ON EXACT, PERFECT FLUID SOLUTIONS OF THE EINSTEIN FIELD EQUATIONS.

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LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L’AVONS RÉCU
ON EXACT, PERFECT FLUID SOLUTIONS
OF THE EINSTEIN FIELD EQUATIONS

by

David Alan Wilkinson

A Dissertation
submitted to the Faculty of Graduate Studies
through the Department of Physics
in Partial Fulfillment of the
Requirements for the Degree
of Doctor of Philosophy at
The University of Windsor

Windsor, Ontario, Canada

1979
ABSTRACT

ON EXACT, PERFECT FLUID SOLUTIONS
OF THE EINSTEIN FIELD EQUATIONS

by

David Alan Wilkinson

Perfect fluid spacetimes are studied in two contexts: as models of isolated steady-state systems and secondly, as cosmological models.

It is proved that stationary, axially symmetric spacetimes containing matter in non-convective flow must be either Petrov type I or D.

A new derivation of the Wahlquist (1968) solution, describing a finite, stationary, axially symmetric, rigidly rotating perfect fluid, is presented which allows an increased number of arbitrary constants.

It is shown that a particular complexification of some steady-state solutions results in metric forms suitable to describe non-stationary perfect fluid cosmologies. The exact cosmological solution obtained is shown to describe a new locally rotationally symmetric spacetime with non-zero shear and expansion. This solution is shown to belong to Class IIIb of Stewart and Ellis (1968), and is the first non-vacuum solution found belonging to this class. It is also the first known Bianchi Type VIII perfect fluid solution.
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NOTATION

We will normally follow the notation and conventions of Glass (1977) except in Chapter VI and Appendix C where we follow the notation of Ellis and MacCallum (1969).

Greek indices $\omega, \beta, \mu, \ldots$ run over the four spacetime coordinate labels 0, 1, 2, 3.

There is a summation over repeated indices (Einstein summation convention.)

Covariant derivatives are denoted by $\nabla_\mu$ or a semicolon.

Partial derivatives are denoted by $\partial_\mu$ or a comma.

Parentheses around indices denote symmetrization, square brackets around indices denote antisymmetrization.

The dual operator is $\frac{1}{2} \eta_{\mu\nu}$ and the dual is represented by an asterisk.

The Lie derivative with respect to $\mathbf{q}$ is denoted by $\mathcal{L}_q$.

An overhead caret denotes a unit vector.

Ricci's Identity,

$$2 \, q_{\mu;[\alpha \beta]} \equiv R^{\nu}_{\mu\nu\beta} q_{\nu}$$

Ricci Tensor,

$$R_{\mu\nu} := R^{\alpha}_{\mu\nu\alpha}$$

Curvature Scalar

$$R := R^{\alpha}_{\mu\nu\alpha}$$

In Chapter VI and Appendix C we have:

Latin indices $i, j, k, \ldots$ run from 0 to 3 and label coordinates.

Latin indices $a, b, c, \ldots$ run from 0 to 3 and are tetrad indices.
Greek indices $\alpha, \beta, \gamma, \pi, \ldots$ run over 1, 2, 3.

Commutator

$$\{x, y\} f := x(y f) - y(x f)$$

for all functions $f$.

Ricci Tensor,

$$R_{\alpha\beta} := R_{\alpha\beta}^c$$

Three index alternating symbol.

$$\varepsilon_{\mu\nu\sigma} = \gamma_{\mu\nu\sigma\tau} U^\tau$$

Perfect fluid energy-momentum tensor,

$$T_{\mu}^\nu = (w + p) u_{\mu} u^\nu - \kappa \rho \delta_{\mu}^\nu$$

where $w$ is the energy density, $p$ is the isotropic pressure and $u^\mu$ is the unit 4-velocity of the fluid.

Field Equations,

$$R_{\alpha\nu} = -\kappa \left( T_{\alpha}^\nu - \frac{1}{2} \delta_{\alpha}^\nu T \right)$$

where $\kappa = 1$ (Chapters 4 and 5) $\kappa = -1$ (Chapter 6)
CHAPTER I
INTRODUCTION

Since the advent of Einstein's theory of general relativity in 1916 considerable effort has been expended in the search for exact solutions. Apart from the aesthetic appeal of such solutions they provide us with a valuable tool with which to gain insight into the mathematical possibilities and physical interpretations of the theory.

In this work we will be primarily concerned with solutions that describe self-gravitating matter within the fluid picture. In the first part of the work we investigate isolated, steady-state, axisymmetric "interior" systems, while in the last chapter we will present a non-stationary, anisotropic, cosmological model.

There are several exact solutions describing rotating fluid bodies known in Newtonian physics, the Maclaurin spheroids and the Jacobi ellipsoids being prime examples. The situation in general relativity, however, is somewhat different and such solutions have proved extremely elusive. This is due, in large part, to the considerable difficulty inherent in solving the highly complicated "interior" field equations.

The early work in this field, which concentrated on the static problem, yielded the well known Schwarzschild fluid ball. Other spherically symmetric, static fields with perfect fluid sources have since been constructed (Buchdahl 1959, Glass and Goldman 1978).

In 1937 Van Stockum obtained the general interior solution describing rigidly rotating dust. As a special case of which
he presented the metric of a rigidly rotating cylinder of dust which he was able to show could be matched to the Levi-Civita (1918) cylindrical vacuum field. Further work on rigidly rotating, pressure-free matter was done by Ehlers (1957; 1959) who presented a mapping yielding the general field of uniformly rotating dust from the general static vacuum field. Recently, using a formalism developed by Geroch (1972), Winicour (1975) has obtained all the stationary, axisymmetric, rotating dust solutions. These solutions, useful for studying the effects of differential rotation, contain the Van Stockum solutions as a special case.

The first interior solution describing a finite rotating body was discovered by Wahlquist (1968). This solution, both stationary and axially symmetric, describes a rigidly rotating perfect fluid bounded by a finite surface of zero pressure. Unfortunately, the level surfaces of constant pressure and density turn out to be prolate and hence it is unlikely that any matching vacuum exterior solution can be found. This solution has been rederived, from a different starting point, by Bonanos (1976). In Chapter IV we will present a derivation of the Wahlquist metric with an extended range of arbitrary constants. This solution is obtained using different assumptions than either Wahlquist or Bonanos. In particular, we do not assume that the gravitational field be Petrov type D.

While the number of solutions describing a finite rotating body remains few, the same cannot be said of the number of cosmological models. Indeed, one criticism that has been leveled against relativistic cosmology is that the theory allows too many solutions.

The first relativistic model of the universe was
proposed by Einstein himself in 1917. His model, the so-called "Einstein Universe", depended for its existence upon the inclusion, in the field equations, of the famous (infamous?) cosmological term. Postulated by Einstein in order to obtain a static model—a feature thought to be imperative at that time—the term's inclusion in the field equations is still today a bone of contention among cosmologists.

Later that same year the Einstein model was followed by the de Sitter universe. This model, also a solution of the modified field equations, was interesting in as much as it provided the first theoretical indications that the universe may be expanding.

Non-static models were studied first by the Russian mathematician Friedmann (1922) and later by Lemaitre (1927) and Eddington (1929). Robertson (1936) and Walker (1936) provided the work with a rigorous mathematical formulation.

The Friedmann-Robertson-Walker (F.R.W.) models, later given the name "Big Bang", required the universe to be in an overall state of evolution, either contracting or expanding. This aspect of the theory was given dramatic confirmation only a few years later when in 1929 Hubble announced his empirical discovery of the "expansion of the universe".

It should be noted, however, that the application of general relativity to cosmology has not gone unchallenged. There have been several attempts made at providing alternative cosmological theories, the most notable of these being the "kinematical relativity" of E. A. Milne (1935) and the steady-state theory of Bondi and Gold (1948) and Hoyle (1948). These theories, although well received at the time, have fallen from grace somewhat and observational evidence
seems to be slowly but surely ruling them out as acceptable models of the universe.\(^1\)

The "Big Bang" cosmological models begin with an initial singular state: the explosion. Gamow (1949) argued that, if this explosion really took place, then remnants of it might still be observable. The discovery in 1965 of the cosmic microwave background\(^2\) by Penzias and Wilson and its subsequent interpretation as the radiation predicted by Gamow (Dicke, Roll, Wilkinson and Peebles 1965) gave the theory a tremendous boost.

The high degree of isotropy of the background radiation (Partridge and Wilkinson 1967, Conklin and Bracewell 1967, Penzias and Wilson 1967) coupled with measurements of the isotropy of the X-ray background (Silk 1970) and certain discrete sources (Kristian and Sachs 1966) has served to strengthen the consensus that today's observed universe has an F.R.W. geometry.

However, the last decade or so has seen a marked increase in the study of anisotropic models. Misner (1968) has suggested that the present state of the universe should be largely independent of the "initial conditions" and he has attempted to show that any initial anisotropy would be dissipated in the course of evolution. The possible existence of a primordial magnetic field has been discussed by Zeldovitch (1966) and Thorne (1966) amongst others.

\(^2\) There have been suggestions that this background radiation may not be of cosmological origin. See e.g. F. Hoyle and N.C. Wickramasinghe, Nature, 218, 1124, (1968).
Measurements of a low helium content in the surfaces of some stars (Greenstein 1966, Sargent and Searle 1966) have prompted some authors (e.g. Hawking and Tayler 1966, Thorne 1967) to investigate helium production in anisotropic "Big Bangs". Kantowski and Sachs (1966) have presented some spatially homogeneous, anisotropic dust solutions.

Spatially homogeneous models, both isotropic and anisotropic, have been studied extensively. The most general of such studies are those of Taub (1950), Saunders (1967), Ellis and Mac Callum (1969) and Collins and Stewart (1971). In addition, locally rotationally symmetric space-times containing perfect fluid have been classified by Stewart and Ellis (1968).

The anisotropic, perfect fluid cosmology presented in Chapter VI is an example of a locally rotationally symmetric, perfect fluid space-time. It belongs to Class III b of Stewart and Ellis (1968) and is, as far as the author is aware, the first non-vacuum solution found belonging to this class.

In Chapter II we give a general review of the kinematics of matter flows whose orbit tangents are a combination of the axial and timelike Killing vectors. In this case, where the groups of motions generated by the two Killing vectors is orthogonally transitive, a natural basis for steady-state configurations is constructed. The question of coordinates is discussed and particular metric forms are exhibited for various coordinate systems. In particular, for perfect matter in rigid rotation a natural coordinate system on the 2-space of Killing orbits is shown to exist.

Chapter III extends the work of Chapter II to include differentially rotating configurations. A Newman-Penrose
tetrad is chosen and the spin coefficients and Weyl tensor components evaluated. It is shown that, under the condition of orthogonal transitivity, the matter region must be either Petrov type I or D. This extends a result due to Lind (1974) who has shown that the stationary, axisymmetric, asymptotically flat vacuum region outside the matter can also only be type I or D.

In Chapter IV we begin the search for exact solutions. Starting with the assumption of multiplicative separability for the Killing scalar which acts as a pressure potential, we are able to identically satisfy one of the field equations. This then determines the functional form of the remaining metric components in terms of four unknown functions of a single variable. The separation and solution of the remaining equations is carried out in Appendix B. Using a particular separation technique the remaining field equations are reduced to a set of ordinary differential equations for these four functions. Two of these functions are shown to be coordinates on the 2-space. The general solution for the remaining two functions is given. This exact solution turns out to be the solution discovered by Wahlquist (1968) with an increased number of arbitrary constants. These extra constants, however, do not modify the equation of state which remains $w + 3p = \text{constant}$.

The assumption of multiplicative separability is extended in Chapter V to include all metric components. It is shown that under this assumption the possible solutions divide up into two distinct classes. The introduction of an imaginary constant into a particular subset of one of these classes enables us to obtain a metric form for a non-stationary fluid.
The metric form obtained in Chapter V is studied in Chapter VI. The conditions under which the metric describes a perfect fluid are derived. The resulting metric is shown to describe a locally rotationally symmetric space-time with non-zero shear and expansion. Expressions for the pressure and density are derived and, although the pressure is negative, it is shown that the weak, dominant and strong energy conditions are all satisfied.

In Appendix A we give a brief summary of the Newman-Penrose null tetrad formalism. A null tetrad is chosen for the form of the metric studied in Chapter V and explicit expressions for the spin coefficients and Weyl tensors components are given.

In Appendix C we give a review of the orthonormal tetrad technique we used to study the metric in Chapter VI.
CHAPTER II

STEADY-STATE, SELF GRAVITATING MATTER CONFIGURATIONS

In this chapter we shall examine space-times which are generally considered as models of axisymmetric, rotating, steady-state configurations of self gravitating matter. The presentation follows that of Glass (1977). We shall assume that the space-time \((M,g)\) is stationary, axially symmetric and asymptotically flat at spatial infinity. These assumptions imply the existence of two Killing vector fields on \(M\). A vector field \(\xi^\mu\) which is everywhere timelike, asymptotically unit- and has open timelike lines as orbits and a vector field \(\eta^\mu\) which is spacelike, has closed trajectories (topologically circles) and is asymptotically orthogonal to \(\xi^\mu\).

The vector field \(\eta^\mu\) will be zero on a two-dimensional axis of rotation.

Under the above conditions one finds that the vector fields \(\xi^\mu\) and \(\eta^\mu\) commute.

\[
\mathcal{L}_\xi \eta^\mu = - \mathcal{L}_\eta \xi^\mu = 0 \tag{2.1}
\]

We impose the further requirement that the matter occupy a single simply-connected region as in the case of a galactic nucleus or an ordinary star.

The matter flow is assumed to be non-convective with tangent

\[
u^\mu = \alpha \xi^\mu + \beta \eta^\mu \tag{2.2}
\]

where

\[
\mathcal{L}_\xi \nu^\mu = \mathcal{L}_\eta \nu^\mu = 0 \tag{2.3}
\]
and is normalised such that

\[ U^\mu U_\mu = 1 \]  \hspace{1cm} (2.4)

from which it follows

\[ \alpha^2 \lambda_{00} + 2\alpha \beta \lambda_{01} + \beta^2 \lambda_{11} = 1 \]  \hspace{1cm} (2.5)

where

\[ \lambda_{00} := \xi^\mu \xi_\mu \quad \lambda_{01} := \xi^\mu \eta_\mu \quad \lambda_{11} := \eta^\mu \eta_\mu \]

It is now useful to construct a spacelike vector \( V^\mu \) which is orthogonal to \( U^\mu \).

We define

\[ V^\mu := \gamma \xi^\mu + \delta \eta^\mu \]  \hspace{1cm} (2.6)

where the scalars (Boyer 1966) \( \gamma \) and \( \delta \) are given by

\[ \gamma := \eta^\mu U_\mu = \alpha \lambda_{01} + \beta \lambda_{11} \]  \hspace{1cm} (2.7a)

\[ \delta := \xi^\mu U_\mu = \alpha \lambda_{00} + \beta \lambda_{01} \]  \hspace{1cm} (2.7b)

The vectors \( (\xi^\mu, \eta^\mu) \) and \( (U^\mu, V^\mu) \) are related by the transformation

\[ \begin{pmatrix} U^\mu \\ V^\mu \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \xi^\mu \\ \eta^\mu \end{pmatrix} \]  \hspace{1cm} (2.8)

which is a Lorentz transformation in the tangent space of the 2-surface spanned by \( \xi^\mu \) and \( \eta^\mu \).

(2.4) requires

\[ \alpha \delta - \beta \gamma = 1 \]  \hspace{1cm} (2.9)
The norm of the spacelike vector $V^\alpha$ is
\[ V^2 = - V^\alpha V_\alpha = \lambda_{00}^2 - \lambda_{00} \lambda_{11} \]  \hspace{1cm} (2.10)
and it is assumed that $V$ is zero only on the axis of rotation.

The orbits of $\xi^\alpha$ and $\eta^\sigma$ form a two-dimensional manifold (hereafter referred to as the "2-space") with induced metric
\[ h_{\mu\nu} = \eta_{\mu\nu} - U_{\mu} U_{\nu} + \hat{U}_\mu \hat{U}_\nu \]  \hspace{1cm} (2.11)
h_{\mu\nu} acting on free indices projects all tensor fields onto the 2-space where they have the property
\[ \xi (\tau^{r...r} e^{...e} \sigma) = 0 \]
\[ \eta (\tau^{r...r} e^{...e} \sigma) = 0 \]
In general, however, only covariant tensor fields on the 2-space are Lie derived with respect to $U^\alpha$ and $V^\alpha$.

The alternating tensor on the 2-space is given by
\[ \epsilon_{\mu\nu} := \eta_{\mu\nu e\sigma} U^e \hat{V}^\sigma = V^{-1} \eta_{\mu\nu e\sigma} \tau^e \eta^\sigma \]  \hspace{1cm} (2.12)
with the property
\[ \epsilon_{\mu\rho} \epsilon^{\mu\sigma} = h_{\rho} \sigma \]

It is possible to express the Einstein field equations as a set of differential equations involving only scalar and tensor fields on this 2-manifold. Here, however, since certain physical quantities connected with the matter flow (e.g. the shear and vorticity) have components outside the 2-space, we will not restrict ourselves
entirely to the space of Killing orbits.

We have, so far, constructed eight scalar fields; $\alpha$, $\beta$, $\gamma$, $\delta$, $\lambda_{00}$, $\lambda_{01}$, $\lambda_{11}$, and $\psi$, connected by four relations, equations (2.2), (2.7) and (2.10). We choose the set $\alpha$, $\beta$, $\gamma$, $\delta$ and $\psi$ (of $\alpha$, $\beta$, $\gamma$, and $\delta$ only three are independent) to describe the properties of the spacetime. (Other authors, e.g., Hansen and Winicour 1975, have chosen the set $\lambda_{00}$, $\lambda_{01}$, $\lambda_{11}$ and $\eta^2 = \beta/\alpha$).

The relationships between these scalars can be inverted to give

\begin{align}
\lambda_{00} &= \delta^2 - \beta^2 \psi^2, \\
\lambda_{01} &= \alpha \beta \psi^2 - \gamma \delta, \\
\lambda_{11} &= \gamma^2 - \alpha^2 \psi^2.
\end{align}

(2.13a) (2.13b) (2.13c)

We can now proceed to describe the kinematical quantities associated with the matter orbits in terms of this set of scalars.

Recall the matter orbits have unit tangent

\[ u^\mu = \alpha \xi^\mu + \beta \eta^\mu. \]

(2.2)

The rate of change of $U^\mu$ is given by

\[ u_{\nu;\mu} = a_{\mu} u_{\nu} + \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{\Theta}{3} \gamma_{\mu\nu}. \]

(2.14)

where

\[ \gamma_{\mu\nu} := \partial_{\mu\nu} - u_{\mu} u_{\nu}, \]

(2.15)

\[ a_{\mu} := u_{\mu;\nu} u^{\nu}, \]

(2.16)

\[ \Theta := u_\mu;\nu^{\mu}. \]

(2.17)
\[ W_{\mu \nu} := u [ \gamma_{\mu} ; \gamma_{\nu} ] - a [ \gamma_{\mu} \gamma_{\nu} ] \]  
(2.18)

\[ \sigma_{\mu \nu} := u (\gamma_{\mu} \gamma_{\nu}) - a (\gamma_{\mu} \gamma_{\nu}) - \frac{3}{5} \gamma_{\mu \nu} \]  
(2.19)

An immediate consequence of (2.2) is that

\[ \theta = 0 \]  
(2.20)

The tensor \( \gamma_{\mu \nu} \) is the metric tensor on the 3-space quotient to \( U^a \).

The vector \( a_{\mu} \) is the acceleration vector where \( (2.4) \rightarrow a_{\mu} U^a = 0 \).

The fact that \( s^\mu \) and \( \eta^\mu \) are commuting Killing vectors implies the well-known relationships

\[ s^\mu \nabla_\mu s_\nu = -\frac{1}{2} \nabla_\nu (\lambda_{\nu}) \]  
(2.21a)

\[ \eta^\mu \nabla_\mu \eta_\nu = -\frac{1}{2} \nabla_\nu (\lambda_{\nu}) \]  
(2.21b)

\[ s^\mu \nabla_\mu \eta_\nu = \eta^\mu \nabla_\mu s_\nu = -\frac{1}{2} \nabla_\nu (\lambda_{\nu}) \]  
(2.21c)

Use of these relationships together with equations (2.5) and (2.7) enables us to write

\[ a_{\mu} = \delta a_{\mu} - \gamma_{\beta} \beta_{,\mu} \]

\[ = \alpha^{-1} a_{,\mu} - \alpha \gamma W_{\mu} \]  
(2.22)

where \( W_{\mu} \), the "differential rotation vector", is defined as

\[ W_{\mu} := \left( \frac{\beta}{\alpha} \right)_{,\mu} \]  
(2.23)

When \( W_{\mu} = 0 \) the rotation is said to be rigid. This implies \( \beta = \Sigma_{o} \alpha \) \((\Sigma_{o} = \text{constant})\), and in this case (2.22) reduces to

\[ a_{\mu} = \left( \log \alpha \right)_{,\mu} \]  
(2.24)
The tensor \( \sigma_{\mu \nu} \) is the rate of shear tensor which, using equations (2.2) and (2.6) together with (2.23), can be expressed in the form

\[
\sigma_{\mu \nu} = \alpha^2 \, V_{(\mu} \, W_{\nu)} \quad (2.25)
\]

from which it can be seen that the shear vanishes at a given point if and only if the rotation is rigid.

The shear scalar is

\[
\sigma^2 := \frac{1}{2} \, \sigma_{\mu \nu} \, \sigma^{\mu \nu} = \frac{1}{4} \, \alpha^2 \, V^2 \, W^2 \quad (2.26)
\]

where

\[
W^2 = - W_{\mu} \, W^{\mu} \quad (2.27)
\]

Before we consider the vorticity tensor \( \omega_{\mu \nu} \), it is useful to make a short digression to discuss the action of the isometry group generated by \( \xi^\mu \) and \( \eta^\mu \). In particular we wish to consider the conditions under which the group action is orthogonally transitive i.e. that there exist a family of two-dimensional surfaces which are orthogonal to the surfaces of transitivity of the group. These conditions are contained in a theorem due to Carter (1969). We shall show that a system containing perfect matter, with non-convective flow, and which has an axis of rotation, satisfies both of Carter's criteria.

Carter's first condition requires that the quantity

\[
(\iota_i) \, F^\mu = (\iota_i) \, \lambda^\nu \, T_{\nu}^{\mu} \quad (2.28)
\]

called the flux vector of the group - where \((i) \, \lambda \) are the generators
of the isometry group and \( T_{\mu \nu} \) is the stress energy tensor of the matter - be tangent to the surfaces of transitivity of the group. For perfect matter with \( U^\nu = u^\nu \delta^\nu_\mu + \rho \eta^\nu \) this condition is clearly satisfied.

Carter's second condition requires that the surfaces of transitivity become degenerate on a non-empty subset \( \mathcal{F} \) where the group has fixed points. This condition is satisfied by virtue of the existence of a rotation axis. Hence both of Carter's criteria have been satisfied and thus orthogonal transitivity has been proved. In the remainder of this section we will assume the group action to be orthogonally transitive.

We will now proceed to discuss the vorticity tensor \( \omega_{\mu \nu} \). Using equations (2.2), (2.6) and (2.21) together with the equation

\[
\omega_{\mu \nu} \xi^\nu = - \left( \frac{\beta}{\lambda} \right) \omega_{\mu \nu} \eta^\nu \tag{2.29}
\]

it is possible to write \( \omega_{\mu \nu} \) as

\[
\omega_{\mu \nu} = \delta^{\nu}_{\sigma} V^{-2} \nabla_{[\sigma} X_{\nu]} \tag{2.30}
\]

where the vector \( X_{\mu} \) is defined as

\[
X_{\mu} := \left( \frac{\gamma}{\delta} \right), \mu \tag{2.31}
\]

The vorticity vector lies in the 2-space and is given by

\[
\omega_{\mu} := \omega_{\mu \nu} U^\nu = \frac{1}{2} \delta^2 V^{-1} \varepsilon_{\mu \nu} X^\nu \tag{2.32}
\]

3. This quantity transforms as an ordinary vector with respect to \( \mu \) in the manifold and as a covariant vector with respect to (i) under a change of basis of the Lie algebra of the group.
from this expression it is clear that the vector $X^\mu$ is always orthogonal to the vorticity vector,

$$\omega_\mu X^\mu = 0$$  \hspace{1cm} (2.33)

The fact that the vorticity vector defines the instantaneous axis of local rotation leads to the interpretation of $X^\mu$ as a "local rotation arm".

The vorticity scalar is

$$\omega^2 = -\omega_\mu \omega^\mu = \frac{1}{2} \omega_{\mu\nu} \omega^{\mu\nu} = \frac{1}{4} \delta^{\nu\rho} \nabla^2 X^\nu$$  \hspace{1cm} (2.34)

The scalars $\sigma$ and $\omega$ vanish if, and only if, the corresponding tensors vanish.

Since the vorticity vector and the local rotation arm are always orthogonal an orthonormal tetrad has been constructed which provides a natural basis for the spacetime. Thus the metric can be written,

$$g_{\mu\nu} = u^\mu u_\nu - \hat{V}^\mu \hat{V}_\nu - \hat{X}^\mu \hat{X}_\nu - \hat{\omega}^\mu \hat{\omega}_\nu$$  \hspace{1cm} (2.35)

where the orientation is given by

$$\eta^\nu_{\mu\nu\rho} u^\nu \hat{X}^\rho \hat{\omega}^\sigma = 1$$  \hspace{1cm} (2.36)

It now remains to choose a coordinate system.

(A) Killing coordinates.

Choose $x^0 = t$ as a group parameter along the trajectories of the timelike Killing vector $\xi^\mu$ and $x^1 = \varphi$ as a group parameter along the orbits of the spacelike Killing vector $\eta^\mu$. 
\[ \xi^\mu = \delta^\mu_0, \quad \eta^\mu = \delta^\mu_1 \]  

(2.37)

The line element can now be written

\[ ds^2 = \lambda_{00} \, dt^2 + 2 \lambda_{0i} \, dt \, d\psi + \lambda_{ii} \, d\psi^2 
- (\dot{\chi}_\mu \dot{\chi}_\nu + \omega_\mu \omega_\nu) \, dx^\mu \, dx^\nu \]  

(2.38)

(B) Comoving coordinates.

Choose \( U^\mu = \alpha \xi^\mu \), which is equivalent to the coordinate transformation from Killing coordinates to comoving coordinates given by

\[ a \, \tau = dt \]  

(2.39a)

\[ d \bar{\varphi} = d\varphi - (\dot{\beta}/\alpha) \, dt \]  

(2.39b)

where \( d \bar{\varphi} \) is exact only in the case of rigid rotation (\( \dot{\beta} = \lambda_{00} \)).

The line element now becomes

\[ ds^2 = \alpha^{-2} \left( a \, \tau - \alpha \, Y \, d \bar{\varphi} \right)^2 - \alpha^2 \, \gamma^{ij} \, d \bar{\varphi}^2 
- \chi^{-1} \left( \chi_i \chi_j + 4 \gamma^{ij} \delta_{\bar{\nu}} \omega_\nu \right) \, dx^i \, dx^j \]  

(2.40)

(C) Rigid rotation coordinates.

For rigid rotation the matter 4-velocity is proportional to a timelike Killing vector field

\[ U^\mu = \alpha \left( \xi^\mu + \lambda_{00} \eta^\mu \right) \]  

(2.41)

Substituting this into Ricci's identity yields the equation

\[ \omega_{[\mu \nu]} - 2 \omega_{[\mu \alpha \nu]} \sigma = -\frac{1}{2} \eta_{\mu \nu \sigma \rho} R^{\sigma \rho} U^\rho \]  

(2.42)

Now since \( a_\mu = a^{-1} \, \xi_\mu \), this equation becomes

\[ \alpha^2 (\alpha^{-2} \omega_{\mu \nu}) ; \nu = -\frac{1}{2} \eta_{\mu \nu \rho \sigma} R^{\sigma \rho} U^\rho \]  

(2.43)
Hence it follows that for perfect matter in rigid rotation $\alpha^{-2} \omega_\mu$ is locally a gradient. In this case there exists a natural coordinate system on the 2-space.

We choose

$$\alpha^{-2} \omega_\mu = \Lambda_0 \nabla_\mu \xi \quad (-\infty < \xi < \infty) \quad (2.44a)$$

$$\gamma/\xi = \Lambda_0 \gamma^2 \left( 1 + \Lambda_0 \gamma^2 \right)^{-1} \quad (-\infty < \gamma < \infty) \quad (2.44b)$$

where $\Lambda_0 > 0$.

In comoving coordinates the line element is given by

$$dS^2 = \alpha^{-2} \left( dt - \Lambda_0 \gamma^2 d\phi \right)^2 - \alpha^{-2} \gamma^2 d\phi^2 \quad (2.45)$$

$$- 4 \Lambda^{-1} \left( 1 + \Lambda \gamma^2 \right)^{-4} \Lambda_0^2 \left( \gamma^2 d\gamma^2 + \alpha^2 \gamma^2 d\xi^2 \right)$$

The choice of coordinates for $\gamma/\xi$ is restricted by the inequalities of Hansen and Winicour (1975)

$$\lambda_\infty + \Lambda_0 \lambda_\infty > 0 \quad (2.46a)$$

$$\lambda_\infty + \Lambda_0 \lambda_\infty \leq 0 \quad (2.46b)$$

Substituting into these inequalities from (2.13) leads to

$$\gamma/\xi > 0 \quad (2.47)$$

The choice $\Lambda_0 y^2 (1 + \Lambda_0 y^2)^{-1}$ allows a particularly simple form for some of the metric components.

It is of interest, in rotating systems, to determine the velocity of the matter as measured by a locally non-rotating observer. In any stationary spacetime it is possible to construct a closed 1-form, $t_\mu dx^\mu$, which, in a simply connected region, labels a
family of spacelike hypersurfaces whose orthogonal trajectories are the world lines of locally non-rotating observers, (Glass 1977).

Beginning with the timelike Killing vector \( \xi^\nu \) we define a twist vector and a twist bivector by

\[
\Omega^\mu := \frac{1}{2} \eta^{\mu \nu \rho \sigma} \xi_\nu \xi_\rho \xi_\sigma \quad (2.48a)
\]

\[
\mathcal{L}_{\mu \nu} := \lambda^{-\frac{1}{2}} \eta_{\mu \nu \rho \sigma} \xi^\rho \Omega^\sigma \quad (2.48b)
\]

The vector \( \mathcal{L}_\mu \) is curl free and is zero if, and only if, the spacetime is static.

Let \( S_\mu \) be a solution of the curl equation

\[
S_{[\mu ; \nu]} = \lambda^{-\frac{1}{2}} \Omega_{\mu \nu} \quad (2.49)
\]

with \( S_\mu \) and \( \xi^\nu \) orthogonal. Solutions always exist (locally) since

\[
( \eta^{\mu \nu \rho \sigma} \lambda^{-\frac{1}{2}} \mathcal{L}_{\mu \nu} ) ; \sigma = 0 \quad (2.50)
\]

which follows from

\[
\frac{1}{2} \mathcal{L}^\mu = 0 \quad (2.51)
\]

and

\[
\mathcal{L}^\mu : \mu = 2 \lambda^{-1} \lambda_{\alpha \beta \gamma} \mathcal{L}^\nu = 0 \quad (2.52)
\]

The closed 1-form is then constructed as

\[
\xi_\mu dx^\mu := \lambda^{-1} \xi_\mu dx^\mu - S_\mu dx^\mu \quad (2.53)
\]

Closure follows from the fact that

\[
\xi_{[\mu ; \nu]} = \lambda^{-\frac{1}{2}} \xi_{[\mu \lambda_\nu \nu]} + \lambda^{-\frac{1}{2}} \mathcal{L}_{\mu \nu} \quad (2.54)
\]

For spacetimes possessing an orthogonally transitive
pair of commuting Killing vectors $\xi^\mu$ and $\eta^\mu$ we have

$$\xi_\mu \, dx^\mu = \, dt$$  \hspace{1cm} (2.55a)

and

$$S_\mu \, dx^\mu = \left( \frac{\lambda^0}{\lambda_{\infty}} \right) \, d\varphi$$  \hspace{1cm} (2.55b)

in Killing coordinates, and generally

$$\xi_\mu = \sqrt{-} \left( - \lambda_\infty \, \xi_\mu + \lambda_0 \, \eta_\mu \right)$$  \hspace{1cm} (2.56a)

and

$$S_\mu = \sqrt{-} \left( \frac{\lambda^0}{\lambda_{\infty}} \right) \left( \lambda_\infty \, S_\mu - \lambda_0 \, \eta_\mu \right)$$  \hspace{1cm} (2.56b)

The 4-velocity $U^\mu$ can now be written in the Minkowski-like form

$$U^\mu = \left( 1 - v^2 \right)^{-1/2} \left( \xi^\mu + \nu \, \eta^\mu \right)$$  \hspace{1cm} (2.57)

The speed of matter, $\nu$, measured by a locally non-rotating observer is given by

$$\left( 1 - \nu^2 \right)^{-1/2} = U^\mu \xi_\mu$$  \hspace{1cm} (2.58)

or

$$\nu^2 = \sqrt{-} \frac{\lambda^2}{\lambda_{\infty}} \left( \frac{\beta}{\alpha} + \frac{\lambda^0}{\lambda_{\infty}} \right)^2$$  \hspace{1cm} (2.59)

in agreement with Bardeen(1970).
CHAPTER III
Petrov Classification of Rotating, Steady-State Matter Systems.

In the last chapter it was shown that the assumption of orthogonally transitive action of the group of motions generated by \( \hat{f}^\alpha \) and \( \eta^\mu \), together with the assumption of comoving coordinates, allows the line element to be written as

\[
\begin{align*}
\mathbf{ds}^2 &= \alpha^{-2} \left( dT - \alpha \sigma d\Phi \right)^2 - \kappa^2 V^2 d\bar{\Phi}^2 \\
&\quad - \chi^{-2} \left( \chi_\mu \chi_\nu + 4V^2 \delta^{-4} \omega_\mu \omega_\nu \right) dx^\mu dx^\nu 
\end{align*}
\]  
(3.1)

where the coordinates \( T \) and \( \Phi \) are defined by

\[
\begin{align*}
\eta^\mu &= \alpha \delta^\mu_T, \\
\eta^\mu &= \delta^\mu_\Phi
\end{align*}
\]  
(3.2)

As we saw in the last chapter, this coordinate system is useful in the rigid rotation case as it allows the line element to be written in a particularly simple form. However, in the case of differential rotation, comoving coordinates are inappropriate since the metric components become dependent on the time coordinate in addition to the 2-space coordinates.

With Killing coordinates the line element is written

\[
\mathbf{ds}^2 = \lambda_{00} dt^2 + 2 \lambda_{01} dt d\theta + \lambda_{02} d\theta^2 + \lambda_{03} d\varphi^2 + \lambda_{04} d\varphi^2 
\]  
(3.3)

It therefore remains to choose coordinates on the 2-space. One possible choice is isothermal coordinates

\[
\begin{align*}
\eta_{\mu\nu} dx^\mu dx^\nu &= -G^{-2} \left( dp^2 + dq^2 \right) 
\end{align*}
\]  
(3.4)

where the coordinates \( p \) and \( q \) are defined by the Beltrami conditions (Spivak 1975), which are the necessary and sufficient conditions for
the existence of isothermal coordinates
\[ h^{\mu \nu} p_{\mu \nu} q_{\nu \nu} = 0 \]  
(3.5)
\[ h^{\mu \nu} p_{\mu \nu} p_{\nu \nu} = h^{\mu \nu} q_{\mu \nu} q_{\nu \nu} \]  
(3.6)

With this choice, the metric contains four unknown functions; \( \lambda_{00} \), \( \lambda_{1i} \), \( \lambda_{ii} \) and \( Q \), of the coordinates \( p \) and \( q \).

However, for this choice, the explicit relationship between the physical vectors of the matter flow and the 2-space is lost. (It is, of course, recovered when a solution of the field equations is obtained).

As was shown in the last section, it can be useful to construct coordinates from the vectors \( X_{\mu} \) and \( \omega_{\mu} \) since they can be eigenvectors of the Ricci tensor. Hansen and Winicour (1977) have shown that in the case of differential rotation,
\[ \lambda_{00} > 0 \ \Rightarrow \ \lambda_{00} + \Omega \lambda_{0i} > 0 \]  
(3.7a)
and
\[ \Omega > 0 \ \Rightarrow \ \lambda_{0i} / \lambda_{ii} < 0 \]  
(3.7b)

where \( \Omega := \beta / \alpha \).

Inequality (3.7a) holds outside any ergoregions of the matter (in this work \( \beta^{\mu} \) is considered to be timelike everywhere) and (3.7b) holds when the matter has a uniform sense of rotation. It follows from (2.13) that these inequalities imply
\[ \gamma / \delta > 0 \]  
(3.8)

One can now make the choice
\[ \gamma / \delta = \Sigma_0 y^2 \quad (0 \leq y < \infty) \]  
(3.9)
with \( \mathcal{N} \) a positive constant.

Thus

\[ X_\mu \, dx^\mu = 2 \mathcal{N} \Omega \, dy \]  

(3.10)

Since any covariant vector on the 2-space is proportional to a gradient, the vorticity vector can be written as

\[ \omega_\mu \, dx^\mu = S_0 \delta^2 \cdot E \, dz \quad (-\infty < z < \infty) \]  

(3.11)

thus defining the \( z \) coordinate and the proportionality function \( S_0 \delta^2 \cdot E \).

Hence we can now write the line element (3.1) as

\[ ds^2 = S_0 \, dt^2 + 2 \lambda \, dt \, d\phi + \lambda \, d\phi^2 \]

\[- 4 \mathcal{N} \delta \left( \frac{v^2}{2} \, dy^2 + V^2 \, E^2 \, dz^2 \right) \]  

(3.12)

We thus have five unknown functions appearing in the metric.

In the case of perfect matter there is an equation for the curl of \( \omega_\mu \),

\[ \mathcal{R}_{\mu
u} \omega_\mu \, \hat{V}^\nu = (\delta^2 \cdot V^{-1} \cdot X^\nu)_{;\nu} + \delta^2 \cdot (V^{-1})_{;\nu} \cdot X^\nu \]

\[- (\alpha^2 \cdot V \cdot W^\nu)_{;\nu} - \alpha^2 \cdot V_{,\nu} \cdot W^\nu \]

\[- 2 \delta^2 \cdot V^{-1} \cdot a_\nu \cdot X^\nu - 2 \alpha^2 \cdot V_{,\nu} \cdot W^\nu = \mathcal{O} \]  

(3.13)

which, using equation (2.32), can be rewritten as

\[ 2 \epsilon^{\mu\nu} \omega_\mu ;\nu = 2 \delta^2 \cdot V^{-1} \cdot a_\mu \cdot X^\mu + \alpha^2 \cdot V_{,\mu} \cdot W^\mu \]

\[ + (\alpha^2 \cdot V \cdot W^\mu)_{;\mu} + 2 \alpha^2 \cdot V \cdot a_\mu \cdot W^\mu \]  

(3.14)

This provides us with a condition on the function \( E \) which appears in the metric (3.12).
Using equation (3.11) we have

\[ E^\mu E,\mu X^\mu = V^2 \delta^{-2} \left\{ \alpha^2 W^\mu ;,\mu + 2\alpha^2 V^\mu V,\mu W^\mu + 2\alpha^2 a,\mu W^\mu \right\} + 2\alpha \delta \Omega X^2 + 4a,\mu X^\mu \]  

(3.15)

The metric (3.12) reduces to a rigid rotation metric when

\[ \Omega = \mathcal{R}_0, \quad E = \alpha^2 \delta^{-2} \]

In this case equation (3.15) is satisfied identically.

With either isothermal or \((y,z)\) coordinates on the 2-space, \(\mathcal{R}_0\) is one of the unknown functions, often given in terms of an equation of state. For \((y,z)\) coordinates it appears explicitly in equation (3.15) and in isothermal coordinates its appearance is necessary to identify the pressure and density parts of the Ricci tensor.

We now proceed to determine the possible Petrov types of matter configurations for which the fluid flow vector \(U^\mu\) is given by (2.2), and where the action of the isometry group generated by \(\xi^\mu\) and \(\eta^\mu\) is orthogonally transitive. In this case the vorticity vector and the "local rotation arm" provide a basis for the 2-space and the full metric can be written as

\[ g_{\mu\nu} = U_\mu U_\nu - \hat{\xi}_\mu \hat{\xi}_\nu - \hat{\chi}_\mu \hat{\chi}_\nu - \hat{\omega}_\mu \hat{\omega}_\nu \]  

(2.35)

We now choose a Newman–Penrose (1962) tetrad of null vectors

\[ \xi^\mu = \frac{1}{\sqrt{2}} \left( U^\mu - \hat{\chi}_\mu \right) \]  

(3.16a)

\[ \eta^\mu = \frac{1}{\sqrt{2}} \left( U^\mu + \hat{\chi}_\mu \right) \]  

(3.16b)

\[ \mathcal{M}^\mu = \frac{1}{\sqrt{2}} \left( \hat{\Omega}_\mu + i \hat{\omega}_\mu \right) \]  

(3.16c)
with the orthogonality properties
\[ l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1 \]
\[ l_\mu l^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = \eta_\mu m^\mu = 0 \]
\[ l_\mu m^\mu = l_\mu \bar{m}^\mu = m_\mu m^\mu = m_\mu \bar{m}^\mu = 0 \]

These equations imply the completeness relation
\[ \varrho_{\mu\nu} = 2 l(\mu n_\nu) - 2 m_\mu \bar{m}_\nu \]

(3.18)

Consider the tetrad transformation
\[ l^\mu \rightarrow -l^\mu \]
\[ n^\mu \rightarrow -n^\mu \]

(3.20)

At any point \( x \), this transformation reflects through the origin the subspace of the tangent space at \( x \) spanned by the Killing fields and leaves the description of the geometry invariant. (This transformation is equivalent to the coordinate transformation, in Killing coordinates, \( t \rightarrow -t, \phi \rightarrow -\phi \). All the Newman-Penrose scalars (see Newman-Penrose 1962 or Appendix A) which change sign under this transformation must be zero
\[ \sigma = \lambda = \rho = \mu = \gamma = \epsilon = 0 \]
\[ \overline{\Phi}_1 = \overline{\Phi}_3 = 0 \]
\[ \Phi_{01} = \Phi_{12} = 0 \]

One more spin coefficient can be obtained. Since \( l^\mu \) and \( n^\mu \) are surface forming it follows that
\[ \tau + \overline{\tau} = 0 \]

(3.21)
The vanishing of \( \Psi_1 \) and \( \Psi_2 \) shows the tetrad choice (3.16) to be a natural one in the sense that it automatically provides an eigenvector of the Weyl tensor

\[
C^{\mu\nu}_{\alpha\beta} M_{\mu\nu}^{\alpha\beta} = 4 \Psi_2 \Psi^{\mu\nu}
\]

(3.22)

where

\[
M_{\mu\nu}^{\alpha\beta} = 2 \epsilon^{[\mu}_{\eta} \eta^{\beta]} - 2 \eta_{\mu} \epsilon^{\alpha\beta}
\]

(3.23)

(see Appendix A)

The Petrov classification is given by the multiplicity of the roots of

\[
\Psi_0 b^4 + 4 \Psi_1 b^3 + 6 \Psi_2 b^2 + 4 \Psi_3 b + \Psi_4 = 0
\]

(3.24)

\( \Psi_0 \neq 0 \).

This equation is a quartic with algebraic expressions as coefficients.

Defining (D'Inverno and Russell-Clark 1971)

\[
I := \Psi_0 \Psi_4 - 4 \Psi_1 \Psi_3 + 3 \Psi_2^2
\]

(3.25)

\[
G := \Psi_0^2 \Psi_3 - 3 \Psi_0 \Psi_1 \Psi_2 + 2 \Psi_3 \Psi_4
\]

(3.26)

\[
J := \begin{vmatrix}
\Psi_0 & \Psi_1 & \Psi_2 \\
\Psi_1 & \Psi_2 & \Psi_3 \\
\Psi_2 & \Psi_3 & \Psi_4 \\
\end{vmatrix}
\]

(3.27)

\[
H := \begin{vmatrix}
\Psi_0 & \Psi_1 \\
\Psi_1 & \Psi_2 \\
\end{vmatrix}
\]

(3.28)
\[ K := \Psi_0^2 I - 12 H^2 \]  \hspace{1cm} (3.29)

We see the necessary and sufficient conditions for (3.24) to have

(i) at least two equal roots is \( I^3 = 27J^2 \)

(ii) at least three equal roots is \( I = J = 0 \)

(iii) four equal roots is \( G = H = I = 0 \)

(iv) two pairs of equal roots is \( G = K = 0 \)

For spacetimes under consideration here, with \( \Psi_1 = \Psi_3 = 0 \) we have

\[ I = \Psi_0 \Psi_4 + 3 \Psi_2^2 \]  \hspace{1cm} (3.30)

\[ J = \Psi_2 (\Psi_0 \Psi_4 - \Psi_2^2) \]  \hspace{1cm} (3.31)

\[ G = 0 \]  \hspace{1cm} (3.32)

\[ H = \Psi_0 \Psi_2 \]  \hspace{1cm} (3.33)

\[ K = \Psi_0^2 (\Psi_0 \Psi_4 - 9 \Psi_2^2) \]  \hspace{1cm} (3.34)

Three cases can be distinguished (\( \Psi_0 \neq 0 \))

(a) \( \Psi_0 \neq 0, \) \( \Psi_4 \neq 0 \) implies type I or D

Note that \( I^3 = 27J^2 \) can be written

\[ \Psi_0 \Psi_4 (\Psi_0 \Psi_4 - 9 \Psi_2^2) = 0 \]  \hspace{1cm} (3.35)

(b) \( \Psi_0 = 0, \) \( \Psi_4 \neq 0 \) or \( \Psi_4 = 0, \) \( \Psi_0 \neq 0 \)

implies type II
(c) $\psi_0 - \psi_4 = 0$ implies type D

Cases (b) and (c) can be eliminated. In case (b), $\xi^\mu$ is one of the principal null vectors (p.n.v.) when $\psi_0 = 0$, $\psi_4 \neq 0$ and $n^\mu$ is one of the p.n.v. when $\psi_4 = 0$, $\psi_0 \neq 0$. In case (c) both $\xi^\mu$ and $n^\mu$ are p.n.v.. Since $\xi^\mu$ and $n^\mu$ are each a linear combination of the two Killing vectors $\xi^\mu$ and $n^\mu$ it follows that on the axis of rotation ($\eta^\mu = 0$) both $\xi^\mu$ and $n^\mu$ are aligned with $\xi^\mu$ and are therefore timelike on the rotation axis. Thus $\xi^\mu$ and $n^\mu$ can never be principal null vectors of the Weyl tensor.¹

When $\psi_2 = 0$ the matter region is conformally flat or type I. Collinson (1976) has shown that every conformally flat, axisymmetric spacetime (with orthogonal transitivity) is necessarily static.

The rotating, steady-state configurations belonging to case (a) are as follows.

(i) Type I with

$$\psi_0 \psi_4 = 9 \psi_2^2 \quad \psi_0, \psi_4, \psi_2 \neq 0$$

(ii) Type D with

$$\psi_0 \psi_4 = 9 \psi_2^2 \quad \psi_0, \psi_4, \psi_2 \neq 0$$

The remaining case is

(iii) Type I with $\psi_2 = 0$ and $\psi_0 \neq 0$, $\psi_4 \neq 0$.

When $\psi_2 = 0$ the electric and magnetic parts of the Weyl tensor with respect to $U^\mu$ have components only in the 2-space. (see Appendix A)

Lind (1974) has shown that the stationary, axially
symmetric, vacuum region outside the matter also can only be either Petrov type I or D (Kerr) when one imposes the condition of asymptotic flatness.

Much of the material in this chapter appeared in Glass and Wilkinson (1978).
CHAPTER IV

EXACT, PERFECT FLUID INTERIOR SOLUTION

We now turn our attention towards finding exact solutions of the Einstein field equations for a stationary, axially symmetric, rigidly rotating perfect fluid. Our starting point will be the metric \((2.45)\), (with a slight change in notation \(\alpha \rightarrow f\), \(x \rightarrow \omega\)) however, we will choose isothermal coordinates \(X\) and \(Y\) \((3.4)\) on the 2-space.

The metric therefore takes the form, \((t, \phi, x, y) = (0, 1, 2, 3)\)

\[
d s^2 = f^2 (d t - \omega d \phi)^2 - f^{-2} \nu^2 d \phi^2 \\
- f^{-2} Q^2 (d x^2 + d y^2)
\]  \((4.1)\)

where the functions \(f, \omega, \nu\) and \(Q\) are functions of \(X\) and \(Y\) only.

The function \(\nu\) is required to vanish on the axis of rotation and to this end we assume \(\nu\) to be of the form

\[
\nu^2 = h_2(x) \cdot h_3(y)
\]  \((4.2)\)

where the function \(h_3(y)\) is defined such that \(h_3(y) = 0\) gives the axis of rotation.

We now extend the range of independent variables of the remaining metric functions such that they also depend explicitly on \(h_2(x)\) and \(h_3(y)\).

i.e. \(f = f(x, y, h_2, h_3)\), \(\omega = \omega(x, y, h_2, h_3)\), \(Q = Q(x, y, h_2, h_3)\)

The coordinates \(x\) and \(y\) are related to the coordinates \(X\) and \(Y\) via the coordinate transformations

\[
dx = \frac{dx}{H(x, h_2)} \quad , \quad dy = \frac{dy}{N(y, h_3)}
\]  \((4.3)\)
This coordinate transformation enables us to simplify certain of the field equations later and could have been introduced at that time, however, we prefer to introduce it at this point.

Before we attempt to solve the Einstein field equations (The relevant field equations are given in Appendix B), certain notational definitions are required

\[
\frac{\partial}{\partial x} = M \frac{\partial}{\partial x} = M \left\{ \frac{\partial}{\partial h_2} h_2 + \frac{\partial}{\partial x} \right\} \\
\frac{\partial}{\partial y} = N \frac{\partial}{\partial y} = N \left\{ \frac{\partial}{\partial h_3} h_3 + \frac{\partial}{\partial y} \right\}
\]

where \( h_2 := \partial_2 h_2 := \partial h_2/\partial x \), \( h_3 := \partial_3 h_3 := \partial h_3/\partial y \)

\[\frac{\partial}{\partial x}\] means differentiation with respect to \( x \) as an explicit variable, similarly for \( \frac{\partial}{\partial y} \).

As an example

\[
\n^{-1} \nabla^2 \nabla/\partial x \partial y = \frac{1}{4} h_2^{-1} h_3^{-1} NN h_2 h_3
\]

In solving the Einstein field equations we first examine the \( R_{23} = 0 \) equation. It is known (e.g. Hansen and Winicour 1975) that a solution of the remaining field equations identically satisfies the \( R_{23} = 0 \) equation. (This allows, at least in principle, for the explicit choice of an equation of state.) We therefore seek functional forms which satisfy the \( R_{23} = 0 \) equation identically.

Using the notation (4.3) the \( R_{23} = 0 \) equation can be written,

\[
h_2 h_3 \left\{ 2 f - 2 h_2^{''} h_3^{''} + \frac{\partial f}{\partial h_2} \frac{\partial}{\partial h_2} + \frac{\partial f}{\partial h_3} \right\} + \frac{1}{4} h_2^{''} h_3^{''} + \frac{1}{2} f_2 h_2^{''} h_3^{''} f - \frac{\partial f}{\partial h_2} \frac{\partial}{\partial h_2} - \frac{1}{2} h_2^{''} h_3^{''} \frac{\partial}{\partial h_2} - \frac{1}{2} h_2^{''} h_3^{''} \frac{\partial}{\partial h_3}
\]
\[ + h_2 \left\{ 2 f^{-2} h_2^{-1} h_3^{-1} \frac{\partial f}{\partial h_2} \frac{\partial f}{\partial h_3} - \frac{1}{2} h_2^{-1} h_3^{-1} f^{4} \frac{\partial \omega}{\partial h_2} \frac{\partial \omega}{\partial h_3} \right\} \\
+ h_3 \left\{ 2 f^{-2} h_2^{-1} h_3^{-1} \frac{\partial f}{\partial h_2} \frac{\partial f}{\partial h_3} - \frac{1}{2} h_2^{-1} h_3^{-1} f^{4} \frac{\partial \omega}{\partial h_5} \frac{\partial \omega}{\partial h_x} \right\} \\
+ \left\{ 2 f^{-2} h_2^{-1} h_3^{-1} \frac{\partial f}{\partial h_2} \frac{\partial f}{\partial h_3} - \frac{1}{2} h_2^{-1} h_3^{-1} f^{4} \frac{\partial \omega}{\partial h_5} \frac{\partial \omega}{\partial h_x} \right\} = 0 \]

(4.4)

We seek functions \( f, Q \) and \( \omega \) which identically satisfy (4.4) without restricting \( h_2 \) and \( h_3 \). In order to do this we set each of the terms in the braces equal to zero. This gives us a set of four coupled partial differential equations for the functions \( f, Q \) and \( \omega \).

\[ h_2 h_3 \frac{\partial (f^{-2})}{\partial h_2} \frac{\partial (f^{-2})}{\partial h_3} - \frac{\partial \omega}{\partial h_2} \frac{\partial \omega}{\partial h_3} - h_2 f^{-4} Q^{-1} \frac{\partial Q}{\partial y} = 0 \]  

(4.5a)

\[ h_2 h_3 \frac{\partial (f^{-2})}{\partial h_2} \frac{\partial (f^{-2})}{\partial h_3} - \frac{\partial \omega}{\partial h_2} \frac{\partial \omega}{\partial h_3} - h_2 f^{-4} Q^{-1} \frac{\partial Q}{\partial x} = 0 \]  

(4.5b)

\[ 2 h_2 h_3 \frac{\partial (f^{-2})}{\partial h_2} \frac{\partial (f^{-2})}{\partial h_3} - 2 \frac{\partial \omega}{\partial h_2} \frac{\partial \omega}{\partial h_3} - 2 h_2 f^{-4} Q^{-1} \frac{\partial Q}{\partial h_2} = 0 \]  

(4.5c)

\[ - 2 h_2 f^{-4} Q^{-1} \frac{\partial Q}{\partial h_3} + f^{-4} = 0 \]
\[ h_2 \frac{\partial \omega}{\partial x} \left( \frac{\partial f}{\partial x} \right) + \frac{\partial \omega}{\partial y} \left( \frac{\partial f}{\partial y} \right) = 0 \]  

(4.5d)

In order to solve this system of equations we have to make certain assumptions concerning the functional relationships between the functions \( f \) and \( \omega \).

We satisfy equation (4.5d) by setting

\[ \frac{\partial \omega}{\partial x} = h_3 \frac{\partial (f^{-2})}{\partial x} \neq 0 \quad (4.6a) \]

\[ \frac{\partial \omega}{\partial y} = h_2 \frac{\partial (f^{-2})}{\partial y} \neq 0 \quad (4.6b) \]

Substituting these relationships into equations (4.5a) and (4.5b) gives

\[ h_2 \frac{\partial (f^{-2})}{\partial y} \left\{ h_3 \frac{\partial (f^{-2})}{\partial h_2} - \frac{\partial \omega}{\partial h_2} \right\} - h_3 f^{-4} Q^{-1} \frac{\partial \omega}{\partial y} = 0 \quad (4.7a) \]

\[ h_3 \frac{\partial (f^{-2})}{\partial x} \left\{ h_2 \frac{\partial (f^{-2})}{\partial h_3} - \frac{\partial \omega}{\partial h_3} \right\} - h_2 f^{-4} Q^{-1} \frac{\partial \omega}{\partial x} = 0 \quad (4.7b) \]

We now make the further assumption that \( Q \) has no explicit \( x \) or \( y \) dependence

\[ \frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} = 0 \quad (4.8) \]

This gives us the equations

\[ h_3 \frac{\partial (f^{-2})}{\partial h_2} = \frac{\partial \omega}{\partial h_2} \quad (4.9a) \]

\[ h_2 \frac{\partial (f^{-2})}{\partial h_3} = \frac{\partial \omega}{\partial h_3} \quad (4.9b) \]

We are now able to integrate equations (4.6) and (4.9) to give
\[ \omega = \lambda_2 f^{-2} + \lambda_4 (h_3, x, y) \] (4.10a)

\[ \omega = \lambda_3 (y, h_2, h_3) \] (4.10b)

\[ \omega = \lambda_2 (h_3, x, y) \] (4.10c)

\[ \omega = h_3 f^{-2} + \lambda_4 (h_3, x, y) \] (4.10d)

Comparing (4.10b) with (4.10d) and (4.10a) with (4.10c) gives

\[ \lambda_3 = \lambda_4 = \lambda_3 (y, h_3) = \lambda_3 (y) \]

\[ \lambda_1 = \lambda_2 = \lambda_2 (x, h_2) = \lambda_2 (x) \]

hence we have

\[ \omega = h_2 f^{-2} + \lambda_2 \] (4.11a)

\[ \omega = h_3 f^{-2} + \lambda_3 \] (4.11b)

Rearranging these expressions gives the functional form of \( f^{-2} \) and \( \omega \) as

\[ \omega = \frac{h_2 \lambda_3 - h_3 \lambda_2}{h_2 - h_3} + \omega_0 \] (4.12)

\[ f^{-2} = \frac{\lambda_3 - \lambda_2}{h_2 - h_3} \] (4.13)

where we have used the freedom of adding a constant \( \omega_0 \) to \( \omega \) to ensure that \( \omega \) vanishes on the axis of rotation.

We define

\[ \omega_0 := -\lambda_3 \bigg|_{h_3 = 0} \] (4.14)

We now return to equation (4.5c) where, using the functional forms
(4.12) and (4.13), this equation reduces to

\[ 2 \cdot h_2 \frac{\partial Q}{\partial h_2} + 2 \cdot h_3 \frac{\partial Q}{\partial h_3} - Q = 0 \]  

(4.15)

Integrating this equation gives

\[ Q = h_2^{k} h_3^{-k} (a h_2^n + b h_3^n)^{1/2n} \]  

(4.16)

where k, n, a and b are arbitrary constants.

We have thus completely specified the metric components in terms of four functions of a single variable; \( h_2(x), h_3(y), \lambda_2(x) \) and \( \lambda_3(y) \). However, there are still three partial differential equations (i.e. six ordinary differential equations) remaining to be satisfied.

Thus we have six equations for four unknowns and the system is over-specified. We can remove this difficulty by a suitable choice of the arbitrary parameters in the function Q. 

For the choice, \( k = 0 \), \( b = -a \), and \( n = 1 \)

i.e.

\[ Q = a(h_2 - h_3)^{1/2} \]  

(4.17)

we find that two of the field equations become identical up to a multiplicative factor

\[ R^3_3 - R^2_2 = \frac{h_2}{h_3 - h_2} (R^1 - R^0_0) \]  

(4.18)

We are now left with two partial differential equations for four unknown functions of a single variable.

(i) \( R^3_3 - R^2_2 = 0 \)  

(4.19)

(ii) \( R^1_0 = 0 \)  

(4.20)

---

4. It can be shown that other choices lead to "constraint equations" (8.12c) that are inconsistent.
Equations (4.12), (4.13) and (4.17) now give the general form of the metric as,

\[ ds^2 = \frac{h_2 - h_3}{\lambda_2 - \lambda_3} \left\{ ds^2 - \left( \frac{h_2 \lambda_3 - h_2 \lambda_2}{h_2 - h_3} + w_0 \right) d\phi \right\}^2 \]

\[ - \left( \frac{\lambda_3 - \lambda_2}{h_2 - h_3} \right) h_2 h_3 d\phi^2 - \alpha^2(\lambda_3 - \lambda_2) \left\{ \frac{dx^2}{N_1^2} + \frac{dy^2}{N_2^2} \right\} \]

where the functions \( M(x) \) and \( N(y) \) remain to be chosen and the functions \( h_2(x) \), \( h_3(y) \), \( \lambda_2(x) \) and \( \lambda_3(y) \) are to be determined by equaions (4.19) and (4.20).

The separation and solution of these equations is carried out in Appendix B. It is shown that the functions \( \lambda_2(x) \) and \( \lambda_3(y) \) may be considered as coordinates on the 2-space, and the choice \( \lambda_2(x) = x^2 \) and \( \lambda_3(y) = y^2 \) is made. The functions \( M(x) \) and \( N(y) \) are then determined as

\[ M(x)^2 = \frac{1}{4x^2} \left( a_2 x^2 + k_0 x^4 + c_0 \right) h_2(x) \]

\[ N(y)^2 = \frac{1}{4y^2} \left( -a_2 y^2 - k_0 y^4 + c_0 \right) h_3(y) \]

where \( a_2 \), \( k_0 \), and \( c_0 \) are arbitrary constants, not all positive.

The remaining equations then give the general solution for \( h_2(x) \) and \( h_3(y) \) as

\[ h_2(x) = \frac{\Theta_1}{\Delta} + \frac{\Theta_2}{\Delta} x + \frac{3a_2^2 c_0}{2k_0} \left( c_0 + 2a_2 x + 2k_0 x^2 \right) \int \frac{dx}{(c_0 + 2a_2 x + 2k_0 x^2)^2} \]

\[ - (B.34) \]
\[ h_3(y) = \frac{\Theta_1}{\Delta} + \frac{\Theta_2}{\Delta} y + \frac{3a_0 A_0}{2k_0} (c_0 + 2a_0 y + 2k_0 y^2) \frac{1}{(c_0 + 2a_0 y + 2k_0 y^2)^{\frac{1}{2}}} \int \frac{dy}{(c_0 + 2a_0 y + 2k_0 y^2)^{\frac{1}{2}}} \]  
-B.35

where

\[ \Theta_1 = 4c_0 (a_0 + 2x c_0) + 2a_0 b_0 - 6a_0^2 c_0 A_0 \]  
-B.36a

\[ \Theta_2 = 4a_0 (a_0 + 2x c_0) + 4k_0 b_0 - 12a_0 A_0 (c_0 k_0 - a_0^2) \]  
-B.36b

\[ \Delta = 4(2c_0 k_0 - a_0^2) \]  
-B.36c

\[ x = x^2, \quad y = y^2 \]

The general form of the line-element is therefore given by

\[ ds^2 = \frac{h_2 - h_3}{y - x} \left\{ \frac{\left( (y_2 - x_2) \right)}{h_2 - h_3} + \omega \right\}^{\frac{1}{2}} \frac{d\theta}{h_2 - h_3} + \frac{(y - x)}{h_2 - h_3} \frac{d\phi^2}{h_2 h_3 d\phi^2} \]

\[-a^2 (y - x) \left\{ \frac{d\chi^2}{(a_2 x + k_0 x^2 + \chi c_0) h_2} + \frac{d\eta^2}{(-a_2 y - k_0 y^2 - \eta c_0) h_3} \right\} \]

-4.22

The equation of state can be determined from the equation

\[ -16\pi p = -\nabla \cdot \nabla \log V = \frac{1}{V \sqrt{h}} \partial_{\alpha} \left\{ h^{\alpha \beta} \partial_{\beta} \sqrt{h} \right\} \]  
4.23

where \( h_{\alpha \beta} \) is the metric on the 2-space and \( h = \det(h_{\alpha \beta}) \)

together with the equation

\[ (\log f^{-1})_{\nu} = \frac{h_{\nu \mu} p_{,\mu}}{w + p} \]  
4.24

These two equations together give

\[ w + 3p = \frac{3a_0 A_0}{16\pi a_0^2} = \text{constant} \]  
4.25
Thus the equation of state is the same as that of the Wahlquist (1968) solution and in fact our solution becomes equivalent to the Wahlquist solution upon setting $c_0 = 0$. Further details of this solution can be found in the Wahlquist (1968) reference.
CHAPTER V

FURTHER EXACT SOLUTIONS

In the previous chapter we saw that the assumption of separability of the metric function \( V \), together with several other assumptions, led to sufficient simplification of the field equations as to allow exact solutions to be obtained. In this section we will pursue the question of separability of metric components further by proposing that all the metric functions be multiplicatively separable. In order that such an assumption be meaningful we require that the coordinate system in which the metric components assume the separable form be one which is physically defined. To this end we adopt the rigid rotation \((y, z)\) coordinate system (2.45) of Chapter II, in which the metric takes the form

\[
\begin{align*}
\,ds^2 &= \alpha^{-2} \left\{ dt - \lambda_0 y^2 d\phi \right\}^2 - \alpha^2 \nu^2 d\phi^2 \\
&\quad - \Lambda^{-2} \left( y^2 dy^2 + \alpha^8 \nu^2 dz^2 \right)
\end{align*}
\] (5.1)

where we have set

\[
\Lambda = \frac{1}{2} X (1 + \lambda_0 y^2)^2 \Lambda_0^{-1}
\] (5.2)

It is useful, when imposing multiplicative separability on the metric functions, to employ the exponential form of the metric.

We define,

\[
\begin{align*}
\alpha^2 &= e^A - e^{A_2(y)} + A_3(z) \\
\nu^2 &= e^B - e^{B_2(y)} + B_3(z) \\
\Lambda^2 &= e^D - e^{D_2(y)} + D_3(z)
\end{align*}
\]

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Using this notation we can write the field equations as follows:

The pressure isotropy equations are, \((t, \theta, y, z) \rightarrow (0, 1, 2, 3)\)

\[
R^3_3 - R^2_2 = 0
\]

\[
y^{-2}(\ddot{A}_2 \ddot{B}_2 - \dot{A}_2 \dot{B}_2 + \dot{A}_2^2 \ddot{B}_2 - \frac{1}{y} \dot{B}_2 - 4A_2 \ddot{y} e^{-2A - B})
- e^{-4A - B}(A_3 \ddot{B}_3 - \dot{A}_3 \dot{B}_3 + \dot{B}_3 \ddot{B}_3) = 0
\]

\[
(5.3a)
\]

\[
R^1_1 - R^2_2 = 0
\]

\[
y^{-2}(2A_2 \ddot{B}_2 - 2\dot{A}_2 \dot{B}_2 + 7A_2 \ddot{B}_2 - 2\dot{B}_2 + 2A_2 \ddot{B}_2 + \frac{1}{y} \dot{B}_2 - \frac{1}{y} \dot{B}_2 - \frac{3}{y} \dot{A}_2)
- e^{-4A - B}(A_3 \ddot{B}_3 + \dot{A}_3 \dot{B}_3 - 2A_3 \ddot{B}_3 - 2\dot{A}_3 \dot{B}_3 - 2A_3 \dot{B}_3) = 0
\]

\[
(5.3b)
\]

and the remaining field equations become

\[
R^2_{23} = 0
\]

\[
2A_2 \dot{A}_3 + \dot{A}_3 \ddot{B}_3 - 3A_2 \ddot{B}_3 + \ddot{B}_2 \dot{B}_3 + \ddot{B}_2 \ddot{B}_3 = 0
\]

\[
(5.3c)
\]

\[
R^1_{12} = 0
\]

\[
(5.3d)
\]

The pressure and density are given by

\[
32\pi p = e^D y^{-2}(B_2 + \dot{B}_2^2 + 2A_2 \ddot{B}_2 - \frac{1}{y} \dot{B}_2) - e^{D-4A-B}(2A_3 \ddot{B}_3 - \ddot{B}_3)
\]

\[
(5.4a)
\]

\[
32\pi \nu = e^D y^{-2}(16A_2 \ddot{y} e^{-2A-B} - 10A_2 \ddot{B}_2 - 8A_2 \ddot{B}_2 - 2A_2 \ddot{B}_2 - 4A_2 + \frac{3}{y} \dot{B}_2
+ \frac{4}{y} \dot{A}_2)
- e^{D-4A-B}(3B_3 + 4A_3 \ddot{B}_3 - 6A_3 \ddot{B}_3 - 8A_3 \ddot{B}_3)
\]

\[
(5.4b)
\]

here \(\cdot\) indicates differentiation with respect to the subscript index.

i.e. \(\dot{A}_2 := \frac{dA_2}{dy}\).

Differentiating equation (5.3a) with respect to \(z\) we obtain the equation...
\[ 4A_e^2 (2A_3 + B_3) e^{2A} + (4A_3 + B_3) (A_3^2 - A_3 B_3 + B_3 + B_3 D_3) \]
\[ - (A_3 - A_3 B_3 + B_3 + B_3 D_3)^3 = 0 \]  
(5.5)

From this equation we can see that the solutions of equations (5.3) divide into two distinct classes,
either  
\[ (i) \quad A = A(z) \]
(5.6)
or \[ (ii) \quad 2A + B = \text{function of } y \text{ only.} \]  
(5.7)

We shall first investigate solutions belonging to Class (ii).

Class (ii) solutions.

Substituting condition (5.7) into equation (5.3c) gives
\[ B_2 (A_3 - D_3) + 2A_3 (4A_2 - D_2) = 0 \]
(5.8)
which implies
\[ \dot{D}_3 = (2k - 1)A_3 \]
(5.9a)
\[ \dot{D}_2 = 4A_2 + kB_2 \]  
(5.9b)
where \( k \) is a separation constant.

Substitution of conditions (5.7) and (5.9) into equations (5.3a) and (5.3b) yields the system of separated equations;

\[ x - \text{equations.} \]
\[ 2A_3^2 + (4k - 5)A_3^2 + \lambda_1 e^{2A_3} = 0 \]  
(5.10a)
\[ 2(2k - 1)A_3 - 4(k - 1)A_3^2 - \lambda_1 e^{2A_3} = 0 \]  
(5.10b)

\[ y - \text{equations.} \]
\[ B_2 + 3A_2 B_2 + kB_2^2 + A_2^2 - \frac{1}{y} B_2 - 4A_2 y^2 e^{-2A_2 - B_2} \]
\[ - \lambda_1 y^2 e^{-4A_2 - B_2} = 0 \]  
(5.11a)
\[ (1 - k)B_2 + 2(1 - k)A_2 B_2 - A_2 - A_2^2 - \frac{(1 - k)}{y} B_2 + \frac{1}{y} A_2 \]
\[ - \lambda_2 y^2 e^{-4A_2 - B_2} = 0 \]  
(5.11b)

\( \lambda_1 \) and \( \lambda_2 \) are separation constants.
We can solve equations (5.10) simultaneously by the choice
\[ A_3 = \ln \left\{ \frac{1-k}{2\lambda_2^2} \left( \frac{\lambda_2}{1-k} \right) \right\}^{-2} \]  
(5.12)
where \( c_0 \) is a constant of integration, and
\[ \lambda_2 = \frac{2\lambda_1 (1-k)}{3-4k}. \]

It should be noted that the choice \( k = 1 \) is not allowed in this coordinate system. To see this we set \( k = 1 \) in equations (5.10) and (5.11), these then reduce to
\[ \dddot{A}_3 - \dot{A}_3^2 + \lambda_1 e^{2A_3} = 0 \]
\[ \dddot{A}_2 + \dot{A}_2^2 - \frac{1}{y} \dot{A}_2 = 0 \]
\[ \dddot{B}_2 + 3\dot{A}_2 \ddot{B}_2 + \dot{B}_2^2 + \dot{A}_2^2 - \frac{1}{y} \ddot{B}_2 - 4A_2 y^2 e^{-2A_2-B_2} - A_1 y^2 e^{-4A_2-B_2} = 0 \]

Now substituting these equations, together with conditions (5.7) and (5.9) \((k=1)\) into the expressions for \( \Phi_0, \Phi_4 \) and \( \Phi_2 \) (Appendix A) we find
\[ \Phi_0 = \Phi_4 \quad \text{and} \quad \Phi_2 - \Phi_2 \]

These two conditions are necessary and sufficient for the magnetic part of the Weyl tensor (Appendix A) to vanish
i.e.
\[ B_{\mu\nu} = 0 \]

A theorem due to Glass (1975) shows the necessary and sufficient condition for a shear-free perfect fluid to be irrotational is that the Weyl tensor be pure electric type.

Hence in this case we have
\[ \omega_{\mu\nu} = 0 \]
However, the vorticity vector \( \omega \) forms one of the basis vectors of the spacetime and hence we have a contradiction \( \Rightarrow k \neq 1 \).

The simultaneous equations (5.11) have proved difficult to solve and so far no solutions to them have been found. The Class (i) system of equations however, have proved more amenable.

**Class (i) solutions.**

In this case we have \( \omega = \omega(z) \) from which it follows that

\[
p = p(z), \quad w = w(z)
\]

Since the pressure is a function of the single variable \( z \), only, it is clear that the \( p = 0 \) surfaces will not be closed surfaces and hence it is unlikely that any physical solutions belonging to this class will be found. However, it will be shown that a particular complexification of a certain subset of these solutions allows us to obtain non-stationary fluid solutions which may be of cosmological significance.

Using condition (5.6), equation (5.3c) reduces to

\[
\dot{A}_3 B_2 + \dot{B}_2 D_3 + \dot{D}_2 B_3 = 0 \quad (5.13)
\]

which implies

\[
\dot{A}_3 + \dot{D}_3 = \text{constant} \times B_3 = (\mu - 1) B_3
\]

and

\[
\dot{D}_2 = -(\mu - 1) B_2.
\]

Substituting these expressions into equation (5.3b) yields the equation

\[
\mathcal{M} \left\{ y^{-2} \left( \frac{\dot{B}_2}{B_2} - \frac{1}{y} \frac{\dot{B}_2}{B_2} \right) - e^{-B_2} y^{-4} A_3 B_3 (\dot{B}_3 - 2 A_3 B_3) \right\} = 0 \quad (5.14)
\]

hence the choice \( \mu = 0 \), enables us to satisfy this equation identically. (It turns out that the solutions for which \( \mu \neq 0 \) are contained in the \( \mathcal{M} = 0 \) solutions.)
Equation (5.13) now implies
\[ \dot{A}_3 + \dot{B}_3 + \dot{B}_3 = 0 \]
and
\[ \dot{B}_2 - \dot{B}_2 = 0 \]
Substituting these expressions into equation (5.2a) gives
\[ e^{B_2} y^{-2} (\dot{B}_2 - \dot{B}_2) - \frac{1}{y} \dot{B}_2 = \lambda_0 \tag{5.15a} \]
and
\[ 4 \lambda_0 e^{-2A_3 - B_3} + e^{-4A_3 - B_3} (\dot{B}_3 - 2A_3 + B_3 + A_3^2) = \lambda_0 \tag{5.15b} \]
where \( \lambda_0 \) is a separation constant.

We can integrate (5.15a) to give
\[ e^{B_2} = k_1^2 \left( \frac{\lambda_2}{\kappa_1 y^2} \right)^2 \tag{5.16} \]
Equation (5.15b) is more difficult to handle. This is a second order non-linear differential equation in two unknowns and no general solution of the type \( B_3 = B_3(A_3) \) can be found. Hence we have to either choose a functional form for one of the unknowns or specify a functional relationship between them. We shall investigate the set of solutions for which \( B_3 \) is proportional to \( A_3 \).

i.e.
\[ A_3 = \sigma B_3 \]
\[ \sigma = \text{constant} \]
Equation (5.15b) then becomes
\[ \dot{B}_3 + (\sigma^2 - 2 \sigma - 1) B_3^2 + 4 \lambda_0 e^{2 \sigma B_3} - \lambda_0 e^{(4 \sigma + 1) B_3} = 0 \tag{5.17} \]
A first integral of which is given by
\[ \dot{B}_3 = \left\{ \tau_0 e^{2 \sigma B_3} + \tau_1 e^{(4 \sigma + 1) B_3} \right\}^{1/2} \tag{5.18} \]
where \( \tau_0 \) and \( \tau_1 \) are constants of integration constrained by
\[ \tau_0 \left( \sigma^2 - \sigma - 1 \right) + 4 \lambda_0^2 = 0 \tag{5.19a} \]
and
\[ \tau_1(2\sigma^2 - 1) - 2\lambda_0 = 0 \]  \hspace{1cm} (5.19b)

We now define a new variable, \( x \), by
\[ x := s e^{y B_3} \]  \hspace{1cm} (5.20)

\( s \) and \( y \) are constants.

(5.18) now becomes
\[ \frac{dx}{dz} = s y(x/s)(r^2 + y)/y[\tau_0 + \tau_1(x/s)(1 + 2\sigma)/y]^{1/2} \]  \hspace{1cm} (5.21)

If we let
\[ r = (\tau_1/\tau_0)^{1/2}(1 + 2\sigma)/y \]  \hspace{1cm} (5.22)

we have
\[ \frac{dx}{dz} = y s^{-1}(1 + 4\sigma)/2y x(4\sigma + 2y + 1)/2y(1 + 2\sigma)^{1/2} \tau_1[1 + rx^{-1}(1 + 2\sigma)/y]^{1/2} \]  \hspace{1cm} (5.23)

We can now write the metric functions as
\[ a^2 = e^{A_1} = (x/s)^{\sigma/y} \]  
\[ v^2 = e^{B_2} + B_3 = k_1^2(k_2 + \lambda_0 k_1^2 y^2)(x/s)^{1/y} \]  
\[ \Lambda^2 = e^{D_2} + D_3 = k_1^2(k_2 + \lambda_0 k_1^2 y^2)(x/s)^{-(\sigma + 1)/y} \]

The metric (2.45) then takes the form
\[ ds^2 = (x/s)^{-\sigma/y}(dt - \Omega_0 y^2 d\phi)^2 - k_1^2(k_2 + \lambda_0 k_1^2 y^2)(x/s)^{(\sigma + 1)/y} \frac{x}{y^2} \tau_1 \frac{\tau_1}{1 + rx^{-1}(1 + 2\sigma)/y} \]  \hspace{1cm} (5.24)

This metric now satisfies the Einstein field equations for a perfect fluid, where the pressure and the density are given by,
\[ 32\pi p = -\sigma^2 \tau_1 (1 + \sigma)^3 / y - (1 + \sigma)^3 / y \tau_0 (2 + 3\sigma)^2 / y - (2 + 3\sigma)^2 / y \]  
\[ - (5.25) \]
\[ 32\pi \omega = -\sigma (3\sigma + 2) \tau_1 (1 + \sigma)^3 / y - (1 + \sigma)^3 / y \]
\[ + (4 + 7\sigma) \tau_0 (2 + 3\sigma)^2 / y - (2 + 3\sigma)^2 / y \]  
\[ (5.26) \]

We can see from the expression for the pressure that the \( p = 0 \) surfaces can not be finite closed surfaces and hence such solutions are unsuitable for describing finite rotating fluid bodies. However, it is possible to transform these solutions into solutions which can be considered as cosmological models. In order to do this we first make use of the freedom to choose the values of \( \sigma \) and \( \gamma \) in (5.24). We choose

\[ \sigma = \frac{-4}{9}, \quad \gamma = \frac{-1}{18} \]

which gives the metric in the form

\[ ds^2 = (x/s)^{-8} (dt - 3N_0 y^2 d\theta)^2 - k_1^2 (k_2 + \frac{N_0}{k_1 \sqrt{y}})^2 y^2 (x/s)^{-10} d\phi^2 \]
\[ - k_1^2 (k_2 + \frac{N_0}{k_1 \sqrt{y}})^2 y^2 (x/s)^{-10} dy^2 - \frac{(18)^2 s^{10} x^{-12}}{\tau_1 (1 + rx^2)} dx^2 \]  
\[ (5.27) \]

We now make the complex transformation

\[ s \rightarrow iS \]  
\[ (5.28) \]

and, at the same time, demand that \( \tau_1 \) be negative, for which we make the simplifying choice

\[ \tau_1 = -\frac{8 x 162}{49} k_1^2 N_0^2 \]  
\[ (5.29) \]

The resulting metric, having factored out an \( S^{+8} \), is
\[ ds^2 = x^{-8} (dt - \Omega_0 y^2 d\theta)^2 + k_1^2 (k_2 + \Omega_0 y^2)^2 x^{-10} d\theta^2 \]
\[ + k_1^{-2} (k_2 + \Omega_0 y^2)^{-2} x^{-10} y^2 dy^2 - \frac{49s^2}{4k_1^2 \Omega_0^2} \frac{x^{-12}}{(1 + rx^2)} dx^2 \]

where
\[ r = \frac{4 \times 29}{49} k_1^2 s^2 \]

We now perform the coordinate transformation
\[ x \to t'^{-1} \]
\[ t \to z' \]

under which the metric becomes
\[ ds^2 = -a^2 t'^{10} \frac{dt'^2}{(1 + q^2 t'^2)} + b^2 y^2 dy^2 + t'^8 (dz' - \Omega_0 y^2 d\theta)^2 \]
\[ + c^2 t'^{10} (k_2 + \Omega_0 y^2)^2 d\theta^2 \]

where
\[ a^2 = \frac{(49)^2}{29 \times 16} k_1^{-4} \Omega_0^2 \]
\[ b^2 = k_1^{-2} s^2 \]
\[ c^2 = k_1^2 s^2 \]
\[ q^2 = \frac{49}{2 \times 29} k_1^2 s^2 \]

We now drop the primes and allow t to be considered as a real time coordinate. The resulting metric can then be considered as a possible candidate to describe a non-stationary fluid. However, in its present form, this metric is not a solution of the Einstein field equations for which the source of the gravitational field is a perfect fluid. In an attempt to find such solutions we allow the constants a, b, c and q to become independent (i.e. no longer satisfying (5.32)).
In the next chapter we will derive the conditions under which the line-element (5.31) describes a perfect fluid cosmology and examine the properties of the resulting solution.
CHAPTER VI

NON-STATIONARY, ANISOTROPIC, PERFECT FLUID COSMOLOGY

In discussing the metric (5.31) we will adopt the orthonormal tetrad technique developed, primarily, by Ellis (1964) and by Estabrook and Wahlquist (1964). (A review of the elements of this technique is given in Appendix C).

Since the line element (5.31) is required to describe a perfect fluid we must solve the field equations

\[ R_{ab} - \frac{1}{2}g_{ab} = T_{ab} \]  \hspace{1cm} (6.1)

where \( T_{ab} \) is the stress energy tensor of a perfect fluid

\[ T_{ab} = wU_a U_b + p(g_{ab} + U_a U_b) \]  \hspace{1cm} (6.2)

where \( U^a \) is the normalised four-velocity \( (U^a U_a = -1) \), \( w \) is the density and \( p \) is the isotropic pressure of the fluid.

Taking the trace of equation (6.1) we can write these equations as

\[ R_{ab} = T_{aa} - \frac{1}{2}g_{ab} T \]  \hspace{1cm} (6.3)

where \( T := T^a_a = 3p - w \).

Now combining (6.3) and (C.13) we have

\[ R_{ab} = \partial_c \Gamma_{ca}^{\ c} - \partial_b \Gamma_{ca}^{\ c} + \Gamma_{cd} \Gamma_{ba}^{\ d} - \Gamma_{ca} \Gamma_{db}^{\ c} \]

\[ = \frac{1}{2}(w + 3p)U_a U_b + \frac{1}{2}(w - p)\gamma_{ab} \]  \hspace{1cm} (6.4)

where \( \gamma_{ab} = g_{ab} + U_a U_b \)

These equations together with the Jacoby identities (C.11) and the conservation equations

\[ w_{,a} U^a + (w + p)U^a_{;a} = 0 \]  \hspace{1cm} (6.5)

\[ (w + p)U_{a;b} U^b + \gamma_{a;p,b} = 0 \]  \hspace{1cm} (6.6)
form the system of equations to be satisfied.

We are now in a position to determine the conditions under which the line element (5.31) represents a perfect fluid.

We have, \((t, y, z, \phi) = (0, 1, 2, 3)\)

\[
ds^2 = -a^2 \frac{t^{10} dt^2}{(1 + q^2 t^2)} + b^2 t^{10} \Sigma^{-2} y^2 dy^2 + c^2 t^{10} \Sigma^{-2} d\phi^2 \\
+ t^8 (dz - \Omega_0 y^2 d\phi)^2
\]  

(6.7)

where \(\Sigma = k_2 + \Omega_0 y^2\) and \(k_2, a, b, c, q, \text{ and } \Omega_0\) are constants.

The covariant and contravariant metric tensors are

\[
\begin{pmatrix}
-a^2 t^{10} (1 + q^2 t^2)^{-1} & 0 & 0 & 0 \\
0 & b^2 t^{10} \Sigma^{-2} y^2 & 0 & 0 \\
0 & 0 & t^8 & -t^8 \Omega_0 y^2 \\
0 & 0 & -t^8 \Omega_0 y^2 & c^2 t^{10} \Sigma^2 + t^8 \Omega_0^2 y^4
\end{pmatrix}
\]

(6.8)

\[
\begin{pmatrix}
-a^{-2} t^{-10} (1 + q^2 t^2) & 0 & 0 & 0 \\
0 & b^{-2} t^{-10} \Sigma^{-2} y^{-2} & 0 & 0 \\
0 & 0 & t^{-8} + c^{-2} t^{-10} \Sigma^{-2} \Omega_0^{-2} y^4 & c^{-2} t^{-10} \Sigma^{-2} \Omega_0^{-2} y^2 \\
0 & 0 & c^{-2} t^{-10} \Sigma^{-2} \Omega_0^{-2} y & c^{-2} t^{-10} \Sigma^{-2} \Omega_0^{-2} y^2
\end{pmatrix}
\]

(6.9)

We choose as an orthonormal basis \(^4\) (not unique), the vectors

\[
e_0 = U = a^{-1} t^{-5} (1 + q^2 t^2)^{1/2} \frac{\partial}{\partial t} \\
e_0 = -at^{-5} (1 + q^2 t^2)^{-1/2} dt
\]

(6.10a)

(6.10b)

---

4. Following Ellis (1967) the tetrad has been chosen such that \(e_0\) is the flow tangent \(U\) and \(e_1\) is the unique shear eigenvector.
\[ e_1 = t^{-4} \, \partial y \]  
(6.10c)

\[ e_1 = t^4 (dz - N_0 y^2 d\theta) \]  
(6.10d)

\[ e_2 = b^{-1} t^{-5} \Sigma y^{-1} \partial y \]  
(6.10e)

\[ e_2 = b t^5 \Sigma^{-1} y dy \]  
(6.10f)

\[ e_3 = c^{-1} t^{-5} \Sigma^{-1} (N_0 y^2 \partial x^2 + \partial y \phi) \]  
(6.10g)

\[ e_3 = c t^5 \Sigma d\phi \]  
(6.10h)

Direct calculation from (C.3) gives the 14 non-zero Ricci rotation coefficients as,

\[ \Gamma^{1}_{10} = 4a^{-1} t^{-6} (1 + q^2 t^2)^{1/2} \]  
(6.11a)

\[ \Gamma^{0}_{11} = 4a^{-1} t^{-6} (1 + q^2 t^2)^{1/2} \]  
(6.11b)

\[ \Gamma^{2}_{13} = N_0^{-1} c^{-1} t^{-6} \]  
(6.11c)

\[ \Gamma^{3}_{12} = -N_0^{-1} c^{-1} t^{-6} \]  
(6.11d)

\[ \Gamma^{2}_{20} = 5a^{-1} t^{-6} (1 + q^2 t^2)^{1/2} \]  
(6.11e)

\[ \Gamma^{3}_{21} = -N_0^{-1} c^{-1} t^{-6} \]  
(6.11f)

\[ \Gamma^{0}_{22} = 5a^{-1} t^{-6} (1 + q^2 t^2)^{1/2} \]  
(6.11g)

\[ \Gamma^{1}_{23} = N_0^{-1} c^{-1} t^{-6} \]  
(6.11h)

\[ \Gamma^{3}_{30} = 5a^{-1} t^{-6} (1 + q^2 t^2)^{1/2} \]  
(6.11i)

\[ \Gamma^{2}_{31} = N_0^{-1} c^{-1} t^{-6} \]  
(6.11j)
\[ \Gamma_{32}^1 = -\mathcal{A}_0 b^{-1} c^{-1} t^{-6} \]  
(6.11k)

\[ \Gamma_{32}^3 = 2\mathcal{A}_0 b^{-1} t^{-5} \]  
(6.11l)

\[ \Gamma_{33}^0 = 5a^{-1} t^{-6}(1 + q^2 t^2)^{1/2} \]  
(6.11m)

\[ \Gamma_{33}^2 = -2\mathcal{A}_0 b^{-1} t^{-5} \]  
(6.11n)

Now using equation (C.13) we can compute the tetrad Ricci tensor components. The non-vanishing tetrad components of the Ricci tensor are

\[ R_{00} = 18a^{-2} t^{-12} + 4a^{-2} q^2 t^{-10} \]  
(6.12a)

\[ R_{11} = 32a^{-2} t^{-12} + 36a^{-2} q^2 t^{-10} + 2\mathcal{A}_0 c^{-2} b^{-2} t^{-12} \]  
(6.12b)

\[ R_{22} = 40a^{-2} t^{-12} + 45a^{-2} q^2 t^{-10} - 2\mathcal{A}_0 c^{-2} b^{-2} t^{-12} \]  
- \[ 4\mathcal{A}_0 b^{-2} t^{-10} \]  
(6.12c)

\[ R_{33} = R_{22} \]  
(6.12d)

Substituting these expressions into equations (6.4) gives the field equations as,

\[ (0, 0) \quad 18a^{-2} t^{-12} + 4a^{-2} q^2 t^{-10} = \frac{3}{2} p + \frac{1}{2} w \]  
(6.13)

\[ (1, 1) \quad 32a^{-2} t^{-12} + 36a^{-2} q^2 t^{-10} + 2\mathcal{A}_0 c^{-2} b^{-2} t^{-12} \]  
\[ = -\frac{1}{2} p + \frac{1}{2} w \]  
(6.14)

\[ (2, 2) \quad 40a^{-2} t^{-12} + 45a^{-2} q^2 t^{-10} - 2\mathcal{A}_0 c^{-2} b^{-2} t^{-12} - 4\mathcal{A}_0 b^{-2} t^{-10} \]  
\[ = -\frac{1}{2} p + \frac{1}{2} w \]  
(6.15)

\[ (3, 3) = (2, 2) \]  
(6.16)
From the (1,1) and (2,2) equations we find that the necessary conditions for the metric (6.7) to describe a perfect fluid are

\[ \int_0^\infty c^{-2} b^{-2} = 2a^{-2} \]  
\[ 4\int_0^\infty b^{-2} = 9a^{-2} q^2 \]  
(6.17a)  
(6.17b)

Substituting these conditions into the (0,0) and (1,1) equations we find the pressure and density to be

\[ p = -a^{-2} t^{-12} (9 + 16q^2 t^2) \]  
(6.18)  
\[ w = 7a^{-2} t^{-12} (9 + 8q^2 t^2) \]  
(6.19)

It is now a straightforward calculation to show that the pressure and density satisfy the conservation equations (6.5) and (6.6).

We see from these expressions that the pressure in this model is negative, we will return to discuss this point later. However, there are certain energy conditions which must be satisfied if this model is to be considered as being physically reasonable. These conditions are,

1. Weak energy condition: \( w \geq 0, \ w + p \geq 0 \)
2. Dominant energy condition: \( w \geq 0, \ w \leq p \leq w \)
3. Strong energy condition: \( w + p \geq 0, \ w + 3p \geq 0 \)

It is a simple calculation using (6.18) and (6.19) to show that each of these conditions is satisfied for all values of \( t \). Thus the line element (6.7) together with the conditions (6.17) describes a perfect fluid satisfying physically reasonable energy conditions.

It now remains to,

(A) Investigate the kinematical properties of the solution.
(B) Classify the solution as to possible Bianchi type and characterise the corresponding symmetry groups. For this purpose we use the group classification techniques developed by Schucking, Kundt and Behr as described by Ellis and MacCallum (1969).

(A) Kinematics.

In any fluid filled spacetime the fluid flow vector $U^a$ determines the tensor

$$
\gamma_{ab} := g_{ab} + U_a U_b \tag{6.20}
$$

which projects into the instantaneous rest space of an observer comoving with the fluid. The velocity gradient $U_{a;b}$ can be expressed as

$$
U_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{2}{3} \gamma_{ab} - \dot{U}_a U_b \tag{6.21}
$$

where

$$
\omega_{ab} := \left[ U_{a;b} \right] + \left[ U_{a} U_b \right] \tag{6.22}
$$

is the vorticity tensor,

$$
\sigma_{ab} := \left[ U_{(a;b)} \right] + \left[ U_{(a} U_{b)} \right] - \frac{2}{3} \delta^{c}_{a} \gamma_{cb} \tag{6.23}
$$

is the trace-free shear tensor,

$$
\Theta := U^a_{;a} \tag{6.24}
$$

is the expansion and

$$
\ddot{U}_a := U_{a;b} U^b \tag{6.25}
$$

is the acceleration of the fluid.

We define the vorticity vector $\omega^a$ by

$$
\omega^a := \frac{1}{2} \epsilon^{abcd} U_b \omega_{cd} \tag{6.26}
$$

5. The signs of these quantities differ slightly from those in Chapter II since the metric signature in this chapter is +2 rather than -2.
The vorticity and shear scalars are defined respectively by

\[ \omega^2 := \frac{1}{2} \omega_{ab} \omega^{ab} \]  
\hspace{1cm} (6.27)

and

\[ \sigma^2 := \frac{1}{2} \sigma_{ab} \sigma^{ab} \]  
\hspace{1cm} (6.28)

Clearly

\[ \sigma_{ab} = 0 \Leftrightarrow \sigma = 0 \]

and

\[ \omega_{ab} = 0 \Leftrightarrow \omega = 0 \]

We have chosen our orthonormal tetrad (6.10) such that the timelike vector \( e^0 \) is the fluid flow vector \( U \). In this case the tetrad components of \( U \) are

\[ U^a = \delta^a_0, \quad U_a = -\delta^0_a \]  
\hspace{1cm} (6.29)

Now direct calculation using (6.11) and the definition

\[ v_{;b}^a := v_{;b}^a e^1_b + \Gamma_{bc}^a v^c \]

gives us

\[ \omega_{ab} = 0 \]  
\hspace{1cm} (6.30)

and

\[ \dot{U}_a = 0 \]  
\hspace{1cm} (6.31)

(6.30) and (6.31) together \( \Rightarrow \) locally a function \( T \) such that

\[ U_a dx^a = -dT \]

where \( T \) is said to measure proper time along the world lines of the fluid. This function \( T \) is a redefinition of our coordinate \( t \) given by

\[ T = a \int \frac{t^5 dt}{(1 + q^2 t^2)^{1/2}} \]

where we require \( T(t = 0) = 0 \).
hence we have

\[ T = \frac{a(1 + q^2 t^2)^{\frac{1}{2}}}{15q^6} \left( 3q^4 t^4 - 4q^2 t^2 + 8 \right) - \frac{8a}{15q^6} \]

A graphical representation of this function is given by

Clearly the surfaces of constant \( T \) and surfaces of constant \( t \) coincide
and hence the fluid flow vector \( U^a \) is orthogonal to the surfaces
\( (t = \text{constant}) \)

The shear tensor and the expansion are non-zero and are given by

\[ \sigma_{ab} = \text{diag.}(0, (-2/\sqrt{3})\sigma, (1/\sqrt{3})\sigma, (1/\sqrt{3})\sigma) \]

where

\[ \sigma = (1/\sqrt{3})a^{-1}t^{-6}(1 + q^2 t^2)^{\frac{1}{2}} \]

and

\[ \Theta = 14a^{-1}t^{-6}(1 + q^2 t^2)^{\frac{1}{2}} \]

We see from these expressions that

(i) \( \Theta > 0 \iff a > 0 \)

therefore, in order for this model to be expanding we require

\[ a > 0 \]

(ii) The ratio of the expansion and the shear scalar is

a constant

\[ (\sigma / \Theta) = \text{constant} \approx 0.0412 \]
We will return to discuss this point later.

It is possible (Ehlers 1961) to define an average distance scale \( \Omega(t) \), corresponding to the length scale \( R(t) \) of the Robertson - Walker models, by the equation

\[
\frac{1}{l} = \left( \frac{1}{3} \right) \Theta
\]  
(6.35)

\( l \) is thus determined, up to a factor, by the volume behavior of the space. From this function we can, at any time \( t \), define

(a) the Hubble parameter

\[
H := \frac{\dot{l}}{l} = \left( \frac{1}{3} \right) \Theta
\]  
(6.36)

(b) the deceleration parameter

\[
Q := -(\frac{\ddot{l}}{l^2}) = -\Theta^{-2}(3\dot{\Theta} + \Theta^2)
\]  
(6.37)

If \( Q \) is positive then the expansion is slowing down. Direct calculation from (6.33) gives

\[
Q = \frac{4 + q t^2}{14(1 + q t^2)} > 0
\]  
(6.38)

Hence in this model the expansion is slowing down.

One further quantity of kinematical interest is the vector

\[
\mathcal{N}^a := \kappa \Gamma^{abcd}_{\ bce} \dot{e}^c \cdot \dot{e}^d
\]  
(6.39)

This vector is the angular velocity, in the rest frame of an observer with 4-velocity \( U^a \), of the triad \( \{ e_d \} \) with respect to a set of Fermi-propagated axes. In our case we have

\[
\mathcal{N}^a = 0
\]  
(6.40)

(B) Group Classification.

Returning to the basis (6.10) and using the definition (C.5) together with the conditions (6.17) we can calculate the
commutators of the basis vectors to be,

\[
\begin{align*}
[e_0, e_1] &= -4a^{-1}t^{-6}(1 + q^2 t^2)^{\frac{1}{2}} e_1 & (6.41a) \\
[e_0, e_2] &= -5a^{-1}t^{-6}(1 + q^2 t^2)^{\frac{1}{2}} e_2 & (6.41b) \\
[e_0, e_3] &= -5a^{-1}t^{-6}(1 + q^2 t^2)^{\frac{1}{2}} e_3 & (6.41c) \\
[e_2, e_3] &= 2\sqrt{2}a^{-1}t^{-6} e_1 - 3a^{-1}q t^{-5} e_3 & (6.41d) \\
[e_1, e_2] &= 0 & (6.41e) \\
[e_1, e_3] &= 0 & (6.41f)
\end{align*}
\]

Hence the non-zero \( \gamma^c_{ab} \) are, \(( \gamma^c_{ab} - \gamma^c_{[ab]} )\)

\[
\begin{align*}
\gamma^1_{01} &= -4a^{-1}t^{-6}(1 + q^2 t^2)^{\frac{1}{2}} & (6.42a) \\
\gamma^2_{02} &= -5a^{-1}t^{-6}(1 + q^2 t^2)^{\frac{1}{2}} & (6.42b) \\
\gamma^3_{03} &= -5a^{-1}t^{-6}(1 + q^2 t^2)^{\frac{1}{2}} & (6.42c) \\
\gamma^1_{23} &= 2\sqrt{2}a^{-1}t^{-6} & (6.42d) \\
\gamma^3_{23} &= -3a^{-1}q t^{-5} & (6.42e)
\end{align*}
\]

We therefore have

\[
\gamma^a_{bc} = \gamma^a_{bc}(t) & (6.43)
\]

One further quantity we require is the expansion tensor

\[
\theta^a_{ab} := \sigma^a_{ab} + (1/3) \theta \sigma^a_{ab} & (6.44)
\]

whose components are

\[
\theta^a_{ab} = \text{diag} \left( 0, 4a^{-1}t^{-6}(1+q^2 t^2)^{\frac{1}{2}}, 5a^{-1}t^{-6}(1+q^2 t^2)^{\frac{1}{2}}, 5a^{-1}t^{-6}(1+q^2 t^2)^{\frac{1}{2}} \right) & (6.45)
\]
We now, following Schucking (1962) and Kundt (1963), separate $\chi_{\nu\sigma}^{\mu}$ into a symmetric part $\eta_{\nu\sigma}^{\mu}$ and an antisymmetric part represented by $a_{\nu\rho}$.

These are defined by
\[ \eta_{\nu\sigma}^{\mu} := \frac{1}{2} \chi_{\nu\sigma}^{\mu} + \nu \sigma \]

and
\[ a_{\nu\rho} := \frac{1}{2} \chi_{\nu\rho}^{\mu}, \]

The inverse of these relations being
\[ \eta_{\mu\nu}^{\sigma} = \epsilon_{\mu}^{\nu\sigma} \eta_{\nu\sigma}^{\mu} + \delta_{\nu}^{\sigma} a_{\mu} - \delta_{\mu}^{\sigma} a_{\nu}. \]

From (6.42) we can calculate the explicit components of $\eta_{\nu\sigma}^{\mu}$ and $a_{\nu\rho}$ as
\[ \eta_{\mu\nu} = \begin{pmatrix} 2\sqrt{2}a^{-1} & -6 & 0 & -(3/2)a^{-1} & -qt^{-5} \\ 0 & 0 & 0 & 0 & 0 \\ -3(2)a^{-1} & qt^{-5} & 0 & 0 & 0 \end{pmatrix} \]
\[ a_{\nu\rho} = (0, -(3/2)a^{-1}qt^{-5}, 0) \]

We note that
\[ \partial_{\nu} \eta_{\mu\nu} = 0, \quad \partial_{\nu} a_{\nu\rho} = 0. \]

Ellis and MacCallum (1969) give the necessary and sufficient condition for a fluid filled spacetime to be "locally rotationally symmetric" (L.R.S.) as being the existence of a tetrad in which the rotation coefficients have the form,
\[ \Theta_{\mu\nu} = \text{diag.}(\alpha, \beta, \beta) \]
\[ \lambda_{\mu} = -(\lambda_{0}, 0, 0) \]
\[ a_{\mu} = (a_{1}, a_{2}, a_{3}) \]
\[
N_{\mu \nu} = \begin{pmatrix}
    m & -a_3 & a_2 \\
    -a_3 & n & 0 \\
    a_2 & 0 & n
\end{pmatrix}
\] (6.51d)

where \( \nabla_2 N = \nabla_3 N = 0 \), for each coefficient \( N \).

Spatially homogenous, locally rotationally symmetric spaces are spaces in which there exists a group of motions \( G_r \) with \( r = 4 \) or 6, multiply transitive (Eisenhart 1961) on space-like three-surfaces, (in this case these are the surfaces \( \{ t \text{ = constant} \} \)), and having, with one exception \( ^6 \), at least one three-parameter subgroup simply transitive on these surfaces.

Comparing (6.51) with (6.50), (6.49), (6.45) and (6.40) shows our model to be one which is locally rotationally symmetric with

\[
a_1 = a_3 = n = \omega = 0
\] (6.52a)

and

\[
\alpha = 4a^{-1}t^{-6}(1 + q^2t^2)^{1/2}
\] (6.52b)

\[
\beta = 5a^{-1}t^{-6}(1 + q^2t^2)^{1/2}
\] (6.52c)

\[
m = 2\sqrt{2}a^{-1}t^{-6}
\] (6.52d)

\[
a_2 = -(3/2)a^{-1}qt^{-5}
\] (6.52e)

Hence the spacetime contains a group of motions \( G_4 \) or \( G_6 \) multiply transitive on the space-like surfaces \( \{ t \text{ = constant} \} \), with a subgroup \( G_3 \) simply transitive on these surfaces. (If \( r = 6 \) then the space is Robertson - Walker, characterised by the conditions \( \sigma = \omega = U_a = 0 \).

Since for our model this is not the case we must have \( r = 4 \).
If we denote the generators of this subgroup by \( \{ \xi_\mu \} \) then it is possible to find a triad \( \{ e_\nu \} \) lying in the three-surfaces of transitivity which satisfies

\[
\left[ e_\nu, \xi_\mu \right] = 0
\]  

(6.53)

The vectors \( \{ e_\nu \} \) generate a simply-transitive group of transformations in each \( t = \text{constant} \) surface. This is the group reciprocal to the group of motions. Further, since \( e_0 (= \not U) \) is uniquely defined we also have \( \left[ e_0, \xi_\mu \right] = 0 \) and hence

\[
\left[ e_\alpha, \xi_\mu \right] = 0
\]  

(6.54)

It is possible that there exists a second three-parameter subgroup \( G_3 \) which is also simply transitive on these same three-surfaces. If there are two such subgroups then each set of Killing vectors gives rise to a tetrad field satisfying (6.54). Each of these tetrads gives rise to rotation coefficients satisfying (6.43). If there are, indeed, two such subgroups there will be a position dependent rotation relating the tetrads which generate the reciprocal groups of these groups of motions. Ellis and MacCallum (1969) have shown that the quantities \( m, a_1 \) (from 6.51) and

\[ T := 4(a_2^2 + a_3^2) - mn \]

are left invariant by this rotation and they have classified the possible group types according to the values of these parameters. This classification is given in the following table, (reproduced from Ellis and MacCallum 1969)

---

6. This is the special case of Kantowski and Sachs (1966) which admits a \( G_3 \) multiply transitive on two-dimensional surfaces. Their solution requires \( \omega = \gamma_3 = 0 \) which is not the case here.
\[
\begin{array}{cccccc}
\begin{array}{cccc}
 a_1 & m & T & \text{Group type} & n & 2a_2 + a_3^2 \\
 \hline
 0 & 0 & 0 & I & 0 & 0 \\
 & & & \text{VII}_0 & + & 0 \\
 & + & III (n^g \neq 0) & 0 & + & \\
 & & \neq 0 & 0 & 0 & \text{VII}_h \\
 & + & II & 0 & 0 & \text{VIII} \\
 & & & \text{III} (n^g \neq 0) & 0 & + \\
 0 & \neq 0 & 0 & \text{IX} & & - \\
\end{array}
\end{array}
\]

(6.55)

Now, directly from (6.52) we have,

\[ m \neq 0, \ a_1 = 0, \ n^g \neq 0, \ n = 0 \]

and

\[ T = 4a_2^2 = 9\alpha^2 q t^{-10} \geq 0 \]

Comparison of these results with table (6.55) shows our space to be invariant under a one-parameter family of groups of type III \((n^g \neq 0)\).

In this case, however, there exists a second three-parameter subgroup which is also simply transitive on the \((t = \text{constant})\) surfaces. This second subgroup is related to the first by a rotation in these surfaces, the form of which is given by equations (7.5) of Ellis and MacCallum.

This rotation transforms groups of type VIII into groups of type III, (its inverse does the opposite). Hence, the space is invariant under both a simply transitive group of Bianchi type VIII and a one-parameter family of simply transitive groups of Bianchi type III \((n^g \neq 0)\).

The generators of the group of motions \(G_4\) are,

\[ S_1 = \frac{\partial}{\partial \phi} \]  

(6.56a)

\[ S_2 = \frac{\partial}{\partial z} \]  

(6.56b)
$$\mathcal{S}_3 = -2\Lambda_0 \phi \partial/\partial \phi + \Sigma y^{-1} \partial/\partial y + 2k_2\Lambda_0 \phi \partial/\partial z \quad (6.56c)$$

$$\mathcal{S}_4 = (-\Lambda_0 \phi^2 + \frac{b^2}{4c^2\Lambda_0 z^2}) \partial/\partial \phi + \phi \Sigma y^{-1} \partial/\partial y$$

$$+ \left\{ k_2\Lambda_0 \phi^2 + \frac{b^2(k_2+2\Lambda_0 y^2)}{4c^2\Lambda_0 z^2} \right\} \partial/\partial z \quad (6.56d)$$

With the Lie algebra

$$\left[ \mathcal{S}_1, \mathcal{S}_2 \right] = \left[ \mathcal{S}_2, \mathcal{S}_3 \right] = \left[ \mathcal{S}_2, \mathcal{S}_4 \right] = 0$$

$$\left[ \mathcal{S}_4, \mathcal{S}_3 \right] = 2\Lambda_0 \mathcal{S}_4$$

$$\left[ \mathcal{S}_1, \mathcal{S}_3 \right] = -2\Lambda_0 \mathcal{S}_1 + 2k_2\Lambda_0 \mathcal{S}_2$$

$$\left[ \mathcal{S}_1, \mathcal{S}_4 \right] = \mathcal{S}_3$$

The possible three-parameter subgroups and their Bianchi classification are,

Bianchi type III (nφ φ = 0)  Bianchi type III (nφ φ = 0)

$$\{ \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \} \quad \{ \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4 \}$$

Bianchi type VIII (nφ φ = 0)  Bianchi type VIII (nφ φ = 0)

$$\{ \mathcal{S}_3, \mathcal{S}_4, \lambda \} \quad \{ \mathcal{S}_3, \mathcal{S}_4, \mu \}$$

where \( \lambda := 2\Lambda_0 (k_2\mathcal{S}_2 - \mathcal{S}_1) \), \( \mu := -2\lambda + \varepsilon \mathcal{S}_3 + \mathcal{S}_4 \), \( \varepsilon = \text{constant} \).

It is possible to classify this solution according to one further scheme, that of Stewart and Ellis (1968). They have developed a classification
scheme for exact perfect fluid solutions which exhibit local rotational symmetry. An examination of our solution shows it to belong to their Class IIIb. While Brill (1964) has obtained a vacuum electromagnetic field solution belonging to this class, the present solution is, so far as the author is aware, the first non-vacuum solution found belonging to this class.

We have thus presented a new locally rotationally symmetric spacetime containing a perfect fluid with non-zero shear and expansion.

A few concluding remarks are in order. It has been shown that the ratio of the shear scalar and the expansion is a constant. This being the case, the present day value \( (\sigma_0 / \theta_0) \) of this ratio is given as \( (\sigma_0 / \theta_0) \approx 0.0412 \). While this figure is well within the limit \( (\sigma_0 / \theta_0) \leq 1/4 \), obtainable by direct observation, it is larger than the figure produced by indirect arguments concerning the isotropy of the background radiation, which have placed this limit around \( (\sigma_0 / \theta_0) \leq 10^{-3} \).

While the pressure in this model is negative, a feature which prevents it from being considered as a realistic model of the universe, it has been shown that certain physically reasonable energy conditions are satisfied. A model like this may be better suited than empty space models (Kasner 1921, Misner 1969) to describe the early universe with respect to possible anisotropy damping (Matzner and Misner 1973). Solutions like this one may also prove useful in investigating the possible effects of anisotropy in fluid perturbations in cosmological models.
APPENDIX A

NEWMAN-PENROSE FORMALISM

The formalism of Newman and Penrose is well known (Newman and Penrose 1962) so we will not go into details. At each point in spacetime we construct four null vectors $l^\mu$, $n^\mu$, $m^\mu$, and $\bar{m}^\mu$ which satisfy the orthogonality relations

$$l^\mu n_\mu = 1 = -m^\mu \bar{m}_\mu$$

with all other contractions vanishing.

The vectors $l^\mu$ and $n^\mu$ are both real while the vector $m^\mu$ is complex.

Equation (A.1) implies the completeness relation

$$g_{\mu\nu} = 2 l_{(\mu} n_{\nu)} - 2 m_{(\mu} \bar{m}_{\nu)}$$

Directional derivatives are introduced by:

$$\mathcal{D} := l^\mu \nabla_\mu$$

$$\Delta := n^\mu \nabla_\mu$$

$$\delta := m^\mu \nabla_\mu$$

$$\bar{\delta} := \bar{m}^\mu \nabla_\mu$$

where $\nabla_\mu$ is the tensor covariant derivative.

Newman and Penrose have defined twelve complex spin coefficients:

$$\kappa := l_{\mu;\nu} m^\mu \bar{l}^\nu$$

$$\pi := -n_{\mu;\nu} \bar{m}^\mu \bar{l}^\nu$$

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\[ e := \frac{1}{2} (e_{\mu\nu} m^\mu n^\nu - m_{\mu\nu} \bar{m}^\mu \bar{n}^\nu) \]  
(A.4c)

\[ \rho := e_{\mu\nu} m^\mu \bar{m}^\nu \]  
(A.4d)

\[ \lambda := - e_{\mu\nu} \bar{m}^\mu \bar{m}^\nu \]  
(A.4e)

\[ \alpha := \frac{1}{2} (e_{\mu\nu} m^\mu \bar{n}^\nu - m_{\mu\nu} \bar{m}^\mu \bar{n}^\nu) \]  
(A.4f)

\[ \sigma := e_{\mu\nu} m^\mu n^\nu \]  
(A.4g)

\[ \mu := - e_{\mu\nu} \bar{m}^\mu n^\nu \]  
(A.4h)

\[ \beta := \frac{1}{2} (e_{\mu\nu} m^\mu n^\nu - m_{\mu\nu} \bar{m}^\mu \bar{n}^\nu) \]  
(A.4i)

\[ \nu := - e_{\mu\nu} \bar{m}^\mu \bar{n}^\nu \]  
(A.4j)

\[ \gamma := \frac{1}{2} (e_{\mu\nu} m^\mu \bar{n}^\nu - m_{\mu\nu} \bar{m}^\mu \bar{n}^\nu) \]  
(A.4k)

\[ \tau := e_{\mu\nu} m^\mu \bar{n}^\nu \]  
(A.4l)

In Newman–Penrose formalism the tetrad components of the Weyl tensor are defined by,

\[ \Psi_0 := - C_{\mu\nu\alpha\beta} l^\mu m^\nu l^\alpha m^\beta \]  
(A.5a)

\[ \Psi_1 := - C_{\mu\nu\alpha\beta} l^\mu n^\nu l^\alpha m^\beta \]  
(A.5b)

\[ \Psi_2 := - C_{\mu\nu\alpha\beta} \bar{m}^\mu \bar{n}^\nu l^\alpha m^\beta \]  
(A.5c)
\[ \gamma_{t} := - \xi_{\mu} \xi_{\nu} \bar{\eta}^{\mu} \eta_{\nu} \] \hspace{1cm} (A.5d)

\[ \gamma_{b} := - \xi_{\mu} \xi_{\nu} \bar{\eta}^{\mu} \eta_{\nu} \] \hspace{1cm} (A.5e)

The tetrad components of the Ricci tensor are given by

\[ \Phi_{00} := - \frac{1}{2} R_{\mu \nu} \eta^{\mu} \eta^{\nu} \] \hspace{1cm} (A.6a)

\[ \Phi_{11} := - \frac{1}{4} R_{\mu \nu} (\eta^{\mu} \eta^{\nu} + \bar{\eta}^{\mu} \bar{\eta}^{\nu}) \] \hspace{1cm} (A.6b)

\[ \Phi_{22} := - \frac{1}{2} R_{\mu \nu} \eta^{\mu} \eta^{\nu} \] \hspace{1cm} (A.6c)

\[ \Lambda := \frac{R}{12} \] \hspace{1cm} (A.6d)

\[ \Phi_{01} = \Phi_{10} := - \frac{1}{2} R_{\mu \nu} \eta^{\mu} \eta^{\nu} \] \hspace{1cm} (A.6e)

\[ \Phi_{02} = \Phi_{20} := - \frac{1}{2} R_{\mu \nu} \bar{\eta}^{\mu} \bar{\eta}^{\nu} \] \hspace{1cm} (A.6f)

\[ \Phi_{03} = \Phi_{30} := - \frac{1}{2} R_{\mu \nu} \eta^{\mu} \bar{\eta}^{\nu} \] \hspace{1cm} (A.6g)

With the tetrad choice (3.16) the tetrad orientation is given by

\[ \eta_{\mu \nu} \eta_{\rho} \eta^{\rho} \eta^{\sigma} \bar{\eta}^{\sigma} \bar{\eta}^{\rho} \bar{\eta}^{\rho} = - i \] \hspace{1cm} (A.7)

Three self-dual bivectors are defined by,

\[ V^{\mu \nu} := 2 \xi^{[\mu} \xi^{\nu]} \] \hspace{1cm} (A.8a)

\[ M^{\mu \nu} := 2 \xi^{[\mu} \eta^{\nu]} - 2 \bar{\xi}^{[\mu} \bar{\eta}^{\nu]} \] \hspace{1cm} (A.8b)

\[ U^{\mu \nu} := 2 \bar{\xi}^{[\mu} \eta^{\nu]} \] \hspace{1cm} (A.8c)

which are self-dual in the sense that
\[ V^{\mu \nu} = -i \, V^{\mu \nu}, \quad M^{\mu \nu} = -i \, M^{\mu \nu}, \quad \bar{U}^{\mu \nu} = -i \, U^{\mu \nu} \] (A.9)

for the tetrad orientation (A.7). The bivectors and their complex conjugates satisfy the relations

\[ U^{\mu \nu} V_{\mu \nu} = \bar{U}^{\mu \nu} \bar{V}_{\mu \nu} = 2 \] (A.10a)
\[ M^{\mu \nu} M_{\mu \nu} = \bar{M}^{\mu \nu} \bar{M}_{\mu \nu} = -4 \] (A.10b)

with all other contractions vanishing. They satisfy the completeness relation

\[ \frac{1}{2} \eta^{\alpha \beta \mu \nu} + \frac{1}{2} g^{\alpha \beta \mu \nu} = u^{\mu \beta} v_{\mu \nu} + v^{\mu \beta} u_{\mu \nu} - \frac{1}{2} m^{\mu \beta} m_{\mu \nu} \] (A.11)

where

\[ g^{\alpha \beta \mu \nu} := g^{\alpha \beta \mu \nu} - g^{\alpha \nu} g^{\beta \mu} \] (A.12)

We define the self-dual Weyl tensor as

\[ \bar{C}^{\alpha \beta \mu \nu} := \frac{1}{2} \left( C^{\alpha \beta \mu \nu} + i \, \tilde{C}^{\alpha \beta \mu \nu} \right) \] (A.13)

It then follows from (A.5) and (A.8) that

\[ \bar{C}^{\alpha \beta \mu \nu} = -4_0 \, u^{\alpha \beta} u_{\mu \nu} + 4_1 \left( u^{\alpha \beta} m_{\mu \nu} + m^{\alpha \beta} u_{\mu \nu} \right) \]
\[ - 4_2 \left( u^{\alpha \beta} v_{\mu \nu} + v^{\alpha \beta} u_{\mu \nu} + m^{\alpha \beta} m_{\mu \nu} \right) \]
\[ + 4_3 \left( v^{\alpha \beta} m_{\mu \nu} + m^{\alpha \beta} v_{\mu \nu} \right) - 4_4 \, v^{\alpha \beta} v_{\mu \nu} \] (A.14)

The invariants of the Weyl tensor are

\[ \bar{C}_{\alpha \beta \mu \nu} \bar{C}^{\alpha \beta \mu \nu} = 8 \, \Gamma \] (A.15a)
\[ \bar{C}^{\alpha \beta \mu \nu} \bar{C}^{\nu \sigma \rho \beta} - \bar{C}_{\rho \sigma \mu \beta} = -48 \, \Sigma \] (A.15b)
where I and J are defined by (3.25) and (3.27).

When \( \psi_1 = \psi_3 = 0 \), it follows from

\[
C^{\alpha \beta \mu \nu} = C^{(-) \alpha \beta \mu \nu} + C^{(+) \alpha \beta \mu \nu}
\]  

(A.16)

and relations (A.10) that \( M^{\alpha \beta} \) is an eigenvector,

\[
C^{\alpha \beta \mu \nu} M^{\mu \nu} = 4 \psi_2 \ M^{\alpha \beta}
\]  

(A.17)

We can express the Weyl tensor in terms of its electric and magnetic parts by

\[
2 C^{(-) \alpha \beta \mu \nu} U^{\alpha} U^{\beta} = E_{\alpha \beta} + i B_{\alpha \beta}
\]  

(A.18)

For the case \( \psi_1 = \psi_3 = 0 \) and the tetrad choice (3.16) the electric and magnetic parts are (using A.14) respectively,

\[
2 E_{\alpha \beta} = - (\psi_0 + \psi_4) \ m_\alpha \ m_\beta - (\psi_0 + \psi_4) \ m_\alpha \ m_\beta
\]

\[
- (\psi_2 + \psi_2) \left[ \Lambda_{\alpha \beta} + n_\alpha n_\beta - g_{\alpha \beta} - 4 m_{(\alpha} m_{\beta)} \right]
\]  

(A.19)

\[
2 i B_{\alpha \beta} = (\psi_4 - \psi_0) \ m_\alpha \ m_\beta + (\psi_4 - \psi_0) \ m_\alpha \ m_\beta
\]

\[
+ (\psi_2 - \psi_2) \left[ \Lambda_{\alpha \beta} + n_\alpha n_\beta - g_{\alpha \beta} - 4 m_{(\alpha} m_{\beta)} \right]
\]  

(A.20)

We choose as a suitable null tetrad for the metric (5.1) the null vectors,
\[ \sqrt{2} \, \ell^\mu = \left\{ \left( \alpha - \frac{\eta_0 y^2}{\alpha v} \right), -\frac{1}{\alpha v}, 0, 0 \right\} \quad (A.21a) \]

\[ \sqrt{2} \, n^\mu = \left\{ \left( \alpha + \frac{\eta_0 y^2}{\alpha v} \right), \frac{1}{\alpha v}, 0, 0 \right\} \quad (A.21b) \]

\[ \sqrt{2} \, m^\mu = \left\{ 0, 0, -\frac{\Lambda}{y}, -\frac{i \Lambda}{\alpha^4 v} \right\} \quad (A.21c) \]

\[ \sqrt{2} \, l^\mu = \left\{ \frac{1}{\alpha}, -\frac{\eta_0 y^2}{\alpha} + \alpha v, 0, 0 \right\} \quad (A.21d) \]

\[ \sqrt{2} \, n_\mu = \left\{ \frac{1}{\alpha}, -\frac{\eta_0 y^2}{\alpha} - \alpha v, 0, 0 \right\} \quad (A.21e) \]

\[ \sqrt{2} \, m_\mu = \left\{ 0, 0, \frac{y}{\Lambda}, i \frac{\alpha^4 v}{\Lambda} \right\} \quad (A.21f) \]

We can then calculate the spin coefficients to give

\[ \sigma = \rho = \gamma = \epsilon = \mu = \lambda = 0 \quad (A.22a) \]

\[ 2\Pi = \bar{m}' \left( V^{-1} V_1 y \right) + \bar{m} \left( V^{-1} V_1 z \right) \quad (A.22b) \]

\[ 2\tau = -m' \left( V^{-1} V_1 y \right) - m \left( V^{-1} V_1 z \right) \quad (A.22c) \]
\[ 2 \psi = -\bar{m}^2 \left( 2 \alpha^{-1} \psi_{y} + \psi^{-1} \psi_{y} + 2 \alpha^{-2} \Lambda_{\psi} \psi \psi^{-1} \right) - \bar{m}^3 \left( 2 \alpha^{-1} \psi_{z} + \psi^{-1} \psi_{z} \right) \]  
(A.22d)

\[ 2 \kappa = m^2 \left( 2 \alpha^{-1} \psi_{y} + \psi^{-1} \psi_{y} - 2 \alpha^{-2} \Lambda_{\psi} \psi \psi^{-1} \right) + m^3 \left( 2 \alpha^{-1} \psi_{z} + \psi^{-1} \psi_{z} \right) \]  
(A.22e)

\[ 2 \alpha = \bar{m}^2 \left( \Lambda^{-1} \Lambda_{y} - \psi^{-1} \psi_{y} - 4 \alpha^{-1} \psi_{y} - \alpha^{-2} \Lambda_{\psi} \psi \psi^{-1} \right) + \bar{m}^3 \left( \Lambda^{-1} \Lambda_{z} \right) \]  
(A.22f)

\[ 2 \beta = m^2 \left( 4 \alpha^{-1} \psi_{y} + \psi^{-1} \psi_{y} - \Lambda^{-1} \Lambda_{y} - \alpha^{-2} \Lambda_{\psi} \psi \psi^{-1} \right) - m^3 \left( \Lambda^{-1} \Lambda_{z} \right) \]  
(A.22g)

where \( m^2 = -\Lambda (y z) \) and \( m^3 = -\Lambda (x y z) \).

The explicit components of the Weyl tensor are then given by,

\[ 2 \psi_0 = m^3 m^2 \left( A_1 + A_2 \right) + m^2 m^3 \left( B_1 + B_2 \right) \]  
(A.23a)

\[ 2 \psi_4 = \bar{m}^2 \bar{m}^2 \left( A_1 - A_2 \right) + \bar{m}^2 \bar{m}^3 \left( B_1 - B_2 \right) \]  
(A.23b)
\[ \Psi_2 = m^2 \bar{m}^2 (D) + \bar{m}^2 m^3 (G) + m^3 \bar{m}^3 (H) \] (A.23c)

where

\[ A_1 = -g^{-1}g_{yy} + 3g^{-2}g_{,y}^2 - 2g^{-1}g_{,y} \Lambda^{-1}\Lambda_{,y} + \frac{1}{3} g^{-1}g_{,y} \] (A.24a)

\[ -2g^{-1}g_{,y} \Lambda^{-1}V_{,y} + 4g^{-2} \Lambda^2 y^2 \]

\[ A_2 = \Lambda_0 g^{-1}y (-g^{-1}g_{,y} + 4\Lambda^{-1}\Lambda_{,y} + 4V^{-1}V_{,y}) \] (A.24b)

\[ B_1 = 6g^{-2}g_{,y}g_{,zz} - 2g^{-1}g_{,y}g_{,z} - 2g^{-1}g_{,z} \Lambda^{-1}\Lambda_{,y} \]

\[ -3g^{-1}g_{,z} \Lambda^{-1}V_{,y} - 2g^{-2}g_{,y} \Lambda^{-1}\Lambda_{,z} - V^{-1}V_{,z}g^{-1}g_{,y} \] (A.24c)

\[ B_2 = \Lambda_0 g^{-1}y (-4g^{-1}g_{,z} + 4\Lambda^{-1}\Lambda_{,z} + 2V^{-1}V_{,z}) \] (A.24d)

\[ C = -g^{-1}g_{,zz} + 3g^{-2}g_{,z}^2 - 2g^{-1}g_{,z} \Lambda^{-1}V_{,z} \]

\[ -2g^{-1}g_{,z} \Lambda^{-1}\Lambda_{,z} \] (A.24e)

\[ G = -6 \Lambda_0 y g^{-2}g_{,z} \] (A.24f)
\[ D = 4 g^{-1} g_{zz} - 2 \Lambda^{-1} \Lambda, z z + 2 \Lambda^{-2} \Lambda, z^2 - 3 V^{-1} V, y y + 6 V^{-2} V, y^2 + 8 g^{-2} g, y^2 + 2 \Lambda^{-1} \Lambda, y V^{-1} V, y - 4 g^{-1} g, y \Lambda^{-1} \Lambda, y - 10 g^{-1} g, y V^{-1} V, y + \frac{2}{y} V^{-1} V, y - \frac{4}{y} g^{-1} g, y + \frac{2}{y} \Lambda^{-1} \Lambda, y + 4 \Lambda^2 q^2 g^2 \]  

\[ H = - V^{-1} V, z z - 2 \Lambda^{-1} \Lambda, z z + 2 \Lambda^{-2} \Lambda, z^2 - q^{-2} g, z^2 + 2 g^{-1} g, z V^{-1} V, z + 4 g^{-1} g, z \Lambda^{-1} \Lambda, z - 2 \Lambda^{-1} \Lambda, z V^{-1} V, z \]  

where \( g = \alpha^2 V \).
APPENDIX B

SEPARATION AND SOLUTION OF THE FIELD EQUATIONS

The Einstein field equations written out for the metric (4.1) are;

\[ R^1_{1} - R^2_{2} = 0 \]
\[ Q^{-1}Q'V^{-1}V' - Q^{-1}QV^{-1}V + 2f^{-2}f'^2 - V^{-1}V'' - Q^{-2}Q'2 - Q^{-2}Q'^2 \]
\[ + Q^{-1}Q'' + Q^{-1}Q + \frac{1}{2} \omega^2 f'f^2 V^{-2} = 0 \] (B.1)

\[ R^3_{3} - R^2_{2} = 0 \]
\[ 2Q^{-1}Q'V^{-1}V' - V^{-1}V'' - 2f^{-2}f'^2 + \frac{1}{2} \omega^2 f^2 V^{-2} + 2f^{-2}f'^2 + V^{-1}V' \]
\[ - 2Q^{-1}QV^{-1}V - \frac{1}{2} \omega^2 f'f^2 V^{-2} = 0 \] (B.2)

\[ R^1_{23} = 0 \]
\[ 2f^{-2}f'C - Q^{-1}Q'V^{-1}V' + Q^{-1}QV^{-1}V' - \frac{1}{2} \omega^2 f^2 V^{-2} + V^{-1}V' = 0 \] (B.3)

\[ R^1_{0} = 0 \]
\[ (f^2V^{-1}V')' + (f^2V^{-1}V) - 0 \] (B.4)

where ' indicates \( \frac{\partial}{\partial x} \) and ' indicates \( \frac{\partial}{\partial y} \).

Substituting the functional forms (4.2), (4.12), (4.13), and (4.17) into these equations reduces the number of equations remaining to be solved to two, these being the \( R^3_{3} - R^2_{2} = 0 \) equation and the \( R^1_{0} = 0 \) equation. Now, using the notation (4.3) these equations can be written;

\[ R^3_{3} - R^2_{2} = 0 \]
\[ h^{-2}h'^2 - 2\lambda_2 h^{-1}h_2 - 2h^{-1}h_2 h_2 + 4\lambda_2^{-2} \lambda_2^{-1} h^{-2}h_2 \lambda_2 - \]
\[ 2\lambda_2^{-2} \lambda_2^{-1} h^{-1}h_2 - h_2^{-2}h_3 N^2 + 2N \lambda_2^{-1} h^{-1} h_3 + \]
\[ 2N \lambda_2^{-1} h^{-1}h_3 + 4N (\lambda_2^{-1} h^{-1}h_3 h_3 \lambda_3^{-1} - 2\lambda_2^{-1} \lambda_3^{-1} h^{-1}h_3 h_3 h_3 \lambda_3^{-2} (h_2 - h_3) \lambda_3^{-1} = 0 \] (B.5)
\[ R_0^1 = 0 \]

\[
2N h_2^{-1} \dot{h}_2 + 2N h_2^{-1} \dot{h}_3 - M h_2^{-1} \dot{\lambda}_2 + M h_2^{-1} \dot{\lambda}_3 - 2N h_2^{-1} \dot{\lambda}_2 + 4N h_2^{-1} (\lambda_2 - \lambda_3)^{-2} (h_2 - h_3) \ddot{\lambda}_2 \\
- (\lambda_2 - \lambda_3)^{-1} \left[ 4N h_2^{-1} \dot{h}_2 \ddot{\lambda}_2 + 2N h_2^{-1} (h_2 - h_3) \dot{\lambda}_2 - N h_2^{-1} (h_2 - h_3) \dot{\lambda}_2 \right] \\
+ 2N h_2^{-1} (h_2 - h_3) \dot{\lambda}_2 + 4N h_2^{-1} (h_2 - h_3) (\lambda_2 - \lambda_3)^{-2} \ddot{\lambda}_3 + N h_3^{-1} \dot{\lambda}_3^2 \\
- 2N h_3^{-1} (h_2 - h_3) \dot{\lambda}_3 + (\lambda_2 - \lambda_3)^{-1} \left[ 2N h_3^{-1} (h_2 - h_3) \dot{\lambda}_3 - 4N h_3^{-1} \dot{\lambda}_3 \right] \\
+ h_3^{-2} N h_3^{-1} (h_2 - h_3) \ddot{\lambda}_3 + 2N h_3^{-1} (h_2 - h_3) \dot{\lambda}_3 \right] = 0 \quad (B.6)
\]

where \( N = \frac{dM}{dx} \) and \( \dot{N} = \frac{dN}{dy} \).

Adding equations (B.5) and (B.6) yields the useful equation

\[
2N h_2^{-1} (\lambda_2 - \lambda_3)^{-1} \ddot{\lambda}_2 - 2N h_2^{-1} \dot{\lambda}_2 + h_2^{-2} \dot{\lambda}_2 + 2N h_3^{-1} \dot{\lambda}_3 \\
- 2N h_2^{-1} \lambda_2 + 2N h_3^{-1} (\lambda_2 - \lambda_3)^{-1} \lambda_3 - 2N h_3^{-1} \dot{\lambda}_3 - h_3^{-2} \dot{\lambda}_3 + 0 \quad (B.7)
\]

This equation could, of course, be solved identically by setting \( \dot{\lambda}_2 = \dot{\lambda}_3 = 0 \) (i.e. \( \lambda_2 = \text{constant}, \lambda_3 = \text{constant} \) however, this possibility will be considered as a separate case later and in what follows we take \( \dot{\lambda}_2 \neq 0, \dot{\lambda}_3 \neq 0 \).

We are now in a position to make use of the functions \( M(x) \) and \( N(y) \).

We define two new functions \( \alpha(x) \) and \( \beta(y) \) by

\[
\alpha(x) := N h_2^{-1} \quad (B.8a)
\]

\[
\beta(y) := N h_3^{-1} \quad (B.8b)
\]

Substituting these expressions into equation (B.7) gives

\[
2 \left( \alpha \ddot{\lambda}_2 + \beta \ddot{\lambda}_3 \right) = (\lambda_2 - \lambda_3) \left( \dot{\alpha} \dot{\lambda}_2 + 2 \alpha \ddot{\lambda}_2 - \dot{\beta} \dot{\lambda}_3 - 2 \beta \ddot{\lambda}_3 \right) \quad (B.9)
\]

This equation has now become independent of the \( h_i \)'s and therefore we have, in a sense, decoupled this field equation from the remaining one.

There still however, remains the problem of separating equation (B.9)
into its x and y components. To this end we introduce four new functions, \( n(x), F(x), g(y) \) and \( G(y) \), purely as a notational device.

We define,

\[
\begin{align*}
n(x) &= \alpha \dot{\lambda}_2^2, & g(y) &= \beta \dot{\lambda}_3^2 \\
F(x) &= \alpha \ddot{\lambda}_2 + 2\alpha \dot{\lambda}_2, & G(y) &= -\beta \ddot{\lambda}_3 - 2\beta \dot{\lambda}_3
\end{align*}
\]

Equation (B.9) can then be written as

\[
2(n + g) = (F + G)(\lambda_2 - \lambda_3)
= \lambda_2^F + \lambda_2^G - \lambda_3^F - \lambda_3^G
\]  
\text{(B.10)}

The left hand side of this equation is merely the sum of a function of \( x \) and a function of \( y \), therefore the right hand side must be this also. The terms which appear to cause some difficulty are the terms \( \lambda_2^G \) and \( \lambda_3^F \), the remaining terms already being of the required form. At first sight it appears that we require

\[
\lambda_2^G = \lambda_3^F = 0
\]

however, this leads to inconsistencies later; nor is it the most general decomposition of equation (B.10). In order to achieve generality we adopt a convention from complex analysis, only here instead of splitting complex functions into real and imaginary parts we split functions of a single variable into a "pure" part plus a constant part

i.e.

\[
\lambda_2 = \lambda_2^0 + a_2
\]  
\text{(B.11a)}

where \( \lambda_2^0 \) contains all the "pure" \( x \) dependence of \( \lambda_2 \) and \( a_2 \) is the "associated" constant, which may or may not be zero.

In the same way we write

\[
\lambda_3 = \lambda_3^0 + a_3
\]  
\text{(B.11b)}
\[ G = G^0 + a_G \quad \text{(B.11c)} \]
\[ F = F^0 + a_F \quad \text{(B.11d)} \]

We can now write \( \lambda_2 G - \lambda_3 F \) as
\[ \lambda_2^0 G^0 - \lambda_3^0 F^0 + a_G \lambda_2^0 \lambda_2 + a_F \lambda_3^0 \lambda_3 - a_3 K^0 - a_G a_2 - a_F a_3 \]

We are now in a position to separate equation (B.9). Using the definitions of \( n(x), F(x), g(y) \) and \( G(y) \) we can write equation (B.9) as the system of equations,
\[ \dot{\lambda}_2 \left( a_3 - \lambda_2 \right) + 2 \dot{\lambda}_2 \left( \lambda_2^2 - \lambda_2 \lambda_2 + a_3 \lambda_2 \right) - a_G \lambda_2 + a_F a_3 = c_0 \quad \text{(B.12a)} \]
\[ \dot{\lambda}_3 \left( a_3 - \lambda_3 \right) + 2 \dot{\lambda}_3 \left( \lambda_3^2 - \lambda_3 \lambda_2 + a_2 \lambda_3 \right) - a_F \lambda_3 + a_G a_2 = c_0 \quad \text{(B.12b)} \]

where \( c_0 \) is a separation constant.

Together with the equation
\[ \lambda_2^0 G^0 - \lambda_3^0 F^0 = 0 \quad \text{(B.12c)} \]

This equation acts as a constraint on solutions of equations (B.12a) and (B.12b).

In order to solve this system of equations we will first examine equation (B.12a), which on division by \( \dot{\lambda}_2 \left( a_3 - \lambda_2 \right) \) becomes
\[ \dot{x} + x \left\{ \frac{2 \lambda_2}{\lambda_2} + \frac{2 \lambda_2}{a_3 - \lambda_2} \right\} = \frac{a_G \lambda_2 - a_F a_3 + c_0}{\lambda_2 \left( a_3 - \lambda_2 \right)} \quad \text{(B.13)} \]

Treating this as a first order linear ordinary differential equation for \( \dot{x} \) we find an integrating factor
\[ I = \frac{\dot{\lambda}_2}{\left( a_3 - \lambda_2 \right)^2} \quad \text{(B.14)} \]

Hence we are able to obtain the general solution
\[ \alpha = \frac{1}{\lambda^2} \left\{ a_0 \lambda_2 + k_0 (a_3 - \lambda_2)^2 + \frac{a_3}{2} (a_f - a_G) + c_0 \right\} \] (B.15)

where \( k_0 \) is a constant of integration.

One is similarly able to find the general solution to equation (B.12b)

\[ \beta = \frac{1}{\lambda^3} \left\{ -a_0 \lambda_3 + k_0 (a_2 - \lambda_3)^2 + \frac{a_2}{2} (a_f - a_G) - \frac{c_0}{2} \right\} \] (B.16)

where \( K_0 \) is a constant of integration.

Application of the constraint equation (B.12c) leads to

\[ K_0 = -k_0 \] (B.17a)

and

\[ 2(a_2 - a_3)k_0 = a_f - a_G \] (B.17b).

Thus we have the complete solution to one half of the field equations.

From (4.1), (4.3) and (B.8) the metric on the 2-space is

\[ ds^2 = -f^{-2} Q^2 \left( \frac{dx^2}{\alpha \lambda^2} + \frac{dy^2}{\beta \lambda^3} \right) \]

Now, using (B.15)

\[ \frac{dx^2}{\alpha \lambda^2} = \frac{d\lambda^2}{\left( a_0 \lambda_2 + k_0 (a_3 - \lambda_2)^2 + \frac{a_3}{2} (a_f - a_G) + \frac{c_0}{2} \right) \lambda^2} \]

and similarly from (B.16) and (B.17)

\[ \frac{dy^2}{\beta \lambda^3} = \frac{d\lambda^2}{\left( -a_0 \lambda_3 - k_0 (a_2 - \lambda_3)^2 + \frac{a_2}{2} (a_f - a_G) - \frac{c_0}{2} \right) \lambda^3} \]

Therefore, \( \lambda_2 \) and \( \lambda_3 \) have become coordinates on the 2-space and hence, without loss of generality, we can set

\[ a_2 = a_3 = 0 \] (B.18a)

which implies \( a_G = a_F \), \( k_0 \) arbitrary (B.18b)

\( \alpha \) and \( \beta \) therefore reduce to
$$\alpha = \frac{1}{\lambda_2^2} \left( a_g \lambda_2 + k_0 \lambda_2^2 + c_0 \lambda_2^3 \right) \tag{B.19}$$

and

$$\beta = \frac{1}{\lambda_3^3} \left( -a_g \lambda_3 - k_0 \lambda_3^2 - c_0 \lambda_3^3 \right) \tag{B.20}$$

In order to preserve the signature of the metric, not all of $a_g$, $k_0$ and $c_0$ can be positive.

We now return to equation (B.5), which, using (B.8), (B.19) and (B.20), together with a little algebraic manipulation can be written as

$$(H + J)(\lambda_2 - \lambda_3) + 2(1 + m) \left( 2 h_2 + h_3 \right) \left[ a_g + k_0 (\lambda_2 + \lambda_3) \right] = 0 \tag{B.21}$$

where similarly to equation (B.9) we have introduced the functions

$$H(x) = -\dot{\alpha} h_2 - 2\lambda \dddot{h}_2 \tag{B.22a}$$

$$l(x) = 2\lambda \ddot{\lambda} \dot{h}_2 \tag{B.22b}$$

$$J(y) = \dot{\beta} h_3 + 2\beta \dddot{h}_3 \tag{B.22c}$$

$$m(y) = 2\beta \lambda_3 \dddot{h}_3 \tag{B.22d}$$

Proceeding as before we introduce the associated constants by

$$H = H^0 + a_H \tag{B.23a}$$

$$J = J^0 + a_J \tag{B.23b}$$

$$h_2 = h_2^0 + a_4 \tag{B.23c}$$

$$h_3 = h_3^0 + a_5 \tag{B.23d}$$

Substituting these expressions into equation (B.21) gives rise to the following system of equations:
\[ x \text{- equation} \]
\[ H \lambda_2 + 21 - 2a_G h_2 - 2k_0 \lambda_2 h_2 + a_j \lambda_2 + 2k_0 a_5 \lambda_2 = b_0 \] (B.24)

\[ y \text{- equation} \]
\[ J \lambda_3 - 2a_G h_3 - 2k_0 \lambda_3 h_3 + a_H \lambda_3 + 2k_0 a_5 \lambda_3 = b_0 \] (B.25)

where \( b_0 \) is the separation constant.

**Constraint equation**
\[ \xi \lambda_2 (J^0 + 2k_0 h_3^0) - \lambda_3 (J^0 + 2k_0 h_2^0) = 0 \] (B.26)

We now make a choice for the coordinates \( \lambda_2 \) and \( \lambda_3 \). We choose,
\[ \lambda_2 = x^2 \]
\[ \lambda_3 = y^2 \]

with these choices, equations (B.24) and (B.25) respectively, become
\[ -\xi (a_G x^2 + k_0 x^4 + c_0/2) h_2 + (3k_0 x^3/2 + 2a_G x + 5c_0/4x) h_2 \]
\[ - (2a_G + 2k_0 x^2) h_2 + a_j x^2 + 2k_0 a_5 x^2 - b_0 = 0 \] (B.27)

and
\[ -\xi (a_G y^2 + k_0 y^4 + c_0/2) h_3 + (3k_0 y^3/2 + 2a_G y + 5c_0/4y) h_3 \]
\[ - (2a_G + 2k_0 y) h_3 + a_R y^2 + 2a_4 k_0 y^2 - b_0 = 0 \] (B.28)

Multiplying (B.27) by \(-4x^{-5}\) gives
\[ (2a_G x^{-3} + 2k_0 x^{-1} + c_0 x^{-5}) h_2 - (6k_0 x^{-2} + 8a_G x^{-4} + 5c_0 x^{-6}) h_2 \]
\[ + (8a_G x^{-5} + 8k_0 x^{-3}) h_2 = 4a_j x^{-3} + 8k_0 a_5 x^{-3} - 4b_0 x^{-5} \]

which admits the first integral
\[ (2a_G x^{-3} + 2k_0 x^{-1} + c_0 x^{-5}) h_2 - (2a_G x^{-4} + 4k_0 x^{-2}) h_2 \]
\[ = -2a_j x^{-2} - 4k_0 a_5 x^{-2} + 6x^{-4} + 6a_G A_0 \] (B.29)

Similarly when one multiplies equation (B.28) by \(-4y^{-5}\) one is able to obtain the first integral
\[(2a_Gy^{-3} + 2k_0y^{-1} + c_0y^{-5})h_3 - (2a_Gy^{-4} + 4k_0y^{-2})h_3\]
\[= -2a_Hy^{-2} - 4k_0a_4y^{-2} + b_0y^{-4} + 6a_G\beta_0.\]  
(B.30)

where \(A_0\) and \(B_0\) are constants of integration.

Making use of these two first integrals we are able to write the functions \(H(x)\) and \(J(y)\) as
\[H(x) = -2k_0^2 h_2 - 6a_GA_0x^2 + a_J + 2k_0a_5\]  
(B.31)
\[J(y) = -2k_0^2 h_3 - 6a_G\beta_0y^2 + a_H + 2k_0a_4\]  
(B.32)

Substituting these expressions into the constraint equation (B.26) yields
\[A_0 = B_0\]  
(B.33a)
and
\[a_H - a_J = 2k_0(a_5 - a_4)\]  
(B.33b)

We are now able to integrate equations (B.29) and (B.30) to give
\[h_2 = \frac{\Theta_1}{\Delta} + \frac{\Theta_2}{\Delta} x + \frac{3a_GA_0}{2k_0} (c_0 + 2a_Gx + 2k_0x^2)^{\frac{1}{2}} \int \frac{dX}{(c_0 + 2a_Gx + 2k_0x^2)^{\frac{1}{2}}}\]  
(B.34)
\[h_3 = \frac{\Theta_1}{\Delta} + \frac{\Theta_2}{\Delta} y + \frac{3a_G\beta_0}{2k_0} (c_0 + 2a_Gy + 2k_0y^2)^{\frac{1}{2}} \int \frac{dY}{(c_0 + 2a_Gy + 2k_0y^2)^{\frac{1}{2}}}\]  
(B.35)

where
\[\Theta_1 = 4c_0(a_J + 2a_5k_0) + 2a_Gb_0 - 6a_G^2 c_0 A_0\]  
(B.36a)
\[\Theta_2 = 4a_G(a_J + 2a_5k_0) + 4k_0b_0 - 12a_GA_0(c_0k_0 - a_G^2)\]  
(B.36b)
\[\Delta = 4(2c_0k_0 - a_G^2)\]  
(B.36c)

\[X = x^2, \quad Y = y^2\]

The evaluation of the last term in each expression depends upon the signs of \(\Delta\) and \(k_0\). However, we can deduce from (B.34) and (B.35) that
\[ a_5 = a_4 \quad \text{(B.37a)} \]

which implies, from (B.33b)
\[ a_H = a_j, \quad k_0 \text{ arbitrary} \quad \text{(B.37b)} \]

For completeness sake we now examine the case
\[ \lambda_2 = \text{constant} \]
\[ \lambda_3 = \text{constant} \]

In this case equation (B.7) is satisfied identically and equation (B.5) reduces to
\[ h_2^{-2}h_2^2M^2 - 2Mh_2h_2^{-1}h_2^{-1} - h_3^{-2}h_3^2N^2 + 2Nh_3^{-1}h_3^{-1}h_3^{-1}h_3^{-1} = 0 \quad \text{(B.38)} \]

We can separate this equation into its 'x' and 'y' parts to give
\[ h_2^{-2}h_2^2M^2 - 2Mh_2h_2^{-1}h_2^{-1} = \ell_0 \quad \text{(B.39a)} \]
\[ h_3^{-2}h_3^2N^2 - 2Nh_3h_3^{-1}h_3^{-1} = \ell_0 \quad \text{(B.39b)} \]

where \( \ell_0 \) is a separation constant.

These equations can be integrated to give
\[ M^2 = (\ell_0 - \ell_1 h_2^2)h_2^{-2} \quad \text{(B.40a)} \]
\[ N^2 = (\ell_0 - \ell_1 h_3^2)h_3^{-2} \quad \text{(B.40b)} \]

where \( \ell_1 \) is a constant of integration.

The metric can now be written as
\[ ds^2 = \frac{h_2 - h_3}{3} \left( dt - \left( \frac{\lambda_3 - \lambda_2}{h_2 - h_3} \right) \frac{1}{a^2} \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi} \right)^2 - \frac{(\lambda_3 - \lambda_2) h_2 h_3}{h_2 - h_3} \frac{d\theta^2}{d\phi} \]

\[ - a^2 (\lambda_3 - \lambda_2) \left( \frac{dh_2^2}{h_2 (1 - l_1 h_2)} + \frac{dh_3^2}{h_3 (1 - l_1 h_3)} \right) \]  \hspace{1cm} (B.41)

for causality we require \( \lambda_3 > \lambda_2 \).

Using equations (4.23) and (4.24) we find the equation of state as

\[ p = \frac{-l_{11}}{32a^2 (\lambda_3 - \lambda_2)} = \text{constant} \]  \hspace{1cm} (B.42)

\[ w + p = 0 \]  \hspace{1cm} (B.43)

Thus the only physical solution obtainable in this case is when \( l_{11} = 0 \).

\( \Rightarrow p = w = 0 \). In this case however, a simple calculation shows that the metric reduces to flat space.
APPENDIX C

ORTHONORMAL TETRAD TECHNIQUE

At each point of spacetime we introduce a local orthonormal tetrad of vectors \( \{ e_a \} \). Since this tetrad will not, in general, be parallely transported from point to point we restrict ourselves to purely local considerations. We further introduce a local coordinate system denoted by \( \{ x^i \} \).

The components of \( e_a \) referred to the basis \( \partial / \partial x^i \) are

\[
e_a = e_a^i \frac{\partial}{\partial x^i}, \quad \text{det} (e_a^i) \neq 0
\]

so

\[
\partial_a f = e_a f = f \cdot e_a^i
\]

The tetrad components of the metric tensor are

\[
g_{ab} = e_a \cdot e_b = e_a^i e_b^j g_{ij} \tag{C.1}
\]

Since the tetrad is orthonormal

\[
g_{ab} = e_a^i e_b^i = \text{diag} (-1, +1, +1, +1) \tag{C.2}
\]

The metric components \( g^{ab} \) are numerically equal to the \( g_{ab} \). Tetrad indices are raised and lowered by \( g^{ab} \) and \( g_{ab} \) respectively.

We define the Ricci rotation coefficients by

\[
\Gamma_{abc} := e_a \cdot \nabla_b e_c = e_a^i e_c^j ; e_b^i \tag{C.3}
\]

where since the \( g_{ab} \) are constant

\[
\Gamma_{abc} + \Gamma_{cba} = 0 \tag{C.4}
\]

Thus there are 24 independent real Ricci rotation coefficients. They can be regarded as the "tetrad components" of the Christoffel symbols in that

\[
\nabla_{a;b} = \nabla_i e_a^{i;j} = \nabla_a e_b^{i;j} - \Gamma_{bca}^c v_c
\]
for any vector \( V \).

Similarly we can obtain the explicit tetrad components of any tensor equation.

The commutators \( [e_a, e_b] \) of the basis vectors are defined by

\[
[e_a, e_b] \phi := \partial_a (\partial_b \phi) - \partial_b (\partial_a \phi) - \gamma^c_{ab} \partial_c \phi
\]  
(C.5)

where \( \gamma^c_{ab} = \gamma^c_{[ab]} \).

The quantities \( \gamma^c_{ab} \) and \( \Gamma^c_{ab} \) are related by

\[
\gamma^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba}
\]  
(C.6)

from which, using (C.4) we have

\[
\Gamma^c_{abc} = 2(\gamma^c_{abc} + \gamma^c_{cab} - \gamma^c_{bca})
\]  
(C.7)

Thus the 24 independent \( \gamma^c_{bc} \) are completely equivalent to the 24 independent \( \Gamma^c_{bc} \).

The Ricci identity for a general vector \( v^a \) is

\[
\nabla^b_{;cd} v^b_{;dc} = -\Gamma^b_{acd} v^a
\]  
(C.8)

from which we can obtain the tetrad components of the Riemann tensor

\[
R^b_{ecd} = \partial_c \Gamma^b_{de} - \partial_d \Gamma^b_{ce} + \Gamma^b_{ca} \Gamma^a_{de} - \Gamma^b_{da} \Gamma^a_{ce} + \Gamma^b_{ae} \gamma^a_{dc}
\]  
(C.9)

The Riemann tensor components are not all independent, they are related algebraically by the cyclic identities

\[
R^a_{[bcd]} = 0
\]  
(C.10)

From (C.7) these equations are the conditions

\[
\partial_{[a} \gamma^a_{cb]} - \gamma^a_{e[a} \gamma^e_{cb]} = 0
\]  
(C.11)

which using (C.5) are seen to be the components of the Jacoby identities

\[
[e_b, [e_c, e_d]] + [e_d, [e_b, e_c]] + [e_c, [e_d, e_b]] = 0
\]  
(C.12)
for the basis vectors $e^a$.

Contraction of (C.9) gives the tetrad Ricci tensor components

$$R_{ab} := R^c_{\ acb} - \partial_c \Gamma_{ba}^c - \partial_b \Gamma_{ca}^c + \Gamma_{cd}^c \Gamma_{ba}^d - \Gamma_{ca}^d \Gamma_{db}^c$$  \hspace{1cm} (C.13)
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VITA AUCTORIS

David Alan Wilkinson was born July 5, 1949 in Macclesfield, Cheshire, England. In 1969 he graduated with 'A' levels from Oldham College of Technology, Lancashire, England. From 1969 to 1972 he attended Queen Elizabeth College, University of London, England and was awarded a B.Sc. Honours in Mathematics. From 1973 to 1974 he was a student at King's College, University of London, England and received an M.Sc. in Applied Mathematics. He entered the Faculty of Graduate Studies at the University of Windsor in 1974.