Operator spaces.

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OPERATOR SPACES

by

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Abstract

An operator space is defined as a linear space which is equipped with a sequence of matrix norms satisfying two natural axioms. There are some natural operator space structures for all Banach spaces. In this thesis we investigate operator space structures more concretely and their connections with the classical Banach space theory. In particular, some Banach space isometries can be described as the operator space identifications.
사랑하는 나의 어머니께...
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1. INTRODUCTION

The theory of operator spaces is very recent and can be characterized as a non-commutative Banach space theory. We define an operator space $V$ to be a linear space equipped with a sequence $\| \cdot \|_n$ of matrix norms for the $n \times n$ matrix spaces $\mathbb{M}_n(V)$ over $V$ satisfying two abstract conditions which are characterized by Ruan (1988).

In the classical operator theory, we are primarily interested in operator spaces $V$ which are complete with the norm on $\mathbb{M}_1(V) = V$. An operator space $V$ can be described as a Banach space with an isometric embedding into $\mathcal{B}(H)$ of all bounded linear operators on some Hilbert space $H$. Since an $n \times n$ matrix over $\mathcal{B}(H)$ can be considered as an operator on $H^n$ (i.e., $\mathbb{M}_n(\mathcal{B}(H)) \cong \mathcal{B}(H^n)$), there is a corresponding norm on each matrix space $\mathbb{M}_n(V)$.

The morphisms for operator spaces are the completely bounded linear mappings and complete isometries instead of bounded linear mappings and isometries in the classical operator theory.

Throughout this thesis, we frequently use the classical theory of operators in Functional Analysis to characterize the theory of operator spaces.
This thesis is divided into four chapters. The first chapter deals with conventions on operator spaces such as the definitions of operator spaces and completely bounded mappings. We also include the Ruan’s representation theorem in this chapter.

The second chapter contains some constructions of new operator spaces when some operator spaces $V$ are given. Especially, we emphasize the operator space structure on the dual space $V^*$ of an operator space $V$ and investigate the concept of duality for operator spaces.

The third chapter introduces the Arveson-Wittstock theorem which is an operator-valued Hahn-Banach extension theorem analogous to the classical theory.

The last chapter is devoted to the projective tensor product $V \otimes^\gamma W$ of operator spaces. In fact, there are three tensor products in the operator space theory: the operator space projective and injective tensor products which are parallel to the Banach space projective and injective tensor products, respectively, and the Haagerup tensor product which has some new properties not found for tensor product of Banach spaces. In this chapter, we first define the algebraic tensor product of linear spaces and describe two tensor products of Banach spaces. Then we discuss the operator space projective tensor product. Finally, as an example, we obtain the Banach space identification $L_1(X) \otimes^\gamma L_1(Y) \cong L_1(X \times Y)$, where $X = (X, \mathcal{M}, \mu)$ and $Y = (Y, \mathcal{N}, \nu)$ are $\sigma$-finite measure spaces and $\otimes^\gamma$ denotes the Banach space projective tensor product. The corresponding operator space identification also holds.

This thesis is mainly based on the book [7] by Effros and Ruan. Throughout the thesis we suppose that all linear spaces are over the complex field $\mathbb{C}$. 
2. Operator Spaces and Completely Bounded Linear Mappings

We define an abstract operator space by introducing two axioms (Ruan, 1988) about the abstract matrix norm. Comparing with the normed spaces, we discuss the completely bounded linear mappings and the complete linear isometries for operator spaces. We will introduce a representation theorem which shows that an operator space $V$ is isometrically embedded in $B(H)$ of all bounded linear operators on a Hilbert space $H$ with respect to their matrix norms.

2.1. Concrete and Abstract Operator Spaces. Let $X$ be a complex normed space. We will denote $X_{\| \cdot \| \leq 1}$ the unit ball in $X$. By the classical Hahn-Banach theorem, for each $x \in X$, there is a linear functional $f \in X^*$ with $\|f\| = 1$ such that $|f(x)| = \|x\|$. This leads to an isometry $\Phi : X \to l_\infty(X^*_{\| \cdot \| \leq 1})$ defined by $\Phi(x)(f) = f(x)$ for $f \in X^*_{\| \cdot \| \leq 1}$.

Thus, we define a concrete function space on a set $S$ to be a linear subspace $E$ of $l_\infty(S)$, and we see that every Banach space is isometric to such a concrete function space. If we replace functions by operators and $l_\infty(S)$ by $B(H)$, the Banach space of all bounded linear operators on a Hilbert space $H$, we arrive at the concept of a concrete operator space to be defined shortly.

A matrix norm on a linear space $V$ is an assignment of a norm $\| \cdot \|_n$ on each matrix space $M_n(V)$, the linear space of $n \times n$ matrices with entries in $V$. We usually denote the corresponding normed space by $M_n(V)$. If $H$ is a Hilbert space and $V$ is a subspace of $B(H)$, there is a natural way to obtain a matrix norm on $V$. For each $n \in \mathbb{N}$, $M_n(B(H))$ is linearly isomorphic to $B(H^n)$, so the norm on $B(H^n)$
induces a norm on $M_n(B(H))$, and we let $\| \cdot \|_n$ be the restriction of this norm to $M_n(V)$.

**Definition 1.** A concrete operator space is a complex linear subspace $V$ of $B(H)$ together with the matrix norm $\| \cdot \|$ on $V$ induced by the norms on the spaces $B(H^n)$, in the manner just described.

Our first goal is to characterize concrete operator spaces in terms of properties of the matrix norm.

We denote by $M_{m,n}$ the linear space $M_{m,n}$ of $m \times n$ matrices over $\mathbb{C}$ with the norm $\| \cdot \| = \| \cdot \|_\infty$, where $\| \cdot \|$ is determined by the usual identification of $M_{m,n}$ with $B(l^2_m, l^2_n)$. Also, $T_{m,n} = (M_{m,n}, \| \cdot \|_1)$ with the trace class norm $\| \cdot \|_1$, defined by $\| \alpha \|_1 = \text{trace}(|\alpha|)$, where $|\alpha| = (\alpha^* \alpha)^{1/2}$ and $\alpha^*$ denotes the conjugate transpose of the matrices $\alpha$, $HS_{m,n} = (M_{m,n}, \| \cdot \|_2)$ with the Hilbert-Schmidt norm $\| \cdot \|_2$, defined by $\| \alpha \|_2 = \left[ \text{trace}(\alpha^* \alpha) \right]^{1/2} = \left[ \sum_{i,j} |\alpha_{i,j}|^2 \right]^{1/2}$.

There are two natural operations on the finite matrix spaces $M_{m,n}(V)$, where $V$ is a complex linear space. Given $v \in M_{m,n}(V)$ and $w \in M_{p,q}(V)$, we define the direct sum $v \oplus w \in M_{m+p,n+q}(V)$ by

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \in M_{m+p,n+q}(V),$$

and for all $\alpha \in M_{m,p}$, $v \in M_{p,q}(V)$ and $\beta \in M_{q,n}$, we define the matrix product $\alpha v \beta \in M_{m,n}(V)$ by

$$\alpha v \beta = \left[ \sum_{k,l} \alpha_{i,k} v_{k,l} \beta_{l,j} \right] |1 \leq i \leq m, 1 \leq j \leq n|.$$

Now we define an abstract operator space by introducing two axioms about these natural operations.
Definition 2. An abstract operator space is a linear space $V$ with a matrix norm $\| \cdot \|$ such that

[M1] $\| v \oplus w \|_{m+n} = \max\{\|v\|_m, \|w\|_n\}$ and

[M2] $\| \alpha v \beta \|_n \leq \|\alpha\| \|v\|_m \|\beta\|$, for all $v \in M_m(V), w \in M_n(V)$ and $\alpha \in M_{n,m}, \beta \in M_{m,n}$.

We let the normed space $M_n(V)$ denote $M_n(V)$ with the given norm $\| \cdot \|_n$. A matrix norm $\| \cdot \|$ is called an operator space matrix norm if it satisfies [M1] and [M2]. For any abstract operator space $V$, from [M1], the natural mapping $v \mapsto v \oplus 0 = \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}$ is an isometry of $M_n(V)$ into $M_{n+1}(V)$, where $v \in M_n(V)$ and thus we obtain a norm on $M_{n}^{\text{fin}}(V) = \bigcup_{n=1}^{\infty} M_n(V)$ which determines the operator space structure on $V$.

In this section we use the tensor product briefly. The definition and some basic properties of the tensor product are given in Section 5.1.

The following Proposition shows that any concrete operator space is an abstract operator space and $B(H)$ is an abstract operator space.

Proposition 1. Let $H$ be a Hilbert space. Then for all $a \in M_m(B(H)), b \in M_n(B(H)), \alpha \in M_{n,m}$ and $\beta \in M_{m,n}$, we have

[M1] $\| a \oplus b \| = \max\{\|a\|, \|b\|\}$ and

[M2] $\| \alpha a \beta \| \leq \|\alpha\| \|a\| \|\beta\|$. 

Proof. Since we have the natural identification $M_m(B(H)) \cong B(H^m)$ determined by the matrix multiplication, we regard $a$ and $b$ as bounded linear operators on $H^m$.
and $H^n$ respectively. Let $H = H^m \oplus H^n$ be the algebraic direct sum of $H^m$ and $H^n$.

Then, $a \oplus b : H \to H$ is a bounded operator with $\|a \oplus b\| = \max\{\|a\|, \|b\|\}$. 

(M1)

Indeed,

$$\|(a \oplus b)(x \oplus y)\| = \|ax \oplus by\| = \sqrt{\|ax\|^2 + \|by\|^2}$$

$$\leq \sqrt{(\|a\| \|x\|)^2 + (\|b\| \|y\|)^2}$$

$$\leq M \sqrt{\|x\|^2 + \|y\|^2} = M \|x \oplus y\|,$$

where $M = \max\{\|a\|, \|b\|\}$. Thus, we get $\|a \oplus b\| \leq M$.

Conversely, $\sup\{\|(a \oplus b)x\| : x \in H^m, \|x\| \leq 1\} = \sup\{\|ax\| : x \in H^m, \|x\| \leq 1\} = \|a\| \leq \|a \oplus b\|$. Similarly, we can get $\|b\| \leq \|a \oplus b\|$.

Thus, $M = \max\{\|a\|, \|b\|\} \leq \|a \oplus b\|.$

There is a linear isomorphism between $H^m$ and the algebraic tensor product $C^m \otimes H$. This becomes a Hilbert space isomorphism if $C^m \otimes H$ is given the inner product such that $\langle \eta \otimes \xi, \eta' \otimes \xi' \rangle = \langle \eta, \eta' \rangle \langle \xi, \xi' \rangle$ for all $\eta, \eta' \in C^m$ and $\xi, \xi' \in H$.

For a bilinear mapping $\varphi : B(C^m, C^n) \times B(H) \to B(C^m \otimes H, C^n \otimes H)$ defined by $\varphi(\alpha, T)(\eta \otimes \xi) = \alpha \eta \otimes T \xi$ for $\alpha \in B(C^m, C^n) \cong M_{n,m}$, $T \in B(H)$, $\eta \in C^m$, $\xi \in H$,

there is a unique linear mapping $\Phi : B(C^m, C^n) \otimes B(H) \to B(H^m, H^n)$ such that $\Phi(\alpha \otimes T) = \varphi(\alpha, T)$ and $\|\alpha \otimes T\| = \|\alpha\||\|T\||$. If $I$ is the identity operator in $B(H)$, $\alpha \otimes I$ corresponds to $\alpha$ and $\|\alpha \otimes I\| = \|\alpha\||\|I\|| = \|\alpha\|$. Similarly, this is true for $\beta$.

Thus, $\|\alpha a \beta\| = \|(\alpha \otimes I)a(\beta \otimes I)\| \leq \|\alpha\| \|a\| \|\beta\| \|I\| = \|\alpha\| \|a\| \|\beta\|$. 

(M2)

Given Hilbert spaces $H$ and $K$, if we use the natural linear identification defined by the matrix multiplication

$$M_n(B(H, K)) \cong B(H^n, K^n)$$

(1)
to determine a matrix norm on $B(H, K)$, then from the argument for Proposition 1, the matrix norm on $B(H, K)$ satisfies [M1] and [M2] and thus $B(H, K)$ is an abstract operator space.

Let $V$ be an abstract operator space and let $v \in M_n(V)$. If $\mu \in M_n$ is unitary, then

$$\|\mu v\| \leq \|\mu\| \|v\| = \|\mu^{-1} \mu v\| \leq \|\mu^{-1}\| \|\mu v\| = \|\mu v\| \quad \text{(since } \|\mu\| = \|\mu^{-1}\| = 1).$$

Hence, we get $\|\mu v\| = \|v\mu\| = \|v\|$ (unitary invariance). Thus, we can permute rows and columns of $v$ without changing its norm since such an operation corresponds to multiplication by a permutation matrix.

We state the polar decomposition theorem without proof (see [7,Theorem 1.2.1]).

**Theorem 1.** For any $\alpha \in M_{m,n}$, there exists a partial isometry $\nu : \mathbb{C}^n \to \mathbb{C}^m$ such that $\alpha = \nu|\alpha|$. Moreover, $\nu$ maps $(\ker \alpha)^\perp$ onto range $\alpha$, $\ker \nu = \ker \alpha$ and $\nu^* \alpha = |\alpha|$. For any $\alpha \in M_{m,n}$ (resp. $\alpha \in T_{m,n}$ or $\alpha \in HS_{m,n}$), $|\alpha| \in M_n$ (resp. $|\alpha| \in T_n$ or $|\alpha| \in HS_n$) has the same norm as $\alpha$. If $m = n$, then there exists a unitary $\mu \in M_n$ such that $\alpha = \mu|\alpha|$.

Let $V$ be a linear space over $\mathbb{C}$. We have linear isomorphisms

$$V^n \cong \mathbb{C}^n \otimes V \cong V \otimes \mathbb{C}^n$$

defined by $v = (v_{i,j}) \mapsto \sum_{i=1}^n \varepsilon_i \otimes v_i$, and $v = (v_{i,j}) \mapsto \sum_{i=1}^n v_i \otimes \varepsilon_i$, where $\varepsilon_i = (0, \cdots, 0, 1_i, 0, \cdots, 0)$ is the usual basis vector for $\mathbb{C}^n$.

Let $V$ be an abstract operator space. We have an identification

$$M_m(V) \cong M_m \otimes V (\cong V \otimes M_n) : v = [v_{i,j}] \mapsto \sum_{i,j} \varepsilon_{i,j} \otimes v_{i,j},$$
where \( \{e_{i,j} = \begin{bmatrix} 0 & \cdots & 0 \\ \cdots & 1_{i,j} & \cdots \\ 0 & \cdots & 0 \end{bmatrix} : 1 \leq i, j \leq m \} \) is a vector basis for \( M_m \). Conversely, if \( \alpha = [\alpha_{i,j}] \in M_m \) and \( v_0 \in V \), then the corresponding elementary tensor \( \alpha \otimes v_0 \) is given by

\[
\alpha \otimes v_0 = \sum_{i,j} \alpha_{i,j} e_{i,j} \otimes v_0 = \sum_{i,j} \epsilon_{i,j} \otimes (\alpha_{i,j} v_0) = [\alpha_{i,j} v_0].
\]

Thus, for all \( v \in M_m(V) \), \( v \) can be represented as a sum of elementary tensors, i.e.,

\[
v = \sum_{k=1}^{n} \alpha_k \otimes v_k.
\]

Let \( M_{mn}(V \otimes W) \) be the linear space of \( mn \times mn \) matrices over the tensor product \( V \otimes W \) of linear spaces. For given \( v \in M_m(V) \) and \( w \in M_n(W) \), we define the Kronecker product \( v \otimes w \in M_{mn}(V \otimes W) \) by \( (v \otimes w)_{(i,k),(j,l)} = v_{i,j} \otimes w_{k,l} \).

Alternatively, the Kronecker product can be defined in terms of elementary tensors. That is, for given \( v = \alpha \otimes v_0 \in M_m \otimes V \cong M_m(V) \) and \( w = \beta \otimes w_0 \in M_n \otimes W \cong M_n(W) \), \( v \otimes w \in M_{mn}(V \otimes W) \cong M_{mn} \otimes V \otimes W \) is defined by

\[
(\alpha \otimes v_0) \otimes (\beta \otimes w_0) \cong (\alpha \otimes \beta) \otimes (v_0 \otimes w_0).
\]

For \( \alpha = [\alpha_{i,j}] \in M_m \) and \( v = [v_{k,l}] \in M_n(V) \), \( \alpha \otimes v = [\alpha_{i,j} v_{k,l}] \cong [v_{k,l} \alpha_{i,j}] \cong v \otimes \alpha \).

**Proposition 2.** Let \( V \) be an abstract operator space. Then, for any \( v \in M_n(V) \) and \( \alpha \in M_p \),

\[
\|v \otimes \alpha\| = \|\alpha \otimes v\| = \|v\| \|\alpha\|.
\]

**Proof.** By Theorem 1, we can write \( \alpha = \mu |\alpha| \), where \( \mu \) is unitary. From the finite-dimensional spectral theorem, there is a unitary matrix \( \lambda \in M_p \) and scalars \( c_1 \geq \cdots \geq c_p \geq 0 \) such that \( \|\alpha\| = c_1 \) and \( |\alpha| = \lambda^*(c_1 \oplus \cdots \oplus c_p) \lambda \).
Then, we get
\[\|\alpha \otimes v\| = \|\mu \lambda^*(c_1 \oplus \cdots \oplus c_p)\lambda \otimes v\|\]
\[= \|\mu(\lambda \otimes I_n)(c_1 v \oplus \cdots \oplus c_p v) (\lambda \otimes I_n)\|\]
\[= \|\mu(\lambda \otimes I_n)(c_1 v \oplus \cdots \oplus c_p v) (\lambda \otimes I_n)\|\]
\[= \|c_1 v \oplus \cdots \oplus c_p v\|\]
\[= \max\{|c_1 v|, \ldots, |c_p v|\}\]
\[= c_1 \|v\| = \|\alpha\| \|v\|\].

Since \(v \otimes \alpha \cong \alpha \otimes v\), we obtain \(\|v \otimes \alpha\| = \|\alpha \otimes v\| = \|\alpha\| \|v\|\).

Let \(V\) be an abstract operator space. For any \(m, n \in \mathbb{N}\), a norm on a rectangular matrix space \(M_{m,n}(V)\) can be obtained by regarding \(M_{m,n}(V)\) as a subspace of \(M_p(V)\) by adding zero entries to obtain a square matrix, where \(p = \max\{m, n\}\). We let \(M_{m,n}(V)\) denote the linear space \(M_{m,n}(V)\) with this norm. It is easy to see that [M1] and [M2] are satisfied for rectangular matrices.

For any \(v = [v_{i,j}] \in M_n(V)\), we have that
\[
v_{i,j} = [0 \cdots 1_i \cdots 0] [v_{k,l}] 1_j = E_i v E_j^*
\]
and \(v = \sum_{i,j} E_i^* v_{i,j} E_j\), where \(E_i = [0 \cdots 0 1_i 0 \cdots 0]\). Thus,
\[
\|v_{i,j}\| = \|E_i v E_j^*\| \leq \|v\| \quad \text{and} \quad \|v\| = \|\sum_{i,j} E_i^* v_{i,j} E_j\| \leq \sum_{i,j} \|v_{i,j}\|.
\]

From the inequality (3), we can see that any two matrix norms on \(M_n(V)\) are equivalent if they are equivalent on \(V = M_1(V)\) and that a sequence \(v_k\) in \(M_n(V)\) converges if and only if the entries \((v_k)_{i,j}\) converge in \(V\). It follows that if \(W\) is a closed subspace of an abstract operator space \(V\), then \(M_n(W)\) is closed in \(M_n(V)\) for each \(n \in \mathbb{N}\). Also, \(V\) is complete if and only if each \(M_n(V)\) is complete.
2.2. Completely Bounded Linear Mappings. Let $V$ and $W$ be abstract operator spaces and $\varphi : V \rightarrow W$ a linear mapping. For each $n \in \mathbb{N}$, there exists a corresponding linear mapping $\varphi_n : M_n(V) \rightarrow M_n(W)$ defined by $\varphi_n(v) = [\varphi(v_{i,j})]$ for all $v = [v_{i,j}] \in M_n(V)$. We define the completely bounded norm of $\varphi$ by

\begin{equation}
\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\}.
\end{equation}

For $v \in M_n(V)$ and $w \in M_m(V)$,

$$\varphi_{n+m}(v \oplus w) = [\varphi(v_{i,j})] \oplus [\varphi(w_{k,l})] = \varphi_n(v) \oplus \varphi_m(w),$$

and $\alpha \in M_{n,p}$, $\beta \in M_{p,n}$, $v \in M_p(V)$,

$$\varphi_n(\alpha v \beta) = \varphi[\sum_{i,j} \alpha_{i,k}v_{k,l}\beta_{l,j}] = [\sum_{i,j} \alpha_{i,k}\varphi(v_{k,l})\beta_{l,j}] = \alpha \varphi_p(v) \beta.$$

From [M1], we obtain that

\begin{equation}
\|\varphi\| \leq \|\varphi_2\| \leq \cdots \leq \|\varphi_n\| \leq \cdots \leq \|\varphi\|_{cb}.
\end{equation}

If $\|\varphi\|_{cb} < \infty$ [resp. $\|\varphi\|_{cb} \leq 1$], we say that $\varphi$ is completely bounded [resp. completely contractive]. And we denote the space of completely bounded linear mappings from $V$ to $W$ by $CB(V, W)$.

We define $\varphi : V \rightarrow W$ to be a complete isometry if each mapping $\varphi_n : M_n(V) \rightarrow M_n(W)$ is an isometry and we denote $V \hookrightarrow W$ to be a complete isometry.

**Lemma 1.** Given $m, n \in \mathbb{N}$ with $m \geq n$ and a vector $\eta \in \mathbb{C}^m \otimes \mathbb{C}^n$, there exists an isometry $\beta : \mathbb{C}^n \hookrightarrow \mathbb{C}^m$ and a vector $\bar{\eta} \in \mathbb{C}^n \otimes \mathbb{C}^n$ such that $(\beta \otimes I_n)(\bar{\eta}) = \eta$.

**Proof.** There exist unique vectors $\eta_j \in \mathbb{C}^m (j = 1, 2, \cdots, n)$ such that

$$\eta = \sum_{j=1}^{n} \eta_j \otimes \varepsilon_j^{(n)}.$$
If $F \subseteq \mathbb{C}^m$ is the subspace spanned by the vectors $\eta_j$, then $\dim F \leq n \leq m$. Thus we can find an isometry $\beta : \mathbb{C}^n \hookrightarrow \mathbb{C}^m$ with image containing $F$. For each $j$, there is a unique vector $\tilde{\eta}_j \in \mathbb{C}^n$ such that $\beta(\tilde{\eta}_j) = \eta_j$. Thus if $\tilde{\eta} = \sum_{j=1}^n \tilde{\eta}_j \otimes \xi_j^{(n)}$, we have that
\[
(\beta \otimes I_n)(\tilde{\eta}) = (\beta \otimes I_n)(\sum_{j=1}^n \tilde{\eta}_j \otimes \xi_j^{(n)})
\]
\[
= \beta(\sum_{j=1}^n \tilde{\eta}_j) \otimes I_n(\xi_j^{(n)})
\]
\[
= \sum_{j=1}^n \beta(\tilde{\eta}_j) \otimes \xi_j^{(n)}
\]
\[
= \sum_{j=1}^n \eta_j \otimes \xi_j^{(n)} = \eta.
\]

\[\Box\]

**Proposition 3.** If $V$ is an abstract operator space and $\varphi : V \rightarrow M_n$ is a linear mapping, then $\|\varphi\|_{cb} = \|\varphi_n\|$.

**Proof.** By the definition of $\|\cdot\|_{cb}$, we have $\|\varphi_n\| \leq \|\varphi\|_{cb}$. We need to show that for any integer $m \geq n$, $\|\varphi_m\| \leq \|\varphi_n\|$. Given $\varepsilon > 0$, consider $v \in M_m(V)$ with $\|v\| \leq 1$ such that $\|\varphi_m\| - \varepsilon < \|\varphi_m(v)\|$. We select unit vectors $\eta, \xi \in (\mathbb{C}^n)^m = \mathbb{C}^m \otimes \mathbb{C}^n$ such that $\|\varphi_m\| - \varepsilon < \|\langle \varphi_m(v)\eta, \xi \rangle\|$, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{C}^m \otimes \mathbb{C}^n$.

From Lemma 1, there exist isometries $\alpha, \beta : \mathbb{C}^n \hookrightarrow \mathbb{C}^m$ and unit vectors $\tilde{\xi}, \tilde{\eta} \in \mathbb{C}^n \otimes \mathbb{C}^n$ such that $\tilde{\xi} = (\alpha \otimes I_n)(\tilde{\xi})$ and $\tilde{\eta} = (\beta \otimes I_n)(\tilde{\eta})$. Thus, we have
\[
\|\varphi_m\| - \varepsilon < \|\langle \varphi_m(v)(\beta \otimes I_n)(\tilde{\eta}), (\alpha \otimes I_n)(\tilde{\xi}) \rangle\|
\]
\[
= \|\langle (\alpha \otimes I_n)^* \varphi_m(v)(\beta \otimes I_n)(\tilde{\eta}), \tilde{\xi} \rangle\|
\]
\[
= \|\langle (\alpha^* \otimes I_n)\varphi_m(v)(\beta \otimes I_n)(\tilde{\eta}), \tilde{\xi} \rangle\|
\]
\[
= \|\langle \alpha^* \varphi_m(v)\beta(\tilde{\eta}), \tilde{\xi} \rangle\|
\]
\[
= \|\varphi_n(\alpha^* v \beta)(\tilde{\eta}, \tilde{\xi})\|
\]
\[
\leq \|\varphi_n(\alpha^* v \beta)\|\|\tilde{\eta}\|\|\tilde{\xi}\|
\]
\[
\leq \|\varphi_n\|\|\alpha^* v \beta\| \leq \|\varphi_n\|.
\]
Since $\varepsilon > 0$ is arbitrary, we get $\|\varphi_m\| \leq \|\varphi_n\|$. Therefore, $\|\varphi\|_{cb} = \|\varphi_n\|$. \[\Box\]
Corollary 1. Let $V$ be an abstract operator space. Then for each linear functional $f : V \to \mathbb{C}$, $\|f\|_{cb} = \|f\|$. 

Proof. Since $\mathbb{C} \cong M_1$ and $f_1 = f$, we obtain $\|f\|_{cb} = \|f\|$ from Proposition 3. \qed

If $V$ is an abstract operator space and $v \in V$ with $\|v\| = 1$, then the mapping

$$\theta_v : \mathbb{C} \to V : \alpha \mapsto \alpha v$$

is a complete isometry. To see this, let $n \in \mathbb{N}$ and we consider the corresponding linear mapping

$$(\theta_v)_n : M_n \to M_n(V).$$

For $\alpha \in M_n$, we have $(\theta_v)_n(\alpha) = [\theta_v(\alpha_{i,j})] = [\alpha_{i,j}v] = \alpha \otimes v$. Thus,

$$\|(\theta_v)_n(\alpha)\| = \|\alpha \otimes v\| = \|\alpha\|\|v\| = \|\alpha\|.$$ (6)

That is, for each $n \in \mathbb{N}$, $(\theta_v)_n$ is an isometry and hence $\theta_v$ is a complete isometry. In particular, there is only one operator space of dimension 1.

Let $V$ be a $n$-dimensional Banach space. An Auerbach basis for $V$ is a vector basis $v_1, v_2, \ldots, v_n$ with $\|v_i\| = 1 (i = 1, 2, \ldots, n)$ for which there exist a basis $f_1, \ldots, f_n$ in $V^*$ with $\|f_i\| = 1$ and $f_i(v_j) = \delta_{i,j}$. Every finite-dimensional Banach space has an Auerbach basis.

Corollary 2. Let $V$ and $W$ be abstract operator spaces with $\dim V = n$ or $\dim W = n$. Then any linear mapping $\varphi : V \to W$ satisfies $\|\varphi\|_{cb} \leq n \cdot \|\varphi\|$.

Proof. Suppose that $\dim W = n$. Let $\{w_1, w_2, \ldots, w_n\}$ be an Auerbach basis for $W$ with $\|w_i\| = 1$. Then there exists a basis $\{g_1, g_2, \ldots, g_n\}$ in $W^*$ such that $\|g_i\| = 1$
and \( g_j(w_i) = \delta_{i,j} \). For any \( w \in W \),

\[
\sum_{j=1}^{n} \theta_{w_j} \circ g_j(w) = \sum_{j=1}^{n} \theta_{w_j} \circ g_j(\sum_{i=1}^{n} \alpha_i w_i) \\
= \sum_{j=1}^{n} \theta_{w_j}(\sum_{i=1}^{n} \alpha_i g_j(w_i)) \\
= \sum_{j=1}^{n} \theta_{w_j}(\alpha_j) \\
= \sum_{j=1}^{n} \alpha_j w_j = w,
\]

where \( \theta_{w_j} : \mathbb{C} \to W \) are the complete isometries such that \( \theta_{w_j}(\alpha) = \alpha w_j \). Therefore, \( \sum_{j=1}^{n} \theta_{w_j} \circ g_j \) is an identity operator on \( W \) denoted by \( \text{id}_W \). Thus, we have that

\[
\varphi = \text{id}_W \circ \varphi = \sum_{j=1}^{n} \theta_{w_j} \circ g_j \circ \varphi.
\]

Note that each \( g_j \circ \varphi \) is a bounded linear functional on \( V \) and hence, by Corollary 1, \( \|g_j \circ \varphi\|_{cb} = \|g_j \circ \varphi\| \leq \|\varphi\| \). It follows that

\[
\|\varphi\|_{cb} = \| \sum_{j=1}^{n} \theta_{w_j} \circ g_j \circ \varphi \|_{cb} \\
\leq \sum_{j=1}^{n} \|\theta_{w_j}\|_{cb} \|g_j \circ \varphi\|_{cb} \\
= \sum_{j=1}^{n} \|g_j \circ \varphi\| \\
\leq n \cdot \|\varphi\|.
\]

Thus, we get \( \|\varphi\|_{cb} \leq n \cdot \|\varphi\| \).

Similarly, if \( V \) is \( n \)-dimensional then we may replace \( W \) by \( \varphi(V) \). \( \square \)

**Corollary 3.** If \( V \) and \( W \) are \( n \)-dimensional abstract operator spaces, then there exists a linear isomorphism \( \varphi : V \to W \) such that \( \|\varphi\|_{cb} \|\varphi^{-1}\|_{cb} \leq n^2 \).

**Proof.** Let \( v_i \in V \) and \( w_i \in W(i = 1, 2, \cdots, n) \) be Auerbach bases and \( f_i \in V^* \) and \( g_i \in W^*(i = 1, 2, \cdots, n) \) be the associated dual bases. We let

\[
\varphi : V \to W : v \mapsto \sum_{i=1}^{n} f_i(v)w_i.
\]
Then
\[ \varphi^{-1} : W \rightarrow V : w \mapsto \sum_{i=1}^{n} g_i(w) u_i. \]

Since \( \varphi(v) = \sum_{i=1}^{n} f_i(v) w_i = \sum_{i=1}^{n} \theta_{w_i}(f_i(v)) \), we get \( \varphi = \sum_{i=1}^{n} \theta_{w_i} \circ f_i \). Thus,
\[ \|\varphi\| \leq \sum_{i=1}^{n} \|\theta_{w_i}\| \|f_i\| = n. \]

Similarly,
\[ \|\varphi^{-1}\| \leq \sum_{i=1}^{n} \|\theta_{v_i}\| \|f_i\| = n. \]

Therefore, we obtain \( \|\varphi\| \|\varphi^{-1}\| \leq n^2 \).

For any abstract operator space \( V \) and a linear functional \( \varphi : V \rightarrow \mathbb{C} \), by Corollary 1, \( \|\varphi\| = \|\varphi\| \). So, we have the Banach space identification

\[ (7) \quad V^* = B(V, \mathbb{C}) = CB(V, \mathbb{C}). \]

Moreover, there are other abstract operator spaces \( W \) for which

\[ (8) \quad B(V, W) = CB(V, W). \]

We will investigate these spaces in Section 3.3. We now prove that if \( W \) is any commutative \( C^* \)-algebra, then (8) holds.

**Proposition 4.** Let \( V \) be an abstract operator space and let \( \mathcal{A} \) be a commutative \( C^* \)-algebra. Then any bounded linear mapping \( \varphi : V \rightarrow \mathcal{A} \) satisfies \( \|\varphi\| = \|\varphi\| \).

**Proof.** We know that \( \mathcal{A} \) is \( * \)-isomorphic to the commutative \( C^* \)-algebra \( C_0(\Omega) \), where \( \Omega \) is a locally compact Hausdorff space and \( C_0(\Omega) \) is the space of continuous complex-valued functions on \( \Omega \) vanishing at \( \infty \).
We can identify $M_n(C_0(\Omega))$ with $C_0(\Omega, M_n)$, the corresponding matrix valued functions: if $f = [f_{i,j}] \in M_n(C_0(\Omega))$ with $f_{i,j} \in C_0(\Omega)$ and $\omega \in \Omega$, then $f(\omega) = [f_{i,j}(\omega)] \in M_n$. So, $||f|| = \sup\{||f(\omega)|| : \omega \in \Omega\} = \sup\{||f_{i,j}(\omega)|| : \omega \in \Omega\}$.

Let $v \in M_n(V)$ and consider $\varphi_n : M_n(V) \to M_n(C_0(\Omega))$. Then

$$||\varphi_n(v)|| = ||[\varphi(v_{i,j})]||$$

$$= \sup\{||\varphi(v_{i,j})(\omega)|| : \omega \in \Omega\}$$

$$= \sup\{||\varphi(v_{i,j})(\omega)|| \alpha, \beta \in \mathbb{C}^n\text{ and } ||\alpha||_2 = ||\beta||_2 = 1\}$$

$$= \sup_{\omega, \alpha, \beta} \{\sum_{i,j} \beta_i \varphi(v_{i,j})(\omega) \alpha_j\}\}.$$

If we let $\alpha$ and $\beta$ be column matrices, then

$$||\varphi_n(v)|| = \sup_{\omega, \alpha, \beta} \{|\varphi(\sum_{i,j} \beta_i v_{i,j} \alpha_j)(\omega)|\}$$

$$= \sup_{\omega, \alpha, \beta} \{|\varphi(\beta^* v \alpha)(\omega)|\}$$

$$\leq ||\varphi|| \cdot \sup_{\alpha, \beta} \{|\beta^* v \alpha|| \alpha||\}$$

$$\leq ||\varphi|| \cdot \sup_{\alpha, \beta} \{|\beta^* v|| \alpha||\}$$

$$= ||\varphi|| ||v||,$$

i.e., $||\varphi_n|| \leq ||\varphi||$ for all $n \in \mathbb{N}$. Thus, $||\varphi||_{cb} \leq ||\varphi||$ and hence $||\varphi||_{cb} = ||\varphi||$. \hfill \Box

For any $C^*$-algebras $A$ and $B$, a $*$-homomorphism $\varphi : A \to B$ is completely contractive. To see this, consider the corresponding linear mapping

$$\varphi_n : M_n(A) \to M_n(B).$$

Let $n \in \mathbb{N}$ and $v = [v_{i,j}]$, $w = [w_{i,j}] \in M_n(A)$. Then $vw = \sum_{k=1}^n v_{i,k} w_{k,j}$ and

$$\varphi_n(vw) = [\varphi(\sum_{k=1}^n v_{i,k} w_{k,j})]$$

$$= \sum_{k=1}^n \varphi(v_{i,k} w_{k,j})$$

$$= \sum_{k=1}^n \varphi(v_{i,k}) \varphi(w_{k,j})$$

$$= \varphi(v_{i,j})[\varphi(w_{i,j})]$$

$$= \varphi_n(v)\varphi_n(w).$$
Also,
\[
\varphi_n(v^*) = \varphi_n([v_{i,j}^*])
= [\varphi(v_{j,i}^*)]
= [\varphi(v_{i,j})^*]
= [\varphi(v_{i,j})]^*
= \varphi_n(v)^*.
\]
Thus, \(\varphi_n\) is also a \(*\)-homomorphism for each \(n \in \mathbb{N}\).

Since
\[
\|\varphi_n(v)\|^2 = \|\varphi_n(v)^*\varphi_n(v)\|
= \|\varphi_n(v^*v)\|
\leq \|\varphi_n\|\|v^*v\|
= \|\varphi_n\|\|v\|^2,
\]
\(\|\varphi_n\|^2 \leq \|\varphi_n\|\) and we obtain \(\|\varphi_n\| \leq 1\). Therefore, \(\varphi\) is completely contractive.

Let \(V\) be an abstract operator space. Given contractions \(\mu \in M_{m,n}, \gamma \in M_{n,m}\),
\[
\varphi : M_n(V) \to M_m(V) : v \mapsto \mu v \gamma
\]
is completely contractive. To see this, let \(r \in \mathbb{N}\) and consider
\[
\varphi_r : M_r(M_n(V))(\cong M_{r \times n}(V)) \to M_r(M_m(V))(\cong M_{r \times m}(V)).
\]
For any \(v \in M_{r \times n}(V)\), \(\varphi_r(v) = [\varphi(v_{i,j})] = [\mu v_{i,j} \gamma] = (I_r \otimes \mu)v(I_r \otimes \gamma)\) and hence
\[
\|\varphi_r(v)\| = \|(I_r \otimes \mu)v(I_r \otimes \gamma)\|
\leq \|I_r \otimes \mu\|\|v\|\|I_r \otimes \gamma\|
\leq \|v\|.
\]
Thus, we get \(\|\varphi_r\| \leq 1\) for all \(r\) and \(\varphi\) is completely contractive.

It is easy to see that if \(\varphi_i : V \to W_i\) are completely contractive, then
\[
\varphi(v) = \varphi_1(v) \oplus \cdots \oplus \varphi_n(v)
\]
is also completely contractive.
Now, consider the diagonal truncation $D_n : M_n(V) \rightarrow M_n(V)$ such that

$$D_n(v) = D_n \left( \begin{bmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{n,1} & \cdots & v_{n,n} \end{bmatrix} \right) = \begin{bmatrix} v_{1,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & v_{n,n} \end{bmatrix}.$$ 

Since $D_n(v) = E_1 v E_1^* + \cdots + E_n v E_n^*$, where $E_i = [0, \cdots, 1_i, \cdots, 0]$, by the above two paragraphs, we have that $D_n$ is completely contractive.

**Proposition 5.** For any $n \in \mathbb{N}$, the transpose mapping $t : M_n \rightarrow M_n$ is an isometry with $\|t\|_{cb} = n$.

**Proof.** For all $\alpha \in M_n$,

$$\|\alpha\| = \sup \{ |\langle \alpha \xi, \eta \rangle| : \xi, \eta \in \mathbb{C}^n \text{ and } \|\xi\| = \|\eta\| = 1 \}$$

$$= \sup \{ |\langle \bar{\eta}, \bar{\alpha} \xi \rangle| : \xi, \eta \in \mathbb{C}^n \text{ and } \|\xi\| = \|\eta\| = 1 \}$$

$$= \sup \{ |\langle \alpha^{tr} \bar{\eta}, \bar{\xi} \rangle| : \xi, \eta \in \mathbb{C}^n \text{ and } \|\xi\| = \|\eta\| = 1 \}$$

$$= \|\alpha^{tr}\|.$$ 

Thus, $t$ is an isometry.

Any $\alpha \in M_n$ can be written as a sum of $n$ generalized diagonal matrices. That is,

$$\alpha = \begin{bmatrix} \alpha_{1,1} & 0 & \cdots & 0 \\ 0 & \alpha_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \alpha_{n,n} \end{bmatrix} + \begin{bmatrix} 0 & \alpha_{1,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \alpha_{n-1,n} \\ \alpha_{n,1} & \cdots & \cdots & 0 \end{bmatrix} + \cdots.$$ 

Let $\pi = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$ be the cyclical permutation matrix. Then

$$\alpha = D_n(\alpha) + D_n(\alpha \pi) \pi^{-1} + D_n(\alpha \pi^2) \pi^{-2} + \cdots + D_n(\alpha \pi^{n-1}) \pi^{-(n-1)}$$

$$= \sum_{k=0}^{n-1} D_n(\alpha \pi^k) \pi^{-k},$$ 

where the diagonal truncation $D_n : M_n(V) \rightarrow M_n(V)$ is completely contractive.

Now,

$$t(\alpha) = t(\sum_{k=0}^{n-1} D_n(\alpha \pi^k) \pi^{-k}) = \sum_{k=0}^{n-1} D_n(\alpha \pi^k)^{tr} \pi^{-k}$$

$$= \sum_{k=0}^{n-1} (\pi^{-k})^{tr} D_n(\alpha \pi^k) = \sum_{k=0}^{n-1} \pi^k D_n(\alpha \pi^k).$$
since \((\pi^{-k})^{tr} = ((\pi^{-1})^{tr})^{tr} = (\pi^{-1})^{tr} = \pi^{k}\). So, \[\|t\|_{cb} = \sum_{k=0}^{n-1} \|\alpha \mapsto \pi^{k} D_n(\alpha \pi^{k})\|_{cb} \leq n.\]

In the following, we need to show that \(\|t\|_{cb} \geq n\). Let \(\epsilon = [\epsilon_{i,j}] \in M_n(M_n)\), where 
\[\epsilon_{i,j} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1_{i,j} & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.\] Then

\[
\bar{\epsilon} = t_n(\epsilon) = \begin{bmatrix} \epsilon_{11} & \epsilon_{21} & \cdots & \cdots \\ \epsilon_{12} & \epsilon_{22} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\
\end{bmatrix}.
\]

We can see that \(\bar{\epsilon}\) is a permutation matrix and hence \(\bar{\epsilon}\) is unitary.

Let \(\epsilon_i = [0 \cdots 0 1_i 0 \cdots 0]^{tr}\). Then

\[
\epsilon_{i,j} = \epsilon_i \epsilon_j^{*}, \quad t_n(\bar{\epsilon}) = \epsilon = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix} = \begin{bmatrix} \epsilon_1^* \\ \cdots \\ \epsilon_n^* \end{bmatrix}.
\]

So
\[
\|t_n(\bar{\epsilon})\| = \|\epsilon\| = \|\begin{bmatrix} \epsilon_1^* & \cdots & \epsilon_n^* \end{bmatrix}\| \leq \| \begin{bmatrix} \epsilon_1^* \\ \cdots \\ \epsilon_n^* \end{bmatrix} \| 
\leq \| \begin{bmatrix} \epsilon_1 \\ \cdots \\ \epsilon_n \end{bmatrix} \| = n \leq \|t_n\| \|\bar{\epsilon}\| = \|t_n\| \|t_n\| \geq n.
\]

\[
\text{i.e., } \|t_n\| \geq n. \text{ Therefore, } \|t\|_{cb} = \|t_n\| \geq n. \quad \square
\]

2.3. The **Representation Theorem.** We know that any normed space can be represented as a function space by the Hahn-Banach theorem. In this section, we will represent an abstract operator space as an operator space on a Hilbert space.
We will relate matrix-valued functionals on a space \( V (\varphi : V \to M_n) \) to scalar functionals on the matrix spaces over \( V (F : M_n(V) \to \mathbb{C}) \) and then we apply the classical Hahn-Banach theorem.

**Definition 3.** Let \( K \) be a convex subset of a real linear space \( V \). Then \( e : K \to \mathbb{R} \) is affine if \( e(\alpha x + (1 - \alpha)y) = \alpha e(x) + (1 - \alpha)e(y) \) for all \( \alpha \in [0, 1] \) and \( x, y \in K \).

**Lemma 2.** Suppose that \( \mathcal{E} \) is a cone of real continuous affine functions on a compact convex subset \( K \) of a topological linear space \( E \) and that for each \( e \in \mathcal{E} \), there exists a corresponding point \( k_e \in K \) with \( e(k_e) \geq 0 \). Then there is a point \( k_0 \in K \) such that \( e(k_0) \geq 0 \) for all \( e \in \mathcal{E} \).

**Proof.** For each \( e \in \mathcal{E} \), let \( K(e) = \{ k \in K : e(k) \geq 0 \} \). We will show that \( \bigcap_{e \in \mathcal{E}} K(e) \neq \emptyset \). By the hypotheses, for each \( e \in \mathcal{E} \), \( K(e) \neq \emptyset \) and \( K(e) \) is a closed subset of the compact set \( K \) since \( e \) is continuous. Now, suppose that \( \bigcap_{e \in \mathcal{E}} K(e) = \emptyset \). Then there exists \( e_1, e_2, \cdots, e_n \in \mathcal{E} \) such that \( K(e_1) \cap \cdots \cap K(e_n) = \emptyset \).

Let \( \theta : K \to \mathbb{R}^n \) be defined by \( \theta(k) = (e_1(k), \cdots, e_n(k)) \). Then \( \theta \) is continuous and affine. Since \( K \) is a compact convex set, \( \theta(K) \) is a compact convex set in \( \mathbb{R}^n \). Since there doesn't exist \( k \in K \) such that \( e_i(k) \geq 0 \) for \( i = 1, 2, \cdots, n \), we have \( \theta(K) \cap (\mathbb{R}^n)^+ = \emptyset \).

From the usual geometric separation, there exists a linear functional \( f \) on \( \mathbb{R}^n \) such that \( f((\mathbb{R}^n)^+) \geq 0 \) and \( f(\theta(K)) < 0 \), where \( f(x_1, \cdots, x_n) = c_1 x_1 + \cdots + c_n x_n \) for some constants \( c_i \geq 0 \). Since

\[
f(\theta(k)) = f((e_1(k), \cdots, e_n(k))) = c_1 e_1(k) + \cdots + c_n e_n(k) < 0
\]
and $E$ is a cone, we get $e = f \circ \theta = c_1 e_1 + \cdots + c_n e_n \in E$. Thus $K(e) = \emptyset$, contradicting the hypothesis $K(e) \neq \emptyset$. Therefore, there exists $k_0 \in K$ such that $e(k_0) \geq 0$ for all $e \in E$. \hfill \Box

**Lemma 3.** If $V$ is an abstract operator space and $F \in [M_n(V)]^*$ such that $\|F\| = 1$, then there exist states $p_0$ and $q_0$ on $M_n$ such that

$$|F(\alpha v \beta)| \leq p_0(\alpha \alpha^*)^{1/2} \|v\| q_0(\beta^* \beta)^{1/2}$$

for all $\alpha \in M_{n,r}, \beta \in M_{r,n}$ and $v \in M_r(V)$ ($r \in \mathbb{N}$).

**Proof.** Let $r \in \mathbb{N}$, $\alpha \in M_{n,r}, \beta \in M_{r,n}$, and $v \in M_r(V)$ with $\|v\| = 1$. Let $S_n = \{f : M_n \rightarrow \mathbb{C} \text{ is positive and } \|f\| = 1\}$ be the state space of $M_n$. Note that there exists a $\theta > 0$ such that $\operatorname{Re} F(e^{i\theta} \alpha v \beta) = |F(e^{i\theta} \alpha v \beta)| = |F(\alpha v \beta)|$. So, if we replace $\alpha$ by $e^{-i\theta} \alpha$, then $\operatorname{Re} F(\alpha v \beta) = |F(\alpha v \beta)|$. That is, it suffices to find $p_0, q_0 \in S_n$ such that

$$\operatorname{Re} F(\alpha v \beta) \leq p_0(\alpha \alpha^*)^{1/2} q_0(\beta^* \beta)^{1/2}.$$  

In fact, we can replace (*) by

$$\operatorname{Re} F(\alpha v \beta) \leq \frac{1}{2} [p_0(\alpha \alpha^*) + q_0(\beta^* \beta)].$$

To see this, let us replace $\alpha$ by $t^{1/2} \alpha$ and $\beta$ by $t^{-1/2} \beta$ for $t > 0$. Then

$$\operatorname{Re} F(\alpha v \beta) \leq \frac{1}{2} [tp_0(\alpha \alpha^*) + t^{-1} q_0(\beta^* \beta)].$$

If $p_0(\alpha \alpha^*) \neq 0$ and $q_0(\beta^* \beta) \neq 0$, then by letting $t = p_0(\alpha \alpha^*)^{-1/2} q_0(\beta^* \beta)^{1/2}$, we get (*). If $p_0(\alpha \alpha^*) = 0$ and $q_0(\beta^* \beta) \neq 0$(or $p_0(\alpha \alpha^*) \neq 0$ and $q_0(\beta^* \beta) = 0$), then by letting $t \rightarrow \infty$, (*) holds. If $p_0(\alpha \alpha^*) = 0$ and $q_0(\beta^* \beta) = 0$, then $\operatorname{Re} F(\alpha v \beta) = 0$.

Thus, in all cases we have (*).

To apply Lemma 2, let $K = S_n \times S_n$. Since $S_n$ is a weak* compact and convex subset of $M_n^*$, $K$ is a weak* compact and convex subset of $(M_n \oplus M_n)^*$.
Let $A(K)$ be the linear space of continuous real-valued affine functions on $K$.

Given $r \in \mathbb{N}, \alpha \in M_{n,r}, \beta \in M_{r,n}$ and $\nu \in M_{r}(V)$ with $\|\nu\| = 1$, we define a corresponding function $e_{\alpha,\nu,\beta} \in A(K)$ by $e_{\alpha,\nu,\beta}(p, q) = p(\alpha \alpha^*) + q(\beta^* \beta) - 2ReF(\alpha \nu \beta)$.

Let $\mathcal{E} = \{e_{\alpha,\nu,\beta} \in A(K) : \alpha \in M_{n,r}, \beta \in M_{r,n}, \nu \in M_{r}(V) \text{ and } \|\nu\| = 1\}$. For each $e \in \mathcal{E}$, consider $(p_{e}, q_{e}) \in K$ such that

$$p_{e}(\alpha \alpha^*) = \|\alpha \alpha^*\| = \|\alpha\|^2 \quad \text{and} \quad q_{e}(\beta^* \beta) = \|\beta^* \beta\| = \|\beta\|^2.$$ 

Then

$$e(p_{e}, q_{e}) = \|\alpha\|^2 + \|\beta\|^2 - 2ReF(\alpha \nu \beta) \geq 0$$

since $ReF(\alpha \nu \beta) \leq |F(\alpha \nu \beta)| \leq \|\alpha \nu \beta\| \leq \|\alpha\||\nu\||\beta\| \leq \frac{1}{2}(\|\alpha\|^2 + \|\beta\|^2)$. Moreover, it is easy to see that $\mathcal{E}$ is a cone in $A(K)$. Thus, since the hypotheses of Lemma 2 are satisfied, we can find $(p_{0}, q_{0}) \in K$ such that $e(p_{0}, q_{0}) \geq 0$ for all $e \in \mathcal{E}$, i.e., there exist states $p_{0}$ and $q_{0}$ such that $ReF(\alpha \nu \beta) \leq \frac{1}{2}[p_{0}(\alpha \alpha^*) + q_{0}(\beta^* \beta)]$ for all $\alpha \in M_{n,r}, \beta \in M_{r,n}, \nu \in M_{r}(V)$ with $\|\nu\| = 1(r \in \mathbb{N})$. \hfill \qed

A representation of a $C^*$-algebra $\mathcal{A}$ is a pair $(\pi, H)$, where $H$ is a Hilbert space and $\pi : \mathcal{A} \rightarrow \mathcal{B}(H)$ is a $*$-homomorphism. We usually omit $H$ and say that $\pi$ is a representation of $\mathcal{A}$. A representation $\pi$ of a $C^*$-algebra $\mathcal{A}$ is cyclic if there is a vector $e \in H$ such that $cl(\pi(\mathcal{A})e) = H$; here $e$ is called a cyclic vector for the representation $\pi$.

Let $\pi : \mathcal{A} \rightarrow \mathcal{B}(H)$ be a cyclic representation with cyclic vector $e$. If we define $f : \mathcal{A} \rightarrow \mathbb{C}$ by $f(a) = \langle \pi(a)e, e \rangle$, then $f$ is a bounded linear functional on $\mathcal{A}$ with $\|f\| \leq \|e\|^2$ (since $|f(a)| \leq \|\pi(a)||e||^2 \leq \|a||e||^2$).
We now state two theorems in functional analysis without proof which are useful later (see [1, page 28 and 257]).

**Theorem 2.** Let \((X, \mathcal{M}, \mu)\) be a \(\sigma\)-finite measure space and \(H = L_2(X, \mathcal{M}, \mu) = L_2(X)\) the Hilbert space of square \(\mu\)-integrable functions on \(X\). If we define \(M_\varphi : L_2(X) \to L_2(X)\) by \(M_\varphi(f) = \varphi \cdot f\) for all \(\varphi \in L_\infty(X)\), then \(M_\varphi \in \mathcal{B}(L_2(X))\) and 
\[
\|M_\varphi\| = \|\varphi\|_\infty, \text{ where } \|\varphi\|_\infty = \inf\{t > 0 : \mu(\{x \in X : |\varphi(x)| > t\}) = 0\}.
\]

It follows that \(\pi : L_\infty(X) \to \mathcal{B}(L_2(X))\) defined by \(\pi(\varphi) = M_\varphi\) is a representation of the \(C^*\)-algebra \(L_\infty(X)\). \(M_\varphi\) in \(\mathcal{B}(L_2(X))\) is called a **multiplication operator**.

**Theorem 3.** (Gelfand-Naimark-Segal Construction) Let \(\mathcal{A}\) be a \(C^*\)-algebra.

(i) If \(f\) is a positive linear functional on \(\mathcal{A}\), then there is a cyclic representation 
\((\pi_f, H_f)\) of \(\mathcal{A}\) with cyclic vector \(e\) such that \(f(a) = \langle \pi_f(a)e, e \rangle\) for all \(a \in \mathcal{A}\).

(ii) If \((\pi, H)\) is a cyclic representation of \(\mathcal{A}\) with cyclic vector \(e\) and \(f(a) = \langle \pi(a)e, e \rangle\) and if \((\pi_f, H_f)\) is constructed as in (i), then \(\pi\) and \(\pi_f\) are equivalent.

**Lemma 4.** Let \(V\) be an abstract operator space. Given a linear functional \(F \in [M_n(V)]^*\) with \(\|F\| = 1\), there exists a complete contraction \(\varphi : V \to M_n\) and unit vectors \(\eta, \xi \in (\mathbb{C}^n)^n\) such that \(F(v) = \langle \varphi_n(v)\eta, \xi \rangle\) for all \(v \in M_n(V)\).

**Proof.** Step 1. Let \(p_0\) and \(q_0\) be the states on \(M_n\) (as in Lemma 3) such that 
\[
|F(\alpha v \beta)| \leq \sqrt{\alpha} \sqrt{\beta} \langle v \rangle \|q_0(\beta^* \beta)^{1/2} \rangle^{1/2}
\]
for all \(\alpha \in M_n, \beta \in M_r, v \in M_r(V)\) \((r \in \mathbb{N})\). Since \(M_n\) is a \(C^*\)-algebra, by the GNS theorem, there are representations \(\pi : M_n \to \mathcal{B}(H)\) and \(\theta : M_n \to \mathcal{B}(K)\) and
cyclic vectors $\xi_0 \in H$ and $\eta_0 \in K$ such that

\[ p_0(\alpha') = \langle \pi(\alpha')\xi_0, \xi_0 \rangle, \quad q_0(\alpha') = \langle \theta(\alpha')\eta_0, \eta_0 \rangle \]

for all $\alpha' \in M_n$, where $H$ and $K$ are finite-dimensional Hilbert spaces.

Given a row matrix $\alpha = [\alpha_1 \cdots \alpha_n] \in M_{1,n}$, we define $\tilde{\alpha} \in M_n$ by

\[ \tilde{\alpha} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 0 & \cdots & \cdots & 0 \\ \vdots & \cdot & \cdot & \cdot \\ 0 & \cdots & \cdots & 0 \end{bmatrix}. \]

Let $\tilde{M}_{1,n} = \{ \tilde{\alpha} : [\alpha_1 \cdots \alpha_n] \in M_n \}, \ H_0 = \pi(\tilde{M}_{1,n})\xi_0 \subseteq H$ and $K_0 = \theta(\tilde{M}_{1,n})\eta_0 \subseteq K$.

If we fix $v \in V$, the sesquilinear form $B_v : K_0 \times H_0 \to \mathbb{C}$ defined by

\[ B_v(\theta(\tilde{\beta})\eta_0, \pi(\tilde{\alpha})\xi_0) = F(\alpha^*v\beta) \]

is bounded, since

\[ |F(\alpha^*v\beta)| \leq p_0(\alpha^*\alpha)^{1/2}\|v\|q_0(\beta^*\beta)^{1/2} \]
\[ = \langle \pi(\alpha^*\alpha)\xi_0, \xi_0 \rangle^{1/2}\|v\|\langle \theta(\beta^*\beta)\eta_0, \eta_0 \rangle^{1/2} \]
\[ = \langle \pi(\tilde{\alpha}^*)\pi(\tilde{\alpha})\xi_0, \xi_0 \rangle^{1/2}\|v\|\langle \theta(\tilde{\beta})^*\theta(\tilde{\beta})\eta_0, \eta_0 \rangle^{1/2} \quad \text{(since $\alpha^*\alpha = \tilde{\alpha}^*\tilde{\alpha}$)} \]
\[ = \|\pi(\tilde{\alpha})\xi_0\|\|v\|\|\theta(\tilde{\beta})\eta_0\|. \]

Thus, there is a bounded linear operator $\varphi_0(v) : K_0 \to H_0$ such that

\[ B_v(\theta(\tilde{\beta})\eta_0, \pi(\tilde{\alpha})\xi_0) = \langle \varphi_0(v)\theta(\tilde{\beta})\eta_0, \pi(\tilde{\alpha})\xi_0 \rangle = F(\alpha^*v\beta). \]

We see that the corresponding mapping $\varphi_0 : V \to B(K_0, H_0)$ is linear.

Let $\dim K_0 = k$ and $\dim H_0 = h$. Since $k \leq n$ and $h \leq n$, $K_0$ and $H_0$ can be identified with $\mathbb{C}^k \oplus 0_{n-k}$ and $\mathbb{C}^h \oplus 0_{n-h}$, respectively. If $E$ is the projection of $\mathbb{C}^n$ onto $K_0$ and we define $\varphi(v) = \varphi_0(v)E : \mathbb{C}^n \to \mathbb{C}^n$ ($v \in V$), then the mapping $\varphi : V \to M_n$ satisfies

\[ F(\alpha^*v\beta) = \langle \varphi(v)\theta(\tilde{\beta})\eta_0, \pi(\tilde{\alpha})\xi_0 \rangle. \]
Given $v = [v_{i,j}] \in M_n(V)$, from (10), we have

$$F(v) = \sum_{i,j} F(E^*_i v_{i,j} E_j) = \sum_{i,j} \langle \phi(v_{i,j}) \theta(E_j) \eta_0, \pi(E_i) \xi_0 \rangle = \langle \varphi_n(v) \eta, \xi \rangle,$$

where

$$\xi = \begin{bmatrix} \pi(E_1) \xi_0 \\ \vdots \\ \pi(E_n) \xi_0 \end{bmatrix}, \quad \eta = \begin{bmatrix} \theta(E_1) \eta_0 \\ \vdots \\ \theta(E_n) \eta_0 \end{bmatrix} \in (\mathbb{C}^n)^n.$$

**Step 2.** Since $\|\pi(E_i) \xi_0\|^2 = \langle \pi(E_i)^* \pi(E_i) \xi_0, \xi_0 \rangle = \langle \pi(E_i^* E_i) \xi_0, \xi_0 \rangle = p_0(E_i^* E_i)$,

$$\|\xi\|^2 = \sum_i \|\pi(E_i) \xi_0\|^2 = \sum_i p_0(E_i^* E_i) = p_0(\sum_i E_i^* E_i) = p_0(I) = 1.$$

Thus, $\xi$ is a unit vector and similarly $\eta$ is also a unit vector.

**Step 3.** Finally we show that $\varphi$ is completely contractive.

Let $\xi_1 = \begin{bmatrix} \pi(\tilde{\alpha}_1) \xi_0 \\ \vdots \\ \pi(\tilde{\alpha}_n) \xi_0 \end{bmatrix} \in H^*_0$ and $\eta_1 = \begin{bmatrix} \theta(\tilde{\beta}_1) \eta_0 \\ \vdots \\ \theta(\tilde{\beta}_n) \eta_0 \end{bmatrix} \in K^*_0$, where $\alpha_i, \beta_j \in M_{1,n}$.

We have

$$\|\xi_1\|^2 = \sum_i \|\pi(\tilde{\alpha}_i) \xi_0\|^2 = \sum_i p_0(\alpha_i^* \alpha_i) = p_0(\sum_i \alpha_i^* \alpha_i) = p_0(\alpha^* \alpha),$$

and similarly, $\|\eta_1\|^2 = q_0(\beta^* \beta)$, where $\alpha = [\alpha_1 \cdots \alpha_n]^tr$, $\beta = [\beta_1 \cdots \beta_n]^tr \in M_n$.

For all $v = [v_{i,j}] \in M_n(V)$, we have

$$\langle (\varphi_0)_n(v) \eta_1, \xi_1 \rangle = \langle (\varphi_0(v_{i,j})) \eta_1, \xi_1 \rangle = \sum_{i,j} \langle \varphi_0(v_{i,j}) \theta(\tilde{\beta}_j) \eta_0, \pi(\tilde{\alpha}_i) \xi_0 \rangle = \sum_{i,j} F(\alpha_i^* v_{i,j} \beta_j) = F(\alpha^* v \beta).$$
Thus,
\[ |\langle (\varphi_0)_n(v), \xi_1 \rangle| = |F(\alpha^* v \beta)| \]
\[ \leq p_0 (\alpha^* \alpha)^{1/2} \|v\| q_0 (\beta^* \beta)^{1/2} \]
\[ = \|v\| \|\xi_1\| \|\eta_1\|. \]
It follows that \( \|\langle (\varphi_0)_n(v) \rangle\| \leq \|v\| \) for all \( v \in M_n(V) \) and hence \( \|\varphi_0\| \leq 1 \). Since \( \|\varphi_n\| \leq \|\langle (\varphi_0)_n \rangle\| \), we obtain \( \|\varphi_n\| \leq 1 \). Therefore, \( \|\varphi\|_{cb} = \|\varphi_n\| \leq 1 \) (Proposition 3), i.e., \( \varphi \) is completely contractive. \( \square \)

The following is the matrix-valued analogue of the classical Hahn-Banach theorem which is used to prove the representation theorem for abstract operator spaces.

**Lemma 5.** Let \( V \) be an abstract operator space. For all \( v \in M_n(V) \), there exists a complete contraction \( \varphi : V \to M_n \) such that \( \|\varphi_n(v)\| = \|v\| \).

**Proof.** Let \( v \in M_n(V) \). By the Hahn-Banach Theorem, there exists an \( F \in [M_n(V)]^* \) such that \( |F(v)| = \|v\| \) and \( \|F\| = 1 \). From Lemma 4, there is a complete contraction \( \varphi : V \to M_n \) unit vectors \( \eta, \xi \in (\mathbb{C}^n)^n \) such that \( F(v) = \langle \varphi_n(v) \rangle \eta, \xi \). Since
\[ \|v\| = |F(v)| = |\langle \varphi_n(v) \rangle \eta, \xi| \leq \|\varphi_n(v)\| \leq \|\varphi_n\| \|v\| \leq \|v\|, \]
we have \( \|\varphi_n(v)\| = \|v\| \). \( \square \)

**Theorem 4.** *(The Representation Theorem)* If \( V \) is an abstract operator space, then there is a Hilbert space \( H \), a concrete operator space \( W \subseteq B(H) \) and a complete isometry \( \Phi \) of \( V \) onto \( W \).

**Proof.** For each \( n \in \mathbb{N} \), let \( S_n = CB(V, M_n) \|_{\alpha \leq 1} \) and \( S = \bigcup_{n \in \mathbb{N}} S_n \).

Let \( H = \bigoplus_{\varphi \in S} \mathbb{C}^{n(\varphi)} \), where \( n(\varphi) \) is the integer \( n \) with \( \varphi \in S_n \), and we define
\[ \Phi : V \to B(H) : v \mapsto \langle \varphi(v) \rangle_{\varphi \in S}. \]
Then, by definition,
\[
\Phi_n : M_n(V) \rightarrow M_n(B(H)) \cong B(H^n) : v \mapsto (\varphi_n(v))_{\varphi \in \mathcal{S}}.
\]
Since each \( \varphi \in \mathcal{S} \) is a complete contraction, \( \Phi \) is also a complete contraction.

Given \( v \in M_n(V) \), from Lemma 5, there is a complete contraction \( \varphi_0 \in \mathcal{S}_n \) such that \( \|(\varphi_0)_n(v)\| = \|v\| \). This implies that \( \|\Phi_n(v)\| \geq \|(\varphi_0)_n(v)\| = \|v\| \). On the other hand, \( \|\Phi_n(v)\| \leq \|\Phi_n\|\|v\| \leq \|v\| \). Therefore, \( \|\Phi_n(v)\| = \|v\| \), i.e., \( \Phi \) is a complete isometry from \( V \) onto \( W(=(\varphi(v))_{\varphi \in \mathcal{S}} : v \in V) \) \( \subseteq B(H) \).

By Proposition 1 and Theorem 4, we shall not distinguish between abstract and concrete operator spaces.

The following proposition is an alternative way to determine operator spaces which is easier to use.

**Proposition 6.** Suppose that \( V \) is a linear space and that we are provided with mappings \( \| \cdot \| : M_n(V) \rightarrow [0, \infty) \) for all \( n \in \mathbb{N} \), which satisfy

[M1'] \( \|v \oplus w\|_{m+n} \leq \max\{\|v\|_m, \|w\|_n\} \),

[M2] \( \|\alpha v|\beta\|_n \leq \|\alpha\|\|v\|_m\|\beta\| \)

for all \( v \in M_m(V), w \in M_n(V), \alpha \in M_{n,m}, \beta \in M_{m,n} \). Then these mappings are seminorms which satisfy [M1] and [M2]. Furthermore, if \( \| \cdot \|_1 \) is a norm, then each \( \| \cdot \|_n \) is also a norm \( (n \in \mathbb{N}) \) and they determine an operator space structure on \( V \).

**Proof.** It is obvious that \( \|v\|_n \geq 0 \) for all \( v \in M_n(V) \). Given \( v, w \in M_n(V) \) and \( \varepsilon > 0 \), let \( s = \|v\|_n + \varepsilon \) and \( t = \|w\|_n + \varepsilon \). Then \( v = sv \) and \( w = tw \) with \( \|\hat{v}\|_n, \|\hat{w}\|_n < 1 \). We have
\[
v + w = \gamma \begin{bmatrix} \hat{v} & 0 \\ 0 & \hat{w} \end{bmatrix} \gamma^*,
\]
where \( \gamma = [s^{1/2}I_n \ t^{1/2}I_n] \) satisfies \( \|\gamma\| \|\gamma^*\| = \|\gamma \gamma^*\| = \|(s + t)I_n\| = s + t. \)
From \([M1']\) and \([M2]\), we have
\[
\|v + w\|_n \leq \|\gamma\| \|\gamma^*\| \max\{\|v\|_n, \|w\|_n\} < s + t = \|v\|_n + \|w\|_n + 2\varepsilon.
\]
Since \(\varepsilon > 0\) is arbitrary, we have \(\|v + w\|_n \leq \|v\|_n + \|w\|_n\), i.e., \(\|\cdot\|_n\) is subadditive.

Now, for \(\alpha(\neq 0) \in \mathbb{C}\), since \(\alpha v = (\alpha I_n)v\), we have
\[
\|\alpha v\|_n = \|(\alpha I_n)v\|_n \leq \|\alpha I_n\| \|v\|_n = |\alpha| \|v\|_n = |\alpha| \|\frac{1}{\alpha}(\alpha I_n)v\|_n \leq \|\alpha v\|_n.
\]
Thus, we get \(\|\alpha v\|_n = |\alpha| \|v\|_n\). Therefore, \(\|\cdot\|_n\) is a seminorm on \(M_n(V)\) for each \(n \in \mathbb{N}\).

If \(v \in M_m(V)\) and \(w \in M_n(V)\), then
\[
\|v\|_m = \| [I_m, 0_{m,n}](v \oplus w) [I_m \ 0_{n,m}] \|_m \leq \|v \oplus w\|_{m+n}
\]
and similarly, \(\|w\|_n \leq \|v \oplus w\|_{m+n}\). Thus, \(\|v \oplus w\|_{m+n} \geq \max\{\|v\|_m, \|w\|_n\}\).

From (*) and \([M1']\), we get \(\|v \oplus w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}\), i.e., \([M1]\) holds.

If \(\|\cdot\|_1\) is a norm, then for all \(v_{i,j} \neq 0(\in V), \|v_{i,j}\|_1 > 0.\) Since for any \(v = [v_{i,j}] \in M_n(V), \|v_{i,j}\|_1 = \|E_i v E_j^*\| \leq \|v\|_n\), we obtain that \(\|v\|_n > 0\) for all \(v \neq 0(\in M_n(V))\).
Thus, \(\|\cdot\|_n\) is an operator space matrix norm and determines an operator space structure on \(V\). \(\Box\)
3. Constructions of Operator Spaces

In this chapter we construct other operator spaces from the given ones. In order to show that they are operator spaces, either we will use axioms \([\text{M1}']\) and \([\text{M2}]\) in Proposition 6 or we will apply the representation theorem (Theorem 4).

3.1. Subspaces, Quotients, Products and Conjugates. We divide this section into four subsections.

I. Operator Subspaces and Operator Matrix Spaces.

Let \(V\) be an operator space. If \(W\) is a linear subspace of \(V\), then \(M_n(W)\) is a linear subspace of \(M_n(V)\) and a norm on \(M_n(W)\) is obtained by restricting the operator space matrix norm for \(M_n(V)\) to \(M_n(W)\). It is easy to check that the matrix norm on \(W\) satisfies \([\text{M1}]\) and \([\text{M2}]\) and thus it is an operator space matrix norm on \(W\). We let \(M_n(W)\) denote \(M_n(W)\) with the relative norms and \(W\) is called an operator subspace of \(V\).

For each \(p \in \mathbb{N}\), we use the identification \(M_n(M_p(V)) = M_{np}(V)\) to determine a matrix norm for \(M_p(V)\). Given \(\alpha \in M_{n,m}, \beta \in M_{m,n}\) and \(v \in M_m(M_p(V)) = M_{mp}(V), w \in M_n(M_p(V)) = M_{np}(V)\), we have

\[
\|v \oplus w\|_{m+n} = \|v \oplus w\|_{mp+np} \\
= \max\{\|v\|_{mp}, \|w\|_{np}\} \\
= \max\{\|v\|_m, \|w\|_n\},
\]

and

\[
\|\alpha v \beta\|_n = \|(\alpha \otimes I_p)v(\beta \otimes I_p)\|_{np} \\
\leq \|\alpha \otimes I_p\| \|v\|_{mp} \|\beta \otimes I_p\| \\
= \|\alpha\| \|v\|_m \|\beta\|.
\]

Thus, we can see that \(M_p(V)\) is an operator space. Moreover, if \(\varphi : V \hookrightarrow \mathcal{B}(H)\) is a completely isometric injection, then \(\varphi_p : M_p(V) \hookrightarrow M_p(\mathcal{B}(H)) = \mathcal{B}(H^p)\) is a
corresponding completely isometric injection of $M_p(V)$.

II. Quotient Operator spaces.

Let $V$ be an operator space. We have that if $N$ is a closed subspace of $V$, then $M_n(N)$ is a closed subspace of $M_n(V)$ for each $n \in \mathbb{N}$. To determine the matrix norm on $V/N$, we consider the linear isomorphism

$$M_n(V/N) \cong M_n(V)/M_n(N) : [v_{i,j} + N] \mapsto [v_{i,j}] + M_n(N),$$

where $M_n(V)/M_n(N) = \{v + M_n(N) : v = [v_{i,j}] \in M_n(V)\}$ and $\|v + M_n(N)\| = \inf\{\|v + w\| : w \in M_n(N)\}$. This isomorphism defines a norm on $M_n(V/N)$. We will denote this normed space by $M_n(V/N)$.

Let $\pi : V \to V/N$ be the quotient mapping. Then for each $n \in \mathbb{N}$, the mapping

$$\pi_n : M_n(V) \to M_n(V/N) : v = [v_{i,j}] \mapsto [v_{i,j} + N] = \tilde{v}$$

is a surjection and for $\tilde{v} \in M_n(V/N)$,

$$\|\tilde{v}\| = \inf\{\|v\| : v \in M_n(V), \pi_n(v) = \tilde{v}\}.$$ (11)

The next proposition shows that the matrix norm on $V/N$ satisfies [M1'] and [M2] and thus $V/N$ is an operator space.

**Proposition 7.** If $N$ is a closed subspace of an operator space $V$, then $V/N$ is an operator space.

**Proof.** Let $\pi : V \to V/N$ be the quotient mapping. Given $\alpha \in M_{n,m}, \beta \in M_{m,n}$ and $\tilde{v} \in M_m(V/N)$, there exists $v \in M_m(V)$ such that $\pi_m(v) = \tilde{v}$ and $\|v\| < \|\tilde{v}\| + \varepsilon$. It follows that $\pi_n(\alpha \nu \beta) = \alpha \pi_m(v) \beta = \alpha \tilde{v} \beta$ and $\|\alpha \tilde{v} \beta\| \leq \|\alpha\| \|\nu\| \|\beta\| \leq \|\alpha\| \|\nu\| \|\beta\|$. Therefore, $V/N$ is an operator space.
\[ \|\alpha\|(\|\tilde{v}\| + \varepsilon)\|\beta\|. \] Since \( \varepsilon > 0 \) is arbitrary, we get \( \|\alpha \tilde{v} \beta\| \leq \|\alpha\|\|\tilde{v}\|\|\beta\| \), i.e., \([M2]\) holds.

Given \( \tilde{w} \in M_n(V/N) \) and \( w \in M_n(V) \) such that \( \pi_n(w) = \tilde{w} \) and \( \|w\| < \|\tilde{w}\| + \varepsilon \), we have \( \pi_{m+n}(v \oplus w) = \pi_{m}(v) \oplus \pi_{n}(w) = \tilde{v} \oplus \tilde{w} \) and thus,

\[ \|\tilde{v} \oplus \tilde{w}\| \leq \|v \oplus w\| = \max\{\|v\|, \|w\|\} \leq \max\{\|\tilde{v}\|, \|\tilde{w}\|\} + \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, we obtain \( \|\tilde{v} \oplus \tilde{w}\| \leq \max\{\|\tilde{v}\|, \|\tilde{w}\|\} \), i.e., \([M1']\) is also true.

From Proposition 6, \( V/N \) is an operator space.

\[ \square \]

III. Product Operator Spaces.

If \( \{X_s : s \in S\} \) is a family of normed spaces, then we let \( \prod_{s \in S} X_s \) denote the normed space \( l_\infty(S; X_s) = \{(x_s)_{s \in S} : x_s \in X_s \text{ and } \sup\{\|x_s\| : s \in S\} < \infty\} \) with the norm \( \|(x_s)_{s \in S}\|_\infty = \sup\{\|x_s\| : s \in S\} \).

Let \( \{V_s : s \in S\} \) be a family of operator spaces. We define the product operator space \( \prod_{s \in S} V_s \) to be the space \( l_\infty(S; V_s) \) equipped with the matrix norm determined by the identification \( M_n(\prod_{s \in S} V_s) \cong \prod_{s \in S} M_n(V_s) : [(v_{s}^{ij})_{s \in S}] \mapsto ([v_{s}^{ij})_{s \in S}. \) We can see that the matrix norm on \( \prod_{s \in S} V_s \) satisfies \([M1']\) and \([M2]\). Thus, \( \prod_{s \in S} V_s \) is an operator space.

Alternatively, we can use the representation theorem to show this. That is, given a completely isometric injection \( V_s \hookrightarrow \mathcal{B}(H_s) \) for each \( s \in S \), we obtain a corresponding completely isometric injection \( \prod_{s \in S} V_s \hookrightarrow \mathcal{B}(H), \) where \( H = \bigoplus_{s \in S} H_s. \)

IV. Conjugate Operator Spaces.

If \( V \) is an arbitrary linear space, we define a conjugate space \( \overline{V} \) to be \( V \) with the usual addition and the scalar multiplication \( (\alpha, v) \mapsto \overline{\alpha} v. \) Let \( V \to \overline{V} : v \mapsto \overline{v} \) be
the identity mapping. If \( V \) is a normed space, then there is a natural norm on \( \overline{V} \) defined by \( \|\overline{v}\| = \|v\| \).

Let \( V \) be an operator space and we fix a conjugate space \( \overline{V} \). The linear isomorphisms \( M_n(V) \cong M_n(\overline{V}) : v = [v_{i,j}] \mapsto \overline{v} = [\overline{v}_{i,j}] \ (n \in \mathbb{N}) \) define a matrix norm on \( \overline{V} \) and \( \overline{V} \) becomes an operator space.

Let \( H \) be a Hilbert space with a conjugate space \( \overline{H} \) equipped with the inner product \( \langle \overline{v}, \overline{w} \rangle = \langle w, v \rangle \). For each \( T \in \mathcal{B}(H) \), we define \( T : \overline{H} \to \overline{H} \) by \( \overline{T}(\overline{v}) = \overline{T(v)} \ (v \in H) \). Then \( \overline{T} \in \mathcal{B}(\overline{H}) \) and \( \|\overline{T}\| = \|T\| \). If \( \pi : V \hookrightarrow \mathcal{B}(H) \) is a completely isometric injection, then \( \overline{\pi} : \overline{V} \hookrightarrow \mathcal{B}(\overline{H}) : \overline{v} \mapsto \pi(v) \) is also a completely isometric injection.

3.2. Dual Spaces and Mapping Spaces. There is a natural operator space structure on the mapping space \( CB(V, W) \) for any operator spaces \( V \) and \( W \) which will be described in the following. First, we have the dual space \( V^* = \mathcal{B}(V, \mathbb{C}) = CB(V, \mathbb{C}) \).

In this section, we will define \( M_n(V^*) \) by introducing an appropriate norm on \( M_n(V^*) \) and we will investigate the notion of duality for operator spaces.

If \( E \) is a normed space and \( n \in \mathbb{N} \), then each \( f = (f_1, f_2, \ldots, f_n) \in (E^*)^n \) determines a function \( f : E \to \mathbb{C}^n \) by \( f(v) = (f_1(v), f_2(v), \ldots, f_n(v)) \). This correspondence defines an isometric identification \( l^0_\infty(E^*) = \mathcal{B}(E, l^0_\infty) \).

Similarly, each \( f = [f_{i,j}] \in M_n(V^*) \) determines a linear mapping \( f : V \to M_n \) such that \( f(v) = [f_{i,j}(v)] \). It follows that we have a linear isomorphism \( M_n(V^*) \cong CB(V, M_n) \) and we will use it to define the norm on \( M_n(V^*) \). Let \( M_n(V^*) \) be the
corresponding normed space. Then we get the isometric identification

\[ M_n(V^*) = CB(V, M_n). \]

(12)

We recall that if \( V^* \) is the dual space of a linear space \( V \) with the usual duality pairing \( \langle v, f \rangle = f(v) \), where \( v \in V \), \( f \in V^* \), then the matrix pairing

\[ \langle \cdot, \cdot \rangle : M_p(V) \times M_q(V^*) \to M_{p \times q} \]

is given by \( \langle v, f \rangle = [f_{k,l}(v_{i,j})] = [f(v_{i,j})] = f_p(v) \).

The matrix pairing is the more natural one for operator spaces in some respects. We will use the matrix pairing to determine the norms on \( M_n(V^*) \) and \( M_n(V) \), respectively, where \( V \) is an operator space.

For any \( f \in M_n(V^*) = CB(V, M_n) \), \( \|f\|_{cb} = \|f_n\| \) and

\[
\|f\| = \sup\{\|f_n(v)\| : v \in M_n(V), \|v\| \leq 1\}
\]

\[
= \sup\{\|\langle f(v_{i,j}) \rangle \| : v = [v_{i,j}] \in M_n(V), \|v\| \leq 1\}
\]

\[
= \sup\{\| \langle f, v \rangle \| : v \in M_n(V), \|v\| \leq 1\}.
\]

Although \( M_n(V^*) \) is not defined to be the Banach dual of \( M_n(V) \) just as \( l^\infty_\infty(E^*) \neq l^\infty_1(E^*) \) for a Banach space \( E \), we see from the above that the norm on \( M_n(V) \) really determines the norm on \( M_n(V^*) \).

Conversely, from Lemma 5, for any \( v \in M_n(V) \), there exists a complete contraction \( f : V \to M_n \) such that \( \|f_n(v)\| = \|v\| \), and thus

\[
\|v\| = \sup\{\|f_n(v)\| : f \in CB(V, M_n), \|f\|_{cb} \leq 1\}
\]

\[
= \sup\{\| \langle f, v \rangle \| : f \in CB(V, M_n), \|f\|_{cb} \leq 1\}.
\]

It follows that the norm on \( M_n(V^*) \) determines that on \( M_n(V) \).

Now, let us show that the matrix norm on \( V^* \) determines an operator space. Let

\( f \in M_m(V^*), \alpha \in M_{n,m} \) and \( \beta \in M_{m,n} \). Then

\[
\| (\alpha f \beta) \| = \| (\alpha \otimes I_r) f_r (\beta \otimes I_r) \|
\]

\[
\leq \|\alpha \otimes I_r\| \|f_r\| \|\beta \otimes I_r\|
\]

\[
\leq \|\alpha\| \|f\|_{cb} \|\beta\|,
\]
and hence $\| \alpha f \beta \|_{cb} \leq \| \alpha \| \| f \|_{cb} \| \beta \|$, i.e., [M2] holds. On the other hand, given $f \in M_n(V^\ast), g \in M_n(V^\ast)$ and $v \in M_r(V)$ with $\| v \| \leq 1$, we have
\[
\|(f \oplus g)_r(v)\| = \| (f_{ij} \oplus g_{ij})_r(v_{ij}) \| \\
= \| f_r(v) \oplus g_r(v) \| \\
\leq \max \{ \| f_r(v) \|, \| g_r(v) \| \} \\
\leq \max \{ \| f \|_{cb}, \| g \|_{cb} \},
\]
and thus
\[
\| f \oplus g \|_{cb} \leq \max \{ \| f \|_{cb}, \| g \|_{cb} \},
\]
i.e., [M1'] is true. From Proposition 6, we conclude that $V^\ast$ is an operator space.

Given an operator space $V$, we let $\iota_V : V \hookrightarrow V^{\ast\ast}$ be the canonical inclusion defined by $\langle \iota_V(v), f \rangle = \langle f, v \rangle$ ($v \in V$ and $f \in V^\ast$).

**Proposition 8.** For any operator space $V$, the canonical inclusion (embedding)
\[ \iota_V : V \hookrightarrow V^{\ast\ast} \] is a complete isometry.

**Proof.** For any $v \in M_n(V)$ and $f \in M_n(V^\ast) = CB(V, M_n)$, we have
\[
((\iota_V)_n(v))_n(f) = \langle \iota_V(v)(f_{ij}), v_{ij} \rangle \\
= \langle \iota_V(v_{ij})(f_{ij}), v_{ij} \rangle \\
= \langle f_{ij} v_{ij} \rangle \\
= \langle f, v \rangle.
\]
It follows that
\[
\| (\iota_V)_n(v) \|_{cb} = \sup \{ \| (\iota_V)_n(v)(f) \| : f \in CB(V, M_n), \| f \|_{cb} \leq 1 \} \\
= \sup \{ \| \langle f, v \rangle \| : f \in CB(V, M_n), \| f \|_{cb} \leq 1 \} \\
= \| v \|.
\]
Hence, $(\iota_V)_n$ is an isometry for each $n$ and $\iota_V$ is a complete isometry. \hfill \Box

Let $V$ and $W$ be normed spaces. Each bounded linear mapping $\varphi : V \rightarrow W$ determines a corresponding dual linear mapping $\varphi^\ast : W^\ast \rightarrow V^\ast$ defined by
\[
\varphi^\ast(f)(v) = f(\varphi(v)) \quad \text{for } f \in W^\ast \text{ and } v \in V.
\]
(13) 

From the Hahn-Banach theorem, we obtain $\| \varphi^\ast \| = \| \varphi \|$. 
By analogy, if \( \varphi : V \rightarrow W \) is a completely bounded mapping of operator spaces, then we let \( \varphi^* : W^* \rightarrow V^* \) be the dual Banach space mapping. For any \( v \in M_n(V) \) and \( g \in M_{m}(W^*) = CB(W, M_m) \), we have
\[
\langle g, \varphi_n(v) \rangle = [g_{k,i}(\varphi(v_{i,j}))] \\
= [\varphi^*(g_{k,i})(v_{i,j})] \\
= \langle (\varphi^*)_m(g), v \rangle.
\]

**Proposition 9.** Let \( V \) and \( W \) be operator spaces and let \( \varphi : V \rightarrow W \) be a completely bounded mapping. Then, for all \( n \in \mathbb{N} \), \( \|(\varphi^*)_n\| = \|\varphi_n\| \) and \( \|\varphi^*\|_{cb} = \|\varphi\|_{cb} \).

**Proof.** We consider the mapping
\[
(\varphi^*)_n : M_n(W^*) \rightarrow M_n(V^*) : g \mapsto (\varphi^*)_n(g) \in M_n(V^*) = CB(V, M_n).
\]
We have
\[
\|(\varphi^*)_n\| = \sup\{\|(\varphi^*)_n(g)\| : g \in M_n(W^*), \|g\|_{cb} \leq 1\} \\
= \sup_{\|g\|_{cb} \leq 1}\{(\varphi^*)_n(g), v \rangle : v \in M_n(V), \|v\| \leq 1\} \\
= \sup_{\|v\| \leq 1, \|g\|_{cb} \leq 1}\{\|\langle g, \varphi_n(v) \rangle\|\} \\
= \|\varphi_n\|,
\]
and hence
\[
\|\varphi^*\|_{cb} = \|\varphi\|_{cb}.
\]

Given a completely bounded mapping \( \varphi \in CB(V, W) \), its second adjoint mapping \( \varphi^{**} : V^{**} \rightarrow W^{**} \) is also completely bounded with
\[
\|(\varphi^{**})_V\| = \|\varphi\|_{cb} \text{ and } \varphi^{**}|_V = \varphi.
\]

Given an operator space \( V \), a net \( f_{\lambda} = [f_{k,i}^{\lambda}] \in M_m(V^*) = CB(V, M_m) \), where \( \lambda \) is in a directed index set \( \Lambda \), converges in the weak* topology to \( f = [f_{k,i}] \in M_m(V^*) \) if for all \( n \in \mathbb{N} \) and \( v \in M_n(V) \), the net \( \langle f_{\lambda}, v \rangle = [f_{k,i}^{\lambda}(v_{i,j})] \) (\( \lambda \in \Lambda \)) converges in
the norm topology to \( \ll f, v \gg = [f_k, l(v_{i,j})] \). In other words, \( f_\lambda \to f \) in the weak* topology if and only if, for each pair \((k, l)\), the net \((f^\lambda_{k,l})_{\lambda \in \Lambda}\) converges weakly* to \( f_{k,l} \).

Thus, the weak* topology on \( V^* \) determines the weak* topology on each \( M_n(V^*) \).

If \( W \) is a weak* closed subspace of \( V^* \), then \( M_n(W) \) is weak* closed in \( M_n(V^*) \) for all \( n \in \mathbb{N} \).

It can be seen that a mapping \( f = [f_{i,j}] : V \to M_n \) is continuous if and only if the linear functionals \( f_{i,j} : V \to \mathbb{C} \) are continuous. Similarly, a mapping \( F = [F_{i,j}] : V^* \to M_n \) is continuous in the weak* topology if and only if each \( F_{i,j} : V^* \to \mathbb{C} \) is weak* continuous. The latter implies that \( F_{i,j} = \iota_V(v_{i,j}) \) for some \( v_{i,j} \in V \) and thus \( F = (\iota_V)_n(v) \). If we let \( CB^p(V^*, W^*) \) denote the space of the weak* continuous completely bounded mappings \( \varphi : V^* \to W^* \), and \( \iota_V : V \hookrightarrow V^{**} \) be the canonical injection, then \( CB^p(V^*, M_n) = (\iota_V)_n(M_n(V)) \).

Given a Hilbert space \( H \) with an orthonormal basis \( \{e_n\} \), let \( B(H) \) be the algebra of all bounded linear operators on \( H \) and let \( T(H) = \{b \in B(H) : \|b\|_1 = \text{trace}(|b|) < \infty\} \) be the space of trace class operators on \( H \). For \( a \in B(H) \), if we define \( F_a : T(H) \to \mathbb{C} \) by

\[
F_a(b) = \text{trace}(ab) = \text{trace}(ba) = \sum_n \langle a e_n, e_n \rangle
\]

for all \( b \in T(H) \), then \( F_a \) is a linear functional on \( T(H) \). Since

\[
\|F_a\| = \sup\{|F_a(b)| : b \in T(H), \|b\|_1 \leq 1\} \\
= \sup\{|\text{trace}(ab)| : b \in T(H), \|b\|_1 \leq 1\} \\
\leq \sup\{\sum_n |\langle a e_n, e_n \rangle| : b \in T(H), \|b\|_1 \leq 1\} \\
\leq \sup\{\|a b\|_1\} \leq \sup\{\|a\| \|b\|_1\} \leq \|a\|,
\]

we obtain that \( F_a \) is bounded. In fact, \( \|F_a\| = \|a\| \) for all \( a \in B(H) \) and every bounded linear functional on \( T(H) \) is of this form. Thus, if we define \( \rho : B(H) \to \)
$T(H)^*$ by $\rho(a) = F_a$, then $\rho$ is an isometric isomorphism, i.e.,

(15) \quad B(H) \cong T(H)^*.

Now, we let $\mathcal{K}(H)$ be the space of all compact linear operators on $H$. For $b \in T(H)$, if we define $\Phi_b : \mathcal{K}(H) \to \mathbb{C}$ by

$$\Phi_b(c) = \text{trace}(bc) = \text{trace}(cb) = \sum_n \langle bce_n, e_n \rangle$$

for all $c \in \mathcal{K}(H)$, then $\Phi_b$ is a linear functional on $\mathcal{K}(H)$. Again, $\|\Phi_b\| = \|b\|$ for all $b \in T(H)$ and all bounded linear functionals on $T(H)$ are of this form. Therefore, we have

(16) \quad T(H) \cong \mathcal{K}(H)^*.

These two Banach space identifications are actually operator space dualities. To see this, we need to show that the isometries $B(H) \to T(H)^*$ and $T(H) \to \mathcal{K}(H)^*$ are complete isometries.

**Theorem 5.** Given Hilbert space $H$, we have the operator space dualities

(17) \quad B(H) \cong T(H)^* \text{ and } T(H) \cong \mathcal{K}(H)^*.

**Proof.** First, we show that the isometry $B(H) \to T(H)^*$ is a complete isometry.

Let $b_0 \in M_n(B(H))$. Recall that $\|b_0\|$ is determined by the norm on $M_n(B(H)^*)$ by

$$\|b_0\| = \sup \{ \| \ll b_0, w \gg \| : w \in M_n(B(H)^*) = CB(B(H), M_n), \|w\|_{cb} \leq 1 \}. \quad \text{(18)}$$

Since the norm on $M_n(T(H)^*)$ is determined by the norm on $M_n(T(H))$, it is sufficient to prove that

$$\|b_0\| = \sup \{ \| \ll b_0, w \gg \| : w \in M_n(T(H)), \|w\| \leq 1 \}. \quad \text{(18)}$$
For any $w \in M_n(T(H))$ with $\|w\| \leq 1$,
\[ \| \ll b_0, w \gg \| = \|w_n(b_0)\| \leq \|w\|_0 \|b_0\| \leq \|b_0\|. \quad \cdots (\ast) \]

On the other hand, given $\varepsilon > 0$, there exist unit vectors $\eta = (\eta_i), \xi = (\xi_i) \in H^n$ such that $|\langle b_0 \eta, \xi \rangle| \geq \|b_0\| - \varepsilon$. Let $H_1$ (resp. $H_2$) be the linear span of $\eta_i \in H$ (resp. $\xi_i \in H$) and we fix isometries $s_k$ of $H_k$ into $C^n$ ($k = 1, 2$). Consider the linear functional $w : B(H) \to M_n : b \mapsto r_{2br_1^*}$, where $r_k = s_k e_k$ and $e_k$ is the projection of $H$ onto $H_k$. Then $w$ is a weak* continuous complete contraction and, for any $b \in M_n(B(H))$, $w_n(b) = r_{2br_1^*}$, where $r_k^{(n)} = r_k \oplus \cdots \oplus r_k$. Since $\eta \in H_1^n$ and $\xi \in H_2^n$, \[ \| \ll b_0, w \gg \| = \|w_n(b_0)\| = \|r_{2br_1^*}b_0r_1^{(n)}\| \geq |\langle b_0 \eta, \xi \rangle| \geq \|b_0\| - \varepsilon. \quad \cdots (\ast \ast) \]

From $(\ast)$ and $(\ast \ast)$, it follows that (18) holds.

Next, let us show the second operator space duality. If we combine the Banach space dualities $B(\mathcal{H}) \cong T(\mathcal{H})^*$ and $T(\mathcal{H}) \cong \mathcal{K}(\mathcal{H})^*$, then we have $B(\mathcal{H}) \cong \mathcal{K}(\mathcal{H})^{**}$. It follows that the unit ball of $M_n(\mathcal{K}(H)) = \mathcal{K}(\mathcal{H})$ is weak* dense in the unit ball of $M_n(B(H)) = B(\mathcal{H})$. Thus, by the preceding result, for any $w \in M_n(T(H))$, we have \[ \|w\| = \sup \{ \| \ll b, w \gg \| : b \in M_n(B(H)), \|b\| \leq 1 \} \]
\[ = \sup \{ \| \ll k, w \gg \| : k \in M_n(\mathcal{K}(H)), \|k\| \leq 1 \}. \]
Therefore, $T(H) \to \mathcal{K}(H)^*$ is a complete isometry. \qed

Let $V$ and $W$ be linear spaces and let $\mathcal{L}(V, W)$ be the space of linear mappings from $V$ to $W$. We can identify each $[\varphi_{i,j}] \in M_r(\mathcal{L}(V, W))$ ($\varphi_{i,j} \in \mathcal{L}(V, W)$) with a linear map $\varphi : V \to M_r(W)$ defined by $\varphi(v) = [\varphi_{i,j}(v)]$ and thus we have a canonical linear space identification $M_r(\mathcal{L}(V, W)) \cong \mathcal{L}(V, M_r(W))$. 
Similarly, if $V$ and $W$ are operator spaces, then we have the linear isomorphism $M_m(CB(V,W)) \cong CB(V,M_m(W))$. If we define a norm on $M_m(CB(V,W))$ to be the completely bounded norm on $CB(V,M_m(V))$ and let $M_m(CB(V,W))$ denote the resulting normed space, then we have the isometric identification

\[(19) \quad M_m(CB(V,W)) = CB(V,M_m(W)).\]

The way to prove that $CB(V,W)$ is an operator space is the same as the argument for the dual space $V^*$. 

From the above discussion, we see that we only use the operator space structure on $W$ to define the operator space $CB(V,W)$. In fact, if $E$ is any normed space and $W$ is an operator space, then $B(E,W)$ becomes an operator space under the isometric identification $M_n(B(E,W)) = B(E,M_n(W))$ ($n \in \mathbb{N}$).

**Proposition 10.** Let $V$ and $W$ be operator spaces. If $W$ is complete, then $CB(V,W)$ is complete.

**Proof.** Let $\{\varphi_n\}$ be any Cauchy sequence in $CB(V,W)$. Then $\{\varphi_n\}$ is a Cauchy sequence in $B(V,W)$. Since $B(V,W)$ is complete, there exists a bounded linear mapping $\varphi : V \to W$ such that $\varphi_n \to \varphi$ in the norm topology.

Given $\varepsilon > 0$. Since $\{\varphi_n\}$ is a Cauchy sequence in $CB(V,W)$, there exists an $N \in \mathbb{N}$ such that whenever $n, m > N$, we have $\|\varphi_n - \varphi_m\|_{cb} < \varepsilon$. Let $p \in \mathbb{N}$ and $v = [v_{i,j}] \in M_p(V)$. Then

\[\|((\varphi_n - \varphi_m)_p)(v)\| \leq \|\varphi_n - \varphi_m\|_{cb} \|v\| < \varepsilon \|v\|\]
for all $n, m > N$. Since $\varphi_m(v_{i,j}) \to \varphi(v_{i,j})$ as $m \to \infty$, we get $\|(\varphi_n - \varphi)_p(v)\| \leq \varepsilon\|v\|$ for all $p \in \mathbb{N}$ and $v \in M_p(V)$ whenever $n > N$. Thus $\|\varphi_n - \varphi\|_{\text{cb}} \leq \varepsilon$ for all $n > N$.

It follows that $\varphi \in \mathcal{CB}(V, W)$ and $\varphi_n \to \varphi$ in $\mathcal{CB}(V, W)$. \qed

3.3. The Minimal and Maximal Quantizations. In this section we investigate the minimal and maximal quantizations of the Banach space $E$: the minimal and the maximal matrix norms on $E$ satisfying [M1] and [M2].

For any Banach space $E$, let $b_r(E) = \mathcal{B}(E, M_r)_{\|\cdot\|_{\leq 1}}$ and $b(E) = \bigcup_{r \in \mathbb{N}} b_r(E)$. For $x \in M_n(E)$, if we define $\|x\|_{\text{min}}$ and $\|x\|_{\text{max}}$ by

\begin{equation}
\|x\|_{\text{min}} = \sup\{|f_n(x)| : f \in b_1(E) = E^*_n, \|\cdot\|_{\leq 1}\}
\end{equation}

and

\begin{equation}
\|x\|_{\text{max}} = \sup\{|f_n(x)| : f \in b(E)\},
\end{equation}

then they are operator space matrix norms on $E$ satisfying [M1] and [M2].

Instead of verifying [M1] and M[2] directly, let us see this by some natural complete isometric embeddings. Consider the linear isometric injections

\[ E \hookrightarrow l_\infty(b_1(E)) : x \mapsto (f(x)) \]

and

\[ E \hookrightarrow \prod_{r \in \mathbb{N}} l_\infty (b_r(E), M_r) : x \mapsto ((f(x))_{f \in b_r(E)})_{r \in \mathbb{N}}, \]

respectively. We have the natural operator space identifications

\[ l_\infty (b_1(E)) = \prod_{b_1(E)} \mathbb{C} \]

and

\[ \prod_{r \in \mathbb{N}} l_\infty (b_r(E), M_r) = \prod_{r \in \mathbb{N}} \prod_{b_r(E)} M_r. \]
And the relative matrix norms on the subspace $E$ are exactly $\| \cdot \|_{\text{min}}$ and $\| \cdot \|_{\text{max}}$ as defined in (20) and (21), respectively. So, $\|x\|_{\text{min}}$ (resp. $\|x\|_{\text{max}}$) determines an operator space which will be denoted by $\text{min } E$ (resp. $\text{max } E$). We refer to $\text{min } E$ (resp. $\text{max } E$) as the \textit{minimal} (resp. \textit{maximal}) \textit{quantization} of $E$.

If $V$ is an operator space and $v \in M_n(V)$, then

$$\|v\| = \sup\{\|f_n(v)\| : f \in CB(V, M_n)_{\|\| \leq 1}\}.$$

Since

$$b_1(V) = B(V, C)_{\|\| \leq 1} \subseteq CB(V, M_n)_{\|\| \leq 1} \subseteq b(V),$$

we get $\|v\|_{\text{min}} \leq \|v\| \leq \|v\|_{\text{max}}$ for all $v \in M_n(E)$.

For any normed space $E$ and operator space $V$, we have the isometric identification

$$(22) \quad CB(V, \text{min } E) = B(V, E)$$

or equivalently, for any linear mapping $\varphi : V \to E$,

$$\|\varphi : V \to \text{min } E\|_{\text{cb}} = \|\varphi : V \to E\|.$$ 

We only have to show that $\|\varphi : V \to \text{min } E\|_{\text{cb}} \leq \|\varphi : V \to E\|$. To see this, let $v \in M_n(V)$ and $\|v\| \leq 1$. Then

$$\|\varphi_n(v)\|_{\text{min}} = \sup\{\|f_n \circ \varphi_n(v)\| : \|f : E \to C\| \leq 1\}$$

$$= \sup\{\|(f \circ \varphi)_n(v)\| : \|f : E \to C\| \leq 1\}$$

$$\leq \sup\{\|f \circ \varphi\| : \|f : E \to C\| \leq 1\}$$

$$= \sup\{\|f \circ \varphi\| : \|f : E \to C\| \leq 1\} \text{ (since } CB(V, C) = V^*)$$

$$\leq \|\varphi\|$$

for all $n \in \mathbb{N}$. Therefore, $\|\varphi : V \to \text{min } E\|_{\text{cb}} \leq \|\varphi : V \to E\|$.

For any normed space $E$ and operator space $W$,

$$(23) \quad CB(\text{max } E, W) = B(E, W).$$
That is, for any linear mapping $\varphi : E \to W$,

$$\|\varphi : \max E \to W\|_{cb} = \|\varphi : E \to W\|.$$ 

Again, we only have to show that $\|\varphi : \max E \to W\|_{cb} \leq \|\varphi : E \to W\|$. Suppose

$$\|\varphi\| \leq 1 \text{ and } v \in M_n(\max E).$$

Then

$$\|\varphi_n(v)\| = \sup \{\|f \circ \varphi_n(v)\| : \|f : W \to M_r\|_{cb} \leq 1, \ r \in \mathbb{N}\}$$

$$\leq \sup \{\|f \circ \varphi_n(v)\| : \|f : W \to M_r\| \leq 1, \ r \in \mathbb{N}\}$$

$$\leq \sup \{\|g_n(v)\| : \|g : E \to M_r\| \leq 1, \ r \in \mathbb{N}\} = \|v\|_{\max}$$

for all $n \in \mathbb{N}$. It follows that $\|\varphi : \max E \to W\|_{cb} \leq \|\varphi : E \to W\|$.

Let $E$ be a normed space and let $B^\sigma(E^*, M_n)$ be the space of weak* continuous mappings from $E^*$ to $M_n$. Then we have the linear isomorphism $M_n(V) \cong B^\sigma(E^*, M_n)$ given by $v \mapsto (f \mapsto [f(v_{i,j})])$. If $v = [v_{i,j}] \in M_n(V)$, then

$$\|v\|_{\min} = \sup \{\|f_n(v)\| : f \in B(E, C), \|f\| \leq 1\}$$

$$\quad = \sup \{\|f_{\{v_{i,j}\}}\| : f \in B(E, C), \|f\| \leq 1\}.$$ 

This implies that we have the isometric identification

$$M_n(\min E) = B^\sigma(E^*, M_n).$$

Furthermore, any linear mapping $f : E \to M_n$ has an extension $f^{**} : E^{**} \to M_n$ which is weak* continuous and this gives us a natural identification

$$B(E, M_n) = B^\sigma(E^{**}, M_n).$$

Thus from (12), (23), (24) and (25), we have

$$M_n((\max E)^*) = CB(\max E, M_n) = B(E, M_n)$$

$$= B^\sigma(E^{**}, M_n) = M_n(\min E^*)$$

for all $n \in \mathbb{N}$. Therefore, we obtain

$$\max E)^* = \min(E^*).$$

Let $V$ be an operator space. If $D$ is a subset of $V^*_{\|\cdot\| \leq 1}$ such that the absolutely convex hull $\text{co}(D)$ (i.e. the smallest absolutely convex set containing $D$) is weak*
dense in $V_{\| \cdot \| \leq 1}$, then for any $v \in M_n(V)$,

\[(27) \quad \|v\|_{\min} = \sup \{ \|f_n(v)\| : f \in D \}.\]

To see this, it suffices to prove that $\|v\|_{\min} \leq \sup \{ \|f_n(v)\| : f \in D \}$ since by
definition, $\|v\|_{\min} = \sup \{ \|g_n(v)\| : g \in V_{\| \cdot \| \leq 1} \} \geq \sup \{ \|f_n(v)\| : f \in D \}$. Suppose
that $\|f_n(v)\| \leq 1$ for all $f$ in $D$. For any $g \in V_{\| \cdot \| \leq 1}$, there is a net $(g_\beta)$ in $|co|(D)$
such that $g_\beta \to g$ in the weak* topology. It follows that $g_\beta(v_{i,j}) \to g(v_{i,j})$ and the
matrices $(g_\beta)_n(v) \to g_n(v)$ in the norm topology. If $g_\beta = \sum_k t_k f_k$ with $f_k \in D$ and
$\sum_k |t_k| \leq 1$, then

\[\|(g_\beta)_n(v)\| = \| \sum_k t_k f_k(v) \| \leq \sum_k |t_k| \leq 1.\]

We get $\|g_n(v)\| \leq 1$ and thus $\|v\|_{\min} \leq 1$, i.e., $\|v\|_{\min} \leq \sup \{ \|f_n(v)\| : f \in D \}$.

An operator space $V$ is called minimal or abelian if $V = \min V$ and maximal if $V = \max V$.

Let $\mathcal{A}$ be a commutative $C^*$-algebra. Then we can identify $\mathcal{A}$ with $C_0(\Omega)$ which
is the commutative $C^*$-algebra of complex continuous functions vanishing at $\infty$ on
a locally compact Hausdorff space $\Omega$. Now we have a natural mapping $\delta : \Omega \to \mathcal{A}^*$
such that $\delta(x)(g) = g(x)$ for all $x \in \Omega$ and $g \in \mathcal{A} = C_0(\Omega)$. By the bipolar theorem,
$|co|(\delta(\Omega))$ is weak* dense in $\mathcal{A}_{\| \cdot \| \leq 1}$ and from (27), for any $a = [a_{i,j}] \in M_n(\mathcal{A}) \cong
C_0(\Omega, M_n)$, we have

\[\|a\|_{\min} = \sup \{ \|f(a_{i,j})\| : f \in \delta(\Omega) \} = \sup \{ \|a_{i,j}(x)\| : x \in \Omega \} = \sup \{ \|a(x)\| : x \in \Omega \} = \|a\|.\]

Thus, we see that as an operator space, $\mathcal{A}$ is the minimal quantization, i.e., $\mathcal{A} = \min \mathcal{A}$. 
Combining the above with (22), we get another proof of Proposition 4. Similarly, if \( E \) is a normed space, then each isometric injection \( E \hookrightarrow \mathcal{A} \) determines a completely isometric injection \( \min E \hookrightarrow \mathcal{A} \).

Let \( E \) be a normed space and let \( \mathcal{A} \) be a commutative \( C^* \)-algebra. Given an isometric injection \( E \hookrightarrow \mathcal{A} \), we have an isometric extension \( E^{**} \hookrightarrow \mathcal{A}^{**} \). Note that \( \mathcal{A}^{**} \) is also a commutative \( C^* \)-algebra. So, we have a completely isometric injection \( \min(E^{**}) \hookrightarrow \mathcal{A}^{**} \) and hence \( \min(E^{**}) = (\min E)^{**} \). From (26), we have

\[
(\max E^*)^* = \min(E^{**}) = (\min E)^{**}.
\]

Since these identifications are compatible with the dualities, we have the complete isometry

\[(28) \quad \max(E^*) = (\min E)^* \.
\]

**Proposition 11.** Let \( V \) be an operator space. Then \( V \) is minimal if and only if \( V \) is completely isometric to a subspace of a commutative \( C^* \)-algebra.

**Proof.** Suppose that \( V \) is minimal. If \( V \hookrightarrow \mathcal{A} \) be an isometric embedding of \( V \) into a commutative \( C^* \)-algebra \( \mathcal{A} \), then \( V = \min V \hookrightarrow \min \mathcal{A} = \mathcal{A} \) is a complete isometry. Conversely, if we have a complete isometric injection \( V \hookrightarrow \mathcal{A} \) of an operator space \( V \) into a commutative \( C^* \)-algebra \( \mathcal{A} \), then since \( \min V \hookrightarrow \min \mathcal{A} = \mathcal{A} \) is also a complete isometry, we have that \( V = \min V \). \( \Box \)
4. The Extension Theorem

In this chapter we will introduce the Arveson-Wittstock-Hahn-Banach theorem and discuss the injectivity for operator spaces.

4.1. The Arveson-Wittstock Theorem. Let $E$ and $F$ be normed spaces. A bounded linear mapping $\varphi : E \to F$ is a quotient mapping if the induced mapping $\bar{\varphi} : E / \ker \varphi \to F$ is an isometry, or equivalently, $\varphi$ maps $E_{\| \cdot \| < 1}$ onto $F_{\| \cdot \| < 1}$. A linear mapping $\varphi : E \to F$ is an exact quotient mapping if $\varphi$ maps $E_{\| \cdot \| \leq 1}$ onto $F_{\| \cdot \| \leq 1}$.

From the classical Hahn-Banach theorem, we have that given normed spaces $E \subseteq F$ and $n \in \mathbb{N}$, if $f \in B(E, l^n_{\infty}), f = (f_1, \cdots , f_n) : E \to l^n_{\infty}$, then there exists an isometric extension $\tilde{f} = (\tilde{f}_1, \cdots , \tilde{f}_n) : F \to l^n_{\infty}$, where $\tilde{f}_i$ is an isometric extension of $f_i$ $(i = 1, 2, \cdots , n)$.

Let $\rho : B(F, l^n_{\infty}) \to B(E, l^n_{\infty})$ be the restriction mapping. Then $\rho$ maps $B(F, l^n_{\infty})_{\| \cdot \| \leq 1}$ onto $B(E, l^n_{\infty})_{\| \cdot \| \leq 1}$. Also, since $l^n_1(E)^* = l^n_{\infty}(E^*) = B(E, l^n_{\infty})$ and the inclusion $i : l^n_1(E) \hookrightarrow l^n_1(F)$ is isometric, $\rho = i^* : l^n_1(F)^* \to l^n_1(E)^*$ is an exact quotient mapping. Now we have the commutative diagram

$$
\begin{array}{ccc}
B(F, l^n_{\infty}) & \xrightarrow{\rho} & B(E, l^n_{\infty}) \\
\| & & \| \\
l^n_1(F)^* & \xrightarrow{\rho} & l^n_1(E)^*.
\end{array}
$$

To apply this result to operator spaces, we need to define a norm $\| \cdot \|_1$ on $M_n(V)$ which corresponds to the norm $\| \cdot \|_1$ on $E^n$ for normed spaces $E$ and we will denote the resulting normed space $(M_n(V), \| \cdot \|_1)$ by $T_n(V)$.

For $f \in M_n(V^*) = CB(V, M_n)$, we have

$$
\| f \| = \sup \{ \| f(r \tilde{v}) \| : \| \tilde{v} \| \leq 1, \tilde{v} \in M_r(V), r \in \mathbb{N} \}
= \sup \{ \| \langle f, \tilde{v} \rangle \| : \| \tilde{v} \| \leq 1, \tilde{v} \in M_r(V), r \in \mathbb{N} \},
$$
where \( \langle f, \tilde{v} \rangle \) = \([f_{k,i}(\tilde{v}_{i,j})] = [f(\tilde{v}_{i,j})] = f_r(\tilde{v}) \in M_{n \times r} \). If we let \( D_{r \times n} \) be the closed unit ball of \( l^r_{2 \times n} (= C^{r \times n} \text{ with } \| \cdot \|_2) \), then

\[
\| f \| = \sup \{ \| \langle f, \tilde{v} \rangle \eta, \xi \| : \eta, \xi \in D_{r \times n} \} \\
= \sup \{ \left\| \sum_{i,j,k,l} f_{k,i}(\tilde{v}_{i,j})\eta(j,k)\tilde{\xi}(i,k) \right\| : \eta, \xi \in D_{r \times n} \} \\
= \sup \{ \left\| \sum_{k,l} (f_{k,i}, \sum_{i,j} \tilde{\xi}(i,k)\tilde{v}_{i,j}\eta(j,k)) \right\| : \eta, \xi \in D_{r \times n} \},
\]

where the supremum is taken over all \( \tilde{v} \in M_r(V) \) with \( \| \tilde{v} \| \leq 1 \) and \( r \in \mathbb{N} \). Given vectors \( \eta, \xi \in D_{r \times n} \), we let \( \alpha_{k,i} = \tilde{\xi}(i,k) \) and \( \beta_{j,l} = \eta(j,l) \). Then the matrices \( \alpha = [\alpha_{k,i}] \in HS_{n,r}, \beta = [\beta_{j,l}] \in HS_{r,n} \) satisfy \( \| \alpha \|_2 = \| \xi \| \) and \( \| \beta \|_2 = \| \eta \| \) and we have

\[
\| f \| = \sup \{ \left\| \sum (f_{k,i}, (\alpha \tilde{v} \beta)_{i,j}) \right\| : \| \tilde{v} \|, \| \alpha \|_2, \| \beta \|_2 \leq 1 \} \\
= \sup \{ \left\| \langle f, \alpha \tilde{v} \beta \rangle \right\| : \| \tilde{v} \|, \| \alpha \|_2, \| \beta \|_2 \leq 1 \} \\
= \sup \{ \left\| \langle f, v \rangle \right\| : v = \alpha \tilde{v} \beta, \| \tilde{v} \|, \| \alpha \|_2, \| \beta \|_2 \leq 1 \},
\]

where each supremum is taken over all \( \tilde{v} \in M_r(V) \) with \( \| \tilde{v} \| \leq 1 \) and \( r \in \mathbb{N} \). Thus, if we define \( \| \cdot \|_1 : M_n(V) \to [0, \infty) \) by

\[
(29) \quad \| v \|_1 = \inf \{ \| \alpha \|_2 \| \tilde{v} \| \| \beta \|_2 : v = \alpha \tilde{v} \beta \},
\]

where \( \alpha \in HS_{n,r}, \beta \in HS_{r,n} \) and \( \tilde{v} \in M_r(V) \) with \( r \) arbitrary, then we obtain that

\[
(30) \quad \| f \| = \sup \{ \left\| \langle f, v \rangle \right\| : \| v \|_1 \leq 1 \}.
\]

Now, we prove that \( \| \cdot \|_1 \) is a norm and automatically \( T_n(V) = (M_n(V), \| \cdot \|_1) \) is the predual of \( M_n(V^*) \).

**Lemma 6.** Let \( V \) be an operator space and \( n \in \mathbb{N} \). Then \( \| \cdot \|_1 \) is a norm on \( M_n(V) \) and we have isometric identifications

\[
(31) \quad T_n(V)^* \cong M_n(V^*)
\]

and

\[
(32) \quad M_n(V)^* \cong T_n(V^*).
\]
Proof. (i) Given \( v \in M_n(V) \) with \( \|v\|_1 < 1 \), let \( v = \alpha \tilde{v} \beta \), where \( \alpha \in HS_{n,r}, \beta \in HS_{r,n} \) and \( \tilde{v} \in M_r(V) \) with \( \|\tilde{v}\|, \|\alpha\|_2, \|\beta\|_2 < 1 \). Then, it follows that \( \|v\| \leq \|\alpha\|_2 \|\tilde{v}\| \|\beta\|_2 \leq \|\alpha\|_2 \|\tilde{v}\| \|\beta\|_2 < 1 \). Thus for any \( v \in M_n(V) \), \( \|v\| \leq \|v\|_1 \). If \( \|v\|_1 = 0 \), then \( \|v\| = 0 \) which implies that \( v = 0 \).

(ii) For any \( \lambda \in \mathbb{C} \) and \( v = \alpha \tilde{v} \beta \in M_n(V) \), \( \lambda v = \alpha (\lambda \tilde{v}) \beta \) and \( \|\lambda v\|_1 \leq \|\alpha\|_2 \|\lambda \tilde{v}\| \|\beta\|_2 \).

\[
\|v\|_1 = \|\lambda^{-1} \lambda v\|_1 \leq \|\lambda^{-1}\| \|\lambda v\|_1, \text{ i.e., } \|\lambda v\|_1 \leq \|\lambda v\|_1. \text{ Thus, } \|\lambda v\|_1 = \|\lambda\| \|v\|_1.
\]

(iii) Given \( v_1, v_2 \in M_n(V) \) and \( \varepsilon > 0 \), let \( v_i = \alpha_i \tilde{v}_i \beta_i \) with \( \|\tilde{v}_i\| \leq 1 \), \( \|\alpha_i\|_2 = \|\beta_i\|_2 < (\|v_i\|_1 + \varepsilon)^{1/2} \) (\( i = 1, 2 \)). If we let \( \alpha = [\alpha_1 \alpha_2], \beta = [\beta_1 \beta_2]^T \) and \( \tilde{v} = \tilde{v}_1 \oplus \tilde{v}_2 \), then we have

\[
\|\alpha\|_2^2 = \|\alpha_1\|_2^2 + \|\alpha_2\|_2^2,
\]

\[
\|\beta\|_2^2 = \|\beta_1\|_2^2 + \|\beta_2\|_2^2
\]

and \( \|\tilde{v}\| = \max\{\|\tilde{v}_1\|, \|\tilde{v}_2\|\} \leq 1 \). Since \( v_1 + v_2 = \alpha_1 \tilde{v}_1 \beta_1 + \alpha_2 \tilde{v}_2 \beta_2 = \alpha \tilde{v} \beta \), it follows that

\[
\|v_1 + v_2\|_1 \leq \|\alpha\|_2 \|\beta\|_2
\]

\[
\leq \frac{1}{2}(\|\alpha\|_2^2 + \|\beta\|_2^2)
\]

\[
= \frac{1}{2}(\|\alpha_1\|_2^2 + \|\beta_1\|_2^2 + \|\alpha_2\|_2^2 + \|\beta_2\|_2^2)
\]

\[
< \|v_1\|_1 + \|v_2\|_1 + 2\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, we get \( \|v_1 + v_2\|_1 \leq \|v_1\|_1 + \|v_2\|_1 \), i.e., we have subadditivity. Therefore, from (i), (ii) and (iii), \( \|\cdot\|_1 \) is a norm.

The duality (31) follows from (30) since the norm on \( M_n(V^*) \) is determined by the norm on \( T_n(V) \).

Finally, we prove the second duality (32). Given \( f \in M_n(V)^* \) with \( \|f\| < 1 \), from Lemma 4, there exists a complete contraction \( \tilde{f} \in M_n(V^*) = CB(V, M_n) \) with
\[ \| f \|_c \leq 1 \text{ and } \xi, \eta \in \mathbb{C}^n \text{ with } \eta = (\eta_{ij}) \text{ and } \xi = (\xi_{i,j,k}) \text{ such that} \]
\[
  f(v) = \langle \tilde{f}_n(v) \eta, \xi \rangle = \sum_{i,j,k,l} \tilde{f}_{i,j,k,l}(v_{i,j,k}) \eta_{i,j,k,l} \xi_{i,j,k,l} = \sum_{i,j,k,l} \xi_{i,j,k,l} \tilde{f}_{i,j,k,l}(v_{i,j,k}).
\]

If we let \( \alpha = [\alpha_{i,k}] = [\xi_{i,k}] \) and \( \beta = [\beta_{i,j}] = [\eta_{i,j}] \), then we get \( f = \alpha \tilde{f} \beta \) and \( \| \alpha \|_2, \| \beta \|_2 < 1 \). It follows that \( \| f \|_1 \leq \| \alpha \|_2 \| \tilde{f} \|_c \| \beta \|_2 \leq 1 \). Therefore, \( \| f \|_1 \leq \| f \| \) for all \( f \in M_n(V)^* \).

Conversely, let \( f = \alpha \tilde{f} \beta \in M_n(V)^* \) with \( \| \tilde{f} \|_c \leq 1 \) and \( \| \alpha \|_2, \| \beta \|_2 < 1 \). Then \( |f(v)| = |\langle \tilde{f}_n(v) \eta, \xi \rangle| \leq \| \tilde{f}_n(v) \| \leq \| v \| \) for all \( v \in M_n(V) \). Thus, \( \| f \| \leq 1 \) and it follows that \( \| f \| \leq \| f \|_1 \). Therefore, \( M_n(V)^* \cong T_n(V^*) \).

**Lemma 7.** Let \( V \) be an operator space and \( v \in M_n(V) \). Then

\[
  \| v \|_1 < 1 \text{ if and only if } v = \alpha \hat{v} \beta,
\]

where \( \hat{v} \in M_n(V) \), \( \alpha \in HS_n \) and \( \beta \in HS_n \) satisfy \( \| \hat{v} \| < 1 \), \( \| \alpha \|_2 < 1 \) and \( \| \beta \|_2 < 1 \), and furthermore we can suppose that \( \alpha \) and \( \beta \) are invertible matrices.

**Proof.** If \( v = \alpha \hat{v} \beta \) with \( \hat{v} \in M_n(V) \), \( \alpha \in HS_n \) and \( \beta \in HS_n \) satisfying \( \| \hat{v} \|, \| \alpha \|_2, \| \beta \|_2 < 1 \), by definition, it is evident that \( \| v \|_1 < 1 \).

On the other hand, we suppose that \( \| v \|_1 < 1 \) and \( v = \alpha_1 w \beta_1 \), \( w \in M_p(V) \), \( \alpha_1 \in HS_{n,p} \) and \( \beta_1 \in HS_{p,n} \) with \( \| w \| < 1 \), \( \| \alpha_1 \|_2 < 1 \) and \( \| \beta_1 \|_2 < 1 \). From the polar decomposition theorem, there is a partial isometry \( \nu : \mathbb{C}^n \to \mathbb{C}^p \) such that \( \beta_1 = \nu |\beta_1| \), where \( \beta_1 : \mathbb{C}^n \to \mathbb{C}^p \) is regarded as a linear mapping and \( |\beta_1| = (\beta_1^* \beta_1)^{1/2} \), and we have that \( |\beta_1| \in HS_n, \| |\beta_1||_2 = \|\beta_1\|_2 < 1 \) and \( \nu^* \beta_1 = |\beta_1| \).
If we let $P$ be the projection of $\mathbb{C}^n$ onto the range of $|\beta_1|$, then $\nu(I - P) = 0$, where $I$ is the $n \times n$ identity matrix and for $\varepsilon > 0$, $\beta = |\beta_1| + \varepsilon(I - P)$ is an invertible $n \times n$ matrix with $\beta_1 = \nu \beta$. We may assume that $\|\beta\|_2 < 1$ by taking sufficiently small $\varepsilon$.

Similarly, $\alpha_1^* = \rho_1|\alpha_1^*|$, where $\rho_1 : \mathbb{C}^n \to \mathbb{C}^p$ is a partial isometry and thus there is a partial isometry $\rho$ such that $\alpha_1 = \alpha \rho$ and $\alpha$ is an invertible $n \times n$ matrix with $\|\alpha\|_2 < 1$. It follows that $v = \alpha_1 w \beta_1 = \alpha \rho \nu \beta = \alpha \bar{\nu} \beta$, where $\bar{\nu} = \rho \nu$, is the desired decomposition of $v$. \hfill \Box

**Corollary 4.** If $V$ is a subspace of an operator space $W$, the inclusion mapping $T_n(V) \hookrightarrow T_n(W)$ is an isometry for each $n \in \mathbb{N}$.

**Proof.** If $v \in M_n(V)$, then $\|v\|_{T_n(W)} \leq \|v\|_{T_n(V)}$ since there are more decompositions for $v$ in $W$. On the other hand, given $v \in M_n(V)$ with $\|v\|_{T_n(W)} < 1$, let $v = \alpha \bar{\nu} \beta$ be a decomposition of $v$ as in Lemma 7. Since $\alpha$ and $\beta$ are invertible, $\bar{w} = \alpha^{-1} \nu \beta^{-1} \in M_n(V)$ and thus $\|v\|_{T_n(V)} < 1$. Therefore, $\|v\|_{T_n(V)} = \|v\|_{T_n(W)}$. \hfill \Box

Let $V$ and $W$ be normed spaces. From the classical Hahn-Banach theorem, we have that given a bounded linear mapping $\varphi : V \to W$, $\varphi$ is an isometry $\iff \varphi^*$ is a quotient mapping $\iff \varphi^*$ is an exact quotient mapping. Furthermore, $\varphi$ is a quotient mapping $\implies \varphi^*$ is an isometry. If $V$ is complete, then $\varphi$ is a quotient mapping $\iff \varphi^*$ is an isometry.

**Corollary 5.** Let $V$ be a subspace of an operator space $W$. Then any completely bounded mapping $\varphi : V \to M_n$ has an extension $\tilde{\varphi} : W \to M_n$ satisfying $\|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$. 
Proof. Since the inclusion mapping $\rho : T_n(V) \to T_n(W)$ is isometric for each $n \in \mathbb{N}$ (Corollary 4), $\rho^* : T_n(W)^* \to T_n(V)^*$ is an exact quotient mapping.

Under the identifications $T_n(W)^* = M_n(W^*) = CB(W, M_n)$ and $T_n(V)^* = M_n(V^*) = CB(V, M_n)$, $\rho^*$ is corresponding to the restriction mapping $\Phi : CB(W, M_n) \to CB(V, M_n)$. So, $\Phi$ is also an exact quotient mapping, i.e., $\Phi$ maps $CB(W, M_n)_{||\cdot|| \leq 1}$ onto $CB(V, M_n)_{||\cdot|| \leq 1}$.

It follows that for $\varphi \in CB(V, M_n)$, there exists $\tilde{\varphi} \in CB(W, M_n)$ such that $\Phi(\tilde{\varphi}) = \varphi$ and $||\tilde{\varphi}||_{cb} = ||\varphi||_{cb}$ (since if $||\varphi||_{cb} = 1$, then $1 = ||\varphi||_{cb} \leq ||\tilde{\varphi}||_{cb} \leq 1$). \(\square\)

Corollary 5 is a special case of the Arveson-Wittstock-Hahn-Banach theorem. We are now ready to prove the Arveson-Wittstock-Hahn-Banach theorem in the general form.

Theorem 6. Let $V$ be a subspace of an operator space $W$ and let $H$ be a Hilbert space. Then any complete contraction $\varphi : V \to B(H)$ has a completely contractive extension $\psi : W \to B(H)$.

Proof. If $H = \mathbb{C}^n$ and hence $B(H) \cong M_n$, this is proved in Corollary 5. Let $H$ be an arbitrary Hilbert space and let $\mathcal{F} = \{ F \in B(H) : F \text{ is a finite-rank orthogonal projection} \}$. If $\dim F(H) = n$, then we may identify $B(F(H))$ with $M_n$. It follows that the completely contractive mapping $F\varphi F : V \to B(F(H))(\subseteq B(H))$, defined by $v \mapsto F\varphi(v)F$, has a completely contractive extension $\psi_F : W \to B(H)$.

If we order $\mathcal{F}$ by $F_1 \preceq F_2$ iff $F_1(H) \subseteq F_2(H)$, then $\{\psi_F\}_{F \in \mathcal{F}}$ is a net of contractions in $CB(W, B(H))$. Since $B(H)_{||\cdot|| \leq 1}$ is compact in the weak operator topology, $CB(V, B(H))_{||\cdot|| \leq 1}$ is compact in the point-weak operator topology. It follows that
there exists a cluster point \( \psi \) of the net \( \{ \psi_F \}_{F \in \mathcal{F}} \) in the point-weak operator topology on \( CB(W, B(H)) \). For each \( v \in V \) and \( \xi \in H \), we let \( F_0 \) be the projection of \( H \) onto \( C\xi + C\varphi(v)\xi \). If \( F \geq F_0 \), then

\[
\psi_F(v)\xi = F\varphi(v)F\xi = F\varphi(v)\xi = \varphi(v)\xi.
\]

Since \( \langle \psi_F(v)\xi, \eta \rangle \to \langle \psi(v)\xi, \eta \rangle \) for all \( \eta \in H \), it follows that \( \langle \psi(v)\xi, \eta \rangle = \langle \varphi(v)\xi, \eta \rangle \) for all \( \eta \in H \). Hence, \( \psi(v)\xi = \varphi(v)\xi \) for all \( v \in V \) and \( \xi \in H \), i.e., \( \psi(v) = \varphi(v) \) for all \( v \in V \). Therefore, \( \psi \) is a completely contractive extension of \( \varphi \).

\( \square \)

4.2. **Injectivity.** A Banach space \( V \) is said to be **injective** if for any Banach spaces \( W_0 \subseteq W \), every bounded linear mapping \( \varphi_0 : W_0 \to V \) has a linear extension \( \varphi : W \to V \) with \( \|\varphi\| = \|\varphi_0\| \). From the classical Hahn-Banach theorem, \( C \) is an injective Banach space. By analogy, an operator space \( V \) is **injective** if for any operator spaces \( W_0 \subseteq W \), every completely bounded linear mapping \( \varphi_0 : W_0 \to V \) has a linear extension \( \varphi : W \to V \) with \( \|\varphi\|_{cb} = \|\varphi_0\|_{cb} \). From Theorem 6, \( B(H) \) is an injective operator space. Let \( B \) be a linear space, then a linear mapping \( \Phi : B \to B \) is called a **projection** if \( \Phi^2 = \Phi \).

**Proposition 12.** If \( B \) is an injective operator space and \( \Phi : B \to B \) is a completely contractive projection, then \( V = \Phi(B) \) is also injective. Conversely, if \( V \) is an injective operator space and \( V \subseteq B(H) \), then there is a completely contractive projection of \( B(H) \) onto \( V \).

**Proof.** Let \( W_0 \) be a subspace of an operator space \( W \). If \( B \) is injective, then for every completely bounded linear mapping \( \varphi_0 : W_0 \to \Phi(B) \subseteq B \), there exists a linear extension \( \varphi : W \to B \) such that \( \|\varphi\|_{cb} = \|\varphi_0\|_{cb} \). We have the commutative
Let $\psi = \Phi \circ \varphi$. Then $\psi : W \rightarrow \Phi(B)$ is a linear extension of $\varphi_0$ and
\[
\|\varphi_0\|_{cb} \leq \|\psi\|_{cb} \leq \|\Phi\|_{cb}\|\varphi\|_{cb} \leq \|\varphi\|_{cb} = \|\varphi_0\|_{cb},
\]
i.e., $\|\psi\|_{cb} = \|\varphi_0\|_{cb}$. It follow that $\Phi(B)$ is injective. Conversely, if $V$ is injective and $V \subseteq B(H)$, the diagram
\[
\begin{array}{c}
B(H) \\
\downarrow \Phi \\
\text{id} : V \longrightarrow V \subseteq B(H)
\end{array}
\]
shows that $\Phi$ is a completely contractive projection of $B(H)$ onto $V$. \hfill \square

**Lemma 8.** An operator space $V$ is injective if and only if for any operator spaces $W_0 \subseteq W$, the restriction mapping $\rho : CB(W,V) \rightarrow CB(W_0,V)$ is an exact complete quotient mapping.

**Proof.** By definition, $\rho : CB(W,V) \rightarrow CB(W_0,V)$ is an exact complete quotient mapping if and only if for each $n \in \mathbb{N}$, $\rho_n : CB(W,M_n(V)) \rightarrow CB(W_0,M_n(V))$ is an exact quotient mapping. The latter is equivalent to the fact that $M_n(V)$ is injective for all $n \in \mathbb{N}$. Thus, it suffices to prove that $V$ is injective if and only if $M_n(V)$ is injective for all $n \in \mathbb{N}$. Given a completely isometric representation $V \hookrightarrow B(H)$, the corresponding mapping $M_n(V) \hookrightarrow M_n(B(H))$ is a complete isometry.

Suppose $V$ is injective. Let $P : B(H) \rightarrow V$ be a completely contractive projection of $B(H)$ onto $V$ (see Proposition 12). Then $P_n : M_n(B(H)) \rightarrow M_n(V)$ is also a completely contractive surjective projection. Since $M_n(B(H)) = B(H^n)$ is injective, from Proposition 12, $P_n(B(H^n)) = M_n(V)$ is also injective. \hfill \square
Let $V$ and $W$ be operator spaces and let $\varphi : V \to W$ be a bounded linear mapping. For each $n \in \mathbb{N}$, let $\varphi_n : M_n(V) \to M_n(W)$ and $T_n(\varphi) : T_n(V) \to T_n(W)$ be the corresponding linear mappings. Then $(\varphi^*)_n : M_n(W^*) \to M_n(V^*)$ and $T_n(\varphi^*) : T_n(W^*) \to T_n(V^*)$. Since $T_n(V)^* = M_n(V^*)$ and $T_n(V^*) = M_n(V)^*$, we have

\[(33) \quad T_n(\varphi)^* = (\varphi^*)_n \quad \text{and} \quad (\varphi_n)^* = T_n(\varphi^*).\]

**Theorem 7.** Let $V$ and $W$ be operator spaces and let $\varphi : V \to W$ be a bounded linear mapping. Then

(i) $\|T_n(\varphi)\| = \|\varphi_n\|$

(ii) the following are equivalent:

(a) $\varphi_n$ is an isometric injection,
(b) $T_n(\varphi)$ is an isometric injection,
(c) $(\varphi^*)_n$ is a quotient mapping,
(d) $T_n(\varphi^*)$ is a quotient mapping;

(iii) if $V$ is complete, then the followings are equivalent:

(a) $\varphi_n$ is a quotient mapping,
(b) $T_n(\varphi)$ is a quotient mapping,
(c) $(\varphi^*)_n$ is an isometric injection,
(d) $T_n(\varphi^*)$ is an isometric injection.

**Proof.** (i) From Proposition 9 and (33), we have

$$\|\varphi_n\| = \|(\varphi^*)_n\| = \|T_n(\varphi)^*\| = \|T_n(\varphi)\|.$$ 

(ii) Firstly, let us prove that if $\varphi_n$ is a quotient mapping, then $T_n(\varphi)$ is also a quotient mapping. Suppose that $\varphi_n$ is a quotient mapping. For any $w = \alpha \tilde{w} \beta \in T_n(W)$ with $\|w\|_1 < 1$, where $\tilde{w} \in M_n(W)$ and $\alpha, \beta \in HS_n$ with $\|\tilde{w}\|, ||\alpha||, ||\beta||_2 < 1$, there is $\tilde{v} \in M_n(V)$ with $\|\tilde{v}\| < 1$ such that $\varphi_n(\tilde{v}) = \tilde{w}$. If we let $v = \alpha \tilde{v} \beta$, then $\|v\|_1 < 1$ and $T_n(\varphi)(v) = \alpha \varphi_n(\tilde{v}) \beta = \alpha \tilde{w} \beta = w$. Thus, $T_n(\varphi)$ is a quotient mapping. It follows
that 
\[ \varphi_n \text{ is an isometric injection} \implies T_n(\varphi) \text{ is an isometric injection (from Corollary 4)}. \]
\[ \implies (\varphi^*)_n (= T_n(\varphi)^*) \text{ is a quotient mapping}. \]
\[ \implies T_n(\varphi^*) (= (\varphi_n)^*) \text{ is a quotient mapping}. \]
\[ \implies \varphi_n \text{ is an isometric injection}. \]

(iii) We have that if \( V \) is complete, then each \( M_n(V) \) is also complete. And
\[ \varphi_n \text{ is a quotient mapping} \implies T_n(\varphi) \text{ is a quotient mapping (from (ii))}. \]
\[ \implies (\varphi^*)_n (= T_n(\varphi)^*) \text{ is an isometric injection}. \]
\[ \implies T_n(\varphi^*) (= (\varphi_n)^*) \text{ is an isometric injection}. \]
\[ \implies \varphi_n \text{ is a quotient mapping (since } M_n(V) \text{ is complete)}. \]
\[ \square \]
5. The Projective Tensor Product

The tensor product $V \otimes W$ of two operator spaces is an operator space whose structure is derived from the operator space structure of $V$ and $W$. In this section, we characterize the operator space projective tensor product which has the corresponding properties for the projective tensor product of Banach spaces.

Firstly, we define the algebraic tensor product of linear spaces and characterize the tensor product of Banach spaces and then we will turn to operator spaces.

5.1. The Algebraic Tensor Product of Linear Spaces. We need the following concept of free linear space to define the algebraic tensor product of two linear spaces.

**Definition 4.** Let $X$ be any nonempty set. A linear space $\mathcal{F}_X$ over $\mathbb{C}$ is called the free linear space on $X$ if $\mathcal{F}_X = \{ \sum_{i=1}^{n} \alpha_i x_i : x_i \in X, \alpha_i \in \mathbb{C} \}$, where $\alpha x_i + \beta x_i = (\alpha + \beta)x_i$, $\alpha(\beta x_i) = (\alpha \beta)x_i$. That is, $\mathcal{F}_X$ is the set of all formal finite linear combinations of elements of $X$.

The term free comes from the fact that there is no relationship between $x_i \in X$. We see that any linear space $V$ is the free linear space on any basis for $V$.

Let $U$ and $V$ be linear spaces over $\mathbb{C}$ and $U \times V = \{(u, v) : u \in U, v \in V\}$ the Cartesian product which is the set of all ordered pairs with no algebraic structure.

Then, by definition, the free linear space $\mathcal{F}_{U \times V}$ on $U \times V$ is given by

$$\mathcal{F}_{U \times V} = \{ \sum_{i=1}^{n} \alpha_i (u_i, v_i) : (u_i, v_i) \in U \times V, \alpha_i \in \mathbb{C} \},$$

where $\alpha(u_i, v_i) + \beta(u_i, v_i) = (\alpha + \beta)(u_i, v_i)$, $\alpha(\beta(u_i, v_i)) = (\alpha \beta)(u_i, v_i)$. We need to know that $\alpha(u, v) \neq (\alpha u, \alpha v)$ and $(u, v) + (u', v') \neq (u + u', v + v')$. 

Definition 5. Let $U$ and $V$ be linear spaces over $\mathbb{C}$. If $T$ is the subspace of the free linear space $\mathcal{F}_{U \times V}$ generated by all vectors of the form

$$\alpha(u, v) + \beta(u', v) - (\alpha u + \beta u', v) \text{ and } \alpha(u, v) + \beta(u, v') - (u, \alpha v + \beta v'),$$

where $\alpha, \beta \in \mathbb{C}$, $u, u' \in U$ and $v, v' \in V$, then the quotient space $\mathcal{F}_{U \times V}/T$ is called the tensor product of $U$ and $V$ and is denoted by $U \otimes V$.

By definition, for all $w \in U \otimes V$, $w = \sum_{i=1}^{n} \alpha_i(u_i, v_i) + T$ and if $w = 0$, then $\sum_{i=1}^{n} \alpha_i(u_i, v_i) \in T$. Usually we denote the coset $(u, v) + T$ by $u \otimes v$. Thus, $w$ can be represented as a finite sum; i.e., $w = \sum_{i=1}^{n} u_i \otimes v_i$, where

\begin{equation}
\alpha(u \otimes v) + \beta(u' \otimes v) = (\alpha u + \beta u') \otimes v \text{ and}
\end{equation}

\begin{equation}
\alpha(u \otimes v) + \beta(u \otimes v') = u \otimes (\alpha v + \beta v').
\end{equation}

Therefore, $\sum_{i=1}^{n} u_i \otimes v_i = \sum_{j=1}^{m} x_j \otimes y_j$ if and only if one expression is obtained from the other one by a finite number of replacements using (34) and (35).

We can characterize the tensor product via a universal property (see [2, Theorem 14.3]).

Theorem 8. Let $U, V$ and $W$ be linear spaces over $\mathbb{C}$. The pair $(U \otimes V, t)$, where $t : U \times V \to U \otimes V$ is the bilinear mapping defined by $t(u, v) = u \otimes v$, has the following universal property. If $f : U \times V \to W$ is any bilinear mapping, then there exists a unique linear mapping $\varphi : U \otimes V \to W$ such that the diagram

\[
\begin{array}{ccc}
U \times V & \xrightarrow{t} & U \otimes V \\
\downarrow f & & \uparrow \varphi \\
W & & \\
\end{array}
\]

commutes, i.e., $\varphi \circ t = f$ and $\varphi(u \otimes v) = f(u, v)$. 
Moreover, if a pair \((X, s)\) has this universal property, then \(X\) is isomorphic to \(U \otimes V\). Therefore, \(U \otimes V\) is unique up to a linear isomorphism.

Let \(B(U \times V, W)\) be the space of all bilinear mappings from \(U \times V\) to \(W\) and \(L(U \otimes V, W)\) be the space of all linear mappings from \(U \otimes V\) to \(W\), where \(U, V\) and \(W\) are linear spaces over \(\mathbb{C}\). From Theorem 8, for each bilinear mapping \(f : U \times V \to W\), there corresponds a unique linear mapping \(\varphi : U \otimes V \to W\) such that \(f = \varphi \circ t\), i.e., \(f(u, v) = \varphi(u \otimes v)\) for all \(u \in U\) and \(v \in V\).

Consider \(\rho : B(U \times V, W) \to L(U \otimes V, W)\) such that \(\rho(f)(u \otimes v) = f(u, v)\). Then \(\rho\) is well-defined and \(\rho\) is linear, since for all \(f, g \in B(U \times V, W)\),

\[
[\alpha \rho(f) + \beta \rho(g)](u \otimes v) = \alpha f(u, v) + \beta g(u, v) = (\alpha f + \beta g)(u, v) = \rho(\alpha f + \beta g)(u \otimes v),
\]

and thus we get \(\alpha \rho(f) + \beta \rho(g) = \rho(\alpha f + \beta g)\).

Also, \(\rho\) is surjective. To see this, let \(\varphi : U \otimes V \to W\) be any linear mapping. Then \(f = \varphi \circ t : U \times V \to W\) is bilinear and we have \(\rho(f) = \varphi\). If \(\rho(f) = 0\), then \(f = \rho(f) \circ t = 0\). This implies that \(\rho\) is injective. Therefore, we have the linear isomorphism

\[
(36) \quad B(U \times V, W) \cong L(U \otimes V, W).
\]

The following basic properties of tensor products can be seen easily from the definition and the universal property of the tensor product. We summarize them in the following theorem without the proof.

**Theorem 9.** (1) If \(u_i\) are linearly independent in \(U\) and \(v_i\) are arbitrary in \(V\) (\(i = 1, 2, \cdots, n\)), then \(\sum_{i=1}^{n} u_i \otimes v_i = 0\) implies \(v_i = 0\) for all \(i\).
(2) If $u \neq 0$ and $v \neq 0$, then $u \otimes v \neq 0$.

(3) Let $\{ e_i : i \in I \}$ be a basis for $U$ and $\{ f_j : j \in J \}$ a basis for $V$. Then $\{ e_i \otimes f_j : i \in I, j \in J \}$ is a basis for $U \otimes V$.

(4) For any finite dimensional spaces $U$ and $V$, $\dim(U \otimes V) = \dim U \cdot \dim V$.

Example 1. (i) Let $V$ be a linear space over $\mathbb{C}$. Then $\mathbb{C} \otimes V \cong V : \alpha \otimes v \mapsto \alpha v$ and $V \otimes \mathbb{C} \cong V : v \otimes \alpha \mapsto \alpha v$. Thus, we have $\mathbb{C} \otimes V \cong V \otimes \mathbb{C} \cong V$.

Moreover, $\mathbb{C}^n \otimes V \cong V \otimes \mathbb{C}^n \cong V^n : (\alpha_1, \ldots, \alpha_n) \otimes v \mapsto (\alpha_1v, \ldots, \alpha_nv)$.

(ii) Let $M_{m,n}$ be the space of all $m \times n$ matrices over $\mathbb{C}$. Then, $\mathbb{C}^m \otimes \mathbb{C}^n \cong M_{m,n}$.

(iii) Let $V$ be a vector space over $\mathbb{C}$ and $M_n(V)$ the space of all $n \times n$ matrices over $V$. Then, $M_n \otimes V \cong M_n(V)$ via $[\alpha_i,j] \otimes v \mapsto [\alpha_i,jv]$.

Let $f : V \rightarrow V'$ and $g : W \rightarrow W'$ be linear mappings. Then there is a unique linear mapping $f \otimes g : V \otimes W \rightarrow V' \otimes W'$ such that

\[(f \otimes g)(v \otimes w) = f(v) \otimes g(w).\]

(37)

To prove this, consider the bilinear mapping $\varphi : V \times W \rightarrow V' \otimes W'$ defined by $\varphi(v, w) = f(v) \otimes g(w)$. Then there exists a unique linear mapping $f \otimes g : V \otimes W \rightarrow V' \otimes W'$ such that $\varphi(v, w) = (f \otimes g)(v \otimes w)$. Since $\varphi(v, w) = f(v) \otimes g(w)$, we have $(f \otimes g)(v \otimes w) = f(v) \otimes g(w)$. The linear mapping $f \otimes g$ is called the tensor product of $f$ and $g$. In particular, if $V' = W' = \mathbb{C}$, then $(f \otimes g)(v \otimes w) = f(v)g(w)$ for all $v \in V$ and $w \in W$. 

5.2. The Tensor Product of Banach Spaces. In this section, we will introduce the tensor product of two Banach spaces equipped with two reasonable norms called the least crossnorm and the greatest crossnorm.

Definition 6. Let $E \otimes F$ be the algebraic tensor product of Banach spaces $E$ and $F$. A norm $\| \cdot \|_\alpha : E \otimes F \to [0, \infty)$ is called a reasonable crossnorm if it satisfies the following conditions.

C1. $\|x \otimes y\|_\alpha \leq \|x\| \cdot \|y\|$ for all $x \in E$ and $y \in F$.

C2. If $f \in E^*$ and $g \in F^*$, then $f \otimes g \in (E \otimes F, \| \cdot \|_\alpha)^*$ and $\|f \otimes g\| \leq \|f\| \cdot \|g\|$.

Proposition 13. Let $\| \cdot \|_\alpha$ be a reasonable crossnorm on $E \otimes F$. Then

(i) $\|x \otimes y\|_\alpha = \|x\| \cdot \|y\|$ for all $x \in E$ and $y \in F$.

(ii) If $f \in E^*$ and $g \in F^*$, then $\|f \otimes g\| = \|f\| \cdot \|g\|$.

Proof. To prove (i), let $x \in E$ and $y \in F$. We choose $f \in E^*$, $g \in F^*$ such that $f(x) = \|x\|$, $g(y) = \|y\|$ and $\|f\| = \|g\| = 1$. From (C2), $f \otimes g \in (E \otimes F, \| \cdot \|_\alpha)^*$ and $\|f \otimes g\| \leq \|f\| \cdot \|g\| = 1$. Thus, we have

$$\|x\| \cdot \|y\| = \|f(x)\| \cdot \|g(y)\| = \|(f \otimes g)(x \otimes y)\| \leq \|f \otimes g\| \cdot \|x \otimes y\|_\alpha \leq \|x \otimes y\|_\alpha.$$ 

Therefore, together with (C1), it follows that $\|x \otimes y\|_\alpha = \|x\| \cdot \|y\|$.

To prove (ii), let $f \in E^*$, $g \in F^*$. Choose sequences $(x_n)$ in $X$ and $(y_n)$ in $Y$ such that

$$\|x_n\| = \|y_n\| = 1, \|f\| = \lim_{n \to \infty} |f(x_n)| \text{ and } \|g\| = \lim_{n \to \infty} |g(y_n)|.$$
Then,
\[ \|f\|_g = \lim_{n \to \infty} |f(x_n)||g(y_n)| = \lim_{n \to \infty} |(f \otimes g)(x_n \otimes y_n)| \leq \|f \otimes g\| \|x_n \otimes y_n\|_\alpha \leq \|f \otimes g\|. \]

Thus, together with (C2), it follows that \( \|f \otimes g\| = \|f\|_g \). \( \Box \)

Let \( E \) and \( F \) be Banach spaces. Now, we construct the least reasonable crossnorm and the greatest reasonable crossnorm on \( E \otimes F \).

For \( u \in E \otimes F \), if we define \( \|u\|_\lambda \) by
\[ \|u\|_\lambda = \sup\{|(f \otimes g)(u)| : f \in E^*, g \in F^*, \|f\|, \|g\| \leq 1\}, \]
then \( \| \cdot \|_\lambda \) is a norm on \( E \otimes F \). To see this, suppose \( u = \sum_{i=1}^{n} x_i \otimes y_i \neq 0 \), where \( x_i \neq 0 \) and \( y_i \) are linearly independent. We can choose \( f \in E^*, g \in F^* \) such that \( g(y_i) \neq 0 \), \( g(y_i) = 0 \ (i = 2, 3, \ldots, n) \) and \( f(x_1) \neq 0 \). Then \( |(f \otimes g)(u)| = |\sum_{i=1}^{n} f(x_i)g(y_i)| = |f(x_1)g(y_1)| \neq 0 \) and thus \( \|u\|_\lambda \neq 0 \). Also we have
\[ \|u + v\|_\lambda = \sup\{|(f \otimes g)(u + v)| : f \in E^*, g \in F^*, \|f\|, \|g\| \leq 1\} \]
\[ = \sup\{|(f \otimes g)(u) + (f \otimes g)(v)| : f \in E^*, g \in F^*, \|f\|, \|g\| \leq 1\} \]
\[ \leq \|u\|_\lambda + \|v\|_\lambda, \]
and
\[ \|\alpha u\|_\lambda = \sup\{|(f \otimes g)(\alpha u)| \} = |\alpha| \sup\{|(f \otimes g)(u)| \} = |\alpha| \|u\|_\lambda, \]
where the supremum is taken over \( f \in E^*, g \in F^*, \|f\|, \|g\| \leq 1 \).

**Proposition 14.** The norm \( \| \cdot \|_\lambda \) is a reasonable crossnorm on \( E \otimes F \) and if \( \| \cdot \|_\alpha \)
is any reasonable crossnorm on \( E \otimes F \), then \( \|u\|_\lambda \leq \|u\|_\alpha \) for all \( u \in E \otimes F \).

**Proof.** Let \( x \in E \) and \( y \in F \). Then
\[ \|x \otimes y\|_\lambda = \sup\{|(f \otimes g)(x \otimes y)| : f \in E^*, g \in F^*, \|f\|, \|g\| \leq 1\} \]
\[ = \sup\{|f(x)g(y)| : f \in E^*, g \in F^*, \|f\|, \|g\| \leq 1\} \]
\[ \leq \|x\|\|y\|. \]
So, (C1) holds. If \( f \in E^\ast \) and \( g \in F^\ast \), then, for all \( u \in E \otimes F \),

\[
| (f \otimes g)(u) | = ||f|| \cdot ||g|| \cdot | (f/||f|| \otimes g/||g||)(u) | \leq ||f|| \cdot ||g|| \cdot ||u||_\lambda.
\]

Thus, \( f \otimes g \in (E \otimes F, || \cdot ||_\lambda)^* \) and \( ||f \otimes g|| \leq ||f|| ||g|| \), i.e., (C2) holds. Therefore, \( || \cdot ||_\lambda \) is a reasonable crossnorm on \( E \otimes F \).

If \( || \cdot ||_\alpha \) is any reasonable crossnorm on \( E \otimes F \), then for \( f \in E^\ast, g \in F^\ast \) with \( ||f||, ||g|| \leq 1 \), we have \( f \otimes g \in (E \otimes F, || \cdot ||_\alpha)^* \) and for all \( u \in E \otimes F \),

\[
| (f \otimes g)(u) | \leq ||f \otimes g|| \cdot ||u||_\alpha \leq ||u||_\alpha.
\]

Thus, by taking supremum over \( f \) and \( g \) with \( ||f||, ||g|| \leq 1 \) on both sides, we get \( ||u||_\lambda \leq ||u||_\alpha \). Therefore, \( || \cdot ||_\lambda \) is the least reasonable crossnorm on \( E \otimes F \). \( \Box \)

We denote \( E \otimes_\lambda F = (E \otimes F, || \cdot ||_\lambda) \) and since \( E \otimes_\lambda F \) is usually incomplete, its completion under \( || \cdot ||_\lambda \) is denoted by \( E \otimes^\lambda F \) called the injective tensor product of \( E \) and \( F \).

The least reasonable crossnorm \( || \cdot ||_\lambda \) can be described in another way. Let \( B(E \times F) \) be the Banach space of all bounded bilinear functionals on \( E \times F \) with the norm \( || \cdot || \) defined by \( ||\varphi|| = \sup\{|\varphi(x,y)| : x \in E, y \in F, ||x||, ||y|| \leq 1 \} \) for all \( \varphi \in B(E \times F) \).

For each \( u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F \), \( u \) can be regarded as an element \( \hat{u} \) of \( B(E^\ast \times F^\ast) \) defined by \( \hat{u}(f,g) = (f \otimes g)(u) = \sum_{i=1}^n f(x_i)g(y_i) \) for all \( f \in E^\ast, g \in F^\ast \). Then,

\[
||\hat{u}|| = \sup\{|\hat{u}(f,g)| : f \in E^\ast, g \in F^\ast, ||f||, ||g|| \leq 1 \}
\]

\[
= \sup\{|(f \otimes g)(u)| : f \in E^\ast, g \in F^\ast, ||f||, ||g|| \leq 1 \}
\]

\[
= ||u||_\lambda.
\]

It follows that there is a natural isometry of \( E \otimes^\lambda F \) into \( B(E^\ast \times F^\ast) \).
Now, let us turn to the greatest reasonable crossnorm on $E \otimes F$. If we define $\| \cdot \|_\gamma$ on $E \otimes F$ by
\[
\|u\|_\gamma = \sup\{|\varphi(u)| : \varphi \in B(E \times F), \|\varphi\| \leq 1\}
\]
for $u \in E \otimes F$, where $\varphi \in B(E \times F)$ is regarded as an element of $L(E \otimes F, \mathbb{C})$ via (36), then $\| \cdot \|_\gamma$ is a seminorm on $E \otimes F$.

To see this, it is obvious that $\|u\|_\gamma \geq 0$ for all $u \in E \otimes F$. For $u, v \in E \otimes F$,
\[
\|u + v\|_\gamma = \sup\{|\varphi(u + v)| : \varphi \in B(E \times F), \|\varphi\| \leq 1\}
\leq \sup\{|\varphi(u)| + |\varphi(v)| : \varphi \in B(E \times F), \|\varphi\| \leq 1\}
\leq \sup\{|\varphi(u)|\} + \sup\{|\varphi(v)|\}
= \|u\|_\gamma + \|v\|_\gamma.
\]
Also, we have for $\alpha \in \mathbb{C}$,
\[
\|\alpha u\|_\gamma = \sup\{|\alpha \varphi(u)| : \varphi \in B(E \times F), \|\varphi\| \leq 1\}
= |\alpha| \sup\{|\varphi(u)| : \varphi \in B(E \times F), \|\varphi\| \leq 1\}
= |\alpha| \|u\|_\gamma.
\]

**Proposition 15.** The seminorm $\| \cdot \|_\gamma$ is a reasonable crossnorm on $E \otimes F$. If $\| \cdot \|_\alpha$ is any reasonable crossnorm on $E \otimes F$, then $\|u\|_\alpha \leq \|u\|_\gamma$ for all $u \in E \otimes F$.

**Proof.** Since $E^* \otimes \lambda F^*$ is isometric to a closed subspace of $B(E^{**} \times F^{**})$, we can write $\|f \otimes g\|_{B(E^{**} \times F^{**})} = \|f\| \|g\|$ for $f \in E^*$, $g \in F^*$ and thus for the restriction of $f \otimes g$ to $E \otimes F$, we have $\|f \otimes g\|_{E \otimes F} \leq \|f\| \|g\|$. So, if $u \in E \otimes F$, then
\[
\|u\|_\lambda = \sup\{|(f \otimes g)(u)| : f \in E^*, g \in F^*, \|f\|, \|g\| \leq 1\}
\leq \sup\{|\varphi(u)| : \varphi \in B(E \times F), \|\varphi\| \leq 1\} = \|u\|_\gamma.
\]
This implies that $\| \cdot \|_\gamma$ is a norm on $E \otimes F$.

Also, if $x \in E$ and $y \in F$, then
\[
\|x \otimes y\|_\gamma = \sup\{|\varphi(x, y)| : \varphi \in B(E \times F), \|\varphi\| \leq 1\}
= \sup\{|\varphi(x/y)| : \varphi \in B(E \times F), \|\varphi\| \leq 1\}
= \|x\| \|y\|.
\]
Since $\| \cdot \|_\lambda \leq \| \cdot \|_\gamma$, if $f \in E^*$ and $g \in F^*$, then, for all $u \in E \otimes F$,
\[
|(f \otimes g)(u)| \leq \|f\| \|g\| \|u\|_\lambda \leq \|f\| \|g\| \|u\|_\gamma.
\]
Thus, \( f \otimes g \in (E \otimes F, \| \cdot \|_\gamma)^* \) and \( \| f \otimes g \| \leq \| f \| \| g \| \). Therefore, \( \| \cdot \|_\gamma \) is a reasonable crossnorm on \( E \otimes F \).

Let \( \| \cdot \|_\alpha \) be any reasonable crossnorm on \( E \otimes F \) and \( u \in E \otimes F \). Select an \( f \) in \( (E \otimes F, \| \cdot \|_\alpha)^* \) such that \( f(u) = \| u \|_\alpha \), and \( \| f \|_{(E \otimes F, \| \cdot \|_\alpha)^*} = 1 \). Define the bilinear functional \( \varphi \) on \( E \times F \) by \( \varphi(x, y) = f(x \otimes y) \). Then

\[
|\varphi(x, y)| = |f(x \otimes y)| \leq \| x \otimes y \|_\alpha = \| x \| \| y \| \text{ for all } x \in E, y \in F.
\]

Therefore, \( \varphi \in B(E \times F) \) with \( \| \varphi \| \leq 1 \) and we obtain

\[
\| u \|_\alpha = |f(u)| = |\varphi(u)| \leq \| u \|_\gamma.
\]

\( \square \)

The norm \( \| \cdot \|_\gamma \) is called the projective tensor product norm on \( E \otimes F \) and we denote \( E \otimes_\gamma F = (E \otimes F, \| \cdot \|_\gamma) \). The completion of \( E \otimes F \) under \( \| \cdot \|_\gamma \) is denoted by \( E \otimes^\gamma F \) and called the projective tensor product of \( E \) and \( F \). The completed norm in \( E \otimes^\gamma F \) is still denoted by \( \| \cdot \|_\gamma \).

The following proposition gives a useful alternative way to describe \( \| \cdot \|_\gamma \).

**Proposition 16.** If \( u \in E \otimes F \), then

\[
\| u \|_\gamma = \inf \left\{ \sum_{i=1}^{n} \| x_i \| \| y_i \| : x_i \in E, y_i \in F, u = \sum_{i=1}^{n} x_i \otimes y_i \right\},
\]

where the infimum is taken over all representations of \( u \).

*Proof.* Let \( \| u \|_\alpha = \inf \left\{ \sum_{i=1}^{n} \| x_i \| \| y_i \| : x_i \in E, y_i \in F, u = \sum_{i=1}^{n} x_i \otimes y_i \right\} \). Then \( \| \cdot \|_\alpha \) is a seminorm on \( E \otimes F \). To see this, it is evident that \( \| u \| \geq 0 \) for all \( u \in E \otimes F \). For
\( u, v \in E \otimes F, \)
\[
\|u + v\|_{\alpha} = \| \sum_{i=1}^{n} x_i \otimes y_i + \sum_{j=1}^{m} v_j \otimes w_j \|_{\alpha} \\
\leq \sum_{i,j} (\|x_i\|_{\alpha} + \|v_j\|_{\alpha}) \\
= \sum_{i=1}^{n} \|x_i\|_{\alpha} |y_i| + \sum_{j=1}^{m} \|v_j\|_{\alpha} |w_j|.
\]

By taking infimum over all representations of \( u \) and \( v \), respectively, we obtain \( \|u + v\|_{\alpha} \leq \|u\|_{\alpha} + \|v\|_{\alpha} \).

Furthermore, for \( \lambda \in \mathbb{C} \) and \( u \in E \otimes F \),
\[
\|\lambda u\|_{\alpha} = \| \sum_{i=1}^{n} \lambda x_i \otimes y_i \|_{\alpha} \leq |\lambda| \sum_{i=1}^{n} \|x_i\| |y_i|.
\]

Thus, we get \( \|\lambda u\|_{\alpha} \leq |\lambda| \|u\|_{\alpha} \).

For \( \lambda(\neq 0) \in \mathbb{C} \), \( \|u\|_{\alpha} = \| \frac{1}{\lambda} \cdot \lambda u\|_{\alpha} \leq \frac{1}{|\lambda|} \|\lambda u\|_{\alpha} \), i.e.,
\[|\lambda| \|u\|_{\alpha} \leq \|\lambda u\|_{\alpha}.\]

Therefore, for all \( \lambda \in \mathbb{C} \), \( \|\lambda u\|_{\alpha} = |\lambda| \|u\|_{\alpha}. \)

If \( u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F \), then
\[
\|u\|_{\gamma} = \| \sum_{i=1}^{n} x_i \otimes y_i \|_{\gamma} \leq \sum_{i=1}^{n} \|x_i \otimes y_i\|_{\gamma} = \sum_{i=1}^{n} \|x_i\| |y_i|.
\]

It follows that \( \|u\|_{\gamma} \leq \|u\|_{\alpha} \) and hence \( \cdot \|_{\alpha} \) is a norm on \( E \otimes F \). Since \( \cdot \|_{\gamma} \) is the greatest reasonable crossnorm, we have \( \|u\|_{\gamma} = \|u\|_{\alpha}. \)

Let \( E, F, G \) and \( H \) be Banach spaces. Let \( S : E \rightarrow G \) and \( T : F \rightarrow H \) be bounded linear mappings. Consider the linear mapping \( S \otimes T : E \otimes F \rightarrow G \otimes H \) defined by
\[
(S \otimes T)(\sum_{i=1}^{n} x_i \otimes y_i) = \sum_{i=1}^{n} Sx_i \otimes Ty_i.
\]

If \( E \otimes F \) and \( G \otimes H \) are equipped with \( \cdot \|_{\gamma} \) respectively, then \( S \otimes T \) is bounded.

In fact, we have
\[
\|(S \otimes T)(\sum_{i=1}^{n} x_i \otimes y_i)\|_{\gamma} = \| \sum_{i=1}^{n} Sx_i \otimes Ty_i \|_{\gamma} \\
\leq \sum_{i=1}^{n} \|Sx_i\| \|Ty_i\| \\
\leq \|S\| \|T\| \sum_{i=1}^{n} \|x_i\| \|y_i\|,
\]
i.e., \(\|(S \otimes T)(u)\|_\gamma \leq \|S\|\|T\|\|u\|_\gamma\) for all \(u \in E \otimes F\). It follows that there is a unique bounded linear extension \(S \otimes^\gamma T : E \otimes^\gamma F \to G \otimes^\gamma H\) with \(\|S \otimes^\gamma T\| \leq \|S\|\|T\|\).

Let \(\mathcal{B}(E \times F, G)\) be the space of all bounded bilinear mappings from \(E \times F\) to \(G\) equipped with the norm \(\| \cdot \|\) defined by \(\|\varphi\| = \sup\{\|\varphi(x, y)\| : x \in E, y \in F, \|x\|, \|y\| \leq 1\}\) for all \(\varphi \in \mathcal{B}(E \times F, G)\). In the following, let us show the isometric isomorphisms

\[
\mathcal{B}(E \otimes^\gamma F, G) \cong \mathcal{B}(E \times F, G) \cong \mathcal{B}(E, \mathcal{B}(F, G)).
\]

(38)

Let \(\varphi : E \times F \to G\) be a bounded bilinear mapping. Then

\[\|\varphi\| = \sup\{\|\varphi(x, y)\| : x \in E, y \in F, \|x\|, \|y\| \leq 1\}.
\]

We have a unique linear mapping \(\psi : E \otimes F \to G\) such that

\[
\psi\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} \psi(x_i \otimes y_i) = \sum_{i=1}^{n} \varphi(x_i, y_i) \quad (x_i \in E, y_i \in F).
\]

Since \(\|\psi\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\| \leq \|\varphi\| \sum_{i=1}^{n} \|x_i\|\|y_i\|\), \(\psi\) is a bounded linear mapping from \((E \otimes F, \| \cdot \|_\gamma)\) to \(G\) and \(\|\psi\| \leq \|\varphi\|\).

Conversely, if \(\psi : E \otimes^\gamma F \to G\) is a bounded linear mapping, then \(\psi\) corresponds to a bilinear mapping \(\varphi : E \times F \to G\) defined by \(\varphi(x, y) = \psi(x \otimes y) \quad (x \in E, y \in F)\).

Since \(\|\varphi(x, y)\| = \|\psi(x \otimes y)\| \leq \|\psi\|\|x \otimes y\|_\gamma = \|\psi\|\|x\|\|y\|\), \(\varphi\) is bounded and \(\|\varphi\| \leq \|\psi\|\). It follows that \(\|\psi\| = \|\varphi\|\). Therefore, \(\mathcal{B}(E \otimes^\gamma F, G) \cong \mathcal{B}(E \times F, G)\).

To prove the second isomorphism in (38), we notice that every bounded bilinear mapping \(\varphi : E \times F \to G\) induces a bounded linear mapping \(\rho : E \to \mathcal{B}(F, G)\) defined by \(\rho(x)(y) = \varphi(x, y) \quad (x \in E, y \in F)\), since \(\|\rho(x)(y)\| = \|\varphi(x, y)\| \leq \|\varphi\|\|x\|\|y\|\) and hence \(\|\rho\| \leq \|\varphi\|\).
On the other hand, let \( \rho \in \mathcal{B}(E, \mathcal{B}(F, G)) \). We define the bilinear mapping \( \varphi : E \times F \to G \) by \( \varphi(x, y) = \rho(x)(y) \). Then \( |\varphi(x, y)| = |\rho(x)(y)| \leq \|\rho(x)\| \|y\| \leq \|\rho\| \|x\| \|y\| \).

Thus, \( \varphi \) is bounded and \( \|\varphi\| \leq \|\rho\| \). Therefore, \( \mathcal{B}(E \times F, G) \cong \mathcal{B}(E, \mathcal{B}(F, G)) \).

In particular, if \( G = \mathbb{C} \) in (38), then we have \( (E \otimes F)^* \cong \mathcal{B}(E \times F) \cong \mathcal{B}(E, F^*) \).

### 5.3. The Operator Space Projective Tensor Product

Let \( V \) and \( W \) be operator spaces and \( M_{mn}(V \otimes W) \) the linear space of \( mn \times mn \) matrices over the tensor product \( V \otimes W \). We recall that given \( v = \alpha \otimes v_0 \in M_m \otimes V \) and \( w = \beta \otimes w_0 \in M_n \otimes W \), where \( \alpha \in M_m \), \( \beta \in M_n \), \( v_0 \in V \), \( w_0 \in W \), the Kronecker product

\[
(39) \quad v \otimes w \in M_{mn}(V \otimes W) \cong M_{mn} \otimes V \otimes W
\]

is given by \( (\alpha \otimes v_0) \otimes (\beta \otimes w_0) \cong (\alpha \otimes \beta) \otimes (v_0 \otimes w_0) \) (see § 2.1).

Moreover, for any \( u \in M_n(V \otimes W) \), \( u \) can always be decomposed as \( u = \alpha(v \otimes w)\beta \), where \( \alpha \in M_{np}, \beta \in M_{pq}, v \in M_p(V) \) and \( w \in M_q(W) \) with \( p, q \in \mathbb{N} \) arbitrary.

From now on, we assume that an operator space \( V \) is complete with respect to the norm \( \| \cdot \|_1 \) and thus \( M_n(V) \) is complete.

An operator space matrix norm \( \| \cdot \|_\mu \) on \( V \otimes W \) is said to be a subcross matrix norm (resp. a cross matrix norm) if \( \|v \otimes w\|_\mu \leq \|v\| \|w\| \) (resp. \( \|v \otimes w\|_\mu = \|v\| \|w\| \)) for all \( v = [v_{i,j}] \in M_p(V) \) and \( w = [w_{k,l}] \in M_q(W) \).

For \( u \in M_n(V \otimes W) \), we define

\[
(40) \quad \|u\|_\wedge = \inf \{\|\alpha\| \|v\| \|w\| \|\beta\| : u = \alpha(v \otimes w)\beta\},
\]

where the infimum is taken over all decompositions of \( u \) with \( v \in M_p(V), w \in M_q(W), \alpha \in M_{npq} \) and \( \beta \in M_{pq} \) with \( p, q \in \mathbb{N} \) arbitrary.
In the next theorem, we will show that $\| \cdot \|_\wedge$ is an operator space matrix norm.

We denote $V \otimes_\Lambda W = (V \otimes W, \| \cdot \|_\wedge)$ and define the (operator space) projective tensor product $V \otimes^\wedge W$ to be the completion of $V \otimes_\Lambda W$.

**Theorem 10.** For any operator spaces $V$ and $W$, $\| \cdot \|_\wedge$ is the greatest operator space subcross matrix norm on $V \otimes W$.

**Proof.** First, let us show that $\| \cdot \|_\wedge$ is a seminorm satisfying [M1] and [M2]. For given $u_1 \in M_m(V \otimes W)$, $u_2 \in M_n(V \otimes W)$ and $\varepsilon > 0$, consider decompositions $u_i = \alpha_i (v_i \otimes w_i) \beta_i$ with $\| v_i \| = \| w_i \| = 1$ and $\| \alpha_i \| = \| \beta_i \| \leq (\| u_i \|_\wedge + \varepsilon)^{1/2}$. If $v = v_1 \oplus v_2$ and $w = w_1 \oplus w_2$, then

$$v \otimes w = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix} \otimes \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix} = \begin{bmatrix} v_1 \otimes w_1 & 0 & 0 & 0 \\ 0 & v_1 \otimes w_2 & 0 & 0 \\ 0 & 0 & v_2 \otimes w_1 & 0 \\ 0 & 0 & 0 & v_2 \otimes w_2 \end{bmatrix}.$$  

Thus,

$$u_1 \oplus u_2 = \begin{bmatrix} \alpha_1 (v_1 \otimes w_1) \beta_1 & 0 \\ 0 & \alpha_2 (v_2 \otimes w_2) \beta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & \alpha_2 & 0 \\ \alpha_2 & 0 & 0 & 0 \end{bmatrix} (v \otimes w) \begin{bmatrix} \beta_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \beta_2 \end{bmatrix}$$

$$= \alpha (v \otimes w) \beta,$$

where $\| v \| = 1$, $\| w \| = 1$ and $\alpha = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 \end{bmatrix}$, $\beta = \begin{bmatrix} \beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 \end{bmatrix}^t$, and

$$\| u_1 \oplus u_2 \|_\wedge \leq \| \alpha \| \| \beta \| = \| \alpha \alpha^* \|^{1/2} \| \beta^* \beta \|^{1/2}$$

$$= \| \alpha_1 \alpha_1^* + \alpha_2 \alpha_2^* \|^{1/2} \| \beta_1 \beta_1^* + \beta_2 \beta_2^* \|^{1/2}$$

$$\leq (\max\{\| \alpha_i \|^2\})^{1/2} (\max\{\| \beta_i \|^2\})^{1/2}$$

$$\leq \max\{\| u_i \|_\wedge \} + \varepsilon.$$  

Since $\varepsilon > 0$ is arbitrary, we have [M1'].
For [M2], let $\gamma \in M_{p,m}$ and $\delta \in M_{m,p}$. Then

$$\gamma u_1 \delta = (\gamma \alpha_1 (v_1 \otimes w_1)) \beta_1 \delta = (\gamma \alpha_1) (v_1 \otimes w_1) (\beta_1 \delta)$$

and $\|\gamma u_1 \delta\|_\Lambda \leq \|\gamma \alpha_1\| \|\beta_1 \delta\| \leq \|\gamma\| \|\delta\| (\|u_1\|_\Lambda + \varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we get [M2]. From Proposition 6, $\| \cdot \|_\Lambda$ is a seminorm which satisfies [M1] and [M2].

Next, we show that the seminorm $\| \cdot \|_\Lambda$ on $V \otimes W$ is a norm. Suppose $u \neq 0$. Then we can let $u = \sum_{i=1}^{n} v_i \otimes w_i \in V \otimes W$ such that $v_i \neq 0$ for all $i$ and $w_i$ are linearly independent. If we choose $g_i \in W^*_i \| \cdot \| \leq 1$ with $g_i(w_j) = \delta_{i,j}$ and $f \in V^*_i \| \cdot \| \leq 1$ with $f(v_1) \neq 0$ and let $g = g_1$, then

$$(f \otimes g)(u) = (f \otimes g)(\sum_{i=1}^{n} v_i \otimes w_i) = \sum_{i=1}^{n} (f \otimes g)(v_i \otimes w_i) = \sum_{i=1}^{n} f(v_i) g(w_i) = f(v_1) \neq 0.$$ 

Now, suppose $u_1 = \alpha(v \otimes w) \beta$ with $v \in M(m(V), w \in M_n(W)$. Then, since $\|f_m\| = \|f\| \leq 1$ and $\|g_n\| = \|g\| \leq 1$,

$$|(f \otimes g)(u)| = |(f_m \otimes g_n)(u_1)| = |(f_m \otimes g_n)(\alpha(v \otimes w) \beta)| = |\alpha(f_m(v) \otimes g_n(w)) \beta| \leq \|\alpha\| \|f_m(v)\| \|g_n(w)\| \|\beta\| \leq \|\alpha\| \|v\| \|w\| \|\beta\|.$$ 

Thus, $0 \neq |(f \otimes g)(u)| \leq \|u_1\|_\Lambda$ and it follows that $\| \cdot \|_\Lambda$ on $V \otimes W$ is a norm.

Therefore, $\| \cdot \|_\Lambda$ is an operator space matrix norm on $V \otimes W$.

Finally, if $v \in M_m(V)$ and $w \in M_n(W)$, then $v \otimes w = I_{mn}(v \otimes w) I_{mn}$ and thus $\|v \otimes w\|_\Lambda \leq \|v\| \|w\|$, i.e., $\| \cdot \|_\Lambda$ is a subcross matrix norm. If $\| \cdot \|_\mu$ is any subcross matrix norm on $V \otimes W$, then, for all $u \in M_n(V \otimes W)$, $\|u\|_\mu \leq \|\alpha\| \|v\| \|w\| \|\beta\|$ and it follows that $\|u\|_\mu \leq \|u\|_\Lambda$. □
Given operator spaces $V, W$ and $X$, if $\varphi : V \times W \to X$ is a bilinear mapping, then for each $m, n \in \mathbb{N}$, there is a corresponding bilinear mapping

$$\varphi_{m,n} : M_m(V) \times M_n(W) \to M_{mn}(X)$$

such that $\varphi_{m,n}(v, w) = [\varphi(v_{i,j}, w_{k,l})]$, or equivalently, by elementary tensors,

$$\varphi_{m,n}(\alpha \otimes v_0, \beta \otimes w_0) = [\varphi(\alpha_{i,j}v_0, \beta_{k,l}w_0)] = [\alpha_{i,j} \beta_{k,l} \varphi(v_0, w_0)] = [\alpha_{i,j} \beta_{k,l} \varphi(v_0, w_0)] = \alpha \otimes \beta \otimes \varphi(v_0, w_0) .$$

We define the \textit{completely bounded norm} of $\varphi$ by

$$\|\varphi\|_{cb} = \sup\{\|\varphi_{m,n}\| : m, n \in \mathbb{N}\} = \sup\{\|\varphi_{n,m}\| : n \in \mathbb{N}\} ,$$

and $\varphi$ is said to be \textit{completely bounded} (resp.\textit{completely contractive}) if $\|\varphi\|_{cb} < \infty$ (resp. $\|\varphi\|_{cb} \leq 1$). It is evident that $\|\cdot\|_{cb}$ is a norm on the linear space $CB(V \times W, X)$ of all completely bounded bilinear mappings.

We now define a matrix norm on $CB(V \times W, X)$ via the natural isometric identification

$$M_n(CB(V \times W, X)) \cong CB(V \times W, M_n(X))$$

by $\varphi(v, w) = [\varphi_{i,j}(v, w)]$, where $\varphi_{i,j} \in CB(V \times W, X)$ and $\varphi \in CB(V \times W, M_n(X))$.

Corresponding to (38), we have the following completely isometric isomorphisms.

\textbf{Proposition 17.} If $V, W$ and $X$ are operator spaces, then there are natural completely isometric identifications

$$CB(V \otimes^\wedge W, X) \cong CB(V \times W, X) \cong CB(V, CB(W, X)) .$$

\textit{Proof.} Given a bilinear mapping $\varphi : V \times W \to X$, there is a corresponding linear mapping $\bar{\varphi} : V \otimes W \to X$ such that $\bar{\varphi}(v \otimes w) = \varphi(v, w)$ for all $v \in V$ and $w \in W$. 


If \( u = \alpha (v \otimes w) \beta \in M_n(V \otimes W) \), where \( v \in M_p(V) \) and \( w \in M_q(W) \), for the corresponding mapping \( \tilde{\varphi}_n : M_n(V \otimes W) \to M_n(X) \), we have

\[
\tilde{\varphi}_n(u) = \tilde{\varphi}_n(\alpha (v \otimes w) \beta) = \alpha \tilde{\varphi}_{p,q}(v \otimes w) \beta \\
= \alpha [\varphi(v_{i,j} \otimes w_{k,l})] \beta = \alpha [\varphi(v_{i,j}, w_{k,l})] \beta \\
= \alpha \varphi_{p,q}(v, w) \beta,
\]

and thus, \( \|\tilde{\varphi}_n(u)\| \leq \|\alpha\| \|\varphi_{p,q}\| \|v\| \|w\| \|\beta\| \leq \|\varphi\|_{cb} \|\alpha\| \|v\| \|w\| \|\beta\| \). It follows that
\[
\|\tilde{\varphi}_n(u)\| \leq \|\varphi\|_{cb} \|u\|_\Lambda \text{ for all } n \in \mathbb{N} \text{ and } u \in M_n(V \otimes W) \text{ and hence } \|\tilde{\varphi}\|_{cb} \leq \|\varphi\|_{cb}.
\]

Conversely, given \( \varepsilon > 0 \), we can choose \( p, q \in \mathbb{N}, v \in M_p(V) \| \| \leq 1 \) and \( w \in M_q(W) \| \| \leq 1 \) such that \( \|\varphi\|_{cb} \leq \|\varphi_{p,q}(v, w)\| + \varepsilon \). Since \( \|v \otimes w\|_\Lambda \leq \|v\| \|w\| \leq 1 \), we get
\[
\|\varphi\|_{cb} \leq \|\tilde{\varphi}_{p,q}(v \otimes w)\| + \varepsilon \leq \|\varphi\|_{cb} + \varepsilon.
\]
It follows that \( \|\varphi\|_{cb} \leq \|\tilde{\varphi}\|_{cb} \) and thus \( \|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb} \). Since \( \varphi \) has a completely bounded linear extension \( \Phi : V \otimes^\Lambda W \to X \) such that \( \|\Phi\|_{cb} = \|\varphi\|_{cb} \), we obtain the first completely isometric identification.

For any bounded bilinear mapping \( \varphi : V \times W \to X \), there is a corresponding linear mapping \( \tilde{\varphi} : V \to B(W, X) \) defined by \( \tilde{\varphi}(v)(w) = \varphi(v, w) \) \((v \in V, w \in W)\). If \( \varphi \in CB(V \times W, X) \), then, for \( v \in V, w \in M_q(W) \),

\[
\|\tilde{\varphi}(v)_q(w)\| = \|[\tilde{\varphi}(v)(w_{k,l})]\| \\
= \|[\varphi(v, w_{k,l})]\| \\
= \|\varphi_{1,q}(v, w)\| \\
\leq \|\varphi\|_{cb} \|v\| \|w\|.
\]

This implies that \( \tilde{\varphi}(v) \) is completely bounded (i.e., \( \tilde{\varphi}(V) \subseteq CB(W, X) \)) and \( \|\tilde{\varphi}(v)\|_{cb} \leq \|\varphi\|_{cb} \|v\| \). If \( v \in M_p(V) \) and \( w \in M_q(W) \), then, since \( \tilde{\varphi}_p(v) \in CB(W, X) \),

\[
(\tilde{\varphi}_p(v))_q(w) = [\tilde{\varphi}_p(v)(w_{k,l})] \\
= [\tilde{\varphi}(v_{i,j})(w_{k,l})] \\
= [\varphi(v_{i,j}, w_{k,l})] = \varphi_{p,q}(v, w).
\]

Thus, we obtain \( \|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb} \) and the isometric injection

\[
\theta : CB(V \times W, X) \to CB(V, CB(W, X)) : \varphi \mapsto \tilde{\varphi}.
\]
To see that \(\theta\) is surjective, let \(\psi \in CB(V, CB(W, X))\). We can define a bilinear mapping \(\varphi : V \times W \to X\) by \(\varphi(v, w) = \psi(v)(w)\). By the above discussion, we have \(\varphi \in CB(V \times W, X)\) and \(\theta(\varphi) = \psi\). Therefore, we have the isometry
\[
CB(V \times W, X) \cong CB(V, CB(W, X)).
\]
For each \(r \in \mathbb{N}\), replacing \(X\) by \(M_r(X)\), we have the isometry
\[
CB(V \times W, M_r(X)) \cong CB(V, CB(W, M_r(X))),
\]
which is equivalent to that
\[
\theta_r : M_r(CB(V \times W, X)) \to M_r(CB(V, CB(W, X)))
\]
is an isometry. Therefore, \(\theta : CB(V \times W, X) \to CB(V, CB(W, X))\) is a complete isometry, i.e., the second complete isometric identification holds.

**Corollary 6.** Let \(V, V_1, W\) and \(W_1\) be operator spaces. Given complete contractions \(\varphi : V \to V_1\) and \(\psi : W \to W_1\), the corresponding mapping \(\varphi \otimes \psi : V \otimes W \to V_1 \otimes W_1\) extends to a complete contraction \(\varphi \otimes \psi : V \otimes^\wedge W \to V_1 \otimes^\wedge W_1\).

**Proof.** Let \(\varphi \times \psi\) be a bilinear mapping \(\varphi \times \psi : V \times W \to V_1 \otimes^\wedge W_1 : (v, w) \mapsto \varphi(v) \otimes \psi(w)\). Then \(\varphi \times \psi\) is a complete contraction since for all \(v = [v_{i,j}] \in M_p(V)\) and \(w = [w_{k,l}] \in M_q(W)\),
\[
\|((\varphi \times \psi)_{p,q} (v, w))\| \leq \|((\varphi \times \psi)(v_{i,j}, w_{k,l}))\| \leq \|\varphi(v_{i,j}) \otimes \psi(w_{k,l})\| \leq \|\varphi_p(v) \otimes \psi_q(w)\| \leq \|\varphi_p(v)\| \|\psi_q(w)\| \leq \|v\| \|w\|.
\]
From Proposition 17, the linear extension \(\varphi \otimes \psi : V \otimes W \to V_1 \otimes^\wedge W_1\) can be extended to a complete contraction \(\varphi \otimes \psi : V \otimes W \to V_1 \otimes^\wedge W_1\). 
\(\Box\)
Let $V$ and $W$ be linear spaces. Then we have

\[(43) \quad V \otimes W \cong W \otimes V. \quad \text{(Commutativity)}\]

To see this, consider the bilinear mappings $\varphi : V \times W \to W \otimes V : (v, w) \mapsto w \otimes v$ and $\psi : W \times V \to V \otimes W : (w, v) \mapsto v \otimes w$. Then, there is a unique linear mapping $\lambda_1 : V \otimes W \to W \otimes V$ such that $\lambda_1(v \otimes w) = \varphi(v, w) = w \otimes v$. Similarly, there is a unique linear mapping $\lambda_2 : W \otimes V \to V \otimes W$ such that $\lambda_2(w \otimes v) = \psi(w, v) = v \otimes w$.

It follows that $\lambda_1 \circ \lambda_2 = I$ and $\lambda_2 \circ \lambda_1 = I$, where $I$ is an identity mapping. Thus, $\lambda_1$ and $\lambda_2$ are inverse isomorphisms.

Furthermore, for another linear space $X$, we have linear isomorphisms

\[
V \otimes W \otimes X \cong (V \otimes W) \otimes X : v \otimes w \otimes x \mapsto (v \otimes w) \otimes x
\]

and $V \otimes W \otimes X \cong V \otimes (W \otimes X) : v \otimes w \otimes x \mapsto v \otimes (w \otimes x)$. Thus, we have

\[(44) \quad (V \otimes W) \otimes X \cong V \otimes (W \otimes X). \quad \text{(Associativity)}\]

We can extend (43) and (44) to the projective tensor product. Turning to operator spaces, for each $n \in \mathbb{N}$, we have the matrix level isomorphisms

\[
M_n(V \otimes W) \cong M_n \otimes (V \otimes W) \cong M_n \otimes (W \otimes V) \cong M_n(W \otimes V)
\]

and $M_n((V \otimes W) \otimes X) \cong M_n(V \otimes (W \otimes X))$. We can also extend these to the operator space projective tensor product $V \otimes^\wedge W$ and we have the following Proposition 18.

**Proposition 18.** Given operator spaces $V, W$ and $X$, we have the completely isometric isomorphisms

\[
V \otimes^\wedge W \cong W \otimes^\wedge V \quad \text{and} \quad (V \otimes^\wedge W) \otimes^\wedge X \cong V \otimes^\wedge (W \otimes^\wedge X).
\]
Let $V$ and $W$ be operator spaces. If we take $X = C$ in Proposition 17, we obtain a natural completely isometric identification $\lambda : (V \otimes^\wedge W)^* \cong CB(V, W^*)$, and also a complete isometric identification $\rho : (V \otimes^\wedge W)^* \cong CB(W, V^*)$ by Proposition 18, where the mappings $\lambda(u) : V \to W^*$ and $\rho(u) : W \to V^*$ $(u \in (V \otimes^\wedge W)^*)$ are defined by $(\lambda(u)(v))(w) = \langle u, v \otimes w \rangle = (\rho(u)(w))(v)$. Thus, we conclude the completely isometric identifications

$$
(V \otimes^\wedge W)^* \cong CB(V, W^*) \cong CB(W, V^*).$

Since $T_n^* = M_n$, $T_n$ has a natural operator space structure and we get the complete isometric isomorphism

$$
(T_n \otimes^\wedge V)^* \cong CB(V, M_n) = M_n(V^*).
$$

If $V = T_r$, then $(T_n \otimes^\wedge T_r)^* \cong M_n(T_r^*) \cong M_n(M_r) = M_{n \times r} = T_{n \times r}^*$ and thus $T_n \otimes^\wedge T_r \cong T_{n \times r}$ as operator spaces.

Furthermore, since $T_n(V)^* \cong M_n(V^*)$ as Banach spaces (see Lemma 6), we have a natural isometric identification,

$$
T_n(V) \cong T_n \otimes^\wedge V.
$$

Now, if $u \in M_n \otimes V$, then

$$
\|u\|_{T_n \otimes^\wedge V} = \sup\{|\langle u, f \rangle| : f \in M_n(V^*), \|f\|_{cb} \leq 1\} = \|u\|_{T_n(V)}.
$$

Since $T_n$ is an operator space, we may use (47) to define an operator space matrix norm on $T_n(V)$. Then we have the complete isometries

$$
T_n(V)^* \cong (T_n \otimes^\wedge V)^* \cong CB(V, M_n) = M_n(V^*),
$$

i.e., the isomorphism $T_n(V)^* \cong M_n(V^*)$ is an operator space identification.
Combining (47) with the isometric identification \( T_n(V^*) \cong M_n(V)^* \), we have a natural isometry

\[ T_n \otimes^\wedge V^* \rightarrow M_n(V)^*. \]

To show that this is also a complete isometry, by Theorem 7 and (47), it suffices to prove that, for all \( r \in \mathbb{N} \), the corresponding mapping \( T_r \otimes^\wedge T_n \otimes^\wedge V^* \rightarrow T_r \otimes^\wedge M_n(V)^* \) is isometric. This follows immediately from the following isometric identifications:

\[
T_r \otimes^\wedge T_n \otimes^\wedge V^* \cong T_{r \times n} \otimes^\wedge V^* \cong T_{r \times n}(V^*) \\
\cong [M_{r \times n}(V)]^* \cong [M_r(M_n(V))]^* \\
\cong T_r(M_n(V)^*) \cong T_r \otimes^\wedge M_n(V)^*.
\]

Summarizing the above, we have the following Proposition 19.

**Proposition 19.** Let \( V \) be an operator space and \( n \in \mathbb{N} \). Then we have the completely isometric identifications

\[ T_n(V)^* \cong M_n(V^*), \ M_n(V)^* \cong T_n(V^*), \text{ and } M_n(V^{**}) \cong M_n(V^{**}). \]

5.4. **Examples.** In this section, for a given measure space, we will show some examples of (complete) isometric isomorphisms on the projective tensor product. To do this, we need some definitions and theorems which we now state without proof.

Let \((X, \mathcal{M}, \mu)\) be a measure space and \(1 \leq p < \infty\). We define \(L_p(X)\) to be the space of all measurable complex functions \(f\) on \(X\) such that \(\int_X |f|^p d\mu < \infty\) with the norm \(\|f\|_p = \left( \int_X |f|^p d\mu \right)^{1/p}\).

We identify functions that are equal almost everywhere with respect to the measure \(\mu\). Then, since \(\|f\|_p = 0\) if and only if \(f = 0\) almost everywhere, we can consider that \(\|f\|_p = 0\) only for \(f = 0\).
We have that the spaces $L_p(X)$ for $1 \leq p < \infty$ are Banach spaces with the norm $\| \cdot \|_p$. When $p = 2$, $L_2(X)$ is a Hilbert space with inner product $\langle f, g \rangle = \int_X f \overline{g} d\mu$ for $f, g \in L_2(X)$.

In the special case when $X = \mathbb{N}, \mathcal{M} = \mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$) and $\mu$ is taken as the counting measure, the space $L_p(X)$ reduces to the sequence space

$$l_p = \{(x_n) : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$$

with the norm $\|(x_n)\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$.

**Definition 7.** Let $(X, \mathcal{M}, \mu)$ be a measure space. For any measurable complex function $f$ on $X$, define the essential supremum (or $\infty$-norm) of $f$ by

$$\|f\|_\infty = \text{ess sup}|f(x)| = \inf\{t > 0 : \mu\{x : |f(x)| > t\} = 0\}.$$  

If $\|f\|_\infty < \infty$, $f$ is called an essentially bounded function. We denote $L_\infty(X)$ to be all measurable complex functions defined on $X$ such that $\|f\|_\infty < \infty$. We identify functions that are equal almost everywhere with respect to the measure $\mu$.

Then, since $\|f\|_\infty = 0$ if and only if $f$ vanishes almost everywhere, we can consider that $\|f\|_\infty = 0$ only for $f = 0$. We have that $L_\infty(X)$ is a Banach space with the norm $\|f\|_\infty$.

In the special case when $X = \mathbb{N}, \mathcal{M} = \mathcal{P}(\mathbb{N})$ and $\mu$ is taken as the counting measure, the space $L_\infty(X)$ reduces to the sequence space

$$l_\infty = \{(x_n) : \sup_{n \in \mathbb{N}} |x_n| < \infty\}$$

with the supremum norm $\|(x_n)\|_\infty = \sup_{n \in \mathbb{N}} |x_n|$.

We recall that a measure space $(X, \mathcal{M}, \mu)$ is finite if $\mu(X) < \infty$ and it is $\sigma$-finite if $X = \bigcup_{n=1}^{\infty} X_n$ with $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$.

There is a natural duality between the spaces $L_p(X)$ and $L_q(X)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p < \infty$. The following theorem called the Riesz representation theorem
shows that bounded linear functionals on $L_p(X)$ are represented by functions in $L_q(X)$

**Theorem 11.** Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space and let $p, q$ be conjugate indices with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p < \infty$. Then, for each bounded linear functional $\Phi : L_p(X) \to \mathbb{C}$, there is a function $g \in L_q(X)$ such that

$$\Phi(f) = \int_X fgd\mu \text{ for all } f \in L_p(X) \text{ and } ||\Phi|| = ||g||_q.$$ 

Thus we have $L_p(X)^* \cong L_q(X)$ (see [8, Theorem 13.18]).

More generally, let $(X, \mathcal{M}, \mu)$ be a finite measure space and $E$ a Banach space.

A function $f : X \to E$ is called **simple** if there exists $x_1, x_2, \cdots, x_n \in E$ and $A_1, A_2, \cdots, A_n \in \mathcal{M}$ such that $f = \sum_{i=1}^n x_i \chi_{A_i}$, where $\chi_{A_i}$ denotes the characteristic function of $A_i$.

A function $f : X \to E$ is called **$\mu$-measurable** if there exists a sequence of simple functions $(f_n)$ such that $\lim_{n \to \infty} ||f_n - f|| = 0$ $\mu$-almost everywhere. So, every simple function is $\mu$-measurable and $||f(\cdot)||$ is $\mu$-measurable if $f : X \to E$ is $\mu$-measurable. Also, if $f, g : X \to E$ are $\mu$-measurable and $\alpha \in \mathbb{C}$, then $\alpha f + g : X \to E$ is $\mu$-measurable.

**Definition 8.** Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $E$ a Banach space. A $\mu$-measurable function $f : X \to E$ is called Bochner integrable if there exists a sequence of simple functions $(f_n)$ such that $\lim_{n \to \infty} \int_X ||f_n - f||d\mu = 0$.

For $E \in \mathcal{M}$, $\int_E f d\mu = \lim_{n \to \infty} \int_E f_n d\mu$, where $\int_E f_n d\mu$ is defined in the obvious way.
There is another way to describe Bochner integrable functions through the following theorem (see [9, page 45, Theorem 2]).

**Theorem 12.** A \( \mu \)-measurable function \( f : X \to E \) is Bochner integrable if and only if \( \int_X \|f\|d\mu < \infty \).

Let \( (X, \mathcal{M}, \mu) \) be a finite measure space and \( E \) a Banach space. Let \( 1 \leq p < \infty \).

We denote \( L_p(X, E) \) to be the space of all (equivalence class of) \( E \)-valued Bochner integrable functions \( f \) on \( X \) with \( \int_X \|f\|^p d\mu < \infty \). The norm \( \|\cdot\|_p \) is defined by \( \|f\|_p = \left( \int_X \|f\|^p d\mu \right)^{1/p} \). We have that \( L_p(X, E) \) is a Banach space under \( \|\cdot\|_p \) and simple functions are norm dense in \( L_p(X, E) \) for \( 1 \leq p < \infty \). In this thesis we will only deal with \( p = 1 \).

**Example 2.** Let \( (X, \mathcal{M}, \mu) \) be a finite measure space and \( L_1(X) \otimes^\gamma E \) be the projective tensor product of \( L_1(X) \) and \( E \). Then we have an isometric isomorphism

\[
L_1(X) \otimes^\gamma E \cong L_1(X, E),
\]

where \( \|\cdot\|_\gamma \) is the projective tensor product norm on \( L_1(X) \otimes E \). In particular, we have \( l^n_1 \otimes^\gamma E \cong l^n_1(E) \) via \( (\alpha_1, \cdots, \alpha_n) \otimes x \mapsto (\alpha_1 x, \cdots, \alpha_n x) \).

To see this, consider the natural inclusion mapping \( \varphi : L_1(X) \otimes_\gamma E \to L_1(X, E) \) defined by \( \varphi(\sum_{i=1}^n f_i \otimes x_i)(t) = \sum_{i=1}^n f_i(t)x_i \). It is easy to check that \( \varphi \) is linear. Since,

\[
\|\varphi(\sum_{i=1}^n f_i \otimes x_i)\|_1 = \int_X \|\varphi(\sum_{i=1}^n f_i \otimes x_i)(t)\|d\mu(t)
\]

\[
= \int_X \|\sum_{i=1}^n f_i(t)x_i\|d\mu(t)
\]

\[
\leq \sum_{i=1}^n \|f_i\|_1 \|x_i\|,
\]

we have \( \varphi(\sum_{i=1}^n f_i \otimes x_i)(t) = \sum_{i=1}^n f_i(t)x_i \).
by taking infimum on both sides over all representations $\sum_{i=1}^{n} f_i \otimes x_i$ of $u$ in $L_1(X) \otimes_{\gamma} E$, we get $\|\varphi(u)\|_1 \leq \|u\|_\gamma$ and thus $\varphi$ is bounded with $\|\varphi\| \leq 1$.

Since each $f_i \in L_1(X)$ can be approximated by simple functions, the subspace of $L_1(X) \otimes_{\gamma} E$ consisting of elements of the form $\sum_{i=1}^{n} \chi_{A_i} \otimes x_i$, where $A_i$ are pairwise disjoint sets in $\mathcal{M}$ and $x_i \in X$ for all $i$, is norm dense in $L_1(X) \otimes_{\gamma} E$.

Moreover, $\varphi$ maps this dense subspace of $L_1(X) \otimes_{\gamma} E$ onto a dense subspace of simple functions in $L_1(X, E)$. So, to show that $\varphi$ is an isometry, it suffices to prove that

$$\|\sum_{i=1}^{n} \chi_{A_i} \otimes x_i\|_\gamma \leq \|\varphi(\sum_{i=1}^{n} \chi_{A_i} \otimes x_i)\|_{L_1(X, E)}.$$ 

Now,

$$\|\sum_{i=1}^{n} \chi_{A_i} \otimes x_i\|_\gamma \leq \sum_{i=1}^{n} \|\chi_{A_i} \otimes x_i\|_\gamma = \sum_{i=1}^{n} \|\chi_{A_i}\|_1 \|x_i\|$$

$$= \sum_{i=1}^{n} \mu(A_i) \|x_i\| = \int_X \sum_{i=1}^{n} \|x_i\| \chi_{A_i} d\mu$$

$$= \|\sum_{i=1}^{n} x_i \chi_{A_i}\|_{L_1(X, E)} = \|\varphi(\sum_{i=1}^{n} \chi_{A_i} \otimes x_i)\|_{L_1(X, E)}.$$ 

Thus, $\varphi$ can be uniquely extended to an isometric linear mapping of $L_1(X) \otimes_{\gamma} E$ onto $L_1(X, E)$.

Let $(X, \mathcal{M}_1, \mu_1)$ and $(Y, \mathcal{M}_2, \mu_2)$ be $\sigma$-finite measure spaces. We let $\mathcal{M}_1 \otimes \mathcal{M}_2$ denote the $\sigma$-algebra generated by the family $\{A_1 \times A_2 : A_1 \in \mathcal{M}_1, A_2 \in \mathcal{M}_2\}$. It is called the product $\sigma$-algebra. There is a natural way to define a measure on $\mathcal{M}_1 \otimes \mathcal{M}_2$ associated with $\mu_1$ and $\mu_2$, which is called the product measure of $\mu_1$ and $\mu_2$ and denoted by $\mu_1 \otimes \mu_2$.

We merely state the following product measure theorem and Fubini's theorem which will be used in Example 3.
Theorem 13. Let \((X, \mathcal{M}_1, \mu_1)\) and \((Y, \mathcal{M}_2, \mu_2)\) be \(\sigma\)-finite measure spaces. Then there is exactly one measure \(\mu = \mu_1 \otimes \mu_2\) on \(\mathcal{M}_1 \otimes \mathcal{M}_2\) such that \(\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)\) for all \(A_1 \in \mathcal{M}_1, A_2 \in \mathcal{M}_2\).

Theorem 14. Let \((X, \mathcal{M}_1, \mu_1)\) and \((Y, \mathcal{M}_2, \mu_2)\) be \(\sigma\)-finite measure spaces and \(\mu = \mu_1 \otimes \mu_2\) be the product measure on \(\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2\), and let \(f\) be an \(\mathcal{M}\)-measurable function. Then

(i) For each \(x \in X\), \(f(x, \cdot)\) is \(\mathcal{M}_2\)-measurable and for each \(y \in Y\), \(f(\cdot, y)\) is \(\mathcal{M}_1\)-measurable.

(ii) If \(f \geq 0\) or \(f\) is \(\mu\)-integrable, then

\[\int_Y \int_X f(x, y) d\mu_1(x) d\mu_2(y) = \int_{X \times Y} f d\mu = \int_X \int_Y f(x, y) d\mu_2(y) d\mu_1(x).\]

Example 3. Let \((X, \mathcal{M}, \mu)\) and \((Y, T, \nu)\) be \(\sigma\)-finite measure spaces. We have

\[L_1(X) \otimes^\gamma L_1(Y) \cong L_1(X \times Y),\]

where \(\| \cdot \|_\gamma\) is the projective tensor product norm on \(L_1(X) \otimes L_1(Y)\).

To see this, for \(f \in L_1(X), g \in L_1(Y)\), we define

\[f \times g : X \times Y \to \mathbb{C} \text{ by } (f \times g)(x, y) = f(x)g(y).\]

For the bilinear mapping \(\varphi : L_1(X) \times L_1(Y) \to L_1(X \times Y) : (f, g) \mapsto f \times g\), there exists a unique linear mapping \(\bar{\varphi} : L_1(X) \otimes^\gamma L_1(Y) \to L_1(X \times Y)\) such that
\( \varphi(f \otimes g) = f \times g. \) Since,
\[
\| \varphi(\sum_{i=1}^{n} f_i \otimes g_i) \|_1 = \int_{X \times Y} \| \varphi(\sum_{i=1}^{n} f_i \otimes g_i)(x, y) \| d(\mu \otimes \nu)(x, y) \\
= \int_{X \times Y} \| \sum_{i=1}^{n} (f_i \times g_i)(x, y) \| d(\mu \otimes \nu)(x, y) \\
\leq \sum_{i=1}^{n} \int_{X \times Y} |f_i(x)||g_i(y)| d(\mu \otimes \nu)(x, y) \\
= \sum_{i=1}^{n} \int_{X} \int_{Y} |f_i(x)||g_i(y)| d\mu(x) d\nu(y) \\
= \sum_{i=1}^{n} \| f_i \| \| g_i \|,
\]
by taking infimum on both sides over all representations \( \sum_{i=1}^{n} f_i \otimes g_i \) of \( u \) in \( L_1(X) \otimes_{\gamma} L_1(Y) \), we get \( \| \varphi(u) \|_1 \leq \| u \|_{\gamma} \) and thus \( \varphi \) is bounded with \( \| \varphi \| \leq 1. \)

Moreover, \( \varphi \) maps the dense subspace of \( L_1(X) \otimes_{\gamma} L_1(Y) \) consisting of elements of the form \( \sum_{i=1}^{n} a_i \chi_{A_i} \otimes \chi_{B_i} \), where \( a_i \in \mathbb{C} \) and \( A_i \in \mathcal{M}_1 \) with \( \mu(A_i) < \infty \) (resp. \( B_i \in \mathcal{M}_2 \) with \( \nu(B_i) < \infty \)) are pairwisely disjoint, onto a dense subspace of simple functions in \( L_1(X \times Y) \). Since,
\[
\| \sum_{i=1}^{n} a_i \chi_{A_i} \otimes \chi_{B_i} \|_{\gamma} \leq \sum_{i=1}^{n} \| a_i \chi_{A_i} \otimes \chi_{B_i} \|_{\gamma} = \sum_{i=1}^{n} \| a_i \chi_{A_i} \|_{1} \| \chi_{B_i} \|_{1} \\
= \sum_{i=1}^{n} |a_i| \mu(A_i) \nu(B_i) = \int_{X} \int_{Y} \sum_{i=1}^{n} |a_i| \chi_{A_i} \chi_{B_i} d\mu d\nu \\
= \| \sum_{i=1}^{n} a_i \chi_{A_i} \otimes \chi_{B_i} \|_{L_1(X \times Y)} = \| \varphi(\sum_{i=1}^{n} a_i \chi_{A_i} \otimes \chi_{B_i}) \|_{L_1(X \times Y)}.
\]
It follows that \( \varphi \) is an isometry and thus \( \varphi \) can be uniquely extended to an isometric linear mapping of \( L_1(X) \otimes_{\gamma} L_1(Y) \) onto \( L_1(X \times Y) \).

More specifically, let \( B(\mathbb{R}) \) be the Borel \( \sigma \)-algebra of \( \mathbb{R} \) generated by all open sets in \( \mathbb{R} \). Then the family of Borel sets of \( \mathbb{R}^2 \) is the product \( \sigma \)-algebra \( B(\mathbb{R}) \otimes B(\mathbb{R}) \). If \( \lambda_1 \) is the one-dimensional Lebesgue measure, the two dimensional Lebesgue measure \( \lambda_2 \) satisfies \( \lambda_2(A_1 \times A_2) = \lambda_1(A_1) \lambda_1(A_2) \) for \( A_1 \in B(\mathbb{R}), A_2 \in B(\mathbb{R}) \) and the restriction of the two dimensional Lebesgue measure to \( B(\mathbb{R}) \otimes B(\mathbb{R}) \) is the product of two copies of
the one dimensional Lebesgue measure on $B(\mathbb{R})$, i.e., $\lambda_2|_{B(\mathbb{R})} \otimes \lambda_1|_{B(\mathbb{R})} = \lambda_1|_{B(\mathbb{R})} \times \lambda_1|_{B(\mathbb{R})}$.

Thus, we have

$$L_1(\mathbb{R}) \otimes^\gamma L_1(\mathbb{R}) \cong L_1(\mathbb{R} \times \mathbb{R}).$$

Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space. Then $L_\infty(X)$ can be isometrically embedded into $B(L_2(X))$ and hence $L_\infty(X)$ has its natural operator space structure.

As the predual of $L_\infty(X)$, $L_1(X)$ has also a natural operator space structure. It can be seen that the complete isometry $L_1(X) \otimes^\wedge L_1(Y) \cong L_1(X \times Y)$ holds, where $(Y, \mathcal{N}, \nu)$ is a $\sigma$-finite measure space. In particular, we have the operator space identification $L_1(\mathbb{R}) \otimes^\wedge L_1(\mathbb{R}) \cong L_1(\mathbb{R}^2)$. 
REFERENCES


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