POINTED REPRESENTATIONS OF SIMPLE LIE ALGEBRAS.

MARIA MAGDOLNA. PAP

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LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L'AVONS RÉCU
POINTED REPRESENTATIONS OF SIMPLE
LIE ALGEBRAS

by

Maria Magdolna Pap

A Dissertation,
submitted to the Faculty of Graduate Studies
through the Department of Mathematics
in Partial Fulfillment of the requirements
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Windsor, Ontario, Canada

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Abstract

Pointed Representations of Simple Lie Algebras

Let $L$ denote a simple Lie algebra over $F$, an algebraically closed field of characteristic zero. Let $H$ be a fixed Cartan subalgebra of $L$.

This thesis deals with algebraically irreducible representations of $L$ which admit a one-dimensional weight space. Any such representation will be called pointed. The family of all pointed representations includes all dominated irreducible representations and is included in the family of all Harish-Chandra modules which are $H$-finite.

Let $C(L)$ denote the centralizer of $H$ in the universal enveloping algebra $U(L)$ of $L$. For each algebra homomorphism $\phi: C(L) \to F$, one can construct a unique irreducible representation $(\rho, V)$ of $L$ which admits a weight space decomposition relative to $H$ such that the weight space corresponding to $\phi + H \in H^*$ is one-dimensional. Conversely, if $(\rho, V)$ is a pointed representation of $L$ admitting $V_\lambda$ as a one-dimensional weight space for some $\lambda \in H^*$, then there exists a unique algebra homomorphism $\phi: C(L) \to F$ which extends $\lambda$ such that $(\rho, V)$ is equivalent to the representation constructed from $\phi$.

We present a study of the pointed representations of $L$ via their associated algebra homomorphisms. Our aim is to "label" the equivalence classes of pointed representations by elements from the family of algebra homomorphisms $\phi: C(L) \to F$ in analogy to the technique of labelling the dominated irreducible
representations with their highest weight. In particular, we succeed in classifying, up to equivalence, all pointed representations of the Lie algebras $A_n$ (over the complex number field) for $n = 1, 2, 3$. We also present some partial results toward the classification of pointed representations of $A_n$ for $n > 3$, and some conjectures regarding the case of arbitrary simple Lie algebras.
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TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>v</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>vi</td>
</tr>
<tr>
<td>LIST OF APPENDICES</td>
<td>viii</td>
</tr>
<tr>
<td>LIST OF NOTATIONS</td>
<td>ix</td>
</tr>
<tr>
<td>TERMINOLOGY</td>
<td>xi</td>
</tr>
<tr>
<td>CHAPTER</td>
<td></td>
</tr>
<tr>
<td>I. BASIC DEFINITIONS AND THEOREMS</td>
<td>1</td>
</tr>
<tr>
<td>§1. Basic Definitions and Theorems</td>
<td>1</td>
</tr>
<tr>
<td>§2. Homomorphisms and Representations</td>
<td>4</td>
</tr>
<tr>
<td>§3. Cartan's Criteria for Solvability and Semisimplicity</td>
<td>7</td>
</tr>
<tr>
<td>§4. Cartan Subalgebras</td>
<td>10</td>
</tr>
<tr>
<td>§5. Weight Functions and Representations</td>
<td>12</td>
</tr>
<tr>
<td>§6. Simple Roots, Cartan Basis</td>
<td>15</td>
</tr>
<tr>
<td>§7. The Dynkin Diagram of a Lie Algebra</td>
<td>19</td>
</tr>
<tr>
<td>§8. The Weyl Group</td>
<td>21</td>
</tr>
<tr>
<td>§9. Automorphisms of Simple Lie Algebras</td>
<td>23</td>
</tr>
<tr>
<td>II. THE UNIVERSAL ENVELOPING ALGEBRA OF A LIE ALGEBRA</td>
<td>26</td>
</tr>
<tr>
<td>§1. Basic Definitions and Properties</td>
<td>26</td>
</tr>
<tr>
<td>§2. Representations of L and U(L)</td>
<td>30</td>
</tr>
<tr>
<td>§3. The Cycle Subalgebra of U(L)</td>
<td>32</td>
</tr>
<tr>
<td>III. POINTED REPRESENTATIONS</td>
<td>37</td>
</tr>
<tr>
<td>§1. Pointed Representations and Mass Functions</td>
<td>37</td>
</tr>
<tr>
<td>§2. Equivalence of Mass Functions</td>
<td>41</td>
</tr>
<tr>
<td>§3. Weak Equivalence</td>
<td>43</td>
</tr>
</tbody>
</table>
IV. DOMINATED AND COMPLETE REPRESENTATIONS ........................................... 48
   §1. Construction of Algebra Homomorphisms for the Cycle Subalgebra of U(L) ........ 48
   §2. Dominated Irreducible Representations .................................................. 57
   §3. Complete Representations ............................................................................. 61
V. STANDARD HOMOMORPHISMS ........................................................................ 70
   §1. The Lie Algebra $A_n$ .................................................................................... 71
   §2. Standard Homomorphisms ................................................................................ 72
   §3. Boundary Weights of $g$-Standard Representations ........................................ 80
   §4. The "Labelling" of $g$-Standard Homomorphisms ........................................... 91
VI. POINTED REPRESENTATIONS OF $A_n$ .............................................................. 98
   §1. Pointed Representations of $A_1$ ..................................................................... 98
   §2. Pointed Representations of $A_2$ ..................................................................... 101
   §3. The Structure of $C(A_n), \ n \geq 3$ ................................................................. 114
   §4. Algebra Homomorphisms on $C(A_3)$ ........................................................... 121
   §5. Conjectures ...................................................................................................... 130

APPENDIX ............................................................................................................. 131

BIBLIOGRAPHY ..................................................................................................... 136

VITA AUCTORIS .................................................................................................... 138
LIST OF APPENDICES

Appendix A: Operation of some elements of $\text{Aut}(A_3;H)$ on the generators of $C(A_3)$ 131

Appendix B: Some calculations involved in constructing all algebra homomorphisms $\phi: C(A_3) \to C$ 133
NOTATION

\textbf{F:} algebraically closed field of characteristic 0
\textbf{C:} the field of complex numbers
\textbf{Z:} the set of integers
\textbf{L:} Lie algebra of finite dimension over \( F \)
\textbf{\{,\}:} Lie product
\textbf{\( \rho \):} representation map
\textbf{V:} representation space
\textbf{H:} Cartan subalgebra of \( L \) (cf. p.10.)
\textbf{\( H^* \):} dual space of \( H \)
\textbf{\( < , > \):} Cartan-Killing form (cf. p.8.)
\textbf{\( V_\lambda \):} \( \lambda \)-weight space of a representation (cf. p.12.)
\textbf{\( L_\alpha \):} \( \alpha \)-root space of \( L \) (cf. p.14.)
\textbf{\( \Delta \):} set of non-zero roots
\textbf{\( \Delta_0 \):} set of all roots
\textbf{\( \Delta^+ \):} set of positive roots (cf. p.16.)
\textbf{\( \Delta^{++} \):} set of simple roots (cf. p.16.)
\textbf{\( X_\beta, Y_\beta, H_\beta \):} cf. p.18.
\textbf{\( S_\alpha \):} cf. p.21.
\textbf{\( W \):} cf. p.22.
\textbf{U}(L): universal enveloping algebra of \( L \) (cf. p.26.)
\textbf{Ad} \( x \): adjoint representation of \( L \) in \( U(L) \) (cf. p.33.)
\textbf{U}_x: cf. p.33.
\textbf{C}(L): the cycle subalgebra of \( L \) (cf. p.34.)
\textbf{P}_L: the family of pointed representations of \( L \) (cf. p.38.)
$\text{Aut}(L:H)$: cf. p.43.
$\Gamma$: cf. p.50.
$C(\Gamma), \overline{C}(\Gamma)$: cf. p.54.
$\overline{\Phi}$: cf. p.55.
$\Gamma_0, \Gamma_c$: cf. p.61.
$F_L, \hat{F}_L$: cf. p.78.
$\sigma_\alpha$: cf. p.82.
<table>
<thead>
<tr>
<th><strong>TERMINOLOGY</strong></th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td>boundary weight function</td>
<td>81</td>
</tr>
<tr>
<td>characteristic weight function</td>
<td>62</td>
</tr>
<tr>
<td>closed set of roots</td>
<td>50</td>
</tr>
<tr>
<td>complete algebra homomorphism</td>
<td>75</td>
</tr>
<tr>
<td>complete representation</td>
<td>62</td>
</tr>
<tr>
<td>complete set of roots</td>
<td>50</td>
</tr>
<tr>
<td>cycle subalgebra of $U(L)$</td>
<td>34</td>
</tr>
<tr>
<td>disconnected subsets of the simple roots</td>
<td>62</td>
</tr>
<tr>
<td>dominant weight function</td>
<td>57</td>
</tr>
<tr>
<td>elementary cycle</td>
<td>34</td>
</tr>
<tr>
<td>extreme algebra homomorphism</td>
<td>78</td>
</tr>
<tr>
<td>$g$-standard algebra homomorphism</td>
<td>77</td>
</tr>
<tr>
<td>mass function</td>
<td>38</td>
</tr>
<tr>
<td>pointed representation</td>
<td>37</td>
</tr>
<tr>
<td>standard representation</td>
<td>74</td>
</tr>
</tbody>
</table>
Chapter One

§1. Basic definitions and theorems

Let \( F \) denote an algebraically closed field of characteristic 0. We adopt the convention that whenever we write "linear space", "algebra", "Lie algebra" etc., we mean "linear space over \( F \)", "algebra over \( F \)" etc.

Definition 1.1.1. An algebra \( A \) is a linear space with an \( F \)-bilinear product \( * : A \times A \to A \).

An associative algebra \( A \) is an algebra satisfying
\[
x * (y * z) = (x * y) * z \quad \text{for all} \quad x, y, z \in A.
\]

A Lie algebra is an algebra satisfying the identities
\[
(1) \quad x * x = 0 \\
(2) \quad x * (y * z) + y * (z * x) + z * (x * y) = 0 \quad \text{for all} \quad x, y, z \in L. \quad \text{(Identity (2) is called the Jacobi-identity)}.
\]

Notation: The binary operation in a Lie algebra is generally denoted by \([x, y]\) and is called "the bracket product of \( x \) and \( y\)."

Remark: Any associative algebra \( A \) can be given a Lie algebra structure by defining the bracket product \([x, y]\) of any two elements \( x \) and \( y \) of \( A \) to be \( x * y - y * x \) where \( x * y \) denotes their product in \( A \). The Lie algebra obtained in this way is called "the Lie algebra of the associative algebra \( A \)" and is denoted by \( A_L \).

Definition 1.1.2. A subspace \( I \) of a Lie algebra \( L \) is called an ideal of \( L \) if \( x \in L, y \in I \) imply \([x, y] \in I\).
Note: Since the bracket product is anti-commutative (i.e. \([x,y] = -[y,x]\), for all \(x, y \in L\)), there is no distinction between "left" and "right" ideals of a Lie algebra.

**Definition 1.1.3.** If \(L\) is a Lie algebra, the set of all linear combinations of commutators \([x,y] \ (x,y \in L)\) is clearly an ideal. It is denoted by \([L,L]\) and is called the "derived algebra" of \(L\).

**Note:** \(L\) is abelian if and only if \([L,L] = 0\).

**Definition 1.1.4.** If a Lie algebra \(L\) has no proper ideals and if, moreover, \([L,L] \neq 0\), we call \(L\) simple. (Clearly, if \(L\) is simple, \(L = [L,L]\)).

**Definition 1.1.5.** Let \(L\) be a Lie algebra which is not simple and let \(I\) be a proper ideal of \(L\). We define the quotient algebra \(L/I\) in the following way: as a vector space, \(L/I\) is just the quotient space, while Lie multiplication is defined by \([x+I, y+I] = [x,y] + I\).

**Definition 1.1.6.** Let \(S_1\) and \(S_2\) be subsets of a Lie algebra \(L\). We denote by \([S_1,S_2]\) the smallest subalgebra of \(L\) containing the set \(\{(x,y) | x \in S_1, y \in S_2\} \).

**Note:** If \(I\) and \(J\) are two ideals of \(L\), \([I, J]\) is also an ideal. The derived algebra \([L,L]\) is just a special case of this construction.
Definition 1.1.7. Let $L_1$ and $L_2$ be two subalgebras of the Lie algebra $L$. We say that $L$ is the direct sum of $L_1$ and $L_2$ (and write $L = L_1 \oplus L_2$) if the following two conditions are satisfied:

1. $L = L_1 \oplus L_2$ (direct sum of linear spaces)
2. $[L_1, L_2] = 0$.

Definition 1.1.8. Let $L$ be a Lie algebra and for each integer $n \geq 1$ define $L^n$ by $L^1 = L$ and $L^{n+1} = [L, L^n]$. It is easily seen that, for any $n$, $L^n \supseteq L^{n+1}$ and $L^n$ is an ideal of $L$. The sequence $\{L^n\}$ of ideals of $L$ is called the "descending central series of $L$".

Definition 1.1.9. A Lie algebra $L$ is called nilpotent if there exists an integer $k$ such that $L^k = \{0\}$.

Definition 1.1.10. Let $L$ be a Lie algebra, and define, for each integer $n \geq 1$, a sequence of ideals $\{L^{(n)}\}$ by setting $L^{(1)} = L$, $L^{(n+1)} = [L^{(n)}, L^{(n)}]$. $\{L^{(n)}\}$ is called the "descending derived series of $L$".

Definition 1.1.11. A Lie algebra $L$ is said to be solvable if, for some integer $k$, $L^{(k)} = \{0\}$.

Remark: It is clear that, for all $i$, $L^{(1)} \subseteq L^1$, so nilpotent Lie algebras are solvable. The converse however is false: for example, let $L$ be the Lie algebra with basis $\{e, f\}$ such that $[ef] = e = -[fe]$. This is clearly solvable but not nilpotent.
Definition 1.1.12. A Lie algebra $L \neq \{0\}$ is called semisimple if it does not admit a nonzero solvable ideal. Equivalently, we may define $L \neq 0$ to be semisimple if it has no nonzero abelian ideals.

Since the center $Z(L) = \{z \in L \mid [x,z] = 0 \text{ for all } x \in L\}$ of $L$ is an abelian ideal of $L$, it follows that the center of a semisimple Lie algebra is zero.

§2. Homomorphisms and representations

Definition 1.2.1. Let $L_1$ and $L_2$ be Lie algebras. A linear map $\rho : L_1 \rightarrow L_2$ is called a (Lie-algebra) homomorphism if $\rho(x+y) = [\rho(x), \rho(y)]$ for all $x, y \in L$. $\rho$ is called an isomorphism if it is one-to-one and onto. An automorphism of a Lie algebra $L$ is an isomorphism from $L$ to $L$. $\text{Aut } L$ denotes the group of all automorphisms of $L$.

Remark: If $\rho : L_1 \rightarrow L_2$ is a Lie algebra homomorphism, $\ker \rho = \{x \in L \mid \rho(x) = 0\}$ is an ideal of $L_1$. Also, $\text{Im } \rho$ is a subalgebra of $L_2$. As in other algebraic theories, there is a one-to-one correspondence between homomorphisms and ideals of a Lie algebra $L$. To $\rho$ is associated $\ker \rho$, and to an ideal $I$ is associated the canonical homomorphism $x \mapsto x + I$ of $L$ onto $L/I$. We also have the standard homomorphism theorem: If $\rho : L_1 \rightarrow L_2$ is a Lie algebra homomorphism then $L_1/\ker \rho \cong \text{Im } \rho$. If $I$ is any ideal of $L$, contained in $\ker \rho$, there exists a unique
homomorphism \( \theta : L_1/I \to L_2 \) such that the following diagram commutes (where \( \pi \) denotes the canonical homomorphism):

\[
\begin{array}{ccc}
L_1 & \xrightarrow{\phi} & L_2 \\
\downarrow{\pi} & & \downarrow{\theta} \\
L_1/I & & \\
\end{array}
\]

**Definition 1.2.2.** A representation of a Lie algebra \( L \) is a pair \((\rho, V)\), where \( V \) is a linear space (not necessarily finite dimensional) and \( \rho \) is a Lie algebra homomorphism from \( L \) into \( gl(V) \), where \( gl(V) \) denotes the algebra \( \text{End} \ V \) of all endomorphisms of \( V \) viewed as a Lie algebra under the bracket product. (Thus, if \((\rho, V)\) is a representation of \( L \), \( \rho([x, y]) = [\rho(x), \rho(y)] = \rho(x) \rho(y) - \rho(y) \rho(x) \) for all \( x, y \in L \).)

The dimension of a representation \((\rho, V)\) of \( L \) is defined to be the dimension of the linear space \( V \).

**Remark:** A representation of an associative algebra \( A \) is a pair \((\rho, V)\), where \( V \) is a linear space and \( \rho \) is an algebra homomorphism from \( A \) into the associative algebra \( \text{End} \ V \); that is, \( \rho(xy) = \rho(x)\rho(y) \) for all \( x, y \in A \).

**Definition 1.2.3.** A representation \((\rho, V)\) of the Lie algebra \( L \) is said to be faithful if the homomorphism \( \rho \) is injective.

**Definition 1.2.4.** Let \((\rho, V)\) be a representation of the Lie algebra \( L \). A subspace \( W \) of \( V \) is called \( \rho \)-invariant if \( \rho(x)W \subseteq W \) for each \( x \in L \).

If \( W \) is a \( \rho \)-invariant subspace of \( V \), we may define
a new representation \((\rho', W)\) by setting, for each \(w \in W, x \in L\), 
\(\rho'(x)w = \rho(x)w\). The representation \((\rho', W)\) of \(L\) obtained in this way is called the restriction of \((\rho, V)\) to the \(\rho\)-invariant subspace \(W\).

**Definition 1.2.5.** A representation \((\rho, V)\) is called algebraically irreducible (or simply irreducible) if the only \(\rho\)-invariant subspaces of \(V\) are \(\{0\}\) and \(V\), it is said to be reducible if \(V\) has a proper \(\rho\)-invariant subspace; and completely reducible if for every \(\rho\)-invariant subspace \(W\) of \(V\) there exists a \(\rho\)-invariant subspace \(W'\) of \(V\) such that \(V = W \oplus W'\).

**Remark:** Let \((\rho, V)\) be a completely reducible representation of the Lie algebra \(L\). Then \(V\) can be expressed as a direct sum of \(\rho\)-invariant subspaces \(\{V_i\}_{i \in I}\) such that the restricted representations \((\rho', V_i)\) are irreducible. The representations \((\rho', V_i)\) are called the irreducible components of \((\rho, V)\).

**Definition 1.2.6.** Two representations \((\rho_1, V_1)\) and \((\rho_2, V_2)\) of a Lie algebra \(L\) are said to be equivalent, if there exists an isomorphism \(\phi : V_1 \rightarrow V_2\) such that 
\[\rho_2(x) \circ \phi = \phi \circ \rho_1(x)\] for all \(x \in L\).

We now single out an important representation of Lie algebras.
Definition 1.2.7. The map $\text{Ad}_L$ from $L$ into $\text{gl}(L)$ (where $L$ denotes the underlying linear space of $L$) defined by setting for each $a \in L$

$$\text{Ad}_L(a) : L \rightarrow L$$

$$: x \mapsto [a,x]$$

is called the adjoint map determined by $L$. The pair $(\text{Ad}_L, L)$ is a representation of the Lie algebra $L$, called the adjoint representation.

Note: It follows directly from the above definition that the kernel of the adjoint map is the center of $L$, and hence the adjoint representation of any semi-simple Lie algebra $L$ is faithful. The corresponding special representation of associative algebras is the left regular representation:

Definition 1.2.8. If $A$ is an associative algebra, the map $R_A : A \rightarrow \text{End}_A$ defined by setting for each $a \in A$,

$$R_A(a) : A \rightarrow A$$

$$: x \mapsto ax$$

is called the left regular map determined by $a$. The pair $(R_A, A)$ is called the left regular representation of $A$.

§3. Cartan's Criteria for Solvability and Semi-Simplicity

In the remainder of this chapter, $L$ will denote a fixed but arbitrary finite dimensional Lie algebra over
F (an algebraically closed field of characteristic zero).

Definition 1.3.1. The Cartan-Killing form of \( L \) is a mapping from \( L \times L \) to \( F \) given by

\[
\langle x, y \rangle = \text{tr} (\text{Ad}_x(y) \circ \text{Ad}_y(x)),
\]

where "tr" denotes the usual trace operator defined on \( \text{End}_L \).

The Cartan-Killing form is a symmetric bilinear form on \( L \), which is also "associative" in the sense that

\[
\langle [x,y], z \rangle = \langle x, [y,z] \rangle \quad \text{for all } x, y, z \in L.
\]

The Cartan-Killing form is also invariant under automorphisms \( Q \) of \( L \):

\[
\langle Q(x), Q(y) \rangle = \langle x, y \rangle \quad \text{for all } x, y \in L.
\]

Theorem 1.3.1. (Cartan's Criterion for Solvability)

A Lie algebra \( L \) is solvable if and only if \( L \) is orthogonal to \([L,L]\) with respect to the Cartan-Killing form, i.e. if and only if \( \langle [x,y], z \rangle = 0 \) for all \( x, y, z \in L \).


Theorem 1.3.2. (Levi)

Every finite dimensional Lie algebra \( L \) can be decomposed into the direct sum of a solvable Lie algebra and a semisimple Lie algebra. The solvable part is uniquely determined: it is the maximal solvable ideal of \( L \) (called the (solvable) radical of \( L \)); the semisimple part is not determined uniquely, but is determined up to an automorphism on \( L \).
Thus, the study of Lie algebras naturally leads to a study of solvable and semisimple Lie algebras. The following theorem gives necessary and sufficient conditions for a Lie algebra to be semi-simple.

**Theorem 1.3.3.** (Cartan's Criterion for Semi-Simplicity)
A Lie algebra $L$ is semi-simple if and only if the Cartan-Killing form on $L$ is non-degenerate. (i.e. $\langle x, y \rangle = 0$ for all $y \in L$ implies $x = 0$).


The following two theorems reduce the study of semi-simple Lie algebras and their finite dimensional representations to the study of simple Lie algebras and their finite dimensional irreducible representations.

**Theorem 1.3.4.** (Structure theorem for semi-simple Lie algebras)
Let $L$ be a semi-simple Lie algebra. Then there exist ideals $L_1, L_2, \ldots, L_k$ of $L$ which are simple (as Lie algebras), such that $L = L_1 \oplus L_2 \oplus \ldots \oplus L_k$. Every simple ideal of $L$ coincides with one of the $L_i$. Moreover, the Cartan-Killing form of $L_i$ is the restriction of the Cartan-Killing form of $L$ to $L_i \times L_i$.


**Theorem 1.3.5.** (Weyl)
Every finite dimensional representation of a semi-simple Lie algebra is completely reducible.

§4. Cartan Subalgebras

Definition 1.4.1. If $B$ is a subalgebra of a Lie algebra $L$, the normalizer $N_L(B)$ of $B$ is the set

$$\{x \in L \mid \{x, B\} \subseteq B\}.$$ 

Remark: It follows from the above definition and the Jacobi identity that $N_L(B)$ is a subalgebra containing $B$, and $B$ is an ideal of $N_L(B)$. (In fact, $N_L(B)$ is the largest subalgebra of $L$ in which $B$ is contained as an ideal.)

Definition 1.4.2. A subalgebra $H$ of a Lie algebra $L$ is called a Cartan subalgebra if

1. $H$ is nilpotent, and
2. $H$ is its own normalizer, i.e. $H = N_L(H)$.

Let $L$ be a Lie algebra over $F$ (algebraically closed field, char $F = 0$). Suppose $x \in L$ is such that ad $x$ is nilpotent ($(\text{ad} \ x)^{k+1} = 0$). Then the usual exponential power series for a linear transformation makes sense over $F$ because it has finitely many terms:

$$\exp(\text{ad} \ x) = 1 + \text{ad} \ x + \frac{(\text{ad} \ x)^2}{2} + \ldots + \frac{(\text{ad} \ x)^k}{k!}.$$

Using the "Leibniz rule" for derivations, and the above expansion; one can see that $\exp(\text{ad} \ x) \in \text{Aut} \ L$. The automorphisms of the form $\exp(\text{ad} \ x)$ (ad $x$ nilpotent) generate a normal subgroup of $\text{Aut} \ L$. This subgroup is denoted by $\text{Int} \ L$, and its elements are called inner automorphisms.
It can be shown, (cf. [12] S.S.L. §9) that every Lie algebra (over an algebraically closed field of characteristic 0) admits at least one Cartan subalgebra. For semi-simple Lie algebras, the following result holds:

**Theorem 1.4.1.** If \( H_1 \) and \( H_2 \) are Cartan subalgebras of a semi-simple Lie algebra \( L \), then there exists an automorphism \( Q \in \text{Int } L \) such that \( Q(H_1) = H_2 \).


**Remark:** The above theorem implies in particular that all Cartan subalgebras of a semi-simple Lie algebra have the same dimension.

**Theorem 1.4.2.** Let \( H \) be a fixed Cartan subalgebra of the semi-simple Lie algebra \( L \). Then the restriction of the Cartan-Killing form of \( L \) to \( H \) is non-degenerate.

**Proof:** [12] - S.S.L. p.9-06.

**Theorem 1.4.3.** Let \( H \) be a fixed Cartan subalgebra of the semi-simple Lie algebra \( L \), and let \( H^* \) denote the dual space of \( H \). Then, associated with each element \( \lambda \in H^* \), there exists a unique element \( h_\lambda \in H \) such that \( \lambda(h) = \langle h_\lambda, h \rangle \) for all \( h \in H \).


The above theorem leads us to the following definition.

**Definition 1.4.2.** The dual Cartan-Killing form is a map from \( H^* \times H^* \) into \( F \) defined by setting for any two
elements \( \lambda, \mu \in \mathfrak{h}^* \),
\[
\langle \lambda, \mu \rangle = \langle h^\lambda, h^\mu \rangle.
\]

**Remark:** The dual Cartan-Killing form is symmetric, bilinear and non-degenerate on \( \mathfrak{h}^* \).

§5. **Weight functions of representations**

**Note:** For the remainder of this chapter (unless otherwise noted), \( L \) shall denote a semi-simple finite dimensional Lie algebra, and \( H \) will be an arbitrarily chosen but fixed Cartan subalgebra of \( L \).

**Definition 1.5.1.** Let \( (\rho, V) \) be a representation of \( L \). Then for any \( \lambda \in \mathfrak{h}^* \) define \( V_\lambda = \{ v \in V \mid \rho(h)v = \lambda(h)v \text{ for all } h \in H \} \). Clearly \( V_\lambda \) is a subspace of \( V \). If \( V_\lambda \neq \{0\} \), then \( \lambda \) is said to be a weight function of \( (\rho, V) \), and we call \( V_\lambda \) a weight space of \( (\rho, V) \).

**Remark:** More precisely, we should speak of "the weights and weight spaces "with respect to \( H " \), however, since we have fixed a Cartan subalgebra \( H \), this dependence on \( H \) can be suppressed.

**Theorem 1.5.1.** Every finite dimensional representation \( (\rho, V) \) of \( L \) has at least one weight function.

**Proof:** [2], Bouwer, p.26.

**Remark:** The above theorem is not true, in general, for infinite dimensional representations. (cf. Lemire [10d]).
**Definition 1.5.2.** The set of all weight functions of a given representation \((\rho, V)\) of \(L\) is called the **weight set** of \(\rho\) and is denoted by \(W(\rho)\).

**Theorem 1.5.2.** Equivalent representations of \(L\) have identical weight sets.


**Definition 1.5.3.** The weight functions of the adjoint representation of \(L\) are called the **roots** of \(L\) and the corresponding weight spaces are called the **root spaces** of \(L\). We shall denote the set of non-zero roots by \(\Delta\).

In the next theorem, we list the most important properties of the roots and root spaces of a semi-simple Lie algebra \(L\) with Cartan subalgebra \(H\).

**Theorem 1.5.3.**

1. There exists at least one root of \(L\) (since \(L\) is assumed to be finite dimensional).
2. If \(\alpha \in H^*\) is a root of \(L\), then \(-\alpha\) is also a root; moreover the only \(F\)-multiples of \(\alpha\) which are roots are \(\alpha, 0, \) and \(-\alpha\).
3. If \(\alpha\) and \(\beta\) are any two roots and \(\alpha + \beta \neq 0\), then the root spaces \(L_\alpha\) and \(L_\beta\) are orthogonal relative to the Cartan-Killing form.
4. For any two roots \(\alpha, \beta\) of \(L\), we have \([L_\alpha, L_\beta] \subseteq L_{\alpha + \beta}\).
5. For any two non-zero roots \(\alpha, \beta\) of \(L\),

\[
\frac{-2<\alpha, \beta>}{<\beta, \beta>} \text{ is an integer.}
\]

6. For any non-zero root \(\alpha\) of \(L\), \(L_\alpha\) is one dimensional.
(7) There are \( l \) linearly independent roots where 
\( l = \dim H \). (hence, there exists a basis of \( H^* \) consisting of roots.)

(8) \( L \) can be decomposed as the direct sum 
\( L = H \oplus \sum_{\alpha \neq 0} L_\alpha \).


Theorem 1.5.5. There exists a 1-1 correspondence between
automorphisms \( Q \) of \( L \) that preserve \( H \) and linear
transformations \( Q^* \) of \( H^* \) that
(a) preserve the dual Cartan-Killing form on \( H^* \), and
(b) permute the roots of \( L \).

The correspondence is established by the mapping \( Q \leftrightarrow Q^* \)
where \( Q \) and \( Q^* \) are related by 
\( Q(h^\lambda_\alpha) = h^\lambda_{Q^*(\alpha)} \)
for any \( \lambda \in H^* \).


Definition 1.5.4. Let \((\rho, V)\) be a representation of \( L \),
\( \lambda \) be a weight function of \((\rho, V)\), and \( \alpha \) a non-zero root
of \( L \). The set of all weight functions of \((\rho, V)\) of the
form \( \lambda + m\alpha \) where \( m \) is an integer, is called the
\( \alpha \)-chain through \( \lambda \).

Definition 1.5.5. If \( \alpha \) and \( \beta \neq 0 \) are roots and \( \alpha - m\beta, \alpha - (m-1)\beta, \ldots, \alpha, \alpha + \beta, \ldots, \alpha + n\beta \) is the \( \beta \)-string
through \( \alpha \), we define the Cartan integer \( A_{\alpha, \beta} = m - n \).

The following lemma summarizes the basic properties Cartan integers:
**Lemma 1.5.1.** Let $\alpha, \beta$ be any roots. Then

(a) $A_{\alpha, \alpha} = 0$

(b) $A_{-\alpha, \beta} = -A_{\alpha, \beta} = A_{\alpha, -\beta}$

(c) $A_{\alpha, \alpha} = 2$ (provided $\alpha \neq 0$)

(d) $-3 \leq A_{\alpha, \beta} \leq 3$

(e) If $A_{\alpha, \beta} \leq -2$ then $\alpha - \beta$ is not a root.

(f) If $\beta \neq 0$, $A_{\alpha, \beta} = \frac{2\alpha(h_\beta')}{\beta(h_\beta')} = \frac{2\langle h_\alpha', h_\beta' \rangle}{\langle h_\beta', h_\beta' \rangle}$, where $h_\beta'$ is as in Theorem 1.4.3.

(g) If $\alpha, \beta \neq 0$, then $A_{\alpha, \beta} = 0$ iff $A_{\beta, \alpha} = 0$

(h) If $\alpha, \beta, \gamma \neq 0$ and $\gamma \neq 0$ are all roots, then $A_{\alpha + \beta, \gamma} = A_{\alpha, \gamma} + A_{\beta, \gamma}$.


§6. Simple roots, Cartan basis

From now on, we shall identify the prime subfield of $F$ with the rational numbers $\mathbb{Q}$ and let $H_0^*$ stand for the $\mathbb{Q}$-space spanned by the roots. We have:

**Lemma 1.6.1.**

(i) $\dim H_0^* = \dim H$

(ii). $\langle \rho, \delta \rangle$ is a rational number for each $\rho, \delta \in H_0^*$ and $\langle \rho, \delta \rangle$ is a positive symmetric bilinear form on $H_0^*$.

We now introduce an ordering in the rational vector space $H_0^*$ by picking an ordered basis of $H_0^*$ in $\Delta$ and lexicographically ordering the coefficients. Thus if

$$\sigma = \sum_{i=1}^{r} a_i \alpha_i$$

and

$$\rho = \sum_{i=1}^{r} b_i \alpha_i$$

where $\{\alpha_i\}$ is the basis in $\Delta$ and $a_i, b_i$ are rational numbers ($i=1, 2, \ldots, r$)

then $\sigma < \rho$ if $a_1 < b_1$ or if $a_j = b_j$ for $1 \leq j \leq q$ but $a_{q+1} < b_{q+1}$.

It is easy to see that this is a total ordering and if $\sigma < \rho$ then $\sigma + \tau < \rho + \tau$ for all $\tau \in H_0^*$. Also, if $\sigma < \rho$, then $a\sigma < a\rho$ or $a\sigma > a\rho$ accordingly as $a > 0$ or $a < 0$.

Note: This ordering is dependent on the choice of basis $\{\alpha_i\}$.

Lemma 1.6.2. Let $\rho_1, \ldots, \rho_k \in H_0^*$ and suppose that $\rho_i > 0$ and $<\rho_i, \rho_j> < 0$ if $i \neq j$. Then the $\rho_i$'s are linearly independent over $\mathbb{Q}$.

Proof: [9], Jacobson, p.119.

Definition 1.6.1. Relative to the ordering defined on $H_0^*$ we call a root $\alpha$ simple if $\alpha > 0$ and $\alpha$ cannot be written as a sum $\beta + \gamma$ where $\beta$ and $\gamma$ are positive roots.

The most important properties of simple roots are summarized in the following theorem:
Theorem 1.6.1. (L as in §5)

Let \( \pi \) be the collection of simple roots relative to a fixed lexicographic ordering on \( H_0^* \). Then

1. \( \alpha, \beta \in \pi \) and \( \alpha \nparallel \beta \) implies \( \alpha - \beta \notin \Delta \).
2. \( \alpha, \beta \in \pi \) and \( \alpha \nparallel \beta \) implies \( \langle \alpha, \beta \rangle \leq 0 \).
3. \( \pi \) forms a basis for \( H_0^* \).
4. \( \beta \in \Delta \) and \( \beta > 0 \) implies \( \beta = \sum a_1 \alpha_1 \) where the \( a_1 \) are non-negative integers and \( \alpha_1 \in \pi \) for all \( i \).
5. If \( \beta \) is a positive non-simple root, then there is some \( \alpha \in \pi \) such that \( \beta - \alpha \) is a positive root.

Proof: [9], Jacobson, p.120.

Definition 1.6.2. We call \( \pi = \{\alpha_1, \ldots, \alpha_r\} \) the simple system of roots for \( L \) relative to \( H \) and the given ordering on \( H_0^* \). The matrix \( (A_{ij}) \) is called the Cartan matrix for \( L \) relative to \( H \) where \( A_{ij} = A_{\alpha_i \alpha_j} \) for \( i, j = 1, 2, \ldots, r \).

Remark: The diagonal entries \( A_{ii} \) are all 2 and the off-diagonal entries are 0, -1, -2 or -3 by Lemma 1.5.1.

Theorem 1.6.2. The semi-simple Lie algebra \( L \) is determined up to isomorphism by its associated Cartan matrix.

Proof: [9], Jacobson, p.127. (cf. also §9).

Definition 1.6.3. A simple system of roots \( \pi = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \) is said to be indecomposable if it is impossible to partition \( \pi \) into non-empty, disjoint sets \( \pi', \pi'' \) such that \( A_{ij} = 0 \).
for every $\alpha_i \in \pi'$, $\alpha_j \in \pi''$.

**Theorem 1.6.3.** A semi-simple Lie algebra is simple if and only if the associated simple system of roots is indecomposable.

**Proof:** [9], Jacobson, p.128.

We now construct a basis for the semi-simple Lie algebra $L$ which is naturally induced by a given system

$\pi = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots of $L$ .

**Theorem 1.6.4.** Let $\beta$ be a positive root of $L$ and let $X_\beta, Y_\beta$ be two fixed non-zero elements of $L_\beta$ and $L_{-\beta}$ respectively such that $\langle X_\beta, Y_\beta \rangle = 1$. Let $H_\beta$ be the element $\frac{2}{\langle \beta, \beta \rangle} h'_\beta$ (where $h'_\beta$ is as in Theorem 1.4.2).

Then the set $\Omega = \{h'_\alpha, X_\beta, Y_\beta \mid \alpha \in \pi, \beta \in \Delta_+\}$ forms a basis for $L$ (here $\Delta_+$ denotes the set of positive roots).

**Proof:** [12], S.S.L. §10

**Definition 1.6.4.** The basis constructed in Theorem 1.6.4 is called a Cartan basis of $L$.

The Cartan basis - elements of $L$ not only provide a linear basis for the vector space $L$ but also completely determine the multiplicative structure of $L$ via the following relations:

For arbitrary positive roots $\beta, \gamma$, and an arbitrary simple root $\alpha$, we have (with notation as in Theorem 1.6.4)
\[ [H_\alpha, X_\beta] = 2 \frac{\alpha, \beta}{\beta, \beta} X_\beta \]

\[ [H_\alpha, Y_\beta] = -2 \frac{\alpha, \beta}{\beta, \beta} Y_\beta \]

\[ [X_\beta, Y_\gamma] = \begin{cases} 
\text{a non-zero multiple of } X_{\beta-\gamma} \text{ if } \beta-\gamma \text{ is a positive root} \\
\text{a non-zero multiple of } Y_{\beta-\gamma} \text{ if } \beta-\gamma \text{ is a positive root} \\
0 \text{ otherwise.}
\end{cases} \]

\[ [X_\beta, X_\gamma] = \begin{cases} 
\text{a non-zero multiple of } X_{\beta+\gamma} \text{ if } \beta+\gamma \text{ is a positive root} \\
0 \text{ otherwise.}
\end{cases} \]

\[ [Y_\beta, Y_\gamma] = \begin{cases} 
\text{a non-zero multiple of } Y_{\beta+\gamma} \text{ if } \beta+\gamma \text{ is a positive root} \\
0 \text{ otherwise.}
\end{cases} \]

§7. The Dynkin diagram of a Lie algebra

The problem of classifying the simple Lie algebras can be reduced, in light of the results of the last two sections, to first associating with each system of simple roots of a simple Lie algebra a certain diagram, and then proving that to each "admissible" diagram there exists a simple Lie algebra.

**Definition 1.7.1.** Let \( \pi = \{\alpha_1, \ldots, \alpha_n\} \) be a system of simple roots of \( \mathfrak{L} \). We choose \( n \) points (label them by \( \alpha_1, \ldots, \alpha_n \)), and connect \( \alpha_1 \) to \( \alpha_j \) \((i \neq j)\) by \( A_{ij}A_{ji} \) lines. The resulting diagram is called the root diagram associated with \( \mathfrak{L} \).
Theorem 1.7.1. In terms of the root diagrams, all possible fundamental systems are given in the following table (where the integers over each "point" $a_i$ represent the normalized length of the corresponding root in the Euclidean metric space $H^*_0$).

<table>
<thead>
<tr>
<th>Root diagram</th>
<th>Notation</th>
<th>Dim H</th>
<th>Dim L</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\text{l}}{a_1} \frac{\text{l}}{a_2} \ldots \frac{\text{l}}{a_{n-1}} \frac{\text{l}}{a_n}$</td>
<td>$A_n$</td>
<td>$n$</td>
<td>$n(n+2)$</td>
</tr>
<tr>
<td>$\frac{\text{2}}{a_1} \frac{\text{2}}{a_2} \ldots \frac{\text{2}}{a_{n-1}} \frac{\text{1}}{a_n}$</td>
<td>$B_n$</td>
<td>$n$</td>
<td>$n(2n+1)$</td>
</tr>
<tr>
<td>$\frac{\text{l}}{a_1} \frac{\text{l}}{a_2} \frac{\text{l}}{a_3} \ldots \frac{\text{l}}{a_{n-1}} \frac{\text{2}}{a_n}$</td>
<td>$C_n$</td>
<td>$n$</td>
<td>$n(2n+1)$</td>
</tr>
<tr>
<td>$\frac{\text{l}}{a_1} \frac{\text{l}}{a_2} \frac{\text{l}}{a_{n-3}} \frac{\text{l}}{a_{n-2}} \frac{\text{l}}{a_{n-1}}$</td>
<td>$D_n$</td>
<td>$n$</td>
<td>$n(2n-1)$</td>
</tr>
<tr>
<td>$\frac{\text{3}}{a_1} \frac{\text{1}}{a_2}$</td>
<td>$G_2$</td>
<td>$2$</td>
<td>$14$</td>
</tr>
<tr>
<td>$\frac{\text{1}}{a_1} \frac{\text{1}}{a_2} \frac{\text{2}}{a_3} \frac{\text{2}}{a_4}$</td>
<td>$F_4$</td>
<td>$4$</td>
<td>$52$</td>
</tr>
<tr>
<td>$\frac{\text{1}}{a_1} \frac{\text{1}}{a_2} \frac{\text{l}}{a_3} \frac{\text{l}}{a_4} \frac{\text{l}}{a_5}$</td>
<td>$E_{6}$</td>
<td>$6$</td>
<td>$78$</td>
</tr>
<tr>
<td>$\frac{\text{l}}{a_1} \frac{\text{l}}{a_2} \frac{\text{l}}{a_3} \frac{\text{l}}{a_4} \frac{\text{l}}{a_5} \frac{\text{l}}{a_6}$</td>
<td>$E_{7}$</td>
<td>$7$</td>
<td>$133$</td>
</tr>
</tbody>
</table>
It can also be shown, that to every root diagram on the above list, there corresponds a simple Lie algebra.

Every simple Lie algebra (over an algebraically closed field of characteristic zero) is isomorphic to one of the Lie algebras $A_\ell (\ell \geq 1), B_\ell (\ell \geq 2), C_\ell (\ell \geq 3), D_\ell (\ell \geq 4)$, or to an "exceptional" Lie algebra $G_2, E_6, E_7, E_8, F_4$ (these algebras are all pairwise non-isomorphic.)

Thus we have a complete classification of simple Lie algebras (over algebraically closed fields of characteristic zero).


§8. The Weyl Group

We now introduce another structure associated with each semi-simple Lie algebra.

Definition 1.8.1. Let $\alpha$ be a non-zero root of the semi-simple Lie algebra $L$. Then the map $S_\alpha : H^*_0 \to H^*_0$

$$S_\alpha : \xi \to \xi - 2 \frac{\langle \xi, \alpha \rangle}{\langle \alpha, \alpha \rangle} \cdot \alpha$$

is called the Weyl reflection with respect to $\alpha$. 

Since $<,>$ is a non-degenerate bilinear form on $H^*_0$, the map $S_\alpha$ has the following properties:

1) $S_\alpha$ is a linear transformation on $H^*_0$ which leaves fixed every vector in the hyperplane orthogonal to $\alpha$, and sends $\alpha$ to $-\alpha$.

2) If $\lambda$ is a weight of $H$ in a finite dimensional representation of $L$, then

$$S_\alpha(\lambda) = \lambda - \frac{2<\lambda,\alpha>}{<\alpha,\alpha>} \alpha$$

is also a weight. In particular, $S_\alpha$ permutes the roots of $L$ among themselves.

Definition 1.8.2. The group of linear transformations on $H^*_0$ generated by the reflections $S_\alpha$ ($\alpha \in \Delta$) is called the Weyl group of $L$, denoted by $W$.

Remark: The Weyl group plays an important role in the representation theory of $L$. We have just noted that the set of roots of $L$ is invariant under the Weyl group. If two elements of $W$ produce the same permutation of the roots, they are identical since the roots span $H^*_0$. Since there are a finite number of roots it follows that $W$ is a finite group.

Some important properties of $W$ are listed in the following theorem:

Theorem 1.8.1. Let $\pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a system of simple roots for the semi-simple Lie algebra $L$.
Then

1) The Weyl group $W$ is generated by the reflections $S_{a_i}$ ($a_i \in \pi$).

2) If $\alpha$ is any non-zero root, there exists a $\sigma \in W$ such that $\sigma(\alpha) \in \pi$.

3) If $\pi_1$ is another system of simple roots for $L$, there exists a $\sigma \in W$ such that $\sigma(\pi) = \pi_1$.

4) If $\sigma(\pi) = \pi$ for $\sigma \in W$, then $\sigma =$ identity.


9. Automorphisms of simple Lie algebras

In this section we shall study the groups of automorphisms of a simple Lie algebra $L$ over $F$ (where $F$, as usual, denotes an algebraically closed field with char $F = 0$).

As we have noted before, if $H_1$ and $H_2$ are Cartan subalgebras of $L$, then there exists an automorphism $\eta \in \text{Int } L$ such that $\eta(H_1) = H_2$. This allows us to talk about "the" Cartan subalgebra of $L$.

Automorphisms will also play an important role in our study of irreducible representations of simple Lie algebras.

We have already noted (Theorem 1.6.2) that a semi-simple Lie algebra is determined up to isomorphism by its Cartan matrix. More explicitly, we have the following theorem (which also leads to the construction of some automorphisms of $L$):
Theorem 1.9.1. Let \( L \) and \( L' \) be simple Lie algebras over \( F \), with Cartan subalgebras \( H \) and \( H' \), and corresponding root systems \( \Delta \), \( \Delta' \) respectively. Suppose there is an isomorphism of \( \Delta \) onto \( \Delta' \). This isomorphism extends uniquely to an isomorphism of vector spaces \( \psi : H^* \to H'^* \), which, in turn, induces an isomorphism \( \phi : H \to H' \) via the Cartan-Killing form identification of \( H \) and \( H' \) with their duals. Fix a basis (consisting of simple roots) \( \pi = \Delta \), so \( \pi' = \{ \alpha' \mid \alpha \in \pi \} \) is a basis of \( \Delta' \). For each \( \alpha \in \pi \), \( \alpha' \in \pi' \), choose arbitrary (non-zero) \( x_\alpha \in L_\alpha \), \( x'_\alpha \in L'_\alpha \). (ie. choose an arbitrary Lie algebra isomorphism \( \phi_\alpha : L_\alpha \to L'_\alpha \). Then there exists a unique isomorphism \( \phi : L \to L' \) extending \( \phi : H \to H' \) and extending all the \( \phi_\alpha (\alpha \in \pi) \).

Proof: [8], Humphreys, p.75.

Note: The above theorem easily extends to semi-simple Lie algebras.

Theorem 1.9.1 can be used to prove the existence of automorphisms of a simple Lie algebra (with \( H, \Delta \) as above): any automorphism of \( \Delta \) determines an automorphism of \( H \) which can be extended to \( L \).

Thus, we shall now consider \( \text{Aut}(\Delta) \) (the group of automorphisms of the root system \( \Delta \)) in some detail.

The Weyl group \( W \) of \( L \) is a normal subgroup of \( \text{Aut}(\Delta) \). Let \( \phi = \{ \sigma \in \text{Aut}(\Delta) \mid \alpha(\pi) = \pi \} \) (where \( \pi \) is a fixed basis of \( \Delta \) consisting of simple roots). Clearly
If \( \tau \in \text{Aut} \Delta \), and \( \alpha, \beta \in \Delta \), then \( \langle \alpha, \beta \rangle = \langle \tau(\alpha), \tau(\beta) \rangle \).

Therefore, each \( \tau \in \phi \) determines an automorphism of the Dynkin diagram of \( \Delta \). On the other hand, each automorphism of the Dynkin diagram clearly determines an automorphism of \( \Delta \), so \( \phi \) may be identified with the group of **automorphisms**. Thus, the Weyl group of \( L \) accounts for most of the automorphisms of \( \Delta \). If \( \sigma \in W \), the extension of \( \sigma \) to an automorphism of \( L \) maps \( L_\alpha \) to \( L_{\alpha \sigma^{-1}} \).

For later use, we mention that the simple Lie algebra \( A_n \) has \( \binom{n+1}{2} \) positive roots, its Weyl group \( W \) is isomorphic to \( S_{n+1} \) (the symmetric group on \( n+1 \) elements) and hence \( W \) has order \( (n+1)! \). \( \phi \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \) (for \( n \geq 2 \)).
Chapter Two

The Universal Enveloping Algebra of a Lie algebra

The notion of the Universal Enveloping algebra of a Lie algebra is extremely important in representation theory. By embedding the Lie algebra into its universal enveloping algebra $U(L)$ we shall see that there is a one-to-one correspondence between representations of $L$ and the associative representations of $U(L)$. Moreover, this correspondence preserves irreducibility. Thus we may consider the associative representations of $U(L)$ instead of representations of $L$.

This chapter again contains well-known material. Therefore we merely present a summary of the definitions and main results which will be needed later.

§1. Basic definitions and properties

Definition 2.1.1. Let $L$ be a Lie algebra. A pair $(U, i)$ (where $U$ is an associative algebra with unit and $i$ is a homomorphism $i : L \to U_L$) is called a universal enveloping algebra of $L$ if the following condition holds:

If $R$ is any associative algebra with unit and $f$ is any homomorphism of $L$ into $R_L$, then there is a unique homomorphism $f'$ of $U$ to $R$ such that the following diagram commutes:

$$
\begin{array}{ccc}
L & \xrightarrow{i} & U \\
\downarrow{f} & & \downarrow{f'} \\
R & \rightarrow & R_L
\end{array}
$$
The following theorem summarizes the most important properties of the universal enveloping algebra:

**Theorem 2.1.1.** (1) The universal enveloping algebra of a Lie algebra is unique up to isomorphism; if \((U, i)\) and \((V, j)\) are both universal enveloping algebras for the Lie algebra \(L\), then there exists a unique isomorphism \(j^*\) of \(U\) onto \(V\) such that \(j = i \circ j^*\).

(2) \(U\) is generated by the image \(i(L)\).

(3) Let \(L_1, L_2\) be Lie algebras with universal enveloping algebras \((U_1, i_1)\) \((U_2, i_2)\) respectively and let \(\theta\) be a homomorphism from \(L_1\) into \(L_2\). Then there exists a unique homomorphism \(\tilde{\theta}\) from \(U_1\) to \(U_2\) such that the following diagram is commutative:

![Diagram](image)

(4) Let \(B\) be an ideal of the Lie algebra \(L\) and let \(R\) be the ideal of \(U\) generated by \(i(B)\). Then \(U/R\) together with the map \(\tilde{j} : L/B \to U/R\)

\[
\tilde{j} : \lambda + B \mapsto i(\lambda) + R
\]

is the universal enveloping algebra of \(L/B\).


We now give a construction of the universal enveloping algebra for a Lie algebra \(L\).

Let \(T\) be the tensor algebra based on the linear space
(Recall that
\[ T = T^0 \oplus T^1 \oplus T^2 \oplus \ldots \oplus T^n \oplus \ldots, \] where \( T^0 = F \cdot 1 \),
\[ T^1 = L \oplus L \oplus \ldots \oplus L \] (n times). Here "\( \oplus \)" denotes tensor multiplication characterized by
\[ (x_1 \oplus \ldots \oplus x_i) \oplus (y_1 \oplus \ldots \oplus y_k) = x_1 \oplus x_2 \oplus \ldots \oplus x_i \oplus y_1 \oplus \ldots \oplus y_k \]

Next, let \( B \) be the ideal in \( T \) generated by all the elements of the form
\[ [a, b] = a \otimes b + b \otimes a \ (a, b \in L). \]
Let \( U = T/B \). Let \( i \) denote the restriction to \( L \) of the canonical homomorphism of \( T \) onto \( U \).

**Theorem 2.1.2.** \( i \), as defined above, is a homomorphism of \( L \) into \( U_L \), and \( (U, i) \) is a universal enveloping algebra of \( L \).


To make good use of the universal enveloping algebra \( U \) of \( L \), we must have more information about how \( L \) "sits" in \( U \):

**Theorem 2.1.3.** (Poincaré-Birkhoff-Witt)

Suppose the Lie algebra \( L \) has a totally ordered basis \( \{ e_i \mid i \in I \} \) where \( I \) is a suitable index set. Let \( \sigma \) denote a finite sequence \( \{ \sigma_1, \ldots, \sigma_n \} \) of elements of \( I \) such that \( \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_n \). Then let \( x_\sigma \) be the element
\[ x_\sigma = e_{\sigma_1} e_{\sigma_2} \ldots e_{\sigma_n}. \]
The set of all such elements \( x_\sigma \)
(where \( \sigma \) ranges over all possible finite, increasing sequences of elements of \( I \)) forms a basis for the linear
space \( U(L) \).


Remark: It follows from the above theorem, that the mapping \( i: L \rightarrow U(L) \) is an injection, that is, the linear space \( L \) is embedded in the linear space \( U(L) \). Using this embedding we shall assume from now on that \( L \) is a linear subspace of \( U(L) \).

Definition 2.1.2. Let \( L \) be a Lie algebra. The linear space \( L \) together with the binary operation \( \ast : L \times L \rightarrow L \) given by \( x \ast y = 0 \) for all \( x, y \in L \) is again a Lie algebra, called "the abelian Lie algebra" associated with \( L \). It is denoted by \( L_a \). Its universal enveloping algebra is called the "symmetric algebra" of \( L \) and is denoted by \( S(L) \).

Theorem 2.1.4. Let \( \{ e_i \mid i \in I \} \) be a totally ordered basis of \( L \). Then the symmetric algebra \( S(L) \) of \( L \) is isomorphic to the algebra of commutative polynomials of finite degree in the unknowns \( e_i \). The set of all commutative monomials of finite degree in the unknowns \( e_i \) forms a basis for \( S(L) \).


With the aid of Theorem 2.1.4, the P.B.W. theorem can be restated in the following form:

Theorem 2.1.5. For any Lie algebra \( L \), the linear space \( U(L) \) is isomorphic to the linear space \( S(L) \).
§2. Representations of $L$ and $U(L)$.

Let $(\rho, V)$ be a representation of the Lie algebra $L$. Then, by the universal property of $U(L)$, (stated in Definition 2.1.1) there exists a unique homomorphism $\bar{\rho} : U(L) \rightarrow \text{End}(V)$ such that $\rho = \bar{\rho} \circ 1$.

It is easily verified that $(\bar{\rho}, V)$ is an (associative) representation of $U(L)$. Conversely, given a representation $(\bar{\rho}, V)$ of $U(L)$, we obtain a representation of $L$ by simply restricting the map $\bar{\rho}$ to the subspace $L$ of $U(L)$.

If $(\bar{\rho}, V)$ is an irreducible representation of $L$, then the representation $(\bar{\rho}, V)$ of $U(L)$ is also irreducible, since any proper (non-zero) $\bar{\rho}$-invariant subspace $W$ of $V$ is also invariant under the restriction of $\bar{\rho}$ to $L \subseteq U(L)$. Conversely, if a representation $(\bar{\rho}, V)$ of $U(L)$ is irreducible, then the corresponding representation $(\bar{\rho}, V)$ of $L$ is also irreducible. Thus, we have the following theorem:

**Theorem 2.2.1.** There is a one-to-one correspondence between representations of a Lie algebra $L$ and representations of its universal enveloping algebra $U(L)$. This correspondence preserves irreducibility.


The above theorem allows us, in studying the representations of $L$, to concern ourselves only with the representations of the associative algebra $U(L)$. 

Theorem 2.2.2. Let \((\rho, V)\) be a cyclic representation of the universal enveloping algebra \(U(L)\) of \(L\). Then there exists a left ideal \(I\) of \(U(L)\) such that \((\rho, V)\) is equivalent to the left regular representation of \(U(L) \mod I\). If, in particular, \((\rho, V)\) is irreducible, there exists a maximal left ideal \(M\) of \(U(L)\) such that \((\rho, V)\) is equivalent to the left regular representation of \(U(L) \mod M\).

Moreover, if \(M'\) is any maximal left ideal of \(U(L)\), then the left regular representation of \(U(L) \mod M'\) is irreducible.


Definition 2.2.1. Let \((\rho, V)\) be any representation of \(U(L)\). Let \(v\) be any element of \(V\). The annihilator ideal of \(v\) is defined to be the set
\[
\text{Ann}(v) = \{u \in U(L) \mid \rho(u)v = 0\}
\]
Clearly \(\text{Ann}(v)\) is a left ideal of \(U(L)\).

Theorem 2.2.3. Let \((\rho, V)\) be any irreducible representation of \(U(L)\), and let \(0 \neq v_0 \in V\). Then
1. \(\rho(U(L))v_0 = V\)
2. \(\text{Ann}(v_0)\) is a maximal left ideal of \(U(L)\)
3. \((\rho, V)\) is equivalent to the left regular representation of \(U(L) \mod \text{Ann}(v_0)\).

§3. The Cycle Subalgebra of \( U(L) \)

In this section we define the cycle subalgebra of a universal enveloping algebra. This concept was first introduced by Chevalley [4] in his algebraic proof of the existence of Lie algebras corresponding to various Cartan matrices. We shall first establish the relationship between the cycle subalgebra of \( U(L) \) and the weight functions of irreducible representations of \( U(L) \), then we shall characterize the representations of \( L \) admitting a one dimensional weight space.

The material of this section is due to Bouwer [2] and Lemire [10a].

Lemma 2.3.1. There exists one and only one anti-automorphism \( \psi \) of the algebra \( U(L) \) such that \( \psi(x) = -x \) for all \( x \in L \).

Proof: [5a] Dixmier, p.73.

Definition 2.3.1. The anti-automorphism given by the above lemma is called the principal anti-automorphism of \( U(L) \) and is denoted by \( u \rightarrow u^T \).

For all \( u \in U(L) \), let \( L(u) \) and \( R(u) \) be the mappings \( v \rightarrow u \cdot v \) and \( v \rightarrow v \cdot u \) of \( U(L) \) into itself. The mapping \( u \rightarrow L(u) \) is a representation of \( U(L) \) in \( U(L) \) called the left regular representation of \( U(L) \). The corresponding representation of \( L \) (i.e. the mapping \( x \rightarrow L(x) \) (\( x \in L \)) is termed the left regular representation of \( L \) in \( U(L) \).

The mapping \( u \rightarrow R(u^T) \) is a representation of \( U(L) \) in \( U(L) \). The corresponding representation of \( L \) (i.e. the
mapping \( x \mapsto R(x) (x \in L) \) is called the \textbf{right regular representation of} \( L \) \textbf{in} \( U(L) \).

It is easily checked that the mapping 
\( x \mapsto \rho(x) = \mathcal{L}(x) - R(x) \) is again a representation of \( L \) \textbf{in} \( U(L) \) \textbf{. It is called the adjoint representation of} \( L \) \textbf{in} \( U(L) \), and is often denoted by \( \text{Ad} \). For \( x \in L \) and \( u \in U(L) \), we have \( \rho(x)(u) = \text{Ad} x(u) = [x,u] \).

For any \( \xi \in H^* \) define the set 
\( U_\xi = \{ u \in U(L) \mid \text{Ad}(u) = \xi(h)u \text{ for all } h \in H \} \). Clearly, for any \( \xi \in H^* \), \( U_\xi \) is a subspace of \( U(L) \), and 
\[ U(L) = \bigoplus_{\xi \in H^*} U_\xi. \]

\textbf{Note:} It can be shown (cf. [2] Bouwer, p.108) that if \( \xi \) is not an integral linear combination of the simple roots, \( U_\xi = \{0\} \).

Let \( \Delta \) denote the non-zero roots of \( L \), \( \Delta^+ \) the positive roots, and \( \Delta^{++} \) the simple roots of \( L \). (cf. p.16)

Using the P.B.W. Theorem, one observes that the universal enveloping algebra \( U(L) \) of \( L \) admits a linear basis called a \textbf{Cartan basis of} \( U(L) \), consisting of all elements of the form
\[ \bigoplus_{\beta \in \Delta^+} \mathcal{X}^m(\beta) \bigoplus_{\gamma \in \Delta^+} \mathcal{Y}^k(\gamma) \bigoplus_{\delta \in \Delta} \mathcal{X}^\lambda(\delta) \bigoplus_{\beta \in \Delta} \mathcal{X}^\beta \]
(\text{where the exponents are non-negative integers and the products preserve some fixed order of terms}). It is easy to verify that the sub-
space $U_\xi$ is generated by all elements of the form $\otimes$
which also satisfy the condition $\sum_{\beta \in A^+} (n(\beta) - m(\beta)) \beta = \xi$.

**Proposition 2.3.1.** For any $\xi_1, \xi_2 \in H^*$, $U_{\xi_1} \cap U_{\xi_2} \subseteq U_{\xi_1 + \xi_2}$.

**Proof:** Let $x_1 \in U_{\xi_1}$, $x_2 \in U_{\xi_2}$, $h \in H$. Then

\[ \text{Adh}(x_1 x_2) = [h x_1 x_2] = h x_1 x_2 - x_1 x_2 h = h x_1 x_2 - x_1 x_2 h + \]

\[ + x_1 h x_2 - x_1 h x_2 = h x_1 x_2 + x_1 (h x_2 - x_2 h) - \]

\[ - x_1 h x_2 = (h x_1 - x_1 h) x_2 + x_1 (h x_2 - x_2 h) = \]

\[ = x_1^t \xi_2(h) x_2 + \xi_1(h) x_1 x_2 = (\xi_1(h) + \xi_2(h)) x_1 x_2 \]

thus, $x_1 x_2 \in U_{\xi_1 + \xi_2}$.

**Remark:** The above proposition implies that the subspace

$U_0 = \{ u \in U(L) \mid \text{Adh}(u) = 0 \ \forall h \in H \}$

is actually a subalgebra of $U(L)$ (containing 1). It is called the cycle subalgebra, denoted by $C(L)$. The elements of $C(L)$ are called cycles.

Clearly $C(L)$ is the centralizer of $H$ in $U(L)$.

**Definition 2.3.2.** A basis element $e \in C(L)$ of the form $\otimes$
is called an elementary cycle of $U(L)$, if, when considered as a commutative monomial, $e$ contains no cycles other than 1 and itself.

**Proposition 2.3.2.** The elementary cycles of $U(L)$ generate the cycle subalgebra $C(L)$.

**Proof:** Since (by the P.B.W. theorem) the symmetric
algebra $S(L)$ and $U(L)$ are isomorphic as linear spaces, it suffices to show that any Cartan basis element $u \in C(L)$ can be written as a product of elementary cycles. Let $u$ be such an element. If $u$ is not an elementary cycle, one can express it as a product of two cyclic Cartan basis elements in $S(L)$. Since $u$ contains a finite number of factors, the result can be established by repeating the above process a finite number of times.

Remark: Lemire has shown in his thesis [10a] that $C(L)$ is finitely generated. We shall not need this result in its generality, however we shall explicitly compute the generators of $C(A_n)$ in Chapter six.

In the following, we shall establish a close connection between the spaces $U_\xi$ and the weight spaces of representations of $L$. In particular we shall show that if $\lambda_0$ is a fixed weight of an irreducible representation $(\rho, V)$ of $L$ and $0 \not= v_0 \in V_{\lambda_0}$, then $V_{\lambda_0} = \rho(C(L))v_0$; i.e. the cycle subalgebra generates the $\lambda_0$-weight space by acting on an arbitrary non-zero vector $v_0 \in V_{\lambda_0}$.

Lemma 2.3.2. Let $(\rho, V)$ be a representation of $L$ having at least one weight function $\lambda$, and let $\xi \in H^*$. Let $(\overline{\rho}, V)_{\overline{\rho}}$ be the corresponding representation of $U(L)$. Then

$$\overline{\rho}(U_\xi)V_\lambda \subseteq V_{\lambda + \xi}.$$ Moreover, if $\rho$ is irreducible, we have equality.

Proof: Let $x \in U_\xi$, $v \in V_\lambda$, $h \in H$. Then
\[ \tilde{\rho}(h)\tilde{\rho}(x)(v) = \tilde{\rho}(hx)(v) = \tilde{\rho}(xh + [h,x])\lambda(v) = \tilde{\rho}(xh + \xi(h)x)(v) = \lambda(h)\tilde{\rho}(x)(v) + \xi(h)\tilde{\rho}(x)(v) = (\lambda+\xi)(h)\tilde{\rho}(x)(v) \] 

that is, 
\[ \tilde{\rho}(x)v \in V_{\lambda+\xi} \] 

Also, if \((\rho,V)\) is irreducible, then \(V = \bigoplus \lambda V_{\lambda}\) and therefore we have \(\tilde{\rho}(U_\xi) V_\lambda = V_{\lambda+\xi}\).

Remark: Lemma 2.3.2 implies, in particular, that if \(\lambda\) is any weight of an irreducible representation \((\rho,V)\) of \(L\) and \(0 \not= v \in V_\lambda\), then \(V_\lambda = \tilde{\rho}(C(L))v\).
Chapter Three

Pointed Representations

In this chapter, we begin our study of the main problem of this thesis, that of the classification of irreducible representations which admit a one-dimensional weight space. (Such representations will be called pointed.)

We shall establish a correspondence between pointed representations of a simple Lie algebra $L$ and certain algebra homomorphisms from the cycle subalgebra $C(L)$ to the base field $F$. However, this correspondence is not one-to-one since it may happen that different algebra homomorphisms yield equivalent pointed representations.

In section 2, we shall investigate conditions under which two mass functions characterize equivalent representations. Finally, in section 3 we introduce an equivalence relation on the family of algebra homomorphisms $\phi: C(L) \to F$. As we shall see in Chapter 5, the representations corresponding to representatives of the equivalence classes "essentially" cover all pointed representations.

Some of the results are based on works of Cartan [3], Chevalley [4], Harish-Chandra [7], Bouwer [2] and Lemire [10a].

§1. Pointed representations and mass functions

Definition 3.1.1. An irreducible representation $(\rho, V)$ of a simple Lie algebra $L$ is called pointed iff $(\rho, V)$
admits a one-dimensional weight space. The family of all pointed representations of \( L \) will be denoted by \( P_L \).

**Remark:** \( P_L \) includes all dominated irreducible representations, in particular all finite dimensional irreducible representations of \( L \). [cf. Ch. 4 §2.]. Moreover, \( P_L \) is included in the family of all Harish-Chandra modules which are \( H \)-finite. (cf. Gindikin, Kirillov and Fuks, [13]).

**Definition 3.1.2.** Let \( (\rho, V) \) be a representation of \( U(L) \). An algebra homomorphism \( \phi : C(L) \rightarrow F \) for which there exists a non-zero vector \( v \in V \) such that \( \rho(c)v = \phi(c)v \) for all \( c \in C(L) \) is called a mass function of the representation \( (\rho, V) \).

**Theorem 3.1.1.** Let \( (\rho, V) \) be an irreducible representation of \( U(L) \) admitting a one-dimensional weight space \( V_\lambda \). Then the weight function \( \lambda : H \rightarrow F \) can be extended to an algebra homomorphism \( \phi : C(L) \rightarrow F \) such that \( \phi \) is a mass function of \( (\rho, V) \). Conversely, if \( \phi : C(L) \rightarrow F \) is a mass function of the irreducible representation \( (\rho, V) \) of \( U(L) \), then the restriction of \( \phi \) to the Cartan subalgebra \( H \) is a weight function of \( (\rho, V) \) and the corresponding weight space is one-dimensional.

**Proof:** First let \( V_\lambda \) be a one-dimensional weight space of \( (\rho, V) \) and let \( 0 \neq v \in V_\lambda \). Then, since \( (\rho, V) \) is irreducible, by the Remark after Lemma 2.3.2, \( V_\lambda = \rho(C(L))v \), and since \( \dim V_\lambda = 1 \), for all \( c \in C(L) \), \( \rho(c)v = t_cv \) for
some $t_c$.

Define a map $\phi : C(L) \to F$ by $\phi(c) = t_c$ where $t_c v = \rho(c)v$. [this definition does not depend on the choice of $v$.] Since $\lambda$ is a weight function of $(\rho, V)$, $\phi$ is obviously an extension of $\lambda$. The fact that $\phi$ is an algebra homomorphism follows directly from the properties of the representation $\rho$.

Conversely, if $\phi : C(L) \to F$ is a mass function of $(\rho, V)$ then the restriction $\phi|_H$ is a weight function of $(\rho, V)$. Let $\phi|_H = \lambda$, and let $0 \neq v \in V_\lambda$. Then $V_\lambda = \rho(C(L))v = \lambda(C(L))v = F \cdot v$ i.e. $\dim V_\lambda = \dim(F \cdot v) = 1$.

Thus we have obtained a correspondence between mass functions of an irreducible representation $(\rho, V)$ of $U(L)$ and one-dimensional weight spaces of $(\rho, V)$.

Next, we shall show that given any algebra homomorphism $\phi : C(L) \to F$, there exists an irreducible representation of $U(L)$ admitting $\phi$ as a mass function.

**Theorem 3.1.2.** Let $\phi : C(L) \to F$ be any algebra homomorphism. Then there exists a unique irreducible representation of $U(L)$ admitting $\phi$ as a mass function.

**Proof:** Let $I_\phi$ denote the left ideal of $U(L)$ generated by $\{c - \phi(c) \cdot 1 | c$ is an elementary cycle of $U(L)\}$. Then $I_\phi$ is a proper left ideal of $U(L)$, and hence there exists at least one maximal left ideal $M_\phi$ of
U(L) containing \( I_\phi \). \( M_\phi \) is in fact unique: it is determined as the largest proper \( \phi \)-invariant subspace of the left regular representation \( \rho \) of \( U(L) \mod I_\phi \).

Now the left regular representation of \( U(L) \mod M_\phi \) is irreducible, since \( M_\phi \) is maximal. Moreover, \( 1 + M_\phi \not\subset M_\phi \) and for any \( c \in C(L) \) we have \( c \cdot (1 + M_\phi) = c + M_\phi = \phi(c) \cdot 1 + M_\phi = \phi(c) \cdot (1 + M_\phi) \), i.e. \( \phi \) is a mass function of the representation.

Finally suppose that \((\rho,V)\) is an irreducible representation of \( U(L) \) admitting \( \phi \) as a mass function. Then \( \phi \cdot \mathcal{H} \) is a weight function. Let \( v \) be any non-zero element in the \( \phi \cdot \mathcal{H} \) weight space.

Since \((\rho,V)\) is assumed to be irreducible, \( V = \rho(U) \cdot v \).

Define \( \psi : V \rightarrow U(L) / M_\phi \) by \( \psi(\rho(u)v) = u + M_\phi \). Then \( \psi \) is a well-defined linear space isomorphism. Let \( v_1 = \rho(u_1)v \) be any element of \( V \). Then for any \( u \in U(L) \), we have
\[
\psi(\rho(u)v_1) = \psi(\rho(u)(\rho(u_1)v)) = \psi(\rho(uu_1)v) = uu_1 + M_\phi,
\]
thus \((\rho,V)\) is equivalent to the left regular representation of \( U(L) \mod M_\phi \), via the isomorphism \( \psi \).

Remark: In the following, for any algebra homomorphism \( \phi : C(L) \rightarrow F \), \( M_\phi \) will denote the unique maximal left ideal of \( U(L) \) containing \( \ker \phi \).
§2. **Equivalence of Mass Functions**

In the previous section we have established a correspondence between the set of (non-zero) algebra homomorphisms \( \phi : C(L) \to F \) onto the set of equivalence classes of pointed representations of \( U(L) \). This correspondence is many-to-one, however (that is, different algebra homomorphisms may yield equivalent irreducible representations.) We shall now consider some conditions under which two mass functions characterize equivalent irreducible representations.

First we need the following result:

**Theorem 3.2.1.** Let \((\rho, V)\) be an irreducible representation of \( U(L) \) admitting at least one weight function \( \lambda_0 \). Let \( 0 \neq v_0 \in V_{\lambda_0} \), and let \( \lambda \) be an arbitrary weight function of \((\rho, V)\). Then there exist integers \( m_1, \ldots, m_n \) such that

\[
\lambda = \lambda_0 + \sum_{i=1}^{n} m_i a_i \quad \text{(where \( \{a_i\}_{i=1}^{n} \) is the set of simple roots)}.
\]

**Proof:** Since \((\rho, V)\) is irreducible, \( \rho(U(L))v_0 = V \).

Moreover, \( V \) is the direct sum of the spaces \( \rho(U_{\xi})v_0 \) where \( \xi \) ranges over all integral linear combinations of simple roots of \( L \). Thus, if \( 0 \neq v \in V_{\lambda} \), then

\( v \in \rho(U_{\xi})v_0 \) for some \( \xi \). Also, by Lemma 2.3.2,

\( \rho(U_{\xi})v_0 \leq V_{\lambda_0 + \xi} \); and hence \( \lambda = \lambda_0 + \xi = \lambda_0 + \sum_{i=1}^{n} m_i a_i \).
Theorem 3.2.2. Let $\phi_1, \phi_2 : C(L) \to F$ be non-zero algebra homomorphisms corresponding to equivalent pointed representations $\rho_1$ and $\rho_2$. Then $(\phi_1 - \phi_2)^\prime_H = \sum_{i=1}^{k} m_i \alpha_i$ where the $\alpha_i$'s are the simple roots of $L$ and the $m_i$'s are integers.

Proof: By Theorem 3.2.1 the weights of $(\rho_2, V_2)$ are of the form $\phi_2 + \sum_{i=1}^{k} m_i \alpha_i$, ($m_i$'s are integers). Since by assumption $\rho_1$ and $\rho_2$ are equivalent, $\phi_1^H$ is a weight function of $(\rho_2, V_2)$ (cf. Theorem 1.5.2). Thus there exist integers $m_1, \ldots, m_k$ such that

$$\phi_1^H = \phi_2^H + \sum_{i=1}^{k} m_i \alpha_i,$$

and hence

$$(\phi_1 - \phi_2)^\prime_H = \sum_{i=1}^{k} m_i \alpha_i.$$

Theorem 3.2.3. Let $(\rho_1, V_1)$ and $(\rho_2, V_2)$ be equivalent irreducible representations of $U(L)$ admitting $\phi_1$ and $\phi_2$ as mass functions respectively. Assume $\phi_1^H = \phi_2^H$. Then $\phi_1 = \phi_2$.

Proof: Let $\psi : V_1 \to V_2$ be an isomorphism establishing the equivalence of $\rho_1$ and $\rho_2$. Then we have, for all $v \in V_1$ and $c \in C(L)$

$$\rho_2(c) \circ \psi(v) = \psi \circ \rho_1(c)(v).$$

Now let $v_1$ be a non-zero element of the $\phi_1^H$-weight space of $(\rho_1, V_1)$. Then $\psi(v_1)$ is a non-zero element of the
$\phi_2$-weight space of $(\rho_2, V_2)$, and for any $c \in C(L)$ we have:

$$\phi_1(c) \psi(v_1) = \psi(\phi_1(c)v_1) = \psi \circ \rho_1(c)(v_1) = \rho_2(c) \circ \psi(v_1) = \phi_2(c) \psi(v_1).$$

Thus $\phi_1 = \phi_2$.

§3. Weak equivalence

Let $\sigma$ be an automorphism of $L$. Then $\sigma$ can be uniquely extended to an automorphism of $U(L)$. (We shall also denote this extension by $\sigma$.) Let $\text{Aut}(L; H)$ denote the group of automorphisms of $L$ that leave the Cartan subalgebra $H$ fixed; i.e., $\text{Aut}(L; H) = \{\sigma \in \text{Aut}(L) \mid \sigma(H) \subseteq H\}$.

Lemma 3.3.1. Let $\sigma \in \text{Aut}(L; H)$, $\xi \in H^*$. Then $\sigma$ maps $U_\xi$ into $U_{\xi \sigma^{-1}}$.

Proof: Let $x \in U_\xi$, $h \in H$. Then $\text{Ad} h(\sigma(x)) = [h, \sigma(x)] = [\sigma^{-1}(h), \sigma(x)] = \sigma([\sigma^{-1}(h), x]) = \sigma(\xi(\sigma^{-1}(h)) x) = \xi(\sigma^{-1}(h)) \sigma(x)$, that is $\sigma(x) \in U_{\xi \sigma^{-1}}$.

Remark: The above lemma implies, in particular, that if $\sigma \in \text{Aut}(L; H)$, $\sigma$ maps $C(L)$ into $C(L)$. (The restriction of $\sigma$ to $C(L)$ will also be denoted by $\sigma$.) Thus, if $\phi : C(L) \to F$ is an algebra homomorphism, so is $\phi \circ \sigma$.

Definition 3.3.1. Let $\phi_1 : C(L) \to F$ and $\phi_2 : C(L) \to F$ be algebra homomorphisms. We say that $\phi_1$ is weakly equivalent.
to $\phi_2$ iff there exists $\sigma \in \text{Aut}(L;H)$ such that $\phi_1 = \phi_2 \circ \sigma$.

Note: 1. Weak equivalence is clearly an equivalence relation.
2. If $\phi_1 : C(L) \to F$ and $\phi_2 : C(L) \to F$ are weakly equivalent, their corresponding pointed representations are "Q-associates" in the sense of Bouwer. (cf. [2], p.154.)

In order to get more information about the relationship between pointed representations corresponding to weakly equivalent algebra homomorphisms, we first need the following result:

**Lemma 3.3.2.** Let $\phi : C(L) \to F$ be an algebra homomorphism, $M_\phi$ the maximal left ideal of $U(L)$ containing $\ker \phi$. Then

$$M_\phi = \sum_{\xi \in H^*} \Theta(U_\xi \cap M_\phi), \text{ and } u \in U_\xi \cap M_\phi$$

if and only if $U_{-\xi} u \subseteq \ker \phi$.

Proof: Recall that $U(L) = \sum_{\xi \in H^*} \Theta U_\xi$ and $U_{\xi_1} U_{\xi_2} \subseteq U_{\xi_1 + \xi_2}$. Moreover, since $M_\phi$ is an ideal,

$$U(L) \cap M_\phi = M_\phi = \sum_{\xi \in H^*} \Theta (U_\xi \cap M_\phi).$$

It remains to prove that $u \in U_\xi \cap M_\phi$ iff $U_{-\xi} u \subseteq \ker \phi$.

Assume first that $u \in U_\xi \cap M_\phi$. Take $v \in U_{-\xi}$ and assume that $\phi(vu) \neq 0$. Then $vu - \phi(vu) \in \ker \phi \subseteq M_\phi$.

and, since $vu \in M_\phi$, $\phi(vu) \in M_\phi$, thus $1 \in M_\phi$ which
contradicts the fact that $M_\phi$ is a maximal left ideal.

Conversely, let $u \in U_\xi$ and assume that for all $v \in U_{-\xi}$, $vu \in \ker \phi$.

Define, for each $\xi \in H^*$, $U_\xi' = \{w \in U_\xi \mid (\forall v \in U_{-\xi})vw \in \ker \phi\}$

and let $N = \bigoplus_{\xi \in H^*} U_\xi'$.

Then $u \in N$ (by definition of $N$). Moreover, $N$ contains $\ker \phi$ since it contains $U_\xi \cap M_\phi$ for all $\xi \in H^*$ (by the first part of the proof). $N$ is clearly a subalgebra of $U(L)$. We shall show that $N$ is actually a left ideal of $U(L)$: let $u' \in N$, $x \in U(L)$. Then

$$u' = \sum_{i=1}^{l} u_{\xi_i}^i \quad (u_{\xi_i}^i \in U_{\xi_i}'$, $i = 1, 2, \ldots, l)$$

and

$$x = \sum_{j=1}^{k} u_{\xi_j} \quad (u_{\xi_j} \in U_{\xi_j}'$, $j = 1, 2, \ldots, k),$$

so

$$xu' = u_{\xi_1}^1 u_{\xi_1} + u_{\xi_1}^1 u_{\xi_2} + \ldots + u_{\xi_k}^1 u_{\xi_l}^l.$$ (some of the terms may be zero.) Consider the term $u_{\xi_i}^1 u_{\xi_j}^i$. It clearly belongs to $U_{\xi_1 + \xi_j}$, moreover, if $v \in U_{-(\xi_1 + \xi_j)}$,

$$v(u_{\xi_1}^1 u_{\xi_j}^i) = (vu_{\xi_1}^1)u_{\xi_j}^i$$

and, since $vu_{\xi_1}^1 \in U_{-\xi_j}$,

$$\phi((vu_{\xi_1}^1)u_{\xi_j}^i) = 0 \text{ by definition of } U_{\xi_j}' \text{.}$$ (If there is another term, say, $u_{\xi_m}^n u_{\xi_1 + \xi_j}$, we must have $\xi_m + \xi_n = \xi_1 + \xi_j$, and so for any $v \in U_{-(\xi_1 + \xi_j)}$)
Thus each term \( u_{i,j} u'_{i,j} \) on the right-hand side of the expression for \( xu' \) belongs to \( U_{i,j} + U'_{i,j} \), and hence \( xu' \in N \),

Thus \( N \) is a left ideal of \( U(L) \) containing \( \ker \phi \), hence \( N \subseteq M_{\phi} \), and since \( u \in N \), \( u \in M_{\phi} \) as well.

**Lemma 3.3.3.** Let \( \phi : C(L) \to F \) be an algebra homomorphism, \( \sigma \in \text{Aut}(L:H) \). Then for any \( \xi \in H^* \), \( M_{\phi} \cap U_{\xi} \subseteq \sigma^{-1}(M_{\phi} \cap U_{\xi}) \).

**Proof:** Let \( u \in M_{\phi} \cap U_{\xi} \). Then by Lemma 3.3.1,

\[
\sigma^{-1}(u) \in U_{\xi_{\sigma}}.
\]

Moreover, for any \( v \in U_{\xi_{\sigma}} \), \( \sigma(v) \in U_{\xi} \) and hence \( \phi(\sigma(v)u) = 0 \) by Lemma 3.3.2. But

\[
\phi(\sigma(v)u) = \phi \sigma(v) \sigma^{-1}(u).
\]

Thus \( \sigma^{-1}(u) \in \ker(\phi \sigma) \), so \( \sigma^{-1}(u) \in M_{\phi_{\sigma}} \cap U_{\xi_{\sigma}} \); i.e., \( \sigma^{-1}(M_{\phi} \cap U_{\xi}) \subseteq M_{\phi_{\sigma}} \cap U_{\xi_{\sigma}} \).

The reverse inclusion can be proved in a similar manner.

**Lemma 3.3.4.** Let \( \phi_1 : C(L) \to F \) be an algebra homomorphism, \( \sigma \in \text{Aut}(L:H) \), \( \phi_2 = \phi_1 \circ \sigma \), and let \( \lambda \) be a weight of the left regular representation of \( U(L) \mod M_{\phi_1} \). Then \( \lambda \circ \sigma \)
is a weight of the left regular representation of \( U(L) \mod M_{\phi_2} \);
moreover, the corresponding weight spaces have the same dimension.

Proof: Define a map $\hat{\sigma} : U(L)/M_{\phi_1} + U(L)/M_{\phi_2}$ by setting $\hat{\sigma}(u + M_{\phi_1}) = \sigma^{-1}(u) + M_{\phi_2}$. This is a well-defined linear isomorphism by Lemma 3.3.3. Now let $u + M_{\phi} \in (U(L)/M_{\phi_1}$ and $\lambda \circ \sigma$ of the left regular representation of $U(L)$ mod $M_{\phi_2}$. In fact, for any $h \in H$,

$$h(\sigma^{-1}(u) + M_{\phi_2}) = \hat{\sigma}(\sigma(h)(u) + M_{\phi_1})$$

$$= \hat{\sigma}(\lambda(\sigma(h))u + M_{\phi_1}) = \lambda \circ \sigma(h)\hat{\sigma}(u + M_{\phi_1})$$

$$= \lambda \circ \sigma(h)(\sigma^{-1}(u) + M_{\phi_2}) .$$

Finally, if $u_1 + M_{\phi_1}$ and $u_2 + M_{\phi_1}$ are linearly independent in $U(L)/M_{\phi_1}$, then clearly $\sigma^{-1}(u) + M_{\phi_2}$ and $\sigma^{-1}(u_2) + M_{\phi_2}$ are also linearly independent, thus the dimension of weight spaces is preserved under the map $\hat{\sigma}$. 


Chapter Four

Dominated and Complete Representations

In this chapter, we present a brief study of two important subfamilies of pointed representations of a simple Lie algebra $L$.

In §1, we give a method of constructing algebra homomorphisms for the cycle subalgebra $C(L)$ of $U(L)$. The results of this section are due to Lemire [10c].

In §2, we define and characterize the irreducible representations of $U(L)$ that admit a "dominant weight function". The main results concerning these representations are well known and will be stated without proof; however we shall reset them in the framework of mass functions for the purpose of later generalizations.

In §3, we shall study the family of "complete" representations of $U(L)$, first introduced by Bouwer (cf. [2a]) under the term "standard representations".

§1. Construction of algebra homomorphisms for the cycle subalgebra of $U(L)$

For the remainder of this thesis, $L$ shall denote a simple Lie algebra over the complex number field $C$. (This assumption merely serves to simplify the computations, most of the definitions and theorems can be easily extended to Lie algebras over an arbitrary algebraically closed field...
of characteristic 0 .)

Let \( \Delta \) denote the set of non-zero roots of \( L \) with a fixed order defined on it. Denote by \( \Delta_0 \) the set of all roots, by \( \Delta_+ \) the positive roots, and by \( \Delta_{++} \) the simple roots (with respect to the ordering given on \( \Delta_0 \)). We shall now introduce a particular basis (called Chevalley basis) for \( L \), which will be used in a number of proofs in the rest of this thesis.

**Lemma 4.1.1.** It is possible to choose root vectors \( X_\alpha \in L_\alpha \) \( (\alpha \in \Delta) \) satisfying

(a) \( [X_\alpha, X_{-\alpha}] = H_\alpha \) (for the definition of \( H_\alpha \), cf. Theorem 1.6.4.)

(b) If \( \alpha, \beta, \alpha + \beta \in \Delta \), \( [X_\alpha, X_\beta] = c_{\alpha, \beta} X_{\alpha + \beta} \), then

\[ c_{\alpha, \beta} = -c_{-\alpha, -\beta}. \]

**Proof:** Humphreys, [8], p.144.

**Definition 4.1.1.** Let \( \Delta_{++} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \) be a system of simple roots of \( L \). Any basis

\[ \{X_\alpha | \alpha \in \Delta \} \cup \{H_i = H_{\alpha_i} \mid \alpha_i \in \Delta_{++}\} \]

for which the \( X_\alpha \) satisfy properties a) and b) of Lemma 4.1.1, is called a Chevalley basis for \( L \).

**Theorem 4.1.1.** Let \( \{X_\alpha, H_i \mid \alpha \in \Delta, 1 \leq i \leq l\} \) be a Chevalley basis for \( L \). Then the structure constants are integers.

More precisely:
(a) \([H_i, H_j] = 0, i \leq 1, j \leq \ell\)

(b) \([H_i, X_\alpha] = \frac{2 \langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle} X_\alpha, 1 \leq i \leq \ell, \alpha \in \Delta\)

(c) \([X_\alpha, X_{-\alpha}] = H_\alpha\) is an integral linear combination of \(H_1, H_2, \ldots, H_\ell\).

(d) If \(\alpha, \beta\) are independent roots, \(\beta = r\alpha, \ldots, \beta + qa\) the \(\alpha\)-string through \(\beta\), then \([X_\alpha, X_\beta] = 0\) if \(q = 0\) while \([X_\alpha, X_\beta] = \pm (r+1)X_{\alpha+\beta}\) iff \(\alpha + \beta \in \Delta\).


Remark: For convenience, we shall denote \(\alpha \in \Delta_+\), \(X_{-\alpha}\) by \(Y_\alpha\).

We shall now provide a method of constructing algebra homomorphisms \(\phi : C(L) \rightarrow C\). First we require some definitions:

Definition 4.1.2. A subset \(\Gamma\) of \(\Delta_0\) is said to be closed in \(\Delta_0\), if

(i) \(0 \in \Gamma\)

(ii) \(\alpha \in \Gamma\) implies \(-\alpha \in \Gamma\)

(iii) \(\alpha, \beta \in \Gamma\) and \(\alpha + \beta \in \Delta_0\) together imply \(\alpha + \beta \in \Gamma\).

Definition 4.1.3. A subset \(\Gamma\) of \(\Delta_0\) is called complete in \(\Delta_0\) if

(i) \(\Gamma\) is closed in \(\Delta_0\)

(ii) \(\alpha, \beta \in \Delta_+\) and \(\alpha + \beta \in \Gamma\) together imply \(\alpha, \beta \in \Gamma\)

(iii) \(\Delta_+ \cap \Gamma\) is a basis of \(\Gamma\).
Definition 4.1.4. Two complete subsets $\Gamma_1$ and $\Gamma_2$ of $\Delta_0$ are said to be disconnected, if

$$\alpha \in \Gamma_1, \beta \in \Gamma_2, \alpha, \beta \neq 0 \implies \alpha + \beta \not\in \Delta_0.$$  

Remark: It easily follows from the above definitions, that if $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ are pairwise disconnected complete subsets of $\Delta_0$, then $\bigcup_{i=1}^k \Gamma_i$ is again a complete subset of $\Delta_0$.

Let $\{Y_\beta, X_\beta, H_\alpha \mid \beta \in \Delta_+, \alpha \in \Delta_{++}\}$ be a Chevalley basis of $L$. By the Poincare-Birkhoff-Witt theorem, a linear basis of $U(L)$ is provided by the set of all monomials of the form

$$\bigotimes_{\beta \in \Delta_+} Y_{n(\beta)} \otimes_{\beta \in \Delta_+} X_{m(\beta)} \otimes_{\alpha \in \Delta_{++}} H_k(\alpha),$$

where the exponents $n(\beta), m(\beta), k(\alpha)$ are non-negative integers and the products preserve a fixed order.

Next, let $\xi = \sum_{\alpha \in \Delta_{++}} \ell(\alpha)\alpha$, where the $\ell(\alpha)$ are integers.

A linear basis for $U_\xi$ (cf. p.33) then consists of all monomials of the form $\bigotimes$ satisfying $\sum_{\beta \in \Delta_+} (m(\beta) - n(\beta))\beta = \xi$.

(This follows from the definition of $U_\xi$ and the commutation relations given in Theorem 4.2.1.)

In particular, a linear basis for $C(L)$ is provided by the set of all monomials of the form $\bigotimes$ satisfying
\[ \sum_{\beta \in \Delta_+} (m(\beta) - n(\beta)) \beta = 0. \]

**Lemma 4.1.2.** Let \( \Gamma \) be a complete subsystem of \( \Delta_0 \). Let \( u \) be a basis element of \( U(L) \) of the form \( \circledast \) (cf. p. 51), such that \( m(\beta) \neq 0 \) for at least one \( \beta \in \Delta_+ \setminus \Gamma \). Let \( v \) be any basis element of \( U(L) \) of the form \( \circledast \). Then \( vu \) is a linear combination of basis elements of \( U(L) \) of the form \( \circledast \), each containing at least one factor \( X_\gamma \) with \( \gamma \in \Delta_+ \setminus \Gamma \).

**Proof:** Let us call the property of containing a factor \( X_\gamma \) with \( \gamma \in \Delta_+ \setminus \Gamma \) "property \( P \)". We shall prove the Lemma by induction on the degree of \( u \) (as a monomial).

Let \( u = X_\gamma \) with \( \gamma \in \Delta_+ \setminus \Gamma \), and let \( v \) be any basis element of \( U(L) \) of the form

\[ v = \prod_{\beta \in \Delta_+ \setminus \Gamma} \gamma^n(\beta) \prod_{\beta \in \Delta_+} x^m(\beta) \prod_{\alpha \in \Delta_{++}} h^k(\alpha). \]

Then \( \nu \cdot u = (\prod_{\beta \in \Delta_+ \setminus \Gamma} \gamma^n(\beta) \prod_{\beta \in \Delta_+} x^m(\beta) \prod_{\alpha \in \Delta_{++}} h^k(\alpha)) \cdot X_\gamma \). Now, since

\[ [H_\alpha, X_\gamma] = \frac{2<\gamma, \alpha>}{<\gamma, \gamma>} X_\gamma \text{ (theorem 4.1.1b)}, \]

the factor \( X_\gamma \) can be commuted past the \( H_\alpha \)'s to obtain a linear combination of monomials each having property \( P \). In each term of that linear combination, \( X_\gamma \) can be put in its "proper" place, (i.e. commuted past some of the \( X_\beta \)). This results in a linear combination
of basis elements of the form \( \circ \), each having property \( P \). In fact, as \( \Gamma \) is complete, for any \( \beta \in \Delta_+ \), \( \beta \circ \gamma \not\in \Delta \) or \( \beta \circ \gamma \in \Delta_+ \setminus \Gamma \), and thus by Theorem 4.1.1 (d), \( v u \) is of the required form.

Next, assume that the result holds for any basis element of \( U(L) \) of degree \( \leq k \). Let \( u \) be a \( k+1 \)-degree basis element of \( U(L) \) with property \( P \), and let \( v \) be any basis element of \( U(L) \) of the form \( \circ \).

If \( u \) has at least one factor \( H_{\alpha} (\alpha \in \Delta_+) \) then
\[
u = u_{\perp} \cdot H_{\alpha}, \quad \text{where} \quad u_{\perp} \quad \text{is of degree} \quad k \quad \text{having property} \quad P, \quad \text{thus by the inductive assumption,} \quad \nu u = (v u_{\perp}) H_{\alpha} \quad \text{is a linear combination of basis elements of the form} \quad \circ \quad \text{each having property} \quad P. \quad \text{If} \quad u \quad \text{is of the form}
\[
\prod_{\beta \in \Delta_+} y_{n(\beta)}^n \prod_{\beta \in \Delta_+} x_{m(\beta)}^m,
\]
then \( u = u_{\perp} \cdot X_{\beta_0} \) where \( u_{\perp} \) is a basis element of \( U(L) \) of the form \( \circ \), and \( X_{\beta_0} \) is the first factor from the right in \( u \). Now, if \( \beta_0 \in \Delta_+ \setminus \Gamma \), we can apply the argument for first-degree monomials to show that \( \nu u = v (u_{\perp} \cdot X_{\beta_0}) = (v u_{\perp}) \cdot X_{\beta_0} \) is a linear combination of basis elements of \( U(L) \) each having property \( P \). If \( \beta_0 \in \Gamma \), then \( u_{\perp} \) has property \( P \) and is of degree \( k \), and hence, by the inductive assumption, \( v u_{\perp} \) is
a linear combination of basis elements each having property $P$. Then $vu = vu_1 \cdot x_{\beta_0}$ is also a linear combination of basis elements each having property $P$, (by Theorem 4.1.1 (b) and (d), and the fact that $\mathfrak{R}$ is complete.)

Thus, by induction, the statement holds for all basis elements $u$ with property $P$.

Corollary 4.1.1. Let $\Gamma$ be a complete subsystem of $\Delta_0$. Let $u$ be a basis element of $U(L)$ of the form $u$ with $n(\beta) \neq 0$ for at least one $\beta \in \Delta_+ \setminus \Gamma$. Then for any basis element $v \in U(L)$ of the form $u$, $u \cdot v$ is a linear combination of basis elements of the form $u'$ each having at least one factor $Y_{\gamma}$ with $\gamma \in \Delta_+ \setminus \Gamma$.

The proof is analogous to that of Lemma 4.1.2.

Let $\Gamma$ be a complete subsystem of $\Delta_0$. Denote by $C(\Gamma)$ the linear subspace of $C(L)$ generated by all basis elements of $C(L)$ for which $m(\beta) = n(\beta) = 0$ for all $\beta \in \Delta_+ \setminus \Gamma$, and let $\overline{C(\Gamma)}$ be the linear subspace of $C(L)$ generated by all basis elements of $C(L)$ not in $C(\Gamma)$. With this notation, we have:

Lemma 4.1.3. $\overline{C(\Gamma)}$ is a subalgebra of $C(L)$.

Proof: Since $\mathfrak{R}$ is closed, the commutant of any two
elements from the set 

\[ B(\Gamma) = \{ Y_\beta, X_\beta, H_\alpha \mid \beta \in \Delta_+ \cap \Gamma, \alpha \in \Delta_+ \} \]

is either zero or can be expressed as a linear combination of elements from \( B(\Gamma) \). Thus \( C(\Gamma) \) is closed under multiplication and hence is a subalgebra of \( C(L) \).

**Lemma 4.1.4.** \( \bar{C}(\Gamma) \) is a two-sided ideal of \( C(L) \).

**Proof:** By definition, any basis element \( u \) of \( \bar{C}(\Gamma) \) must contain at least one factor \( X_\beta \) with \( m(\beta) \neq 0 \) \((\beta \in \Delta_+ \setminus \Gamma)\)

and at least one factor \( Y_\gamma \) with \( n(\gamma) \neq 0 \) \((\gamma \in \Delta_+ \setminus \Gamma)\)

Thus, by Lemma 4.1.2 and Corollary 4.1.1, \( vu \) and \( uv \) both belong to \( \bar{C}(\Gamma) \) for any \( v \in C(L) \).

**Theorem 4.1.2.** Let \( \Gamma \) be a complete subsystem of \( \Delta_0 \) and let \( \phi : \bar{C}(\Gamma) \to C \) be an algebra homomorphism. Then \( \phi \) can be extended to an algebra homomorphism \( \bar{\phi} : C(L) \to C \).

**Proof:** Since \( C(L) = C(\Gamma) \oplus \bar{C}(\Gamma) \) (as linear spaces), and since \( \bar{C}(\Gamma) \) is a two-sided ideal of \( C(L) \), the extension \( \bar{\phi} \) of \( \phi \) defined by setting \( \bar{\phi} \) equal to zero on \( \bar{C}(\Gamma) \) is an algebra homomorphism.

**Remark:** The extension \( \bar{\phi} \) of \( \phi \) obtained in this way will be called the "trivial extension of \( \phi \)."

**Theorem 4.1.2** enables us to construct mass functions on \( C(L) \) by extending algebra homomorphisms on suitable sub-
algebras $C(\Gamma)$ of $C(L)$. The next theorem provides a
method of constructing mass functions of $C(L)$ by "combining"
 algebra homomorphisms on different subalgebras of $C(L)$.

**Theorem 4.1.3.** Let $\Gamma_1$ and $\Gamma_2$ be two disconnected
complete subsystems of $\Delta_0$: Let $\phi_1 : C(\Gamma_1) \to \mathbb{C}$ and
$\phi_2 : C(\Gamma_2) \to \mathbb{C}$ be algebra homomorphisms. If $\phi_1 = \phi_2$ on
$C(\{0\})$, then $\phi_1$ and $\phi_2$ admit a common extension to a
mass function on $C(L)$.

Proof: As we have noted before, $\Gamma_1 \cup \Gamma_2$ is again a
complete subsystem of $\Delta_0$. Thus, by Theorem 4.1.2, it
suffices to find a common extension of $\phi_1$ and $\phi_2$ to
$C(\Gamma_1 \cup \Gamma_2)$.

Since $\Gamma_1$ and $\Gamma_2$ are disconnected, we have, for all
$\beta_1 \in \Gamma_1 \cap \Delta_+ \text{ and } \beta_2 \in \Gamma_2 \cap \Delta_+$, that

$$[X_{\beta_1}, X_{\beta_2}] = [X_{\beta_1}, Y_{\beta_2}] = [Y_{\beta_1}, Y_{\beta_2}] = 0.$$ 

Thus any basis element of $C(\Gamma_1 \cup \Gamma_2)$ can be expressed as a
commuting product of basis elements $c_1 \in C(\Gamma_1)$ \text{ and }
$c_2 \in C(\Gamma_2)$. This representation is unique up to factors
from $C(\{0\})$. Since by assumption $\phi_1 = \phi_2$ on $C(\{0\})$, we
can define $\phi : C(\Gamma_1 \cup \Gamma_2) \to \mathbb{C}$ by setting for any basis
element $c \in C(\Gamma_1 \cup \Gamma_2)$, $\phi(c) = \phi_1(c_1) \cdot \phi_2(c_2)$ (where $c_1$
and $\omega_2$ are as above), and extending linearly to all of $C(\Gamma_1 \cup \Gamma_2)$. Then $\phi$ is an algebra homomorphism on $C(\Gamma_1 \cup \Gamma_2)$.

The following result will be useful later:

**Lemma 4.1.5.** Let $\Gamma$ be a complete subsystem of $\Delta_0$ and let $\phi : C(L) \to C$ be an algebra homomorphism such that $\phi = 0$ on $\overline{C}(\Gamma)$. Then, if $u$ is any basis element of $U(L)$ of the form $\bigotimes$ (cf. p. 51) for which there is a $\gamma \in \Delta_+ \setminus \Gamma$ with $m(\gamma) \neq 0$, then $u \in M_\phi$.

**Proof:** Let $u = \bigotimes_{\beta \in \Delta_+} Y_\beta \cdot \bigotimes_{\beta \in \Delta_+} X_\beta \cdot \bigotimes_{\alpha \in \Delta_+} H_\alpha^k$ with $m(\gamma) \neq 0$ for some $\gamma \in \Delta_+ \setminus \Gamma$, and let

$$\sum_{\beta \in \Delta_+} (m(\beta) - n(\beta)) \beta = \eta.$$ If $v \in U_{-\eta}$ is any basis element of $U(L)$ of the form $\bigotimes$, then by Lemma 4.1.2 $vu \in \overline{C}(\Gamma)$, as by assumption $\phi(vu) = 0$. This implies that $u \in M_\phi$.

§2. Dominated Irreducible Representations

**Definition 4.2.1.** A weight function $\lambda : H \to C$ of a representation $(\rho, V)$ of $U(L)$ is said to be a dominant weight function (with respect to the fixed ordering on $\Delta$), if, for each positive root $\beta$ of $L$, $\lambda + \beta$ is not a weight function. A representation admitting a dominant weight function is said to be dominated.
The following theorem establishes the most important properties of the weight spaces of dominated irreducible representations.

**Theorem 4.2.1.** Let \((\rho, V)\) be a dominated irreducible representation of \(U(L)\) with dominant weight function \(\lambda\). Then:

(i) each weight space is finite dimensional,

(ii) \(V_\lambda\) is one dimensional

(iii) \(V\) is the direct sum of its weight spaces

(iv) \(\lambda\) is the unique dominant weight of \((\rho, V)\)

(v) All weights of \((\rho, V)\) are of the form \(\lambda - \sum a_i \Delta_i\),

where the \(m_i\) are non-negative integers.

(vi) \((\rho, V)\) is finite dimensional if and only if \(\lambda(\Delta_i)\) is a non-negative integer for each simple root \(\Delta_i\) of \(L\).


**Remark:** By Theorem 4.2.1 a dominated irreducible representation of \(U(L)\) is pointed.

We shall now use the results of §1 to establish a one-to-one correspondence between dominated irreducible representations of \(U(L)\) and certain mass functions of \(C(L)\).

\(\Gamma = \{0\}\) is a complete subsystem of \(\Delta_0\). Since the Cartan subalgebra \(H\) of \(L\) is abelian, \(C(\{0\})\) is an abelian subalgebra of \(C(L)\). Thus the algebra homomorphisms
\( \phi : C(\{0\}) \to \mathbb{C} \) are in one-to-one correspondence with the linear functionals \( \lambda : H \to \mathbb{C} \).

For any linear functional \( \lambda \in H^* \), we shall denote the corresponding algebra homomorphism of \( C(\{0\}) \) also by \( \lambda \), and the trivial extension [cf. Theorem 4.1.2] of \( \lambda \) to all of \( C(L) \) by \( \overline{\lambda} \).

**Theorem 4.2.2.** Let \((\rho, V)\) be a dominated irreducible representation of \( U(L) \) with \( \lambda : H \to \mathbb{C} \) as dominant weight function. Then \((\rho, V)\) is equivalent to the left regular representation of \( U(L) \mod \mathbb{C} \), where \( \overline{\lambda} \) is the trivial extension of \( \lambda \).

**Proof:** By Theorem 4.2.1 (ii), the weight space \( V_\lambda \) is one-dimensional. Thus, by Theorem 3.1.1 the map \( \phi : C(L) \to \mathbb{C} \) defined by \( \rho(c)v_0 = \phi(c)v_0 \) (0 \# \( v_0 \in V_\lambda \)) is a mass function of \((\rho, V)\), and by Theorem 3.1.2, \((\rho, V)\) is equivalent to the left regular representation of \( U(L) \mod M_\phi \), where \( M_\phi \) is the unique maximal left ideal of \( U(L) \) containing \( \ker \phi \).

Take any basis element \( u \in \overline{C(\{0\})} \). Then \( u \) contains a factor \( X_\beta \) with \( m(\beta) \# 0 \), and we must have \( \rho(u)v_0 = 0 \) since otherwise \( \lambda + \beta \) would be a weight function, contradicting the fact that \( \lambda \) is a dominant weight of the representation.

Thus \( \phi \) must be identically zero on \( \overline{C(\{0\})} \) i.e. \( \phi = \overline{\lambda} \), and so \((\rho, V)\) is equivalent to the left regular representation of \( U(L) \mod \mathbb{C} \).
Theorem 4.2.3. Let \( \phi : C(L) \to C \) be an algebra homomorphism such that \( \phi(c) = 0 \) for all \( c \in C(\{0\}) \). Then there exists a dominated irreducible representation of \( U(L) \) admitting \( \lambda = \phi \cdot H \) as a dominant weight function.

Proof: By Theorem 3.1.2 the left regular representation of \( U(L) \) mod \( M_\phi \) is an irreducible representation of \( U(L) \) admitting \( \phi \) as a mass function. It remains to prove that \( \lambda = \phi \cdot H \) is a dominant weight function of this representation.

Let \( \beta \in \Delta_+ \) and \( u \in U_\beta \). We may assume that \( u \) is a basis element of \( U(L) \). Then \( u \) satisfies the conditions of Lemma 4.1.4 (with \( \Gamma = \{0\} \) as a complete subsystem of \( \Delta_0 \)) and hence \( u \in M_\phi \). This implies that \( \lambda + \beta \) is not a weight function of the left regular representation of \( U(L) \) mod \( M_\phi \).

Thus \( \lambda \) is a dominant weight function of this representation.

In order to obtain more information about the weight space structure of dominated irreducible representations of \( U(L) \), we need the following results:

Lemma 4.2.1. For any simple root \( \alpha \) of \( L \), the following identities hold in \( U(L) \):

1. \( X_{a-a}^{m_x^n} \) \( \prod_{j=1}^{m} [Y_{a} X_{\alpha} - (m+1-j)(m-j-H_{\alpha})] \)
2. \( Y_{a-a}^{m_x^n} \) \( \prod_{j=1}^{m} [Y_{a} X_{\alpha} - (m-j)(m-j+1-H_{\alpha})] \)
where $m$ is an integer $\geq 1$.


**Theorem 4.2.4.** Let $(\rho, V)$ be a dominated irreducible representation of $U(L)$, with dominant weight function $\lambda : H \to \mathbb{C}$. Then for any simple root $\alpha$ such that $\alpha(H_a) \neq 0$ a non-negative integer, the $\alpha$-chain through $\lambda$ is (singly) infinite. More precisely, $\lambda - m\alpha$ is a weight function of $(\rho, V)$ for every non-negative integer $m$.

Proof: Let $\overline{\lambda} : C(L) \to \mathbb{C}$ be the trivial extension of $\lambda$. Then for any simple root $\alpha$ of $L$, $\overline{\lambda}(Y_{\alpha}X_{\alpha}) = 0$, and if $\lambda(H_a) = \overline{\lambda}(H_a) \neq 0$ a non-negative integer, then by Lemma 4.2.1, $X_{\alpha}^mY_{\alpha}^m \equiv M$. Therefore (by Lemma 2.3.2) $\lambda - m\alpha$ is a weight function of $(\rho, V)$ for every integer $m \geq 1$.

§3. Complete Representations

Let $\Gamma_0$ denote an arbitrary subset of the simple roots of $L$. In this section we shall define and briefly study the "$\Gamma_0$-complete" representations of $U(L)$. This concept was originally introduced by Bouwer [2a] and it provided a generalization of the concept of a dominated irreducible representation.

**Lemma 4.3.1.** Let $\Gamma_0$ be a non-empty set of simple roots of $L$ and let $\Gamma_0^c$ be the closure of $\Gamma_0$ (in $\Delta_0$) with respect to addition
and subtraction. Then $\Gamma_c$ is a complete subsystem of the roots (called the completion of $\Gamma_0$).

Proof: Clearly, $\Gamma_c$ is closed and $\Gamma_0 \cap \Gamma_c$ is a basis for $\Gamma_c$. Thus if $\alpha = \sum a_i \epsilon_{\alpha_i}, \beta = \sum b_i \epsilon_{\alpha_i}, a_i \epsilon \Delta_{++}$ and $b_i \epsilon \Delta_{++}$ are two positive roots (i.e. $a_i > 0, b_i > 0$ for all $i$), and $\alpha + \beta \epsilon \Gamma_c$, then we must have $a_i = b_i = 0$ for all $\epsilon_{\alpha_i} \epsilon \Delta_{++} \setminus \Gamma_0$, that is, $\alpha, \beta \epsilon \Gamma_c$. Thus $\Gamma_c$ is complete.

Remark: The completion of $\Gamma_0$ is easily seen to be the smallest complete subsystem of $\Delta_0$ containing $\Gamma_0$. This interpretation allows us to define the completion of $\emptyset$ to be $\{0\}$.

Definition 4.3.1. Two subsets $\Gamma_0^1$ and $\Gamma_0^2$ of the simple roots are said to be disconnected, iff their corresponding completions $\Gamma_c^1$ and $\Gamma_c^2$ are disconnected. (cf. Definition 4.1.4.)

Definition 4.3.2. Let $\Gamma_0$ be a fixed subset of the simple roots of $L$, $\Gamma_c$ the completion of $\Gamma_0$. An irreducible representation $(\rho, V)$ of $U(L)$ is said to be complete with respect to $\Gamma_0$ (or simply $\Gamma_0$-complete), if $(\rho, V)$ admits a weight function $\lambda : H \rightarrow \mathbb{C}$ (called a characteristic weight function) such that
(1) \(\dim V_{\lambda} = 1\).

(11) \(\lambda = \sum a_i a_i\), where for each \(a_i \in \Gamma_0\), the real part of the corresponding coefficient \(a_i\) satisfies \(0 \leq \text{Re}(a_i) < 1\).

(iii) For each \(a_j \in \Gamma_0\), \(\lambda + m a_j\) occurs as a weight function of \((\rho, \mathfrak{V})\) for every integer \(m\).

(iv) For each positive root \(\beta \in \Delta_+ \setminus \Gamma_0\), \(\lambda + \beta\) is not a weight function of \((\rho, \mathfrak{V})\).

The cardinality of \(\Gamma_0\) is called the order of the representation \((\rho, \mathfrak{V})\).

Remarks: 1. Property (i) of Definition 4.3.1 implies that a \(\Gamma_0\)-complete representation is pointed.

2. As we have noted in the introduction, Bouwer [2], [2a] used the term "\(\Gamma_0\)-standard" for the representations given by Definition 4.3.1. We shall reserve that terminology for another class of representations. (cf. Chapter Five).

Property (iii) of Definition 4.3.1 can be expressed by saying that for every simple root \(a \in \Gamma_0\), the \(a\)-chain through \(\lambda\) is "doubly infinite". This was the motivation for choosing the term "\(\Gamma_0\)-complete".

Property (iii) also implies that, unless \(\Gamma_0 = \emptyset\), a \(\Gamma_0\)-complete representation is infinite dimensional.

3. If \(\Gamma_0 = \emptyset\), the set of \(\emptyset\)-complete representation coincides with the set of dominated irreducible representations.
4. Let \((\rho, V)\) be a \(\Gamma_0\)-complete representation with characteristic weight function \(\lambda : H \to \mathcal{C}\). Since \(\dim V_\lambda = 1\), \((\rho, V)\) admits a mass function \(\phi : C(L) \to \mathcal{C}\) given by \(\rho(c)v_0 = \phi(c)v_0\) (where \(0 \neq v_0 \in V_\lambda\)). The algebra homomorphism \(\phi\) must be identically zero on \(\mathcal{C}(\Gamma_c)\), since otherwise there would exist a \(\beta \in \Delta_+ \setminus \Gamma_c\) such that \(\lambda + \beta\) is a weight function of \((\rho, V)\).

**Theorem 4.3.1.** Let \((\rho, V)\) be a \(\Gamma_0\)-complete representation of \(U(L)\) with characteristic weight function \(\lambda : H \to \mathcal{C}\). Then every weight function of \((\rho, V)\) is of the form

\[
\lambda + \sum_{\alpha \in \Delta_+ \setminus \Gamma_0} p(\alpha)\alpha - \sum_{\alpha \in \Delta_+ \setminus \Gamma_0} r(\alpha)\alpha,
\]

where the coefficients \(p(\alpha)\) are integers and the coefficients \(r(\alpha)\) are non-negative integers.

**Proof:** Let \(\phi : C(L) \to \mathcal{C}\) be the algebra homomorphism given by \(\phi(c)v_0 = \rho(c)v_0\) (\(0 \neq v_0 \in V_\lambda\)). Then, \(\phi\) must be zero on \(\mathcal{C}(\Gamma_c)\). Let \(\eta = \lambda + \sum_{\alpha \in \Delta_+ \setminus \Gamma_0} \ell(\alpha)\alpha\) and \(\xi = \sum_{\alpha \in \Delta_+ \setminus \Gamma_0} \ell(\alpha)\alpha\) where the \(\ell(\alpha)\) are integers. Consider any basis element \(u \in U_\xi\). If \(\ell(\alpha) > 0\) for some \(\alpha \in \Delta_+ \setminus \Gamma_0\), then there must exist some \(\beta \in \Delta_+ \setminus \Gamma_c\) such that \(m(\beta) = 0\) in \(u\), and hence by Lemma 4.1.4, \(u \in M_\phi\). Thus in order for \(\eta\) to be a weight function of \((\rho, V)\), we must have that \(\ell(\alpha) \leq 0\) for all \(\alpha \in \Delta_+ \setminus \Gamma_0\) so \(\eta\) can be
written in the form \( \lambda + \sum_{\alpha \in \Gamma_0} p(\alpha)\alpha - \sum_{\alpha \in \Delta_++ \backslash \Gamma_0} r(\alpha)\alpha \) where the \( r(\alpha) \) are non-negative integers.

**Theorem 4.3.2.** The characteristic weight function of a \( \Gamma_0 \)-standard representation \((\rho, V)\) of \( U(L) \) is unique.

Proof: Assume that \( \lambda_1 : H \rightarrow C \) and \( \lambda_2 : H \rightarrow C \) are characteristic weight functions for \((\rho, V)\). By Theorem 4.3.1, \( \lambda_2 = \lambda_1 + \sum_{\alpha \in \Gamma_0} p(\alpha)\alpha - \sum_{\alpha \in \Delta_++ \backslash \Gamma_0} r(\alpha)\alpha \) where the \( p(\alpha) \) are integers and the \( r(\alpha) \) are non-negative integers. By condition (ii) of Definition 4.3.2, \( p(\alpha) = 0 \) for all \( \alpha \in \Gamma_0 \) and by condition (iv) of Definition 4.3.2, \( r(\alpha) = 0 \) for all \( \alpha \in \Delta_++ \backslash \Gamma_0 \).

Thus \( \lambda_2 = \lambda_1 \).

**Theorem 4.3.3.** Let \( \Gamma_0 \) be a subset of the simple roots of \( L \), \( \Gamma_c \) its completion. Let \( \phi : C(L) \rightarrow C \) be an algebra homomorphism such that

1. \( \phi = 0 \) on \( \overline{C}(\Gamma_c) \)
2. \( \phi \big|_{\Delta_+} = \sum_{\alpha \in \Gamma_0} a_1 \alpha_1 \) where, for each \( \alpha_1 \in \Gamma_0 \), \( 0 \leq \text{Re}(a_1) < 1 \)
3. For each \( \alpha_1 \in \Gamma_0 \), \( \phi(Y_{\alpha_1}X_{\alpha_1}) \neq r(r+1) + \phi(H_{\alpha_1}) \) for all integers \( r \).

Then the left regular representation of \( U(L) \) mod \( M_\phi \) is complete with respect to \( \Gamma_0 \).
Proof: By Lemma 3.1.2, the left regular representation of $U(L) \mod M_\phi$ is irreducible and admits $\lambda = \phi_+ H$ as an one-dimensional weight function. To show that $\lambda = \phi_+ H$ is a characteristic weight function of this representation, it remains to verify that conditions (iii) and (iv) of Definition 4.3.2 are satisfied.

By Lemma 4.2.1, we have, for any $a_i \in \Gamma_0$, that

$$x_{a_i}^m y_{a_i}^m = \prod_{j=1}^{m} \left[ \left( Y_{a_i} x_{a_i} - (m+1-j)(m-j-H_{a_i}) \right) \right].$$

Applying condition (3), we see that for any $a_i \in \Gamma_0$, $\phi(x_{a_i}^m y_{a_i}^m) \neq 0$ and hence

$$y_{a_i}^m \notin M_\phi,$$

thus $\lambda - ma_i$ is a weight function of the representation for all positive integers $m$.

Similarly, using the identity

$$y_{a_i}^m x_{a_i}^m = \prod_{j=1}^{m} \left[ \left( Y_{a_i} x_{a_i} - (m-j)(m-j+1+H_{a_i}) \right) \right] (a_i \in \Gamma_0)$$

and condition (3), we obtain that $x_{a_i}^m \notin M_\phi$, hence

$$\lambda + ma_i$$

is a weight function for all positive integers $m$.

Thus condition (iii) of Definition 4.3.2 is satisfied.

Finally, let $\beta \in A_+ \setminus \Gamma_c$, and let $u \in U_\phi$ be a basis element. Then, since $\phi = 0$ on $\overline{C}(\Gamma_c)$, the conditions of Lemma 4.3.4 are satisfied, and so $u \in M_\phi$. Thus $\lambda + \beta$ is not a weight function of the representation.
Remark: We shall call an algebra homomorphism \( \phi : C(L) \to \mathbb{C} \) satisfying the conditions of Theorem 4.3.3 \( \Gamma_0 \)-admissible.

If \( \Gamma_0 = \{\alpha\} \) where \( \alpha \) is any simple root of \( L \), then 
\[
C(\Gamma_c) \text{ is generated by the set } B = \{1, Y_{\alpha}X_{\alpha}, H_\beta(\beta \in \Delta_{++}) \}
\]
and is clearly an abelian subalgebra of \( C(L) \). Therefore if we set \( \phi(1) = 1, \phi(H_\beta) = \lambda(H_\beta) \) for all \( \beta \in \Delta_{++} \) and 
\[
\phi(Y_{\alpha}X_{\alpha}) = p \neq r(r+1+\lambda(H_\alpha)),
\]
then \( \phi \) can be uniquely extended to an algebra homomorphism \( \phi : C(\Gamma_c) \to \mathbb{C} \), which can then be trivially extended (cf. Theorem 4.1.2) to an algebra homomorphism \( \bar{\phi} : C(L) \to \mathbb{C} \). We can always define linear functionals \( \lambda : H \to \mathbb{C} \) satisfying condition (2) of Theorem 4.3.3 (with \( \Gamma_0 = \{\alpha\} \)), and the extension \( \bar{\phi} : C(L) \to \mathbb{C} \) obtained above from such linear functionals will be \( \Gamma_0 \)-admissible. Thus Theorem 4.3.3 establishes the existence of complete representations of order 1. Theorem 4.1.3 allows us to construct complete representations of higher order by extending admissible homomorphisms defined for disconnected subsets of the simple roots.

Corollary 4.3.1. Let \((\rho, V)\) be a \( \Gamma_0 \)-complete representation. Then \((\rho, V)\) admits at most one \( \Gamma_0 \)-admissible mass function.

Proof: Assume that \( \phi_1 : C(L) \to \mathbb{C} \) and \( \phi_2 : C(L) \to \mathbb{C} \) are \( \Gamma_0 \)-admissible mass functions of \((\rho, V)\). Then 
\[
\phi_1^{\downarrow H} \quad \text{and} \quad \phi_2^{\downarrow H}
\]
are both characteristic weight functions of
and hence by Theorem 4.3.2: \( \phi^1_H = \phi^2_H \). Then by Theorem 3.2.2 \( \phi_1 = \phi_2 \).

We conclude this chapter by constructing an algebra homomorphism such that the associated pointed representation is not complete with respect to any non-empty subset of the simple roots.

Let \( L \) be the simple Lie algebra \( A_3 \) with simple roots \( \Lambda = \{\alpha, \beta, \gamma\} \). Let \( r_0^{(1)} = \{\alpha\} \), \( r_0^{(2)} = \{\gamma\} \). Then \( r_0 = r_0^{(1)} \cup r_0^{(2)} \) is a disconnected system of simple roots.

Its completion is \( \Gamma_c = \{0, \pm\alpha, \pm\gamma\} \). \( C(\Gamma_c) \) is generated by \( \{1, H_\alpha, H_\beta, H_\gamma, Y_\alpha X_\alpha, Y_\gamma X_\gamma\} \). Define a map \( \phi: C(\Gamma_c) \to \mathbb{C} \) by

\[
\phi(1) = 1 \\
\phi(H_\alpha) = \lambda_1, \quad \phi(H_\beta) = \lambda_2, \quad \phi(H_\gamma) = \lambda_3, \quad \text{where the} \\
\lambda_i \quad (i=1,2,3) \quad \text{are arbitrary complex numbers (not integers).} \\
\phi(Y_\alpha X_\alpha) = 5(6 + \lambda_1) \\
\phi(Y_\gamma X_\gamma) = -3(-2 + \lambda_3).
\]

Then, by Theorem 4.1.3., \( \phi \) can be trivially extended to an algebra homomorphism \( \bar{\phi}: C(A_3) \to \mathbb{C} \). Consider the left regular representation of \( U(A_3) \) modulo \( M_\lambda \). \( \lambda = \phi^H \) is a one-dimensional weight function of this representation. Using the identities of Lemma 4.2.1, it is easily seen that
\( \lambda + m \alpha \) (for \( m \leq 5 \)) and \( \lambda + n \gamma \) (for \( n \geq -2 \)) occur as weight functions. However, \( \lambda + 6 \alpha \) and \( \lambda - 3 \gamma \) do not occur as weight functions by Lemma 4.2.1 and Theorem 4.1.4.

Since equivalent representations have identical weight spaces, the pointed representation constructed above is not equivalent to any complete representation.

In Chapter six, we construct all algebra homomorphisms \( \phi : C(A_3) \rightarrow \mathbb{C} \) and show that, in a sense, the algebra homomorphisms yielding complete representations characterize all pointed representations of \( A_3 \).
Chapter Five

Standard Homomorphisms

The example at the end of the previous chapter shows that the class of pointed representations of a simple Lie algebra \( L \) properly includes the family of complete representations. In this chapter, we shall introduce a new family of algebra homomorphisms (called "standard" homomorphisms) which yield pointed representations of \( L \).

In §1, we briefly describe the structure of the simple Lie algebra \( A_n \). In §2, we construct a family of representations for \( A_n \), and using the results of §1, Chapter four, we construct algebra homomorphisms (called "generalized standard homomorphisms") \( \phi : C(L) \to 0 \) for arbitrary simple Lie algebras \( L \). Based on our study of the pointed representations of \( A_1, A_2 \) and \( A_3 \) (to be presented in detail in the next chapter), we conjecture that the pointed representations of a simple Lie algebra \( L \) can be "labelled" by a subfamily of the generalized standard algebra homomorphisms.

In §3, we examine the weight space structure of the pointed representations obtained from standard homomorphisms, and in §4 we prove that our conjecture is true for every Lie algebra \( L \) such that the mass functions corresponding to the pointed representations of \( L \) are weakly equivalent to \( g \)-standard algebra homomorphisms.
§1. The Lie Algebra $A_n$

The Lie algebra $A_n$ consists of all $(n+1) \times (n+1)$ complex matrices of trace zero. With the usual matrix addition and bracket product the diagonal matrices in $A_n$ form a Cartan subalgebra $H$. Let $w_i$ denote the projection of an $(n+1) \times (n+1)$ matrix onto its $(i,i)$-th component. With this notation, we have

Lemma 5.1.1. The set of (non-zero) roots of $A_n$ with respect to the Cartan subalgebra $H$ is \{ $w_i - w_j | (i \neq j) \}$, $i,j = 1,2, \ldots , n+1$.

A simple set of roots $\Delta_+^+$ is given by

\[
\{ e_i - e_{i+1} | i = 1,2, \ldots , n \}
\]

and the set of positive roots is $\Delta_+ = \{ w_i - w_j | 1 \leq i < j \leq n+1 \}$.


For each $i,j = 1,2, \ldots , n$, let $e_{ij}$ denote an $(n+1) \times (n+1)$ matrix with 1 in the $(i,j)$-th position and zeroes elsewhere. Let

$H_{a_i} = e_i \cdot i - e_{i+1} \cdot i+1$ \hspace{1cm} (i=1,2, \ldots , n)

$X_{a_i} + \ldots + a_j = e_{i,j} + 1$ \hspace{1cm} (1 \leq i \leq j \leq n)

$X_{a_i} + \ldots + a_j = e_{j+1,i}$ \hspace{1cm} (1 \leq i \leq j \leq n)

Then the elements $H_{a_i}$ (i=1,2, \ldots , n) form a basis for the Cartan subalgebra $H$, and the set

\[
\{ Y_{a_i} + \ldots + a_j, X_{a_i} + \ldots + a_j | 1 \leq i \leq j \leq n \} \cup H_{a_i} \ (i=1,2, \ldots , n)
\]

is
a Chevalley basis for $A_n$. Moreover, for each 
$\xi = w_i - w_j \in \Delta_+$, the element $x_{\xi} = e^1_j$ belongs to the 
$\xi$-root space, and $y_{\xi} = e^1_j \in \Delta_-$ belongs to the $-\xi$ root space of 
$A_n$ with respect to $\mathfrak{h}$.

The following commutation relations describe the bracket product in $A_n$:

$[H_{\alpha_i}, H_{\alpha_j}] = 0$ for $i, j = 1, 2, \ldots, n$

$[H_{\alpha_i}, X_{\alpha}] = a(H_{\alpha_i})X_{\alpha}$ for $i = 1, \ldots, n$ and $a = w_i - w_j \in \Delta$

$[X_{\alpha}, Y_{\beta}] = H_{\alpha_i} + H_{\alpha_{i+1}} + \ldots + H_{\alpha_{j-1}}$ for $a = w_i - w_j \in \Delta_+$

$[X_{\alpha}, X_{\beta}] = (\delta_{jk} - \delta_{kl})X_{\alpha + \beta}$ for $a + b \in \Delta_+$

$[Y_{\alpha}, Y_{\beta}] = (\delta_{kl} - \delta_{jk})Y_{\alpha + \beta}$

$[X_{\alpha}, Y_{\beta}] = (\delta_{jk} - \delta_{kl})X_{\alpha - \beta}$ if $a - \beta \in \Delta_+$

$= (\delta_{jk} - \delta_{kl})Y_{\beta - \alpha}$ if $\beta - a \in \Delta_+$

§2. Standard Homomorphisms

In this section, we present Lemire's construction (cf. [10a]) of a large family of (not necessarily irreducible)
representations of $A_n$ having a one-dimensional weight space.

These representations will, in turn, be used to define algebra homomorphisms $\phi : C(L) \rightarrow C$ yielding pointed representations of $L$ for arbitrary simple Lie algebras.
Theorem 5.2.1. (Notation as in §1.)

Let $V$ be a vector space over $\mathbb{C}$ with basis $\{v(k) \mid k = (k_1, \ldots, k_n) \text{ where each } k_i \text{ is an integer}\}$. Let $s$ be an arbitrary complex parameter, and let $\lambda \in H^*$ be a linear functional. Define

$$\rho(h) v(k) = \lambda(h) - \sum_{i=1}^{n} k_i v(k)$$

$$\rho(x_i) v(k) = (s - \lambda(h) - \sum_{i=1}^{n} k_i) v(k)$$

$$\rho(x_j) v(k) = (s - \lambda(h) - \sum_{i=1}^{n} k_i) v(k)$$

where $k = (k_1, \ldots, k_n)$ is the $n$-tuple having $1$ in the $i, i+1, \ldots, j-1$ positions and zeroes elsewhere, and, by convention, $h_0 = 0 = k_0 = k_{n+1}$.

The linear extension of $\rho$ to all of $\mathfrak{a}_n$ is a representation on the vector space $V$. Moreover, each weight space of this representation is one-dimensional.

Proof: The fact that $\rho$ preserves the commutation relations given in §1, can be verified by direct computation. Thus the extension of $\rho$ to $\mathfrak{a}_n$ is a Lie-algebra homomorphism.

Now consider a vector $v(k) \in V$ (where $k = (k_1, \ldots, k_n)$). It is easily seen (by the first set of relations defining $\rho$) that $v(k)$ belongs to the $\lambda + \sum_{i=1}^{n} k_i a_i$ weight space. In particular, $v(0)$ is in $V_{\lambda}$. 
Thus, by Theorem 3.2.1, all weights of the representation are of the form \( \lambda + \sum_{i=1}^{n} \lambda_{i} \) where the \( \lambda_{i} \) are integers.

Since the vectors \( v(k) \) form a basis of \( V \) and each basis vector belongs to a different weight space, each weight space must be one dimensional.

**Remark on notation:** For each complex scalar \( s \) and each linear functional \( \lambda \in H^{*} \), the representation \((\rho, V)\) of \( A_{n} \) defined in Theorem 5.2.1 will be denoted by \((\rho, V_{s}, \lambda)\).

**Corollary 5.2.1.** Let \((\rho, V_{s}, \lambda)\) be a representation of \( A_{n} \) defined in Theorem 5.2.1. Then the map \( \phi: C(A_{n}) \to C \) given by \( \phi(c)v(\overline{0}) = \rho(c)v(\overline{0}) \) \((c \in C(A_{n}))\) is an algebra homomorphism.

**Proof:** The vector \( v(\overline{0}) \) belongs to the \( \lambda \)-weight space of \((\rho, V_{s}, \lambda)\). Since this weight space is one dimensional, the map \( \phi \) defined above is an algebra homomorphism by Theorem 3.1.1.

**Definition 5.2.1.** An algebra homomorphism \( \phi: C(A_{n}) \to C \) is called **standard** if there is a representation \((\rho, V_{s}, \lambda)\) of \( A_{n} \) such that \( \phi(c)v(\overline{0}) = \rho(c)v(\overline{0}) \) for all \( c \in C(A_{n}) \).

**Definition 5.2.2.** Let \( \phi: C(A_{n}) \to C \) be a standard algebra homomorphism arising from the representation \((\rho, V_{s}, \lambda)\). If 
\[ s \in \bigcup_{i=0}^{n} \left( \mathbb{Z} + \lambda(H_{a_{1}} + \ldots + H_{a_{i}}) \right) \] and for all \( i, 0 \leq \Re a_{i} < 1 \),
where \( \lambda = \prod_{i=1}^{n} a_i a_i \), then \( \phi \) is called complete.

**Lemma 5.2.1.** Let \( \phi: C(A_n) \to C \) be a complete standard algebra homomorphism. Let \( \Gamma_0 \) denote the set of all simple roots of \( A_n \). Then \( \phi \) is \( \Gamma_0 \)-admissible.

**Proof:** We must show that, for each \( a_i \in \Gamma_0 \),
\[
\phi(Y_{a_i} X_{a_i}) = r(r+1+\lambda(H_{a_i})) \text{ for any integer } r.
\]
By definition
\[
\phi(Y_{a_i} X_{a_i})v(\overline{0}) = \rho(Y_{a_i} X_{a_i})v(\overline{0})
\]
\[
= \rho(Y_{a_i}) (s - \lambda(H_{a_i} + \ldots + H_{a_i_{i-1}}) - 0)v(\overline{0})
\]
\[
= (s - \lambda(H_{a_i} + \ldots + H_{a_i_{i-1}}))\rho(Y_{a_i})v(\overline{0}), \text{ where } \overline{0}
\]
is the \( n \)-tuple having \( 1 \) in the \( i \)-th position and zeroes elsewhere.

Thus
\[
\phi(Y_{a_i} X_{a_i})v(\overline{0}) = (s - \lambda(H_{a_i} + \ldots + H_{a_i_{i-1}}))\rho(Y_{a_i})v(\overline{0}) =
\]
\[
= (s - \lambda(H_{a_i} + \ldots + H_{a_i_{i-1}}))(s - \lambda(H_{a_i} + \ldots + H_{a_i_{i}}) - 1)v(\overline{0}).
\]

Now assume that \( \phi(Y_{a_i} X_{a_i}) = r(r+1+H_{a_i}) \) for some integer \( r \).

Solving the equation
\[
(s - \lambda(H_{a_i} + \ldots + H_{a_i_{i-1}}))(s - \lambda(H_{a_i} + \ldots + H_{a_i_{i}}) - 1) =
\]
\[
= r(r+1+H_{a_i}) \text{ for } s, \text{ we obtain the following two solutions:}
\]
\[
s_1 = \lambda(H_{a_i} + \ldots + H_{a_i_{i}}) + 1 + r \quad \text{and}
\]

s_2 = \lambda (H_{a_1} + \ldots + H_{a_{i-1}}) - r.

Both of these contradict the assumption on s.

Thus, φ is Γ_0-admissible.

Remark: The above Lemma, together with Theorem 4.3.3, shows that if φ: C(A_n) → C is a complete standard algebra homomorphism, the left regular representation of U(A_n) mod M_φ is complete of order n.

Now let L be an arbitrary simple Lie algebra. Let r_0(1) be subsystems of simple roots of L isomorphic to disconnected systems of simple roots of A_{n_1} for some integers n_1. Let r_c(1) denote the completion of r_0(1) for each i.

Then (cf. Humphreys, [8], p. 75, Theorem 14.2) the subalgebra U(r_c(1)) of U(L) generated by the elements

\{1, Y_\beta, X_\beta, H_\alpha \mid \beta \in \Delta_+ \cap r_c(1), \alpha \in \Delta_+ \cap r_c(1)\}

is isomorphic to U(A_{n_1}). Moreover, C(L) \cap U(r_c(1)) is

isomorphic to C(A_{n_1}).

Recall that C(r_c(1)) is the linear subspace (actually a subalgebra of C(L) generated by all basis elements of C(L) (cf. p.51) for which m(\beta) = n(\beta) = 0 for all \beta \in \Delta_+ \setminus r_c(1).

It then follows (by the form of the generators of U(r_c(1)) that
\[ C(r^{(1)}_c) = \{ C(L) \cap U(r^{(1)}_c) \} \cdot U(H). \]

Thus, if \( \phi : C(A_{n_1}) \rightarrow C \) is an algebra homomorphism, then \( \phi \) can be extended to an algebra homomorphism \( \overline{\phi} : C(r^{(1)}_c) \rightarrow C \) by identifying \( C(L) \cap U(r^{(1)}_c) \) with \( C(A_{n_1}) \) and defining \( \phi(H_a) \) arbitrarily for \( a \in A_{++} \setminus r^{(1)}_c \).

Now let \( \phi_i : C(A_{n_1}) \rightarrow C \) be algebra homomorphisms. Construct algebra homomorphisms \( \overline{\phi}_i : C(r^{(1)}_c) \rightarrow C \) as above, setting \( \overline{\phi}_i U(H) = \overline{\phi}_j U(H) \) for all \( i, j \). Then since the \( r^{(1)}_c \) are pairwise disconnected, by Theorem 4.1.3 (extended to a finite number of subalgebras \( C(r^{(1)}_c) \)) the \( \overline{\phi}_i \) admit a common extension to an algebra homomorphism \( \overline{\phi} : C(L) \rightarrow C \) satisfying

1) \( \overline{\phi} = \overline{\phi}_i \) on \( C(r^{(1)}_{c_i}) \) for each \( i \), and
2) \( \phi = 0 \) on \( C(\bigcup_{i} r^{(1)}_c) \).

**Definition 5.2.3**: If \( \overline{\phi} : C(L) \rightarrow C \) is an algebra homomorphism constructed as above from standard algebra homomorphisms \( \phi_i : C(A_{n_1}) \rightarrow C \), then \( \overline{\phi} \) is called generalized standard (or \( g \)-standard) with respect to \( \bigcup_{i} r^{(1)}_c \). (We shall also call the corresponding irreducible representation \( U(L)/M^-_{\phi} \) \( g \)-standard).
Remark: We shall show in Chapter Six, that every algebra homomorphism \( \phi: C(A_n) \to \mathbb{C} \) (for \( n = 1,2,3 \)) is weakly equivalent to a \( g \)-standard algebra homomorphism.

Recall (Theorem 4.2.2) that corresponding to any linear functional \( \lambda: H^* \to \mathbb{C} \), there is a unique (up to equivalence) dominated representation of \( L \) with \( \lambda \) as dominant weight function.

The correspondence between algebra homomorphisms \( \phi: C(L) \to \mathbb{C} \) and pointed representations is not one-to-one. We now wish to specify a subfamily \( \hat{\mathcal{F}}_L \) of the set \( \mathcal{F}_L \) of all algebra homomorphisms \( \phi: C(L) \to \mathbb{C} \), such that members of \( \hat{\mathcal{F}}_L \) "label" the pointed representations of \( L \) in the following sense:

1) If \( \phi_1 \) and \( \phi_2 \) are distinct elements of \( \hat{\mathcal{F}}_L \), then \( U(L)/M_{\phi_1} \) and \( U(L)/M_{\phi_2} \) are not isomorphic as \( L \)-modules.

2) If \((\rho, V) \in \mathcal{F}_L \), then there exist \( \phi \in \hat{\mathcal{F}}_L \) and \( \sigma \in \text{Aut}(L; H) \) such that \( V \cong U(L)/M_{\phi \sigma} \).

Definition 5.2.4. A generalized standard algebra homomorphism \( \phi: C(L) \to \mathbb{C} \) defined relative to \( \bigcup_{i=1}^{g} r_{0}^{(i)} \) is said to be \textbf{extreme} if the restriction \( \phi|_{C(L) \cap U(r_{0}^{(i)})} \) is complete for each \( i \).

Lemma 5.2.2. Let \( \phi: C(L) \to \mathbb{C} \) be an extreme \( g \)-standard algebra homomorphism defined with respect to \( r_{0} = \bigcup_{i=1}^{g} r_{0}^{(i)} \).
Then $\phi$ is $\Gamma_0$-admissible and hence the left regular representation of $U(L) \mod M_\phi$ is $\Gamma_0$-complete with $\lambda = \phi_H$ as characteristic weight function.

Proof: Apply the proof of Lemma 5.2.1 to each restriction $\phi |_{\{C(L) \cap U(\Gamma_c^{(1)})\}}$. Then the result follows from the definition of $\Gamma_0$-admissible homomorphisms (using also the fact that the $\Gamma_c^{(1)}$ are disconnected), and Theorem 4.3.3.

Theorem 5.2.3. Let $\phi_1, \phi_2 \in F_L$ be distinct extreme $g$-standard algebra homomorphisms. Then $U(L)/M_{\phi_1}$ and $U(L)/M_{\phi_2}$ are not isomorphic as $L$-modules.

Proof: Let $\phi_1$ be extreme $g$-standard with respect to $\Gamma_{\phi_1}$, $\phi_2$ extreme $g$-standard with respect to $\Gamma_{\phi_2}$, and assume that $U(L)/M_{\phi_1} \cong U(L)/M_{\phi_2}$. We first show that $\Gamma_{\phi_1} = \Gamma_{\phi_2}$.

In fact, by Lemma 5.2.2 $U(L)/M_{\phi_1}$ is $\Gamma_{\phi_1}$-complete with $\phi_{\phi_1 H}$ as characteristic weight function ($i=1,2$). Thus, if $\alpha \in \Gamma_{\phi_1} \setminus \Gamma_{\phi_2}$ then $\phi_{\phi_1 H} + \alpha$ is a weight function of $U(L)/M_{\phi_1}$ but not of $U(L)/M_{\phi_2}$ (by Definition 4.3.2).
This, however contradicts the assumption that
\[ U/M_{\phi_1} \cong U/M_{\phi_2} \] as \( L \)-modules.

Thus \( U/M_{\phi_1} \) and \( U/M_{\phi_2} \) are both complete with respect to the same set of simple roots. By Theorem 4.3.2, we must have \( \phi_1^H = \phi_2^H \), and since \( U/M_{\phi_1} \cong U/M_{\phi_2} \), \( \phi_1 = \phi_2 \) (by Theorem 3.2.3).

Remark: The above theorem shows that the family of extreme g-standard algebra homomorphisms fulfill the first requirement (cf. p. 78) of being satisfactory "labels" for the pointed representations of \( L \). Thus, we propose the following:

Conjecture: The extreme g-standard algebra homomorphisms "label" the pointed representations of \( L \).

In the next section we show that, provided the mass functions associated with the pointed representations of \( L \) are all weakly equivalent to g-standard algebra homomorphisms, the conjecture is true.

§3. Boundary weights of g-standard representations

Definition 5.3.1. Let \( \phi : C(L) \to \mathcal{C} \) be a g-standard algebra homomorphism defined with respect to \( \Gamma_0 = \bigcup_{i=1}^{\infty} \Gamma_0^{(i)} \). A function of the form \( \phi^H + \sum_{\alpha \in \Gamma_0} \ell_{\alpha} \alpha \) (where the
coefficients \( l_\alpha \) are integers) is called a boundary function of the representation \( U/M_\phi \). If a boundary function occurs as a weight function of \( U/M_\phi \), it is called a boundary weight function.

Remark: Lemire has shown in his thesis [10] that if \( \Gamma_0 \) is totally disconnected (i.e. \( \alpha \in \Gamma_0, \beta \in \Gamma_0 \) implies \( \alpha + \beta \notin \Delta \)) and \( \phi: C(L) \to C \) is a \( \Gamma_0 \)-admissible algebra homomorphism, then all boundary functions occur as weight functions of \( U/M_\phi \) and the associated weight spaces are one dimensional. He also conjectured that this is true for arbitrary \( \Gamma_0 \)-complete representations. We shall show that his conjecture is true for all \( \Gamma_0 \)-complete representations admitting an extreme g-standard mass function.

We first need the following:

**Lemma 5.3.1.** Let \( \phi_1: C(L) \to C \) and \( \phi_2: C(L) \to C \) be algebra homomorphisms. \( U(L)/M_{\phi_1} \cong U(L)/M_{\phi_2} \) if and only if for \( \xi = (\phi_1 - \phi_2),_H \) there exists \( u_0 \in U_\xi \setminus M_{\phi_2} \) such that for all \( c \in C(L) \) and all \( w \in U_\xi \), \( \phi_1(c)\phi_2(wu_0) = \phi_2(wcu_0) \).

**Proof:** Since \( U(L)/M_{\phi_1} \cong U(L)/M_{\phi_2} \), there exists an \( L \)-module homomorphism \( \psi: U(L)/M_{\phi_1} \to U(L)/M_{\phi_2} \). If \( \psi(1 + M_{\phi_1}) = u_0 + M_{\phi_2} \) then \( u_0 \in U_\xi \setminus M_{\phi_2} \). Moreover, for all
Let $c \in C(L)$ and $w_0 \in U_{-\xi}$, we have
\[
\psi(w_0c + M_{\phi_1}) = \psi(w_0\phi_1(c)(1 + M_{\phi_1})) = \psi(\phi_1(c)(w_0 + M_{\phi_1}) = \phi_1(c)\psi(w_0 + M_{\phi_1}) = \phi_1(c)(w_0u_0 + M_{\phi_2}) = \phi_1(c)\phi_2(w_0u_0)(1 + M_{\phi_2})
\]
But also $\psi(w_0c + M_{\phi_1}) = w_0cu_0 + M_{\phi_2} = \phi_2(w_0u_0)(1 + M_{\phi_2})$.

Thus, $\phi_1(c)\phi_2(w_0u_0) = \phi_2(w_0cu_0)$.

Conversely, assume that there exists $u_0 \in U_{\xi} \setminus M_{\phi_2}$ such that $\phi_1(c)\phi_2(wu_0) = \phi_2(wcu_0)$ for all $w \in U_{-\xi}$ and $c \in C(L)$.
Let $M = \{ u \in U(L) \mid u \cdot u_0 \in M_{\phi_2} \}$. Then $M$ is a maximal left ideal of $U(L)$ and $U(L)/M \cong U(L)/M_{\phi_2}$. Moreover, if $c \in \ker \phi_1$, i.e. $\phi_1(c) = 0$ then $\phi_1(c)\phi_2(wu_0) = \phi_2(wcu_0) = 0$ for all $w \in U_{-\xi}$, so $cu_0 \in M_{\phi_2}$, hence $c \in M$.

This implies that $U(L)/M_{\phi_1} = U(L)/M \cong U(L)/M_{\phi_2}$.

Next, we require some notation:

Recall (Chapter I, §8), that for any simple root $\alpha \in \Delta_+$, the map $S_\alpha : H^* \rightarrow H^*$ given by $S_\alpha(\xi) = \xi - \frac{2\langle \xi, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$ is an automorphism of the root system. Thus [cf. Chapter I, §9] the maps $S_\alpha (\alpha \in \Delta_+)$ induce an automorphism (again denoted by $S_\alpha$) of $H$. For each $S_\alpha$, there exists a unique automorphism, (denoted by $S_{\alpha}^L$) of $L$ extending $S_{\alpha} : H \rightarrow H$ and such that
σα(Χβ) ∈ Lσα(β) for all β ∈ Δ++. With this notation, we have:

**Lemma 5.3.2.** Let \( \phi: C(A_n) \to \mathfrak{g} \) be a standard algebra homomorphism corresponding to the representation \( V_{s,\lambda} \). Then

(i) \( \phi \circ \sigma_{a_1} \) is standard parametrized by \( s' = s - \lambda(H_{a_1}) \) and \( \lambda' = \lambda \circ \sigma_{a_1} \).

(ii) \( \phi \circ \sigma_{a_i} \) (\( i = 2, 3, \ldots, n \)) is standard, parametrized by \( s' = s \) and \( \lambda' = \lambda \circ \sigma_{a_1} \).

(iii) If \( \xi = \sum_{i=1}^{n} l_1 \alpha_i \) (where the \( l_1 \) are integers) and \((\phi + \xi)^H \) is a one-dimensional weight function of \( U/M_\phi \), then the algebra homomorphism \( \phi': C(A_n) \to \mathfrak{g} \) associated with \( (\phi + \xi)^H \) is also standard, parametrized by \( s' = s + l_1 \) and \( \lambda' = (\phi + \xi)^H \).

**Proof:**

1. Consider the representation \((\rho', V_{s'}, \lambda')\) with \( s' = s - \lambda(H_{a_1}) \), \( \lambda' = \lambda \circ \sigma_{a_1} \) and \( \rho' \) defined as in §2, where the underlying vector space \( V \) is the same as the one defining \( \phi \). Then it is easily checked (by direct computation) that for any \( c \in C(A_n) \), \( \rho \circ \sigma_{a_1}(c) = \rho'(c) \) and \((\rho', V_{s'}, \lambda') \cong (\rho, V_{s}, \lambda)\) where the equivalence map is
the identity. Thus for any \( c \in C(A_n) \), \((\phi \circ \sigma_{\lambda_1})(c)v(\overline{0}) = \rho \circ \sigma_{\lambda_1}(c)v(\overline{0}) = \rho'(c)v(\overline{0})\), that is, the algebra homomorphism \( \phi \circ \sigma_{\lambda_1} \) is standard, associated with the representation \((\rho', V_{s'}, \lambda')\).

(ii) Consider the representation \((\rho', V_{s'}, \lambda')\) with \( s' = s \), \( \lambda' = \lambda \circ \sigma_{\alpha_1} (i > 2) \). Then \((\rho \circ \sigma_{\alpha_1}, V_s, \lambda) \cong (\rho', V_{s'}, \lambda')\) where the equivalence map is the identity. Thus, for any \( c \in C(A_n) \), \( \phi \circ \sigma_{\alpha_1}(c)v(\overline{0}) = \rho \circ \sigma_{\alpha_1}(c)v(\overline{0}) = \rho'(c)v(\overline{0}) \), i.e. \( \phi \circ \sigma_{\alpha_1} \) is standard, parametrized by \( s' \) and \( \lambda' \).

(iii) Using again the explicit description of \( \rho \), we observe that the representations \((\rho, V_s, \lambda)\) and \((\rho', V_{s'}, \lambda')\) (where \( s' = s + \alpha_1 \) and \( \lambda' = \lambda + \xi = \lambda + \sum_{i=1}^{n} \alpha_1 \)) are equivalent, and the equivalence is given by the map \( \psi(v(k_1, \ldots, k_n)) = v(k_1 - \alpha_1, \ldots, k_n - \alpha_1) \). Moreover, \( U/M_\phi \cong U/M_\psi \), and this isomorphism is given by \( \theta(1 + M_\phi) = u_0 + M_\psi \), where \( u_0 \in U_\xi \setminus M_\phi \)

(obviously we also have that \( u_0 \notin M_\psi \)). We can also assume that \( u_0 \) has been chosen in such a way that \( \rho'(u_0)v(-\alpha_1, \ldots, -\alpha_1) = v(\overline{0}) \). In fact, for any \( u \in U_\xi \setminus M_\phi \), \( \rho'(u)v(-\alpha_1, \ldots, -\alpha_1) = \rho'(u)\psi(v(\overline{0})) = \psi(\rho(u)v(\overline{0})) = \psi(p \cdot v(\alpha_1, \ldots, \alpha_n)) \) for some non-zero scalar \( p \), since \( u \notin M_\phi \).
Also since $u_0 \notin M_{\phi}$, there exists a $v_0 \in U_{-\xi}$ such that 
$\phi(v_0 u_0) = 1$. Now since $U/M_{\phi} \cong U/M_{\phi}$, we have (by 
Lemma 5.3.1) that for all $c \in C(L)$ and $w_0 \in U_{-\xi}$, 
$\phi'(c) \phi'(w_0 u_0) = \phi'(w_0 cu_0)$.

Therefore, for all $c \in C(A_n)$, we have

$$\rho'(c)v(\overline{0}) = \rho'(c)\rho'(u_0)v(-l_1, \ldots, -l_n) = \rho'(cu_0)v(-l_1, \ldots, -l_n)$$
$$= \rho'(cu_0)\psi(v(\overline{0})) = \psi \circ \rho(cu_0)v(\overline{0}) = \rho(w_0 cu_0)v(\overline{0}) = \phi(w_0 cu_0)v(\overline{0})$$
$$= \phi'(c)v(\overline{0})$$, so $\phi'$ is standard, parametrized by $s' = s + l_1$ and $\lambda' = (\phi + \xi)'_{H}$.

**Theorem 5.3.1.** Let $\phi: C(A_n) \to C$ be a standard algebra homo-
morphism parametrized by $s \in C$ and $\lambda \in H^*$. Then the weight
spaces of $U/M_{\phi}$ are at most one dimensional. Moreover, if

$s \notin \lambda(\omega_1 + \ldots + \omega_n) + Z$ for any $i = 0, 1, \ldots, n$, then

$\lambda + \sum_{i=1}^{n} l_i \omega_i$ occurs as a weight function for all integers $\lambda_i$.

**Proof:** Let the representation defining $\phi$ be $(\rho, V_s, \lambda)$
(cf. §2). Let $\xi = \sum_{i=1}^{n} l_i \omega_i \in Z$.

For every $u \in U_{\xi}$, define a scalar $\mu(u)$ by

$\rho(u)v(0) = \mu(u)v(l_1, l_2, \ldots, l_n)$. (This is possible since
all weight spaces of $(\rho, V_s, \lambda)$ are one-dimensional.)
First observe that if \( \mu(u) = 0 \) then \( u \in M_\phi \). In fact, for any \( w \in U_{-\xi} \), \( \phi(wu)v(\overline{0}) = \rho(wu)v(\overline{0}) = \rho(w)\rho(u)v(\overline{0}) = \rho(w)\mu(u)v(l_1, \ldots, l_n) = 0 \), thus \( u \in M_\phi \).

Now assume that \((\phi + \xi)_H\) is a weight function of \( U/M_\phi \).

Take any two elements \( u_1 + M_\phi \) and \( u_2 + M_\phi \) from the corresponding weight space such that \( u_1, u_2 \in U_{-\xi} \). (cf. Lemma 2.3.2.) We claim that the vectors \( u_1 + M_\phi \) and \( u_2 + M_\phi \) are linearly dependent.

If \( u_1 \in M_\phi \), there is nothing to prove. Hence we may assume that \( \mu(u_1) \neq 0 \). Consider the element \( u = u_2 - \frac{\mu(u_2)}{\mu(u_1)} u_1 \).

Then \( u \in U_{-\xi} \) and for any \( w \in U_{-\xi} \) we have

\[
\phi(wu)v(\overline{0}) = \rho(wu)v(\overline{0}) = \rho(w)\rho(u_2 - \frac{\mu(u_2)}{\mu(u_1)} u_1)v(\overline{0})
\]

\[= \rho(w)\mu(u_2) - \frac{\mu(u_2)}{\mu(u_1)} u_1)v(l_1, \ldots, l_n)\]

\[= \rho(w)\mu(u_2) - \frac{\mu(u_2)}{\mu(u_1)} \mu(u_1)v(l_1, \ldots, l_n) = 0 .\]

Thus \( u \in M_\phi \), i.e. \( u_1 + M_\phi \) and \( u_2 + M_\phi \) are linearly dependent.

Next, assume that \( s \lambda (H_{\alpha_1} + \ldots + H_{\alpha_n}) + Z \) for all \( i = 1, 2, \ldots, n \), and let \( \lambda' = \lambda + \sum_{i=1}^{n} \lambda_i \alpha_i \) (\( \alpha_i \in Z \)).
Then by Lemma 4.2.1 (and the condition on s) $\lambda + l_1 \cdot a_1$ is a weight function of $U/M_\phi$, and the corresponding weight space is one dimensional by the first part of this proof. Now the algebra homomorphism $\varphi'$ corresponding to the one-dimensional weight function $\lambda + l_1 a_1$ is also standard, parametrized by $s' = s' + l_1$ and $\lambda' = \lambda + l_1 a_1$. But then $s' \not\mid \lambda'(H_{a_1} + \ldots + H_{a_n}) + Z$ for any $1 = 0, 1, \ldots, n$ and hence (using again Lemma 4.2.1) $\lambda' + l_2 a_2 = \lambda + l_1 a_1 + l_2 a_2$ is a weight function of $U(L)/M_\phi$. But $U(L)/M_\phi \cong U(L)/M_\phi$, so $\lambda + l_1 a_1 + l_2 a_2$ is a weight function of $U(L)/M_\phi$. Applying the above argument $n$ times, we obtain that $\lambda + l_1 a_1 + \ldots + l_n a_n$ is a weight function of $U(L)/M_\phi$.

Lemma 5.3.2. Let $\phi : C(L) \to C$ be a g-standard algebra homomorphism defined with respect to $\Gamma_0 = \bigcup \Gamma_0^{(1)}$. Let $\xi = \sum_{\alpha \in \Gamma_0} \lambda^a \cdot \alpha$, where the $\lambda^a$'s are integers. Then any basis element $u \in U_\xi \setminus M_\phi$ can be expressed as $u = u_1 \cdot u_2 \ldots u_\ell \cdot h$ where $h \in U(H)$ and $u_j \in U_{\xi_j}$, $j = 1, 2, \ldots, \ell$.

Proof: We shall use induction on the degree of $u$ (as a monomial). The statement is trivially true if $u$ is of degree one. Assume that it holds for all monomials of degree $\leq m$, and let $-u \in U_\xi \setminus M_\phi$ be a basis element of degree $m + 1$.
By Lemma 4.1.4, $u$ does not contain any factors $X_\beta$ such that $\beta \in \Delta_+ \setminus U_{(i)}^c$, and consequently (since $\xi \in U_{(i)}^c$), $u$ also cannot contain any factors of the form $Y_\beta$ with $\beta \in \Delta_+ \setminus U_{(i)}^c$. Now if $u$ contains any factor $h \in U(H)$ then we are finished by the inductive assumption. Otherwise, since the $U_{(i)}^c$ are disconnected, $[U(U_{(i)}^c), U(U_{(j)}^c)] = 0$ for $i \neq j$, and therefore $u$ can be expressed in the required form.

**Lemma 5.3.3.** Let $\phi: C(L) \to C$ be $g$-standard with respect to $\bigcup_{i=1}^l U_{(i)}^c$. Then, for any $i = 1, 2, \ldots, l$, $M_{\phi} \cap U(U_{(i)}^c)$ is a maximal left ideal of $U(U_{(i)}^c)$.

Proof: $M_{\phi} \cap U(U_{(i)}^c)$ is a left ideal of $U(U_{(i)}^c)$ which contains $\ker \phi \cap U(U_{(i)}^c)$. To show that it is maximal, we must prove that if $u \in U(U_{(i)}^c) \setminus M_{\phi}$ where $\xi = \sum_{\alpha \in \Delta_+} \lambda_\alpha \cdot \alpha$, then there exists a $v \in U(U_{(i)}^c) \setminus \xi$ such that $\phi(vu) = 0$.

Let $v_0 \in U_{-\xi}$ be a basis element of minimal degree such that $\phi(v_0 u) \neq 0$ (such a $v_0$ exists since $u \notin M_{\phi}$). We claim that $v_0 \in U(U_{(i)}^c) \setminus \xi$.

First observe that $v_0$ cannot contain any factors from $U(H)$ since otherwise we would have $v_0 = wh + \text{lower degree terms}$, and
\[ \phi(v_0 u) = \phi(whu) = \phi(wu)(h) + \xi(h)\phi(wu) = 0. \]

But this contradicts the choice of \( v_0 \).

Next, we note that (since the \( r_c^{(i)} \) are disconnected) if \( v_0 \) contains any factors of the form \( X_\beta \) or \( Y_\beta \) with \( \beta \notin r_c^{(i)} \), \( v_0 = c \cdot v' \) where \( c \in C(L) \) and \( v' \in U(r_c^{(i)})^{-\xi} \).

By the minimality of the degree of \( v_0 \), \( c \) must be a non-zero scalar and hence \( v_0 \in U(r_c^{(i)})^{-\xi} \).

With the aid of Lemmas 5.3.2 and 5.3.3, we are now able to prove the main result of this section.

**Theorem 5.3.2.** Let \( \phi: C(L) \to C \) be a \( g \)-standard algebra homomorphism defined with respect to \( U r_c^{(i)} \). Then the weight space corresponding to each boundary function

\[ \phi_+ + \sum_{\alpha \in \Gamma_0} \lambda_\alpha \cdot a \]

is at most one dimensional. Moreover, if

\[ \xi = \sum_{\alpha \in \Gamma_0} \lambda_\alpha \]

and \( \xi_1 = \sum_{\alpha \in \Gamma_0} \lambda_\alpha \cdot a \), then \( (\phi + \xi)_+ \) is a weight of \( U(L)/M_\phi \) if and only if \( (\phi + \xi_1)_+ \) is a one-dimensional weight of \( U(r_c^{(i)})/M_\phi \cap U(r_c^{(i)}) \) for all \( i = 1, 2, \ldots, \ell \).

**Proof:** Since \( \phi_+ C(L) \cap U(r_c^{(i)}) \) is standard for each \( i \),

\[ \dim(U(r_c^{(i)})/M_\phi \cap U(r_c^{(i)}))_{\phi + \xi_1} \leq 1 \] by Theorem 5.3.1.

Suppose that for some \( j \), \( U(r_c^{(j)})_{\xi_j} \subseteq M_\phi \)
(i.e. \( \dim(U(r^{(j)}_c)/M_\phi \cap U(r^{(j)}_c))_{\phi+\xi_j} = 0 \).

Let \( u \in U_\xi \) be a basis element. Then by Lemma 5.3.2,
\[
u = u_1 \cdot u_2 \cdots u_j \cdots u_k \cdot h \quad \text{with} \quad h \in U(H), \quad u_1 \in U_{\xi_1}.\]
Then (since the \( r^{(1)}_c \) are disconnected),
\[
u = u_{1} u_{2} \cdots u_{j-1} u_{j+1} \cdots u_{k} u_{h} = \]
\[
u = u_{1} u_{2} \cdots u_{j-1} u_{j+1} \cdots u_{k} h u_{j} + \xi_{j}(h) u_{1} \cdots u_{j-1} u_{j+1} \cdots u_{k} u_{j} \in M_\phi.\]
Thus \( U_\xi \subseteq M_\phi \), i.e. \( \dim(U(L)/M_\phi)_{\phi+\xi} = 0 \).

Now assume that for all \( i \), \( (\phi+\xi_1)_H \) is a one-dimensional weight function of \( U(r^{(1)}_c)/M_\phi \cap U(r^{(1)}_c) \).

Let \( u_1 + (M_\phi \cap U(r^{(1)}_c)) \) (i=1,2,...,l) be a generator of the \( \phi+\xi_1 \)-weight space of \( U(r^{(1)}_c)/M_\phi \cap U(r^{(1)}_c) \).

Then \( v = u_1 \cdot u_2 \cdots u_k \in U_\xi \setminus M_\phi \), and any \( u \in U_\xi \) is a non-zero scalar multiple of \( v \), thus \( \dim(U(L)/M_\phi)_{\phi+\xi} = 1 \).

**Corollary 5.3.1.** Let \( (\rho,V) \) be a \( \Gamma_0 \)-complete representation of \( U(L) \), where \( \Gamma_0 = (\bigcup \Gamma^{(1)}_0) \) is a subset of the simple roots of \( L_\xi \), the \( \Gamma^{(1)}_0 \) being disconnected. If the mass function \( \phi: C(L) \to C \) associated with \( (\rho,V) \) is extreme \( \gamma \)-standard with respect to \( \bigcup \Gamma^{(1)}_0 \), then all boundary functions \( \phi + \sum \xi_\alpha \cdot \alpha \) occur as weight functions of \( (\rho,V) \), and...
the corresponding weight spaces are one-dimensional.

Proof: Let \( \xi = \sum_{\alpha \in \Gamma_0} \xi_\alpha \cdot \alpha \). Since \( \Phi^* C(L) \cap U(r^{(i)}_c) \) is complete \( g \)-standard for each \( i = 1, 2, \ldots, \ell \), by Theorem 5.3.1, \( \dim(U(r^{(1)}_c)/M_{\Phi} \cap U(r^{(1)}_c)/\Phi^* \xi_1) = 1 \) (where \( \xi_1 = \sum_{\alpha \in \Gamma_0} \xi_\alpha \cdot \alpha \), \( \xi_\alpha \in \mathbb{Z} \)). Thus, by Theorem 5.3.2,

\[
\dim(U(L)/M_{\Phi})/\Phi^* \xi = 1.
\]

§4. The "labelling" of \( g \)-standard homomorphisms

In this section, we shall prove that for every \( g \)-standard algebra homomorphism \( \Phi: C(L) \to \mathbb{C} \) there exists an extreme \( g \)-standard algebra homomorphism \( \Phi_0: C(L) \to \mathbb{C} \) such that

\[
U(L)/M_{\Phi \circ \sigma} = U(L)/M_{\Phi_0}
\]

for some automorphism \( \sigma \in \text{Aut}(L:H) \).

Lemma 5.4.1. Let \( \Phi: C(A_n) \to \mathbb{C} \) be a standard algebra homomorphism parametrized by \( s \in \mathbb{C} \) and \( \lambda \in H^* \). Assume that \( s - \lambda (H_{a_1} + \ldots + H_{a_j}) \in \mathbb{Z} \) for some \( j \) (0 ≤ j ≤ n).

Then there exists a \( g \)-standard algebra homomorphism \( \Phi^* \) defined with respect to \( r^{(1)}_0 = \{a_1, \ldots, a_{n-1}\} \) or \( r^{(2)}_0 = \{a_2, \ldots, a_n\} \) such that \( U(A_n)/M_{\Phi} = U(A_n)/M_{\Phi_0 \circ \sigma} \) for some \( \sigma \in \text{Aut}(A_n:H) \).

Proof: Assume that \( m = s - \lambda (\sum_{i=0}^{r} H_{a_i}) \) is positive
and that \( m \) is the minimum (by absolute value) among the integers in the set \( \{ s - \lambda(\sum_{i=0}^{p} H_{a}) , p = 0, \ldots, n \} \). For \( r > 1 \), set \( \sigma = \sigma_{a_{r+1}} \circ \sigma_{a_{r+2}} \circ \cdots \circ \sigma_{a_{n}} \). Then \( \phi \circ \sigma \) is standard (parametrized by \( s' = s \) and \( \lambda' = \lambda \circ \sigma \)), and

\[
\begin{align*}
s' - \lambda'(\sum_{i=1}^{n} H_{a}) &= s - \lambda \circ \sigma(\sum_{i=1}^{n} H_{a}) \\
&= s - \lambda(\sum_{i=1}^{n} H_{a}) = m > 0.
\end{align*}
\]

For \( r = 0 \), set \( \sigma = \sigma_{a_{1}} \circ \sigma_{a_{2}} \circ \cdots \circ \sigma_{a_{n}} \). Then \( \phi \circ \sigma \) is standard, parametrized by \( s' = s - \lambda(H_{a_{1}}) \) and \( \lambda' = \lambda \circ \sigma \), and

\[
\begin{align*}
s' - \lambda'(\sum_{i=1}^{n} H_{a}) &= s - \lambda(H_{a_{1}}) - \lambda \circ \sigma(\sum_{i=1}^{n} H_{a}) \\
&= s - \lambda(H_{a_{1}}) - \lambda(H_{a_{1}}) \\
&= s - m = m > 0.
\end{align*}
\]

In both cases (i.e. for \( r = 0 \) and \( r > 1 \)) \( \phi \circ \sigma + (m-1)a_{n} \) is a one-dimensional weight space of \( U(A_{n})/M_{\phi \circ \sigma} \). (This follows by Lemma 4.2.1, and the fact that \( m \) is the smallest, by absolute value, among the integers in the set \( \{ s - \lambda(\sum_{i=1}^{p} H_{a}) \} \). (Thus the cases \( r = 0 \) and \( r > 1 \) need not be considered separately.)

Now let \( \overline{\phi} \) be the mass function associated with the weight function \( \phi \circ \sigma + (m-1)a_{n} \). Then \( \overline{\phi} \) is standard, parametrized by \( s'' = s \) and \( \lambda'' = \lambda \circ \sigma + (m-1)a_{n} \). Then \( \overline{\phi}(c) = 0 \) for any cycle \( c \in C(A_{n}) \) containing a factor of the form
\[ x_{a_1} + a_{i+1} + \ldots + a_n \] (this can be verified directly from the definition of \((\rho, V_s, \lambda')\). (cf. Ch.5.12.) Thus

\[ \bar{\psi} + C(x_{a_1}, \ldots, x_{a_{n-1}}) = 0, \] i.e. \( \bar{\psi} \) is g-standard relative to \( r_0^{(1)} \). Moreover, \( U(A_n)/M_{\bar{\psi}} \cong U/M_{\phi \circ \sigma} \).

Next assume that \( m \leq 0 \). If \( r \neq 0 \),

\[ \sigma = \sigma_1 \circ \ldots \circ \sigma_{a_1} \circ \ldots \circ \sigma_1 \] and \( \phi' = \phi \circ \psi \). Then \( \phi' \) is standard, parametrized by \( s' = s - \lambda(\sum_{i=0}^{n} H_{a_i}) \) and \( \lambda' = \phi \circ \sigma \).

Moreover \( (*) \) (cf. footnote below) \( (\phi \circ \sigma - ma_1)^+ H \) is a one-dimensional weight function of \( U(A_n)/M_{\phi \circ \sigma} \) and therefore the algebra homomorphism \( \bar{\psi} \) associated with \( (\phi \circ \sigma - ma_1)^+ H \) is also standard, parametrized by \( s'' = s' - m = 0 \). Now if \( c \in C(A_n) \) is any cycle containing a factor of the form \( x_{a_1} + \ldots + x_1 \), then \( \bar{\psi}(c) = 0 \) (since \( s'' = 0 \)). Thus, \( \bar{\psi} \) is g-standard with respect to \( r_0^{(2)} \) and \( U(A_n)/M_{\bar{\psi}} \cong U(A_n)/M_{\phi \circ \sigma} \).

If \( r = 0 \), then let \( \bar{\psi} \) be the algebra homomorphism associated with the \( (\phi - ma_1)^+ H \) weight space of \( U(L)/M_{\phi} \).

Then, just as above, we have that \( \bar{\psi} \) is g-standard with respect to \( r_0^{(2)} \).

\[ (*) \] In the special case when \( 0 > m = s - \lambda_1 = -s \), \( (\phi \circ \sigma - ma_1)^+ H \) is not a weight. However, in this case we may choose the positive value for \( m \) and then no such problem arises.
Lemma 5.4.2. Let \( \phi : C(L) \rightarrow C \) be a standard algebra homomorphism defined with respect to \( \bigcup_{1}^{t} \Gamma_{0}^{(1)} (i = 1, 2, \ldots, t) \).

Let \( \xi = \sum \ell_{a}^{(1)} \in \mathbb{Z} \). Assume that \((\phi + \xi)_{+H} \) is a one dimensional weight function of \( U(L)/M_{\phi} \). Then the algebra homomorphism \( \phi' \) associated with \((\phi + \xi)_{+H} \) is also g-standard.

Proof: First note that \( \phi' \) is standard by \( C(L) \cap U(\Gamma_{C}^{(k)}) \).

Lemma 5.3.2. Next, let \( u \in U(\Gamma_{C}^{(1)}) \setminus M_{\phi} \). Then for any \( c \in C(L) \), \( \phi'(c)(u + M_{\phi}) = c(u + M_{\phi}) \). Now if \( c \in C(L) \cap U(\Gamma_{C}^{(j)}) \) for \( j \neq k \), then, as the \( \Gamma_{C}^{(i)} \) are disconnected, we have \( \phi'(c)(u + M_{\phi}) = cu + M_{\phi} = uc + M_{\phi} = \phi(c)(u + M_{\phi}) \) i.e. \( \phi' = \phi \) on \( C(L) \cap U(\Gamma_{C}^{(j)}) \) for all \( j \neq k \).

Finally, let \( c \in \overline{C}(\bigcup_{1}^{t} \Gamma_{C}^{(1)}) \) and \( w \in U_{-\xi} \). Since \( \overline{C}(U_{C}^{(1)}) \) is a two-sided ideal, \( wcu \in \overline{C}(U_{C}^{(1)}) \), and as \( \phi \) is g-standard with respect to \( U_{C}^{(1)} \), \( \phi(wcu) = 0 \). But this means that \( cu \in M_{\phi} \), and thus \( \phi'(c)(u + M_{\phi}) = cu + M_{\phi} = 0(u + M_{\phi}) \) i.e. \( \phi'(c) = 0 \). Thus \( \phi' = 0 \) on \( \overline{C}(U_{C}^{(1)}) \) and hence it is g-standard with respect to \( U_{C}^{(1)} \).

Theorem 5.4.1. Let \( \phi : C(L) \rightarrow C \) be a g-standard algebra homomorphism with respect to \( \bigcup_{1}^{t} \Gamma_{0}^{(1)} (i = 1, 2, \ldots, t) \). Then there exists an extreme g-standard algebra homomorphism \( \overline{\phi} : C(L) \rightarrow C \) such that \( U(L)/M_{\phi} = U(L)/M_{\phi \circ \sigma} \) for some \( \sigma \in \text{Aut}(L:H) \).
Proof: Let \( \| \phi \| \) be the number of simple roots in \( U_0^{(1)} \).
We shall use induction on \( \| \phi \| \). If \( \| \phi \| = 0 \), then \( \phi \) is clearly extreme g-standard.

Assume that the statement is true for all g-standard algebra homomorphisms \( \phi : C(L) \to C \) such that \( \| \phi \| < k \) (\( k > 0 \)), and let \( \phi : C(L) \to C \) be g-standard with \( \| \phi \| = k \). Then \( \phi_j = \phi \circ \chi C(L) \cup U(r_j) \) is standard parametrized by \( s_j \) and \( (j = 1, 2, \ldots, \ell) \). If there exists an \( j \) such that \( \phi_j \) satisfies the conditions of Lemma 5.4.1, then by that Lemma, there exists a g-standard algebra homomorphism \( \phi_j : C(r_j) \to C \) such that \( \| \phi_j \| < \| \phi_j \| \) and \( U(r_j)/M_{\phi_j \circ 1} = U(r_j)/M_\phi \) for some \( \sigma_1 \in \text{Aut}(L:H) \).

Define a map \( \phi' : C(L) \to C \) by setting
\[
\phi'(c) = \begin{cases} 
\phi_j(c) & \text{for } c \in C(r_j) \\
\phi(c) & \text{for } c \in C(L) \setminus C(r_j) 
\end{cases}
\]

Then, as \( C(r_j) \) is an ideal and \( C(L) = C(r_j) \oplus C(L) \), \( \phi' \) is an algebra homomorphism. Clearly it is g-standard and \( \| \phi' \| < k \). We claim that \( U(L)/M_\phi = U(L)/M_{\phi \circ 1} \).

Since \( U(r_j)/M_{\phi_j \circ 1} = U(r_j)/M_{\phi_j} \), for \( \xi = (\phi_j - \phi_j \circ 1) + H \)
\[
= (\phi' - \phi \circ 1) + H,
\]
there exists \( u_0 \in U(r_j) \setminus M_{\phi_j} \) such that \( \phi_j \circ 1(c) \phi_j^{-1}(wuo) = \phi_j(wcu_0) \) for all \( c \in C(r_j) \) and \( w \in U(r_j) \).


Since \( u_0 \notin M_{\phi} \), there exists a \( v_0 \notin U(r_c^{(j)}) \) such that

\[
\text{if } v_0 u_0 = \phi'(v_0 u_0), \text{ and so } u_0 \notin M_{\phi}. \quad \text{Moreover, for all } w \in U(L) \setminus U(r_c^{(j)}) \text{ and all } c \in C(L), \phi \circ \sigma_1(c) \phi'(w_0 u_0) = \phi'(w_0 u_0) = 0, \text{ as can be shown by considering four cases:}
\]

1. \( c \in \mathcal{O}(r_c^{(j)}) \), \( w \in U(r_c^{(j)}) \setminus U(r_f) \);
2. \( c \in C(r_c^{(j)}) \), \( w \in U(L) \setminus U(r_c^{(j)}) \);
3. \( c \in C(r_c^{(j)}) \), \( w \in U(r_c^{(j)}) \setminus U(r_f) \);
4. \( c \in C(r_c^{(j)}) \), \( w \in U(L) \setminus U(r_f) \).

For example, in case 3, \( \sigma_1(c) \in C(r_c^{(j)}) \) (since \( \sigma_1 \) is a product of \( \sigma_a \)'s with \( a \in \Delta + \{ 0 \} \)) of the proof of Lemma 5.4.1 and, since the \( r_{(i)} \) are disconnected, we must have that \( \phi \circ \sigma_1(c) = 0 \) or \( \phi \circ \sigma_1(c) = \phi(c) \). In the first case,

\[
\phi'(w_0 u_0) = \phi'(cw_0) = 0, \quad \text{while in the second, } w_0 u_0 = cw_0
\]

and hence

\[
\phi'(w_0 u_0) = \phi'(cw_0) = \phi'(c)\phi'(w_0) = \phi(c)\phi'(w_0) = \phi \circ \sigma_1(c)\phi'(w_0). \quad \text{Cases 1, 2 and 4 can be handled in a similar manner. Thus, by Lemma 5.3.1, } U(L)/M_{\phi} \equiv U(L)/M_{\phi \circ \sigma_1}.
\]

Now, by the inductive assumption, there exists an extreme \( g \)-standard algebra homomorphism \( \phi : C(L) \to C \) such that

\[
U(L)/M_{\phi} \equiv U(L)/M_{\phi \circ \sigma_2}, \quad \text{for some } \sigma_2 \in \text{Aut}(L:H). \quad \text{Since}
\]

\[
U(L)/M_{\phi} \equiv U(L)/M_{\phi \circ \sigma_1}, \quad \text{for } \xi = (\phi \circ \sigma_1)^H, \text{ there exists}
\]
\[ u_1 \in U_\xi \setminus M_{\phi \sigma_1} \] such that for all \( t \in U_{-\xi} \) and all \( c \in C(L) \),
\[ \phi'(c) \phi \sigma_1(tu_1) = \phi \sigma_1(tc u_1) \]. Then the element \( \sigma_2^n(u_1) \)
satisfies the conditions (cf. Lemma 5.3.1) for \( U(L)/M_{\phi \sigma_1} \) and \( U(L)/M_{\phi \sigma_1 \sigma_2} \) to be isomorphic. Thus
\[ U(L)/M_{\phi} \cong U(L)/M_{\phi \sigma} \], where \( \sigma = \sigma_1 \sigma_2 \in Aut(L:H) \).

Next, assume that for each \( j = 1, 2, \ldots, \ell \),
\[ s_j - \lambda_j (\sum H \alpha) \notin \mathbb{Z} \quad \text{for } i = 0, 1, \ldots, ||\phi_j||. \]
Then, by Theorem 5.3.2,
\[ (\phi + \sum \alpha \cdot a)_{+H} \] is a one-dimensional weight function of \( U(L)/M_{\phi} \) for all integers \( \lambda \). Then, if
\[ \phi_{+H} = \sum a_1^\alpha \], set \( k_\alpha = [\text{Re}(a_1)] \) for all \( a_1^\alpha \in \Delta_+ \), where \( | \cdot | \) denotes the greatest integer function. Then the algebra homomorphism \( \phi \) associated with the one-dimensional weight function
\[ (\phi - \sum k_\alpha a_{+H}) \] is extreme, and \( U(L)M_{\phi} \cong U(L)/M_{\phi} \) by Lemma 5.4.2.
Chapter Six
Pointed Representations of $A_n$

In this chapter, we shall construct all algebra homomorphisms $\phi: C(A_n) \to C$ for $n = 1,2,3$, and analyze their associated pointed representations using the results of the previous chapter. In particular, we establish that every algebra homomorphism $\phi: C(A_n) \to C$ $(n = 1,2,3)$ is weakly equivalent to a g-standard one. We shall also present some partial results towards the classification of the pointed representations of $A_n$ for $n > 3$.

§1. Pointed representations of $A_1$

The representations of the simple Lie algebra $A_1$ have been studied extensively. (cf. For example Bouwer [2a], Arnal and Pinczon [14].) In this section we merely re-state some well known results regarding $A_1$, in our general setting.

$A_1$ has one simple root $\alpha$, and is generated by the elements $H_\alpha$, $Y_\alpha$, $X_\alpha$ satisfying the following commutation relations:

$[H_\alpha, X_\alpha] = 2X_\alpha$ ; $[H_\alpha, Y_\alpha] = -2Y_\alpha$ ; $[X_\alpha, Y_\alpha] = H_\alpha$.

$C(A_1)$ is the subalgebra of $U(A_1)$ independently generated by the elementary cycles $1, H_\alpha, Y_\alpha X_\alpha$. Clearly $C(A_1)$
is abelian. Thus any algebra homomorphism \( \phi: C(A_1) \to \mathbb{C} \) is completely determined by specifying values of \( \phi(H_{\alpha}) \) and \( \phi(Y_{\alpha}X_{\alpha}) \) and extending. Thus we may select arbitrary scalars \( \lambda, s \in \mathbb{C} \) and set \( \phi(H_{\alpha}) = \lambda, \phi(Y_{\alpha}X_{\alpha}) = s(s-1-\lambda) \). Hence any algebra homomorphism \( \phi: C(A_1) \to \mathbb{C} \) is standard. The corresponding pointed representation \( U(A_1)/M_\phi \) has only one-dimensional weight spaces, and all weights are of the form \( \lambda + m\alpha \) \( (\alpha \in \mathbb{Z}) \).

**Proposition 6.1.1.** Let \( \phi: C(A_1) \to \mathbb{C} \) be an algebra homomorphism parametrized by \( s \) and \( \lambda \). Depending on the values of \( s \) and \( \lambda \) the representation \( U(A_1)/M_\phi \) is one of the following three types:

**Type I:** The \( \alpha \)-string through \( \lambda \) is **doubly infinite** if and only if \( s \notin \mathbb{Z}, s - \lambda \notin \mathbb{Z} \). In this case \( U(A_1)/M_\phi \) is equivalent to a complete order one representation.

**Type II:** The \( \alpha \)-string through \( \lambda \) is **singly infinite** if and only if \( s \) and \( \lambda \) satisfy one of the following conditions:

(a) \( s \in \mathbb{Z}, s - \lambda \notin \mathbb{Z} \). Then \( U(A_1)/M_\phi \) is infinite dimensional and has a highest weight (equal to \( \lambda - s\alpha \)) or a lowest weight (equal to \( \lambda + (1-s)\alpha \)) according to \( s \leq 0 \) or \( s > 0 \).

(b) \( s \notin \mathbb{Z}, s - \lambda \in \mathbb{Z} \). Then \( U(A_1)/M_\phi \) is infinite dimensional with highest weight \( \lambda + (s-\lambda-1)\alpha \) if
s - \lambda > 0 , and lowest weight \lambda + (s-\lambda)\alpha if
s - \lambda \leq 0 .

(c) s \in Z and s - \lambda \in Z , having opposite signs. Then
\text{U}(A_1)/M_\phi has a highest weight (equal to
\min\{\lambda + (s-\lambda-1)\alpha , \lambda - s\alpha\}) provided s \leq 0 and
s - \lambda > 0 , and \text{U}(A_1)/M_\phi has a lowest weight (equal
to \max\{\lambda + (s-\lambda+1)\alpha , \lambda + (1-s)\alpha\}) provided s > 0 ,
s - \lambda \leq 0 .

Type III: The \alpha-string through \lambda is \textbf{finite} if and only if
s \in Z and s - \lambda \in Z , both negative or both non-negative.
In the first case \text{U}(A_1)/M_\phi has highest weight
\lambda + (s-\lambda-1)\alpha and lowest weight \lambda - (s-1)\alpha ; in the
second case, the highest weight is \lambda - s\alpha and the lowest
is \lambda + (s-\lambda)\alpha.

Proof: All the statements regarding the type of the
representation \text{U}(A_1)/M_\phi follow by Lemma 5.4.1 and Theorem
5.4.1. The values given for the highest (lowest) weights for
Types II and III can be easily verified using the identities of
Lemma 4.2.1.

Remark: The above listed cases represent all the distinct
equivalence classes of pointed representations of A_1. One
can easily see that for each representation \text{U}(A_1)/M_\phi of
type IIa, there is a unique representation \text{U}(A_1)/M_\phi of
type IIb such that \phi = \phi'_0 \sigma_\alpha . Moreover, the two subcases
under IIc also represent weakly equivalent (via $\sigma_a$) algebra homomorphisms.

In each of the above cases, two algebra homomorphisms $\phi_1: C(A_1) \to C$ (parametrized by $(s_1, \lambda_1)$ and $\phi_2: C(A_1) \to C$ (parametrized by $(s_2, \lambda_2)$) will yield equivalent pointed representations of $A_1$ if and only if $s_1 - s_2 = n$ and $\lambda_1 - \lambda_2 = 2n$ for some integer $n$. (cf. Lemma 5.3.2.)

§2. Pointed representations of $A_2$

In this section, we construct all algebra homomorphisms $\phi: C(A_2) \to C$ and study their associated pointed representations.

A Chevalley basis for $A_2$ is given by the elements

$\{H_a = e_{11} - e_{22}, H_\beta = e_{22} - e_{33}, X_a = e_{12}, X_\beta = e_{23},$ $X_{a+\beta} = e_{13}, Y_a = e_{21}, Y_\beta = e_{32}, Y_{a+\beta} = e_{31}\}$, where $e_{ij}$ denotes the 3x3 matrix with 1 in the $(i,j)$-th position and zeroes elsewhere. (cf. Chapter 5, §1.) It can be shown (by inspection) that $C(A_2)$ has eight elementary cycles, namely

$\{1, H_a, H_\beta, c_1 = Y_aX_a, c_2 = Y_\beta X_\beta, c_3 = Y_{a+\beta}X_{a+\beta}, c_4 = Y_{a+\beta}X_aX_\beta,$ $c_5 = Y_\beta Y_aX_{a+\beta}\}$.

A linear basis of $C(A_2)$ is given by elements of the form

$\{(c_5 \text{ or } c_4)^{(k_1)} (c_3)^{(k_2)} (c_2)^{(k_3)} (c_1)^{(k_4)} H_a^{(k_5)} H_\beta^{(k_6)}\}$.
where the $k_i$ are non-negative integers. (cf. Bouwer [2.1]).

Setting $\phi(H_a) = \lambda_1$, $\phi(H_\beta) = \lambda_2$ and $\phi(c_i) = z_i$ for
$i = 1, 2, \ldots, 6$, $\phi$ can be extended to a linear map on $C(A_2)$.

Bouwer proved in [2b] (Theorem 4.4) that this linear map
is going to be an algebra homomorphism on $C(A_2)$ if and
only if $\phi(c_ic_j) = \phi(c_jc_i)$ for all elementary cycles $c_i, c_j$.

**Lemma 6.2.1.** The following relations hold in $C(A_2)$:

(a) $c_1c_2 = c_2c_1 + c_5 - c_4$

(b) $c_1c_4 = c_4c_1 + c_3c_1 + c_2c_1 + c_5 - c_3 + (c_4 - c_3)(H_a + 1)$

(c) $c_2c_4 = c_4c_2 + c_2c_1 + c_5 - c_3c_2 - c_4H_\beta - c_4$

(d) $c_4c_5 = c_3c_2c_1 + c_3c_2H_\beta + c_3c_1H_\beta + c_3H_\alpha H_\beta + c_5c_3 + 2c_3c_1$
\hspace{2cm} $+ 2c_3H_a + 2c_4 - 2c_3c_2 - c_5H_a - c_5H_\beta - 2c_5 + c_4c_2$
\hspace{2cm} $- c_4c_1 - c_4H_a$.

Proof: direct computations.

Equations (a) - (d) of Lemma 6.2.1 yield the following
relations between the values of an algebra homomorphism
$\phi: C(A_2) \rightarrow \mathbb{C}$:

(1) $z_4 = z_5$ (from (a) above)

(2) $\lambda_1(z_4 - z_3) = z_1(z_3 - z_2)$ (by (b))

(3) $\lambda_2z_4 = z_2(z_1 - z_3)$ (by (c))

(4) $(z_4 - z_3)(z_2 - z_1 - \lambda_1z_4) + z_3(z_2 + \lambda_2)(z_1 + \lambda_1) = 0$ (by (a) and (d))
It can be shown (cf. Bouwer [2a]) that multiplying any other pairs of elementary cycles does not yield new conditions.

Provided \( z_1 \neq 0 \) , \( z_1 \neq -\lambda_1 \) \((i = 1, 2)\), any solution of system (1) - (4) above is also a solution of the following system:

\[
(1') \quad z_4 = z_5 \\
(2') \quad Nz_4 = (z_1 + \lambda_1 - z_2)z_1z_2 \\
(3') \quad Nz_3 = (\lambda_1 + \lambda_2)z_1z_2 \\
(4') \quad N(\lambda_1 + \lambda_2) = (z_2 - z_1 + \lambda_2)(z_2 - z_1 - \lambda_1) , \text{ where} \\
N = z_2^2 + z_2\lambda_1 + \lambda_1\lambda_2 
\]

System \((1') - (4')\) was solved by Bouwer ([2b]) under the tacit assumption that \( \lambda_1 + \lambda_2 \neq 0 \). He obtained the following solution:

\[
\phi(H_\alpha) = \lambda_1 , \quad \phi(H_\beta) = \lambda_2 , \quad \phi(Y_\alpha X_\alpha) = (s-1)(s+\lambda_1) \\
\phi(Y_\beta X_\beta) = s(s-1-\lambda_2) , \quad s' \cdot \phi(Y_{\alpha+\beta} X_{\alpha+\beta}) = \phi(Y_\alpha X_\alpha X_\beta) = \\
= \phi(Y_\beta Y_\alpha X_{\alpha+\beta}) = s \cdot (s-1-\lambda_2)(s+\lambda_1) . \text{ Observe that substituting} \\
s' = s + \lambda_1 \text{, we obtain a standard algebra homomorphism.}
\]

Clearly every solution of system \((1' - 4')\) is also a solution of \((1 - 4)\). Thus in order to determine all possible algebra homomorphisms \( \phi : C(A_n) \rightarrow C \), it suffices to consider the cases \( z_1 = 0 \), \(-\lambda_1 \) \((i = 1, 2)\) and \( \lambda_1 + \lambda_2 = 0 \), and
solve the system 1 - 4 under each of these restrictions.

Assuming first that \( z_1 = 0 \), system 1 - 4 becomes:

\[
\begin{align*}
z_4 &= z_5 \\
\lambda_2 z_4 + z_2 z_3 &= 0 \\
\lambda_1 z_4 &= \lambda_1 z_3 \\
z_4(z_3 + z_2 + \lambda_2 - z_4) + \lambda_1 z_3(z_2 + \lambda_2) &= 0.
\end{align*}
\]

Solving this system, we obtain the following solutions.

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Here \( p_i \) (\( i = 1, 2, 3 \)) denote fixed but arbitrary complex numbers.

Note that \( \phi_2 \) is a particular case of \( \phi_6 \), and \( \phi_3 \) is a particular case of \( \phi_5 \). Moreover, \( \phi_1 \circ \sigma_a = \phi_2 \) and \( \phi_4 \circ \sigma_a = \phi_3 \).
Similarly, the restrictions $z_2 = 0$, $z_1 = -\lambda_1$ (i = 1, 2) each yield six possible solutions, some of which are degenerate.

Finally, if we assume that $\lambda_1 + \lambda_2 = 0$, the equations (1) - (4) become

(i) $z_4 = z_5$

(ii) $-\lambda_1 z_4 = (z_4 - z_3)z_2$

(iii) $\lambda_1 z_4 = z_3(z_1 + \lambda_1) - z_1 z_2$

(iv) $z_4(z_3 - z_1 + z_2 - \lambda_1 - z_4) + z_3(z_1 + \lambda_1)(z_2 - \lambda_1) = 0$.

Adding (ii) and (iii) we obtain

$0 = z_3(z_1 + \lambda_1 - z_2)$ which yields the following subcases:

Case A: $z_3 = 0$. This leads to $z_1 z_2 = 0$ or $(z_1 + \lambda_1)(z_2 - \lambda_1) = 0$. All these possibilities have been listed before.

Case B: $z_2 = z_1 + \lambda_1$. (We can assume $z_1 \neq -\lambda_1$).

Then we have the following possibilities left (separating the cases $\lambda_1 = 0$ and $\lambda_1 \neq 0$):

$B_1$: $z_3 = z_1 = z_2$ and $B_2$: $z_1 = \frac{(z_3 - z_1)(z_1 + \lambda_1)}{\lambda_1}$.

Selecting $z_1 = s(s-1)$ and $z_1 = s(s-\lambda_1-1)$ in cases $B_1$ and $B_2$ respectively, we obtain particular cases of solutions of system $(1') - (4')$. 
Eliminating the overlap between the different cases considered above, we obtain the following (complete) list of algebra homomorphisms $\phi: C(A_2) \to \mathbb{C}$:

<table>
<thead>
<tr>
<th></th>
<th>$T_0$</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$T_3$</th>
<th>$T_4$</th>
<th>$T_5$</th>
<th>$T_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_a$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
</tr>
<tr>
<td>$H_\beta$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
</tr>
<tr>
<td>$c_1$</td>
<td>$s(s-1-\lambda_1)$</td>
<td>$\lambda_1$</td>
<td>$0$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$(s-\lambda_1)(s-1-\lambda_2-1)$</td>
<td>$p$</td>
<td>$0$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
</tr>
<tr>
<td>$c_3$</td>
<td>$s(s-1-\lambda_2-1)$</td>
<td>$p$</td>
<td>$0$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
</tr>
<tr>
<td>$c_4$</td>
<td>$s(s-\lambda_1)(s-1-\lambda_2-1)$</td>
<td>$p$</td>
<td>$p$</td>
<td>$0$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
</tr>
<tr>
<td>$c_5$</td>
<td>$s(s-\lambda_1)(s-\lambda_2-1)$</td>
<td>$p$</td>
<td>$p$</td>
<td>$0$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
<td>$\lambda_1$</td>
</tr>
</tbody>
</table>

Here $\lambda_1$, $\lambda_2$, $s$, $p$ denote fixed but arbitrary complex numbers. Examining the above solutions more closely, we make the following observations.

If $\phi: C(A_n) \to \mathbb{C}$ is of type $T_2$, then $\phi \circ \sigma_\beta$ is of type $T_1$, and if $\phi$ is of type $T_3$, then $\phi \circ \sigma_\alpha \circ \sigma_\beta$ is of type $T_1$. Moreover, if $\phi$ is of type $T_5$, then $\phi \circ \sigma_\alpha$ is of type $T_4$, and if $\phi$ is of type $T_6$, then $\phi \circ \sigma_\beta \circ \sigma_\alpha$ is of type $T_4$.

If $\phi$ is of type $T_0$, then $\phi$ is standard (and hence $\sigma$-standard with respect to $A_0$). If $\phi$ is of type $T_1$ or $T_4$, then $\phi$ is $\sigma$-standard relative to the complete subsets $\Gamma_c^{(1)} = \{ \pm \alpha, 0 \}$ and $\Gamma_c^{(2)} = \{ \pm \beta, 0 \}$, respectively. We summarize
these observations in the following.

**Proposition 6.2.1.** Every algebra homomorphism \( \phi : C(A_2) \to \mathbb{C} \)

is weakly equivalent to a \( g \)-standard one.

We shall now examine the overlap between algebra homomorphisms of types \( T_0 \) and \( T_1 \), and of types \( T_0 \) and \( T_4 \) respectively. By Theorem 5.3.1, if \( \phi : C(A_2) \to \mathbb{C} \) is standard, then all weight spaces of \( U(A_2)/M_\phi \) are at most one dimensional.

We now prove that the converse of this statement is also true.

**Proposition 6.2.2.** If \( \phi : C(A_2) \to \mathbb{C} \) is an algebra homomorphism of type \( T_1 \) or \( T_4 \), having the property that all weight space of \( U(A_2)/M_\phi \) are at most one dimensional, then \( \phi \) is standard.

Proof: Assume first that \( \phi \) is of type \( T_1 \). We must consider two cases:

**Case (i):** \( p + \lambda_1 \neq 0 \). Then, since \( \phi(X_{a+\beta}, Y_{a+\beta}) = \)

\( \phi(Y_\beta Y_a X_{a+\beta} - Y_\beta X_\beta + Y_a X_a + H_a) = p + \lambda_1 \); \( Y_\beta Y_a \notin M_\phi \) and so \( \lambda - (a+\beta) \) is a weight of \( U(A_2)/M_\phi \).

Now \( \lambda - (a+\beta) \) weight space is generated by \( Y_\beta Y_a + M_\phi \) and \( Y_{a+\beta} + M_\phi \). By assumption these generators are dependent mod \( M_\phi \). Thus we must have

1) \( a X_{a+\beta} Y_{a+\beta} + b X_{a+\beta} Y_\beta Y_a \in M_\phi \)

and 2) \( a X_\beta Y_{a+\beta} + b X_\beta Y_\beta Y_a \in M_\phi \)

for some complex numbers \( a, b \), not both zero.
Equations 1) and 2) reduce to

1') \[ a(\lambda_1 + \lambda_2) + b(p + \lambda_1) = 0 \]

2') \[ a(p + \lambda_1) + b(p + \lambda_1)(1 + \lambda_2) = 0 \], and hence

\[ a[p - \lambda_2(1 + \lambda_1 + \lambda_2)] = 0 \], which implies that

\[ p = \lambda_2(1 + \lambda_1 + \lambda_2) \] (since \( a = 0 \) would yield \( b = 0 \)).

Then, setting \( s = 1 + \lambda_1 + \lambda_2 \), \( \phi(c_1) = p = s(s - 1 - \lambda_1) \),

\[ \phi(c_2) = (s - \lambda_1)(s - 1 - \lambda_1 - \lambda_2) = 0 \],

\[ \phi(c_3) = s(s - 1 - \lambda_1 - \lambda_2) = 0 \],

\[ \phi(c_4) = \phi(c_5) = s(s - 1 - \lambda_1)(s - 1 - \lambda_1 - \lambda_2) = 0 \], i.e.

\( \phi \) is standard.

Case (ii): \( p + \lambda_1 = 0 \). If \( \lambda_1 = -p = 0 \), let \( s = 0 \)
to obtain a standard homomorphism. Thus we may assume that

\( \lambda_1 \neq 0 \). Then \( \lambda - \beta \) is a weight, and the corresponding weight
space - generated by \( Y_\beta \) and \( Y_{\alpha + \beta}X_\alpha \) - is one dimensional by
assumption. Thus there exist complex numbers \( A, B \) not both
zero such that

3) \[ AX_\beta Y_\beta + BX_\beta Y_{\alpha + \beta}X_\alpha \in M_\phi \]

4) \[ AY_{\alpha + \beta} Y_\beta + BY_{\alpha + \beta} Y_{\alpha + \beta}X_\alpha \in M_\phi \]

which reduce to

3') \[ \lambda_2 A - \lambda_1 B = 0 \] and

4') \[ -\lambda_1 A - \lambda_1 (1 + \lambda_1 + \lambda_2) B = 0 \] (using \( p = -\lambda_1 \)).
Since $\lambda_1 \neq 0$ and $A, B$ are not both zero, this system yields $\mu = -\lambda_1 + \lambda_2(1 + \lambda_1 + \lambda_2)$, and if we choose $s = 1 + \lambda_1 + \lambda_2$, then $\phi$ parametrized by $\lambda$ and $s$ will be standard.

Similarly if $\phi : C(A_2) \to \mathbb{C}$ is of type $T_4$ and all weight spaces of $U(A_2)/M_\phi$ are at most one dimensional, then $\phi$ is standard.

Remark: If an algebra homomorphism $\phi : C(A_2) \to \mathbb{C}$ is such that all weight spaces of $U(A_2)/M_\phi$ have dimension $\leq 1$, then $\phi \circ \sigma$ also has this property for all $\sigma \in \text{Aut}(A_2:H)$.

Thus we have the following result:

**Corollary 6.2.1.** An algebra homomorphism $\phi : C(A_2) \to \mathbb{C}$ is standard if and only if all the weight spaces of $U(A_2)/M_\phi$ are at most one dimensional.

Proof: By the above theorem and remark, $\phi$ is weakly equivalent to a standard algebra homomorphism $\phi' : C(A_2) \to \mathbb{C}$. Thus, by Lemma 5.3.2, $\phi$ itself is standard. The converse is just Theorem 5.3.1.

We now describe the weight space structure of standard algebra homomorphisms $\phi : C(A_2) \to \mathbb{C}$. 
Let $\phi : C(A_2) \to C$ be standard, parametrized by $s$ and $
lambda$, with $\nlambda(H_a) = \lambda_1$, $\nlambda(H_b) = \lambda_2$.

Case I: $s \notin \mathbb{Z} \cup (\mathbb{Z} + \lambda_1) \cup (\mathbb{Z} + \lambda_1 + \lambda_2)$. Then $U(A_2)/M_\phi$ is equivalent to a complete representation of order 2; $
lambda + k\alpha + \ell\beta$ is a weight function for all integers $k$ and $\ell$, i.e. both the $\alpha$- and the $\beta$-string through $\nlambda$ are doubly infinite.

Case II: $s \in \mathbb{Z} \cup (\mathbb{Z} + \lambda_1) \cup (\mathbb{Z} + \lambda_1 + \lambda_2)$. Then $\phi$ is weakly equivalent to an algebra homomorphism of type $T_1$ ($i = 1, \ldots, 6$).

Now if $\phi_1$ and $\phi_2$ are weakly equivalent, the weight space decompositions of $U(A_2)/M_{\phi_1}$ and $U(A_2)/M_{\phi_2}$ are "similar" in the sense of Lemma 3.3.4. Thus we may assume that $\phi$ itself is of type $T_1$. Then the following subcases may occur:

1. If $p = s(s-1-\lambda_1) \neq -\lambda_1$, then (cf. Theorem 6.2.1.) we have $s = 1 + \lambda_1 + \lambda_2$. Thus, (cf. the analysis for $A_1$) the weight diagram of $U(A_2)/M_\phi$ is one of the following types:

Type 1a): the $\alpha$-string through $\nlambda$ is doubly infinite if and only if $s \notin \mathbb{Z} \cup (\mathbb{Z} + \lambda_1)$. Since $s = 1 + \lambda_1 + \lambda_2$, $\nlambda + k\alpha - \ell\beta$ is a weight function for every $k \in \mathbb{Z}$ and every non-negative integer $\ell$.

The weight diagram is:
Type 1b): the string through $\lambda$ is singly infinite if and only if $s \in \mathbb{Z} \cup (\mathbb{Z} + \lambda_1)$.

(i) if $s \notin \mathbb{Z}$, $s - \lambda_1 \notin \mathbb{Z}$, the weight diagram is

![Diagram](image1)

and

![Diagram](image2)

(ii) if $s \in \mathbb{Z}$, $s - \lambda_1 \notin \mathbb{Z}$, the weight diagram is

![Diagram](image3)

for $s \leq 0$
and

and

(iii) If $s \in \mathbb{Z} \cap (\mathbb{Z} + \lambda_1)$, with $s$ and $s - \lambda_1$ having opposite signs, the weight diagram is

if $s > 0$, $s - \lambda_1 < 0$

Type 1c). The $\alpha$-string through $\lambda$ is finite if and only if $s \in \mathbb{Z} \cup (\mathbb{Z} + \lambda_1)$ with $s$ and $\lambda_1$ both non-positive or both positive. In the first case, (i.e., $s, s - \lambda_1 \leq 0$), the weight diagram is

if $s > 0$, $s - \lambda_1 < 0$. 

for $s > 0$
If \( s, s - \lambda_1 > 0 \), the weight diagram is

Thus, \( \mathbb{U}(A_2)/M_\phi \) is finite dimensional in this case.

2. If \( s = 0 = \lambda_1 \), then weight diagram is one of two types:

Type 2a): If \( 0 < \lambda_2 \in \mathbb{Z} \) or \( \lambda_2 \notin \mathbb{Z} \), then the weight diagram is

Type 2b): If \( 0 > \lambda_2 \in \mathbb{Z} \), the representation \( \mathbb{U}(A_2)/M_\phi \) is finite dimensional and the weight diagram is
Remark: As we noted before, to each of the above weight diagrams five more are associated, corresponding to the algebra homomorphisms weakly equivalent to \( \phi \).

§3. The structure of \( C(A_n) \), \( n \geq 3 \)

Recall (cf. Chapter 5, §1.) that a Chevalley basis for the Lie algebra \( A_n \) is given by
\[
\{Y_{a_1^1 + \ldots + a_j^j}, X_{a_1^1 + \ldots + a_j^j}, (1 \leq i \leq j \leq n), H_{a_1^i} \ (i=1,2,\ldots,n)\}.
\]

Thus, by the Poincare'-Birkhoff-Witt Theorem, a linear basis for \( U(A_n) \) is given by
\[
\{Y_{a_1^1 + \ldots + a_j^j}, X_{a_1^1 + \ldots + a_j^j}, \ (1 \leq i \leq j \leq n), H_{a_1^i} \}
\]
where each product preserves some fixed order on the basis elements of \( A_n \) and the exponents are non-negative integers.

Our aim in this section is to construct all elementary cycles of \( C(A_n) \) and then show that any algebra homomorphism \( \phi: C(A_n) \rightarrow \mathbb{C}^* \) is determined by its values on cycles of degree \( \leq 3 \).

In §2, we listed the elementary cycles of \( C(A_2) \). They are: \( 1, H_a, H_\beta, c_1 = Y_\alpha X_\alpha, c_2 = Y_\beta X_\beta, c_3 = Y_{\alpha+\beta} X_{\alpha+\beta}, c_4 = Y_{\alpha+\beta} X_\alpha X_\beta, c_5 = Y_\beta Y_\alpha X_{\alpha+\beta} \). We observe that the cycles \( c_i \) (\( i=1,2,\ldots,5 \)) can be identified with certain 3x3 matrices.

Consider the array
To each $c_i$ ($i = 1, \ldots, 5$), we associate a $3 \times 3$ matrix having a "1" in each position corresponding to a factor of that cycle and zeroes elsewhere. For example, $Y_{\alpha+\beta}X_{\alpha}X_{\beta}$ corresponds to

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

Examining the matrices obtained in this manner, we note that each row and column contains at most one "1", and if a row has a "1", the corresponding column also has a "1" and vice versa.

The next theorem (due to Lemire) shows that the identification process described above can be generalized for arbitrary $n$. More precisely, we have:

**Theorem 6.3.1.** The algebra $\mathcal{C}(A_n)$ is generated by the set

\[
\{l, H_\alpha, \ldots, H_\alpha \} \cup \{ \mathcal{C}(M) = \prod_{1 \leq i < j \leq n} Y_{\alpha_{i+j-1}} \prod_{1 \leq i < j \leq n} X_{\alpha_{i+j-1}} \}
\]

where $M = (m_{i,j})$ is a non-zero $(n+1)$ x $(n+1)$ matrix of zeroes and "1"-s with $m_{i,i} = 0$ and

\[
\sum_{i=1}^{n+1} m_{i,k} = \sum_{i=1}^{n+1} m_{k,i} = 0 \text{ or } 1 \text{ for each } k, \text{ and } M \text{ cannot be expressed as a} \]
non-trivial sum of two such matrices.

Proof: We must show that every basis monomial of $C(A_n)$ is a linear combination of products of the given generators. We shall proceed by double induction; first, on $n$, and then on the degree of the basis monomials. The statement is trivially true for $A_1$. We now assume that the theorem holds for $A_{n-1}$ ($n \geq 2$) and that every basis monomial $c \in C(A_n)$ of degree less than $k_0$ can be expressed as a linear combination of the given generators.

Let $c \in C(A_n)$ be a basis monomial of degree $k_0$. Clearly we may assume that $c$ has no factor of the form $H_{a_i}^j$, i.e. that $c$ is of the form

$$c = \prod_{1 \leq i < j \leq n} Y_{a_i}^{j+1, i} \prod_{1 \leq i < j \leq n} X_{a_i}^{j+1, i} + \alpha_j$$

and we associate with $c$ the matrix $(a_i, j)$ with $a_i, j = 0$. If $(a_i, j)$ is one of the matrices described in the statement of the theorem, we are finished. If not, then we must have that

$$n+1 \sum_{i=1}^{n+1} s_i, k = \sum_{i=1}^{n+1} s_i k, i \geq 2$$

for some $k$. Without loss of generality, we may assume that this is true for $k = n + 1$. We now factor $c$ into generating elements of $C(A_{n-1}[a_1, \ldots, a_{n-1}])$ by suppressing the index $a_n$. Say $c = c_1 c_2 \cdots c_p + \text{terms of lower degree}$. (Whenever $Y_{a_n}$ or $X_{a_n}$ occur as factors in
c., they are treated as separate factors in this product.)

Since each factor $c_i$ is a generating element of $C(A_{n-1}\{a_1,\ldots,a_{n-1}\})$ or one of the terms $Y_{\alpha_n}$ or $X_{\alpha_n}$, it can contain at most one factor of the form $Y_{\alpha_1+\ldots+\alpha_n}$. Thus, for each $i$, $c_i \in C(A_n)$ or $c_i \in U(A_n)_{-\alpha_n}$.

Since by assumption $\sum_{i=1}^{n+1} \xi_{i,n+1} = \sum_{i=1}^{n+1} \xi_{n+1,i} \geq 2$, the above factorization must contain at least two factors. If there are exactly two factors, then each must contain exactly one term of the form $Y_{\alpha_1+\ldots+\alpha_n}$ and one term of the form $X_{\alpha_j+\ldots+\alpha_n}$, and hence both factors are in $C(A_n)$ and we may apply the inductive hypothesis to each factor. If there are more than two factors, then either all are in $C(A_n)$ in which case we are finished, or at least one (say, $c_1$) is in $U(A_n)_{\alpha_n}$, and another (say, $c_{1'}$) is in $U(A_n)_{-\alpha_n}$. Then $c = (c_1c_{1'})(c_2\ldots) +$ terms of lower degree) and $c_1c_1', c_2 \ldots$ are all in $C(A_n)$.

Applying the inductive hypothesis completes the proof.

Remark: The above proof is essentially a generalization of Bouwer's "inspection" method, used to construct the elementary cycles of $C(A_2)$. (cf. [2a], p. 200).

**Lemma 6.3.1.** The following identities hold in $C(A_n)$:
a) \[ [Y a X a, Y a + \ldots + a X a \ldots X a ] = \]
\[ = Y a_1 + \ldots + a X a_1 \ldots X a_n - Y a_1 Y a_1 + \ldots + a X a_1 \ldots X a_n + a \]

b) \[ [Y a X a, Y a + \ldots + a X a \ldots X a ] = \]
\[ = Y a_2 + \ldots + a X a_1 + a X a_3 \ldots X a_n - Y a_1 + \ldots + a X a_1 \ldots X a_n \]

c) \[ [Y a X a, Y a + \ldots + a X a \ldots X a ] = \]
\[ = Y a_1 + \ldots + a X a_1 \ldots X a_n - \]
\[ - Y a_1 + \ldots + a X a_1 \ldots X a_l + a X a_l + a X a_l + a \]
\[ - Y a_1 + \ldots + a X a_1 \ldots X a_l + a X a_l + a \]
\[ \text{for } i = 2, \ldots, n-1. \]

d) \[ (Y a Y a + \ldots + a X a \ldots X a ) (Y a_1 + \ldots + a X a_1 \ldots X a_n ) \]
\[ = (Y a_1 + \ldots + a X a_1 \ldots X a_l + a Y a_1 + \ldots + a X a_1 \ldots X a_n ) Y a_{n-1} X a_n \]
\[ + Y a_1 + \ldots + a X a_1 \ldots X a_l + a \]
\[ + (Y a_1 + \ldots + a X a_1 \ldots X a_n ) \]

e) \[ (Y a_2 + \ldots + a X a \ldots X a ) (Y a_1 + \ldots + a X a_1 \ldots X a_n ) \]
\[ = Y a_2 + \ldots + a X a_1 \ldots X a_l + a X a_1 \ldots + a X a_2 \ldots X a_n-1 \]
\[ + (Y a_1 + \ldots + a X a_1 \ldots X a_2 + a X a_3 \ldots X a_n-1 ) \]
\[ + (Y a_2 + \ldots + a X a_2 \ldots + a ) (Y a_1 + \ldots + a X a_1 \ldots X a_n-1 ) \]
To prove these identities, one simply has to use the commutation relations given in Chapter 5, §1. With the aid of the above Lemma, we now prove that any algebra homomorphism \( \phi : C(A_n) \to \mathbb{C} \) is determined by its values on a small number of generators. More precisely, we have

**Theorem 6.3.2.** An algebra homomorphism \( \phi : C(A_n) \to \mathbb{C} \) is determined by its values on the generators of \( C(A_n) \) of degree \( \leq 3 \). In particular, if \( \phi = 0 \) on all generators of degree \( \leq 2 \), then \( \phi \equiv 0 \) on \( C(A_n) \).

**Proof:** First note that the automorphisms \( \sigma_{a_{ii}} \in \text{Aut}(A_n) \) can be realized by setting \( \sigma_{a_{ii}}(X) = P_i^{-1}XP_i \) for all \( X \in A_n \), where \( P_i \) is the permutation matrix of the transposition \((i, i+1)\).

To prove the theorem, we shall use induction. The cases \( n = 1 \) and \( n = 2 \) are trivially true. Since every generator of \( C(A_n) \) of degree \( \leq n \) is contained in a subalgebra isomorphic to \( C(A_{n-1}) \), it suffices to show that the values of \( \phi \) on the generators of degree \( n + 1 \) are determined by the values of \( \phi \) on the generators of degree \( \leq n \).

Set \( M_0 = e_{n+1,1} + \sum_{i=1}^{n} e_{i,i+1} \) (where as usual, \( e_{i,j} \) denotes an \((n+1)\times(n+1)\) matrix with "1" in the \((i,j)\)-th position and zeroes elsewhere).
By Lemma 6.2.1 a) and b), we obtain
\[ \phi(c(M)) = \phi(c(P^{-1}M_0 P_n)) = \phi(c(P^{-1}M_0 P_1)) \]
and by part c) of the same lemma, we have
\[ \phi(c(M_0)) = \phi(c(P^{-1}M_0 P_1)) + \phi(a \text{ degree } n \text{ term}) \]
for \( i = 2, 3, \ldots, n-1 \).

Now if \( c(M) \) is any degree \( n+1 \) generator of \( C(A_n) \), then \( M = P^{-1}M_0 P \), where \( P \) is a product of transposition matrices. By applying the corresponding products of automorphisms to the above identities, we have that
\[ \phi(c(M)) = \phi(c(M_0)) + \phi(\text{terms of degree } \leq n). \]
Thus \( \phi \) is determined by its values on generators of degree \( \leq n \) and one generator of degree \( n+1 \), namely \( c(M_0) \).

Now assume that \( \phi = 0 \) on all generators \( (\neq 1) \) of degree \( \leq n \). Applying \( \phi \) to identity d) of Lemma 6.2.1, we obtain \( \phi(c(P^{-1}M_0 P_n))\phi(c(M_0)) = 0 \). But then by \( \phi \), we have \( \phi(c(M_0))^2 = 0 \), and thus \( \phi \) is identically zero on all degree \( n+1 \) generators. Now by §2 of this chapter, we know that if \( \phi: C(A_n) \to C \) is an algebra homomorphism which is zero on degree 1 and 2 generators, then \( \phi \) is also zero on all degree 3 generators and hence is identically zero.

Thus, assume that \( \phi \) is not zero on some generator of degree \( \leq 2 \). We may assume that \( \phi(Y_1 X_0) \neq 0 \). Then applying the map \( \phi \) to identity e) of Lemma 6.2.1, we obtain
that the value of $\phi$ on a generator of degree $n+1$ (namely $\phi(Y_2^{a_2} + \ldots + a_{n-1} Y_1^{a_1} + \ldots + a_0 X_2^{a_2} \ldots X_1^{a_1})$) can be expressed as a rational function of the values of $\phi$ on generators of degree $\leq n$.

We are as yet unable to construct all algebra homomorphisms $\phi: C(A_n) \rightarrow C$. We have succeeded in the case $n = 3$, however. We shall give this construction in the next section, and then, using the above theorem, establish some partial results regarding the general case.

§ 4. Algebra Homomorphisms on $C(A_3)$

We now apply the results of the previous two sections in order to construct all algebra homomorphisms $\phi: C(A_3) \rightarrow C$.

Let $\{a_1, a_2, a_3\} = \{\alpha, \beta, \gamma\}$ be the set of simple roots of $A_3$. Then by Theorem 6.3.1 the elementary cycles of $C(A_3)$ are:

1, $H_{\alpha}$, $H_{\beta}$, $H_{\gamma}$, $c_1 = Y_\alpha X_\alpha$, $c_2 = Y_\beta X_\beta$, $c_3 = Y_\gamma X_\gamma$, $c_4 = Y_{\alpha+\beta} X_{\alpha+\beta}$, $c_5 = Y_{\beta+\gamma} X_{\beta+\gamma}$, $c_6 = Y_{\alpha+\beta+\gamma} X_{\alpha+\beta+\gamma}$, $c_7 = Y_{\alpha+\gamma} X_{\alpha+\gamma}$, $c_8 = Y_{\beta+\gamma} X_{\beta+\gamma}$, $c_9 = Y_{\beta+\gamma} X_{\beta+\gamma}$, $c_{10} = Y_{\gamma} Y_{\beta} X_{\beta+\gamma}$, $c_{11} = Y_{\alpha+\beta+\gamma} X_{\alpha+\beta+\gamma}$, $c_{12} = Y_{\alpha+\beta+\gamma} X_{\alpha+\beta+\gamma}$, $c_{13} = Y_{\gamma} Y_{\alpha+\beta} X_{\alpha+\beta+\gamma}$, $c_{14} = Y_{\alpha+\gamma} X_{\alpha+\gamma}$, $c_{15} = Y_{\alpha+\gamma} X_{\alpha+\gamma}$, $c_{16} = Y_{\gamma} Y_{\beta} X_{\beta+\gamma}$, $c_{17} = Y_{\alpha+\beta+\gamma} X_{\alpha+\beta+\gamma}$, $c_{18} = Y_{\gamma} Y_{\alpha+\beta} X_{\beta+\gamma}$, $c_{19} = Y_{\gamma} Y_{\alpha+\beta} X_{\beta+\gamma}$, $c_{20} = Y_{\beta+\gamma} Y_{\alpha} X_{\alpha+\beta}$.
Let \( \Phi_1 = \{\alpha, \beta\} \), \( \Phi_2 = \{\beta, \gamma\} \), \( \Phi_3 = \{\alpha, \beta, \gamma\} \), \( \Phi_4 = \{\alpha+\beta, \gamma\} \).

Then \( B_1 = \{H_\alpha, H_\beta, Y_\alpha, Y_\beta, Y_{\alpha+\beta}, X_\alpha, X_\beta, X_{\alpha+\beta}\} \),
\( B_2 = \{H_\beta, H_\gamma, Y_\beta, Y_\gamma, Y_{\beta+\gamma}, X_\beta, X_\gamma, X_{\beta+\gamma}\} \),
\( B_3 = \{H_\alpha, H_\beta + H_\gamma, Y_\alpha, Y_{\beta+\gamma}, Y_{\alpha+\beta+\gamma}, X_\alpha, X_{\beta+\gamma}, X_{\alpha+\beta+\gamma}\} \), and
\( B_4 = \{H_\alpha + H_\beta, H_\gamma, Y_{\alpha+\beta}, Y_\gamma, Y_{\alpha+\beta+\gamma}, X_{\alpha+\beta}, X_\gamma, X_{\alpha+\beta+\gamma}\} \) are Cartan bases for subalgebras \( A_2(1) \) (\( i=1, 2, 3, 4 \)) of \( A_3 \), each isomorphic to the Lie algebra \( A_2 \). Thus, if \( \phi: C(A_3) \to \mathbb{C} \) is an algebra homomorphism, then the restriction of \( \phi \) to each of the \( C(A_2(1)) \) must be an algebra homomorphism and hence must coincide with one of the solutions obtained for \( C(A_2) \). Also, we must have
\[ \phi(c_i c_j - c_j c_i) = 0 \] for all elementary cycles \( c_i, c_j \) of \( C(A_3) \).

(Actually, \( \phi \) could be constructed using these relations only, however it is much simpler to follow the method outlined above and use relations (*) to rule out certain possibilities.)

Now to construct an algebra homomorphism \( \phi: C(A_3) \to \mathbb{C} \), we first note that by Theorem 6.3.2, \( \phi \) is determined by its values on \( c_1 - c_1^4 \). First assume that the restriction of \( \phi \) to at least one of the copies \( C(A_2(1)) \) (\( i=1, \ldots, 4 \)) is of type \( T_j \) (\( j=1, \ldots, 6 \)). (cf. §2, p. 106). Without loss of generality, we may assume that \( \phi + C(A_2(\alpha, \beta+\gamma) \) is of type \( T_1 \) or \( T_4 \). In fact, this can always be achieved by applying an
appropriate automorphism \( \sigma \in \text{Aut}(A_3:H) \). For example, if \( \phi^+C(A_2\{a, b\}) \) is of type \( T_2 \), then \( \phi \circ \sigma_b \circ \sigma_y \) is of type \( T_1 \) on \( C(A_2\{a, b+y\}) \). (For a table on the operation of the operation of the Weyl group on the generations of \( C(A_3) \), see Appendix A.)

Thus, assume that \( \phi^+C(A_2\{a, b+y\}) \) is of type \( T_1 \).

Then the possible values of \( \phi \) on the generators of \( C(A_2\{b, y\}) \) and \( C(A_2\{a+b, y\}) \) are provided in the following table:

**Table 6.4.1**

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II_1</th>
<th>II_2</th>
<th>II_3</th>
<th>II_4</th>
<th>III_1</th>
<th>III_2</th>
<th>III_3</th>
<th>III_4</th>
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<tr>
<td>( \phi(H_a) )</td>
<td>( \lambda_1 )</td>
<td>( \lambda_2 )</td>
<td>( \lambda_3 )</td>
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<td>( \phi )</td>
<td>( \phi )</td>
<td>( \phi )</td>
<td>( \phi )</td>
<td>( \phi )</td>
</tr>
<tr>
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<td>( q )</td>
<td>( q )</td>
<td>( r )</td>
<td>( r )</td>
<td>( r )</td>
<td>( r )</td>
<td>( r )</td>
<td>( r )</td>
</tr>
<tr>
<td>( \phi(H_y) )</td>
<td>( p )</td>
<td>( p )</td>
<td>( p )</td>
<td>( p )</td>
<td>( p )</td>
<td>( p )</td>
<td>( p )</td>
<td>( p )</td>
<td>( p )</td>
</tr>
<tr>
<td>( \phi(c_1) )</td>
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<td>( 0 )</td>
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<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
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</tr>
<tr>
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<td>( 0 )</td>
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</tr>
<tr>
<td>( \phi(c_3) )</td>
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<td>( 0 )</td>
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<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \phi(c_4) )</td>
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<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \phi(c_5) )</td>
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<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \phi(c_6) )</td>
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<td>( 0 )</td>
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<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \phi(c_7) = \phi(c_9) )</td>
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<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \phi(c_8) = \phi(c_{10}) )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \phi(c_{11}) = \phi(c_{13}) )</td>
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<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \phi(c_{12}) = \phi(c_{14}) )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \phi(c_{15}) = \ldots = \phi(c_{18}) )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \phi(c_{19}) = \ldots = \phi(c_{21}) )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
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<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>
($\lambda_1, \lambda_2, \lambda_3, p, q, r$ are fixed but arbitrary (cf. restrictions below) complex numbers.)

Remarks: The values in column I represent the assumption that $\Phi^+_{C(A_2(q, \beta + \gamma))}$ is of type $T_\gamma$. Columns $II_1$-$II_4$ and $III_1$-$III_4$ contain the possible values for $\Phi^+_{C(A_2(a + \beta, \gamma))}$, and consistent with $\Phi(c_5) = 0$ (respectively, $\Phi(c_6) = 0$). In columns $II_3$, $II_4$ we also must have $\lambda_2 + \lambda_3 = 0$, and in columns $III_4$ and $III_4$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

The fact that $\Phi(c_7) = \Phi(c_9)$, $\Phi(c_8) = \Phi(c_{10})$ ... $\Phi(c_{19}) = \Phi(c_{20})$ follows from the condition $\Phi(c_ic_j - c jc_i) = 0$.

(Cf. §2. for the justification of the first of these relations.)

Now, combining the conditions given by columns $II_1$-$III_j$ ($i, j = 1, 2$) and then looking at the possible values for $\Phi^+_{C(A_2(a, \beta))}$, we obtain that all solutions coincide with (or are special cases of) one of the following solutions:
Let us label the algebra homomorphism having values listed in columns 1 - 7 above by $\phi_1, \ldots, \phi_7$. We observe that

$\phi_3 \circ \sigma_\beta \circ \sigma_\alpha = \phi_7$, $\phi_4 \circ \sigma_\alpha = \phi_7$, $\phi_5 \circ \sigma_\alpha \circ \sigma_\beta = \phi_2$ and $\phi_6 \circ \sigma_\beta = \phi_2$.

It can also be verified, (using again the commutation relations resulting from the condition $\phi(c_1c_j) = \phi(c_jc_1)$) that assuming that $\phi$ satisfies the conditions given in columns II$_1$ - III$_j$ ($i=1,2,3,4$, $j=3,4$) of Table 6.4.1 does not yield any new solutions. (For an example, cf. Appendix B).
Thus there are only three basic types of algebra homomorphisms 
\( \phi : C(A_3) \to C \) such that \( \phi_C(A_2(\alpha, \beta + \gamma)) \) is of type \( T_i \). Moreover, a similar analysis carried out under the assumption that 
\( \phi_C(A_2(\alpha, \beta + \gamma)) \) is of type \( T_4 \) also does not yield any new solutions, i.e. all the possible homomorphisms obtained that way are weakly equivalent to \( \phi_1, \phi_2 \) or \( \phi_7 \) above.

Now \( \phi_1 = 0 \) on all generators of \( C(A_3) \) in 
\( C(\pm\alpha, \pm(\alpha + \beta), 0) \) of degree \( \leq 3 \). Thus \( \phi_1 \) coincides with the trivial extension of a standard algebra homomorphism 
\( \phi : C(A_2(\alpha, \beta)) \to C \), i.e. \( \phi_1 \) is \( g \)-standard relative to 
\( \Gamma_C = \{0, \pm\alpha, \pm(\alpha + \beta)\} \).

Since \( \phi_2 = 0 \) on all generators of degree \( \leq 3 \) in 
\( C(\pm\alpha, \pm\gamma, 0) \), \( \phi_2 \) coincides with the trivial extension of the algebra homomorphisms \( \phi^{(1)} : C(A_1(\alpha)) \to C \) and \( \phi^{(2)} : C(A_1(\gamma)) \to C \) and hence \( \phi_2 \) is \( g \)-standard relative to \( \Gamma_C^{(1)} \cup \Gamma_C^{(2)} \), where 
\( \Gamma_C^{(1)} = \{0, \pm\alpha\} \), \( \Gamma_C^{(2)} = \{0, \pm\gamma\} \).

Finally, \( \phi_7 \) is \( g \)-standard relative to \( \Gamma_C = \{0, \pm\beta\} \).

Thus, if \( \phi \) is an algebra homomorphism on \( C(A_3) \) such that its restriction to at least one copy of \( C(A_2) \) is of type 
\( T_i \) \((i=1, \ldots, 6)\), then \( \phi \) is weakly equivalent to a \( g \)-standard algebra homomorphism.
It remains to consider the possibility that \( \phi: \mathbb{C}(A_3) \to \mathbb{C} \) is an algebra homomorphism such that the restrictions of \( \phi \) to each of the four copies \( \mathbb{C}(A_2(1)) \) are standard, parametrized as follows:

Table 6.4.3

<table>
<thead>
<tr>
<th>( \phi(H_a) )</th>
<th>( \phi(H_b) )</th>
<th>( \phi(H_c) )</th>
<th>( \phi(c_1) )</th>
<th>( \phi(c_2) )</th>
<th>( \phi(c_3) )</th>
<th>( \phi(c_4) )</th>
<th>( \phi(c_5) )</th>
<th>( \phi(c_6) )</th>
<th>( \phi(c_7)=\phi(c_9) )</th>
<th>( \phi(c_8)=\phi(c_{10}) )</th>
<th>( \phi(c_9)=\phi(c_{13}) )</th>
<th>( \phi(c_{12})=\phi(c_{14}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>( \lambda_2 )</td>
<td>( \lambda_3 )</td>
<td>( s(s-1-\lambda_1) )</td>
<td>( (s-\lambda_1)(s-\lambda_2-1) )</td>
<td>( (t-\lambda_2)(t-\lambda_2-1) )</td>
<td>( v(v-1-\lambda_1-\lambda_2) )</td>
<td>( t(t-1-\lambda_2) )</td>
<td>( s(s-\lambda_1)(s-\lambda_1-\lambda_2-1) )</td>
<td>( s(s-\lambda_1)(s-\lambda_1-\lambda_2-1) )</td>
<td>( t(t-\lambda_2) )</td>
<td>( v(v-1-\lambda_1-\lambda_2) )</td>
<td>( v(v-1-\lambda_1-\lambda_2) )</td>
</tr>
<tr>
<td>( \lambda_1 )</td>
<td>( \lambda_2 )</td>
<td>( \lambda_3 )</td>
<td>( t(t-\lambda_2-1) )</td>
<td>( (t-\lambda_2)(t-\lambda_2-1) )</td>
<td>( (v-\lambda_1-\lambda_2)(v-\lambda_3-1) )</td>
<td>( v(v-1-\lambda_1-\lambda_2) )</td>
<td>( t(t-1-\lambda_2) )</td>
<td>( (t-\lambda_2)(t-\lambda_2-1) )</td>
<td>( (t-\lambda_2)(t-\lambda_2-1) )</td>
<td>( t(t-\lambda_2)(t-\lambda_2-1) )</td>
<td>( v(v-\lambda_1-\lambda_2)(v-\lambda_1-\lambda_2-1) )</td>
<td>( u(u-\lambda_1)(u-\lambda_1-\lambda_2-1) )</td>
</tr>
</tbody>
</table>
In order that \( \phi \) be well defined we must have that the following relations hold among the parameters:

1) \( s = u \) or \( s = 1 + \lambda_1 - u \)
2) \( s = t + \lambda_1 \) or \( s = 1 + \lambda_1 + \lambda_2 - t \)
3) \( t = v - \lambda_1 \) or \( t = 1 + \lambda_1 + 2\lambda_2 + \lambda_3 - v \)
4) \( s = v \) or \( s = 1 + \lambda_1 + \lambda_2 - v \)
5) \( t = u - \lambda_1 \) or \( t = 1 + \lambda_1 + \lambda_2 + \lambda_3 - u \)
6) \( v = u \) or \( v = 1 + \lambda_1 + \lambda_2 + \lambda_3 - u \).

It can be directly verified [cf. Lemire 10b] that choosing the solutions \( s = u = v = t + \lambda_1 \), yields a standard algebra homomorphism \( \phi : C(A_3) \rightarrow \mathbb{C} \). In all the other cases we obtain algebra homomorphisms which are weakly equivalent to (or special cases of) algebra homomorphisms listed in Table 6.4.2.

(For example, the choices \( s = 1 + \lambda_1 - u = 1 + \lambda_1 + \lambda_2 - t \), \( t = 1 + \lambda_1 + 2\lambda_2 + \lambda_3 - v \), \( s = 1 + \lambda_1 + \lambda_2 - v \), \( t = u - \lambda_1 \), \( u = v \) yield \( s = 0 \), i.e. \( \phi \) must be of type \( T_i \) \( (i=1, \ldots, 6) \) on at least one copy of \( C(A_2) \).

Thus every algebra homomorphism \( \phi : C(A_3) \rightarrow \mathbb{C} \) is weakly equivalent to a \( g \)-standard one.

**Theorem 6.4.1.** If \( \phi : C(A_n) \rightarrow \mathbb{C} \) is an algebra homomorphism such that the restriction of \( \phi \) to each isomorphic copy of \( C(A_3) \) in \( C(A_n) \) is standard then \( \phi \) is standard.
Proof: We use induction on \( n \). The statement is trivially true for \( n = 3 \). Assume it is true for \( n-1 \geq 3 \) and let \( \phi: C(A_n) \to C \) satisfy the conditions of the theorem.

Then the restrictions of \( \phi \) to each of the subalgebras
\[
C(A_{n-1}\{a_1, a_2, \ldots, a_{n-1}\}), \quad C(A_{n-1}\{a_1+a_2, a_3, \ldots, a_n\})
\]
\[
C(A_{n-1}\{a_1, a_2+a_3, \ldots, a_n\}), \quad \ldots, \quad C(A_{n-1}\{a_2, \ldots, a_n\})
\]
are standard, parametrized by \( s_1, s_2, \ldots, s_{n+1} \). In order for \( \phi \) to be well defined, we must have \( s_1 = s_2 = \ldots = s_n = s_{n+1} + \phi(h_{a_1}) \).

Thus \( \phi \) coincides on all generators of degree \( \leq 3 \) with a standard algebra homomorphism (since each such generator belongs to at least one of the copies of \( C(A_{n-1}) \) listed above), and hence by Theorem 6.3.1, \( \phi \) must be standard.

**Corollary 6.4.1.** If \((\rho, V)\) is a complete order \( n \) representation of \( A_n \) with characteristic weight \( \lambda \), then the mass function associated with the weight space \( V_{\lambda} \) is a standard complete algebra homomorphism.

Proof: By Definition 4.3.2 we have that \( \lambda + k\alpha_i \) is a weight function of the representation \((\rho, V)\) for each simple root \( \alpha_i \) \((i=1, \ldots, n)\) and every integer \( k \). Let \( \phi \) denote the mass function associated with the \( \lambda \)-weight space of \((\rho, V)\). Then the restriction of \( \phi \) to each copy of \( C(A_2) \) must be standard complete by the above remark, and hence by Theorem 6.4.1 \( \phi \) itself must be standard complete.
Remark: The question whether a complete representation of arbitrary order admits a standard mass function remains open.

§5. Conjectures

Using the results of this chapter as a guide, we now present some conjectures concerning the characterization of pointed representations.

Conjecture 1: If $L$ is a simple Lie algebra and $\phi : C(L) \to \mathcal{C}$ is an algebra homomorphism, then $\phi$ is weakly equivalent to a $g$-standard algebra homomorphism.

Conjecture 2: For any algebra homomorphism $\phi : C(L) \to \mathcal{C}$, there exists an extreme $g$-standard algebra homomorphism $\phi_0 : C(L) \to \mathcal{C}$ such that $U(L)/M_{\phi \circ \sigma} \cong U(L)/M_{\phi_0}$ for some automorphism $\sigma \in \text{Aut}(L;H)$.

Note: As we have proved in Chapter 5, §4, if Conjecture 1 is true for a simple Lie algebra $L$, then Conjecture 2 is also true.

In Chapter 6 we proved that a complete order $n$ representation of $A_n$ admits a standard complete mass function. This leads us to the following:

Conjecture 3: Let $(\rho, V)$ be a $\Gamma_0$-complete representation of $A_n$ where $\Gamma_0$ is an arbitrary set of simple roots. Then $(\rho, V)$ admits an extreme $g$-standard mass function.
Appendix A

Operation of some elements of $\text{Aut}(A_3; H)$ on the generators of $C(A_3)$.

The Weyl group of $A_3$ is generated by the reflections $S_\alpha, S_\beta, S_\gamma$ (cf. Chapter I, §8). Thus (cf. Chapter I, §9) the maps $S_\alpha, S_\beta, S_\gamma$ induce automorphisms of the Cartan subalgebra $H$, which can then be uniquely extended to $U(A_3)$. We denote these extensions (as well as their restrictions to $C(A_3)$) by $\sigma_\alpha, \sigma_\beta, \sigma_\gamma$ respectively.

In order to justify some of the remarks made in Chapter 6, §4, we shall also need (cf. Appendix B) the automorphism $\phi : C(A_3) \to C(A_3)$ which is the extension of the automorphism induced by $\phi_\Delta : \Delta \to \Delta$ of the root system, given by $\phi_\Delta(\alpha) = \gamma$, $\phi_\Delta(\beta) = \beta$, $\phi_\Delta(\gamma) = \alpha$.

We now present the action of $\sigma_\alpha, \sigma_\beta, \sigma_\gamma$ and $\phi$ on the elementary cycles of $C(A_3)$ in a table form.
<table>
<thead>
<tr>
<th>$\sigma_\alpha$</th>
<th>$\sigma_\beta$</th>
<th>$\sigma_\gamma$</th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_a$</td>
<td>$-H_a$</td>
<td>$H_a + H_\beta$</td>
<td>$H_a$</td>
</tr>
<tr>
<td>$H_\beta$</td>
<td>$H_a + H_\beta$</td>
<td>$-H_\beta$</td>
<td>$H_\beta + H_\gamma$</td>
</tr>
<tr>
<td>$H_\gamma$</td>
<td>$H_\gamma$</td>
<td>$H_\beta + H_\gamma$</td>
<td>$-H_\gamma$</td>
</tr>
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<tr>
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<td>$c_4$</td>
<td>$c_2 + H_\beta$</td>
<td>$c_5$</td>
</tr>
<tr>
<td>$c_3$</td>
<td>$c_3$</td>
<td>$c_5$</td>
<td>$c_3 + H_\gamma$</td>
</tr>
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<td>$c_6$</td>
<td>$c_4$</td>
</tr>
<tr>
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<td>$c_{12}$</td>
<td>$-c_8 + c_5$</td>
</tr>
<tr>
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<td>$c_4$</td>
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<td>$c_{10} + c_2 - c_5$</td>
</tr>
<tr>
<td>$c_9$</td>
<td>$c_7$</td>
<td>$c_{7} + c_1 - c_4$</td>
<td>$c_{14}$</td>
</tr>
<tr>
<td>$c_{10}$</td>
<td>$c_{13}$</td>
<td>$c_8$</td>
<td>$c_{8} + c_2 - c_5$</td>
</tr>
<tr>
<td>$c_{11}$</td>
<td>$c_8$</td>
<td>$c_{12}$</td>
<td>$c_{13} + c_4 - c_6$</td>
</tr>
<tr>
<td>$c_{12}$</td>
<td>$c_{14}$</td>
<td>$c_4$</td>
<td>$c_7$</td>
</tr>
<tr>
<td>$c_{13}$</td>
<td>$c_{10}$</td>
<td>$c_{14}$</td>
<td>$c_{11} + c_4 - c_6$</td>
</tr>
<tr>
<td>$c_{14}$</td>
<td>$c_{12}$</td>
<td>$c_{13}$</td>
<td>$c_9$</td>
</tr>
<tr>
<td>$c_{15}$</td>
<td>$c_{17}$</td>
<td>$c_{19} + c_1$</td>
<td>$c_{18} + c_7 - c_{12}$</td>
</tr>
<tr>
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<td>$c_{18}$</td>
<td>$c_{20} + c_4$</td>
<td>$c_{17} + c_9 - c_{14}$</td>
</tr>
<tr>
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<td>$c_{15}$</td>
<td>$c_{18}$</td>
<td>$c_{16} + c_9 - c_{14}$</td>
</tr>
<tr>
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<td>$c_{16}$</td>
<td>$c_{17}$</td>
<td>$c_{15} + c_7 - c_{12}$</td>
</tr>
<tr>
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<td>$c_{20}$</td>
<td>$c_{15} - c_1$</td>
<td>$c_{20}$</td>
</tr>
<tr>
<td>$c_{20}$</td>
<td>$c_{19}$</td>
<td>$c_{16} - c_3$</td>
<td>$c_{19}$</td>
</tr>
</tbody>
</table>
Appendix B

In Chapter 6, §4, we claimed that Table 6.4.2 gives (up to weak equivalence) all algebra homomorphisms \( \phi : C(A_3) \to \mathbb{C} \) such that the restriction of \( \phi \) to at least one copy of \( C(A_2) \) is of type \( T_1 \) (cf. Chapter 6, §3). We now present some of the calculations involved in verifying that claim (we shall use the notation of Appendix A).

Assume, for example, that \( \phi : C(A_3) \to \mathbb{C} \) is an algebra homomorphism satisfying the conditions given in columns I, II, and III of Table 6.4.1, i.e. \( \phi \) has the following values on the first fourteen cycles:

\[
\begin{align*}
H_a & \quad \lambda_1 \\
H_b & \quad \lambda_2 \\
H_Y & \quad \lambda_3 \\
\phi(c_1) & \quad p \\
\phi(c_2) & \quad 0 \\
\phi(c_3) & \quad q \\
\phi(c_4) & \quad -(\lambda_1 + \lambda_2) \\
\phi(c_5) & \quad 0 \\
\phi(c_6) & \quad 0 \\
\phi(c_7) = \phi(c_9) & \quad X \quad \text{to be determined} \\
\phi(c_8) = \phi(c_{10}) & \quad 0 \\
\phi(c_{11}) = \phi(c_{13}) & \quad q \\
\phi(c_{12}) = \phi(c_{14}) & \quad 0
\end{align*}
\]
Note that we also have \( \lambda_1 + \lambda_2 + \lambda_3 = 0 \).

The following identities hold in \( C(A_3) \).

1) \([c_7, c_8] = c_7(c_5 - c_3) + (c_4 - c_1)(c_8 - c_5)\)

2) \([c_1, c_3] = H_a(c_7 - c_4) - c_1(c_4 - c_2)\)

3) \([c_1, c_{15}] = c_1(c_{11} - c_8) - H_a(c_{15} - c_{11})\)

4) \([c_7, c_{10}] = H_\beta(c_{17} - c_{12}) + (c_{13} - c_{12})c_2\)

Since in order for \( \phi \) to be an algebra homomorphism we must have \( \phi[c_i, c_j] = 0 \) for all \( i, j = 1, \ldots, 20 \), equation 1 above implies \( X \cdot q = 0 \) (since \( \phi(c_5) = \phi(c_8) = 0 \)). Now if \( q = 0 \) then \( \phi \) is \( g \)-standard with respect to \( \Gamma_c = \{ \mp a, \pm \beta, 0 \} \). Now consider the case \( X = 0 \). Then by identity 2) above, either a) \( p = -\phi(H_a) = -\lambda \), or b) \( \lambda_1 + \lambda_2 = 0 \).

In case a) \( \phi \circ \sigma_a \circ \phi \) is a \( g \)-standard homomorphism defined relative to \( \Gamma_c = \{ \mp a, \pm \beta, \mp(a+\beta), 0 \} \).

In case b), equation 4) above implies (i) \( \lambda_2 = 0 \) or (ii) \( \phi(c_{17}) = 0 \).

(i) If \( \lambda_2 = 0 \) then \( \lambda_1 = 0 \) (since \( \lambda_1 + \lambda_2 = 0 \)) and hence \( \lambda_3 = 0 \) (since \( \lambda_1 + \lambda_2 + \lambda_3 = 0 \)). Then by identity 3) \( p = 0 \) - or \( q = 0 \).

If \( p = 0 \), then \( \phi \circ \sigma_a \circ \phi \circ \sigma_a \) is \( g \)-standard relative to \( \Gamma_c = \{ \mp a, \pm \beta, \mp(a+\beta), 0 \} \). If \( q=0 \) then \( \phi \) is \( g \)-standard relative to \( \Gamma_c = \{ 0, \mp a \} \).
(ii) If $\phi(c_{17}) = 0$, then $\phi(c_{15}) = \phi(c_{16}) = \ldots = \phi(c_{20}) = 0$.

and by equation 3) above, $p = -\lambda_1$ or $q = 0$. If $p = -\lambda_1$,
we are back in case a). If $q = 0$, then $\phi$ is $g$-standard
relative to $\Gamma_c = \{0, \pm a, \pm b, \pm (a+b)\}$.

Thus all the solutions obtained by assuming that the
conditions given by columns II$_1$ and III$_3$ are satisfied, are
weakly equivalent to a $g$-standard algebra homomorphism.
BIBLIOGRAPHY


VITA AUCTORIS

Maria Pap was born on August 23, 1948 in Budapest, Hungary. She attended Eötvös Loránd University in Budapest from 1966 to 1968. In 1968 she moved to Canada. She obtained a Bachelor of Science Degree in 1971 and a Master of Science Degree in 1972, both from the University of Windsor.