resulting difference equations are cast into the form of equation (3.4), which is suitable for successive line
relaxation with sweep along the Y-direction. It is clear that the tridiagonal matrices resulting from this representation are diagonally dominant, and the presence of the dimensionless permeability enhances this diagonal dominance. These matrices are inverted using the tridiagonal solver (Thomas' algorithm) with successive line over-relaxation for the streamfunction equation and under-relaxation for the vorticity equation.

The solution of the coupled equations together with the pertinent boundary conditions is obtained by implementing the solution procedure which is summarized as follows:

For given $K$ and $Re$,

1. The flow domain is initialized by giving $\psi$ and $\Omega$ some small starting values.

2. The values of $\Omega$ are calculated at the three walls of the channel, excluding the sink location, using the derived expressions for vorticity at the walls (cf. section 4.4).

3. The streamfunction equation is solved for $\psi$, using the tridiagonal solver along each grid line, $i = 2$ to $I_{\text{max}}-1$, and the solution is iterated along each $i$-line using SLOR of the form

$$\psi^{n+1} = \psi^n + \omega (\psi_i^{n+1} - \psi^n) \quad \ldots \ (4.5)$$
where $\Psi_{r}^{n+1}$ is the value obtained by the tridiagonal solver, and $\Psi^{n}$ is the value obtained from the previous iteration. This step is repeated a few times in order to accelerate convergence.

4. $\Omega$ is updated on the boundary using the latest values of $\Psi$.

5. The vorticity equation is solved for $\Omega$ by a similar procedure to that in step 3.

6. Steps 3 to 5 are repeated until the following convergence criterion is met:

$$| F_{i,j}^{n+1} - F_{i,j}^{n} | \leq \delta, \text{ where } \delta = 5 \times 10^{-5} \text{ for } \Psi_{i,j} \text{ and } 5 \times 10^{-4} \text{ for } \Omega_{i,j}.$$

4.4 BOUNDARY CONDITIONS

In this work we consider only the simplest case of a parallel flow entry condition, that is, uniform flow into the channel. In order to satisfy the no-slip condition on the solid walls, we assume a plug flow, constant across the channel but zero on the boundaries. This may be viewed as a uniform flow approaching the opening to the channel from infinity.
The no-slip boundary condition on the solid walls and the uniform entrance condition are ensured by the following conditions on the streamfunction, $\Psi$, when the sink is located at point "c":

$\Psi = 0$ on "ab" and on "bc", except at point "c"
$\Psi = 1$ on "dc", except at point "c"
$\Psi = Y$ on "ad".

Taking different locations of the sink along the line "dc", say at point "e", it can be easily seen that the conditions on $\Psi$ along "dc" take the form:

$\Psi = 0$ between "e" and "c"
$\Psi = 1$ between "d" and "e".

The condition of uniform flow along "ad" also gives the condition on $\Omega$, namely $\Omega = 0$, along the line "ad". No explicit conditions are given for the vorticity on the rest of the boundaries. Vorticity boundary conditions are thus derived in terms of streamfunction values at internal grid points by expressing the streamfunction equation, in its finite difference form, along the lines "ab", "bc" and "dc". This results in the following expressions for $\Omega$ on the above lines:
\[ \Omega = - \frac{2\Psi_{i,1}}{\Delta y^2} \quad \text{for } i = 2, 3, \ldots, I_{\text{max}} - 1 \] ...(4.8)

\[ \Omega_{i, J_{\text{max}}} = - \frac{2}{\Delta y^2} \left[ \Psi_{i, J_{\text{max}} - 1} - 1 \right] \quad \text{for } i = 2, 3, \ldots, I_{\text{max}} - 1 \] ...(4.7)

\[ \Omega_{I_{\text{max}}, j} = - \frac{2}{\Delta y^2} \Psi_{I_{\text{max}} - 1, j} \quad \text{for } j = 2, 3, \ldots, J_{\text{max}} - 1. \] ...(4.8)

It should be noted that corner points do not enter in the numerical procedure and thus no conditions need to be specified at these points. The expression for \( \Omega_{i, J_{\text{max}}} \) is only valid when the sink is located at point "c". If the sink takes different locations, say at point "e"; then the above condition holds along the segment "de" and the vorticity along the segment "ec" can be calculated from:

\[ \Omega_{\text{ec}} = - \frac{2}{\Delta y^2} \Psi_{i, J_{\text{max}} - 1}. \] ...(4.8)

4.5. RESULTS AND DISCUSSION

Solutions have been obtained for three of the models
of flow through porous media. Different values of permeability and Reynolds number have been considered. For the sake of comparison, solutions when the flow is governed by Darcy’s law is also investigated.

Solutions to the Brinkman model are obtained for the range of permeability $K = 0.1, 0.01, 0.001, 0.0001$ and $K \to 1$ when the sink is located at $(X,Y) = (1,1)$. For different sink locations, solutions are obtained for $K = 0.1$ and the results are compared with the results from a Navier-Stokes calculation at $Re = 0$.

In the case of the Darcy-Lapwood-Brinkman model (DLB), solutions are obtained for $K = 0.1, 0.01, 0.001$ and $K \to 1$ for $Re = 1, 10, 50$. For the Darcy-Lapwood modified model results are based on $Re_c = 100$ and $500$, $K = 0.1, 0.01, 0.001$ and $K \to 1$ when $Re = 10$. Results obtained using the DL modified are similar in qualitative behaviour to those obtained using the DLB model, but they differ quantitatively due to the reduction of the viscous effects caused by the introduction of $Re_c$.

4.5.1. Velocity Profiles

Figure 4.2 illustrates the Brinkman horizontal velocity component for $K = 0.001$ on different vertical lines along the channel length. For the fully-developed
flow considered, the increase in this velocity component as we approach the sink location is clear and is due to the greater influence of the sink on the fluid in that region. This behaviour persists for all values of $K$.

In Figs. 4.3(a) and 4.3(b), the horizontal component of velocity along the centreline of the channel is shown for different values of permeability, and $Re = 10$ in the case of DLB model. Comparison is also given with the results of Darcy's law. For low permeability, the fluid undergoes greater sink effect and is attracted to the sink faster in the upper regions of the channel. In lower regions, the horizontal component of velocity is greater for lower permeability. This behaviour in the lower regions might be attributed to the development of secondary corner eddies as the permeability increases. The corner eddies tend to hinder the flow and, therefore, this results in a reduction in the horizontal velocity component. This reduction causes a redistribution of the flow which in turn results in an increase in the horizontal component of velocity in the middle parts of the channel, as the permeability is increased.

The redistribution of the flow also results in an increase in the vertical velocity component, taken along the horizontal centreline of the channel, with increasing permeability, as indicated in Figs. 4.4(a) and 4.4(b).
These figures also illustrate this vertical velocity component for different permeabilities, and Re = 10 in case of DLB model.

In Fig. 4.3(c), the horizontal velocity component along the vertical centreline of the channel is illustrated for the DLB model when K = 0.001 and different Re. In Fig. 4.4(c), the vertical velocity component along the horizontal centreline of the channel is illustrated for the DLB model at K = 0.001 and different Re. Although the increase in Re does not have much influence when the permeability is low it does, however, influence the velocity when the permeability is high, as can be seen from Fig. 4.4(b) where comparison is made with the results obtained using Brinkman's model.

The effect of changing the sink location on the horizontal velocity component, taken along the vertical centreline of the channel, is illustrated in Fig. 4.5 for K = 0.1 in the case of Brinkman's model. Comparison is made in this figure with the Navier-Stokes flow at Re = 0. As the sink location is moved away from the right-hand boundary, an increase in the flow rate occurs. This increase results in the increase of the horizontal velocity component in the upper regions of the channel. Soon after, the fluid is pulled back due to the effect of the sink.
4.5.2. Flow Development

The flow development for a given sink location and a given permeability is illustrated in Figures 4.6 to 4.10.

In Figs. 4.6(a) and 4.6(b), the streamline pattern is illustrated for $K = 0.001$ and, in the case of DLB, $Re = 10$, and indicates that in the upper regions of the channel the fluid is attracted directly into the sink. In the lower regions, the fluid is pulled up slightly as it is attracted to the sink. This behaviour in the lower regions might be attributed to the finite dimensions of the channel or due to the tendency for the corner eddies to appear. As the permeability is decreased, this behaviour becomes unnoticeable, as demonstrated in Fig. 4.7(a), where the streamlines at $K = 0.0001$ and in Darcy's case are relatively straight at the inlet of the channel.

For higher permeability, the effects of the corner eddies are felt in the lower regions of the channel, where the streamlines are pulled up and then attracted to the sink. This is also evident in Figs. 4.7(a) and 4.7(b), which show the opposite effect in the upper regions of the channel, where the fluid seems to be pushed down near the inlet and then attracted towards the sink. This behaviour for high permeability is due to the
development of upper wall eddies near the inlet.

The effect of the sink location on the streamline pattern is illustrated in Figures 4.8, 4.9 and 4.10 when $K = 0.1$ and the flow is governed by Brinkman's model. These figures indicate the higher flow rate associated with the movement of the sink away from the right-hand boundary. When the sink location approaches the centre of the upper wall, upper corner eddies start to develop, in addition to lower corner eddies. The size of the region of separation on the upper wall, near the inlet to the channel, begins to increase as the sink is moved to the left. The development of the secondary wall eddies and the increase in the extent of the region of viscous separation with the movement of the sink location toward the inlet to the channel seems to be supportive of the conclusion that the wall eddies begin to appear mainly due to the finite dimensions, especially the length, of the channel.

4.5.3. Secondary Flow Structure

For the type of flow considered, viscous separation has developed for values of permeability $K > 0.001$, in both the Brinkman and DLB regimes. This is indicated in Figs. 4.7(a) and 4.7(b), where the lower corner eddy is
small in size. In case of the DL modified model, the lower corner eddy is also small in size even for large permeability. This is indicated in Fig. 4.7(c) where for $K = 0.1$, $Re = 10$ and $Re_c = 100$ the secondary streamfunction has value $-0.55 \times 10^{-3}$. This illustrates that in the case of internal flow where separation might occur, the term $\mu \nabla^2 \psi$ which appears in the DLB equation is important. In the case of the DL modified model, only a constant multiple, $1/Re_c$, of this term is taken into account. This might render this model inappropriate for internal flows.

For different $Re$ and permeability $K = 0.01$, DLB results indicate that $Re$ does not have much effect on the size of the viscous separation region. The influence of $Re$, as indicated in Fig. 4.7(b), is to increase the absolute value of the secondary streamfunction, with increasing $Re$. For higher permeability, Fig. 4.7(b) shows the secondary eddies at $Re = 10$ and $K = 0.1$ for the case of DLB.

For Brinkman's model with permeability value $K = 0.1$ and for different sink locations Figs. 4.8, 4.9 and 4.10, show three main regions of viscous separation: the lower right-hand corner region, the upper right-hand corner region and the region near the upper wall of the channel.

The region of viscous separation near the lower
right-hand corner of the channel exists for all of the sink locations. Although the areas spanned by the largest secondary streamlines remain the same in this region for different sink locations, the streamfunction values there increase as the sink is shifted to the left on the upper wall. As the sink is moved to the left the secondary streamlines tend to span larger areas and increase the size of the separation region, but this tendency is restricted by the increase in the flow rate in the lower parts of the channel which accompanies the sink movement to the left and causes the viscous separation region to retain the same size, but with higher streamfunction value.

The second region of separation exists near the upper right-hand corner of the channel. It starts to appear when the sink is located between \( X = 0.625 \) and 0.65 on the upper wall. This region might be very small when the sink is located between \( X = 0.875 \) and 0.9 but does not exist when the sink is located at (1,1).

The third region of viscous separation exists near the top wall of the channel, as shown in Fig. 4.10. This region develops for all cases of sink location and is characterized by the existence of a streamline having a streamfunction value equal to that on the upper wall. This region of viscous separation causes the streamlines
of the main flow to be pushed towards the lower wall in the upstream region. In Fig. 4.10 the sink is located between \( X = 0.625 \) and 0.65, and as the sink is relocated towards the right, the centre of this separated region shifts slightly to the left and towards the upper wall.

4.6. CONCLUSIONS

Three of the porous media flow models have been tested. The focus of the present chapter has been to illustrate the effect of permeability on viscous separation. For a more complete study one should take into account the Forchheimer's model which, for the current study has been left out due to the presence of solid boundaries in the case considered. This by no means ignores the importance of the Forchheimer's model in the flow into a sink.

It has been shown that the DL modified model might not be appropriate in the study of viscous separation. This model might be suitable as a good first approximation for the solution to fluid flow problems with inertia. It is, however, a modification to the purely inertial model known as the Darcy-Lapwood model.

With regard to viscous separation, no study can be made complete without refinement of grid. In this work,
the main features of the flow are illustrated based on the 41x41 grid. This has proven to be fine enough to illustrate some of the important features of the flow considered. For a more thorough understanding of the upper wall eddies, a refinement of the grid is necessary.

In this work the simplest case of a parallel flow entry condition has been employed. For a more general parallel flow the horizontal velocity component \( U (= U(Y)) \) has to be compatible with the flow model considered. For example, in the case of the Darcy-Lapwood-Brinkman's model, the velocity component \( U \) should be of the form

\[
U = -c_1 K Re + \frac{c_1 K Re}{\left( e^{1/\sqrt{K}} + 1 \right)}
\times \left\{ e^{1/\sqrt{K}Y} + e^{1/\sqrt{K}(1-Y)} \right\}
\]

where \( c_1 = dp/dx \) is the constant pressure gradient.

This form of equation satisfies the no-slip condition on the solid boundary, and it has to be used at the entrance to the channel.
CHAPTER 5

DUSTY GAS FLOW MODELS THROUGH POROUS MEDIA

5.1. INTRODUCTION

The study of multi-phase fluid flow through porous media has received considerable attention due to various applications of this type of flow in the physical world, including applications in reservoir engineering and other industrial applications. Dusty fluid flow through porous media, on the other hand, has received less attention in spite of the importance of studies associated with this flow. Dusty fluid flow is usually associated with the design of industrial filters, liquid-dust separators and water purification plants.

As the name implies, the subject matter of dusty fluid flow through porous media deals with the study of a multi-phase flow through porous media when one of the phases is comprised of a cloud of suspension. The other phase is that of the clean fluid-phase. The dust particles are assumed to be suspended in the fluid and this status of suspension is maintained by the assumption
that the phases present neither displace, miscibly or
immiscibly, each other nor do they chemically interact
with each other. Furthermore, the dusty clouds, or
particles, suspended in the fluid could be either solid,
liquid or gas.

In the case of dusty fluid flow in free space, early
investigations were experimental in nature and depended
on observations. The up-to-date work of many
investigators involves the theoretical treatment of dusty
fluid flow based on the equations of motion developed by
Saffman [38], Marble [39] and Soo [40]. These governing
equations are thought of as "equivalent" as long as the
dust concentration, by volume, is very small [40].

In a parallel consideration, one expects that the
study of dusty fluid flow through porous media could be
advanced from the empirical to the phenomenological
stage. The contrary has been the case, as shall be seen
in the following sections which include the objectives
and plan of the current work, the structure of the porous
media considered and the relationship of the current work
with the existing literature.

5.1.1 Objectives of the Current Work

The purpose of this study is to develop simple, yet
realistic, mathematical models to describe the flow through a porous medium of an incompressible fluid containing a small concentration, by volume, of dust particles. The models are phenomenological in nature and are based on the volume averaging technique. They are expected to describe the dusty fluid flow through a porous medium comprised of a sparse distribution of objects (particles) fixed in space. Depending on the way in which the porous matrix is modelled to affect the dusty fluid, four different cases of interest arise. Each of these cases is then classified into different subcases in an attempt to parallel these models with some of the existing models that describe the flow of a single-phase fluid through porous media. The proposed models are then employed in the study of some boundary value problems of interest.

5.1.2. The Structure of the Porous Medium Considered

The applicable physical domain for the proposed dusty fluid models is a porous medium composed of a sparse distribution of objects, or particles, fixed in space or a porous medium having a minimum pore diameter that is larger than the largest dust particle in the suspension. The porous matrix may be viewed as consisting
of a number of dust collectors, fixed in an ordered or disordered manner, in space. This definition of the porous medium excludes compacted porous media or porous media with low permeability where Darcy's law is valid. It also excludes the naturally occurring porous media where the Forchheimer's equation is applicable. This, therefore, restricts the medium to be one in which the Brinkman equation, the Darcy-Lapwood or the Darcy-Lapwood-Brinkman equations are valid.

As in the case of single-phase flow through porous media, the proposed models are expected to describe the macroscopic behaviour of the dusty fluid without giving any information about the pore level behaviour. This stems from the fact that the models are derived through the volume averaging technique, and therefore no distinction is made as to whether the matrix is periodically or randomly composed of the particles in space.

5.1.3. Relationship with Existing Literature

The available literature on the dusty fluid flow through porous media is largely centred around the problems of deep bed filtration of particles in suspension [21,23,82], and the filtration of dilute
emulsions [22, 83, 84, 85]. The methodology adopted in these studies is, therefore, either empirical/semi-empirical in nature or a phenomenological methodology that approaches the problem through the development of partial differential equations that describe the behaviour of deep bed filters. The existing phenomenological models that describe the flow of a dusty fluid in porous media consider the flow to be that of a single-phase fluid through a porous medium of variable porosity that decreases with the increased retention of the dust particles in the bed [23]. This suggests the necessity to develop an alternative approach and a methodology that takes into account the phases present and how the phases interact and affect each other. This is one of the motives behind the current study which, as shall be seen in later sections, replaces the idea of retention with the determination of the dust particle number density.

In the case of flow of suspensions through a deep porous bed, Herzig et.al. [21] gave a review of the available literature and outlined the mechanisms of deposition and the possible capturing processes, which include sedimentation, inertial impaction, direct interception, hydrodynamic effects and diffusion by Brownian motion. In cases where the particle size is greater than one micro-metre, the particle diffusion is
negligible [21], while the effect of inertial impaction is negligible if the fluid-phase is liquid [23]. In addition, if the particles are spherical in shape then the hydrodynamic effects can be neglected [21]. This leaves the interception capture mechanism to be dominant and the particles are captured mainly on the surfaces of the media grains [22]. Settling of particles by sedimentation is also possible, due to the high density associated with the dust particles.

Some of the characteristics that are pertinent to the current study and have been reported in the literature on deep filtration can be summarized as follows. When the fluid enters the porous bed a decrease in the porosity of the medium, accompanied with a decrease in the permeability, is noticed due to the deposition and the retention of the dust particles on the solid grains of the porous matrix and due to clogging of the pores. During the process of filtration, the porosity of the clogged beds is approximately equal to the porosity of the clean bed [23]. It has been reported by Soo et.al. [22] that in the case of suspended solid particles, in a traditional deep filtration process where the ratio of the particle-size to pore-size is small, interception capture mechanism dominates and the particles are captured mainly on the grains surface and,
therefore, flow redistribution in the porous medium is not significantly altered by the presence of the captured particles. The permeability reduction due to the retained particles is not significant and may be ignored [22]. In addition, it can be assumed that the pores are non-pluggable in deep filtration and thus no straining occurs [22]. This is true in cases referred to by Herzig et al. [21] as decolmatage, where distinction is made between spontaneous and provoked decolmatage.

The available literature indicates that the following approaches have been implemented in the study of dusty fluid flows.

1. Dusty fluid flow in free space has been widely studied via the continuum approach. Two phases are assumed to be present, the fluid-phase and the dust-phase, with each phase being treated as a continuum.

2. Dusty fluid flow through porous media with applications to deep filtration has been widely studied [21] via the empirical and semi-empirical approach, and takes into account the optimal design of filters, liquid-dust separation and clogging mechanisms of the pores.

3. Modeling of dilute flow through porous media with applications to emulsion drops has been studied by Soo and Radke [22] with the pores being blocked by the
emulsion drops and flow redistribution has been taken into account.

4. One-dimensional phenomenological models that treat the flow of a single-phase fluid with the porosity and permeability taken as variable [23].

5. A model based on a set of partial differential equations governing the flow through a porous medium where Brinkman's equation is valid was developed by Hamdan and Barron [98].

The lack of phenomenological models based on a set of partial differential equations, the solution to which describes the behaviour of a dusty fluid flowing in a porous medium, gives rise to the current attempt to develop mathematical models that employ the continuum description of the dust-phase. Employing the continuum approach in this context, however, requires certain assumptions on both the type of flow considered and on the porous medium. These assumptions will be discussed in the next section.

The proposed models are expected to describe the flow of a dusty fluid through a porous medium having a sparse structure. The models are not meant to offer a complete replacement to the empirical methodology that has been used in the study of deep filtration due to the presence of some parameters which have to be determined.
experimentally. However, the models offer a means of describing the dynamical behaviour of the fluid flow through porous media and the effects of the presence of a small concentration of dust, by volume, on the flow characteristics. The analogy between the dusty fluid flow models in free space and in porous media is emphasized. The reduction of the proposed models to the single-phase models of flow through porous media in the absence of dust and the reduction to Saffman's dusty gas flow model in free space, when the porous matrix effects are neglected, are used as a cross-check on the validity of the mathematical derivation.

5.2. PHYSICAL ASSUMPTIONS

The models' equations are to be developed based on the following physical assumptions on the porous medium and on the dusty fluid.

1. It is assumed that the dusty fluid flows through a porous matrix consisting of a sparse distribution of particles fixed in space.

2. It is assumed that porosity changes of the medium, due to the retention of dust particles on possible retention sites, are negligible and that clogging of the
pores does not occur and thus straining action is neglected.

3. The permeability to the fluid is assumed to be constant and the reduction in permeability is therefore insignificant.

4. The distribution of the solid grains comprising the solid matrix is uniform.

5. The suspended particles in the viscous fluid are uniform in shape and size with particle diameter large enough so that diffusion by Brownian motion can be neglected, and small enough so that clogging of the pores can be neglected.

6. Flow redistribution in the porous medium is assumed to be unaltered by the presence of captured particles.

7. It is assumed that the main mechanism of particle capture is the direct interception on the surface of the grains of the porous matrix. The effect of settling is ignored in this study by neglecting gravitational forces.

8. It is assumed that the dust behaves as a continuum and that the porous structure and dimensions allow for this behaviour and allow for the possibility of volume averaging over a certain control volume.

9. The dust particles are assumed to be non-interacting, chemically or otherwise, and the dust concentration, by volume, is very small.
Additional assumptions on the interaction of forces exerted by the porous medium and the phases present will be discussed later.

5.3. MODELS DEVELOPMENT

5.3.1. The Averaging Approach

Consider the fluid flow through a porous medium having a constant porosity and a constant permeability to the fluid. The fluid saturating the chosen medium is assumed to consist of two phases: the fluid-phase and the dust-phase. When the fluid-phase and porous medium compressibility effects are neglected, the averaged Navier-Stokes equations, averaged over a control volume of the porous medium, take the following macroscopic form, in the absence of momentum transfer between the phases involved, (see Semrau [10] or Drew [97] for derivation)

\[ \gamma_j \rho_j \left( \frac{\partial \hat{u}_j}{\partial t} + \left( \hat{u}_j \cdot \nabla \right) \hat{u}_j \right) = -\gamma_j \nabla \hat{p}_j \]

\[ + \gamma_j \mu_j \nabla^2 \hat{u}_j + f_j \quad \ldots \quad (5.1) \]
where $\gamma_j$ is the volume fraction of the $j^{th}$ phase in the system, $\tilde{\rho}_j$ is the intrinsic volume-averaged density of phase $j$, $\tilde{p}_j$ is the intrinsic volume-averaged pressure of phase $j$, $\tilde{u}_j$ is the volume-averaged velocity vector of phase $j$, $\mu_j$ is the viscosity of phase $j$ and $\sum_j \tilde{f}_j$ is the sum of external forces exerted on phase $j$.

The volume fraction, $\gamma_j$, satisfies the relation

$$\sum_j \gamma_j = 1 \quad \ldots \ (5.2)$$

and is defined as

$$\gamma_j = \frac{V_j}{V} \quad \ldots \ (5.3)$$

where $V$ denotes the control volume and $V_j$ is the volume occupied by phase $j$.

The macroscopic equation of continuity, in the absence of sources and sinks, is given by

$$\frac{\partial}{\partial t} \gamma_j \tilde{\rho}_j + \nabla \cdot \gamma_j \tilde{\rho}_j \tilde{u}_j = 0 \quad \ldots \ (5.4)$$

where, for the present, compressibility effects have been included.

Since it is assumed that the dust-phase has a very small concentration, by volume, and no miscible or
immiscible displacements occur within the system, it is reasonable to take the volume fraction of each of the phases to be constant. The intrinsic volume-averaged quantities in equation (5.1) are related to the volume-averaged quantities by

\[ \hat{\mathbf{f}}_j = \gamma_j \hat{\mathbf{f}}_j \quad \ldots \quad (5.6) \]

where \( \hat{\mathbf{f}}_j \) refers to a volume-averaged quantity and \( \hat{\mathbf{f}}_j \) refers to an intrinsic volume-averaged quantity.

In light of (5.5), equation (5.1) takes the following form

\[
\hat{\rho}_j \left( \frac{\partial \hat{\mathbf{u}}_j}{\partial t} + (\hat{\mathbf{u}}_j \cdot \nabla) \hat{\mathbf{u}}_j \right) = - \nabla \hat{p}_j + \mu_j \nabla^2 \hat{\mathbf{u}}_j + \hat{\mathbf{f}}_j \quad \ldots \quad (5.8)
\]

where

\[ \hat{\mathbf{f}}_j = f_j / \gamma_j \quad \ldots \quad (5.7) \]

Dropping "\(^\wedge\)" from equations (5.4) and (5.6) and expressing them for the fluid-phase and the dust-phase, we obtain

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for the first phase

equation of continuity:

\[ \frac{\partial \rho_1}{\partial t} + \nabla \cdot \rho_1 u_1 = 0 \] ...

momentum equation:

\[ \rho_1 \left( \frac{\partial u_1}{\partial t} + (u_1 \cdot \nabla) u_1 \right) = -\nabla p_1 + \mu_1 \nabla^2 u_1 + F_1 \] ...

for the second phase

equation of continuity:

\[ \frac{\partial \rho_2}{\partial t} + \nabla \cdot \rho_2 u_2 = 0 \] ...

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momentum equations:

\[
\rho_2 \left( \frac{\partial u^2}{\partial t} + \left( \mathbf{u}_2 \cdot \nabla \right) \mathbf{u}_2 \right) = -\nabla p_2 + \mu_2 \nabla^2 \mathbf{u}_2 + \mathbf{F}, \quad \ldots \quad (5.11)
\]

Neglecting the dust-phase partial pressure and the dust-phase viscosity, and referring to the dust-phase by subscript "2", equation (5.11) takes the form

\[
mN \left( \frac{\partial u^2}{\partial t} + \left( \mathbf{u}_2 \cdot \nabla \right) \mathbf{u}_2 \right) = \mathbf{F}, \quad \ldots \quad (5.12)
\]

where the dust-phase density, \( \rho_2 \), is expressed in terms of the macroscopic particle number density, \( N \), and the mass of a single dust particle, \( m \).

The dust-phase continuity equation (5.10) can then be written in the form
\[ \frac{\partial N}{\partial t} + \nabla \cdot N u = 0 \] ... (S.13)

5.3.2. The Nature of Forces Acting on Each Phase

5.3.2.1. Forces acting on the fluid-phase

The term \( F_1 \) appearing in equation (S.9) represents the sum of external forces exerted on a unit volume of the fluid-phase. The nature of these forces is best understood from the fact that in the flow of a viscous fluid through a porous medium the fluid is subjected to a frictional force, due to the solid matrix of the medium. In addition, when the flow considered is that of a dusty fluid then another force acting on the fluid-phase is that due to the influence of the dust on the clean fluid.

In light of this, let \( F_4 \) be the friction force, per unit volume, on the fluid-phase due to the solid matrix of the porous medium. In order to derive an expression for this friction force it is noted that this force has to balance Darcy's pressure gradient in the medium. In the absence of body forces, Darcy's law takes the form

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\[
\frac{k}{\mu} \nabla p. \quad \text{... (5.14)}
\]

Neglecting inertia, the macroscopic Navier-Stokes equations take the form, [28],

\[
\mu \nabla^2 \hat{u} - \nabla \hat{p} + F = 0 \quad \text{... (5.15)}
\]

where \( F \) is the friction force exerted on the fluid per unit volume, and "\( \hat{\cdot} \)" denotes volume averaging.

Consider now the situation in which the fluid flows through the porous medium in the presence of a small concentration of dust particles. Assuming the relative velocity, \( \hat{q} \), of the fluid-phase with respect to the dust-phase to be constant everywhere, and is given by \( \hat{q} = \hat{u} - \hat{v} = \text{constant} \), where \( \hat{u} \) and \( \hat{v} \) are the fluid-phase and the dust-phase velocity vectors, respectively, equation (5.15) takes the form

\[
- \nabla \hat{p} + F = 0. \quad \text{... (5.16)}
\]

Darcy's law, expressed in terms of the seepage relative velocity, takes the form
\[ u - v = - \frac{k}{\mu_i} \nabla p. \] ... (5.17)

The expression for \( F \) can thus be obtained from equations (5.16) and (5.17). Identifying \( F \) with \( \sim_{ii} \), \( u \) with \( u_i \) and \( v \) with \( u_2 \), we obtain

\[ F_{ii} = - \frac{\mu_i}{k} \left( u_i - u_2 \right). \] ... (5.18)

Although the component of force given by (5.18) is based upon the assumption of relative velocity distortion due to the porous matrix, as depicted by Darcy's law of the form (5.17), an alternative form of the frictional component of force can be obtained by assuming that the porous matrix affects the fluid-phase only. This results in the usual Darcy's law of the form given by equation (5.14) which, in the case of dusty fluid flow through porous media, might be considered to be valid in cases where the dust-phase velocity is small compared to the fluid-phase velocity, and the concentration of dust is very small. This results in \( F_{ii} \) of the form

\[ F_{ii} = - \frac{\mu_i}{k} u_i. \] ... (5.19)
The other contribution to the force that affects the fluid-phase is that due to the influence of dust on the clean fluid. This force is calculated when the two phases move with a variable relative velocity. The assumption of a small concentration of dust, by volume, leads to the following expression for the effect of dust on the clean fluid \([38]\),

\[
F = C_r N (u - u)
\]

\[\frac{{\Delta}}{i}\]

\[\text{(5.20)}\]

in which the effect of dust is represented by a force proportional to the relative velocity of the two phases, and \(F_{\Delta}\) represents the force per unit volume of the fluid-phase due to the dust. \(C_r\) is the coefficient of resistance in the porous medium and \(N\) is the macroscopic number density. It should be noted that \(C_r\) is constant under the assumption of uniform size and distribution of the dust particles. This coefficient is identified with the Stokes coefficient of resistance when the dust particles are spherical and the dusty fluid flow is taken in free space. For the case of dusty fluid through porous media, \(C_r\) should be determined experimentally and is postulated to depend on the porous matrix structure and might be considered as a function of porosity.

If the dust-phase velocity is small compared to the
fluid-phase velocity, the force exerted by the dust particles on the clean fluid may be considered proportional to the fluid-phase velocity. In this case, equation (5.20) takes the form

\[ F_{i2} = -C_r N u_i \]

\[ \ldots (5.21) \]

With the contributions to \( F_i \) given by equations (5.18) and (5.20), the fluid-phase momentum equation (5.9) takes the following form

\[ \rho_i \left( \frac{\partial u_i}{\partial t} + (u_i \cdot \nabla) u_i \right) = -\nabla p_i + \mu_i \nabla^2 u_i \]

\[ + C_r N (u_2 - u_i) + \frac{\mu_i}{k} (u_2 - u_i) \]

\[ \ldots (5.22) \]

The fluid-phase continuity equation (5.8) takes the following form when the fluid is assumed to be incompressible

\[ \nabla \cdot u_i = 0. \]

\[ \ldots (5.23) \]
5.3.2.2. Forces acting on the dust-phase

The term $F_2$, in equation (5.12), represents the sum of external forces exerted on a unit volume of the dust-phase. Although it might be possible to consider that $F_2$ is composed of two forces: one due to the effect of the fluid-phase on the dust, and the other due to the solid matrix, it is reasonable to assume that the latter component is much smaller than the first and, therefore, negligible. This follows from the assumption of a very small bulk concentration of the dust, together with the fact that once the particles are captured and retained on the solid grain surfaces they remain there due to the friction force exerted on them. It is thus reasonable to assume that the friction force exerted by the solid matrix on the dust particles in motion may be neglected. Furthermore, the assumption of small bulk concentration of the dust-phase inhibits the consideration of a separate permeability for the dust-phase, and thus Muskat's relative permeability model [5] is inapplicable for this type of multi-phase flow. The saturation will, therefore, refer to the fluid-phase saturation of the porous medium.

Consequently, the only contribution to the force $F_2$
is due to the fluid-phase influence on the dust and, thus, $F_2$ is given by

$$F_2 = C_r N \left( u_1 - u_2 \right). \quad \cdots (5.24)$$

Substituting equation (5.24) into (5.12), the dust-phase momentum equation will always assume the following form

$$mN \left( \frac{\partial u_2}{\partial t} + \left( u_1 \cdot \nabla \right) u_2 \right) = C_r N \left( u_1 - u_2 \right) \quad \cdots (5.25)$$

while the dust-phase continuity equation is given by (5.13).

In summary, the general equations governing the flow of an incompressible dusty fluid in a homogeneous porous medium are given by equations (5.22) and (5.23), for the fluid-phase, coupled with the dust-phase equations (5.13) and (5.25). These equations constitute a set of eight partial differential equations in the eight unknowns $u_1$, $u_2$, $N$ and $p_1$. 

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5.4. THE EXTENDED MODELS (H-B MODELS)

In the case of single-phase fluid-flow through porous media the governing equations take different forms depending on the type of flow and the type of porous medium considered. The classification of single-phase flow models is based upon the dominancy of inertial effects, viscous effects or both, as has been explained in chapter 1.

In the case of dusty-fluid flow through porous media, the same ideas prevail and thus the classification of different models is based on whether or not the inertial effects are taken into consideration. Further classification of the current dusty fluid equations is based upon the way in which the two phases interact and upon the way in which the porous matrix affects the phases present. It should be noted that the viscous effects of the dust-phase are neglected [39], while the inertia of the dust particles is substantial [38] and, therefore, the inertial terms are always retained in the dust-phase momentum equations.

Since the dust-phase momentum equation always takes the same form, the classification of the dusty fluid flow models through porous media is thus based on the form that the fluid-phase momentum equation takes. It is
easily seen, from equation (5.25), that for the flow of a dusty fluid through porous media the dust-phase behaviour does not depend directly on the permeability. The influence of the permeability on the dust-phase is through the fluid-phase which is coupled with the dust-phase.

5.4.1. The Hamdan-Barron First Extended Models (H-B1)

This model is obtained by substituting equation (5.19) in (5.9) to result in the following fluid-phase momentum equation:

\[
\rho_i \left[ \frac{\partial u_i}{\partial t} + (u_i \cdot \nabla) u_i \right] = -\nabla p_i + \mu_i \nabla^2 u_i - \frac{\mu_i}{k} u_i \quad \ldots (5.26)
\]

while the fluid-phase continuity equation is given by (5.23), and the dust-phase continuity and momentum equations are given, respectively, by (5.13) and (5.25).

Equations (5.13), (5.23), (5.25) and (5.26) represent the equations governing the unsteady flow of an incompressible dusty fluid through porous media (H-B1).
model). It can be seen in this model that the fluid-phase influences the dust-phase but the dust-phase influence on the clean fluid is assumed to be negligible. This case parallels the usual dusty fluid flow in free space when the Reynolds number of the flow is taken to zero, and thus the dust-phase flow characteristics can be determined once the solution to the fluid-phase equations is obtained. The solid matrix is assumed to affect only the fluid-phase in the H-B1 model.

5.4.2. The Hamdan-Barron Second Extended Model (H-B2)

By assuming that the solid matrix affects the relative seepage velocity of the dusty fluid, and assuming that the fluid-phase affects the dust-phase but there is negligible effect of the dust on the clean fluid, the following fluid-phase momentum equation is obtained by substituting equation (5.18) in equation (5.9):

\[
\rho_1 \left\{ \frac{\partial \tilde{u}_1}{\partial t} + (\tilde{u}_1 \cdot \nabla) \tilde{u}_1 \right\} = -\nabla p_1 + \mu_1 \nabla^2 \tilde{u}_1 + \frac{\mu_1}{k} (\tilde{u}_2 - \tilde{u}_1).
\]

\[\cdots (5.27)\]
For the H-B2 model, the governing equations are thus given by equations (5.13), (5.23), (5.25) and (5.27). The two phases influence each other in this model with the coupling between the two phases being through the relative velocity. It should be noted that in this model the fluid-phase does not depend directly on the dust particle number density.

5.4.3. The Hamdan-Barron Third Extended Model (H-B3)

When the fluid-phase affects the dust-phase and the dust-phase affects the fluid-phase, with the porous matrix affecting the fluid-phase only, the fluid-phase momentum equation is obtained by substituting equations (5.19) and (5.20) in equation (5.9) to yield

\[ \rho_s \left( \frac{\partial \mathbf{u}_s}{\partial t} + (\mathbf{u}_s \cdot \nabla) \mathbf{u}_s \right) = - \nabla p_s + \mu_s \nabla^2 \mathbf{u}_s \]

\[ + C_f N (\mathbf{u}_s - \mathbf{u}_d) - \frac{\mu_s}{k} \mathbf{u}_d. \quad \ldots (5.28) \]

The equations governing this model are thus equations (5.13), (5.23), (5.25) and (5.28).
5.4.4. The Hamdan-Barron Fourth Extended Model (H-B4)

Employing equations (5.18) and (5.20) in equation (5.9), the following fluid-phase momentum equation is obtained:

\[ p_1 \frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} (u_i \cdot \nabla) u_j - \nabla p + \mu_1 \nabla^2 u_i = - \nabla p + \mu_1 \nabla^2 u_i \]

\[ \frac{\mu_1}{k} \left( \sum_{j=1}^{n} (u_i - u_j) \right) + \frac{\mu_1}{k} \left( u_i - u_j \right). \quad \ldots (5.29) \]

This equation reflects the effect of the dust-phase on the fluid-phase and the effect of the fluid-phase on the dust-phase, together with the solid matrix effect on the seepage relative velocity. The equations governing this model are thus given by (5.13), (5.23), (5.25) and (5.29). This model represents the most general case of interaction. Furthermore, equation (5.29) reduces to the Navier-Stokes equations when the permeability becomes infinite, i.e. when the porous medium is replaced by free space, and the parameter \( C_r \) is taken to be the Stokes coefficient of resistane for spherical particle. If the particle number density, \( N \), is taken to be zero in equation (5.29) then it reduces to the well-known
Darcy-Lapwood-Brinkman equation.

5.5. CLASSIFICATION OF THE HAMDAN-BARRON EXTENDED MODELS

As in the case of single-phase fluid flow through porous media, dusty fluid flow through porous media can be classified into different types depending on the structure of the medium, its porosity and the type of flow, that is, whether the inertial and viscous effects are accounted for. This gives rise to three different forms for each of the extended H-B models. These three forms for the classification are in parallel with the Brinkman equation, the Darcy-Lapwood equation and the Darcy-Lapwood-Brinkman equation.

5.5.1. Subclassification of the H-B1 Model

5.5.1.1. Darcy-Lapwood-Brinkman-B-H1 Model

In this model, the viscous and inertial effects of the fluid-phase, together with the damping viscous terms, are taken into account and thus the model takes the following form, namely equations (5.13), (5.23), (5.25) and (5.26) which are grouped here for the sake of completeness.
for fluid-phase

\[ \nabla \cdot u_1 = 0. \quad \cdots \quad (5.30) \]

\[ \rho_1 \left( \frac{\partial u_1}{\partial t} + (u_1 \cdot \nabla) u_1 \right) = -\nabla p_1 + \mu_1 \nabla^2 u_1 \]

\[ -\frac{\mu_1}{k} u_1 \]

\[ \cdots \quad (5.31) \]

for dust-phase

\[ \frac{\partial N}{\partial t} + \nabla \cdot N u_2 = 0. \quad \cdots \quad (5.32) \]

\[ mN \left( \frac{\partial u_2}{\partial t} + (u_2 \cdot \nabla) u_2 \right) = C_p N (u_2 - u_1) \quad \cdots \quad (5.33) \]

In the absence of dust effects this model reduces to the Darcy-Lapwood-Brinkman equation.
5.5.1.2. Darcy-Lapwood-H-B1 Model

When the viscous terms are neglected in equation (5.31), and the viscous damping term is retained, the following equations are obtained and are classified as the Darcy-Lapwood-H-B1 model:

for fluid-phase

\[ \nabla \cdot \mathbf{u} = 0 \]  \[ \text{... (5.34)} \]

\[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p - \frac{\mu}{k} \mathbf{u} \]  \[ \text{... (5.35)} \]

for dust-phase

\[ \frac{\partial N}{\partial t} + \nabla \cdot N \mathbf{u} = 0 \]  \[ \text{... (5.36)} \]

\[ mN \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = C_F N \left( \mathbf{u} \cdot \mathbf{u}_{\infty} \right) \]  \[ \text{... (5.37)} \]
It is clear that when the dust effects are eliminated from the above equations the Darcy-Lapwood-H-B1 model reduces to the usual Darcy-Lapwood model, discussed in chapter 1.

5.5.1.3. Brinkman-H-B1 Model

In the absence of inertial effects, and when the flow is steady, the Brinkman-H-B1 model is obtained from (5.31) and takes the form

\[ \nabla \cdot u = 0. \]
\[ \nabla \cdot \mu \nabla u = 0 \]

\[ \nabla p + \mu \nabla^2 u - \frac{\mu}{k} u = 0 \]

\[ \text{for fluid-phase} \]

\[ \text{for dust-phase} \]

\[ \nabla \cdot N u = 0 \]

... (5.38)  
... (5.39)  
... (5.40)
\[ mN (u_2 \cdot \nabla) u_2 = C_r N (u_4 - u_2) \quad \cdots \quad (5.41) \]

When the dust effects are eliminated, this model reduces to the Brinkman model governing the flow of a single-phase fluid flow through porous media.

5.5.2. Subclassification of the H-B2 Model

In a similar method of analysis to the one used to classify the H-B1 model, the H-B2 model is classified into the following three different sub-classes.

5.5.2.1. Darcy-Lapwood-Brinkman-H-B2 Model

Corresponding to equation (5.27), where the viscous and inertial effects are accounted for, the following model is obtained:

\[ \nabla \cdot u_i = 0. \quad \cdots \quad (5.42) \]
\[
\rho_1 \left( \frac{\partial u_1}{\partial t} + (u_1 \cdot \nabla) u_1 \right) = -\nabla p_1 + \mu \frac{\partial^2 u_1}{\partial \tau^2} - \frac{\mu_1}{k} (u_1 - u_2) \quad \ldots \ (5.43)
\]

for dust-phase

\[
\frac{\partial N}{\partial t} + \nabla \cdot N u_2 = 0. \quad \ldots \ (5.44)
\]

\[
mN \left( \frac{\partial u_2}{\partial t} + (u_2 \cdot \nabla) u_2 \right) = C N (u_1 - u_2) \quad \ldots \ (5.45)
\]

5.5.2.2. Darcy-Lapwood-H-B2 Model

for fluid-phase

\[
\nabla \cdot u_1 = 0. \quad \ldots \ (5.46)
\]
\[ \rho_i \left( \frac{\partial u_i}{\partial t} + (u \cdot \nabla) u_i \right) = -\nabla p_i - \frac{\mu_i}{k} (u_i - u_2) \quad \ldots \ (5.47) \]

for dust-phase

\[ \frac{\partial N}{\partial t} + \nabla \cdot N u_2 = 0. \quad \ldots \ (5.48) \]

\[ mN \left( \frac{\partial u_i}{\partial t} + (u \cdot \nabla) u_i \right) = c_r N (u_i - u_2) \quad \ldots \ (5.49) \]

5.5.2.3. Brinkman-H-B2 Model

for fluid-phase

\[ \nabla \cdot u_i = 0. \quad \ldots \ (5.50) \]

\[ -\nabla p_i + \mu_i \nabla^2 u_i - \frac{\mu_i}{k} (u_i - u_2) = 0 \quad \ldots \ (5.51) \]
for dust-phase

\[ \nabla \cdot \mathbf{u}_2 = 0 \]  \hspace{1cm} \cdots (5.52)

\[ mN \left( \frac{u_2 \cdot \nabla}{\sim^2} \right) \frac{u_2}{\sim^2} = C_r N \left( \frac{u_1 - u_2}{\sim^1 \sim^2} \right) \]  \hspace{1cm} \cdots (5.53)

5.5.3. Subclassification of the H-B3 Model

In a similar method of analysis to the one used to classify the H-B1 and H-B2 models, the H-B3 model is classified into the following three different sub-classes.

5.5.3.1. Darcy-Lapwood-Brinkman-H-B3 Model

Corresponding to equation (5.28), where the viscous and inertial effects are accounted for, the following model is obtained:

for fluid-phase

\[ \nabla \cdot \mathbf{u}_1 = 0 \]  \hspace{1cm} \cdots (5.54)
\[
\rho_1 \left[ \frac{\partial u_1}{\partial t} + (u_1 \cdot \nabla) u_1 \right] = -\nabla p_1 + \mu_1 \nabla^2 u_1 - \frac{\mu_1}{k} u_1 + C_r N (u_2 - u_1) \quad \ldots \text{(S.55)}
\]

\[\text{for dust-phase}\]

\[
\frac{\partial N}{\partial t} + \nabla \cdot N u_2 = 0 \quad \ldots \text{(S.56)}
\]

\[
mN \left[ \frac{\partial u_2}{\partial t} + (u_2 \cdot \nabla) u_2 \right] = C_r N (u_1 - u_2) \quad \ldots \text{(S.57)}
\]

5.5.3.2. Darcy-Lapwood-H-B3 Model

\[\text{for fluid-phase}\]

\[\nabla \cdot u_3 = 0 \quad \ldots \text{(S.58)}\]
\[ \rho \left( \frac{\partial u_i}{\partial t} + u_j \cdot \nabla u_i \right) = -\nabla p - \frac{\mu}{k} u_i + C r N (u_i - u_j) \]  \quad \ldots (5.59)

for dust-phase

\[ \frac{\partial N}{\partial t} + \nabla \cdot N u_2 = 0. \]  \quad \ldots (5.80)

\[ m N \left( \frac{\partial u_2}{\partial t} + u_2 \cdot \nabla u_2 \right) = C r N (u_1 - u_2) \]  \quad \ldots (5.81)

5.5.3.3. Brinkman-H-B3 Model

for fluid-phase

\[ \nabla \cdot u_1 = 0 \]  \quad \ldots (5.82)

\[ -\nabla p + \frac{\mu}{k} \nabla^2 u_1 - \frac{\mu}{k} u_1 + C r N (u_2 - u_1) = 0 \]  \quad \ldots (5.83)

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for dust-phase

\[ \nabla \cdot N u_2 = 0 \] \hspace{1cm} \ldots (5.64)

\[ mN ( u_2 \cdot \nabla ) u_2 = C_r N ( u_1 - u_2 ) \] \hspace{1cm} \ldots (5.65)

5.5.4. Subclassification of the H-B4 Model

In a similar method of analysis to the one used to classify the H-B1, H-B2 and H-B3 models, the H-B4 model is classified into the following three different sub-classes.

5.5.4.1. Darcy-Lapwood-Brinkman-H-B4 Model

Corresponding to equation (5.29), where the viscous and inertial effects are accounted for, the following model is obtained:

\textit{for fluid-phase}

\[ \nabla \cdot u_1 = 0. \] \hspace{1cm} \ldots (5.66)
\[ \rho_1 \left( \frac{\partial u_i}{\partial t} + (u_i \cdot \nabla) u_i \right) = -\nabla p_1 + \mu_1 \nabla^2 u_i \]

\[ + \left( \frac{\mu_1}{k} + C_r N \right) (u_2 - u_1) \]  

... (5.87)

for dust-phase

\[ \frac{\partial N}{\partial t} + \nabla \cdot N u_2 = 0. \]  

... (5.88)

\[ mN \left( \frac{\partial u_i}{\partial t} + (u_i \cdot \nabla) u_i \right) = C_r N (u_2 - u_1) \]  

... (5.89)

5.5.4.2. Darcy-Lapwood-H-B4 Model

for fluid-phase

\[ \nabla \cdot u_i = 0 \]  

... (5.70)

\[ \rho_1 \left( \frac{\partial u_i}{\partial t} + (u_i \cdot \nabla) u_i \right) = -\nabla p_1 \]

\[ + \left( \frac{\mu_1}{k} + C_r N \right) (u_2 - u_1) \]  

... (5.71)
for dust-phase

\[ \frac{\partial N}{\partial t} + \nabla \cdot N \: u_2 = 0 \]  \hspace{1cm} ... (5.72)

\[ mN \left( \frac{\partial u^2}{\partial t} + \left. \left( \begin{array}{c} u_2 \cdot \nabla \end{array} \right) \right|_{\sim_2} u_i \right) = C_r N \left( u_i - u_2 \right) \]  \hspace{1cm} ... (5.73)

5.5.4.3. Brinkman-H-B4 Model

for fluid-phase

\[ \nabla \cdot u_i = 0 \]  \hspace{1cm} ... (5.74)

\[ - \nabla p_i + \mu_i \nabla^2 u_i + \left( \frac{\mu_i}{\kappa} + C_r N \right) \left( u_i - u_2 \right) = 0 \]  \hspace{1cm} ... (5.75)

for dust-phase

\[ \nabla \cdot N \: u_2 = 0 \]  \hspace{1cm} ... (5.76)

\[ mN \left( u_2 \cdot \nabla \right) u_2 = C_r N \left( u_i - u_2 \right) \]  \hspace{1cm} ... (5.77)

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5.6. PROPOSED PRACTICAL APPLICATIONS OF THE MODELS

Given a porous medium that fits the description of the media discussed in section 5.1, and dusty fluid that fits the requirements imposed by the assumptions in section 5.2, then for a given permeability and a known coefficient of resistance and known mass of a single dust particle, the following can be accomplished for a given boundary value problem:

1. The characteristics and the dynamics of flow of the dusty fluid can be determined through the appropriate model, which takes into account the type of flow considered and the type of porous flow domain.

2. Given the initial and the boundary distribution of the number density, \( N \), the end states can be determined by solving the appropriate set of model equations.

3. Given some specified end states of \( N \), the flow conditions required to maintain the given distribution can be obtained by solving the appropriate model equations.

4. The models may be used in the study of deep bed filters in which they may offer an initial step in the analyses of filtration in which back-wash is required.
5.7. SUMMARY

Model equations describing the flow of a dusty fluid through porous media have been developed. Due to the new nature of these models, no particular applications have been cited but some possible general practical applications have been proposed in section 5.8. There is no claim that these models actually describe the microscopic detail of flow of a dusty fluid through porous media. They are based on the volume-averaging technique and therefore give a macroscopic description of the flow. Further analyses are necessary to illustrate the possibility of physical implementation of these models, including some experimental evidence and comparison between the theory and experiment. Experiments and/or theoretical analyses are further required to determine the nature of the coefficient of resistance. Furthermore, the way that the interaction between the two phases, and between the porous matrix and the phases present, has been modelled does not rule out the possibility of other ways. This of course requires further analyses, both theoretical and experimental. Although numerical testing has been carried out, as will be discussed in chapters 7 and 8, experimental results are needed to provide a comparison and to validate these models.
CHAPTER 6

ON THE VON MISSE TRANSFORMATION FOR TWO-PHASE FLOW

6.1 INTRODUCTION

A comprehensive treatment, analyses and comparison of the dusty fluid flow models through porous media, developed in chapter 5, necessitates obtaining the solution to a given flow problem when the flow is assumed to be governed by each of the twelve models proposed. In an attempt to demonstrate the deterministic nature and the solubility of the set of partial differential equations governing each of the models, two flow problems are considered.

Two-dimensional flow of a dusty fluid, through a given porous domain, into a line sink will be discussed in chapter 7. The flow channel is assumed to be bounded by solid walls, except along the inlet of the channel and at the point where the sink is located. The vorticity-streamfunction formulation will be adopted to accomplish
this study. Casting the flow equations in vorticity-streamfunction form is the subject matter of section 6.2.

In section 6.3, discussion is given concerning the extension of the von Mises approach that was developed in chapter 2. This extension is not only of great utility in the study of dusty fluid flow through porous media but also represents a major step in the study of general dusty gas flow in free space. One of the main objectives for the von Mises extension in the current work is to facilitate the study of the complicated problem of flow of a dusty fluid over curved boundaries, discussed in chapter 8.

It can also simplify matters in the study of dusty gas flows in rectangular domains and thus might be considered as a replacement to the particle tracking approach.

### 6.2. VORTICITY-STREAMFUNCTION FORMULATION

When the flow of a dusty fluid is considered in two space dimensions, the governing equations for each of the proposed models form a system of six equations in six unknowns \( u_1, u_2, N \) and \( p_1 \), as functions of the cartesian coordinates, \( x, y \), in the plane, and time \( t \).
In a given dusty fluid boundary value problem it is required, therefore, to determine the distribution of the dust particles via the dust-phase number density \( N \), which in turn requires resorting to particle tracking methods and other common techniques. To avoid this complicated procedure, the problem is simplified by taking the particle distribution \( N \) to be constant throughout the flowfield. Taking \( N \) to be constant throughout the flowfield, however, renders an overdetermined system of six equations in the five unknowns \( u_1 \), \( u_2 \) and \( p_1 \). This suggests that when the dust-phase governing equations are cast into vorticity-streamfunction form, with \( N \) taken as constant, the solution might yield dust-phase velocity components that do not necessarily satisfy the dust-phase momentum equations in components form [86]. The dust-phase vorticity-streamfunction formulation is, nevertheless, facilitated by modifying the dust-phase momentum equation to incorporate the dust-phase partial pressure, \( p_2 \). Thus, the dust-phase momentum equation for each of the models discussed in chapter 5 is replaced by

\[
mN \left( \frac{\partial u_2}{\partial t} + (u_1 \cdot \nabla) u_2 \right) = -\nabla p_2 + C N (u_2 - u_1). \quad \ldots (8.1)
\]
The pressure $p_2$ is interpreted as the dust-phase partial pressure required to maintain a uniform distribution of dust particles. Mathematically, it is the variable required to render the set of partial differential equations determinate when $N$ is taken to be constant.

Once the flow variables are determined, $p_2$ can be determined from equation (6.1).

Using the modified dust-phase momentum equation (6.1) in each of the models, the equations of motion governing the flow of a dusty fluid through porous media are expressed in (dimensionless) vorticity-streamfunction form as follows. The dust-phase vorticity-streamfunction equations are written only once, for the Darcy-Lapwood-Brinkman-H-Bi model, since they take the same form for all of the other models. The streamfunction equation is also written once, for the Darcy-Lapwood-Brinkman-H-Bi model, and it takes the same form in all of the other models. The fluid-phase vorticity equation is written for each of the models separately.
### 6.2.1. The H-Bi Vorticity-streamfunction Formulation

#### 6.2.1.1. Darcy-Lapwood-Brinkman-H-Bi Model

**streamfunction equation for fluid-phase:**

\[
\Omega_1 = -\frac{\partial^2 \psi_1}{\partial x^2} - \frac{\partial^2 \psi_1}{\partial y^2}
\quad \text{...(6.2)}
\]

**vorticity equation for fluid-phase:**

\[
\frac{\partial \Omega_1}{\partial t} + \frac{\partial \psi_1}{\partial y} \frac{\partial \Omega_1}{\partial x} - \frac{\partial \psi_1}{\partial x} \frac{\partial \Omega_1}{\partial y} = -\frac{\Omega_1}{K \text{Re}} + \frac{1}{\text{Re}} \nabla^2 \Omega_1 \quad \text{...(6.3)}
\]

**streamfunction equation for dust-phase:**

\[
\Omega_2 = -\frac{\partial^2 \psi_2}{\partial x^2} - \frac{\partial^2 \psi_2}{\partial y^2}
\quad \text{...(6.4)}
\]

**vorticity equation for dust-phase:**

\[
\frac{\partial \Omega_2}{\partial t} + \frac{\partial \psi_2}{\partial y} \frac{\partial \Omega_2}{\partial x} - \frac{\partial \psi_2}{\partial x} \frac{\partial \Omega_2}{\partial y} = D_2 \left[ \Omega_1 - \Omega_2 \right] \quad \text{...(6.5)}
\]
6.2.1.2. Darcy-Lapwood-H-B1 Model

\[
\frac{\partial \Omega_1}{\partial T} + \frac{\partial \psi_1}{\partial Y} \frac{\partial \Omega_1}{\partial X} - \frac{\partial \psi_1}{\partial X} \frac{\partial \Omega_1}{\partial Y} = - \frac{\Omega_1}{K \text{Re}} \quad \ldots(6.6)
\]

6.2.1.3. Brinkman-H-B1 Model

\[-\frac{\Omega_1}{K} + \nabla^2 \Omega_1 = 0 \quad \ldots(6.7)\]

6.2.2. The H-B2 Vorticity-streamfunction Formulation

6.2.2.1. Darcy-Lapwood-Brinkman-H-B2 Model

\[
\frac{\partial \Omega_1}{\partial T} + \frac{\partial \psi_1}{\partial Y} \frac{\partial \Omega_1}{\partial X} - \frac{\partial \psi_1}{\partial X} \frac{\partial \Omega_1}{\partial Y} = \frac{1}{K \text{Re}} \left[ \Omega_2 - \Omega_1 \right]
\]

\[+ \frac{1}{\text{Re}} \nabla^2 \Omega_1 \quad \ldots(6.8)\]

6.2.2.2. Darcy-Lapwood-H-B2 Model

\[
\frac{\partial \Omega_1}{\partial T} + \frac{\partial \psi_1}{\partial Y} \frac{\partial \Omega_1}{\partial X} - \frac{\partial \psi_1}{\partial X} \frac{\partial \Omega_1}{\partial Y} = \frac{1}{K \text{Re}} \left[ \Omega_2 - \Omega_1 \right] \quad \ldots(6.9)
\]

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6.2.2.3. Brinkman-H-B2 Model

\[ \frac{1}{K} [ \Omega_2 - \Omega_1 ] + \nabla^2 \Omega_1 = 0 \] \quad \text{...(6.10)}

6.2.3. The H-B3 Vorticity-streamfunction Formulation

6.2.3.1. Darcy-Lapwood-Brinkman-H-B3 Model

\[ \frac{\partial \Omega_1}{\partial T} + \frac{\partial \Omega_1}{\partial Y} \frac{\partial \Omega_1}{\partial X} - \frac{\partial \Omega_1}{\partial X} \frac{\partial \Omega_1}{\partial Y} = D_1 [ \Omega_2 - \Omega_1 ] - \frac{\Omega_1}{K \text{Re}} \]

\[ + \frac{1}{\text{Re}} \nabla^2 \Omega_1 \] \quad \text{...(6.11)}

6.2.3.2. Darcy-Lapwood-H-B3 Model

\[ \frac{\partial \Omega_1}{\partial T} + \frac{\partial \Omega_1}{\partial Y} \frac{\partial \Omega_1}{\partial X} - \frac{\partial \Omega_1}{\partial X} \frac{\partial \Omega_1}{\partial Y} = D_1 [ \Omega_2 - \Omega_1 ] - \frac{\Omega_1}{K \text{Re}} \] \quad \text{...(6.12)}

6.2.3.3. Brinkman-H-B3 Model

\[ \text{Re} \, D_1 [ \Omega_2 - \Omega_1 ] + \nabla^2 \Omega_1 - \frac{\Omega_1}{K} = 0 \] \quad \text{...(6.13)}
6.2.4. The H-B4 Vorticity-streamfunction Formulation

6.2.4.1. Darcy-Lapwood-Brinkman-H-B4 Model

\[
\frac{\partial \Omega}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \Omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \Omega}{\partial y} = [\Omega_2 - \Omega_1] [D + \frac{1}{K \text{Re}}]
\]
\[+ \frac{1}{\text{Re}} \nabla^2 \Omega \quad \ldots (6.14)
\]

6.2.4.2. Darcy-Lapwood-H-B4 Model

\[
\frac{\partial \Omega}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \Omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \Omega}{\partial y} = [\Omega_2 - \Omega_1] [D + \frac{1}{K \text{Re}}] \ldots (6.15)
\]

6.2.4.3. Brinkman-H-B4 Model

\[
[\text{Re} D + \frac{1}{K}] [\Omega_2 - \Omega_1] + \nabla^2 \Omega_1 = 0. \quad \ldots (6.16)
\]

The dimensionless fluid-phase vorticity, \( \Omega_1 \), and the dust-phase vorticity, \( \Omega_2 \), are defined in terms of the dimensionless fluid-phase and dust-phase velocity.
components, respectively, as

\[
\Omega_1 = \frac{\partial V_1}{\partial x} - \frac{\partial U_1}{\partial y} 
\]

... (6.17)

\[
\Omega_2 = \frac{\partial V_2}{\partial x} - \frac{\partial U_2}{\partial y} 
\]

... (6.18)

where \( U_1, U_2, V_1 \) and \( V_2 \) are the velocity components given in terms of \( \Psi_1 \) and \( \Psi_2 \) by

\[
U_1 = \frac{\partial \Psi_1}{\partial y}, \quad U_2 = \frac{\partial \Psi_2}{\partial y} 
\]

... (6.19)

\[
V_1 = -\frac{\partial \Psi_1}{\partial x}, \quad V_2 = -\frac{\partial \Psi_2}{\partial x}. 
\]

... (6.20)

The governing equations, in vorticity-streamfunction form, were rendered dimensionless with respect to a characteristic length \( L \), a characteristic depth \( D \) and a characteristic velocity \( U_o \), using the following dimensionless variables:

\[ x = XL, \quad y = YD, \quad t = TL / U_o, \quad k = KLD, \]

\[ \psi_1 = \Psi_1 L U_o, \quad \psi_2 = \Psi_2 L U_o, \quad \xi_1 = \Omega_1 U_o / L, \quad \xi_2 = \Omega_2 U_o / L. \]
\[ u_1 = U_1 U, \quad u_2 = U_2 U, \quad v_1 = V_1 U, \quad v_2 = V_2 U, \]
\[ C_r N/\rho_1 = D_{10}/L \quad \text{and} \quad C_r /m = D_{20}/L. \]

Here \( \Psi_1 \) is the fluid-phase streamfunction, \( \Psi_2 \) is the dust-phase streamfunction, \( \Omega_1 \) is the fluid-phase vorticity, \( \Omega_2 \) is the dust-phase vorticity, \( K \) is the permeability, \( \nabla^2 \) is the laplacian in the cartesian plane, with dimensionless cartesian coordinates \( X \) and \( Y \), \( D_1 \) is the dimensionless product \( C_r N/\rho \), \( D_2 \) is the dimensionless quotient \( C_r /m \), \( Re = \rho_1 U_0 /\mu_1 \) is the Reynolds number.

It should be noted that the dimensional terms \( C_r /m \) and \( C_r N/\rho \) have the dimensions of relaxation time and frequency, respectively.

6.3. EXTENSION OF THE von MISES TRANSFORMATION

In chapter 2 the von Mises transformation was extended to include the time derivative so as to facilitate studies of unsteady flow problems over curved boundaries. Successful application of the steady von Mises transformation to fluid flow problems through porous media gives rise to the idea of extending the von Mises transformation even further, in an attempt to offer
a method that is capable of treating the general
two-phase fluid flow through porous media.

Consider the two sets of transformations between the
cartesian coordinates \((X_1, Y_1)\) and time \(T_1\), and the
curvilinear coordinates \((\xi_1, \psi_1)\) and time \(\tau_1\), and between
the cartesian coordinates \((X_2, Y_2)\) and time \(T_2\), and the
curvilinear coordinates \((\xi_2, \psi_2)\) and time \(\tau_2\), defined by

\[
X_1 = \xi_1 
\]

\[
Y_1 = Y_1(\xi_1, \psi_1, \tau_1) 
\]

\[
T_1 = \tau_1 
\]

and

\[
X_2 = \xi_2 
\]

\[
Y_2 = Y_2(\xi_2, \psi_2, \tau_2) 
\]

\[
T_2 = \tau_2 
\]

with \(X_1 = X_2\), \(T_1 = T_2\) and \(Y_1 = Y_2\) in the physical plane.
The Jacobians of the transformations are given by

\[ J_{1s} = \frac{\delta X_1, Y_1}{\delta \bar{s}_1, \bar{\psi}_1} = \frac{\delta Y_1}{\delta \psi_1} \ldots (6.27) \]

\[ J_{2s} = \frac{\delta X_2, Y_2}{\delta \bar{s}_2, \bar{\psi}_2} = \frac{\delta Y_2}{\delta \psi_2} \ldots (6.28) \]

\[ J_{1t} = \frac{\delta X_1, Y_1}{\delta \bar{s}_1, \bar{\tau}_1} = \frac{\delta Y_1}{\delta \tau_1} \ldots (6.29) \]

\[ J_{2t} = \frac{\delta X_2, Y_2}{\delta \bar{s}_2, \bar{\tau}_2} = \frac{\delta Y_2}{\delta \tau_2} \ldots (6.30) \]

where the subscripts \( s \) and \( t \) indicate spatial and temporal, respectively.

It is clear that if \( J_{1s} = 0 \) or is infinite then the first transformation is singular while if \( J_{1t} = 0 \) then the first transformation reduces to the steady von Mises transformation. Similar conclusions can be drawn about the second transformation. If, however, \( 0 < | J_{1s} | < \infty \) and \( 0 < | J_{1t} | < \infty \) then (6.21), (6.22) and (6.23) define a one-to-one transformation. Similarly, if \( 0 < | J_{2s} | < \infty \) and \( 0 < | J_{2t} | < \infty \) then (6.24), (6.25) and (6.26)
define a one-to-one transformation. Applying the chain rule to the functions $X_1$, $Y_1$, and $T_1$ yields

$$\frac{\delta Y_1}{\delta T_1} = - \frac{\delta Y_1}{\delta T_1} / \frac{\delta Y_1}{\delta Y_1}$$

...(6.31)

$$\frac{\delta Y_1}{\delta X_1} = - \frac{\delta Y_1}{\delta X_1} / \frac{\delta Y_1}{\delta Y_1}$$

...(6.32)

$$\frac{\delta Y_1}{\delta Y_1} = 1 / \frac{\delta Y_1}{\delta Y_1}$$

...(6.33)

Similarly, applying the chain rule to the functions $X_2$, $Y_2$, and $T_2$ yields

$$\frac{\delta Y_2}{\delta T_2} = - \frac{\delta Y_2}{\delta T_2} / \frac{\delta Y_2}{\delta Y_2}$$

...(6.34)

$$\frac{\delta Y_2}{\delta X_2} = - \frac{\delta Y_2}{\delta X_2} / \frac{\delta Y_2}{\delta Y_2}$$

...(6.35)

$$\frac{\delta Y_2}{\delta Y_2} = 1 / \frac{\delta Y_2}{\delta Y_2}$$

...(6.36)
Then, using (6.31), (6.32) and (6.33), partial derivatives in the two coordinate systems, in the first transformation, are related by

$$\frac{\partial}{\partial \tau_1} = \frac{\partial}{\partial \xi_1} - \left( \frac{\partial \xi_1}{\partial \tau_1} / \frac{\partial \xi_1}{\partial \psi_1} \right) \frac{\partial}{\partial \psi_1} \quad \ldots (6.37)$$

$$\frac{\partial}{\partial \xi_1} = \frac{\partial}{\partial \tau_1} - \left( \frac{\partial \tau_1}{\partial \xi_1} / \frac{\partial \tau_1}{\partial \psi_1} \right) \frac{\partial}{\partial \psi_1} \quad \ldots (6.38)$$

$$\frac{\partial}{\partial \psi_1} = \left( 1 / \frac{\partial \xi_1}{\partial \psi_1} \right) \frac{\partial}{\partial \xi_1} \quad \ldots (6.39)$$

Similarly, using (6.34), (6.35) and (6.36), partial derivatives of the two coordinate systems, in the second transformation, are related by

$$\frac{\partial}{\partial \tau_2} = \frac{\partial}{\partial \xi_2} - \left( \frac{\partial \xi_2}{\partial \tau_2} / \frac{\partial \xi_2}{\partial \psi_2} \right) \frac{\partial}{\partial \psi_2} \quad \ldots (6.40)$$

$$\frac{\partial}{\partial \xi_2} = \frac{\partial}{\partial \tau_2} - \left( \frac{\partial \tau_2}{\partial \xi_2} / \frac{\partial \tau_2}{\partial \psi_2} \right) \frac{\partial}{\partial \psi_2} \quad \ldots (6.41)$$
\[ \frac{\partial}{\partial y_2} = \left( \frac{1}{1/\partial y_2^2} \right) \frac{\partial}{\partial y_2^2} \] ... \(6.42\)

According to \((6.21)\) and \((6.24)\) we consider \(x_1 = \xi_1\) and \(x_2 = \xi_2\). Together with the fact that \(x_1 = x_2\) in the physical plane, we conclude that \(\xi_1 = \xi_2\) in the computational plane. Likewise, according to \((6.23)\) and \((6.26)\), we take \(T_1 = \tau_1\) and \(T_2 = \tau_2\). Since \(T_1 = T_2\) we must have \(\tau_1 = \tau_2\).

In the case of \(y_1\) and \(y_2\), the situation is different. From \((6.22)\) and \((6.25)\), \(y_1 = y_1(x, \Psi, T)\) and \(y_2 = y_2(x, \Psi, T)\) in the computational plane. The fact that \(y_1 = y_2\) in the physical plane does not necessarily imply that \(y_1 = y_2\) in the computational plane. This is due to the fact that the equations governing \(y_1\) and \(y_2\), in the computational plane, are different. This is clarified further in the following discussion.

The differential operators given by \((6.37)\), \((6.38)\) and \((6.39)\) will thus transform a given differential equation from the physical \((x_1, y_1)\) plane to the computational \((\xi_1, \Psi_1)\) plane. The differential operators given by \((6.40)\), \((6.41)\) and \((6.42)\) will transform a given differential equation from the physical \((x_2, y_2)\) plane to the computational \((\xi_2, \Psi_2)\) plane.
So far no connection has been assumed between the governing equations in the physical plane or between the first and second transformation derived above. In order to make this connection, we assume that we are given, in the physical plane, the coupled partial differential equations governing dusty fluid flow. For the fluid-phase we have a streamfunction equation given in terms of $\Psi_1$ as a function of the two dimensional coordinates $X$ and $Y$ ($=y_1$), and time $T$. This streamfunction equation is coupled with the fluid-phase vorticity equation which is given in terms of $\Omega_1$ as a function of $X$ and $Y$, and time $T$. Similarly, for the dust-phase we assume that we have a streamfunction equation in terms of $\Psi_2$ which is a function of $X$ and $Y$ ($=y_2 = y_4$), and time $T$. The streamfunction equation is coupled with the dust-phase vorticity equation in terms of $\Omega_2$ as a function of $X$, $Y$, and $T$. The two phases are coupled together through the vorticity of each of the phases.

In the physical plane, it is required to solve the coupled equations, simultaneously, to determine the flow variables $\Psi_1(X,Y,T)$, $\Psi_2(X,Y,T)$, $\Omega_1(X,Y,T)$ and $\Omega_2(X,Y,T)$. By using the double transformation, derived above, the flow equations are transformed from the physical plane into a computational plane where $\Psi_1$ becomes an
independent variable when the fluid-phase equations are transformed, and \( \Psi_2 \) becomes an independent variable when the dust-phase equations are transformed. Since in the physical plane the two phases coexist, in other words we only have one physical plane, the one-to-one mapping from/to the physical plane to/from the computational plane necessitates the existence of a single computational domain. Therefore, if the transformation of the governing equations is at all possible then interchanging \( \Psi_1 \) and \( Y \), and \( \Psi_2 \) and \( Y \) must be such that the grid lines \( \Psi_1 = \text{constant} \) and \( \Psi_2 = \text{constant} \) coincide in the computational plane. In this case it is possible to generate a one-to-one mapping from the physical plane to the computational plane.

However, it is important to note that images of \( \Psi_1 = \text{constant} \) and \( \Psi_2 = \text{constant} \) do not coincide in the physical domain. These image lines represent the fluid streamlines and the dust streamlines respectively. Once the roles of the dependent variable \( \Psi_1 \) and the independent variable \( Y \) are interchanged, the fluid-phase governing equations are given in terms of \( \Omega_1 \) and \( Y \), where \( Y = Y(X, \Psi_1), \Omega_1 = \Omega_1(X, \Psi_1) \). Similarly for the dust-phase, where the equations are given in terms of \( \Omega_2 \) and \( Y \), \( Y = Y(X, \Psi_2), \Omega_2 = \Omega_2(X, \Psi_2) \). Thus, we have \( Y = Y(X, \Psi_1) \) where \( Y \)
is governed by the $Y$-equation due to the fluid-phase. Similarly $Y = Y(X, \Psi_2)$, where $Y$ is governed by the $Y$-equation due to the dust-phase. We also know that the lines $\Psi_1 = \text{constant}$ and $\Psi_2 = \text{constant}$ coincide in the computational plane. This might lead to the apparent conclusion that the computed $Y$ is the same. This, of course, is untrue since $Y$ is governed by two different partial differential equations, and therefore the solution to each of the equations renders a value for $Y$ that is different from the other. This can be compared to obtaining a coupled solution to two different flow variables that are functions of the same independent variables. If the partial differential equations are different then the solution for each of the two flow variables is different.

In terms of the physical plane terminology, we can say that: in a given flow problem of a dusty fluid the fluid streamlines are different from the dust streamlines. The governing partial differential equations are to be solved to determine, say, the shape of the streamlines. In terms of the computational domain, we have the following analogy: given $\Psi_1 = \Psi_2 = \text{constant} = \sigma$, determine the dust-phase value of $Y$ on a given grid line and call the solution $Y_2$. Then determine the fluid-phase
value of Y on the same grid line, and call it Y₁. Plotting these two different values of Y gives the shape of the fluid-phase streamline Ψ₁ = σ, and the shape of the dust-phase streamline Ψ₂ = σ.

At this point, in order to implement the transformation equations given by (6.37) through (6.42), and for convenience of notations, Ψ₁ and Ψ₂ are replaced by X, and τ₁ and τ₂ are replaced by T. Equations (6.37) through (6.42) will thus take the following equivalent form:

for the first transformation

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t} - \left( \frac{\partial Y_1}{\partial \Psi_1} \right) \frac{\partial}{\partial \Psi_1} \ldots (6.43)
\]

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial x} - \left( \frac{\partial Y_1}{\partial \Psi_1} \right) \frac{\partial}{\partial \Psi_1} \ldots (6.44)
\]

\[
\frac{\partial}{\partial Y_1} = \left( \frac{1}{\frac{\partial Y_1}{\partial \Psi_1}} \right) \frac{\partial}{\partial \Psi_1} \ldots (6.45)
\]
for the second transformation

\[
\frac{\partial}{\partial T} = \frac{\partial}{\partial T} - \left( \frac{\partial^2 \psi_2}{\partial T \partial \psi_2} \right) \frac{\partial}{\partial \psi_2} \quad \ldots (6.46)
\]

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial x} - \left( \frac{\partial^2 \psi_2}{\partial x \partial \psi_2} \right) \frac{\partial}{\partial \psi_2} \quad \ldots (6.47)
\]

\[
\frac{\partial}{\partial \psi_2} = \left( \frac{1}{\partial^2 \psi_2} \right) \frac{\partial}{\partial \psi_2} \quad \ldots (6.48)
\]

6.4. APPLICATION OF THE EXTENDED von MISES TRANSFORMATION TO DUSTY FLUID FLOW EQUATIONS

The dusty fluid flow equations in vorticity-streamfunction form, given in section 6.2, are transformed in this section by applying the transformation equations (6.43), (6.44) and (6.45) to the fluid-phase equations, and applying the transformation equations (6.46), (6.47) and (6.48) to the dust-phase equations. In this work we shall only be concerned with the study of steady two-dimensional flows so that
derivatives with respect to \( T \) vanish.

The dust-phase streamfunction equation (6.4) is thus transformed to the form:

\[
\left( \frac{\partial \psi_2}{\partial x} \right)^2 \frac{\partial^2 \psi_2}{\partial x^2} - 2 \frac{\partial \psi_2}{\partial x} \frac{\partial \psi_2}{\partial \psi_2} \frac{\partial^2 \psi_2}{\partial x \partial \psi_2} + 
\left( \frac{\partial \psi_2}{\partial x} \right)^2 + 1 \right) \frac{\partial^2 \psi_2}{\partial \psi_2^2} = \left( \frac{\partial \psi_2}{\partial \psi_2} \right)^3 \Omega_2. \quad \ldots (6.49)
\]

The fluid-phase streamfunction equation (6.2) is also transformed to the form:

\[
\left( \frac{\partial \psi_1}{\partial x} \right)^2 \frac{\partial^2 \psi_1}{\partial x^2} - 2 \frac{\partial \psi_1}{\partial x} \frac{\partial \psi_1}{\partial \psi_1} \frac{\partial^2 \psi_1}{\partial x \partial \psi_1} + 
\left( \frac{\partial \psi_1}{\partial x} \right)^2 + 1 \right) \frac{\partial^2 \psi_1}{\partial \psi_1^2} = \left( \frac{\partial \psi_1}{\partial \psi_1} \right)^3 \Omega_1. \quad \ldots (6.50)
\]

The steady-state dust-phase vorticity equation (6.5) is transformed to:
\[
\frac{\partial \Omega_2}{\partial x} = D_2 \frac{\partial \psi_2}{\partial \psi_2} \left( \Omega_1 - \Omega_2 \right). \tag{6.51}
\]

It should be noted that the transformed equations (6.49), (6.50) and (6.51) remain the same for all of the dusty fluid models discussed in section 6.2. For the fluid-phase vorticity equation, for each of the models, we apply transformation equations (6.44) and (6.45) to obtain the following transformed equations.

6.4.1. The H-Bi Fluid-phase Vorticity Equations in Extended von Mises Variables

6.4.1.1. Darcy-Lapwood-Brinkman-H-Bi Model

Applying the steady transformation equations, (6.44) and (6.45), to equation (6.3) we obtain:
\[ \left( \left( \frac{\partial Y_1}{\partial x} \right)^2 + 1 \right) \frac{\partial^2 \Omega_1}{\partial y_1^2} - \Omega_1 \left( \frac{\partial Y_1}{\partial y_1} \right)^2 \frac{\partial \Omega_1}{\partial y_1} - \frac{\partial}{\partial y_1} \left( \frac{\partial Y_1}{\partial y_1} \right)^2 \]

\[ + \left( \frac{\partial Y_1}{\partial y_1} \right)^2 \frac{\partial^2 \Omega_1}{\partial y_1^2} - 2 \frac{\partial Y_1}{\partial y_1} \frac{\partial \Omega_1}{\partial y_1} \frac{\partial^2 Y_1}{\partial x \partial y_1} = \text{Re} \left( \frac{\partial Y_1}{\partial y_1} \right) \frac{\partial \Omega_1}{\partial y_1} \frac{\partial}{\partial x} \]

\[ \ldots (8.52) \]

6.4.1.2. Darcy-Lapwood-H-B1 Model

\[ \frac{\partial \Omega_1}{\partial x} = - \frac{1}{k \text{Re} \frac{\partial Y_1}{\partial y_1}} \Omega_1. \]

\[ \ldots (8.53) \]

6.4.1.3. Brinkman-H-B1 Model

Applying the steady part of transformation equations, (6.44) and (6.45), to equation (6.7) we obtain:
\[
\left( \frac{\partial \psi_1}{\partial x} \right)^2 + 1 \right) \frac{\partial^2 \Omega_1}{\partial \psi_1^2} - \Omega_1 \left( \frac{\partial \psi_1}{\partial \psi_1} \right)^2 \frac{\partial \Omega_1}{\partial \psi_1} - \frac{\Omega_1}{K} \left( \frac{\partial \psi_1}{\partial \psi_1} \right)^2 \\
+ \left( \frac{\partial \psi_1}{\partial \psi_1} \right)^2 \frac{\partial^2 \Omega_1}{\partial x^2} - 2 \frac{\partial \psi_1}{\partial x} \frac{\partial \psi_1}{\partial \psi_1} \frac{\partial^2 \Omega_1}{\partial x \partial \psi_1} = 0 \quad \ldots(8.54)
\]

6.4.2. The H-B2 Fluid-phase Vorticity Equations in Extended von Mises Variables

6.4.2.1. Darcy–Lapwood–Brinkman–H-B2 Model

The steady transformation equations are applied to equation (6.8) to obtain:

\[
\left( \frac{\partial \psi_1}{\partial \psi_1} \right)^2 \frac{\partial^2 \Omega_1}{\partial \psi_1^2} - 2 \frac{\partial \psi_1}{\partial x} \frac{\partial \psi_1}{\partial \psi_1} \frac{\partial^2 \Omega_1}{\partial x \partial \psi_1} + \\
\left( \frac{\partial \psi_1}{\partial x} \right)^2 + 1 \right) \frac{\partial^2 \Omega_1}{\partial \psi_1^2} - \Omega_1 \left( \frac{\partial \psi_1}{\partial \psi_1} \right)^2 \frac{\partial \Omega_1}{\partial \psi_1} \\
- \frac{\Omega_1 - \Omega_1}{K} \left( \frac{\partial \psi_1}{\partial \psi_1} \right)^2 = \text{Re} \frac{\partial \psi_1}{\partial \psi_1} \frac{\partial \Omega_1}{\partial x} . \quad \ldots(8.55)
\]
8.4.2.2. Darcy–Lapwood–H-B2 Model

\[ \frac{\partial \psi_1}{\partial x} = \frac{1}{K \, \text{Re}} \frac{\partial \psi_1^2}{\partial \psi_1} \left[ \Omega_2 - \Omega_4 \right]. \ldots (8.56) \]

8.4.2.3. Brinkman–H-B2 Model

The steady transformation equations are applied to equation (8.10) to obtain:

\[ \left( \frac{\partial \psi_1}{\partial \xi_1} \right)^2 \frac{\partial^2 \psi_1}{\partial x^2} - 2 \frac{\partial \psi_1}{\partial x} \frac{\partial \psi_1}{\partial \psi_1} \frac{\partial^2 \psi_1}{\partial x \partial \psi_1} + \]

\[ \left( \frac{\partial \psi_1}{\partial x} \right)^2 + 1 \right) \frac{\partial^2 \psi_1}{\partial \psi_1^2} - \Omega_1 \left( \frac{\partial \psi_1}{\partial \psi_1} \right)^2 \frac{\partial \psi_1}{\partial \psi_1} \]

\[ - \frac{\Omega_2 - \Omega_4}{K} \left( \frac{\partial \psi_1}{\partial \psi_1} \right)^2 = 0 \ldots (8.57) \]
6.4.3. The H-B3 Fluid-phase Vorticity Equations in
Extended von Mises Variables

6.4.3.1. Darcy-Lapwood-Brinkman-H-B3 Model

The steady transformation equations are applied to equation (6.11) to obtain:

\[
\left( \frac{\dot{Y}_1}{\dot{\psi}_1} \right)^2 \frac{\partial^2 \Omega}{\partial X^2} = 2 \frac{\dot{Y}_1}{\dot{\psi}_1} \frac{\partial \dot{Y}_1}{\partial \dot{\psi}_1} \frac{\partial^2 \Omega}{\partial X \partial \dot{\psi}_1} + \\
\left( \left( \frac{\dot{Y}_1}{\dot{\psi}_1} \right)^2 + 1 \right) \frac{\partial^2 \Omega}{\partial \dot{\psi}_1^2} - \Omega \left( \frac{\partial \dot{Y}_1}{\partial \dot{\psi}_1} \right)^2 \frac{\partial \Omega}{\partial \dot{\psi}_1} - \frac{\Omega}{K} \left( \frac{\dot{Y}_1}{\dot{\psi}_1} \right)^2 \\
+ D_1 \text{Re} \left[ \frac{\dot{\Omega}}{\dot{\psi}_1} - \Omega \right] \left( \frac{\dot{Y}_1}{\dot{\psi}_1} \right)^2 = \text{Re} \frac{\dot{Y}_1}{\dot{\psi}_1} \frac{\partial \Omega}{\partial X}. \quad \ldots(6.58)
\]

6.4.3.2. Darcy-Lapwood-H-B3 Model

\[
\frac{\partial \Omega}{\partial X} = D_1 \frac{\dot{Y}_1}{\dot{\psi}_1} \left[ \frac{\dot{\Omega}}{\dot{\psi}_1} - \Omega \right] - \frac{1}{K \text{Re}} \frac{\dot{Y}_1}{\dot{\psi}_1} \Omega. \quad \ldots(6.59)
\]
6.4.3.3. Brinkman-H-B3 Model

The steady transformation equations are applied to equation (6.13) to obtain:

\[
\left(\frac{\delta \psi_1}{\delta x}\right)^2 \frac{\delta^2 \Omega_1}{\delta x^2} - \Omega_1 \left(\frac{\delta \psi_1}{\delta x}\right)^2 \frac{\delta \psi_1}{\delta x} - \frac{\Omega_1}{K} \left(\frac{\delta \psi_1}{\delta x}\right)^2 \\
+ D_1 \text{Re} \left[\Omega_2 - \Omega_1\right] \left(\frac{\delta \psi_1}{\delta x}\right)^2 = 0. \quad \ldots (6.60)
\]

6.4.4. The H-B4 Fluid-phase Vorticity Equations in Extended von Mises Variables

6.4.4.1. Darcy-Lapwood-Brinkman-H-B4 Model

The steady transformation equations are applied to equation (6.14) to obtain:
\[
\left( \frac{\partial Y}{\partial \Psi} \right)^2 \frac{\partial^2 \Omega}{\partial x^2} - 2 \frac{\partial Y}{\partial x} \frac{\partial Y}{\partial \Psi} \frac{\partial^2 \Omega}{\partial x \partial \Psi} + \\
\left( \frac{\partial Y}{\partial x} \right)^2 - \Omega_1 \left( \frac{\partial Y}{\partial \Psi} \right)^2 \frac{\partial \Omega}{\partial \Psi} + \frac{\Omega_2 - \Omega_1}{K} \left( \frac{\partial Y}{\partial \Psi} \right)^2 + \frac{D_1}{k} \Re \left[ \Omega_2 - \Omega_1 \right] \left( \frac{\partial Y}{\partial \Psi} \right)^2 = \Re \left( \frac{\partial Y}{\partial \Psi} \right) \frac{\partial \Omega}{\partial x} . \quad \ldots (6.61)
\]

6.4.4.2. Darcy-Lapwood-H-B4 Model

\[
\frac{\partial \Omega}{\partial x} = [D_1 + \frac{1}{K \Re}] \frac{\partial Y}{\partial \Psi} \left[ \Omega_2 - \Omega_1 \right] . \quad \ldots (6.62)
\]

6.4.4.3. Brinkman-H-B4 Model

The transformation equations are applied to equation (6.18) to obtain:
\[
\left( \frac{\partial \psi_1}{\partial x} \right)^2 \frac{\partial^2 \omega_1}{\partial x^2} - 2 \frac{\partial \psi_1}{\partial x} \frac{\partial \psi_1}{\partial x} \frac{\partial^2 \omega_1}{\partial x \partial \psi_1} + \\
\left( \frac{\partial \psi_1}{\partial x} \right)^2 \frac{\partial^2 \omega_1}{\partial \psi_1^2} - \Omega_1 \left( \frac{\partial \psi_1}{\partial \psi_1} \right)^2 \frac{\partial \omega_1}{\partial \psi_1} + \frac{\Omega_2 - \Omega_1}{k} \left( \frac{\partial \psi_1}{\partial \psi_1} \right)^2 \\
+ D_1 \text{ Re } [\Omega_2 - \Omega_1] \left( \frac{\partial \psi_1}{\partial \psi_1} \right)^2 = 0 .
\]

\[\text{(8.63)}\]

8.4. **SUMMARY**

In this chapter, the von Mises transformation that was discussed in chapter 2 has been extended in terms of double coordinates in an attempt to provide a method that is applicable to the study of general two-phase flows. In the current work, as will be discussed in chapter 8, this extended von Mises transformation is used in the study of the dusty fluid flow through porous media.

Some of the possible applications of the extended von Mises transformation can be summarised as follows:

1. In the study of dusty gas flows over curved
boundaries which are streamlines, the extended von Mises transformation offers a simple and plausible way to handle the curved boundaries, provided the spatial Jacobians of the transformations remain finite and do not vanish. The method has proven to be easy to implement numerically and is easy to code.

2. In cases where the number density \( N \) is not taken to be constant, it is possible to determine the regions of dust accumulation, by the knowledge of the shape of the dust-phase streamlines which are directly obtained by plotting the \( Y_2 \) values for a given \( Y_2 \).

3. The method is not restricted to dusty fluid flow through porous media. It is easily implemented for the usual dusty gas flow equations in free space.

4. The method is not restricted to the study of dusty gas flows over curved boundaries. It can also be used in the case of straight boundaries, when \( N \) is variable, and thus regions of dust accumulation may be determined, as in point 2 above.
CHAPTER 7
DUSTY FLUID FLOW THROUGH POROUS MEDIA
INTO A TWO-DIMENSIONAL SINK

7.1. INTRODUCTION

The differential equations governing the steady motion of an incompressible dusty fluid flow through porous media are solved numerically to illustrate the effect of introducing a small concentration of dust, by volume, on the fluid-phase characteristics. The fluid is assumed to flow through a finite porous channel into a two-dimensional line sink. A comparison is made between this type of flow and the dusty-fluid flow in a non-porous channel at zero Reynolds number.

The equations governing the flow are given in chapter 6, in vorticity-streamfunction form. In order to achieve a comprehensive treatment of the models derived in chapter 5, all twelve models are analysed in the current chapter. An attempt is made to solve ten of these models numerically in order to illustrate the
similarities and differences between these models. The validity of these models for different flow situations is discussed in this chapter and in chapter 8.

7.2. GOVERNING EQUATIONS

The flow considered here is that of a dusty fluid through a porous channel into a line sink. The channel and the sink are described in chapter 4, where the single-phase flow through the channel into the sink has been treated. Once again, a channel with aspect ratio of unity is considered in the current analyses. For the current problem, we consider the flow to be governed by the H-B1, H-B2, H-B3 and the H-B4 models. All of the sub-models of H-B1 and H-B2 are considered. Out of the H-B3 and H-B4 sub-models, the Brinkman-H-B3 and Brinkman-H-B4 are dropped due to the appearance of Reynolds number in these two sub-models. If, however, they are considered vital, their solutions follow similar analyses to the Darcy-Lapwood-Brinkman-H-B3 and H-B4 models.

In this chapter the Darcy-Lapwood-Brinkman-H-B models will be referred to as the DLB-H-B models, the Darcy-Lapwood-H-B models will be referred to as the
DL-H-B models, and the Brinkman-H-B models will be referred to as the BHB models.

The flow at the inlet of the channel is taken to be uniform, for both of the phases present. The dust-phase tangential velocity components are calculated on the three channel walls via a second order one-sided differencing scheme while the flow equations, in streamfunction-vorticity form, are solved using the second order accurate central differencing scheme.

The models are considered with constant number density, and are expected to serve as a first step in the numerical study of the more general problem of dusty fluid motion with variable number density in confined porous domains.

7.3. **FINITE DIFFERENCE APPROXIMATIONS**

In order to integrate the governing differential equations numerically over the flow domain, a second order accurate 3-point central differencing scheme is used to approximate all of the derivatives over a uniformly discretized grid. The difference equations resulting from such approximations are then cast into relations of the form of equation (3.4), as discussed in
chapter 4, which results in tridiagonal matrices once the
governing equations are expressed at every node \((i,j)\). It
should be noted that diagonal dominance of the resulting
tridiagonal matrices is easy to establish for the
fluid-phase governing equations, and for the dust-phase
streamfunction equation. For the dust-phase vorticity
equation, diagonal dominance of the resulting matrix is
guaranteed provided that the following criterion is met:

\[ \begin{vmatrix} h / M \end{vmatrix} \geq \begin{vmatrix} (V_{z_{i,j}}) \end{vmatrix}, \text{ for } i = 2, 3, \ldots, Imax-1 \text{ and } j = 2, 3, \ldots, Jmax-1, \text{ where } h = \Delta X = \Delta Y \text{ is the step size.} \]

7.4. BOUNDARY CONDITIONS

The boundary conditions used for the fluid-phase are
uniform flow at the entry and no-slip on the three walls
of the channel. These translate into the following
conditions when the sink is located at \((X,Y) = (1,1)\):

\[ \Psi_1 = Y, \ U_1 = 1, \ V_1 = 0, \ \Omega_1 = 0 \text{ for } X = 0, \ 0 \leq Y \leq 1 \]

\[ \Psi_1 = U_1 = V_1 = 0 \text{ for } Y = 0, \ 0 \leq X \leq 1 \]

\[ \Psi_1 = 1, \ U_1 = V_1 = 0 \text{ for } Y = 1, \ 0 \leq X < 1 \]

\[ \Psi_1 = 0, \ U_1 = V_1 = 0 \text{ for } X = 1, \ 0 \leq Y < 1. \]
Since no direct physical conditions are available for the fluid-phase vorticity on the three channel walls, the fluid-phase vorticity conditions are derived by assuming the validity of the flow equations at the boundaries and deriving expressions for the fluid-phase vorticity on the walls in terms of the fluid-phase streamfunction at interior grid points. This of course excludes the singular point where the sink is located.

Using the image line technique, the following second order accurate expressions for the fluid-phase vorticity at the walls are obtained:

at the lower wall:

\[
(\Omega)_{1i,j} = \frac{-2}{\Delta y^2} (\Psi)_{1i,j} \quad \text{for } i = 2, \ldots, I_{\text{max}} - 1
\]

at the right wall:

\[
(\Omega)_{1i,j_{\text{max}}} = \frac{-2}{\Delta x^2} (\Psi)_{1i,j_{\text{max}} - 1} \quad \text{for } j = 1, \ldots, J_{\text{max}} - 1
\]

at the top wall:

\[
(\Omega)_{1i,j_{\text{max}}} = \frac{-2}{\Delta y^2} \left[ (\Psi)_{1i,j_{\text{max}} - 1} - 1 \right] \quad \text{for } i = 2, \ldots, I_{\text{max}} - 1.
\]
With regard to the dust-phase boundary conditions, we also assume that the dust enters the channel uniformly and thus we have the following conditions on the dust-phase at the channel entrance:

\[ \Psi_2 = Y, \ U_2 = 1, \ V_2 = \Omega_2 = 0, \text{ for } X = 0, \ 0 \leq Y \leq 1. \]

However, one cannot assume that the dust-phase velocity components vanish on the channel walls since the dust-phase viscous effects have been neglected and the deposition and collision of some dust particles on the walls usually cause some other particles to slide off resulting in a slip condition. The assumption of constant number density, however, allows the possibility of assuming vanishing normal dust-phase velocity components on the walls, with the non-zero tangential velocity components to be determined. The assumption of vanishing normal velocity at the three walls of the channel and the condition that \( \Psi_2 = Y \) at the inlet lead to the following conditions on the three channel walls:

\[ \Psi_2 = \frac{\partial \Psi_2}{\partial X} = 0 \text{ for } Y = 0 \text{ and } 0 \leq X \leq 1 \]

\[ \Psi_2 = \frac{\partial \Psi_2}{\partial Y} = 0 \text{ for } X = 1 \text{ and } 0 \leq Y < 1 \]
\[ \Psi_2 = 1, \quad \frac{\partial \Psi_2}{\partial X} = 0 \quad \text{for} \quad Y = 1 \quad \text{and} \quad 0 \leq X < 1. \]

The above conditions do not give direct conditions for the dust-phase vorticity on the three channel walls. Vorticity conditions are derived by assuming the validity of the dust-phase flow equations on the walls and deriving expressions for the vorticity on the walls in terms of the dust-phase streamfunction at interior grid points. Using second order accurate forward and backward differencing schemes, together with the fact that \( \Psi_2 = 1 \) on the upper wall and \( \Psi_2 = 0 \) on the other two walls, the following expressions for the dust-phase vorticity conditions are obtained:

at the lower wall:

\[ (\Omega_2)_{i,1} = \frac{2 (\Psi_{i,2} - \Psi_{i,3})}{\Delta Y^2} \quad \text{for} \quad i = 2, \ldots, I_{\text{max}} - 1 \]

at the right wall:

\[ (\Omega_2)_{I_{\text{max}},j} = \frac{2 (\Psi_{I_{\text{max}},j} - \Psi_{I_{\text{max}}-1,j})}{\Delta X^2} \quad \text{for} \quad j = 1, \ldots, J_{\text{max}} - 1 \]
at the top wall:

\[
\Omega_{z,i,\text{Jmax}} = \frac{2(\Psi_{z,i,\text{Jmax}-1} - (\Psi_{z,i,\text{Jmax}-2} - 1)}{\Delta y^2}
\]

for \( i = 2, \ldots, \text{Jmax-1} \).

7.5. Solution Procedure

The resulting tridiagonal system is solved using a tridiagonal solver with successive line relaxation in the Y-direction. The streamfunction equations were over-relaxed and the vorticity equations were under-relaxed.

The solution algorithm employed in the current work is similar to the one employed in chapter 4 and involves first solving for the streamfunctions \( \Psi_1 \) and \( \Psi_2 \). The expressions derived in section 7.4 are then used to compute the fluid-phase and the dust-phase vorticities at the three walls of the channel. Using this data the vorticity equations are then solved for \( \Omega_1 \) and \( \Omega_2 \). This completes one loop of the iterative procedure for given permeability \( K \), and dust parameters \( D_1 \) and \( D_2 \). The above steps are then repeated until the following criterion is satisfied:
\[ |F_{i,j}^{n+1} - F_{i,j}^n| < \delta \]

where \( \delta \) is the error tolerance, taken to be \( 5 \times 10^{-5} \) when \( F_{i,j} \) are the streamfunctions and \( 5 \times 10^{-4} \) when \( F_{i,j} \) are the vorticities.

Once convergence is achieved the velocity components \( U_1, V_1, U_2 \) and \( V_2 \) are calculated in the flow field, using second order accurate central differencing of their respective definitions in terms of \( \Psi_1 \) and \( \Psi_2 \).

The dust-phase velocity component \( U_2 \) is subsequently calculated on the lower and upper walls of the channel using the following second order accurate upwind differencing schemes:

**at the lower wall:**

\[
(U_{2,i,1}) = \frac{4(\Psi_{z,i,2} - \Psi_{z,i,3})}{2 \Delta Y}
\]

**at the upper wall:**

\[
(U_{2,i,J_{\max}}) = \frac{3 - 4(\Psi_{z,i,J_{\max-1}} + \Psi_{z,i,J_{\max-2}})}{2 \Delta Y}
\]
The dust-phase velocity component $v_z$ is calculated on the right wall of the channel using the upwind differencing expression

$$
\left( v_z \right)_{i_{\text{max}}, j} = \frac{\left( \Psi \right)_{2,i_{\text{max}}-2,j} - 4 \left( \Psi \right)_{2,i_{\text{max}}-1,j}}{2 \Delta x}.
$$

7.6. RESULTS AND DISCUSSION

The no-slip condition was utilized for the fluid-phase velocity at the walls of the channel while it was assumed that the dust-phase normal velocity components are zero and the tangential components remained to be determined on the walls of the channel. The tangential components of velocity are discussed in section 7.6.2.

The algorithm was implemented to obtain solutions to the governing equations for different permeability and different dust parameters, for each of the models. The range of dimensionless permeability tested is $K = 0.1, 0.01, 0.001 \text{ and } 0.0001$, for $D_1 = 1, 10 \text{ and } 20$ and for $D_2 = 10, 20, 30, 50 \text{ and } 100$. The range of parameters is chosen so that diagonal dominance of the difference equations resulting from the dust-phase vorticity
transport equation, and the DL-H-B model, is guaranteed.

Solutions were obtained using a grid size of 21x21, which corresponds to step size \( \Delta x = \Delta y = 0.05 \). This coarse grid has proven to be sufficient for the problem at hand, to show the development of secondary vortices near the lower right-hand corner of the channel.

7.6.1. Flow Development and Secondary Eddies

Regions of viscous separation for these types of flow occur near the lower right-hand corner of the channel and in regions close to the sink location. In these regions secondary recirculating eddies develop for the fluid-phase. For the dust-phase, the only secondary eddies that have been detected are those when the Brinkman-H-B2 model has been used. For all of the other models the inclusion of fluid-phase inertia and the assumption of perfect reflection of the dust particles off the solid walls, in addition to ignoring gravitational effects, cause the dust particles to be attracted to the sink in a manner that inhibits the recirculation of the dust.

Fluid-phase corner eddies have developed for the DLB-H-B2 and the DLB-H-B4 models for all of the permeabilities and all of the dust parameters tested, for all values of Reynolds number. When the DLB-H-B1 and
DLB-H-B3 models were used the only secondary eddies detected are those when \( \text{Re} \geq 10, \ K \geq 0.01 \). The dust parameters in this case seem to have minimal effects on the development of secondary eddies.

In the case of Brinkman-H-B1 model no secondary eddies were detected for either of the phases involved. It should be noted that in the case of single-phase fluid flow governed by Brinkman’s equation secondary eddies develop for high enough permeability, as discussed in chapter 4. In the Brinkman-H-B1 case it seems natural that secondary eddies should develop since the fluid-phase affects the dust-phase while the dust-phase effects on the fluid-phase have been neglected. The absence of secondary eddies in this situation may therefore be ascribed to the coarseness of the grid used. Refinement of the grid in this case becomes essential.

In the case of the DL-H-B1 and DL-H-B3, no secondary eddies for either of the phases have been detected. Furthermore, agreement of the solutions up to four significant digits can be noticed. Figure 7.1(a) shows the streamline pattern for this case.

In the case of the Brinkman-H-B2 model and the dusty fluid flow in free space at \( \text{Re} = 0 \), secondary eddies did not appear when \( D_2 > 50 \), for low permeability. When \( K < 0.01 \) the dust-phase secondary eddies were observed for
all the values of \( D_2 \) tested. For \( K = 0.01 \) these secondary eddies start to develop when \( D_2 = 20 \) and they were observed at \( K = 0.1 \) when \( D_2 = 50 \).

In Fig. 7.1(b) the fluid-phase and the dust-phase secondary eddies are illustrated for \( D_2 = 20 \) and \( K = 0.0005 \). The fluid-phase eddies at \( Re = 0 \) and \( D_2 = 20 \) are also illustrated. This figure gives a comparison between the streamline patterns for the cases of \( Re = 0 \) and Brinkman-H-B2. The streamlines are seen to be pulled towards the upper wall as the permeability is reduced.

In Fig. 7.1(c), the effect of the dust parameter \( D_2 \) on the streamline pattern is illustrated for the Brinkman-H-B2 model and indicates that reducing \( D_2 \) results in pulling the lower streamlines towards the upper wall. This indicates that the effect of the sink in this case is to attract the fluid from the upper regions of the channel, thus leaving a larger corner region for the secondary eddies to develop as \( D_2 \) is decreased. This is in agreement with the conclusion that secondary eddies did not develop for \( D_2 > 50 \).

In Fig. 7.1(d) the streamline pattern is compared for the cases of DLB-H-B3 and DLB-H-B4. The fluid-phase streamlines for the case of the DLB-H-B4 model are seen to be pulled towards the upper wall. In the case of DLB-H-B4, the model considers the Darcy resistance to be
in terms of the relative velocity, which is smaller than the fluid-phase velocity. This implies higher flow rate and velocity in the upper regions of the channel. It should be noted that the DLB-H-B2 model behaves in a similar qualitative manner to the DLB-H-B4, and the DLB-H-B1 behaves like the DLB-H-B3. Therefore, these models are not discussed.

Figure 7.1(e) illustrates the secondary eddies for the DLB-H-B1, DLB-H-B3 and DLB-H-B4 models. When the flow is governed by DLB-H-B1 and DLB-H-B3, it is noticed that fluid-phase separation occurs in regions close to the sink and near the right-hand wall. This occurs only when \( \text{Re} = 10, \ K = 0.01, \ D_1 = 1 \) and \( D_2 = 10 \) and 20. Although, in this case, the dust parameters may not have much influence on the formation of secondary eddies, the primary factors are Reynolds number and permeability. The pattern of behaviour might be explained as follows. As the dust enters the channel it moves steadily through the porous medium and it does not undergo any separation in regions close to corners and walls. As the dust particles get reflected from the walls back into the flowfield, and due the influence of the sink, the region of dust accumulation is close to the sink and the dust moves with a higher velocity than the fluid. This in turn forces the fluid-phase to slow down in regions close to the right
wall and close to the sink, and hence results in the fluid-phase separation. With the separated eddies occupying a fairly large region, the fluid is attracted to the sink mainly through the upper regions of the channel.

This pattern of behaviour is also noticed in the DLB-H-B4 and DLB-H-B2 models. Figure 7.1(e) illustrates this behaviour for DLB-H-B4, where the flow at low Re and low permeability, low Re and high permeability and high Re and high permeability behave in a similar manner to the above, except that the eddies occupy larger regions. For low permeability and high Re the situation is different. Corner eddies in the case of Re = 10, \( K = 0.001 \), \( D_1 = 1 \) and \( D_2 = 20 \) have developed. The region of separation in this case is larger than that in the case of Brinkman-HB2 models, which indicates the effect of Re on the size of this region.

### 7.6.2. Dust-phase Velocity Profiles

Figures 7.2(a) through 7.2(e) illustrate the dust-phase velocity profiles at different cross-sections in the flow domain. These velocities are compared for different DLB-H-B dusty fluid models and are given here for the cases of H-B1, H-B2, H-B3 and H-B4 when Re = 10, \( K = 0.001 \), \( D_1 = 1 \) and \( D_2 = 20 \). The results indicate that
the dust velocity components for the DLB-H-B1 model behave in a similar manner to those of the DLB-H-B3 model. The results are very close, numerically, to each other. Results have also indicated that the velocity components for the DLB-H-B2 and DLB-H-B4 are close to each other numerically.

Figures 7.2(a), 7.2(b) and 7.2(c) illustrate the dust-phase velocity components along the top wall, the right-hand wall and along the bottom wall, respectively. These figures indicate that the dust particles velocities are greater, in absolute value, when the DLB-H-B2 and the DLB-H-B4 models are used. The common denominator between the DLB-H-B2 and DLB-H-B4 models is that the definition of the Darcy resistance in terms of the relative velocity between the phases, while in the case of DLB-H-B1 and DLB-H-B3 the Darcy resistance is defined in terms of the fluid-phase velocity only due to the assumption that the dust-phase velocity is small as compared to the fluid-phase velocity. This assumption is of course reflected when calculating the dust-phase velocity components on the solid boundary.

Figure 7.2(d) illustrates the dust-phase vertical velocity component along the centreline of the channel for different DLB-H-B models, while Fig. 7.2(e) illustrates the dust-phase horizontal velocity component.
along the vertical centreline of the channel. Once again, results indicate that for DLB-H-B2 and DLB-H-B4 the obtained values for a particular velocity component are close to each other. Similarly for the DLB-H-B1 and DLB-H-B3 models. Fig. 7.2(d) illustrates the uniformity of the velocity profile, in the case of DLB-H-B2 and DLB-H-B4, over most of the channel cross-section. In case of the DLB-H-B1 and DLB-H-B3, the vertical velocity component increases in regions close to the sink, in a similar way that the clean fluid behaves, as discussed in chapter 4. This behaviour might be expected in light of the fact that ignoring the relative velocity effects, in the Darcy resistance, results in the fluid affecting the dust particles which are then carried with the fluid.

For the dust-phase horizontal velocity component along the vertical centreline of the channel, Fig. 7.2(e) indicates the decrease in this velocity component for the case of DLB-H-B2 and DLB-H-B4 in regions away from the upper wall.

When the flow is governed by the Brinkman-H-B1 model, Figures 7.3(a), 7.3(b) and 7.3(c) illustrate the dust-phase velocity components along the walls when the permeability $K = 0.001$ and for different $D_z$.

Fig. 7.3(a) illustrates the dust-phase horizontal velocity component along the upper wall and indicates the
increase of this component with increasing $D_2$. This increase is attributed to the decrease in the mass of each of the dust particles, with increasing $D_2$, and thus the fluid-phase effect on the dust particles results in increasing their velocity. The increase in the absolute value of the dust-phase vertical velocity component along the right-hand wall, as illustrated in Fig. 7.3(b), with increasing $D_2$ is also clear.

For the case of the Brinkman-H-B2 model, Fig. 7.3(c) illustrates the effect of $K$ on the dust-phase horizontal component of velocity along the lower wall when $D_2 = 20$. It indicates that this component is larger for smaller permeability in regions on the lower wall closer to the inlet of the channel, and it starts to decrease with decreasing permeability as the fluid moves away from the inlet.

In Fig. 7.3(d) the horizontal component of dust-phase velocity along the upper wall of the channel is illustrated for different permeability and different $D_2$. The figure indicates the increase in this component as $K$ is decreased, for a given $D_2$. This component, for a given $K$, decreases with increasing $D_2$.

For a given $D_2$, Fig. 7.3(e) illustrates the vertical dust-phase velocity component along the right-hand wall for different permeabilities, and for a given $K$ and
different $D_2$. It indicates the increase in absolute value of this component with decreasing $K$, for a given $D_2$, and the increase of the absolute value of this component with decreasing $D_2$, for a given $K$.

Figures 7.3(f) and 7.3(g) illustrate the vertical dust-phase velocity component along the horizontal centreline of the channel, and the horizontal dust-phase velocity component along the vertical centreline of the channel, respectively. Fig. 7.3(f) indicates the increase in the dust-phase vertical velocity component as we move away from the inlet, with increasing $D_2$. Fig. 7.3(g) indicates the increase in the horizontal velocity component as we move away from the lower wall.

From the above discussion and from the various figures, it is clear that the Brinkman-H-B2 model lowers the dust-phase components. In certain situations it has the reverse effect, on the dust-phase velocity components, to Brinkman-H-B1. Allowing the dust-phase to interact and influence the fluid-phase seems to dampen out the dust-phase velocities. Near the inlet, the dust-phase horizontal velocity is larger for smaller $D_2$.

7.6.3. **Fluid-phase Velocity Profiles**

For the DLB-H-B models, Figures 7.4(a), (b), (c) and (d) illustrate the fluid-phase horizontal velocity
component along the vertical centreline of the channel.

Fig. 7.4(a) illustrates this component for different permeabilities, different Re and different dust parameters. Although the effect of Re on this component is minimal, for low permeability, the permeability has greater effect on this component, for given dust parameters. The increase of this component with increasing permeability, as we get closer to the top wall, is in agreement with the case of single-phase flow into a line sink. A similar situation occurs in the DLB-H-B3 model, as shown in Fig. 7.4(b).

For the DLB-H-B2 and DLB-H-B4, the effect of increasing $D_1$ on the horizontal fluid-phase velocity component along the vertical centreline of the channel is illustrated in Fig. 7.4(c). The effect of different dust parameters, Re and K is illustrated in Fig. 7.4(d). Fig. 7.4(c) indicates the increase in this velocity component with increasing $D_1$ in regions close to the top wall. Fig. 7.4(d) indicates the increase in this velocity component with increasing permeability.

In Figures 7.5(a) and 7.5(b) the fluid-phase vertical component of velocity along the horizontal centreline of the channel is illustrated for the DLB-H-B1 and DLB-H-B4 models, respectively. In Fig. 7.5(a), this component is shown for different Re and K, for $D_1 = 1$ and
$D_2 = 20$. It indicates that decreasing the permeability, for a given Re, increases this velocity component in regions close to the sink. Increasing Re, for a given permeability, is also seen in Fig. 7.5(a) to increase this component in regions close to the sink.

In Fig. 7.5(b), the vertical component of velocity is illustrated for different dust parameters and different K and Re. It indicates that increasing $D_2$, for given Re and K, does not have much effect on this velocity component, while increasing the permeability is accompanied with a decrease in this component in regions close to the sink. Similarly, increasing Re results in decreasing this component in regions close to the sink.

Velocity components based on the Brinkman-H-B1 model are illustrated in Figures 7.6(a) and 7.6(b). In Fig. 7.6(a) the horizontal velocity component along the vertical centreline of the channel is illustrated to demonstrate that for different $D_2$, the profiles for this velocity component are very close to each other.

In Fig. 7.6(b), the vertical fluid-phase velocity component is illustrated for different $D_2$. The profiles for this component are also close to each other for different $D_2$. Comparison in Fig. 7.6(b) is given with the results of the DL-H-B1 and DL-H-B3 models. The results obtained using the DL-H-B1 and DL-H-B3 models are not all
that accurate in this study. All of the solutions obtained for the different flow parameters agree with each other up to four significant digits. This may be interpreted as a failure of the dusty fluid flow models through porous media based on the Darcy-Lapwood model.

In the case of Brinkman-H-B2, Figure 7.7(a) illustrates the fluid-phase and dust-phase horizontal velocity components, $U_1$ and $U_2$, at different sections of the channel, for $D_2 = 20$ and $K = 0.0005$. It shows $U_1$ and $U_2$ along the lines $X = 0.05$ and 0.5. At $X = 0.05$ the uniform flow at the inlet of the channel still has a great effect while the effect of the sink is very minimal. As we proceed in the downstream direction the effect of the sink starts to dominate and the fluid is attracted towards it.

In the lower 85% of the channel the fluid-phase horizontal component of velocity is slightly greater than the corresponding dust-phase velocity component. Such a situation is reversed in the upper 15% of the channel, since the dust-phase horizontal velocity component must adjust to account for the slip velocity at the upper wall.

For $D_2 = 20$ and $K = 0.1$, Figure 7.7(a) shows that $U_2$ along $X = 0.5$ is smaller than $U_1$ along the same line, in the lower 80% of the channel, and is larger in the
upper 20%. The relative magnitudes of $U_2$ along $X = 0.5$ when $D_2 = 10$ and $K = 0.1$, and when $D_2 = 20$ and $K = 0.1$, are as shown in Figure 7.7(a).

In Figure 7.7(b) the fluid-phase vertical component of velocity is illustrated along the horizontal centreline of the channel for $D_2 = 20$ and $K = 0.1$ and 0.0005. The comparison is also made with the same velocity component at $D_2 = 10$ and $K = 0.1$.

When $D_2 = 20$ the graphs in Figure 7.7(b) demonstrate that $V_4$ is greater when $K = 0.0005$ than when $K = 0.1$, for the intervals $0 < X < 0.4$ and $0.85 < X < 1$. When $K = 0.1$, Figure 7.7(b) shows that $V_4$ is slightly greater when $D_2 = 10$ than when $D_2 = 20$ in the regions near the channel entrance, while the graphs remain close to each other in other regions.

### 7.7. CONCLUSION

In this chapter ten of the dusty fluid flow models have been tested when the fluid flows through a porous medium into a line sink. Regions of viscous separation have been outlined and their dependence on the permeability and on the dust parameters has been discussed. Dusty fluid flow models based on the Darcy-Lapwood model are redeemed as incompatible with the
current flow situation. This may be due to the low permeability associated with the porous medium represented by this model which does not allow for interaction between the dust-phase and the fluid-phase. The type of the dusty fluid flow considered for this model is thus incompatible with the porous medium in question. This leaves the models based on the Darcy-Lapwood-Brinkman model to be more appropriate.
CHAPTER 8

DUSTY FLUID FLOW THROUGH POROUS MEDIA
OVERLYING CURVED SURFACES

8.1 INTRODUCTION

In order to illustrate the possibility of implementing the extended von Mises transformation, discussed in chapter 8, we consider the dusty fluid flow through the porous medium described in chapter 3. The flow, therefore, is considered through a semi-infinite porous medium that is bounded below by a static fluid. The interface between the static fluid and the porous medium is assumed to take the form, in cartesian plane, $Y = f_1(X)$, where $f_1(X)$ is the smooth function given by equation (3.23), namely

$$f_1(X) = \begin{cases} 0.2 \left( 0.25 - X^2 \right)^{1/2}, & -0.5 \leq X \leq 0.5 \\
0, & X < -0.5 \text{ or } X > 0.5. \end{cases}$$

...(8.1)
Since a comprehensive treatment of the different dusty fluid flow models has been given in chapter 7, the current chapter will only consider the most general of these models, that is, the H-B4 group. From this group, the Darcy-Lapwood-Brinkman-H-B4 model is considered. This model is governed by equations (6.49), (6.50), (6.51) and (6.61).

If $D_1 \to 0$ and $Re = 0$, the model reduces to the Brinkman-H-B2 model, for which the governing equations are given by equations (6.57), (6.49), (6.50) and (6.51).

If $D_1 \to 0$ in equation (6.61), the model reduces to the Darcy-Lapwood-Brinkman-H-B2 model, given by equations (6.55), (6.49), (6.50) and (6.51).

Since the Darcy-Lapwood-H-B2 and the Darcy-Lapwood-H-B4 models could not be applied in chapter 7, they are considered for the current flow problem. The Darcy-Lapwood-H-B model, given by equations (6.49), (6.50), (6.51) and (6.62), is considered for different values of $D_1$. If, however, $D_1 \to 0$ in equation (6.62) then it reduces to equation (6.56), that is, the Darcy-Lapwood-H-B4 model, governed by equations (6.49), (6.50), (6.51) and (6.58).

8.2. BOUNDARY CONDITIONS

In this work, the simple case of uniform flow at
infinity, for both the dust-phase and the fluid-phase, is employed. Although the uniform flow at \( X = -\infty \) is plausible, the uniform flow assumption at \( X = +\infty \) seems to be improbable in light of the fact that, in the presence of a boundary, the dust particles may bombard this boundary, deposit and accumulate on this boundary, initiate the motion of other particles already settled on the boundary or bounce off the boundary and reflect back into the flowfield [86].

Although the above is perfectly acceptable in the case of a solid boundary, the current problem avoids this situation by taking the lower boundary of the porous medium to be a static fluid along which the usual flow tangency condition can also be interpreted as a no-penetration condition for the dust particles. This implies that the effect of this boundary on the dust particles is to perfectly reflect the dust particles back into the flowfield [86]. This, of course, is facilitated by the assumption that the dust particle number density is constant throughout the flowfield, which implies that the static interface may be taken as a dust-phase streamline, considered here as the zero-streamline for both of the fluid and the dust. In light of this discussion, and assuming that the flow domain is large enough so that the effect of the hump, \( f_1(X) \), on the
flowfield dies out far downstream, it is possible to impose a uniform flow condition at $X = \pm \infty$.

To summarize, the boundary conditions in the computational domain take the following form:

1. Uniform flow at infinity:

$$Y_1 = \Psi_1 \text{ at } X = \pm \infty \text{ and at } \Psi_1 = \infty \quad \ldots (8.2)$$

$$Y_2 = \Psi_2 \text{ at } X = \pm \infty \text{ and at } \Psi_2 = \infty \quad \ldots (8.3)$$

$$\Omega_1 = 0 \text{ at } X = \pm \infty \text{ and at } \Psi_1 = \infty \quad \ldots (8.4)$$

$$\Omega_2 = 0 \text{ at } X = \pm \infty \text{ and at } \Psi_2 = \infty \quad \ldots (8.5)$$

2. Flow tangency (along the interface):

$$Y_1(x, 0) = f_1(x) \quad -\infty < X < \infty. \quad \ldots (8.6)$$

$$Y_2(x, 0) = f_2(x) \quad -\infty < X < \infty. \quad \ldots (8.7)$$

The vorticity along the interface, for each of the phases present, remains a quantity to be determined.
8.3. **FINITE DIFFERENCE APPROXIMATIONS**

The procedure for obtaining the finite difference approximation in the current case is the same as that discussed in chapter 3. Three-point central differencing of all of the derivatives involved has been employed and the resulting difference equations are then expressed in the form of equation (3.4). The differenced forms of the models are incorporated in the DUSTY Computer Code in Appendix C.

8.4. **SOLUTION ALGORITHM**

The solution procedure and algorithm used in this chapter is an extension to that given in chapter 3, section 3.4. For the sake of completeness, it is summarized as follows.

For a given permeability $K$, Reynolds number $Re$ and dust parameters $D_1$ and $D_2$,

1. The flow domain is initialized by giving $Y_1$, $Y_2$, $\Omega_1$ and $\Omega_2$ small starting values.

2. The $Y_1$-equation is solved for $Y_1$, using the tridiagonal solver, along each grid line, $i = 2$ to Imax-1, and the solution is iterated along each i-line using SLOR of the form

$$Y_1^{n+1} = Y_1^n + \omega \left( Y_1^{n+1} - Y_1^n \right)$$

... (8.8)
where \( Y_{1}^{n+1} \) is the value obtained by the tridiagonal solver, and \( Y_{1}^{n} \) is the value obtained from the previous iteration. This step is repeated a few times in order to accelerate convergence.

3. \( U_{i,1}^{t} \), \( U_{i,2}^{t} \), \( V_{i,1}^{t} \), \( V_{i,2}^{t} \), \( \Omega_{i,1}^{t} \) are calculated, along the static interface, using the derived expressions (cf. section 8.5).

4. The fluid-phase vorticity equation is solved for \( \Omega_{1}^{t} \) by a similar procedure to that in step 2.

5. The \( Y_{2}^{t} \)-equation is solved for \( Y_{2}^{t} \) by a similar procedure to that in step 2.

6. \( U_{i,1}^{t} \), \( U_{i,2}^{t} \), \( V_{i,1}^{t} \), \( V_{i,2}^{t} \), \( \Omega_{i,1}^{t} \) are calculated, along the static interface, using the derived expressions (cf. section 8.5).

7. The dust-phase vorticity equation is solved for \( \Omega_{2}^{t} \) by a similar procedure to that in step 2.

8. Steps 2 to 7 are repeated until the following convergence criterion is met:

\[
\left| F_{i,j}^{n+1} - F_{i,j}^{n} \right| \leq \delta \quad \ldots (8.9)
\]

where \( \delta = 5 \times 10^{-5} \) for \( Y_{1} \) and \( Y_{2} \), and \( 5 \times 10^{-4} \) for \( \Omega_{1} \) and \( \Omega_{2} \).

9. The velocity components for both of the phases are calculated along the static interface and in the flowfield using the derived expressions for velocity (cf. section 8.5).
8.5. VELOCITY AND VORTICITY EXPRESSIONS

Following similar analysis to the one given in section 3.5, we obtain the following expressions:

\[ U_{i,1} = \frac{\Delta \Psi}{Y_{i,2} - f(X_{i})} \]  \hspace{1cm} \ldots(8.10)

\[ U_{i,1} = \frac{\Delta \Psi}{Y_{i,2} - f(X_{i})} \]  \hspace{1cm} \ldots(8.11)

\[ V_{i,1} = U_{i,1} \frac{f'(X_{i})}{1} \]  \hspace{1cm} \ldots(8.12)

\[ V_{i,1} = U_{i,1} \frac{f'(X_{i})}{1} \]  \hspace{1cm} \ldots(8.13)

\[ \Omega_{i,1} = \frac{V_{i+1,1} - V_{i,1}}{\Delta X} - \frac{V_{i,2} - V_{i,1}}{\Delta \Psi} \]

\[ \Omega_{i,1} = \frac{U_{i,1} + U_{i,2}}{\Delta \Psi} \]  \hspace{1cm} \ldots(8.14)
\[ \Omega_{i,1} = \frac{V_{i+1,1} - V_{i,1}}{\Delta x} - \frac{V_{i,2} - V_{i,1}}{\Delta \Psi} \]

\[ - U_{i,1} \frac{U_{i,2} - U_{i,1}}{\Delta \Psi} \ldots (8.15) \]

\[ U_{i,j} = \frac{2 \Delta \Psi}{Y_{i,j+1} - Y_{i,j-1}} \ldots (8.16) \]

\[ U_{i,j} = \frac{2 \Delta \Psi}{Y_{i,j+1} - Y_{i,j-1}} \ldots (8.17) \]

\[ \Psi_{i,j} = \frac{Y_{i+1,j} - Y_{i-1,j}}{Y_{i,j+1} - Y_{i,j-1}} \frac{\Delta \Psi}{\Delta x} \ldots (8.18) \]

\[ V_{i,j} = \frac{Y_{i+1,j} - Y_{i-1,j}}{Y_{i,j+1} - Y_{i,j-1}} \frac{\Delta \Psi}{\Delta x} \ldots (8.19) \]

for \( i = 2, 3, \ldots, I_{\text{max}-1} \) and \( J = 2, 3, \ldots, J_{\text{max}-2} \). In the above expressions, \( \Delta \Psi = \Delta \Psi_1 = \Delta \Psi_2 \).
8.6. RESULTS, ANALYSIS AND DISCUSSION

Solutions have been obtained for Darcy-Lapwood-Brinkman-H-B4 model for $D_2 = 20$ and $D_1 \to 0$, $D_1 = 1$ and 10. The range of $Re = 1, 10$ and 50 has been considered for the values of permeability $K = 0.1, 0.01, 0.001$ and $K \to 1$. When the Brinkman-H-B2 model is considered, solutions are obtained for dust parameter $D_2 = 20$ and 50 and permeability $K = 0.1, 0.01, 0.001$ and $K \to 1$. For the case of Darcy-Lapwood-H-B4 model, $D_1 = 1$ and $D_1 \to 0$ and $D_2 = 20$ and 50, for the range of $Re = 10, 50$ and 100. The permeability range considered is $K = 0.1, 0.01$ and $K \to 1$.

It should be noted that all of the results in the following section are based on a $32 \times 22$ grid. Although it might be argued that this grid is too coarse, it has proven to be fine enough to illustrate the applicability of the method. The question of grid refinement has been addressed in chapter 3 where solutions were also obtained for a $122 \times 22$ grid using the von Mises transformation. In the extended von Mises transformation the situation is not at all different. Here, one deals with differential equations of the same nature as in the case of single-phase flow. The only difference is, of course, the
number of equations involved, and this does not cause any complications. Furthermore, the absence of sharp gradients in the problem considered does not necessitate a refinement of the grid unless the quantitative behaviour is essential.

Comparison between the tested models is made through the information on fluid-phase and dust-phase velocities, fluid-phase vorticity and the flow development patterns given by the streamlines of both phases.

8.6.1. Flow Development

In order to illustrate the differences in the streamline patterns for the fluid-phase and the dust-phase, the requirement of fine scales becomes essential. For this reason the \( Y_1 \) and the \( Y_2 \) are tabulated, for different streamlines and different flow regimes. These are given in Tables 8.1 through 8.5 to illustrate that the extended von Mises method renders the streamlines of the two phases.

In Tables 8.1, 8.2 and 8.3 the \( Y_1 \) and \( Y_2 \) values are given for the streamlines \( \Psi_1 = \Psi_2 = 0.08524, 0.38095 \) and \( 1.3333 \) when the flow is governed by the Darcy-Lapwood-Brinkman-H-B4 model with \( D_2 = 20 \) and different \( D_1 \), Re and K.
In Tables 8.4 and 8.5, the $y_1$ and $y_2$ values are given on the above streamlines for the Brinkman-H-B2 model for different $D_z$.

Once the data in these tables is plotted, the indicated streamlines for the respective phases result.

### 8.6.2. Velocity Profiles

The Darcy-Lapwood-H-B2 and H-B4 Models:

Figure 8.1 illustrates the fluid-phase X-component of velocity along the static interface. The profiles for different dust parameters, different permeability and different Re, have proven to be very close to each other, with difference showing up only in the third and fourth significant digits. These differences cannot be noticed on the scale of the plots employed.

Due to the apparent insignificance of the flow parameters on the flow governed by these models, the Darcy-Lapwood-H-B2 and H-B4 models have to be considered incompatible with the flow of a dusty fluid over curved interfaces. It is worthwhile noting that the validity of these models in closed flow domains, as illustrated in
chapter 7 where these particular models have failed, is also questionable.

The Brinkman-B-H2 model:

Figures 8.2(a) and 8.2(b) illustrate the fluid-phase, X-component, of velocity along the static interface for different permeability and different $D_2$. In Fig. 8.2(a), this velocity component is illustrated for $D_2 = 20$ and different $K$. It indicates that an increase in the permeability results in a decrease of this velocity component over the nonzero part of $f_4(x)$. This behaviour was also noticed in the case of single-phase flow over a static fluid.

In Fig. 8.2(b), this component of velocity is illustrated for $K = 0.1$ and different $D_2$ and indicates that increasing $D_2$ results in decreasing this velocity component over the hump. This is understandable due to the influence of the dust on the fluid-phase, as indicated by the Brinkman-H-B2 model.

For a given permeability, $K = 0.01$, and different $D_2$, Fig. 8.3 illustrates the fluid-phase normal component of velocity over the hump. This profile is representative of this fluid-phase component for different $D_2$ and different permeability. The figure also gives the
comparison, at $D_z = 20$, between the fluid-phase and the
dust-phase normal velocity components.

Figure 8.4 illustrates the fluid-phase $X$-component of velocity along the vertical line passing through a point near the maximum of $f(X)$. The figure indicates that as the permeability is decreased, for a given $D_z$, a fluctuation of this component of velocity occurs as we move away from the static interface. For larger permeability, this oscillation starts to dampen out. Although it is expected that at points far enough from the interface the profile should become more uniform, as indicated by the boundary conditions far upstream, this behaviour might be an indication that the extent of the flow domain is not large enough. This also might be an indication that the permeability associated with the Brinkman-H-B2 model should be large. It has also been noticed that increasing $D_z$ results in damping out of these oscillations, for a given permeability.

The Darcy-Lapwood-Brinkman-H-B4 Model:

Figures 8.5(a) and 8.5(b) illustrate the fluid-phase $X$-component of velocity along the interface for different $K$, and different $Re$, respectively.

In Fig. 8.5(a), this velocity component is
illustrated for $D_1 \to 0$, $Re = 10$ and different permeability and indicates the increase in this velocity component, over the hump, with decreasing permeability. In Fig. 8.5(b), the increase of this velocity component with increasing $Re$, for a given permeability, is indicated.

Figure 8.6 illustrates the effect of permeability on the oscillations in the $X$-component of fluid-phase velocity, where this velocity component is taken along the vertical line passing through a point near the maximum of $f_4(X)$. It indicates that, for a given $Re$, the fluctuations are less when the permeability is higher. Furthermore, a visual conclusion about the effect of $Re$, as can be seen from Fig. 8.6 and Fig. 8.4, it can be concluded that the fluctuations are greater when $Re = 0$ as in the case of Brinkman-H-B2 model.

8.6.3. Fluid-phase Vorticity

Brinkman-H-B2:

Figures 8.7(a) and 8.7(b) illustrate the fluid-phase and the dust-phase vorticities along the static interface, over the hump. In Fig. 8.7(a), the vorticity is illustrated for $D_2 = 20$ and different permeabilities.
From this figure one can conclude that the absolute value of vorticity decreases with decreasing permeability. In Fig. 8.7(b), the fluid-phase and the dust-phase vorticities along the static interface, over the hump, are compared. This figure indicates the increase of the dust-phase vorticity with the fluid-phase vorticity, for a given value of permeability and a given $D_2$. As the value of $D_2$ is increased, for a given permeability, the fluid-phase and the dust-phase vorticities approach each other.

The Darcy-Lapwood-Brinkman-H-Bå Model:

In Fig. 8.8(a), the fluid-phase vorticity along the static interface, over the hump, is illustrated for a given permeability, different $D_1$ and different Re. This figure indicates the increase in vorticity with increasing Re, for a given $D_1$. It also indicates the slight increase in vorticity with increasing $D_1$, for a given Re.

In Fig. 8.8(b), the fluid-phase vorticity along the interface is illustrated for different permeabilities, when $Re = 10$, $D_1 \to 0$ and $D_2 = 20$. This figure also indicates the reduction in the absolute value of vorticity with decreasing permeability.
From the results for the vorticity of the fluid-phase and dust-phase, it can be concluded that reducing the permeability results in increasing the negativity of the vorticity. In addition, decreasing the permeability results in higher fluid-phase X-component of velocity. Furthermore, as seen in Fig. 8.7(b), there is a difference between the fluid-phase and the dust-phase vorticity. This difference represents the forcing term in the H-B dusty fluid flow models through porous media. The higher the difference, between the vorticities, the greater the interaction between the fluid-phase and the dust-phase; in particular, the greater the influence of the dust particles on the fluid-phase. The difference between the two vorticities has been demonstrated to be greater for lower values of $D_2$, as can be seen from Fig. 8.7(b).

The above analyses indicate that reducing the permeability results in increasing the fluid-phase X-velocity component. Although one expects that a reduction in permeability should result in a reduction in velocity, the situation is rather different in case of flow of a dusty fluid through porous media. This is attributed to the increase in the interaction between the phases present with increasing permeability. For higher permeability the dust-phase offers greater effects on the
fluid-phase, resulting in a reduction in the fluid-phase velocity. For lower permeability the interaction is less and the dust particles might not offer much influence on the fluid. It should be noted that the Darcy-Lapwood-H-B models have been rejected for exactly this reason.

8.7. CONCLUSIONS

In this chapter, the extended von Mises transformation has been implemented in the study of dusty fluid flow through a porous medium bounded below by a static fluid. The nature of the problem considered does not allow for definitive conclusions about the models proposed. However, application of the method to more realistic flow problems will not only demonstrate the usefulness of the proposed models but also will give a greater insight in the possibility of implementing the Darcy-Lapwood-H-B2 and H-B4 models which, in the current problem, have been rejected. Furthermore, experiments are required to validate these models.
CHAPTER 9
CONCLUSIONS

9.1. CONTRIBUTIONS

Fluid flow through porous media has been considered. The problems considered have centred around the flow of a single-phase fluid, as governed by different models, and the flow of a dusty fluid for which the governing equations have been proposed in this work.

In the first part of the current work, single-phase fluid flow through porous media overlying curved surfaces and through porous media into a line sink has been treated. The following contributions may be attributed to the analyses given in chapters 2, 3 and 4.

1. The von Mises transformation, which existed in time-independent form, has been extended to the more general time-dependent form in the current work.

2. The applicability of the von Mises transformation to the study of flow through porous media overlying
curved surfaces has been tested in this work for different flow models.

3. Comparison between the different flow models has also been given. The analysis of the different models has led to a proposed modified form of the Darcy-Lapwood equation.

4. Studies of flow through porous media, based on the Darcy-Lapwood-Brinkman and the Darcy-Lapwood modified models, have been carried out when the fluid is being attracted to a line sink. These studies were undertaken to offer further understanding of the phenomenon of viscous separation in porous media.

In the second part of this work, the flow of a dusty fluid through porous media has been considered. In this context, the following contributions have been made.

1. Model equations have been proposed to govern the flow phenomena of a dusty fluid through porous media. The equations have been sub-classified to cover different flow situations in a manner that parallels the flow equations of a single-phase fluid.

2. The von Mises transformation that was extended for time-dependent problems has also been extended in this work to provide a more general double time-dependent
transformation. This extension is capable of transforming the multi-phase flow equations from a single physical plane to a single computational plane.

3. The above extension has been employed in the study of dusty fluid flow through porous media overlying curved surfaces, to illustrate the utility of the transformation.

4. The flow of a dusty fluid through porous media into a line sink has been considered to illustrate the determinate nature of the dusty fluid flow equations proposed in this work. Viscous separation has also been discussed when the flow is governed by the different proposed models.

9.2. RECOMMENDATIONS

1. In using the von Mises approach, the single-phase fluid flow problems considered in this work have been chosen in domain that is bounded below by a static fluid. This choice results in the spatial Jacobian of the transformation being non zero and finite. To achieve full generality of the approach, resolution to the problem of
an infinite Jacobian which arises when the no-slip condition is imposed, needs further clarification.

2. In the case of dusty fluid flow through porous media, the proposed models need further testing. This includes treating more realistic problems involving variable particle number density and different flow problems to investigate the possibility of employing these models in the study of filtration.

3. The extended von Mises transformation to treat multi-phase fluid flow needs to be further tested on multi-phase fluid flow through porous media and in free space in order to demonstrate its practical utility.

4. An extension of the dusty fluid flow models may be achieved by considering a more general form, namely a model that parallels the Darcy-Lapwood-Forchheimer-Brinkman model. This may be written in the following form for the fluid-phase:

\[
\rho_1 \left[ \frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_1 \right] = -\nabla p_1 + \rho_1 g + \mu_1 \nabla^2 \mathbf{v}_1 - \frac{\mu_1}{k} (\mathbf{v}_1 - \mathbf{v}_2) - C_N (\mathbf{v}_1 - \mathbf{v}_2) - \frac{\rho C_d}{\sqrt{k}} \mathbf{v}_1 | \mathbf{v}_1 |.
\]
For the dust-phase, the momentum equation may or may not include the Forchheimer's drag term. This needs further analysis and may require experimental validation.

Further extensions of the proposed models involve taking into account the compressibility effects, the effects of dispersion, diffusion and terms to account for heat convection processes. These suggested extensions will render dusty fluid flow models that offer more generality and greater utility in many situations of practical interest.

5. The study of flow separation of a dusty fluid in porous media, which is introduced in this work, has to be further tested. Refinement of the grid in addition to considering different flow situations might prove to be essential in validating the proposed dusty fluid flow models.
FIGURE SET 1
Fig. 3.1. Representative Sketch: Physical Domain.
Fig. 3.2. Representative Sketch: Computational Domain.
Fig. 3.3. Streamline pattern in the porous medium with a "Joukowski" bump, $Y = f_3(X)$, along the lower surface.
Fig. 3.4. Horizontal velocity component along the interface for different $f(X)$. 
Fig. 3.5. Horizontal velocity component at $X = X_m$, corresponding to position of maximum thickness of $f_3(X)$. 
Fig. 3.6. Vertical velocity component along the interface for different $f(X)$. 
Fig. 3.7. Perturbation velocity potential along the interface for different $f(x)$. 
Fig. 3.8. Velocity potential (pressure) along the interface for different $f(x)$.
Fig. 3.9. Perturbation velocity potential along $j = 2$ for different $f(x)$; no-slip case.
Fig. 3.10. Streamline pattern in the porous medium for Forchheimer's model; \( \text{Re} \sqrt{K} C_d = 1 \).
Fig. 3.11(a). Horizontal velocity component along the interface; Forchheimer and Darcy models.

--- Darcy
--- Forchheimer, $Re \sqrt{K C_d} = 1$
--- Forchheimer, $Re \sqrt{K C_d} = 0$
Fig. 3.11(b). Horizontal velocity component along the interface; Brinkman and Darcy models.

- ■ ■ K = 1
- ■ ■ K = 0.1
- ▲ ▲ K = 0.01
- ● ● K = 0.001 and Darcy's
Fig. 3.12(a). Horizontal velocity component at $X = X_m$; Forchheimer and Darcy models.

- - - - Darcy

- - Forchheimer, $Re \sqrt{K C_d} = 1$

- - Forchheimer, $Re \sqrt{K C_d} = 0$
Fig. 3.12(b). Horizontal velocity component at $X = X_m$:

Brinkman and Darcy models.

- $K \rightarrow 1$
- $K = 0.1$
- $K = 0.01$
- $K = 0.001$ and Darcy
Fig. 3.13. Vertical velocity component along the interface for all models.
Fig. 3.14(a). Vorticity along the interface over the non-zero part of $f(X)$. Forchheimer's model.

- $\text{Re} \sqrt{K C_d} = 1$
- $\Delta \Delta \text{Re} \sqrt{K C_d} = 2$
Fig. 3.14(b). Vorticity along the interface over the non-zero part of f(X). Brinkman's model.

- K = 1
- K = 0.1
- K = 0.01
- K = 0.001
Fig. 3.15(a). Vorticity at $X = X_m$. Forchheimer's model.

- $\bullet\bullet$ Re $\sqrt{K} C_d = 1$
- $\times\times$ Re $\sqrt{K} C_d = 5$
- $\triangle\triangle$ Re $\sqrt{K} C_d = 9$
- $\star\star$ Re $\sqrt{K} C_d = 25$

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Fig. 3.15(b). Vorticity at $X = X_m$. Brinkman's model.

- $\cdot\cdot\cdot K \rightarrow 1$
- $\times\times K = 0.1$
- $\Delta\Delta K = 0.01$
- $\circ\circ K = 0.001$
Fig. 3.16(a). Horizontal velocity component along the interface. DLE model; Re = 10, different K.

- - - K = 1
- - K = 0.01
- - - K = 0.0001
Fig. 3.16(b). Horizontal velocity component along the interface. DLB model; $K = 0.001$, different $Re$.

- - - $Re = 10$
- - - $Re = 500$
Fig. 3.17. Horizontal velocity component at $X = X_m$.
DLB model; $Re = 10$, different $K$.

- $K = 1$
- $K = 0.01$
- $K = 0.0001$
Fig. 3.18(a). Vorticity along the interface. DLB model:
Re = 10, different K.

- $K = 1$
- $K = 0.01$
- $K = 0.001$
- $K = 0.0001$
Fig. 3.18(b). Vorticity along the interface.
DLB model; $K = 0.001$, different $Re$.

- - - - $Re = 10$
- - - - $Re = 500$
Fig. 3.18. Vorticity at $X = X_m$. DLM model; $Re = 10$, different $K$.

- - - $K \rightarrow 1$
- - - $K = 0.01$
- - - $K = 0.0001$
Fig. 3.20. Streamline pattern in the porous medium for DL modified model; $Re = 200$, $Rec = 1000$ and $K = 0.001$. 
Fig. 3.21. Horizontal velocity component along the interface for DL modified model; \( Re = 100 \), \( Rec = 10000 \) and \( K = 0.001 \).
Fig. 3.22. Horizontal velocity component along the interface for DL modified and DLB models; $K = 0.01$, $Re = 100$, $Rec = 500$.

- DL modified, $Rec = 1000$
- DL modified, $Rec = 500$
- DLB model, $Re = 100$, $K = 0.01$
3.23. Horizontal velocity component along $X = X_m$.

DLB and DL modified models; different Re and K, Rec = 1000.

--- DLB model, Re = 100, K = 0.0001

..... DL modified, Re = 100, Rec = 1000, K = 0.01

- - - - DL modified, Re = 200, Rec = 1000, K = 0.0001

- X-X DL modified, Re = 100, Rec = 1000, K = 0.001
Fig. 3.24. Vorticity along the interface for DL modified model; Re = 100, K = 0.01 and different Rec.

- xxxx Rec = 1000
- ---- Rec = 100,000
Fig. 3.25. Vorticity at $X = X_m$ for DL modified model;
Re = 100, $K = 0.01$ and different Rec.
--- Rec = 1000
----- Rec = 100,000
Fig. 4.1. Representative Sketch
Fig. 4.2. Horizontal velocity component along different cross-sections of the channel. Brinkman’s model; $K = 0.001$.

- - - U at $X = 0.025$
- - - U at $X = 0.5$
- - - U at $X = 0.975$
Fig. 4.3(a). Horizontal velocity component along the vertical centreline of the channel. Brinkman's model; different $K$.

- $K \to 1$
- $K = 0.01$
- $K = 0.001$
- $K = 0.0001$
- Darcy's case
Fig. 4.3(b). Horizontal velocity component along the vertical centreline of the channel.
DLB model; Re = 10, different K.

- K → 1
- - - K = 0.01
- - - K = 0.001
- - - - K = 0.0001
- - - - DL modified, Re = 10, Rec = 500, K = 0.1
Fig. 4.3(c). Horizontal velocity component along the vertical centreline of the channel. DLBL model; different Re and K.

- — Re = 50, K = 1
- — Re = 1, K = 0.001
- — Re = 0, K = 0.001
Fig. 4.3kD. Horizontal velocity component along the vertical centreline of the channel.
DL modified model; Re = 10, Rec = 100
and different K.

- - K = 0.1
- - K = 0.01
- - K = 0.001
- - - - K = 0.001, Rec = 1000
Fig. 4.4(a). Vertical velocity component along the horizontal centreline of the channel. Brinkman's model; different K.

- K = 1
- K = 0.01
- K = 0.001
- K = 0.0001
Fig. 4.4(c). Vertical velocity component along the horizontal centreline of the channel. DLB model; Re = 10 and different K.

- K = 1
- K = 0.01
- K = 0.001
- K = 0.0001
Fig. 4.4(c). Vertical velocity component along the horizontal centreline of the channel. DLAB model; different Re and K.

- - - Re = 50, K = 1
- - - - Re = 50, K = 0.001
- - - - - - Re = 1, K = 0.001
- - - - - - - Re = 0, K = 1
Fig. 4.4(d). Vertical velocity component along the horizontal centreline of the channel. DL modified model; Re = 10, Rec = 100, different K.

- - K = 0.1
- - - K = 0.01
- - - - K = 0.001
Fig. 4.5. Horizontal velocity component along the vertical centreline of the channel for different sink locations. Brinkman's model and $Re = 0; K = 0.1$.

- Sink at (1,1):
  - $Re = 0$
  - $K = 0.1$

- Sink near (0.875,1):
  - $K = 0.1$

- Sink near (0.6,1):
  - $Re = 0$
  - $K = 0.1$
Fig. 4.6(a). Streamline pattern for Brinkman’s model at $K = 0.001$.

$\psi = -0.46 \times 10^{-5}$, $K = 0.01$
Fig. 4.8(b). Streamline pattern for D LB model at Re = 10 and K = 0.001.
Fig. 4.8(c). Streamline pattern for DL modified model at $K = 0.001$; $Re = 10$, $Rec = 100$. 

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Fig. 4.7(a). Streamline pattern for Brinkman's model.

- - - - K = 0.001
- - - - K→ 1
- - - - Darcy's case
Fig. 4.7(b). Streamline pattern for DLB model.

- $K = 0.001$

---

Re = 50

---

Re = 10
Fig. 4.7(c). Streamline pattern for DL modified model. 
Reₐ = 500, K = 0.001.

---  Re = 10

-----  Re = 100

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Fig. 4.8. Streamline pattern for Brinkman's model and Re = 0.

Primary flow:  --- Re = 0  ---- K = 0.1

Secondary flow:

--- Re = 0, \( \Psi = -0.39 \times 10^{-4} \)

...... K = 0.1, \( \Psi = -0.28 \times 10^{-4} \)
Fig. 4.9. Streamline pattern for Brinkman's model and Re = 0.

Primary flow: —— Re = 0  #### K = 0.1
Secondary flow:
— Re = 0, Ψ = -0.33x10^{-4}
••••• K = 0.1, Ψ = -0.21x10^{-4}
Fig. 4.10. Streamline pattern for Brinkman's model and Re = 0.

Primary flow: --- Re = 0     K = 0.1
Secondary flow (wall):
--- Re = 0
----- K = 0.1
Secondary flow (lower corner):
--- Re = 0, $\psi = -0.3\times10^{-4}$
----- K = 0.1, $\psi = -0.19\times10^{-4}$
Secondary flow (upper corner):
--- Re = 0, $\psi = -0.57\times10^{-3}$
----- K = 0.1, $\psi = -0.38\times10^{-3}$
Fig. 7.1(a). Fluid-phase streamlines for DL-HB models. $K = 0.001$, $D_1 = 1$, $D_2 = 20$. 
Fig. 7.1(b). Fluid-phase streamlines for $Re = 0$ and Brinkman-HB2 model; $K = 0.0005$, $D_2 = 20$.

Primary streamlines
- $K = 0.0005$
- $Re = 0$

Secondary streamlines
- $K = 0.0005$, $\Psi_1 = - 0.33 \times 10^{-3}$
- $Re = 0$, $\Psi_1 = - 0.11 \times 10^{-3}$
- Dust-phase, $\Psi_2 = - 0.38 \times 10^{-3}$
Fig. 7.1(c). Fluid-phase streamlines for Brinkman-HB2 model; different $D_2$, $K = 0.1$.

- - $D_2 = 20$
- - $D_2 = 10$
Fig. 7.1(d). Fluid-phase streamlines for different DLB-HB models; $K = 0.001$, $Re = 10$, $D_1 = 1$ and $D_2 = 20$.

- DLB-HB3
- DLB-HB4
Fig. 7.1(a). Fluid-phase secondary eddies for different DLB-HB models; $D_2 = 20$.

--- --- DLB-HB3

--- --- DLB-HB4: $Re = 10$, $K = 0.001$, $D_1 = 1$,

$\psi_1 = -0.33 \times 10^{-3}$

$\psi_1 = -0.43 \times 10^{-3}$

$\psi_1 = -0.8 \times 10^{-3}$

--- --- DLB-HB4; $Re = 1$, $K = 0.001$, $D_1 = 1$

--- --- DLB-HB4; $Re = 1$, $K = 0.001$, $D_1 = 20$

--- --- DLB-HB4; $Re = 1$, $K = 0.01$, $D_1 = 1$
Fig. 7.2(a). Dust-phase tangential velocity component along the upper wall for different DLB-HB models. Re = 10, K = 0.001, D₁ = 1, D₂ = 20.

- - - DLB-HB2 and DLB-HB4
- - - DLB-HB1 and DLB-HB3
Fig. 7.2(b). Dust-phase tangential velocity component along the right wall for different DLB-HB models. $Re = 10$, $K = 0.001$, $D_1 = 1$, $D_2 = 20$.

- - DLB-HB2 and DLB-HB4
- - DLB-HB1 and DLB-HB3
Fig. 7.2(c). Dust-phase tangential velocity component along the lower wall for different DLB-HB models. $Re = 10$, $K = 0.001$, $D_1 = 1$, $D_2 = 20$.

- DLB-HB2 and DLB-HB4
- DLB-HB1 and DLB-HB3
Fig. 7.2(d). Dust-phase vertical velocity component along the horizontal centreline of the channel for different DLB-HB models. 
Re = 10, K = 0.001, D_1 = 1, D_2 = 20.

- - - DLB-HB2 and DLB-HB4
- - - DLB-HB1 and DLB-HB3
Fig. 7.2(e). Dust-phase horizontal velocity component along the vertical centreline of the channel for different DLB-HB models. 
$Re = 10$, $K = 0.001$, $D_1 = 1$, $D_2 = 20$.

- DLB-HB2 and DLB-HB4
- DLB-HB1 and DLB-HB3
Fig. 7.3(a). Dust-phase tangential velocity component along the upper wall for Brinkman-HBi model; $K = 0.001$ and different $D_2$.

- $D_2 = 100$
- $D_2 = 50$
- $D_2 = 10$
Fig. 7.3(b). Dust-phase tangential velocity component along the right wall for Brinkman-HB1 model; $K = 0.001$ and different $D_2$.

- $D_2 = 100$
- $D_2 = 30$
- $D_2 = 10$
Fig. 7.3(c). Dust-phase tangential velocity component along the lower wall for Brinkman-HB2 model; $D_2 = 20$ and different $K$.

- - - $K = 0.1$
- - - $K = 0.001$
- - - $K = 0.0005$
Fig. 7.3(d). Dust-phase tangential velocity component along the upper wall for Brinkman-HB2 model; different $D_2$ and different $K$.

$D_2 = 20$:
- $K = 0.1$
- $K = 0.001$
- $K = 0.0005$

$D_2 = 10$, $K = 0.1$
Fig. 7.3(e). Dust-phase tangential velocity component along the right wall for Brinkman-HB2 model; different $D_z$ and different $K$.

- $D_z = 20$:
  - $K = 0.1$
  - $K = 0.0005$
- $D_z = 10$, $K = 0.1$
Fig. 7.3(f). Dust-phase vertical velocity component along the horizontal centreline of the channel for Brinkman-HB2 model; $K = 0.001$ and different $D_2$.

- - - $D_2 = 100$
- - - $D_2 = 30$
- - - $D_2 = 10$
Fig. 7.3(g). Dust-phase horizontal velocity component along the vertical centreline of the channel for Brinkman-HB2 model; \( K = 0.001 \) model different \( D_z \).

- - - - \( D_z = 100 \)
- - - - \( D_z = 30 \)
- - - - \( D_z = 10 \)
Fig. 7.4(a). Fluid-phase horizontal velocity component along the vertical centreline of the channel for DLB-HB1 model; $D_1 = 1$, $D_2 = 20$, and different Re and K.

- $Re = 1$, $K = 0.001$
- $Re = 10$, $K = 0.01$
- $Re = 10$, $K = 0.001$
- $Re = 10$, $K = 0.001$, $D_2 = 10$
Fig. 7.4(b). Fluid-phase horizontal velocity component along the vertical centreline of the channel for DLB-HB3 model; $D_1 = 1$, $D_2 = 20$, and different $Re$ and $K$.

--- Re = 1, $K = 0.001$
--- Re = 10, $K = 0.01$
--- Re = 10, $K = 0.001$
--- Re = 10, $K = 0.001$, $D_2 = 10$
Fig. 7.4(c). Fluid-phase horizontal velocity component along the vertical centreline of the channel for DLB-HB2 model; $D_1 = 20$. $K = 0.001$ and different $Re$ and $D_1$.

- Re = 1, $D_1 = 1$
- Re = 1, $D_1 = 10$
- Re = 10, $D_1 = 1$
Fig. 7.4(d). Fluid-phase horizontal velocity component along the vertical centreline of the channel for DLB-HB4 model; $D_1 = 1$, $D_2 = 20$, and different Re and K.

- $Re = 1$, $K = 0.001$
- $Re = 10$, $K = 0.001$
- $Re = 10$, $K = 0.01$
Fig. 7.5(a). Fluid-phase vertical velocity component along the horizontal centreline of the channel for DLB-HB1 model; \( D_1 = 20 \),
different \( D_2 \). Re and K.

- - - Re = 1, K = 0.001, D_2 = 10

- - - Re = 10, K = 0.001, D_2 = 20

- - - Re = 10, K = 0.01, D_2 = 20
Fig. 7.5(b). Fluid-phase vertical velocity component along the horizontal centreline of the channel for DLB-HBd model; $D_2 = 20$, different $D_1$, Re and K.

- - - $Re = 1, K = 0.001, D_1 = 20$
- - - $Re = 10, K = 0.001, D_1 = 1$
- - - $Re = 10, K = 0.01, D_1 = 1$
Fig. 7.6(a). Fluid-phase horizontal velocity component along the vertical centreline of the channel for Brinkman-HB1 model; $K = 0.001$. 

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Fig. 7.6(b). Fluid-phase vertical velocity component along the horizontal centreline of the channel for Brinkman-HB1 and DL-HB1 models; $K = 0.001$.  

---

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Fig. 7.7(a). Horizontal velocity components at different sections of the channel for Brinkman-HB2 model.

\( K = 0.0005, \, D_2 = 20: \)
- \( U_1 \) at \( X = 0.05 \)
- \( U_2 \) at \( X = 0.05 \)
- \( U_1 \) at \( X = 0.5 \)
- \( U_2 \) at \( X = 0.5 \)

\( K = 0.1, \, D_2 = 20: \)
- \( U_1 \) at \( X = 0.5 \)
- \( U_2 \) at \( X = 0.5 \)
Fig. 7.7(b). Fluid-phase vertical velocity component along the horizontal centreline of the channel for Brinkman-HB2 model; $D_2 = 20$ and different $K$.

- $K = 0.1$
- $K = 0.0005$
- $K = 0.1, D_2 = 10$. 
Fig. 8.1. Fluid-phase horizontal velocity component along the interface. Darcy-Lapwood model; \( K = 0.1 \), \( Re = 10 \).
Fig. 8.2(a). Fluid-phase horizontal velocity component along the interface. Brinkman-HB2 model; $D_2 = 20$, different $K$.

----- $K = 0.1$
***** $K = 0.01$
--- $K = 0.001$
Fig. 8.2(b). Dust-phase horizontal velocity component along the interface. Brinkman-HB2 model; $K = 0.1$, different $D_2$.

--- $D_2 = 50$

--- $D_2 = 20$
Fig. 8.3. Fluid-phase and dust-phase vertical velocity component along the interface. Brinkman-HB2 model; $K = 0.01$, $D_z = 20$.

Fluid-phase
Dust-phase
Fig. 8.4. Fluid-phase horizontal velocity component at $X = X_m$. Brinkman-HB2 model; $D_2 = 20$.

different $K$

--- $K = 0.01$
--- $K = 0.001$
Fig. 8.5(a). Fluid-phase horizontal velocity component along the interface. DIB-HB4 model; \( Re = 10, D_1 \rightarrow 0, D_2 = 20 \), different \( K \).

- \( K = 0.01 \)
- \( K = 0.001 \)
Fig. 8.5(b). Fluid-phase horizontal velocity component along the interface. DLB-HB4 model; 
$D_2 = 20$, $K = 0.01$, different $Re$ and $D_4$.

- $D_4 = 1$, $Re = 10$
- $D_4 = 1$, $Re = 50$
- $D_4 = 10$, $Re = 10$
Fig. 8.6. Fluid-phase horizontal velocity at $X = X_m$.
DLB-HB4 model; $D_1 \to 0$, $D_2 = 20$, $Re = 10$.
different $K$.

--- $K = 0.1$
--- $K = 0.001$
Fig. 8.7(a). Fluid-phase vorticity along the interface. Brinkman-HS2 model; $D_x = 20$, different $K$.

- $K = 0.1$
- $K = 0.01$
- $K = 0.001$
Fig. 8.7(b). Fluid-phase and dust-phase vorticity along the interface. Brinkman-HB2 model; \( K = 0.01 \), different \( D_z \).

- \( D_z = 50, \Omega_1 \)
- \( D_z = 20, \Omega_1 \)
- \( D_z = 50, \Omega_2 \)
- \( D_z = 20, \Omega_2 \)
Fig. 8.8(a). Fluid-phase vorticity along the interface. DNB-HB4 model; $D_2 = 20$, $K = 0.01$, different $Re$ and $D_1$.

- $Re = 50$, $D_1 = 10$
- $Re = 50$, $D_1 = 1$
- $Re = 10$, $D_1 = 1$
Fig. 8.8(b). Fluid-phase vorticity along the interface. DLB-HB4 model; $D_z = 0$.

$D_z = 20$, different $Re$ and $K$.

- $Re = 10$, $K = 0.1$
- $Re = 10$, $K = 0.01$
- $Re = 10$, $K = 0.001$
- $Re = 50$, $K = 0.001$
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Table 8.1

$Y_1$ and $Y_2$ on different streamlines.

DLB-HB4 model; $D_1 = 1$, $D_2 = 20$, $K = 0.1$, $Re = 50$.  

341
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**Table 8.2**

\( Y_1 \) and \( Y_2 \) on different streamlines.

DLB-HB4 model; \( D_1 = 1 \), \( D_2 = 20 \), \( K = 0.1 \), \( Re = 10 \).
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Table 8.3

$Y_1$ and $Y_2$ on different streamlines.

DLB-HB4 model; $D_1 \to 0$, $D_2 = 20$, $K = 0.01$, $Re = 10$. 

343
\[
\begin{array}{cccccccc}
\Psi = 0.09524 & \Psi = 0.38095 & \Psi = 1.3333 \\

X & Y_1 & Y_2 & Y_1 & Y_2 & Y_1 & Y_2 \\
-2 & 0.09524 & 0.09524 & 0.38095 & 0.38095 & 1.3333 & 1.3333 \\
-1.742 & 0.09534 & 0.09532 & 0.38109 & 0.38101 & 1.3306 & 1.3304 \\
-1.484 & 0.09578 & 0.09576 & 0.38244 & 0.38237 & 1.3310 & 1.3309 \\
-1.226 & 0.09650 & 0.09649 & 0.38470 & 0.38465 & 1.3320 & 1.3319 \\
-0.988 & 0.09789 & 0.09782 & 0.38901 & 0.38898 & 1.3335 & 1.3335 \\
-0.709 & 0.10135 & 0.10162 & 0.39788 & 0.39785 & 1.3353 & 1.3353 \\
-0.452 & 0.13749 & 0.13711 & 0.41381 & 0.41379 & 1.3371 & 1.3371 \\
-0.194 & 0.17387 & 0.17385 & 0.42819 & 0.42819 & 1.3383 & 1.3383 \\
0.065 & 0.18015 & 0.18013 & 0.43131 & 0.43132 & 1.3385 & 1.3385 \\
0.323 & 0.16028 & 0.16084 & 0.42187 & 0.42188 & 1.3378 & 1.3376 \\
0.581 & 0.10749 & 0.10707 & 0.40496 & 0.40497 & 1.3358 & 1.3359 \\
0.839 & 0.09891 & 0.09877 & 0.39256 & 0.39259 & 1.3337 & 1.3338 \\
1.097 & 0.09693 & 0.09691 & 0.38630 & 0.38634 & 1.3319 & 1.3319 \\
1.355 & 0.09507 & 0.09507 & 0.38315 & 0.38321 & 1.3305 & 1.3305 \\
1.613 & 0.09557 & 0.09559 & 0.38147 & 0.38154 & 1.3298 & 1.3300 \\
2 & 0.09524 & 0.09524 & 0.38095 & 0.38095 & 1.3333 & 1.3333 \\
\end{array}
\]

Table 8.4

\(Y_1\) and \(Y_2\) on different streamlines.

Brinkman-HB2 model; \(D_2 = 50\), \(K = 0.01\).
\[
\begin{array}{cccccccc}
& \Psi = 0.09524 & \Psi = 0.38095 & \Psi = 1.3333 \\
\hline
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-2 & 0.09524 & 0.09524 & 0.38095 & 0.38095 & 1.3333 & 1.3333 \\
-1.742 & 0.09532 & 0.09523 & 0.38114 & 0.38077 & 1.3310 & 1.3300 \\
-1.484 & 0.09575 & 0.09564 & 0.38259 & 0.38216 & 1.3320 & 1.3309 \\
-1.226 & 0.09649 & 0.09642 & 0.38497 & 0.38484 & 1.3333 & 1.3326 \\
-0.968 & 0.09800 & 0.09808 & 0.38937 & 0.38915 & 1.3350 & 1.3348 \\
-0.709 & 0.10187 & 0.10304 & 0.39825 & 0.39807 & 1.3389 & 1.3367 \\
-0.452 & 0.13702 & 0.13585 & 0.41434 & 0.41412 & 1.3387 & 1.3385 \\
-0.194 & 0.17354 & 0.17325 & 0.42689 & 0.42680 & 1.3398 & 1.3398 \\
0.065 & 0.17998 & 0.17998 & 0.43200 & 0.43202 & 1.3399 & 1.3400 \\
0.323 & 0.16089 & 0.16192 & 0.42229 & 0.42241 & 1.3389 & 1.3390 \\
0.581 & 0.10855 & 0.10825 & 0.40517 & 0.40521 & 1.3370 & 1.3372 \\
0.839 & 0.09673 & 0.09815 & 0.39295 & 0.39297 & 1.3348 & 1.3350 \\
1.097 & 0.09803 & 0.09644 & 0.38679 & 0.38688 & 1.3327 & 1.3330 \\
1.355 & 0.09581 & 0.09777 & 0.38381 & 0.38379 & 1.3311 & 1.3315 \\
1.613 & 0.09539 & 0.09541 & 0.38182 & 0.38208 & 1.3300 & 1.3305 \\
2 & 0.09524 & 0.09524 & 0.38095 & 0.38095 & 1.3333 & 1.3333 \\
\end{array}
\]

Table 8.5

\(Y_1\) and \(Y_2\) on different streamlines.

Brinkman-H2 model; \(D_2 = 20, \ k = 0.01\).
APPENDIX A

(STATIC COMPUTER CODE)
APPENDIX A

COMPUTER CODE STATIC

SINGLE-PHASE FLUID FLOW THROUGH POROUS MEDIA BOUNDED

BELOW BY A STATIC FLUID.

MODELS CONSIDERED: Darcy's law, Brinkman's equation,
Darcy-Forchheimer model,
Darcy-Lapwood-Barron-Handan modified
and Darcy-Lapwood-Brinkman models.

COMMON Y(64,22), X(64), AA1(64,22), AA2(64,22),
WC(64,22), RE1, IMAX, JMAX, IMM1, JMM1, DX, DS, N1,
NN, XMIN, XMAX, PSI_MIN, PSI_MAX, UVC(64,22), JMM2,
PC(64,22), WW1, ERBD1, AB, AC, AD, AE, AF, VC(64,22),
K(64), KK(64), LL(22), LC(22), U1(64,22),
U2(64,22), UC(64,22), AG, AJ, AK, AL, AI, AH, WW2,
BB1(64,22), BB2(64,22), ERBD2, AM, RE, V1(64,22),
W1(64,22), BB3(64,22), BB4(64,22), ERBD3, N3, WW3

THE DIFFERENT VARIABLES AND PARAMETERS, ABOVE, ARE

(I,J): TYPICAL GRID POINT
RE: REYNOLDS NUMBER
RE1: THE DIMENSIONLESS CONSTANT IN DL-B-H MODIFIED MODEL
AK: THE PERMEABILITY K
YCI,J: THE NEW DEPENDENT VARIABLE IN
X(I): THE X-COORDINATE AT GRID LINE i
V1(I,J): THE VALUE OF f AT X(I)
DS: THE STEP SIZE ΔΨ
DX: THE STEP SIZE ΔX
WW1,WW2,WW3: RELAXATION PARAMETERS
ERBD1, ERBD2, ERBD3: ERROR TOLERANCES
XMAX: THE MAXIMUM VALUE OF X
XMIN: THE MINIMUM VALUE OF X
PSI_MAX: THE MAXIMUM VALUE OF ψ
PSI_MIN: THE MINIMUM VALUE OF ψ
IMAX: MAXIMUM NUMBER OF GRID POINTS IN X-DIRECTION
JMAX: MAXIMUM NUMBER OF GRID POINTS IN Y-DIRECTION
IM1: IMAX-1
JMM1: JMAX-1
JMM2: JMAX-2
AB, AC, AD, AE, AF, AG, AH, AI, AJ, AK, AL, and AM: ARE PARAMETERS
N1, N3, NN: REFER TO NUMBER OF ITERATIONS
LC(J): A VECTOR REPRESENTING J-1
LLC(J): A VECTOR REPRESENTING J+1
KCI(J): A VECTOR REPRESENTING I-1
KKI(J): A VECTOR REPRESENTING I+1
UCI(J): X-VELOCITY COMPONENT AT (I,J)
VCI(J): Y-VELOCITY COMPONENT AT (I,J)
WCI(J): VORTICITY AT (I,J)
PCI(J): PERTURBATION POTENTIAL AT (I,J)
UVCI(J): THE EXPRESSION FOR EQUATION OF CONTINUITY
IN (X,Y) COORDINATES
BB1, BB2, BB3, BB4, AA1, AA2, U1, U2, W1: EXPRESSIONS AT (I,J),
AS DEFINED IN THE TEXT, TO IMPROVE THE READABILITY OF THE
COMPUTER CODE

HOW TO RUN THE CODES

IN ORDER TO OBTAIN THE SOLUTION TO THE FLOW GOVERNED BY
DARCY'S LAW, THE SUBROUTINE "DARC2" HAS TO BE RUN BY
ITSELF.

IN ORDER TO RUN THE OTHER SUBROUTINES, THE "DARC2"
RESULTS ARE USED AS INPUT TO THE OTHER SUBROUTINES.

TO OBTAIN THE DARCY-FORCHHEIMER SOLUTION, RUN THE "FORCH"
SUBROUTINE WITH "DARC2".

TO OBTAIN THE DARCY-LAPWOOD-BRINKMAN SOLUTION, RUN THE
"DCLDBN" SUBROUTINE WITH "DARC2".

TO OBTAIN THE BRINKMAN SOLUTION, SET RE = 0 IN THE INPUT
SUBROUTINE AND RUN THE SUBROUTINE "DCLDBN".

TO OBTAIN THE DARCY-LAPWOOD-BARRON-HAMDAN MODIFIED
SOLUTION, GIVE RE1 A VALUE DIFFERENT THAN RE AND RUN THE
SUBROUTINE "DCLDBN" AFTER MODIFYING THE MATRIX
COEFFICIENTS IN "SUBROUTINE DCLDBN".

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THE MAIN PROGRAMME

The following subroutine provides the input data

    CALL INPUT

The following subroutines provide the solution when the flow is governed by Darcy's law.

To solve for \( Y \):

    CALL DARC2

To solve for Darcy's perturbation potential

C    CALL PRES1

The following subroutine provides the solution when the flow is governed by Forchheimer's equation.

C    CALL FORCH

The following subroutine provides the solutions to the Darcy-Lapwood-Brinkman equation, the Brinkman's models and to the Darcy-Lapwood-Barron-Hamdan modified equation.

C    CALL DCLDBN

The following subroutine provides the output data.

    CALL OUTPUT

STOP
END

END OF MAIN PROGRAMME

THE INPUT DATA

SUBROUTINE INPUT
COMMON Y(64,22),X(64),AA1(64,22),AA2(64,22),
\[ WC(64,22), RE1, IMAX, JMAX, IMM1, JMM1, DX, DS, N1, \\
NN, XMIN, XMAX, PSI MIN, PSI MAX, UVC(64,22), JMM2, \\
PCC(64,22), WW1, ERBD1, AB, AC, AD, AE, AF, V1(64,22), \\
KC(64), KK(64), LL(22), LC(22), U1(64,22), \\
U2(64,22), U(64,22), AG, AJ, AL, AI, AH, WW2, \\
BB1(64,22), BB2(64,22), ERBD2, AM, RE, V1(64,22), \\
W1(64,22), BB3(64,22), BB4(64,22), ERBD3, N3, WW3 \\
\]

DATA ERBD1/0.00005, ERBD2/0.00005/, ERBD3/0.00005/ 
DATA WW1/0.93/, WW2/0.23/, WW3/1.66/, AK/0.01/ 
DATA IMAX/64/, JMAX/22/, RE/100.0/, RE1/100.0/ 
DATA IMM1/63/, JMM1/21/, JMM2/20/ 
DATA XMAX/2.0/, XMIN/-2.0/, PSI MAX/2.0/, PSI MIN/0.0/ 

\[ DX=(XMAX-XMIN)/IMM1 \\
DS=(PSI MAX-PSI MIN)/JMM1 \\
\]

\[ AM=DS \times 2 \\
AB=4.0 \times DS \times DS \\
AC=4.0 \times DX \times DX \\
AD=2.0 \times DX \times DS \\
AE=(DX/DS) \times (DX/DS) \\
AF=DX/4.0 \times DS \\
AG=DX \times DX \\
AH=2.0 \times DX \\
AI=2.0 \times DS \\
AJ=DX \times DX/(2.0 \times DS) \\
AL=DX \times DX/\text{AK} \]

**Conditions at the top boundary**

DO 1 I=1,IMAX 
   VC(I, JMAX) = 0.0 
   WC(I, JMAX) = 0.0 
1 CONTINUE

**Conditions at the left & right boundaries**

DO 2 J=1,JMAX 
   VC(1, J) = 0.0 
   VC(MAX, J) = 0.0 
   UC(1, J) = 1.0 
   UC(MAX, J) = 1.0 
   WC(1, J) = 0.0 
   WC(MAX, J) = 0.0 
2 CONTINUE
Conditions on \( Y \) at the left & right boundaries

\[ \text{DO 3 } J=1, \text{JMAX} \]
\[ JJ=J-1 \]
\[ Y(1,J)=JJ\times DS \]
\[ Y(\text{JMAX},J)=JJ\times DS \]

3 CONTINUE

Conditions on \( Y \) at the top boundary

\[ \text{DO 4 } I=2, \text{IMMI} \]
\[ Y(I,\text{JMAX})=\text{PSI MAX} \]

4 CONTINUE

To calculate the values of \( X \) along the lower boundary

\[ \text{DO 5 } I=1, \text{IMAX} \]
\[ II=I-1 \]
\[ X(I)=\text{KMIN}+II\times DX \]

5 CONTINUE

Definition of the lower boundary \( Y = f'(X) \) and definition of \( f'(X) \)

\[ \text{DO 6 } I=25,40 \]
\[ Y(I,1)=0.2\times \text{ABS}(\text{SQRT}(0.25-\text{X}(I)\times X(I))) \]
\[ Y(I,1)=-0.2\times X(I)\times (\text{ABS}(\text{SQRT}(0.25-\text{X}(I)\times X(I)))) \]

6 CONTINUE

\[ \text{DO 7 } I=1,24 \]
\[ Y(I,1)=0.0 \]
\[ V(I,1)=0.0 \]
\[ V(I,1)=0.0 \]

7 CONTINUE

\[ \text{DO 8 } I=41, \text{IMAX} \]
\[ V(I,1)=0.0 \]
\[ V(I,1)=0.0 \]
\[ Y(I,1)=0.0 \]

8 CONTINUE

To initialize the variables in the flowfield

\[ \text{DO 11 } I=2, \text{IMMI} \]
\[ \text{DO 10 } J=2, \text{JMMI} \]
\[ Y(I,J)=0.05\times J \]
\[ W(I,J)=1.0 \]

10 CONTINUE

11 CONTINUE

The vectors \( L, LL, K, KK \)

\[ \text{DO 12 } I=2, \text{IMAX} \]
\[ K(I)=I-1 \]

12 CONTINUE
DO 13 I=1, IMM1
   KKCI)=I+1
13 CONTINUE
DO 14 J=2, JMAX
   L(J,J)=J-1
14 CONTINUE
DO 15 J=1, JMM1
   L(J,J)=J+1
15 CONTINUE
RETURN
END

END OF INPUT SUBROUTINE

THE DARCY'S REGIME SOLVER

SUBROUTINE DARCS
COMMON YC(64,22), XC(64), AA1(64,22), AA2(64,22),
   WC(64,22), RE1, IMAX, JMAX, IMM1, JMM1, DX, DS, N1,
   NN, XMIN, XMAX, PSI_MIN, PSI_MAX, UVC(64,22), JMM2,
   PC(64,22), W1, ERBD1, AB, AC, AD, AE, AF, VC(64,22),
   KC(64), KK(64), LL(22), L(22), U1(64,22),
   U2(64,22), UC(64,22), AG, AJ, AK, AL, AI, AH, WW2,
   BB1(64,22), BB2(64,22), ERBD2, AM, RE, V1(64,22),
   W1(64,22), BB3C(64,22), BB4(64,22), ERBD3, N3, WW3,
   DIMENSION A3(64), B3(64), C3(64), RHS3(64),
   QQ(64), RMAX3(64), YPRC(64)

N3=0
787 IF (N3 .EQ. 200) RETURN
N3=N3+1

DO 52 I=2, IMM1
   DO 51 J=2, JMM1
      AA1(I,J)=YCI, L(I,J)-YCI, C(L))
      AA2(I,J)=YK(I,J), J-Y(KI(I), J)
51 CONTINUE
52 CONTINUE
TO DEFINE THE MATRIX COEFFICIENTS

DO 55 I=2,JMM1
   DO 53 J=2,JMM1

The vector of known quantities

\[ \text{RHS3(J)} = (\text{AA1(I,J)} \times \text{AA1(I,J)} / \text{AB}) \times \]
\[ (\text{YKKK(I,J)} + \text{YKK(I,J)}) \]
\[ - (\text{AA1(I,J)} \times \text{AA2(I,J)} / \text{ADD} \times \text{AF}) \times \]
\[ (\text{YKKK(I,J)} + \text{YKK(I,J)}) \]
\[ - \text{YKK(I,J)} \times \text{YKKK(I,J)} \]

The sub-diagonal elements

\[ \text{A3(J)} = -\text{AE} \times (1.0 + \text{AA2(I,J)} \times \text{AA2(I,J)} / \text{AC}) \]

The diagonal elements

\[ \text{B3(J)} = 2.0 \times \text{AA1(I,J)} \times \text{AA1(I,J)} / \text{AB} + \]
\[ 2.0 \times \text{AE} \times (1.0 + \text{AA2(I,J)} \times \text{AA2(I,J)} / \text{AC}) \]

The super-diagonal elements

\[ \text{C3(J)} = \text{A3(J)} \]
\[ \text{S3} \quad \text{CONTINUE} \]

\[ \text{RHS3(JMM1)} = \text{RHS3(JMM1)} + \]
\[ (\text{AE} \times (1.0 + \text{AA2(I,JMM1)} \times \text{AA2(I,JMM1)} / \text{AC}) \times \text{Y(I,JMAX)}) \]
\[ \text{RHS3(2)} = \text{RHS3(2)} + (\text{AE} \times (1.0 + \text{AA2(I,2)} \times \text{AA2(I,2)} / \text{AC}) \times \text{Y(I,1)}) \]
\[ \text{C3(JMM1)} = 0.0 \]
\[ \text{A3(2)} = 0.0 \]

The tridiagonal solver call

\[ \text{CALL TRID(A3, B3, C3, QQ, RHS3, 2, JMM1)} \]

To calculate the maximum error committed

\[ \text{RMAX3(I)} = 0.0 \]

Internal relaxation

\[ \text{NN} = 0 \]
\[ \text{WHILE (NN .LE. 3)} \]
\[ \quad \text{DO 54 J=2,JMM1} \]
\[ \quad \text{NN} = \text{NN} + 1 \]

353
YPRC(J)=YC(J,J)
YC(J,J)=YC(J,J)+WW3*RHS3(J)-YC(J,J)
ERR3=ABS(YPRC(J)-YC(J,J))
IF (ERR3 .GT. RMAX3(J)) RMAX3(J)=ERR3

54 CONTINUE
ENDWHILE
55 CONTINUE

Convergence check

ERMAX3=0.0
IFLAG3=0

DO 56 I=2,IMM1
   IF (RMAX3(I) .GE. ERBD3) IFLAG3=1
   IF (RMAX3(I) .GT. ERMAX3) ERMAX3=RMAX3(I)
56 CONTINUE

IF (IFLAG3 .EQ. 1) GO TO 787
PRINT 57,N3
PRINT 58,ERMAX3
57 FORMAT('0','A1TR3= ',I3)
58 FORMAT('0','MAX ERR3= ',E15.8)

To calculate the velocities

DO 557 I=2,IMM1

First order accurate
   UCI,1D=DS/(YC(1,2)-YC(1,1))

Second order accurate
   C  UCI,1D=2.0*DS/C.0*YCI,2C-3.0*YCI,1C-YCI,3C)
   C  VCI,1D=V(1,1)*UCI,1C
   C  UCI,JMAX)=(2.0*DS/
                  (C.0*YCI,JMAX)-4.0*YCI,JMM1C+YCI,JMM2C)
557 CONTINUE

DO 558 I=2,IMM1
   DO 558 J=2,JMM1
      UCI,JD=2.0*DS/(YC,I,LL(J)) - YCI,LL(J)
      VCI,JD=(DS/DX)*(YCKI,JD-YCKI,JD)/
               (YC(I,LL(J)) - YCI,LL(J))
558 CONTINUE
558 CONTINUE
RETURN
END

END OF DARC2

THE DARCY'S PC1, JD SOLVER

SUBROUTINE PRES:
COMMON Y(64, 22), X(64), AA1(64, 22), AA2(64, 22),
  W(64, 22), RE1, IMAX, JMAX, IMM1, JMM1, DX, DS, N1,
  NN, XMIN, XMAX, PSI MIN, PSI MAX, UVC(64, 22), JMM2,
  PC(64, 22), WW1, ERBD1, AB, AC, AD, AE, AF, V(64, 22),
  KC(64), KK(64), LLC(22), LC(22), UI(64, 22),
  U2(64, 22), UC(64, 22), AG, AJ, AK, AL, AI, AH, WW2,
  BB1(64, 22), BB2(64, 22), ERBD2, AN, RE, V1(64, 22),
  WI(64, 22), BB3(64, 22), BB4(64, 22), ERBD3, N3, WW3
DIMENSION A2(64), B2(64), C2(64), RHS2(64),
  QQC(64), RMAX2(64), PPRC(64)

To initialize the pressure field

DO 588 I = 1, I MAX
   DO 589 J = 1, J MAX
      PC(I, J) = Y(I, J)
   CONTINUE
589 CONTINUE
588 CONTINUE

DO 572 I = 2, IMM1
   DO 571 J = 2, JMM1
      AA1(I, J) = YCI, LLC(JC) - YCI, LC(JC)
      AA2(I, J) = YCKI(JD) - YCKI, JD
   CONTINUE
571 CONTINUE
572 CONTINUE

N2 = 0
588 IF (N2 .EQ. 800) RETURN
   N2 = N2 + 1
To calculate \( P \) at the lower & upper boundaries

DO 574 I=1,IMM1
   PCI,1)=PCI,2)+DS*V1(I,1)
   PCI,JMAX))=PCI,JMM1)
574 CONTINUE

To calculate \( P \) at left & right boundaries

DO 575 J=1,JMAX
   PCI,J)=PCI,J)
   PCI,JMAX,J)=PCI,JMM1,J)
575 CONTINUE

DO 576 I=2,IMM1
   DO 577 J=3,JMM1
      RHS2(J)=AA1(I,J)*AA1(I,J)/AB
         PK1(J)=PK1(J)+PK1(J,J)
      @ CAA1(I,J)*AA1(I,J)/AC
         PK2(J)=PK2(J)+PK2(J,J)
      @ PK2(I,J)*PK2(I,J)
      @ PK2(J,J)+PK2(I,J)
      @ PK2(I,J)+PK2(I,J)
      @ PK2(I,J)+PK2(I,J)
      A2(J)=A2(J)+AA1(I,J)*AA1(I,J)/AC
      B2(J)=B2(J)+AA1(I,J)*AA1(I,J)/AC
   577 CONTINUE

RHS2(JMM1)=RHS2(JMM1) +
   CA2C1.0+AA2(I,J)*AA2(I,J)/AC*PCI,JMAX)
RHS2(J)=RHS2(J)+CA2C1.0+AA2(I,J)*AA2(I,J)/AC)
   *PCI,J)
   +PC(J,J)
   +2.0AA2(I,J)*AA2(I,J)/AC)
C2(J)=A2(J)
577 CONTINUE

CALL TRIDCA2,B2,C2,QQ,RHS2,3,JMM1)
RMAX=0.0
NN=0
WHILE CNN .LE. 30
   DO 578 J=3,JMM1
      NN=NN+1
      PPR(J)=PCI,J)
      PCI,J=PCI,J+1.399*DS*(RHS2(J)-PCI,J)
      ERR=ABS(PPR(J)-PCI,J)
      IF (ERR .GT. RMAX) RMAX=ERR
   578 CONTINUE
ENDWHILE
578 CONTINUE
ERMAX2=0.0
IFLAG2=0

DO 412 I=2,IMM1
   IF (ERMAX2(I) .GE. ERBD2) IFLAG2=1
   IF (ERMAX2(I) .GT. ERMAX2) ERMAX2=RMAX2(I)
412 CONTINUE

IF (IFLAG2 .EQ. 1) GO TO 888
PRINT 581,N2
PRINT 584,ERMAX2
581 FORMAT('0','MTR2= ',I3)
584 FORMAT('0','MAX ERR2= ',E15.8)

To update P at the boundaries

DO 217 I=1,IMM1
   PC(I,1)=PC(I,2)+DS*V1(I,1)
   PC(I,JMAX)=PC(I,JMM1)
217 CONTINUE

DO 218 J=1,JMAX
   PC(I,J)=PC2(J)
   PC(I,JMAX)=PC(I,JMM1,J)
218 CONTINUE

RETURN
END

END OF PRES1

THE FORCHHEIMER REGIME SOLVER

SUBROUTINE FORCH
COMMON Y(64,22),X(64),AA1(64,22),AA2(64,22),
  WC(64,22),RE1,IMAX,JMAX,IMM1,JMM1,DX,DS,N1,
  NN,XMIN,XMAX,PSMIN,PSIMAX,UVC(64,22),JMM2,
  PC(64,22),WW1,ERBD1,AB,AC,AD,AE,AF,V(64,22),
  KC(64),KK(64),LC(22),LC(22),U1(C64,22),
  U2(C64,22),U3(C64,22),AG,AJ,AK,AL,AI,AH,WM2,
  BB1(C64,22),BB2(C64,22),ERBD2,AM,RE,V1(C64,22),
  W1(C64,22),BB3(C64,22),BB4(C64,22),ERBD3,NS,WW3
DIMENSION A1(64), B1(64), C1(64), RHS1(64), RMAX1(64),
@ YPRC64, A2C64, B2C64, C2C64, R2S2(64),
@ BAAC64, 222, BABC64, 222, BACC64, 222, WPRC64,
@ BAD64, 222, BAE64, 222, QQC64, RMAX2C64

N1 = 0
333 IF (N1 .EQ. 800) RETURN
N1 = N1 + 1

DO 102 I = 2, IMM1
   DO 101 J = 2, JMM1
      AA1(I, J) = YCI(I, LLC(I)) - YCI(I, LCJ)
      AA2I(I, J) = YCKKCI(I, J) - YCKI(I, J)
   101 CONTINUE
   CONTINUE
102 CONTINUE

DO 379 I = 2, IMM1
   DO 378 J = 2, JMM1
      BABC(I, J) = 1.0 + AA2I(I, J) / (4.0 * DX * DX)
      BACC(I, J) = BACC(I, J) * 2
      BB1(I, J) = BACC(I, J) * 8.0 * DS *
         \(2.0 * DX**2 + AA2I(I, J)**2 + AA2I(I, J)**4/\)
      + BACC(I, J) * 4.0 * DX**2 / CREMAA1(I, J)**2
      BB2(I, J) = BABC(I, J) / RE + BACC(I, J) * 2.0 * DS *
      \(1.0 + AA2I(I, J)**2 / (2.0 * DX**2)) / AA1(I, J)**3 *\)
      + BACC(I, J) * ANAA2(I, J) / (2.0 * RE * AA1(I, J))
   378 CONTINUE
   CONTINUE
379 CONTINUE

DO 105 I = 2, IMM1
   DO 103 J = 2, JMM1
      RHS1(J) = BB2I(C, J) * CYCKKCI(I, J) + YCKI(I, J)
      - BB3I(J, J) * CYCKKCI(I, J) + YCKI(I, J) -
      CYCKKCI(I, LLC(J)) + YCKI(I, LCJ) -
      YCKI(I, LLC(J)) - YCKKCI(I, LCJ)
      A1(J) = -BB1I(J, J)
      B1(J) = 2.0 * BB1I(J, J) + BB2I(J, J)
      C1(J) = A1(J)
   103 CONTINUE
RHS1(JMM1) = RHS1(JMM1) + BB1I(J, JMM1) * WY(I, JMAX)
RHS1(2) = RHS1(2) + BB1I(2, J) * WY(I, 1)
C1(JMM1)=0.0
A1(J)=0.0
CALL TRIDCA1,B1,C1,QQ,RHS1,2,JMM1
RMAX1(I)=0.0
NN=0
WHILE (NN .LE. 3)
  DO 104 J=2,JMM1
    NN=NN+1
    YPRC(J)=YCI(J)
    YCI(J)=YCI(J)+WW1*(RHS1(J)-YCI(J))
    ERR1=ABS(YPRC(J)-YCI(J))
    IF (ERR1 .GT. RMAX1(I)) RMAX1(I)=ERR1
  104 CONTINUE
  ENDWHILE

  ERMAX1=0.0
  IFLAG1=0
  DO 106 I=2,IJM1
    IF (RMAX1(I) .GE. ERBD1) IFLAG1=1
    IF (RMAX1(I) .GT. ERMAX1) ERMAX1=RMAX1(I)
  106 CONTINUE
  IF (IFLAG1 .EQ. 1) GO TO 333
  PRINT 107,N1
  PRINT 108,ERMAX1

  107 FORMAT('0','ITRI=',',I3)
  108 FORMAT('0','MAX ERR1=',',E15.8)

  DO 121 I=2,IJM1
    UCI(I)=DS/CYCI(I,2)-YCI(I,1)
  C    UCI(I)=2.0*DS/(4.0*YCI(I,2)-3.0*YCI(I,1)-YCI(I,3))
    VCI(I,1)=VCI(I,1)*UCI(I,1)
  C    UCI,I,MAX0=(2.0*DS)/
    \vspace{2mm}
      (3.0*YCI,I,MAX)-4.0*YCI,I,JMM1)+YCI,I,JMM2)
    UCI,I,MAX=DS/CYCI,I,MAX-YCI,I,JMM1)
  121 CONTINUE

  DO 123 I=2,IJM1
    DO 122 J=2,IJM1
      UCI,I,J)=2.0*DS/(CYCI,I,LLCJ)-YCI,I,LCJ)
      VCI,I,J)=(DS/DX)*YCKI(I,J)-YCKI(I,J)/
        (YCI,I,LLCJ)-YCI,I,LCJ)
  122 CONTINUE
  123 CONTINUE
DO 131 I=2,IMM1

First order accurate vorticity expression

\[
W_{CI,1} = \frac{(V_{CKKI,1}) - (V_{CI,1})}{\Delta X} - \frac{(V_{CI,1}) - (V_{CI,2})}{\Delta S} - \frac{(V_{UCI,1}) - (V_{UCI,2})}{\Delta S}
\]

Second order accurate vorticity expression

\[
W_{CI,1} = \frac{(V_{CKKI,1}) - (V_{CKKI,1})}{\sum_{\Delta X, \Delta S}} - \frac{(V_{CI,1}) - (V_{CI,3})}{\Delta S} - \frac{(V_{UCI,1}) - (V_{UCI,3})}{\Delta S}
\]

131 CONTINUE

To check if continuity equation is satisfied

C DO 133 I=2,IMM1
C   DO 132 J=2,IMM1
C      U_{VCI,J} = (V_{CI,LL(J)} - V_{CI,L(J)})
C      + (U_{CKKI,J} - U_{CKKI,J})
C      (Y_{I,LL(J)} - Y_{I,L(J)})/\Delta X
C      -(Y_{CKKI,J} - Y_{CKKI,J})/\Delta X
C      U_{CI,LL(J)} - U_{CI,L(J)}/\Delta X
C132 CONTINUE
C133 CONTINUE

To calculate the vorticity along the vertical line passing through a point close to the maximum of f(X)

DO 983 J=2,IMM1

\[
W_{C2,1} = \frac{(V_{C32,1}) - (V_{C31,1})}{\Delta X} - \frac{(V_{C32,1}) - (V_{C32,L(J)})}{\Delta S} - \frac{(U_{C32,1}) - (U_{C32,L(J)})}{\Delta S}
\]

983 CONTINUE

RETURN
END

END OF FORCH
THE DARDY-LAPWOOD-BRINKMAN, THE BRINKMAN AND THE  
DARDY-LAPWOOD-BARRON-HAMDAN MODIFIED MODELS

SUBROUTINE DCLDBN
COMMON Y(64,22), X(64), AA1(64,22), AA2(64,22),  
  WC64,22), REI, IMAIM, JMAX, IMM1, IMM2, DX, DS, N1,  
  NN, xmin, xmax, PSMIN, PSIMAX, UVC(64,22), IMMA2,  
  PC(64,22), WAMA, EBD1, AB, AC, AD, AE, AF, VC(64,22),  
  KL(64), KKK(64), LL(22), L(22), U(64,22),  
  U2(64,22), UC(64,22), AG, AJ, AK, AL, AI, AH, WW2,  
  BB(64,22), BA2(64,22), EBD2, AM, RE, V(64,22),  
  W1(64,22), BB3(64,22), BB4(64,22), EBD3, N3, WW3

DIMENSION A1(64), B1(64), C1(64), RHS1(64), RMAX1(64),  
  QQ(64), A2(64), B2(64), C2(64), RHS2(64),  
  RMAX2(64), WPRC(64), YPRC(64)

N1 = 0
777 IF (N1 .EQ. 500) RETURN  
N1 = N1 + 1

DO 302 I = 2, IMM1
  DO 301 J = 2, IMM1
      AA1(I,J) = YCI(LLCJ) - YCI(LCJ)  
      AA2(I,J) = YCKK(I,J) - YCK(I,J)
  CONTINUE
 301  CONTINUE
 302  CONTINUE

Matrix coefficients for the Y-equation

DO 305 I = 2, IMM1
  DO 303 J = 2, IMM1
      RHS1(J) = (AA1(I,J)**2/4.0)*  
        (YCKK(I,J) + YCK(I,J))  
        -AA1(I,J)*AA2(I,J)*  
        (YCKK(I,J,LLCJ) + YCK(I,J,LCJ)) -  
        YCK(I,J,LLCJ) - YCKK(I,J,LCJ) / 8.0  
        -DX*DX*AA1(I,J)**2*WCJ, J**2/8.0  
        A1(J) = -0.25*AA2(I,J)**2 - DX*DX  
        B1(J) = 2.0*DX*DX + 0.5*AA2(I,J)**2+  
        0.5*AA1(I,J)**2  
 303  CONTINUE
RHS1(JM1) = RHS1(JM1) +
(0.25*AA2C1, JM1)*2 + DX*DX)*Y(I, JMAX)
RHS1(2) = RHS1(2) + (0.25*AA2C1, 2)*2 + DX*DX)*Y(I, 1)

C1(JM1) = 0.0
A1(2) = 0.0
CALL TRIDCA1, B1, C1, Q0, RHS1, 2, JM1)
RMAX1(1) = 0.0
NN = 0
WHILE (NN .LE. 3)
   DO 304 J = 2, JM1
      NN = NN + 1
      YPRCJ = YCI, JD
      YCI, JD = YCI, JD + WW1*KRHS1(JD - YCI, JD)
      ERR1 = ABSC(YPRCJ - YCI, JD)
IF (ERR1 .GT. RMAX1(I)) RMAX1(I) = ERR1
   CONTINUE
ENDWHILE
304 CONTINUE
305 CONTINUE

DO 306 I = 2, IM1
C UCI, 1) = DS/(YCI, 2) - YCI, 1)
C UCI, 1) = 2.0*DS/(4.0*YCI, 2) - 3.0*YCI, 1) - YCI, 3)
C VCI, 1) = YCI, 1) + UCI, 1)
C UCI, JMAX) = 2.0*DS/(3.0*YCI, JMAX) -
4.0*YCI, JM1) + YCI, JMAX)
306 CONTINUE

DO 308 I = 2, IM1
DO 307 J = 2, JM1
UCI, JD = 2.0*DS/(YC(J), LLCJD) - YCI, LJ)
VCI, JD = DS/DX)*YC(JCJD) - YCKC(JD) - YCKC(JD)
307 CONTINUE
308 CONTINUE

DO 312 I = 2, IM1
DO 311 J = 2, JM1
AA1C1, JD = YCI, LLCJD) - YCI, LJ)
AA2C1, JD = YCKC(JD) - YCKC(JD)
311 CONTINUE
312 CONTINUE
DO 314  I=2,IM1

First order accurate vorticity expressions

\[ \omega_{i,j} = (v_{i,k}(i,j) - v_{i,j}(i,j))/dx \]
\[ -v_{i,j}(i,j) \ast (v_{i,j}(i,j) - v_{i+k,j}(i,j))/ds \]
\[ -u_{i,j}(i,j) \ast (u_{i,j}(i,j) - u_{i,j+k}(i,j))/ds \]

Second order accurate vorticity expressions

\[ \omega_{i,j} = (v_{i,k}(i,j) - v_{i,j}(i,j))/dx \]
\[ c \ast -v_{i,j}(i,j) \ast (v_{i,j}(i,j) + 4 \ast v_{i,j}(i,j) + v_{i,j+k}(i,j))/2 \]
\[ c \ast -u_{i,j}(i,j) \ast (u_{i,j}(i,j) + 4 \ast u_{i,j}(i,j) + u_{i,j+k}(i,j))/2 \]
\[ c \ast (2 \ast dx \ast ds) \]

314 CONTINUE

Matrix coefficients for the vorticity equation

DO 902  I=2,IM1
DO 900  J=2,IM1

The Darcy-Lapwood-Brinkman vorticity equation

RHS2(J)=0.25*AA1(I,J)*W(KK(I),J)+W(KK(I),J)
\[ -c \ast (w_{i,k}(i,j) - w_{i,j}(i,j))/dx \]
\[ c \ast (w_{i,j}(i,j) + 4 \ast w_{i,j}(i,j) + w_{i,j+k}(i,j))/2 \]
\[ c \ast (2 \ast dx \ast ds) \]
\[ A2(J) = -dx \ast dx - 0.25 \ast AA2(I,J) \]
\[ B2(J) = 2 \ast dx \ast dx + 0.5 \ast AA2(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]

The Darcy-Lapwood-Barron-Hamdan modified model

RHS2(J)=0.25*AA1(I,J)*W(KK(I),J)+W(KK(I),J))/RE1
\[ -c \ast (w_{i,k}(i,j) - w_{i,j}(i,j))/dx \]
\[ c \ast (w_{i,j}(i,j) + 4 \ast w_{i,j}(i,j) + w_{i,j+k}(i,j))/2 \]
\[ c \ast (2 \ast dx \ast ds) \]
\[ A2(J) = (-dx \ast dx - 0.25 \ast AA2(I,J) \]
\[ B2(J) = 2 \ast dx \ast dx + 0.5 \ast AA2(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]
\[ + dx \ast dx \ast AA1(I,J) \]

363
C2(JJ)=A2(JJ)
900    CONTINUE
        C2(JMM1)=0.0
        A2(J2)=0.0

The Darcy-Lapwood-Brinkman case  
C     RHS2C(JMM1)=RHS2C(JMM1)+(DX*DX+0.25*AA2C(J, JMM1)*WCI, JMAX)
C @     RHS2C(2)=RHS2C(2)+(DX*DX+0.25*AA2C(J, 2)*WCI, 1)

The Darcy-Lapwood-Barron-Namdor case  
RHS2C(JMM1)=RHS2C(JMM1)+
  + ((DX*DX+0.25*AA2C(J, JMM1)*WCI, JMAX)  
RHS2C(2)=RHS2C(2)+(DX*DX+0.25*AA2C(J, 2)*WCI, 1)
  + CALL TRIDX(A2, B2, C2, QQ, RHS2, 2, JMM1)
RMAX2(J)=0.0  
NN=0  
WHILE (NN .LE. 3)
    DO 901  J=2, JMM1  
        NN=NN+1  
        WPRC(J)=WCI, J  
        WCI, J=WCJ, J+C+WW2*C*RHS2C(J)-WCI, J  
        ERR2=ABS(WPRC(J)-WCI, J)  
        IF (ERR2 .GT. RMAX2(J)) RMAX2(J)=ERR2  
901    CONTINUE
ENDWHILE
902    CONTINUE

ERMAX1=0.0  
ERMAX2=0.0  
IFLAG1=0  
DO 923  I=2, JMM1  
    IF (CRMAX1(I) .GE. ERBD1) IFLAG1=1  
    IF (CRMAX1(I) .GT. ERMAX1) ERMAX1=CRMAX1(I)  
    IF (CRMAX2(I) .GE. ERBD1) IFLAG1=1  
    IF (CRMAX2(I) .GT. ERMAX2) ERMAX2=CRMAX2(I)  
923    CONTINUE  
IF (IFLAG1 .EQ. 1) GO TO 777
PRINT 139, N1  
PRINT 133, ERMAX1  
PRINT 145, ERMAX2
139 FORMAT ('O', 'X', ITR1 = ', I3)
133 FORMAT ('O', 'MAX ERR1 = ', E15.8)
145 FORMAT ('O', 'MAX ERR2 = ', E15.8)

384
At this point, velocity components and boundary vorticity have to be updated using the expressions in the subroutine, above.

To check if the continuity equation is satisfied.

```
DO 198 I=2,IMM1
   DO 197 J=2,JMM1
      UV(I,J)=(VCI,LL(J,J))-VCI,L(J,J))
            +CYCKCI,J)-UKCi,J))<
            -(YCKCI,J)-YCKCI,J))<2.0*DX
            * (UCI,LL(J,J))-UCi,L(J,J)</2.0*DX)
   197 CONTINUE
   198 CONTINUE
```

RETURN
END

END OF DCLDBN

---

THE TRIDIAGONAL SOLVER (THOMAS' ALGORITHM)

SUBROUTINE TRID(XA,B,C,D,S,NL,NU)
DIMENSION A(64),B(64),C(64),D(64),S(64)

DCNLJ = CCNLJ/BCNLJ
SCNLJ = SCNLJ/BCNLJ
NLP1 = NL + 1
DO 1 N = NLP1,NU
   Z = 1.0/(BCND-A(S)*DCN-1))
   DCND=CCND*Z
1 SCND = (S(CN)-A(S)*SCN-1))<Z
NUPNL = NU + NL
DO 2 NN=NLP1,NU
   N=NUPNL-NN
2 SCND = SCN - DCND*SCN+1)
RETURN
END
THE OUTPUT SUBROUTINE

SUBROUTINE OUTPUT
COMMON Y(64,22), X(64), AA1(64,22), AA2(64,22),
   WC(64,22), RE1, IMAX, JMAX, IMM1, JM1, DX, DS, N1,
   NN, XMIN, XMAX, PSI1MIN, PSI1MAX, UV(64,22), JMM2,
   PC(64,22), WW1, ERBD1, AB, AC, AD, AE, AF, VC(64,22),
   KC(64), KKC(64), LL(22), LC(22), U1(64,22),
   U2(64,22), UC(64,22), AG, AJ, AK, AL, AO, AH, WW2,
   BB1(64,22), BB2(64,22), ERBD2, AM, RE, V1(64,22),
   W1(64,22), BB3(64,22), BB4(64,22), ERBD3, N3, WW3

PRINT 208
208 FORMAT ('1')
PRINT 99, (CWCI, JD, J=1, JMAX), I=1, IMAX)
PRINT 208
PRINT 99, (CYCI, JD, J=1, JMAX), I=1, IMAX)
PRINT 208
PRINT 99, (UCI, JD, J=1, JMAX), I=1, IMAX)
PRINT 208
PRINT 99, (VCI, JD, J=1, JMAX), I=1, IMAX)
PRINT 208
PRINT 99, (PCI, JD, J=1, JMAX), I=1, IMAX)
99 FORMAT (1X, 7(1X, E15.8))

RETURN
END
APPENDIX B

(SINK COMPUTER CODE)
APPENDIX B

COMPUTER CODE: SINK

DUSTY FLUID FLOW THROUGH POROUS MEDIA INTO A LINE SINK

COMMON UC(41,41), VC(41,41), WC(41,41), PSI(41,41),
@ UDX(41,41), VDX(41,41), WDX(41,41), PSI DX(41,41),
@ KC(41), KKC(41), LC(41), LLC(41), X(41), Y(41),
@ ERBD1, ERBD2, ERBD3, ERBD4, WW1, WW2, WW3, WW4,
@ XMAX, XMN, YMAX, YMIN, DX, DY, IMAX, JMAX, I M M 1,
@ J M M 1, A Q, A R, A R R, N 1, N N, R E, A K,
@ D 1, D 2, AA, AB, AC, AD, AE, AF, AG, AH, AL, AM, AN, AP

VARIABLES USED

PSI: FLUID-PHASE STREAMFUNCTION \( \Psi \)
PSID: DUST-PHASE STREAMFUNCTION \( \Psi D \)
U, V: FLUID-PHASE VELOCITY COMPONENTS
UD, VD: DUST-PHASE VELOCITY COMPONENTS
W: FLUID-PHASE VORTICITY
WD: DUST-PHASE VORTICITY
X, Y: THE CARTESIAN (DIMENSIONLESS) COORDINATES
XMAX: MAXIMUM VALUE OF X
YMAX: MAXIMUM VALUE OF Y
XMIN: MINIMUM VALUE OF X
YMIN: MINIMUM VALUE OF Y
DX: STEP SIZE \( \Delta X \) IN HORIZONTAL DIRECTION
DY: STEP SIZE \( \Delta Y \) IN VERTICAL DIRECTION
IMAX: MAXIMUM NUMBER OF GRID POINTS IN X-DIRECTION
JMAX: MAXIMUM NUMBER OF GRID POINTS IN Y-DIRECTION
IM M 1: IMAX - 1
J M M 1: JMAX - 1
AK: THE PERMEABILITY \( K \)
D1, D2: THE DUST PARAMETERS
RE: REYNOLDS NUMBER
N1, NN: REFER TO NUMBER OF ITERATIONS
AA, AB, AC, AD, AE, AF, AG, AH, AL, AM, AN, AP AND AQ: PARAMETERS
AR: ASPECT RATIO
ARR: THE SQUARE OF ASPECT RATIO
L, LL, K, KK: VECTORS REPRESENTING J-1, J+1, I-1, I+1
ERBD1, ERBD2, ERBD3, ERBD4: ERROR TOLERANCES
WW1, WW2, WW3, WW4: RELAXATION PARAMETERS

THE MAIN PROGRAMME

CALL INPUT
CALL SINK1
CALL OUTPUT
STOP
END

THE INPUT DATA

SUBROUTINE INPUT
COMMON U(41, 41), V(41, 41), W(41, 41), PSI(41, 41),
@ UDX(41, 41), VDX(41, 41), WDX(41, 41), PSIDC(41, 41),
@ KC(41), KKC(41), LC(41), LKC(41), XC(41), YC(41),
@ ERBD1, ERBD2, ERBD3, ERBD4, WW1, WW2, WW3, WW4,
@ XMAX, XMIN, YMAX, YMIN, DX, DY, IMAX, JMAX, IMM1,
@ JMM1, , AQ, AR, ARR, N1, NN, RE, AK,
@ D1, D2, AA, AB, AC, AD, AE, AF, AG, AH, AL, AM, AN, AP

The general parameters

DATA ERBD1/0.00005/, ERBD2/0.00005/
DATA ERBD3/0.00005/, ERBD4/0.00005/, RE/1.0/
DATA WW1/1.11/, WW2/0.11/, WW3/1.11/, WW4/0.11/
DATA AK/1.0/, D1/19.0/, D2/20.0/
DATA IMAX/21/, JMAX/21/, IMM1/20/, JMM1/20/
DATA XMAX/0.0/, XMIN/-1.0/, YMAX/1.0/, YMIN/0.0/

Fluid-phase and dust-phase conditions on streamfunctions

DATA (PSIC(I, J), I=1, JMAX)/21*0.0/
DATA (PSIC(I, JMAX), I=1, IMM1)/20*1.0/
DATA (PSIC(IMAX, J), J=2, JMM1)/19*0.0/
DATA (PSIDC(I, J), I=1, IMM1)/20*1.0/
DATA (PSIDC(JMAX), I=1, IMM1)/20*1.0/
DATA (PSIDC(IMAX, J), J=2, JMM1)/19*0.0/

Fluid-phase velocity and vorticity conditions
DATA (WC1,J),J=2,JMM1)/19*0.0/
DATA (UC1,I),I=1,IMAX)/21*0.0/
DATA (VC1,I),I=1,IMAX)/21*0.0/
DATA (UC1,JMAX),I=1,JMM1)/20*0.0/
DATA (VC1,JMAX),I=1,JMM1)/20*0.0/
DATA (UC1,IMAX),J=2,JMM1)/19*0.0/
DATA (VC1,IMAX),J=2,JMM1)/19*0.0/
DATA (UC1,J),J=2,JMM1)/19*0.0/
DATA (VC1,J),J=2,JMM1)/19*0.0/

Dust-phase velocity and vorticity conditions

DATA (WD1,J),J=2,JMM1)/19*0.0/
DATA (UD1,J),J=2,JMM1)/19*0.0/
DATA (VD1,J),J=2,JMM1)/19*0.0/
DATA (VD1,I),I=1,IMAX)/21*0.0/
DATA (VD1,JMAX),I=1,JMM1)/20*0.0/
DATA (UD1,IMAX),J=2,JMM1)/19*0.0/

DX=(CMAX-XMIN)/JMM1
DY=(CMAX-YMIN)/JMM1

AA=DY*DY/AK
AB=DY*RE/2.0
AC=2.0+AA+2.0*ARR
AE=RE*D1*DY*DY
AF=AC+AE
AG=AA+AE
AH=2.0*DY/(RE*AK)
AL=2.0*DY*D1
AM=AL+AH
AN=2.0*DY*D2
AP=DY*DY
AQ=2.0+2.0*ARR

DO 1 I=1,IMAX
   II=I-1
   XCI=XMIN+II*DX
   CONTINUE

DO 2 J=1,JMAX
   JJ=J-1
   YCJ=YMIN+JJ*DY
   CONTINUE

370
Specifying the entry conditions on the streamfunctions

DO 4 J=1,JMAX
    PSI(1,J)=Y(J)
    PSI(D1,J)=PSI(1,J)
4 CONTINUE

Initializing the flow variables

DO 608 I=2,IMM1
    DO 607 J=2,5
        PSI(I,J)=0.1
        PSI(DI,J)=0.1
        WI(J)=2.1
        WD(J)=2.1
    607 CONTINUE
    DO 605 J=6,12
        PSI(I,J)=0.2
        PSI(DI,J)=0.2
        WI(J)=2.2
        WD(J)=2.2
    605 CONTINUE
    DO 606 J=13,20
        PSI(I,J)=0.5
        PSI(DI,J)=0.5
        WI(J)=2.5
        WD(J)=2.5
    606 CONTINUE
    608 CONTINUE

DO 14 I=2,IMAX
    K(I)=I-1
14 CONTINUE
    DO 15 I=1,IMM1
        KK(I)=I+1
15 CONTINUE
    DO 18 J=2,JMAX
        L(J)=J-1
18 CONTINUE
    DO 17 J=1,IMM1
        LL(J)=J+1
17 CONTINUE

RETURN
END
THE DUSTY FLUID SINK FLOW SOLVER

SUBROUTINE SINK
COMMON UC(41,41), VC(41,41), WC(41,41), PSI(41,41),
UDC(41,41), VDC(41,41), WDC(41,41), PSIDC(41,41),
K(41,41), KKC(41,41), L(41,41), LLC(41,41), X(41,41), Y(41,41),
ERBD1, ERBD2, ERBD3, ERBD4, WW1, WW2, WW3, WW4,
XMAX, XMIN, YMAX, YMIN, DX, DY, IMAX, JMAX, IMM1,
JMM1, AQ, AR, ARR, N1, NN, RE, AK,
D1, D2, AA, AB, AC, AD, AE, AF, AG, AH, AL, AM, AN, AP

DIMENSION A1(41), B1(41), C1(41), RHS1(41), RMAX1(41),
QQ(64), ERRRC(41,41), PSIPRC(41), WPRC(41),
A2(41), B2(41), C2(41), RHS2(41), RMAX2(41),
A3(41), B3(41), C3(41), RHS3(41), RMAX3(41),
A4(41), B4(41), C4(41), RHS4(41), RMAX4(41),
PSIDPRC(41), WDPRC(41)

N1 = 0

555 IF (N1 .EQ. 200) RETURN
N1 = N1 + 1

Solving the fluid-phase streamfunction

DO 55 I = 2, IMM1
DO 53 J = 2, JMM1

The matrix coefficients

RHS1(J) = A1(J) + B1(J) + C1(J)

A1(J) = -1.0
B1(J) = AQ
C1(J) = -1.0

53 CONTINUE

RHS1(JMM1) = RHS1(JMM1) + PSI(I, JMAX)
RHS1(2) = RHS1(2) + PSI(I, 1)
C1(JMM1) = 0.0
A1(2) = 0.0
CALL TRID(A1, B1, C1, QQ, RHS1, 2, JMM1)
RMAX1(I) = 0.0
NN = 0

WHILE (NN .LE. 3)
DO 54 J = 2, JMM1

54 NN = NN + 1
PSIPRC(J) = PSI(I, J)
PSIC(J, J) = PSI(I, J) + WW1 * (RHS1(J) - PSI(I, J))
ERR1 = ABS(PSIPRC(J) - PSI(I, J))
Calculating the fluid-phase velocity components and fluid-phase vorticity on the boundary

DO 61 I=2,IMM1
   DO 60 J=2,JMM1
      UC(I,J)=(PSI(I,LLC(J))−PSI(I,L(J)))/2.0×DY
      VC(I,J)=(PSI(KK(I),J)−PSI(KK(I),J))/2.0×DX
   60 CONTINUE
61 CONTINUE

DO 85 I=2,IMM1
   WCI(1)=−2.0×PSI(I,2)/CDY×DY)
   WCI(JMAX)=−2.0×PSI(I,JMM1−1.0)/CDY×DY)
85 CONTINUE

DO 86 J=2,JMM1
   W(JMAX,J)=−2.0×PSI(IMM1,J)/DX×DX)
86 CONTINUE

DO 155 I=2,IMM1
   DO 153 J=2,JMM1

THE FLUID-PHASE VORTICITY EQUATION

FOR THE DLB-HB MODELS:

The RHS and B2 for each of the models:

RHS & B2 for DLB-HB1:

C RHS2(J)=ARR×(WC(KK(I),J)WCK(I),J)
C ±AD×UC(I,J)×WC(KK(I),J)−WC(K(I),J)

C B2C(J)=AC

RHS & B2 for DLB-HB2:

C RHS2(J)=ARR×(WC(KK(I),J)WCK(I),J)
C ±AD×UC(I,J)×WC(KK(I),J)−WC(K(I),J)×AWK(I,J)

C B2C(J)=AC
RHS & B2 for DLB-HB3:

\[ \text{RHS2(C,J,J)} = \text{ARR} \times (\text{WCKK(C,I,J)} + \text{WCK(I,J)} + \text{AEXWDCI,JC}) \]

\[ \text{B2(C,J,J)} = \text{AF} \]

RHS & B2 for DLB-HB4:

\[ \text{RHS2(C,J,J)} = \text{ARR} \times (\text{WCKK(C,I,J)} + \text{WCK(I,J)} + \text{AEXUICL,JC}) \]

\[ \text{B2(C,J,J)} = \text{AF} \]

The coefficients A2 and C2 for all DLB-HB models:

\[ \text{A2(C,J,J)} = -1.0 - \text{ABXWCICL,JC} \]
\[ \text{C2(C,J,J)} = -1.0 + \text{ABXWCICL,JC} \]

FOR THE DL-HB MODELS:

The RHS & B2 for the different DL-HB models

The DL-HB1:

\[ \text{RHS2(C,J,J)} = \text{ARR} \times \text{UCI,J} \times (\text{WCK(C,I,J)} + \text{WCKK(C,I,J)}) \]
\[ \text{B2(C,J,J)} = \text{AH} \]

The DL-HB2:

\[ \text{RHS2(C,J,J)} = \text{AH} \times \text{WDCI,J} + \text{ARR} \times \text{UCI,J} \times (\text{WCKCICL,D,J} + \text{WCKK(C,I,J)}) \]
\[ \text{B2(C,J,J)} = \text{AH} \]

The DL-HB3:

\[ \text{RHS2(C,J,J)} = \text{AL} \times \text{WDCI,J} + \text{ARR} \times \text{UCI,J} \times (\text{WCKCICL,D,J} + \text{WCKK(C,I,J)}) \]
\[ \text{B2(C,J,J)} = \text{AM} \]

The DL-HB4:

\[ \text{RHS2(C,J,J)} = \text{AM} \times \text{WDCI,J} + \text{ARR} \times \text{UCI,J} \times (\text{WCKCICL,D,J} + \text{WCKK(C,I,J)}) \]
\[ \text{B2(C,J,J)} = \text{AM} \]

A2 and C2 coefficients are the same for all DL-HB models.
A2(J) = -V(I,J)
C2(J) = V(I,J)

CONTINUE
C2(JM1) = 0.0
A2(J) = 0.0

The RHS(C2), RHS(CJMM1) for the DLB-HB models

C
RHS2(2) = RHS2(2) + (1.0 + AB*V(I,2)) * WC1,1
C
RHS2(JM1) = RHS2(JM1) - (1.0 - AB*V(I,JM1)) * WC1,JMAX

The RHS(C2), RHS(CJMM1) for the DL-HB models

RHS2(2) = RHS2(2) + V(I,2) * WC1,1
RHS2(JM1) = RHS2(JM1) - V(I,JM1) * WC1,JMAX

CALL TRIEX(A2, B2, C2, QQ, RHS2, 2, JM1)
RMAX2(I) = 0.0
NN = 0
WHILE (NN .LE. 3)
    DO 154 J = 2, JM1
        NN = NN + 1
        WPRC(J) = WC1, J
        WC1, J = WC1, J + WW2 * (RHS2(J) - WC1, J)
        ERR2 = ABS * WPRC(J) - WC1, J
        IF (ERR2 .GT. RMAX2(I)) RMAX2(I) = ERR2
    154 CONTINUE
ENDWHILE
155 CONTINUE

The dust-phase streamfunction equation

DO 255 I = 2, JMM1
    DO 253 J = 2, JMM1
        RHS3(J) = AP * WDC(I, J) + ARR * (PSIDC(KK(I), J) + PSIDC(KI(I), J))
        A3C(J) = -1.0
        B3C(J) = AQ
        C3C(J) = -1.0
    253 CONTINUE
RHS3(JM1) = RHS3(JM1) + PSI DX(I, JMAX)
RHS3(2) = RHS3(2) + PSI DX(I, 1)
C3(JMM1)=0.0
A3(2)=0.0
CALL TRIDX(A3,B3,C3,QQ,RHS3,2,JMM1)
RMAX3(I)=0.0
NN=0
WHILE (NN .LE. 3)
   DO 254 J=2,JMM1
      NN=NN+1
      PSIDPR(J)=PSIDXI(J)
      PSIDCI,J)=PSIDXI,(J)+WW3*(RHS3(J)-PSIDXI,I,J))
      ERR3=ABS(PSIDPR(J)-PSIDXI,I,J))
      IF (ERR3 .GT. RMAX3(I)) RMAX3(I)=ERR3
   CONTINUE
   ENDWHILE
255 CONTINUE

To calculate the dust-phase velocity components

   DO 261 I=2,JMM1
   DO 260 J=2,JMM1
      UDXI,J)=(PSIDXI,L,L(J)J)-PSIDXI,L(J)J)/(2.0*DY)
      VDXI,J)=(PSIDXK(I),J)-PSIDXK(I),J)/2.0*DX)
   CONTINUE
260 CONTINUE
261 CONTINUE

To calculate the dust-phase velocity and vorticity on the boundary

   DO 384 J=2,JMM1
      WDX(IMAX,J)=(2.0*PSIDX(IMM1,J)-PSIDX(IMM1-1,J))/(DX*DX)
   CONTINUE

   DO 285 I=2,JMM1
      WDXI,J)=(2.0*PSIDXI,3)-PSIDXI,3))/(DY*DY)
      WDXI,JMAX)=(2.0*PSIDXI,JMM1)-PSIDXI,JMM1-1,0)/(DY*DY)
   CONTINUE
285 CONTINUE

The dust-phase vorticity equation

   DO 355 I=2,JMM1
   DO 353 J=2,JMM1
      RHS3(J)=AN*WDXI,J)+AR*UDXI,J)*
      (WDXK(I),J)-WDXK(I),J)*)
   CONTINUE
A4(JD) = -VDCI, JD
B4(JD) = AN
C4(JD) = VDCI, JD

353 CONTINUE
RHS4(2D) = RHS4(2D) + VDCI, 2) * WDCI, 1)
RHS4(JMM1) = RHS4(JMM1) - VDCI, JMM1) * WDCI, JMAX)
C4(JMM1) = 0.0
A4(2D) = 0.0
CALL TRIDX A4, B4, C4, QQ, RHS4, 2, JMM1)
RMAX4(CI) = 0.0
NN = 0
WHILE (NN .LE. 3)
   DO 354 J = 2, JMM1
      NN = NN + 1
      WDPRC(JD) = WDCI, JD
      WDCI, JD) = WDCI, JD) + WW4(J) * RHS4(CJ) - WDCI, JDC)
      ERR4 = ABS(WDPRC(JD) - WDCI, JDC)
      IF (ERR4 .GT. RMAX4(CI)) RMAX4(CI) = ERR4
   CONTINUE
354 CONTINUE
ENDWHILE
355 CONTINUE

ERMAX1 = 0.0
ERMAX2 = 0.0
ERMAX3 = 0.0
ERMAX4 = 0.0

IFLAG1 = 0
DO 167 I = 2, JMM1
   IF (RMAX1(I) .GE. ERBD1) IFLAG1 = 1
   IF (RMAX1(I) .GT. ERMAX1) ERMAX1 = RMAX1(I)
   IF (RMAX2(I) .GE. ERBD2) IFLAG1 = 1
   IF (RMAX2(I) .GT. ERMAX2) ERMAX2 = RMAX2(I)
   IF (RMAX3(I) .GE. ERBD3) IFLAG1 = 1
   IF (RMAX3(I) .GT. ERMAX3) ERMAX3 = RMAX3(I)
   IF (RMAX4(I) .GE. ERBD4) IFLAG1 = 1
   IF (RMAX4(I) .GT. ERMAX4) ERMAX4 = RMAX4(I)
167 CONTINUE

IF (IFLAG1 .EQ. 1) GO TO 555
PRINT 171, N1
PRINT 345, ERMAX1
PRINT 347, ERMAX2
PRINT 349, ERMAX3
PRINT 351, ERMAX4
To update the velocity components in the flowfield

DO 461 I=2,IMM1
   DO 460 J=2,JMM1
     UCI,J = (PSI(I,LLC(J)) - PSI(I,LC(J)))/2.0*DY
     VCI,J = (PSI(KCI,J) - PSI(KCI,J))/2.0*DX
     UDIX,J = (PSIDXI,LLC(J)) - PSIDXI,LC(J))/2.0*DY
     VDIX,J = (PSIDXKCI,J) - PSIDXKCI,J))/2.0*DX
   CONTINUE
460 CONTINUE
461 CONTINUE

To update the velocities & vorticities at the boundaries

DO 485 I=2,IMM1
   WCI,1) = -2.0*PSI(I,2)/DY*DY)
   WCI,JMAX) = -2.0*PSI(I,JMAX) - 1.0)/DY*DY)
   WDCI,1) = (2.0*PSIDXI,2) - PSIDXI,3)/DY*DY)
   WDCI,JMAX) = (2.0*PSIDXI,JMAX) - PSIDXI,JMAX - 1) - 1.0)/
     (DY*DY)
   UDCI,1) = (2.0*PSIDXI,2) - PSIDXI,3)/2.0*DY)
   UDCI,JMAX) = (3.0 - 4.0*PSIDXI,JMAX) + PSIDXI,JMAX - 1))/
     (2.0*DY)
485 CONTINUE

DO 488 J=2,JMM1
   WCMAX,J) = -2.0*PSIDXI(JMAX,J)/DX*DX)
   WDCMAX,J) = (2.0*PSIDXI(JMAX) - PSIDXI(JMAX - 1))/
     (DX*DX)
   VDCMAX,J) = (PSIDXI(JMAX - 1) - 4.0*PSIDXI(JMAX))/
     (2.0*DX)
488 CONTINUE

To classify the streamlines

DO 712 KI=1,9
   VAL = KI*0.1
   DO 711 I=2,IMM1
      DO 710 J=2,JMM1
        ERRCI,J) = ABS(PSI(I,J) - VAL)
        IF (ERRCI,J) .LE. 0.01) THEN
THE TRIDIAGONAL SUBROUTINE (THOMAS' ALGORITHM)

SUBROUTINE TRID(A, B, C, D, S, NL, NUD)
DIMENSION AC(41), BC(41), CC(41), DC(41), SC(41)

DCNLJ = CCNLJ / BCNLJ
SCNLJ = SCNLJ / BCNLJ
NLPL = NL + 1
DO 1 N = NLPL, N
      Z = 1.0 / (BCND - ACND * DX(N-1))
      DCND = CCND * Z
 1 SCND = (SCND - ACND * SCN-1) * Z
NUPNL = NU + NL
DO 2 NN = NLP1, NU
      N = NUPNL - NN
 2 SCND = SCND - DCND * SCN+1

RETURN
END

A SAMPLE OUTPUT SUBROUTINE THAT PRINTS OUT Ψ AND ΨD

SUBROUTINE OUTPUT
COMMON UC(41,41), VC(41,41), W(41,41), PSI(41,41),
   & UDC(41,41), VDC(41,41), WDC(41,41), PSIDC(41,41),
   & KC(41), KKC(41), LC(41), LLC(41), XC(41), YC(41),
   & ERBD1, ERBD2, ERBD3, ERBD4, WW1, WW2, WW3, WW4,
   & XMAX, XMIN, YMAX, YMIN, DX, DY, IMAX, JM1, IMM1,
   & JMM1, AQ, AR, ARR, N1, NN, RE, AK,
   & D1, D2, AA, AB, AC, AD, AE, AF, AG, AH, AL, AM, AN, AP

379
PRINT 208
208 FORMAT ('1')
99 FORMAT(1X,7(1X,E15.8))

DO 782 J=1,JMAX
    PRINT 208
    PRINT 99,(PSI(I,J),I=1,IMAX)
782 CONTINUE

DO 882 J=1,JMAX
    PRINT 208
    PRINT 99,(PSID(I,J),I=1,IMAX)
882 CONTINUE

RETURN
END
APPENDIX C

(DUSTY COMPUTER CODE)
APPENDIX C

COMPUTER CODE: DUSTY

DUSTY FLUID FLOW THROUGH POROUS MEDIA BOUNDED BELOW BY A
STATIC FLUID.

Models considered: Darcy-Lapwood-Brinkman-H-B2 and H-B4,
and Brinkman H-B2.

COMMON Y(C64,22), X(C64), AA1(C64,22), AA2(C64,22),
@ WC64,22), WWS, WW4, PSI MIN, PSI MAX, W2(C64,22),
@ IMAX, JMAX, IMM1, JMM1, DX, DS, N1, NN, XMIN, XMAX,
@ WW1, ERBD1, AB, AC, AD, AE, AF, VC64,22),
@ UVC64,22), JMM2, FC64,22), U2(C64,22), UC64,22),
@ KC64), KC64), LLC22), LC22), UI(C64,22),
@ AG, AJ, AK, AL, AI, AH, WW2, ERBD2, AM, BA1(C64,22),
@ BA2(C64,22), RE, V1(C64,22), N3, WW3, N2, D2, D1,
@ W1(C64,22), BB3(C64,22), BB4(C64,22), ERBD3,
@ VFC64,22), YIC64,22), Y2(C64,22), VDC64,22)

VARIABLES AND PARAMETERS USED

(I,J): TYPICAL GRID POINT
Y: THE NEW DEPENDENT VARIABLE IN DARCY'S CASE
Y1: THE FLUID-PHASE NEW DEPENDENT VARIABLE
Y2: THE DUST-PHASE NEW DEPENDENT VARIABLE
X(I,J): THE COORDINATE AT GRID LINE I
WW1, WW2, WW3, WW4, WW5: RELAXATION PARAMETERS
IMAX: MAXIMUM NUMBER OF GRID POINTS IN X-DIRECTION
JMAX: MAXIMUM NUMBER OF GRID POINTS IN Y-DIRECTION
IMM1: IMAX-1
JMM1: JMAX-1
JMM2: JMAX-2
AB, AC, AD, AE, AF, AG, AH, AI, AJ, AL AND AM: PARAMETERS
DX: THE STEP SIZE ΔX
DS: THE STEP SIZE ΔY
ERBD1, ERBD2, ERBD3: ERROR TOLERANCES
N1, N2, N3, NN: REFER TO NUMBER OF ITERATIONS
RE: REYNOLDS NUMBER

382
PSIMAX: THE MAXIMUM VALUE OF \( \Psi \)
PSIMIN: THE MINIMUM VALUE OF \( \Psi \)
XMAX: THE MAXIMUM VALUE OF X
XMIN: THE MINIMUM VALUE OF X
D1, D2: THE DUST PARAMETERS
L(J), L(J), K(I), K(K): VECTORS REPRESENTING \( J-1, J+1, I-1 \) AND \( I+1 \)

AK: PERMEABILITY
U, V: THE DARCY'S VELOCITY COMPONENTS
U1, VF: THE FLUID-PHASE VELOCITY COMPONENTS
U2, VD: THE DUST-PHASE VELOCITY COMPONENTS
W1: THE FLUID-PHASE VORTICITY
W2: THE DUST-PHASE VORTICITY
V1(I,1): \( f'(i) \)

**HOW TO USE THE CODE**


THE SOLUTION TO DARCY'S FLOW IS USED AS THE INITIAL VALUES FOR BOTH THE FLUID- AND THE DUST-PHASE.

THE SOLUTION TO THE DLB-HB4 IS OBTAINED BY CALLING THE "INPUT", "DARC2" AND "DUSTF" SUBROUTINES.

THE SOLUTION TO DL-HB4 IS OBTAINED BY CALLING THE SAME SUBROUTINES, ABOVE, BUT THE MATRIX COEFFICIENTS ARE CHANGED ACCORDINGLY.

THE DL-HB4 VORTICITY EQUATIONS (FLUID AND DUST) ARE RELAXED IN THE I-DIRECTION.

**THE MAIN PROGRAMME**

The following subroutine provides the input data

CALL INPUT

The following subroutine solves the Darcy equation in order to use the results as initial values for the dusty fluid flow variables.

CALL DARC2
The following subroutine solves the dusty fluid equations

CALL DUSTF

The following subroutine prints out the results

CALL SASOUT

STOP

END

THE INPUT DATA

SUBROUTINE INPUT
COMMON YC64,22, XC64, AA1(64,22), AA2(64,22),
@ WC64,22, WW5, WW4, PSI MIN, PSI MAX, W2(84,22),
@ IMAX, JMAX, IMM1, IMM2, DX, DS, N1, NN, XMIN, XMAX,
@ WW1, ERBD1, AB, AC, AD, AE, AF, VC64,22),
@ UVC64,22, JMM2, PC64,22, U2C64,22, UC64,22, KC64,
@ KKC64, LL(22), UC64,22, U1C64,22, AG, AJ, AL, AI, AH, WW2, ERB2, AM, BA1(64,22),
@ BA2(64,22), RE, V1C64,22, N3, WW3, N2, D2, D1,
@ W1C64,22, BB3C64,22, BB4C64,22, ERBD3,
@ VPC64,22, Y1C64,22, Y2C64,22, VDX64,22

DATA ERBD1/0.5/, ERBD2/0.5/, ERBD3/0.00005/
DATA WW1/0.11/, WW2/0.11/, WW3/1.86/, WW4/0.11/
DATA IMAX/32/, JMAX/22/, RE/1.0/, D2/50.0/, D1/1.0/
DATA IMM1/31/, IMM2/21/, JMM2/20/, WW5/0.11/, AK/0.001/
DATA XMAX/2.0/, XMIN/-2.0/, PSI MAX/2.0/, PSI MIN/0.0/

DX=(XMAX-XMIN)/IMM1
DS=(PSI MAX-PSI MIN)/JMM1

AM=DS*DS
AB=4.0*DS*DS
AC=4.0*DX*DS
AD=2.0*DX*DS
AE=(DX/DS)*C(DX/DS)
AF=DX/(4.0*DS)
AG=DX*DX
AH=2.0*DX
AI=2.0*DS
AJ=DX*DX/(2.0*DS)
AL=DX*DX/AK
DO 1  I=1,IMAX
    VC(I,JMAX)=0.0
    WC(I,JMAX)=0.0
    CONTINUE

DO 2  J=1,JMAX
    VC1,JD)=0.0
    VC(IMAX,JD)=0.0
    UC1,JD)=1.0
    UC(IMAX,JD)=1.0
    WC1,JD)=0.0
    WC(IMAX,JD)=0.0
    CONTINUE

DO 3  J=1,JMAX
    JJ=J-1
    YC(J,D)=JJ*DS
    YC(IMAX,JD)=JJ*DS
    CONTINUE

DO 4  I=2,IMM1
    YC(I,JMAX)=PSI MAX
    CONTINUE

DO 5  I=1,IMAX
    II=I-1
    XCLUD=XMIN+II*DX
    CONTINUE

DO 6  I=13,20
    YC(I,1)=0.2*ABS(SQRT(0.25-X(I)*X(I)))
    VCI(I,1)=-0.2*X(I)/(ABS(SQRT(0.25-X(I)*X(I))))
    CONTINUE

DO 7  I=1,12
    YCI(I,1)=0.0
    VCI(I,1)=0.0
    VCI(I,1)=0.0
    CONTINUE

DO 8  I=21,IMAX
    VCI(I,1)=0.0
    VCI(I,1)=0.0
    YCI(I,1)=0.0
    CONTINUE
DO 11 I=2,JMM1
  DO 10 J=2,JMM1
   YCI,J)=0.05*J
  10 CONTINUE
11 CONTINUE

DO 91 I=1,JMAX
  DO 90 J=1,JMAX
    WCI,J)=0.05*J
  90 CONTINUE
91 CONTINUE

DO 12 I=2,JMAX
   KC(I)=I-1
12 CONTINUE

DO 13 I=1,JMM1
   KKCI)=I+1
13 CONTINUE

DO 14 J=2,JMAX
   LC(J)=J-1
14 CONTINUE

DO 15 J=1,JMM1
   LCC(J)=J+1
15 CONTINUE

RETURN
END

THE DARCY REGIME SOLVER

SUBROUTINE DARC2
COMMON YC(64,22),XC(64),AA1(64,22),AA2(64,22),
   WC(64,22),WW5,WW4,PSIMIN,PSIMAX,W2(64,22),
   IMAX,JMAX,IIM1,IJJM1,DX,DS,N1,NN,XMIN,XMAX,
   WW1,ERBD1,AB,AC,AD,AE,AQ,VC(64,22),
   UVC(64,22),JMM2,FC(64,22),U2C(64,22),UC(64,22),
   KC(64),KKC(64),LCC(22),LC(22),U1C(64,22),
   AG,AJ,AK,AL,AI,AH,WW2,ERBD2,AM,BA1(64,22),
   BA2(64,22),RE,V1(64,22),NS,WW3,N2,D2,D1,
   W1C(64,22),BB3C(64,22),BB4C(64,22),ERBD3,
   VFC(64,22),Y1C(64,22),Y2C(64,22),VDC(64,22)
DIMENSION A3(64),B3(64),C3(64),RHS3C(64),RMAT3C(64),
   YPRC(64),QQC(64)
N3=0

787 IF (N3 .EQ. 200) RETURN
N3=N3+1

DO 52  I=2,IMM1
    DO 51  J=2,JMM1
        AA1(I,J)=YCI(LL(J))+YCI(LC(J))
        AA2(I,J)=YCKK(I,J)+YCK(I,J)
    51 CONTINUE

52 CONTINUE

DO 55  I=2,IMM1
    DO 53  J=2,JMM1
        RHS3(JJ)=(AA1(I,J)*AA1(I,J)/AB)*
                 YCKKI(J)+YCKI(J)
                -CAA1(I,J)*AA2(I,J)/AD)*AF*
                YCKI(J)+YCKI(J)-YCKK(I,J)
        A3(JJ)=-AE*CI.0+AA2(I,J)*AA2(I,J)/AC
        B3(JJ)=2.0*CAAA1(I,J)*AA1(I,J)/AB+2.0*AE*
                (1.0+AA2(I,J)**2/AC)
    53 CONTINUE

55 CONTINUE

RHS3(JMM1)=RHS3(JMM1)+
              (AE*CI.0+AA2(I,J)*AA2(I,J)/AC)*YCI(JMAX)

RHS3(2)=RHS3(2)+AE*CI.0+AA2(I,J)*AA2(I,J)/AC)*YCI(1)

C3(JMM1)=0.0

A3(2)=0.0

CALL TRIDX(A3,B3,C3,QQ,RHS3,2,JMM1)

RMAX3(I)=0.0

NN=0

WHILE (NN .LE. 3)
    DO 54  J=2,JMM1
        NN=NN+1
        YPR(J)=YCI(J)
        YCI(J)=YCI(J)+WW3*(RHS3(JJ)+YCI(J))
        ERR3=ABS(YPR(J)-YCI(J))
        IF (ERR3 .GT. RMAX3(I)) RMAX3(I)=ERR3
    54 CONTINUE

ENDWHILE

55 CONTINUE

ERMAX3=0.0

IFLAG3=0

DO 56  I=2,IMM1
    IF (RMAX3(I) .GE. ERBD3) IFLAG3=1
    IF (RMAX3(I) .GT. ERM3) ERMAX3=RMAX3(I)

56 CONTINUE

387
IF (IFLAG3 .EQ. 1) GO TO 787
PRINT 57, N3
PRINT 58, ERMAX3

57 FORMAT('O', ' #ITR3= ', I3)
58 FORMAT('O', ' MAX ERR3= ', E15.8)

DO 557 I = 2, IMM1
   UCI, 1) = DS/CYCI, 2) - YCI, 1)
   VCI, 1) = VI(CI, 1) * UC(I, 1)
   UCI, JMAX) = DS/CYCI, JMAX) - YCI, JMM1)
557 CONTINUE

DO 556 I = 2, IMM1
   DO 555 J = 2, JMM1
      UCI, J) = 2.0 * DS/CYCI, LLC(J)) - YCI, LC(J))
      VCI, J) = (DS/DX) * YCKIC(J), J) - YCKIC(J), J))
      (YCI, LLC(J)) - YCI, LC(J))
555 CONTINUE
556 CONTINUE

RETURN
END

THE DUSTY FLUID REGIME SOLVER

SUBROUTINE DUSTF
COMMON YC(64, 22), XC(64), AA1(64, 22), AA2(64, 22),
   WC(64, 22), WW3, WW4, PSI MIN, PSI MAX, W2(64, 22),
   IMAX, JMAX, IMM1, JMM1, DX, DS, N1, NN, XMIN, XMAX,
   W1, ERBD1, AB, AC, AD, AE, AF, VC(64, 22),
   UV(64, 22), JMM2, FC(64, 22), U2(64, 22), UC(64, 22),
   KC(64), KC(64), LLC(22), LC(22), UL(64, 22),
   AG, AJ, AK, AL, AI, AH, WW2, ERBD2, AM, BA1(64, 22),
   BA2(64, 22), RE, VI(64, 22), WW3, N2, D2, D1,
   WI(64, 22), BB3(64, 22), BB4(64, 22), ERBD3,
   V(64, 22), Y1(64, 22), Y2(64, 22), VD(64, 22)

DIMENSION A1(64), B1(64), C1(64), RHSL(64), RMAX1(64),
   YPR1(64), RMAX2(64), WPR1(64), YPR2(64),
   QQC(64), A2(64), B2(64), C2(64), RHSL(64),
   A3(64), B3(64), C3(64), RMAX3(64),
   A4(64), B4(64), C4(64), RMAX4(64),
   WPR2(64)

398
The initialization of the dusty fluid variables in terms of Darcy's results.

DO 305 I=1,IMAX
   DO 304 J=1,JMAX
      Y1(I,J)=YCI(I,J)
      WCI(I,J)=Y1CI(I,J)
      W1(I,J)=WC1(I,J)
      U1(I,J)=UC1(I,J)
      VFC1(I,J)=VC1(I,J)
      Y2(I,J)=Y1CI(I,J)
      W2(I,J)=W1CI(I,J)
      U2(I,J)=U1CI(I,J)
      VDC1(I,J)=VFC1(I,J)
304 CONTINUE
305 CONTINUE

N1=0
444 IF (N1 .EQ. 500) RETURN
   N1=N1+1

DO 152 I=2,IMM1
   DO 151 J=2,JMM1
      AA1(I,J)=Y1(I,LL(JJ)) - Y1(I,LC(JJ))
      AA2(I,J)=Y1(KK(I),JJ) - Y1(KKI),JJ
151 CONTINUE
152 CONTINUE

Solving the Yi-equation

DO 155 I=2,IMM1
   DO 153 J=2,JMM1
      RHS(I,J)=(AA1(I,J)**2/4.0) * (Y1(KKI),JJ) + Y1(KKI),JJ)
      @ -AA1(I,J)**2*AA2(I,J)**2*Y1(KKI),JJ**2*Y1(KKI),JJ**2-8.0
      @ -DX*DX*AA1(I,J)**2*W1(I,J)/0.000000
      A1(J)=0.25*AA2(I,J)**2-DX*DX
      B1(J)=0.5*AA2(I,J)**2+0.5*AA1(I,J)**2
      C1(J)=A1(J)
153 CONTINUE
RHS(I,JMM1)=RHS(I,JMM1)+
   (0.25*AA2(I,JMM1)**2+DX*DX**2*Y1(I,JMAX)
RHS(I,J)=RHS(I,J)+(0.25*AA2(I,J)**2+DX*DX**2*Y1(I,1)
C1(J,JMM1)=0.0
A1(J)=0.0
CALL TRIDCA1,B1,C1,QQ,RHS1,2,JMM1)
RMAX1(I,J) = 0.0
NN = 0
WHILE (NN .LE. 3)
   DO 154  J = 2, JNM1
      NN = NN + 1
      YPR1(J,J) = Y1(I,J)
      Y1(I,J) = Y1(I,J) + WW1 * (RHS1(J) - Y1(I,J))
      ERR1 = ABS(YPR1(J,J) - Y1(I,J))
      IF (ERR1 .GT. RMAX1(I,J)) RMAX1(I,J) = ERR1
   CONTINUE
ENDWHILE

DO 157  I = 2, IMM1
   U1(I,1) = DS / (Y1(I,2) - Y1(I,1))
   VFC(I,1) = V1(I,1) * U1(I,1)
   U1(I,JMAX) = DS / (Y1(I,JMAX) - Y1(I,IMM1))
CONTINUE

DO 159  I = 2, IMM1
   DO 158  J = 2, JNM1
      U1(I,J) = 2.0 * DS / (Y1(I,LLC(J)) - Y1(I,LC(J)))
      VFC(I,J) = (DS / DX) * (Y1(KK(J)) - Y1(KK(J)))
      @ (Y1(I,LLC(J)) - Y1(I,LC(J)))
   CONTINUE

DO 162  I = 2, IMM1
   DO 161  J = 2, JNM1
      AA1(I,J) = Y1(I,LLC(J)) - Y1(I,LC(J))
      AA2(I,J) = Y1(KK(J)) - Y1(KK(J))
   CONTINUE

DO 163  I = 2, IMM1

First order accurate vorticity expression

\[ W1(i,1) = \frac{(VFC(KK(i)),1) - VFC(i,1))}{DX} \]
@ \[ -VFC(i,1) \times (VFC(i,2) - VFC(i,1)) / DS \]
@ \[ -U1(i,1) \times (U1(i,2) - U1(i,1)) / DS \]

CONTINUE
Solving the Wi-equation: DLB-HB4 case

DO 166 I=2,JMM1
   DO 164 J=2,JMM1
      RHS2(J)=AA1(C),J)+2*W(I,CKCI),J)+WIC(KCI),J)/4.0
      -CDS*DX*RE*AA1(C),J)+WIC(KCI),J)-WIC(KCI),J)/4.0
      -W(I,CKCI),J)+WIC(KCI),J)+WIC(KCI),J)+WIC(KCI),J)/4.0
      -(W(I,CKCI),L(C,J))+WIC(KCI),L(C,J)-WIC(KCI),L(C,J)+AA1(C),J)/8.0
      CREM1+1.0/ACM*DX*DX*AA1(C),J)+2*W2(C),J)+W2(C),J)/4.0
      A2(C),J)=DX*DX+AA2(C),J)/8.0
      C2(S)=A2(C)
      B2(C)=DX*DX+D1*RE+1.0/ACM*AA1(C),J)/4.0
      -2.0*DX*DX+0.5*AA1(C),J)+AA2(C),J)/2.0
      +(W1(C),L(C,J)-W1(C),L(C,J))*DX*DX/C8.0*DISC)*AA1(C),J)/2.0
   CONTINUE
164 RHS2(J,MML1)=RHS2(J,MML1)+CDX*DX+AA2(C),J,MML1)/8.0
   R1W(C,JMAX)
RHS2(C,J)=RHS2(C,J)+CDX*DX+AA2(C),J)/8.0
C2(S)=0.0
A2(C)=0.0
CALL TRIDX(A2,B2,C2,QQ,RHS2,2,JMM1)
RMAX2(C)=0.0
NN=0
WHILE (NN .LE. 3)
   DO 165 J=2,JMM1
      NN=NN+1
      WPR1(J)=W1(C),J)
      W1(C),J)=W1(C),J)+WW2*WHS2(J,J)-W1(C),J)
      ERR2=ABSC*WPR1(J)-W1(C),J)
      IF (ERR2 .GT. RMAX2(C)) RMAX2(C)=ERR2
   CONTINUE
165 CONTINUE
ENDWHILE
166 CONTINUE

Solving the Wi-equation: the DL-HB4 case

C DO 466 J=2,JMM1
C DO 464 I=2,JMM1
C RHS2(C)=D1+1.0/CREM*AA1(C)+Dx*W2(C),J)+AA1(C),J)/DS
C A2(C)=1.0
C C2(C)=1.0
C B2(C)=D1+1.0/CREM*AA1(C),J)/DS
C464 CONTINUE

391
C  C2(IMM1) = 0.0
C  A2(C2) = 0.0
C  CALL TRIX(A2, B2, C2, QQ, RHS2, 2, IMM1)
C  RMAX2(CJ) = 0.0
C  NN = 0
C  WHILE (NN .LE. 3)
C     DO 465 I = 2, IMM1
C        NN = NN + 1
C        WPR1(I) = W1(I, J)
C        W1(I, J) = W1(I, J) * WW4 * (RHS2(I) - W1(I, J))
C        ERR2 = ABS(WPR1(I) - W1(I, J))
C        IF (ERR2 .GT. RMAX2(CJ)) RMAX2(CJ) = ERR2
C    465 CONTINUE
C  ENDWHILE
C  CONTINUE

DO 652 I = 2, IMM1
    DO 651 J = 2, IMM1
        BAI1(I, J) = Y2(I, LLC(J)) - Y2(I, LC(J))
        BAI2(I, J) = Y2(KC(I), J) - Y2(KC(I), J)
    651 CONTINUE
    652 CONTINUE

DO 632 I = 2, IMM1
    BAI1(I, 1) = Y2(I, 2) - Y2(I, 1)
732 CONTINUE

Solving the Y2-equation

DO 655 I = 2, IMM1
    DO 653 J = 2, IMM1
        RHS3(CJ) = (BAI1(I, J) * 2 / 4.0) * 
            (Y2(KC(I), LLC(J)) - Y2(KC(I), LC(J)) - 
            Y2(KC(I), LLC(J)) * Y2(KC(I), LC(J)) / 8.0 - 
            DX * DX * BAI1(I, J) * MM2(I, J) / 0.008)
    A3(CJ) = -0.25 * BAI2(I, J) * 2 * DX * DX
    B3(CJ) = 2.0 / DX * DX + 0.5 * BAI2(I, J) * 2 + 0.5 * BAI1(I, J) * 2
    C3(CJ) = A3(CJ)
    653 CONTINUE

RHS3(JMM1) = RHS3(JMM1) + 
            (0.25 * BAI2(I, JMM1) * 2 * DX * DX * Y2(I, JMAX))
RHS3(2) = RHS3(2) + (0.25 * BAI2(I, 2) * 2 * DX * DX * Y2(I, 1))
C3(JMM1) = 0.0
A3(2) = 0.0
CALL TRIX(A3, B3, C3, QQ, RHS3, 2, JMM1)
RMAX3(I)=0.0
NN=0
WHILE (NN .LE. 3)
   DO 654  J=2,JMM1
      NN=NN+1
      YPR2(I,J) = Y2(I,J)
      Y2(I,J) = Y2(I,J) + WW4*KRHS3(I,J) - Y2(I,J)
      ERR3 = ABS(YPR2(I,J) - Y2(I,J))
      IF (ERR3 .GT. RMAX3(I)) RMAX3(I) = ERR3
   CONTINUE
ENDWHILE
654 CONTINUE
655 CONTINUE

DO 657  I=2,JMM1
   U2(I,1) = DS/Y2(I,2) - Y2(I,1))
   VDI(I,1) = V1(I,1) * U2(I,1)
   U2(I,JMAX) = DS/Y2(I,JMAX) - Y2(I,JMAX)
657 CONTINUE

DO 658  I=2,JMM1
   DO 658  J=2,JMM1
      U2(I,J) = 2.0*DS/Y2(I,LL(J)) - Y2(I,LL(J))
      VDI(I,J) = (DS/DX)*Y2(K(I),J) - Y2(K(I),J)
      Y2(I,LL(J)) = Y2(I,LL(J))
658 CONTINUE
659 CONTINUE

DO 662  I=2,JMM1
   DO 662  J=2,JMM1
      BA1(I,J) = Y2(I,LL(J)) - Y2(I,LL(J))
      BA2(I,J) = Y2(K(I),J) - Y2(K(I),J)
661 CONTINUE
662 CONTINUE

Solving the W2-equation

DO 764  J=2,JMM1
   DO 764  I=2,JMM1
      RHS4(I,J) = D2*DIW1(I,J)*BA1(I,J)/DS
      A4(I,J) = -1.0
      C4(I,J) = 1.0
      B4(I,J) = DS/BA1(I,J)/DS
   CONTINUE
764 CONTINUE
C4(IMM1) = 0.0
A4(2) = 0.0
CALL TRIDXA4,B4,B4,QQ,RHS4,2,IMM1
RMAX4(C,J) = 0.0

383
NN=0
WHILE (NN .LE. 3)
  DO 765 I=2,IMM1
    NN=NN+1
    WPR2C1)=W2(I,J)
    W2(I,J)=W2I(J)+WWS<(RHS4(I)-W2(I,J))
    ERR4=ABS(WPR2C1)-W2I(J)
    IF (ERR4 .GT. RMAX4(J)) RMAX4(J)=ERR4
 765 CONTINUE
ENDWHILE
766 CONTINUE

To check for convergence

ERMAX1=0.0
ERMAX2=0.0
ERMAX3=0.0
ERMAX4=0.0
IFLAG1=0

DO 167 I=2,IMM1
  IF (ERMAXC(I) .GE. ERBD1) IFLAG1=1
  IF (ERMAXC(I) .GT. ERMAX1) ERMAX1=ERMAXC(I)
  IF (ERMAXC(I) .GE. ERBD1) IFLAG1=1
  IF (ERMAXC(I) .GT. ERMAX2) ERMAX2=ERMAXC(I)
167 CONTINUE

DO 767 I=2,IMM1
  IF (ERMAXC(I) .GE. ERBD2) IFLAG1=1
  IF (ERMAXC(I) .GT. ERMAX3) ERMAX3=ERMAXC(I)
767 CONTINUE

DO 778 J=2,IMM1
  IF (ERMAXC(J) .GE. ERBD2) IFLAG2=1
  IF (ERMAXC(J) .GT. ERMAX4) ERMAX4=ERMAXC(J)
778 CONTINUE

IF (IFLAG1 .EQ. 1) GO TO 444
PRINT 171,N1
PRINT 172,ERMAX1
PRINT 173,ERMAX2
171 FORMAT('O','ITR1 = ',I3)
172 FORMAT('O','MAX ERR1 = ',E15.8)
173 FORMAT('O','MAX ERR2 = ',E15.8)
PRINT 271,N2
PRINT 272,ERMAX3
PRINT 273,ERMAX4
271 FORMAT('O','ITR2 = ',I3)
DO 757 I = 2, IMMI
   U2(I,1) = DS/(Y2(I,2) - Y2(I,1))
   VDI(I,1) = VI(I,1) / U2(I,1)
   U2(I,JMAX) = DS/(Y2(I,JMAX) - Y2(I,JMIN))
757 CONTINUE

DO 759 I = 2, IMMI
   DO 758 J = 2, JMMI
      U2(I,J) = 2.0 * DS/(Y2(I,LLC(J)) - Y2(I,LJC(J)) - VDI(I,J) * (Y2(I,LLC(J)) - Y2(I,LJC(J))))
   758 CONTINUE
759 CONTINUE

DO 263 I = 2, IMMI
   W2(I,1) = (VDI(KK(I),1) - VDI(I,1)) / DX
   @
   -VDI(I,1) * (VDI(I,2) - VDI(I,1)) / DS
   @
   -U2(I,1) * (U2(I,2) - U2(I,1)) / DS
263 CONTINUE

DO 257 I = 2, IMMI
   U1(I,1) = DS/(Y1(I,2) - Y1(I,1))
   VFI(I,1) = VI(I,1) / U1(I,1)
   U1(I,JMAX) = DS/(Y1(I,JMAX) - Y1(I,JMIN))
257 CONTINUE

DO 259 I = 2, IMMI
   DO 258 J = 2, JMMI
      U1(I,J) = 2.0 * DS/(Y1(I,LLC(J)) - Y1(I,LJC(J)) - VFI(I,J) * (Y1(I,LLC(J)) - Y1(I,LJC(J))))
   258 CONTINUE
259 CONTINUE

RETURN
END

THE TRIDlgONAL SOLVER (THOMAS' ALGORITHM)

SUBROUTINE TRID(A, B, C, D, S, NL, NU)
DIMENSION AC(NL), BC(NL), CC(NL), DC(NL), SC(NL)

395
THE OUTPUT DATA

SUBROUTINE SASOUT

COMMON Y(C(64,22)), X(C(64)), AA1(C(64,22)), AA2(C(64,22)),
@ WC(64,22), WW5, WW4, PSIMIN, PSIMAX, W2(C(64,22)),
@ IMAX, JMAX, IMM1, JMM1, DX, DS, N1, NN, XMIN, XMAX,
@ WW1, ERBD1, AB, AC, AD, AE, AF, V(C(64,22)),
@ UCS(64,22), JMM2, PC(64,22), U2(C(64,22)), UC(64,22),
@ K(64), KK(C64), LLC(22), LC(22), U1(C(64,22)),
@ AG, AJ, AK, AL, AI, AH, WW2, ERBD2, AM, BA1(C(64,22)),
@ BA2(C(64,22)), RE, V1(C(64,22)), N3, WW3, N2, D2, D1,
@ W1(C(64,22)), BB3(C(64,22)), BB4(C(64,22)), ERBD3,
@ VFC(64,22), Y1(C(64,22)), Y2(C(64,22)), VDX(C(64,22))

PRINT 99, ((Y(I,J), I=1, IMAX), J=1, JMAX)
PRINT 208
PRINT 99, ((Y2(I,J), I=1, IMAX), J=1, JMAX)
PRINT 208
PRINT 99, ((W1(I,J), I=1, IMAX), J=1, JMAX)
PRINT 208
PRINT 99, ((W2(I,J), I=1, IMAX), J=1, JMAX)
PRINT 208
PRINT 99, ((U1(I,J), I=1, IMAX), J=1, JMAX)
PRINT 208
PRINT 99, ((U2(I,J), I=1, IMAX), J=1, JMAX)
PRINT 208
PRINT 99, ((VFC(I,J), I=1, IMAX), J=1, JMAX)
PRINT 208
PRINT 99, ((VDX(I,J), I=1, IMAX), J=1, JMAX)
99 FORMAT (I1,7(E15.8))
208 FORMAT('1.')
RETURN
END
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VITA AUCTORIS

Born in 1959, the author completed his Jordanian GCE in 1979 at Dr. Qadri Touqan Secondary School in the City of Nablus. He then received his Ordinary National Diploma in Engineering at The College, Swindon, Wilts., in 1979. At the University of Windsor, the author received his B.Sc. in 1983 and his M.Sc. in Applied Mathematics in 1985.

During his graduate studies, the author was blessed with his marriage to Carol and the birth of his first daughter Nadia, during his candidacy year in the M.Sc. programme, and the birth of his second daughter Fadia, during his candidacy year in the Ph.D. programme.