1980

SHALLOW FLEXIBLE CYLINDRICAL GRAIN BINS.

ADEL A. MAHMOUD

University of Windsor

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SHALLOW FLEXIBLE CYLINDRICAL GRAIN BINS

BY

C

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A Dissertation
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in Partial Fulfilment for the Requirements of
Doctor of Philosophy
University of Windsor
1979

Windsor, Ontario
To my wife and my parents
The most common on-the-farm grain storage bins are built of shallow cylindrical shells made of cold-formed sheet steel. They constitute a major section of the farm building industry. However, little reliable data is available regarding the resultant loads applied to the walls by the material stored.

The present study is directed to develop additional information regarding the pressures and the pressure-inducing characteristics of the grain stored taking into consideration the interaction between the grain pressure and the deformation of the wall of the bin.

The finite element method is applied considering a composite slice of the grain and the surrounding wall of the bin. It accounts for the flexibility of the wall, the non-linearity of the grain material, and for the variation of grain properties due to the confining pressure and strain level.

A test was conducted on a model of a cylindrical bin filled with sand. The properties of the sand are obtained from a triaxial test. The experimentally measured pressures of the sand are found to be in close agreement with those obtained analytically.

The understanding of interaction between the grain pressures and wall deformations lead to suggesting a new shape for the bin walls. Herein, the wall is curved in the vertical direction towards the grain in order to increase
the strength of the wall through its interaction with the grain pressure.
ACKNOWLEDGEMENTS

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NOMENCLATURE

Any symbol used is generally defined when introduced.

The standard symbols are listed below:

\( a, b, c \)  Experimental constants
\( A \)  Cross-sectional area
\( C \)  Cohesion of the soil
\( d \)  Average diameter of sand used
\( D \)  Diameter of the bin
\( E \)  Modulus of elasticity
\( E_i \)  Initial tangent modulus of elasticity
\( E_t \)  Tangent modulus of elasticity
\( F, G \)  Experimental constants
\( I \)  Moment of inertia
\( h \)  Height of filling
\( H \)  Height of the bin
\( i, j, k, \ell \)  Element node numbering
\( k_f \)  Equivalent hoop stresses spring coefficient
\( K_a \)  Ratio of horizontal to vertical pressure
\( K_n \)  Modulus number
\( K_{I} \)  Shear stiffness number
\( K_n, K_s \)  Normal and tangential stiffness of the interface
\( K_{\text{si}} \)  Initial shear modulus
\( L \)  Length of the interface element
\( n \)  Modulus exponent
\( n_s \)  Shear stiffness exponent
\( n', m' \)  Internal nodes of shell element
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<td>Resultant moment per unit length in the meridional and circumference directions, respectively</td>
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<td>Resultant normal force per unit length in the meridional and circumference directions, respectively</td>
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<td>$P_s, P_r$</td>
<td>Surface loads in the meridional and normal directions respectively</td>
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<td>$P$</td>
<td>Force equivalent to the hoop stresses</td>
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<td>Failure ratio $((\sigma_1-\sigma_3)/\sigma_1)/((\sigma_1-\sigma_3)/\text{ult.})$</td>
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<td>$R_{fs}$</td>
<td>Shear failure ratio</td>
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\( s \) Arc length
\( \bar{s} \) Surface of a body
\( t \) Bin wall thickness
\( t' \) Time
\( T \) Temperature
\( u_f \) Strain energy stored in the spring supports
\( u_r, u_z \) Displacements in \( r \) and \( z \) directions
\( u_1, u_2 \) Displacements in \( \xi \) and \( \eta \) coordinates
\( u^t_s, u^b_s \) Displacement in the tangential direction at the top and the bottom of the interface
\( u^t_n, u^b_n \) Displacement in the normal direction at the top and the bottom of the interface
\( V \) Volume of a body
\( w \) Water content
\( x, y \) The local cartesian coordinates
\( X, Y \) The global cartesian coordinates

**Greek Letters**
\( \alpha \) Latitude angle
\( a_i \) The generalized coordinates
\( \beta \) The slope of the tangent to the curved shell
\( \beta' \) The angle of surcharge
\( \gamma \) Unit weight of soil
\( \gamma_w \) Unit weight of water
\( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \) Parameters showing the coefficients of the displacement functions for beam element
\( \Delta \) Increment of stress or strain
\( \delta_n, \delta_s \) Average normal and tangential displacements
\( \varepsilon_a \) Axial strain
$\varepsilon_r$ Strain in radial direction

$\varepsilon_v$ Strain in $z$-direction

$\varepsilon_x$ Strain in $x$-direction

$\varepsilon_y$ Strain in $y$-direction

$\varepsilon_\ell$ Major principal strain

$\varepsilon_3$ Minor principal strain

$\varepsilon_\theta$ Strain in circumferential direction

$\gamma_{xz}$ Shear strain in polar coordinates

$\gamma_{xy}$ Shear strain in cartizian coordinates

$\phi$ The angle of internal friction

$\phi'$ The angle of friction between sand and steel

$\theta'$ The slope of the bin wall to the vertical

$\vec{z}$ The direction of the beam element

$\eta, \xi$ Dimensionless coordinates

$\rho'$ The hydraulic radius of the bin

$\mu'$ Coefficient of friction between the grain and the bin wall

$\psi$ The angle between the cord and the $z$-axis

$\psi, \mu, \theta, \phi, \rho$ Variable coefficients used in the development of the shell stiffness matrix

$\kappa_\theta, \kappa_s$ The curvature change of the shell

$\chi$ The meridional rotation

$\nu$ Poisson's ratio

$\nu_i$ Initial Poisson's ratio

$\nu_t$ Tangent Poisson's ratio

$\sigma_n$ Average normal stress

$\sigma_r$ Radial stress

$\sigma_z$ Stress in $z$-direction
\( \sigma_x \)  Stress in x-direction
\( \sigma_y \)  Stress in y-direction
\( \sigma_1 \)  Major principal stress
\( \sigma_3 \)  Minor principal stress
\( \sigma_\theta \)  Stress in circumferential direction
\( (\sigma_1 - \sigma_3) \)  Stress difference
\( (\sigma_1 - \sigma_3)_f \)  Stress difference at failure
\( (\sigma_1 - \sigma_3)_{ult} \)  The ultimate stress difference
\( \tau_{rz} \)  Shear stress in polar coordinates
\( \tau_{xy} \)  Shear stress in cartesian coordinates
\( \tau_s \)  Average shear stress
\( \tau_f \)  Failure shear stress
\( \tau_{ult.} \)  The ultimate shear stress
\( \pi_p \)  Total potential energy

**Matrices**

\([A]\)  Matrix contains the element nodal point coordinates
\([\tilde{E}],[C]\)  Constitutive matrices
\([D]\)  An intermediate matrix
\([\{F\}]\)  Load vector
\(\{\tilde{F}\}\)  Condensed load vector
\([\tilde{K}]\)  The condensed stiffness matrix
\([k_c]\)  Joint stiffness matrix
\([k_e]\)  Element stiffness matrix in local coordinates
\([K_e]\)  Element stiffness matrix in global coordinates
\([k_f]\)  Consistent stiffness matrix for the elastic springs
\{q\}  Vector of nodal displacements

[T]  Transformation matrix

[F]  Surface traction vector

\{u\}  The displacement vector

\{F\}  Body forces

\{a\}  Generalized coordinates

\{e\}  The strain vector

\{\phi\}  The interpolation matrix

\{\sigma\}  The stress vector
CHAPTER I

INTRODUCTION

1.1 General

Adequate design of grain bins requires the knowledge of the pressure inducing characteristics of the material being stored and the interaction between the material stored and the structure.

A grain storage is usually classified as either a shallow bin or a deep bin (silo). Shallow bins are defined as those in which their height does not exceed one-and-one-half times the diameter. Deep bins have a ratio of height to diameter greater than that of shallow bins. Shallow bins are the most common farm storage structures, whereas deep bins are built for central elevators or terminal storage.

The pressure due to the grain in bins has been studied for many years. Most of these studies were conducted for deep silos which were considered to be challenging engineering projects for which theories and techniques had to be developed. In the meantime, shallow bins are built on a practical trial-and-correction basis. Some of the information obtained from deep bins had been applied in the design of shallow bins. However, it should be noted that the following basic differences exist between the load characteristics in deep and shallow bins.

1) In deep bins the grain pressure is the result of arching of the grains; however, in shallow bins the pressure is attributed to a sliding wedge mode of action;

2) When unloading, the mass flow of grain is dominant
in deep bins, while funnel flow takes place in shallow bins.

3) The distribution and magnitude of the grain pressure is not affected by the deformations in deep bins since such bins are usually very rigid. However, shallow bins, especially when built of cold formed steel sheets, are very flexible. This flexibility develops an interaction between the grain pressure and wall deformation.

Shallow grain bins were introduced over fifty years ago. The shell theory had been introduced successfully for the analysis of cylindrical grain bins. However, failure of shallow bins has often been reported (10). A better understanding of the characteristics of pressure due to grain will lead to improved design criteria and more reliable structures.

In general, the loading on the bin walls is governed by the grain material properties as well as the geometric configuration and flexibility of the bin wall. The grain material properties include: the bulk density, the angle of internal friction and the coefficient of friction between the stored material and the bin wall. The effect of wall geometry and flexibility can be seen in Figures (1.1) through (1.5).

Figures (1.1.a) and (1.1.b) show the buckling failure of an isotropic shallow bin due to the shear loads of the grain. In order to improve the buckling performance, an orthotropic bin shown in Figures (1.2.a), and (1.2.b) was used to replace the isotropic bin. It can be noted from these figures that these bins also failed due to the development of buckling waves which are repeated at different heights.
Figure (1.1) Buckling Failure of Shallow Bins.
Figure 4.2(a) Filling of Orthotropic Shallow Bin.
Figure (1.2.b) Buckling Failure of Orthotropic Shallow Bin.
This indicates certain interaction characteristics between the bin wall and the stored material. Another set of bins was made of corrugated sheets with horizontal corrugation as shown in Figure (1.3). The strength of these bins was greatly improved, but excessive vertical displacement was observed due to the vertical flexibility of the wall. In order to increase the vertical stiffness of these bins, vertical stiffeners were used as shown in Figures (1.4.a) and (1.4.b). It can be noticed from these figures, that the vertical stiffeners buckled at different levels, due to the vertical shear force of the stored material.

1.2 Objective of Research

The foregoing discussion shows the lack of accuracy in calculating the pressure in shallow grain bins by the present theories.

More basic information is needed on the characteristics of pressure distribution and the effect of interaction between material and structure. Therefore, the proposed study was conducted as follows:

Part I (1) The development of a mathematical model capable of predicting the pressure distribution on the walls of different configurations and rigidities.

(2) To determine the effect of wall stiffness on both the lateral and the vertical pressures.

(3) To determine the stresses on the bin wall directly for the different wall configurations.

Part II (1) Figure (1.5) is a suggested bin developed by
Figure (1.3) Excessive Vertical Deformation of Orthotropic Shallow Bin.
Figure 9(1.4.a) Filling of Stiffened Orthotropic Shallow Bin.
Figure (1.4.b) Buckling Failure of Vertical Stiffeners.
understanding the interaction between the grain and the wall. The analysis of this new bin is carried out in order to study the effect of vertical curvature on the pressure distribution along the wall and the load carrying capacity.

(2) The buckling of such a bin is discussed qualitatively using the existing theories developed for the soil-culvert system.
Figure (1.5) Proposed Bin Configuration.
CHAPTER II
LITERATURE REVIEW

2.1 Grain Storage Research

Deep and shallow grain storage bins have been built for many years; however, there still remains a great number of conflicting theories concerning the best procedure for predicting the lateral and vertical loads in these storage structures. A brief state-of-the-art for each of the deep bin and shallow bin theories is presented in the following sections.

Early investigation of deep bins showed that the lateral wall pressure is attributed to the action of a horizontal arching of the granular material. Janssen (2) in 1875 proposed his grain arch theory for different load distributions in deep bins. Janssen's equation for lateral static grain pressure is:

$$p_h = \frac{\gamma \cdot \rho'}{\mu'} \left[ 1 - e^{-K_a \cdot \mu' \cdot h / \rho'} \right]$$

(2.1)

where, $p_h$ is the horizontal pressure per unit area (psf);
$h$ is the height of filling above level in question, (ft);
$\gamma$ is the unit weight of the filling material (pcf);
$\mu'$ is the coefficient of friction between grain and bin wall material;
$\rho'$ is the hydraulic radius of the bin (area over the perimeter);
$K_a$ is the ratio of the horizontal to vertical pressure.

It should be noticed that Janssen assumed a constant value for
both the coefficient of wall friction $\mu'$ and the ratio of the horizontal to vertical pressure, $K_a$. He defined $K_a$ as:

$$K_a = \frac{1-\sin \phi}{1+\sin \phi}$$

(2.2)

where $\phi$ is the angle of internal friction of the stored material.

Recent work by Reimbert (2) involving deep storage structures considers the coefficient $K_a$ to be a function of the height of the grain, and the coefficient of friction $\mu'$ to be constant. Using these two assumptions and a set of experimental data, the lateral pressure on the bin wall is found to vary hyperbolically as:

$$P_h = \frac{\gamma R}{2 \mu' \nu'} \left[ 1 - \left( \frac{h}{c} + 1 \right)^{-1} \right]$$

(2.3)

where,

$$c = R \left[ \frac{\frac{1+\sin \phi}{2\mu'(1-\sin \phi)} - \tan \beta'}{3} \right]$$

(2.4)

where,

$\beta'$ is the angle of surcharge.

Potyondy (77) presented different shear tests to define the skin friction $\mu'$ between different grains and bin wall materials under different densities and water contents.

Theoretically, the Janssen approach to the problem is sound and relatively simple to use, which probably accounts for its wide use. However, from a fundamental point of view, the results obtained from Janssen's formula differ consistently from experimental values obtained. Ross, et al, (78) suggested that the coefficient of friction $\mu'$, the bulk density
γ, and the lateral to vertical pressure $K_a$, vary within the storage structure and with different types of unloadings. They showed how the variable factors $\mu'$, $\gamma$ and $K_a$ can be taken into account using the basic Janssen approach. They concluded that the variation in $\mu'$ and $K_a$ may result in an over pressure factor of about 25% to 200%. Amundson's (84) experimental results on deep bins, filled with wheat, showed that the pressure distribution was less than those developed by Janssen. Caughey, et al, (10) conducted model tests using several filling materials. They found some cases in which Janssen's equation over predicted lateral pressures, but recommended its use for design purposes. Pieper's et al, (73) experimental research on deep bins showed the inaccuracy of the Janssen formula for predicting the grain pressure and they suggested the use of $K_a$ as:

$$K_a = 1 - \sin\phi$$

with Janssen's formula. The German specifications suggested a value for $K_a$ as 0.5 to be used with Janssen's formula.

Shallow grain bins are characterized by the insignificant arching effect of the stored material and lateral static pressure may be developed due to the wedge action. In 1919, Ketchum (10) reported the work done by the researchers at this time; he concluded that:

1) The theory of Rankine will apply in the case of shallow bins with smooth wall, but not apply to bins with rough walls. Rankine's equation for the active state of pressure is given as:
\[ P_h = \gamma \cdot h \cdot \cos \beta' \cdot \frac{\cos \beta' - \sqrt{\cos^2 \beta' - \cos^2 \phi}}{\cos \beta' + \sqrt{\cos^2 \beta' - \cos^2 \phi}} \]  \( (2.6) \)

2) The ratio of lateral to vertical pressure \( K_a \) is not constant but varies with the depth.

Trepanier and Walford (84) concluded that the Rankine equation can be used to predict the lateral pressure if the correct properties are used and if the boundaries simulated by Rankine are satisfied. Hamilton (34) measured the lateral strains in cylindrical shallow bins with different wall stiffnesses under active and passive pressures. As a result of his work he found that the strains at the bottom of the bin wall with height to diameter ratio of 1.5 are 75% greater than those predicted by Rankine's equation and 41% greater than those predicted by Janssen's equation. Saul (10) concluded that the lateral pressure increases with increasing the rigidity of the wall.

Stewart (84) recommended the use of Coulomb's equation to predict the lateral pressure in shallow rectangular bins. Coulomb's equation is given as:

\[ P_h = \gamma \cdot h \left( \frac{\cos \theta' \cdot \sin(\theta' - \phi)}{\sin(\phi + \theta') + \sqrt{\sin(\phi + \theta') \cdot \sin(\theta' - \beta')}} \right)^2 \cos \phi' \]  \( (2.7) \)

where the angle \( \theta' \) is the slope of the wall surface. Stewart also found that the pressure magnitude and distribution are not affected by the wall stiffness when the wall deformations are sufficient to produce the active state. Abdel-Sayed (2), modified Coulomb's theory to account for the axisymmetric geometry of the cylindrical shallow bins. He considered the
equilibrium of a conical wedge instead of a prismatic wedge. The lateral pressure can be obtained using an iteration technique on the equation:

$$p_h = \frac{1}{2} \gamma R^2 (k_1 + k_2 + k_3) \frac{\sin(\eta - \phi)}{\cos(\phi' + \eta - \theta')} \cos(\phi' - \theta')$$  \hspace{1cm} (2.8)

where,

$$k_1 = \frac{R}{H} \left[ 1 - \frac{1}{3} \frac{R}{H} (\tan \eta - \tan \beta') \right]$$  \hspace{1cm} (2.9)

$$k_2 = \frac{1}{3} \frac{R}{H} \left[ 1 - \frac{R}{H} \frac{h}{\tan \eta - \tan \beta'} \right]^3 (\tan \eta - \tan \beta')$$  \hspace{1cm} (2.10)

$$k_3 = \tan \beta'$$  \hspace{1cm} (2.11)

$$H = \left( R - \frac{h}{\tan \eta - \tan \beta'} \right) (\tan \eta - \tan \beta')$$  \hspace{1cm} (2.12)

Equations (2.1) through (2.12) summarize the different theories applied to the storage structures and it should be noticed that there are notable differences between these theories. However, the ASAE recommends the use of Janssen's equation for deep bins. The Canadian Farm Building Code recommends the use of Janssen's equation for deep bins and the Rankine equation for shallow bins. The American Concrete Institute recommended either Janssen's or Reimbert's equations to predict static bin pressures. No distinction is made between deep and shallow bins.

7.2 Discussion of Current Design Procedure

Reinforced concrete and corrugated metal are the two materials most commonly used in grain bin construction. The reinforced concrete silos are rigid structures while the
corrugated metal bins are flexible. Although there is notable difference in the performance of the rigid and the flexible structures, the above mentioned methods of predicting the loads ignored the flexibility of the grain storage structures.

Janssen's equation is the most recognizable one for predicting the lateral pressure in deep bins, even though there is no unique value for the ratio of the horizontal to the vertical pressure, $K_a$. It is also proven that $K_a$ is not constant but it is a function of the height of the grain which contradicts Janssen's equation.

Although Janssen's formula has been developed assuming an arch action between the grains and the wall, some engineers are using this equation for predicting the pressures in shallow bins. The Canadian Farm Building Code recommends the use of Rankine's equation to predict the pressures in shallow bins although there is noticeable friction between the wall and the stored material especially if the wall is made of corrugated sheets. Although Coulomb's equation or the modified Coulomb equation are the most accurate equation available to predict the lateral pressure in shallow bins, they do not account for the rigidity of the bin wall or the variation of the coefficient of lateral pressure, $K_a$.

Continued buckling failures of shallow bins indicate that there still exists a lack of information about the nature of the developed vertical pressure as well as the interaction between the bin wall and the stored material especially if the wall is flexible.
2.3 Related Work in Soil Mechanics

The lateral loads of backfill into a retaining wall can be predicted using either the classical earth pressure theories such as Rankine's or Coulomb's formulas or by a numerical solution such as the finite element method.

The earth pressure theories indicate an initial hydrostatic pressure ratio of about 0.4 for the at-rest state. This ratio is shown to decrease to the active value with an outward movement of the wall equal to 0.0008h, where h is the height of the fill.

The finite element method provides an added advantage since it can deal with the interaction between the soil pressure and the displacement of the wall. Underground culverts, especially those made of corrugated steel, are the best examples to illustrate the nature of the interaction between soil and structure. The finite element method was used successively to analyze such a structure. A brief review of some of the finite element works are presented in the following.

Clough and Duncan (17) used the plane strain element to study the interaction between the retaining wall and the backfill. Different wall rotations and translations had been applied. Their analysis shows that the active state of pressure is the lower limit of the soil pressure and it can take place only if the wall displacement or rotation is enough to mobilize the soil to the active wedge. Girijavallabhan (38) reported the use of the finite element in the analysis of axisymmetric and plane-strain mass foundation under external
loads. Brown (11) illustrated the use of plane-strain finite element for the analysis of deep flexible culverts. He found that the magnitude and distribution of the pressure on culverts was affected by the stiffness of the culvert and the interface conditions between the soil and the culvert. Smith (82) described the use of axisymmetric finite element in the analysis of circular plates resting on sand. Ellison (27) analysed a deep pile-stiff clay composite system using the axisymmetric analysis. He included the different interface conditions between the clay and the pile. The load-deformation behaviour was one of the major predictions which can be obtained by this method. Nataraja (68) used the plane strain analysis to analyse the soil culvert system under high fills, treating the soil as both geometrically and materially nonlinear. He studied the effect of the relative stiffness between the soil and the culvert on the pressure distribution as well as the load-deformation.

The strength and stability of corrugated metal culverts and arches resting on soil backfill has been studied for a long time by different researchers in order to find the effect of the supporting soil on the stability of the structure. It has been found that the underground conduits and arches have a high load carrying capacity due to the soil-structure interaction which stimulates passive earth pressure at the sites.

Meyerhof and Baikie (65) had extended the theory of stability of plates to estimate the buckling stresses of circular sheets and culverts in elastic media. Baikie (6) extended the research to include arches embedded in the soil.
Abdel-Sayed (1) summarized and evaluated the different approaches developed to deal with buckling of such a problem.
CHAPTER III
MATHEMATICAL FORMULATION

3.1 General

The composite system of grain material and bin wall is treated as a continuum discretized by a finite number of elements. Constant strain triangular, plain strain or axisymmetric elements is employed to the grain material. A general interface element allowing frictional sliding, separation and re bonding of two bodies meeting at a common interface is used to represent the state of contact between the wall and the grain material. Axisymmetric shell or beam on elastic supports elements is applied to the bin wall. The stress-strain relationship for the grain material and the load deformation relationship for the interface element are assumed nonlinear. The stress-strain relationship for the bin wall is assumed to be linear elastic.

3.2 Finite Element Formulation

The static formulation of the finite element method can be derived from virtual work or from variational principles (20). The variational principle is outlined below.

The total potential energy of a structural system can be expressed in matrix form, as:

\[ \pi_p = \frac{1}{2} \int \{\varepsilon\}^T \{\sigma\} d\mathbf{V} - \int \{u\}^T \{\mathbf{x}\} d\mathbf{V} - \int \{u\}^T \{\mathbf{T}\} d\mathbf{s} \]  

where, \( \pi_p \) is the total potential energy

\( \{\sigma\} \) is the stress vector

\( \{\varepsilon\} \) is the strain vector
where, \(\{u\}\) is the displacement vector

\(\{T\}\) is the surface traction vector

\(\{\bar{x}\}\) represents body forces

\(\{\cdot\}^T\) is the transpose vector

\(\bar{s}\) is the surface of a body

\(\psi\) is the volume of a body

The strain energy term in Equation (3.1) can be rewritten in terms of the displacements by the use of constitutive relationship and strain-displacement relationship, as:

\[
\{u\} = [\phi]\{\alpha\} \quad (3.2)
\]

\[
\{\sigma\} = [C]\{\epsilon\} \quad (3.3)
\]

\[
\{\epsilon\} = [B]\{\alpha\} \quad (3.4)
\]

\[
\{q\} = [A]\{\alpha\} \quad (3.5)
\]

where, \(\{q\}\) is the vector of nodal displacements

\(\{\alpha\}\) is the generalized coordinates vector

\(\bar{\phi}\) is a matrix containing the coefficients of the interpolation function

\([C]\) is the constitutive matrix (stress-strain law)

\([B]\) is the constitutive matrix (strain-generalized coordinates relationship)

\([A]\) is the matrix containing the element nodal points coordinates

Substituting Equations (3.2) through (3.5) into Equation (3.1), yield:

\[
\tau_p = \frac{1}{2}\int\{q\}^T[A^{-1}]^T[B]^T[C][B][A^{-1}][q]d\psi

- \int\{q\}^T[A^{-1}]^T[\phi]^T[\bar{x}]d\psi

- \int\{q\}^T[A^{-1}]^T[\phi_s]^T[T]d\psi \quad (3.6)
\]
The \([C]\), \([A]\) and \([B]\) matrices are dependent on the material and kinematic assumptions as will be shown later. Within each element a displacement function is chosen which uniquely defines the state of displacement at all points. The polynomial form of the displacement is most common. The assumed polynomial must contain one unknown for each degree of freedom encountered by the element. By taking the variation of Equation (3.6) with respect to the displacements, the stiffness matrix of an element can be given as:

\[
[k_e] = \int [A^{-1}]^T[B]^T[C][B][A^{-1}]d\mathbf{v} \tag{3.7}
\]

3.3 Material Nonlinearity of Grain Material

The stress-strain behaviour of the grains depends on a number of different factors, including density, water content, structure, strain conditions, confining pressure and shear stress. The stress-strain relations, which can be established from laboratory tests, account for these factors. If the test conditions are adjusted to simulate the corresponding field conditions, it is anticipated that the strains resulting from given stress changes in the laboratory test is representative of the field strains under the same stress changes.

3.3.1 Nonlinear Stress-Strain Models

A literature survey was conducted to find an appropriate soil model which describes the behaviour of the grain stored in bins. It was found that there is no unique soil model which can describe the stress-strain relationship. Prior
to developing particular constitutive forms, some general concepts of soil models are reviewed.

A general relationship for stress is expressed as a functional of different variables as:

$$\sigma = F(\varepsilon(t'), T(t'), w(t'), x_{...}, t')$$ \hspace{1cm} (3.8)

Equation (3.8) simply says, stress, $\sigma$, is functionally related to different variables, such as strain, $\varepsilon(t')$, temperature $T(t')$, moisture content, $w(t')$, anisotropy $x(t)$ and aging ($t'$). It is extremely difficult to identify all the parameters implied in Equation (3.8) over the entire range of state variables. Thus the important variables will be included in the domain of interest, while the unessential variables will be excluded. Accordingly, both the anisotropy and the time dependent, variables discarded.

The time independent nonlinear soil model can be classified into two groups: plasticity models and variable modulus models. The plasticity models require a yield criteria, a hardening rule and a flow rule. They are capable to describe both the failure criteria and the unloading. Details of the plasticity models have been referenced (50, 59, 63, 74, 75 and 95).

There are two major differences among the variable modulus models. First is the relationship used to define the stress-strain relationship. Secondly, the method of updating the constitutive matrix which includes four different techniques. These techniques are: secant method where the load is applied in one step and the solution is iterated to
satisfy both equilibrium and the associated material law. The tangent method where the load is applied in a series of small steps. The modified tangent method which includes the iterative solution with each load step. The chord method is the secant method applied in step-by-step fashion.

Although there are many variable modulus models reported in the literature, all of them illustrate the same trends: the increase of the confining pressure is always increasing the stiffness of the soil, while the increase of the shear strain always decreases the stiffness of the soil. The details of the variable modulus models are given in references (9,17,24,39,52,54,55,56,57,94).

Since the farm grains are classified as cohesionless material, one of the above models can be used to describe the grain behaviour. In the next section a brief review of some of the variable modulus models is described.

Breth (9) has suggested an incremental formula for both the modulus of elasticity and Poisson's ratio as:

\[
E = \frac{(\Delta \sigma_1 + \Delta \sigma_3) \cdot \Delta \sigma_1 - 2\Delta \sigma_3^2}{(\Delta \sigma_1 + \Delta \sigma_3) \cdot \Delta \varepsilon_1 - 2\Delta \sigma_3 \cdot \Delta \varepsilon_3}
\]  
\(3.9\)

\[
\nu = \frac{\Delta \sigma_3 \cdot \Delta \varepsilon_1 - \Delta \sigma_1 \cdot \Delta \varepsilon_3}{(\Delta \sigma_1 + \Delta \sigma_3) \cdot \Delta \varepsilon_1 - 2\Delta \sigma_3 \cdot \Delta \varepsilon_3}
\]  
\(3.10\)

\[
\varepsilon = a + \frac{b}{R - R_f} + \frac{c}{(R - R_f)^2}
\]  
\(3.11\)

where, \(\sigma_1\) is the major principal stress;

\(\sigma_3\) is the minor principal stress
where, $\varepsilon_1$ is the major principal strain; 
$\varepsilon_3$ is the minor principal strain; 
$\varepsilon$ is the axial or lateral strain; 
$\bar{R}$ is the stress ratio $\sigma_3/\sigma_1$; 
$\bar{R}_f$ is the failure ratio or $\sigma_3/\sigma_1$ at failure; 
$\Delta$ is the increment in stress or strain;

$a, b, c$ are constants to be determined from triaxial test.

Hardin suggested a model for different types of soils, to provide a relationship for the secant shear modulus. This model mainly depends on soil type, void ratio, percent saturation and plasticity index.

The hyperbolic stress-strain relationship developed by Wong and Duncan (88) is the most popular, besides, it is simple and can be established from the triaxial test. Since this model is used in the present study, more details are given in the next section.

3.3.2 Hyperbolic Stress-Strain Relationship For Non-Linear Stress Analysis

Kondoner et al, (54, 55), had shown that the nonlinear stress-strain curves for a number of soils could be approximated by a hyperbola. Figure (3.1.a) shows the form of the hyperbola which can be represented by the equation:

$$\sigma_1 - \sigma_3 = \frac{1}{E_i} + \frac{\varepsilon_a}{\varepsilon}$$

$$\frac{\varepsilon_a}{E_i} + \frac{\varepsilon_a}{(\sigma_1 - \sigma_3)_{ult.}}$$

where, $\varepsilon_a$ is the axial strain; 
$E_i$ is the initial tangent modulus.
The value \((\sigma_1 - \sigma_3)_{ult}\) is the asymptotic value of the stress difference which is always greater than the compressive strength of the soil. Equation (3.12) can be transformed into a straight line, as shown in Figure (3.1.b), which can be represented as:

\[
\frac{\varepsilon_a}{\sigma_1 - \sigma_3} = \frac{1}{E_i} \cdot \varepsilon_a \frac{\varepsilon_a}{(\sigma_1 - \sigma_3)_{ult}}.
\] (3.13)

The straight line, given by Equation (3.13) represents the best fit of the hyperbola given by Equation (3.12). For all soils, except fully saturated soils, tested under unconsolidated undrained conditions, if the confining pressure is increased, the stress-strain curve will be steeper with higher strength. In other words, the value of \(E_i\) and \((\sigma_1 - \sigma_3)_{ult}\) both increased with the increase of confining pressure.

Experimental work by Janbu (44) had shown that the relationship between initial tangent modulus and confining pressure may be expressed as:

\[
E_i = K_n \cdot P_a \cdot \frac{\sigma_3}{P_a}^n
\] (3.14)

where, \(K_n\) is an experimental parameter called the modulus number;

\(n\) is an experimental parameter called the modulus exponent;

\(P_a\) is the atmospheric pressure.

The parameters \(K_n\) and \(n\) are dimensionless numbers experimentally determined. Equation (3.14) can be represented as a straight line on a log scale as shown in Figure (3.2).

It should be noticed that Equation (3.14) is valid for any
Figure (3.1.a) Hyperbolic Stress-Strain Relationship.

Figure (3.1.b) Transformed Hyperbolic Stress-Strain Relationship.
Figure (3.2) Relationship Between the Confining Pressure and the Initial Modulus.
system of units as long as the units of the atmospheric pressure, \( P_a \), is adjusted. The relationship between compressive strength and confining pressure is given by Mohr-Coulomb failure criteria (94),

\[
\frac{1}{(\sigma_1 - \sigma_3)_f} = \frac{2c \cdot \cos \phi + 2\sigma_3 \cdot \sin \phi}{1 - \sin \phi}
\]  
(3.15)

in which \( c \) and \( \phi \) are the cohesion intercept and the internal friction angle of the soil, respectively. The compressive strength or the stress difference at failure \((\sigma_1 - \sigma_3)_f\) can be related to the asymptotic stress difference as:

\[
(\sigma_1 - \sigma_3) = R_f (\sigma_1 - \sigma_3)_{ult.}
\]  
(3.16)

in which \( R_f \) is the failure ratio which is always smaller than unity and varies from 0.5 to 0.9 for most soils.

The slope of the tangent to the stress-strain curve can be defined as:

\[
E_t = \frac{3}{\varepsilon_a} (\sigma_1 - \sigma_3)
\]  
(3.17)

By differentiating Equation (3.12) with respect to \( \varepsilon_a \) and substituting Equations (3.14), (3.15) and (3.16) into the resulting expression, the tangent modulus can be expressed as:

\[
E_t = K_h \cdot P_a \left( \frac{\sigma_3}{P_a} \right)^n \left[ 1 - \frac{R_f (\sigma_1 - \sigma_3) (1 - \sin \phi)}{2c \cdot \cos \phi + 2\sigma_3 \cdot \sin \phi} \right]^2
\]  
(3.18)

This equation can be used to calculate the appropriate value of tangent modulus for any stress conditions, if the values of the parameters \( K_h, n, c, \phi \) and \( R_f \) are known. The values of these parameters and the technique for calculating these values will be described in Chapter IV.
3.3.3 Hyperbolic Nonlinear Volume Change

The Poisson's ratio can be determined by measuring the volume change which occurs during a triaxial test. The volumetric change in a triaxial test is given as:

$$\epsilon_v = \epsilon_a + 2\epsilon_r$$  \hspace{1cm} (3.19)

where $\epsilon_r$ is the radial strain. Figure (3.3.a) shows the variation of $\epsilon_a$ with $\epsilon_r$, this curve is approximated by a hyperbolic equation of the form (94):

$$\epsilon_a = \frac{\epsilon_r}{v_i - D\epsilon_r}$$  \hspace{1cm} (3.20)

where $v_i$ is the initial Poisson's ratio and $D$ is an experimental parameter as shown in Figure (3.3.b). The variation of $v_i$ with the confining pressure is shown in Figure (3.4).

By differentiating Equation (3.20) with respect to $\epsilon_r$ and substituting Equations (3.12) through (3.16), the tangent Poisson's ratio is (94):

$$v_t = \frac{G-F \cdot \log_{10}(\frac{\sigma_3}{P_a})}{D(\sigma_1 - \sigma_3)^2} \left[ \frac{1 - \frac{\sigma_3}{2c \cdot \sigma_3 + 2\sigma_3 \cdot \sin \phi}}{K_h \cdot P_a \left( \frac{\sigma_3}{P_a} \right)^n [1 - \frac{R_f (\sigma_1 - \sigma_3)(1 - \sin \phi)}{2c \cdot \sigma_3 + 2\sigma_3 \cdot \sin \phi}] } \right]$$  \hspace{1cm} (3.21)

3.3.4 Incremental Analysis

In the computational applications, both Equations (3.18) and (3.21) are used. Since each of these equations is nonlinear, an incremental approach is advantageous. This incremental procedure is a sequence of linear analyses. Accordingly, the stress-strain relationship given by Equation (3.3) is expressed in an incremental form as:
Figure (3.3) Hyperbolic Volumetric Change

Figure (3.4) Variation of Initial Poisson's Ratio.
\{ \Delta \sigma \} = [c] \{ \Delta \varepsilon \}, \quad (3.22)

where, \{ \Delta \sigma \} is the incremental stress vector;
\{ \Delta \varepsilon \} is the incremental strain vector.

The components of the constitutive matrix, \([c]\), are dependent on the total stress-strain level. It should be noted that for good results, the load increments must be kept small and \([c]\) frequently updated to avoid significant errors.

3.4 Triangular Finite Element Model for Grain Material

3.4.1 Analysis of Axisymmetric Solids Under Axisymmetric Loads

A finite element procedure suitable for the static analysis of axisymmetric solids under axisymmetric loading has been discussed by Girijavallabhan, et al (33). For an axisymmetric continuum, the elastic media is replaced by a finite number of axisymmetric ring elements connected at a finite number of circumferential joints or nodes. Figure (3.5) shows the structural idealization and a typical triangular ring element. When the loading is axisymmetric, the pertinent displacements are those in the radial and longitudinal directions only. Tangential displacements do not exist and the strains and stresses do not vary in the tangential direction. Thus the axisymmetric body can be viewed mathematically as two-dimensional in nature, as indicated by Figure (3.6). For the two dimensional elements, element boundary compatibility is satisfied by assuming linear displacements within the element. These displacements are defined as:
Figure (3.5) Axisymmetric Solid and Typical Ring Element.

Figure (3.6) Axisymmetric Stresses and Strains.
\[ w = a_1 + a_2 \cdot r + a_3 \cdot z \quad (3.23) \]
\[ v = a_4 + a_5 \cdot r + a_6 \cdot z \quad (3.24) \]

where, \( w \) is the displacement in the \( r \)-direction;
\( v \) is the displacement in the \( z \)-direction;
\( a \)'s are the generalized coordinates;
\( r, z \) are the radial and vertical coordinates.

Using Equation (3.5) as well as Equations (3.23) and (3.24), the \([A]\) matrix is obtained. The strain-displacement relationships for an axisymmetric solid of revolution are given by:

\[ \varepsilon_r = \frac{\partial w}{\partial r} \tag{3.25.a} \]
\[ \varepsilon_z = \frac{\partial v}{\partial z} \tag{3.25.b} \]
\[ \varepsilon_\theta = \frac{w}{r} \tag{3.25.c} \]
\[ \gamma_{rz} = \frac{\partial w}{\partial z} + \frac{\partial v}{\partial r} \tag{3.25.d} \]

where, \( \varepsilon_r \) is the radial strain;
\( \varepsilon_z \) is the vertical strain;
\( \varepsilon_\theta \) is the circumference strain;
\( \gamma_{rz} \) is the shear strain.

Using Equations (3.23), (3.24), (3.29) and Equation (3.4), the \([B]\) matrix is obtained. It should be noted that the coefficients of the \([B]\) matrix are not constant for the axisymmetric case. The circumferential strain is a function of position within the element and hence stresses are not constant through the element. It is usual to use an average value for the circumferential strain which yields constant stresses for the axisymmetric triangular element. The stress-strain rela-
The relationships are:

\[
\sigma_r = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu) \varepsilon_r + \nu \varepsilon_z + \nu \varepsilon_\theta \right] \quad (3.26.a)
\]

\[
\sigma_z = \frac{E}{(1+\nu)(1-2\nu)} \left[ \nu \varepsilon_r + (1-\nu) \varepsilon_z + \nu \varepsilon_\theta \right] \quad (3.26.b)
\]

\[
\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)} \left[ \nu \varepsilon_r + \nu \varepsilon_z + (1-\nu) \varepsilon_\theta \right] \quad (3.26.c)
\]

\[
\tau_{rz} = \frac{E}{2(1+\nu)} \gamma_{rz} \quad (3.26.d)
\]

where, \( \sigma_r \) is the radial stress;

\( \sigma_z \) is the vertical stress;

\( \sigma_\theta \) is the circumferential stress;

\( \tau_{rz} \) is the shear stress.

Using Equations (3.26) together with Equation (3.3), the \([C]\) matrix is obtained. The details of matrices \([A]\), \([B]\) and \([C]\) are given in Appendix (A). Substituting the matrices \([A]\), \([B]\) and \([C]\) into Equation (3.7), and performing the integration over the volume of the element, the element stiffness matrix is generated. It should be noticed that the developed element stiffness matrix is given in the global system of axes as shown in Figure (3.5), and this element stiffness matrix includes terms which become infinite in some special cases. The treatment of this singularity and the details of the element stiffness matrix will be discussed in Appendix (A). The nodal load vector due to the gravity load can be obtained from Equation (3.6), the details of this load vector is also given in Appendix (A).
3.4.2 Plane Strain Triangular Element

Some of the designers considered the circular bin structures as a plane strain problem. They take a section of unit width and use either Rankine's or Coulomb's theory to obtain the grain pressure on the wall. To compare between the plane strain and the axisymmetric assumptions, the plane strain analysis is also discussed. The assumed displacement within the element are those given in Equation (3.23) and (3.24), but in the x and y axes. If the w displacement is replaced by the u displacement, then the strain-displacement equations and the stress-strain equations are given by:

\[
\varepsilon_x = \frac{\partial u}{\partial x} \tag{3.27.a}
\]

\[
\varepsilon_y = \frac{\partial v}{\partial y} \tag{3.27.b}
\]

\[
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{3.27.c}
\]

and,

\[
\sigma_x = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu) \varepsilon_x + \nu \varepsilon_y \right] \tag{3.28.a}
\]

\[
\sigma_y = \frac{E}{(1+\nu)(1-2\nu)} \left[ \nu \varepsilon_x + (1-\nu) \varepsilon_y \right] \tag{3.28.b}
\]

\[
\tau_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy} \tag{3.28.c}
\]

where, \( \varepsilon_x \) is the strain in x-direction;

\( \varepsilon_y \) is the strain in y-direction;

\( \gamma_{xy} \) is the shear strain;

\( \sigma_x \) is the stress in x-direction;

\( \sigma_y \) is the stress in y-direction;

\( \tau_{xy} \) is the shear stress.
The matrices $[A]$, $[B]$ and $[C]$ and the load vector are developed as explained in the previous section; they are given in Appendix (A).

It should be noticed that the stress-strain relation given by Equation (3.12) was established from the triaxial test. It is well understood that triaxial test results do not represent a plane-strain condition assumed in this analysis (58, 59), but it has been an accepted practice to use the results of a triaxial test. Clough (16, 17), Nataraja (68) and many other investigators have used similar laboratory curves and they obtained acceptable results.

3.5 **Interface Finite Element Between Grain Material and Bin Wall**

In the finite element analysis, the contact surface between two materials has common nodes which have the same displacements. This means that at the common nodes, there is no licence for relative movement which they hold by virtue to their interface shearing behaviour. In order to overcome this shortcoming in the analysis, a finite element which can handle the interface behaviour of the grain bin continuum at the contact planes and allow for the independent displacement of the two materials is outlined in the following section.

3.5.1 **Linkage Element**

Two different approaches are used with the finite element formulation to treat the interface conditions. First is the method of constraints, and second is the method of stiffness. The method of constraints had been addressed in
references (15,18). A complete finite element formulation for the soil-culvert system using the method of constraints was given by Katona (50). The formulation of the method of constraints needs a bigger stiffness matrix, more computer time, and an iteration technique should be used which sometimes does not converge.

The method of stiffness is basically the simple concept of using 'bar' element across the interface in both the normal and tangential directions. This method has been used successfully in culvert and retaining wall applications (17, 68). Goodman (35) developed one-dimensional finite element to represent the interface behaviour for the jointed rock. An extension of the procedure presented by Ellison (27) for symmetric loading on a pile-soil system. In the method formulated by Goodman (35), the interface between two adjacent locations at a distance L apart may be represented by an element having two pairs of nodes as shown in Figure (3.7.a). Ellison (27) simplified the solution by typifying the interface behaviour over distance L by an element having a pair of nodes as shown in Figure (3.7.b). In this study, the four-noded linkage element is used. Even though the derivation of the stiffness matrix for the linkage element can be found elsewhere (35), for completeness and clarity it will be included subsequently.

The following assumptions are made in the interface analysis of grain-bin systems:

a) The element representing the interface between the grain material and the bin wall is of zero thickness;

b) The element cannot resist any tension, if any
(a) Four Noded Linkage Element

(b) Two Noded Linkage Element

Figure (3.7) Linkage Element.
tension normal to the interface is developed, separation between the two materials will take place;

c) The element offers very high resistance to compression with negligible deformation;

d) The shear strength of the element comes from the friction between the grain material and the wall;

e) The behaviour of the element in shear defines the relative displacement of the two materials.

The linkage element shown in Figure (3.7.a), has the four nodes, i,j,k,l and its local axes located at the centre. The nodes, i and l initially have the same coordinates, and similarly nodes, j and k. The element has length, L, and infinitesimal thickness. The strain energy stored in the element over the unit width may be given as:

\[
U_s = \frac{1}{2} \int_{-L/2}^{L/2} \mathbf{u}_i \cdot \mathbf{P}_i \cdot dx \\
\]  \hspace{1cm} (3.29)

or in matrix notation:

\[
U_s = \frac{1}{2} \int_{-L/2}^{L/2} (\mathbf{u})^T \mathbf{P} \cdot dx \\
\]  \hspace{1cm} (3.30)

where,

\[
\{\mathbf{u}\} = \begin{pmatrix} \mathbf{u}_s^t - \mathbf{u}_s^b \\ \mathbf{u}_n^t - \mathbf{u}_n^b \end{pmatrix} \\
\]  \hspace{1cm} (3.31)

\[
\{\mathbf{P}\} = \begin{pmatrix} \mathbf{P}_s \\ \mathbf{P}_n \end{pmatrix} \\
\]  \hspace{1cm} (3.32)

\(u_s^t\) is the tangential displacement at top of the element;

\(u_s^b\) is the tangential displacement at bottom of the element;
\( u^t_n \) is the normal displacement at top of the element;

\( u^b_n \) is the normal displacement at bottom of the element;

\( P_s \) is the tangential force per unit length;

\( P_n \) is the tangential force per unit length.

The relationship between the force and the displacement is:

\[
\{P\} = [k_c]\{u\} \quad (3.33)
\]

in which, \([k_c]\) is a diagonal matrix expressing the joint stiffness per unit length in the normal and tangential directions given as:

\[
[k_c] = \begin{bmatrix}
  k_s & 0 \\
  0 & k_n
\end{bmatrix} \quad (3.34)
\]

where, \(k_s\) and \(k_n\) are the slopes of the load deformation curves as shown in Figure (3.8). Substituting Equations (3.33) into Equation (3.30), yields:

\[
U_s = \frac{1}{2} \int_{-L/2}^{L/2} \{u\}^T[k_c]\{u\} \, dx \quad (3.35)
\]

The displacement functions for the one-dimensional element are treated in a manner similar to that in the regular two-dimensional analysis, that is:

\[
u^b_s = \frac{1}{s}[u_i(1 - \frac{2x}{L}) + u_j(1 + \frac{2x}{L})] \quad (3.36.a)
\]

\[
u^b_n = \frac{1}{s}[v_i(1 - \frac{2x}{L}) + v_j(1 + \frac{2x}{L})] \quad (3.36.b)
\]

\[
u^t_s = \frac{1}{s}[u_k(1 + \frac{2x}{L}) + u_l(1 - \frac{2x}{L})] \quad (3.36.c)
\]

\[
u^t_n = \frac{1}{s}[v_k(1 + \frac{2x}{L}) + v_l(1 - \frac{2x}{L})] \quad (3.36.d)
\]
The relative displacement vector $u$ can be related to the nodal displacement vector $q$ as:

$$\{u\} = [D]\{q\} \quad (3.37)$$

where,

$$\{q\}^T = \{u, u, u, v, v, v\} \quad (3.38)$$

$$[D] = \begin{bmatrix}
-(1-\frac{2x}{L}) & -(1+\frac{2x}{L}) & (1+\frac{2x}{L}) & (1-\frac{2x}{L}) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -(1-\frac{2x}{L}) & -(1+\frac{2x}{L}) & (1+\frac{2x}{L}) & (1-\frac{2x}{L})
\end{bmatrix} \quad (3.39)$$

Substituting Equation (3.36) into Equation (3.34), yields:

$$U_S = \frac{1}{8} \int_{-L/2}^{L/2} \{q\}^T [D][k_c][D] \{q\} \, dx \quad (3.40)$$

Take the variation of Equation (3.40) integrating over the length of the element; the element stiffness matrix in its local coordinate system may be written as:

$$[k_c] = \frac{L}{6} \begin{bmatrix}
2k_S & k_S & 2k_S & 0 & 0 & 0 & 2k_n \\
k_S & 2k_S & -k_S & 0 & 0 & 0 & k_n \\
-2k_S & -2k_S & 2k_S & 0 & 0 & 0 & 2k_n \\
0 & 0 & 0 & 0 & 2k_n & 2k_n & 0 \\
0 & 0 & 0 & 0 & k_n & -2k_n & 2k_n \\
0 & 0 & 0 & 0 & 0 & 2k_n & -k_n & 2k_n \\
0 & 0 & 0 & 0 & 0 & 0 & 2k_n & 2k_n
\end{bmatrix} \quad (3.41)$$
The variables $k_s$ and $k_n$ are determined experimentally. The stiffness matrix given by Equation (3.41) is assembled to the total stiffness matrix of the structure after the transformation to the global axes using the following transformation matrix:

$$
[T] = \begin{bmatrix}
\cos \theta & 0 & 0 & 0 & \sin \theta & 0 & 0 & 0 \\
0 & \cos \theta & 0 & 0 & 0 & \sin \theta & 0 & 0 \\
0 & 0 & \cos \theta & 0 & 0 & 0 & \sin \theta & 0 \\
0 & 0 & 0 & \cos \theta & 0 & 0 & 0 & \sin \theta \\
-\sin \theta & 0 & 0 & 0 & \cos \theta & 0 & 0 & 0 \\
0 & -\sin \theta & 0 & 0 & 0 & \cos \theta & 0 & 0 \\
0 & 0 & -\sin \theta & 0 & 0 & 0 & \cos \theta & 0 \\
0 & 0 & 0 & -\sin \theta & 0 & 0 & 0 & \cos \theta \\
\end{bmatrix}
$$

(3.42)

It should be noticed that every pair of nodes of the interface element are initially in the same position. Depending on the force applied to the element, these nodes will be displaced by different amounts. The average interface stresses are given by:

$$\sigma_n = k_n \cdot \delta_n$$

(3.43)

$$\tau_s = k_s \cdot \delta_s$$

(3.44)

where, $\sigma_n$ is the average normal stress;

$\tau_s$ is the average shear stress;

$\delta_n$ is the average relative normal displacement;

$\delta_s$ is the average relative shear displacement.
The value of the joint normal stiffness, \( k_n \), would be expected to increase with increasing the value of the compressive stress, and would diminish to zero once tension is developed. In this study, the value of \( k_n \) is considered constant with high value, since there is no experimental program to measure the element normal stiffness. The shear stiffness, \( k_s \), would be expected to increase with the increase in the normal pressure and to decrease with the increase in the shear stress. In other words, the variation of the shear stiffness as regards the shear stress and the normal stress is nonlinear. The mathematical formulation of the nonlinear conduct of the interface element is explained in the next section.

3.5.2 Nonlinear Load-Displacement Model

Clough and Duhcan (17) had shown that the shear stress displacement relationship shown in Figure (3.8) may be represented by a hyperbola of the form:

\[
\tau_s = \frac{\delta_s}{a + b \cdot \delta_s}
\]  

(3.45)

where \( a \) and \( b \) are empirical constants whose values are determined experimentally. The stress displacement relationship given by Equation (3.45) may be transformed into a straight line of the form:

\[
\frac{\delta_s}{\tau_s} = a + b \cdot \delta_s
\]

(3.46)

It may be noticed from Equation (3.46) that the coefficient \( a \) is the intercept while the coefficient \( b \) is the slope of the straight line on the transformed scale. The reciprocal of a
Figure (3.8) Load-Deformation Curves for the Interface Element.
is the initial slope of the shear stress-displacement curve, which is called the initial shear stiffness \( k_{si} \). The reciprocal of \( b \) is the asymptote approached by the shear stress-displacement curve, which is denoted by \( \tau_{ult} \). The ultimate shear is always greater than the failure shear \( \tau_f \); they can be related by the formula:

\[ \tau_f = R_{fs} \cdot \tau_{ult}. \quad (3.47) \]

where \( R_{fs} \) is the failure ratio. The initial shear stiffness, \( k_{si} \), depends on the value of the normal stress on the interface; they connected to each other by Janbu (44) formula, as:

\[ k_{si} = k_I \cdot \gamma_w \left( \frac{\sigma_n}{P_a} \right)^{n_S} \quad (3.48) \]

where, \( k_I \) is the dimensionless stiffness number;

\( n_S \) is the stiffness exponent;

\( \gamma_w \) is the unit weight of water.

The shear strength is expressed in terms of normal stress and the angle of friction between the grain and the wall \( \phi' \), as:

\[ \tau_f = \sigma_n \cdot \tan \phi' \quad (3.49) \]

Substituting Equations (3.47), (3.48), (3.49) into Equation (3.45), yields:

\[ \tau_s = \frac{\delta_s}{\frac{1}{\frac{k_I \cdot \gamma_w}{P_a} n_S} + \frac{R_{fs} \cdot \delta_s}{\sigma_n \cdot \tan \phi'}} \quad (3.50) \]

The shear stiffness \( k_s \) is the slope of the tangent to shear stress-displacement curve. Differentiate Equation (3.50) with respect to \( \delta_s \), the tangent shear stiffness is given as:
\[ k_s = k_{I} \gamma_{W} \left( \frac{\sigma_{n}}{P_{a}} \right)^{n_{s}} \left( 1 - \frac{R_{fs} \cdot T_{s}}{\sigma_{n} \cdot \tan \phi'_{r}} \right)^{2} \]  

The nonlinear shear stiffness, \( k_s \), is expressed in terms of four parameters \( k_{I}, n_{s}, R_{fs} \) and \( \phi' \), which are obtained from the experimental shear box tests.

3.6 Bin Wall Finite Element

The surface of revolution of the bin wall is approximated by curved axisymmetric elements which are connected at its nodal circles. As an alternative approximate solution, the axisymmetric shell can be treated as a beam on elastic supports. In this section, the outlines of the bin wall models are presented.

3.6.1 Finite Element Review of Thin Shells of Revolution

Arbitrary continuous shell of revolution can be divided into a number of short frustums, known as 'shell element.' These frustums are connected at their edges which are called 'nodal circles.' Both equilibrium and compatibility requirements are satisfied at the nodal circles.

The influence coefficients of a shell element are obtained using two different methods. The first method utilizes the homogeneous solution of the governing differential equations. As a result, an element with simple shell geometry is used, such as a circular cylinder, a truncated cone and a spherical cap. The truncated cone element is the most general shape for approximating shell geometry. Meyer (64) used this element for edge loading, and Popov (76) used it for general
axisymmetric loading. The influence coefficients obtained by this method are very complicated and require the evaluation of infinite series which, for some cases, converges very slowly.

The second approach makes use of the direct method of variational problems. The displacement method, the equilibrium method and the mixed method (26) had been advocated. It had been reported (51) that the displacement method was found to be superior to other methods of analysis. In the following paragraph, a summary of some of the displacement models is presented.

Grafton (36) used the conical frustum to analyze the axisymmetric deformation of shells of revolution of cylindrical orthotropic materials. The solution for axisymmetric deformations utilizing Fourier's expansion, was developed by Percy (71). Jones and Strome (49) reported the first attempt to use a curved element which provides the continuity of slopes at the nodal circles, but for an arbitrary shell, the meridional curvatures are not continuous at these nodes. Khojasteh-Bakht (51) reported the use of a curved element assumes a cubical form for the transverse displacement and a linear one for the inplane displacement. This model gives inaccurate results where there is slope discontinuity. Sharifi (81) modified the previous element by assuming a cubical polynomial for both the transverse and the inplane deformations. In the next section, a description of this element will be given and the stiffness matrix will be represented. It should be noticed that this
analysis will be adopted to the small deflection theory and confined to the axisymmetric loading and the support conditions.

3.6.2 Analysis of Rotational Shells

a) Equilibrium Equations

For rotational shell of revolution under axisymmetric loading, there are three equilibrium equations. Adopting the sign convention to be positive as shown in Figure (3.9), the three equilibrium equations are:

\[
\frac{d}{ds}(r_0 N_s) - N_\theta \cdot \cos \alpha - \frac{r_0}{r_1} \cdot Q_s + r_0 p_s = 0 \quad (3.52.a)
\]

\[
\frac{d}{ds}(r_0 Q_s) + N_\theta \cdot \sin \alpha + \frac{r_0}{r_1} \cdot N_s - r_0 p_r = 0 \quad (3.52.b)
\]

\[
\frac{d}{ds}(r_0 M_s) - M_\theta \cdot \cos \alpha + r_0 Q_s = 0 \quad (3.52.c)
\]

where, \( N_s \) and \( N_\theta \) are the resultant normal force per unit length in the meridional and circumference directions, respectively;

\( M_s \) and \( M_\theta \) are the resultant moment per unit length in the meridional and circumference directions, respectively;

\( Q_s \) is the resultant shear force per unit length;

\( p_s \) and \( p_r \) are the surface loads in the meridional and normal directions, respectively;

\( r_0 \) is the radius of a circle normal to the \( z \)-axis;
Figure (3.9) Shell Forces and Moments.
where, \( r_1 \) is the principal radius of curvature in the meridional direction.

Eliminating \( Q_s \) among Equations (3.52), yields:

\[
\frac{d}{ds}(r_0 \cdot N_s) + \frac{1}{r_1} \cdot \frac{d}{ds}(r_0 \cdot M_s) \cdot \cos \alpha + r_0 \cdot p_s = 0
\]

(3.53.a)

\[
\frac{d^2}{ds^2}(r_0 \cdot M_s) - \frac{dM_\theta}{ds} \cdot \cos \alpha + \left( \frac{M_\theta}{r_1} - N_\theta \right) \cdot \sin \alpha - \frac{r_0 \cdot N_s}{r_1} + r_0 \cdot p_r = 0
\]

(3.53.b)

b) **Strain-Displacement Relations**

The filling of axisymmetric bins is assumed to be centric, which indicates an axisymmetric deformation for the bin wall. Correspondingly, the displacement of the bin wall in the circumferential direction vanishes. Thus, only the radial displacement \( w \) and the vertical displacement \( v \) are considered, which gives a strain displacement relations of the form (86),

\[
\varepsilon_\theta = \frac{v}{r_0 \cdot r_1} \cdot \frac{dr}{d\alpha} + \frac{w}{r_2}
\]

(3.54.a)

\[
\varepsilon_s = \frac{1}{r_1} \cdot \frac{dw}{d\alpha} + \frac{w}{r_1}
\]

(3.54.b)

\[
\kappa_\theta = \frac{1}{r_0 \cdot r_1} \left( \frac{v}{r_1} - \frac{dw}{r_1} \right) \cdot \frac{dr_0}{d\alpha}
\]

(3.54.c)

\[
\kappa_s = \frac{1}{r_1} \cdot \frac{dw}{d\alpha} - \frac{v}{r_1}
\]

(3.54.d)

\[
\chi = \frac{1}{r_1} \cdot \frac{dw}{d\alpha} - \frac{v}{r_1}
\]

(3.54.e)

where, \( \varepsilon_\theta \) is the strain of the middle surface in the circumferential direction;

\( \varepsilon_s \) is the strain of the middle surface in the meridional direction;
where, $\kappa_\theta$ is the curvature change in the circumferential direction;

$\kappa_s$ is the curvature change in the meridional direction;

$\chi$ is the meridional rotation.

c) **Stress-Strain Relations**

The stress-strain relationship for isotropic elastic shell is given as:

$$N_\theta = \frac{E \cdot t}{1-v^2} \cdot (\varepsilon_\theta + \nu \cdot \varepsilon_s) \quad (3.55.a)$$

$$N_s = \frac{E \cdot t}{1-v^2} \cdot (\varepsilon_s + \nu \cdot \varepsilon_\theta) \quad (3.55.b)$$

$$M_\theta = \frac{E \cdot t^3}{12(1-v^2)} \cdot (\kappa_\theta + \nu \kappa_s) \quad (3.55.c)$$

$$M_s = \frac{E \cdot t^3}{12(1-v^2)} \cdot (\kappa_s + \nu \kappa_\theta) \quad (3.55.d)$$

where $t$ and $E$ are the shell thickness and modulus of elasticity respectively.

d) **Presentation of Element Geometry**

The meridional shape of an axisymmetric element is replaced by a substitute curve which matches with the original curve at selected points. The polynomial form is used to represent the substitute curve either in the local or the global coordinates (51). Figure (3.10) shows the element geometry, accordingly the meridional curve in $\eta-\xi$ coordinates can be expressed as:

$$\eta = \xi (1-\xi) (a_1 + a_2 \cdot \xi + a_3 \cdot \xi^2 + a_4 \cdot \xi^3) \quad (3.56)$$

where, $a_1 = \tan \beta_i$
Figure (3.10.a)

Figure (3.10.b)

Figure (3.10). Meridional Curve.
where, \( a_2 = \tan \beta_i + \frac{\delta}{2} \eta_i' \) \hfill (3.57.a)

\( a_3 = -(5\tan \beta_i + 4\tan \beta_j) + \frac{\delta}{2} \eta_j' - \eta_i' \) \hfill (3.57.b)

\( a_4 = 3(\tan \beta_j + \tan \beta_i) + \frac{\delta}{2}(\eta_i' - \eta_j') \) \hfill (3.57.c)

\[ \eta'' = \frac{d^2 \eta}{d \xi^2} = \frac{-L}{r_1 \cos \beta} \] \hfill (3.57.d)

\( \eta \) and \( \xi \) are the dimensionless coordinates;
i and \( j \) denotes the end nodal circles;
\( \beta \) is the slope of the tangent to the \( \xi \)-coordinate;
\( L \) is the cord length.

e) Displacement Pattern

The following displacement model, expressed in terms of the local cartesian coordinates, as shown in Figure (3.10) is assumed over each discrete element,

\[ u_1 = \alpha_1 + \alpha_2 \cdot \xi + \alpha_3 \cdot \xi^2 + \alpha_4 \cdot \xi^3 \] \hfill (3.58)

\[ u_2 = \alpha_5 + \alpha_6 \cdot \xi + \alpha_7 \cdot \xi^2 + \alpha_8 \cdot \xi^3 \] \hfill (3.59)

It should be noticed that the number of the generalized coordinates is equal to the total number of internal and external degrees of freedom. The six degrees of freedom at nodes \( i \) and \( j \) are the external degrees of freedom, while the two degrees of freedom at nodes \( m' \) and \( n' \) are the internal degrees of freedom. Displacement compatibility should be maintained at the interelement nodes, i.e., nodes \( i \) and \( j \).

Thus the two degrees of freedom at nodes \( m' \) and \( n' \) are removed by a static condensation process prior to the assemblage of the total structural stiffness matrix.

Expressing the displacements of the middle surface of
a shell in local cartesian coordinates $\xi - \eta$, the following relations between $(v, w)$ and $(u_1, u_2)$, holds:

\[
v = u_1 \cdot \cos \beta + u_2 \cdot \sin \beta
\]

\[
w = -u_1 \cdot \sin \beta + u_2 \cdot \cos \beta
\]

(3.60)

In the following treatment, by applying the chain rule of differentiation, differentiation with respect to the arc length, $s$, is replaced by the differentiation with respect to normalized cord variable, $\xi$. The strain-displacement relations is transformed to the local cartesian coordinates using the relations:

\[
\frac{d}{ds} = \frac{1}{L \cdot \sqrt{1 + \eta'^2}} \cdot \frac{d}{d\xi}
\]

(3.62.a)

\[
\frac{dr_0}{d\alpha} = r_1 \cdot \cos \alpha
\]

(3.62.b)

\[
\cos \beta = \frac{1}{\sqrt{1 + \eta'^2}}
\]

(3.62.c)

\[
\sin \beta = \frac{\eta'}{\sqrt{1 + \eta'^2}}
\]

(3.62.d)

\[
\xi + \psi + \delta = \frac{\pi}{2}
\]

(3.62.e)

The angles $\alpha$, $\psi$ and $\beta$ are defined in Figure (3.10). By using Equation (3.54) and Equations (3.56) through (3.62), the strain-displacement relations are found to be:

\[
\epsilon_0 = \frac{\sin \psi}{r_0} (\alpha_1 + \alpha_2 \cdot \xi + \alpha_3 \cdot \xi^2 + \alpha_4 \cdot \xi^3) + \frac{\cos \psi}{r_0} (\alpha_5 + \alpha_6 \cdot \xi + \alpha_7 \cdot \xi^2 + \alpha_8 \cdot \xi^3)
\]

(3.63.a)

\[
\epsilon_s = \rho (\alpha_2 + 2\alpha_3 \cdot \xi + 3\alpha_4 \cdot \xi^2) + \rho \cdot \eta' (\alpha_6 + 2\alpha_7 \cdot \xi + 3\alpha_8 \cdot \xi^2)
\]

(3.63.b)
\[ \kappa_\theta = \eta' \bar{\psi} (a_2 + 2a_3 \cdot \xi + 3a_4 \cdot \xi^2) - \bar{\psi} (a_6 + 2a_7 \cdot \xi + 3a_8 \cdot \xi^2) \] (3.63.c)

\[ \kappa_s = (1 - \eta'^2) \cdot \phi \cdot a_2 + [2\xi (1 - \eta'^2) \cdot \phi + \Omega] a_3 + [3\xi^2 (1 - \eta'^2) \phi + 3\xi \cdot \Omega] a_4 + 2\eta' \cdot \phi \cdot a_6 + (4\xi \cdot \eta' \cdot \phi - \mu) a_7 + 3\xi (2\xi \cdot \eta' \cdot \phi - \mu) a_8 \] (3.63.d)

where,

\[ \bar{\psi} = \frac{\sin \psi + \eta' \cdot \cos \psi}{L \cdot \pi (1 + \eta'^2)^{3/2}} \] (3.64.a)

\[ \mu = \frac{2}{L^2 (1 + \eta'^2)^{3/2}} \] (3.64.b)

\[ \Omega = \frac{2\eta'}{L^2 (1 + \eta'^2)^{3/2}} \] (3.64.c)

\[ \phi = \frac{\eta'}{L^2 (1 + \eta'^2)^{5/2}} \] (3.64.d)

\[ \rho = \frac{-1}{L (1 + \eta'^2)} \]

\( f \) Assemblage of the Total Stiffness Matrix

The stress-strain relations, Equations (3.55), together with Equations (3.60) through (3.63) are used to form the matrices \([A]\), \([B]\), and \([C]\). Substituting these matrices into Equation (3.7) and integrating, the element stiffness matrix of the shell element in the \( \eta - \xi \) coordinates is obtained. The details of this development is given in Appendix (B). Before going to the static condensation, the element stiffness matrix should be transformed to the global \( r-z \) axes.
The transformation matrix $T$ is developed from Figure (3.10) as:

$$
[T] = \begin{bmatrix}
-sin\psi & -cos\psi & 0 & 0 & 0 & 0 & 0 & 0 \\
\cos\psi & sin\psi & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & sin\psi & -cos\psi & 0 & 0 & 0 \\
0 & 0 & 0 & cos\psi & sin\psi & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

(3.65)

The element stiffness matrix in the global coordinates is of order (8x8), the equilibrium equations can be written as:

$$
[K_e] \{q\} = \{F\}
$$

(3.66)

where,

$$
\{q\}^T = \{ U_r^i U_z^i U_r^j U_z^j U_{m'}^i U_{n'}^i \}
$$

(3.67)

$$
\{F\}^T = \{ F_r^i F_z^i M_i^i F_r^j F_z^j M_j^j F_{m'}^i F_{n'}^i \}
$$

(3.68)

$u_r$ and $u_z$ are the nodal displacements in the $r$ and $z$ directions respectively, while $F_r$ and $F_z$ are the nodal forces in the $r$ and $z$ directions, respectively, and $M$ is the nodal moment.

Equation (3.66) is partitioned to distinguish between the terms corresponding to the internal and the external degrees
of freedom, such that:

\[
\begin{bmatrix}
K_{11} & K_{12} \\
6\times6 & 6\times2 \\
K_{21} & K_{22} \\
2\times6 & 2\times2
\end{bmatrix}
\begin{bmatrix}
q_1 \\
6\times1 \\
q_2 \\
6\times2
\end{bmatrix}
= 
\begin{bmatrix}
F_1 \\
6\times1 \\
F_2 \\
6\times2
\end{bmatrix}
\]

(3.69)

The internal degrees of freedom is removed from the above equation and the condensed equilibrium equation is written as:

\[
[\tilde{K}]{q_1} = \{\tilde{F}\}
\]

(3.70)

where,

\[
[\tilde{K}] = [K_{11}] - [K_{12}] \cdot [K_{22}]^{-1} \cdot [K_{21}]
\]

(3.71.a)

\[
{\tilde{F}} = \{F_1\} - [K_{12}] \cdot [K_{22}]^{-1} \cdot \{F_2\}
\]

(3.71.b)

\[
{q_1}^T = \{u_i^r u_i^z \cdot x_i \cdot u_j^r u_j^z \cdot x_j\}
\]

(3.71.c)

\[
{F_1}^T = \{F_r^i F_z^i M_i^r F_r^j F_z^j M_j^r\}
\]

(3.71.d)

\[
{F_2}^T = \{F_{1}^{n'} F_1^{n'}\}
\]

(3.71.e)

The matrices \( [\tilde{K}] \) and \( \{\tilde{F}\} \) are known as the condensed stiffness matrix and load vector, respectively. The total stiffness matrix and the load vector can be assembled from the stiffness matrices of the element using the direct stiffness method.

Solving the equilibrium equations for the assembled structure, the deformation at the nodal circles is obtained. The displacements for the internal degrees of freedom can be recovered using Equation (3.59). The stress field is determined
using the stress-deformation equations given in Appendix (B).

3.6.3 Shells of Revolution As A Beam on Elastic Supports

The axisymmetric cylindrical shell of revolution under axisymmetric load is treated as a beam on elastic supports, as shown in Figure (3.11). The wall is divided into beam elements connected at their nodal points. The shell action is replaced by elastic springs along the element length as shown in Figure (3.11.c). The value of the spring coefficient together with the equivalent stiffness matrix are developed in this section.

Solving Equations (3.53) through (3.56) together, the hoop and meridional stresses are written as:

\[ N_\theta = \frac{Et}{1-\nu^2} \left[ \frac{v \cdot \cos \alpha + w \cdot \sin \alpha}{R} \right] + \nu \cdot \frac{v \cdot \cos \alpha + w \cdot \sin \alpha}{r_l} \]  
\[ N_\phi = \frac{Et}{1-\nu^2} \left[ \frac{v \cdot \cos \alpha + w \cdot \sin \alpha}{R} \right] + \nu \cdot \frac{v \cdot \cos \alpha + w \cdot \sin \alpha}{r_l} \]  

Solving these two equations together, yields:

\[ N_\theta = \frac{Et}{R} \left[ v \cdot \cos \alpha + w \cdot \sin \alpha \right] + \nu \cdot N_\phi \]  

the displacement \( u_r \) in the \( r \)-direction is written as:

\[ u_r = v \cdot \cos \alpha + w \cdot \sin \alpha \]  

Thus, Equation (3.74) can be written as:

\[ \frac{N_\theta}{u_r} = \frac{Et}{R} + \frac{\nu}{u_r} \cdot N_\phi \]
It should be noticed that the sides of each longitudinal element of the axisymmetric cylindrical shell should remain parallel with the original direction which results in a moment \( M_\theta \) in the circumferential direction equal to:

\[
M_\theta = \nu M_s
\]

(3.77)

This circumferential moment has a stiffness effect on the bending deformation of the longitudinal beam, which is taken into account by increasing either the moment of inertia or the modulus of elasticity by the ratio of \( 1/(1-\nu^2) \), thus:

\[
I_e = \frac{I}{1-\nu^2}
\]

(3.78.a)

or

\[
E_e = \frac{E}{1-\nu^2}
\]

(3.78.b)

Consider the cylindrical shell shown in Figure (3.11.a); if section A-A of unit width is separated as shown in Figure (3.11.b), the resultant force \( P \) per unit length of the perimeter is given by:

\[
P = N_\theta \cdot d_\theta
\]

(3.79)

The corresponding angle \( d_\theta \) is approximated as:

\[
d_\theta = \frac{1}{R}
\]

(3.80)

Using Equations (3.76), (3.79) and (3.80), yields:

\[
\frac{P}{u_r} = \frac{Et}{R^2} + \nu \frac{N_s}{u_r \cdot R}
\]

(3.81)
Figure (3.11) Axisymmetric Shell of Revolution as a Beam on Elastic Springs.
The spring coefficient $k_f$ per unit length is given by:

$$k_f = \frac{E_t}{R^2} + v \cdot \frac{N_S}{u_r \cdot R}$$  \hspace{1cm} (3.82)

As a first approximation, the value of $v \cdot \frac{N_S}{u_r \cdot R}$ is neglected with respect to $\frac{E_t}{R^2}$, then:

$$k_f = \frac{E_t}{R^2}$$  \hspace{1cm} (3.83)

3.6.3.1 The Stiffness Matrix of A Beam on Elastic Springs

The beam element shown in Figure (3.12) is assumed to be prismatic with constant properties along the longitudinal axis. The displacement functions of the beam element in its local coordinates are given as (63):

$$v_o(x) = a_1 \cdot x + a_2$$  \hspace{1cm} (3.84.a)

$$w(x) = a_3 + a_4 \cdot x + a_5 \cdot x^2 + a_6 \cdot x^3$$  \hspace{1cm} (3.84.b)

where,

$$a_1 = \frac{v_j - v_i}{L}$$  \hspace{1cm} (3.85.a)

$$a_2 = v_i$$  \hspace{1cm} (3.85.b)

$$a_3 = w_i$$  \hspace{1cm} (3.85.c)

$$a_4 = \chi_i$$  \hspace{1cm} (3.85.d)

$$a_5 = \frac{1}{L^2} (3w_j - 3w_i - 2\chi_i \cdot L - \chi_j \cdot L)$$  \hspace{1cm} (3.85.e)

$$a_6 = \frac{1}{L^3} (2w_i - 2w_j + \chi_i \cdot L + \chi_j \cdot L)$$  \hspace{1cm} (3.85.f)
Figure (3.12) Beam Element Representation.
The element stiffness matrix, as well as the transformation matrix, is given in Appendix (A).

In order to include the effect of the hoop forces of the axisymmetric shell in the stiffness analysis, two different techniques are used. First, the equivalent isolated spring coefficient at the nodal points, which will end with a diagonal stiffness matrix. Second, the consistent matrix technique which has been proven to be superior to the first technique (66). In this analysis, the consistent matrix technique is used and described in the next section.

The spring reaction at any point along the length of the beam element, i-j, is given by $k_f w(x)dx$. The strain energy stored in the springs along the element is:

$$U_f = \frac{1}{2} \int_0^L w \cdot k_f \cdot w \cdot dx$$  \hspace{1cm} (3.86)  

Using Equations (3.84.b) and (3.85), Equation (3.86) becomes:

$$U_f = \frac{1}{2} \int_0^L \{q\}^T [N]^T [k_f][N]\{q\} dx$$  \hspace{1cm} (3.87)

where,

$$[N]^T = \begin{bmatrix} 0 & 0 & \gamma_1 & \gamma_2 & \gamma_4 \end{bmatrix}$$  \hspace{1cm} (3.88.a)

$$\gamma_1 = 1 - 3 \left(\frac{X}{L}\right)^2 + 2 \left(\frac{X}{L}\right)^3$$  \hspace{1cm} (3.88.b)

$$\gamma_2 = x \left(1 - \frac{X}{L}\right)^2$$  \hspace{1cm} (3.88.c)

$$\gamma_3 = 3 \left(\frac{X}{L}\right)^2 - 2 \left(\frac{X}{L}\right)^3$$  \hspace{1cm} (3.88.d)

$$\gamma_4 = (x-L) \left(\frac{X}{L}\right)^2$$  \hspace{1cm} (3.88.e)
Using Equation (3.7), the consistent stiffness matrix is:

\[
[k_f] = \int_0^L [N]^T[k_f][N]dx \tag{3.89}
\]

Integrating Equation (3.89) over the length, the stiffness matrix of the elastic springs is obtained. The first two rows are zero, while the rest of the elements are given by:

\[
K_{33} = k_{fi}(\frac{13L}{35}) + (k_{f_j} - k_{fi})(\frac{3L}{35})
\]

\[
K_{43} = k_{fi}(\frac{9L}{70}) + (k_{f_j} - k_{fi})(\frac{9L}{140})
\]

\[
K_{44} = k_{fi}(\frac{13L}{35}) + (k_{f_j} - k_{fi})(\frac{2L}{7})
\]

\[
K_{53} = k_{fi}(\frac{11L^2}{210}) + (k_{f_j} - k_{fi})(\frac{L^2}{60})
\]

\[
K_{54} = k_{fi}(\frac{13L^2}{420}) - (k_{f_j} - k_{fi})(\frac{L^2}{60})
\]

\[
K_{55} = k_{fi}(\frac{L^3}{105}) - (k_{f_j} - k_{fi})(\frac{L^3}{280})
\]

\[
K_{63} = -k_{fi}(\frac{13L^2}{420}) - (k_{f_j} - k_{fi})(\frac{L^2}{70})
\]

\[
K_{64} = -k_{fi}(\frac{11L^2}{210}) - (k_{f_j} - k_{fi})(\frac{L^2}{28})
\]

\[
K_{65} = -k_{fi}(\frac{3L^2}{420}) - (k_{f_j} - k_{fi})(\frac{L^3}{280})
\]

\[
K_{66} = k_{fi}(\frac{L^3}{105}) + (k_{f_j} - k_{fi})(\frac{L^3}{68})
\]
The spring coefficient's intensity at nodes \( i \) and \( j \) are \( k_{fi} \) and \( k_{fj} \), respectively. Using Equation (3.83), yields:

\[
k_{fi} = k_{fj} = \frac{Et}{R^2}
\]

Adding this consistent matrix to the familiar beam element stiffness matrix given in Appendix (A), the total local stiffness matrix of a beam on elastic springs is obtained.

3.7 Computer Program

A computer program is developed to carry out the mathematical computations using the method of analysis described above. An automatic generation for the incremental finite element meshes are adjusted in order to simplify the use of the program. Figures (3.13) and (3.14) show the automatically developed meshes for shallow bins with straight wall and curved wall. Figure (3.15) shows a flow chart for the computer program.
Figure (3.13) Mesh Generated for Straight Shell Wall.
Figure (3.14) - Mesh Generated for Curved Shell Wall.
START

NCRE  =  Number of Curved Segments
NLC   =  Number of Load Increments
        (Number of layers)
IESH  =  Variable to Define the Shell
        Element
        if IESH = 1  Exact solution
        = 0  Approximate solution
ICODE =  Code to Define the Type of
        Element
        = 1  Shell Element or Beam
        Element
        = 2  Interface Element
        = 3  Soil Element
IAXI  =  Variable to Define the Integration of the Axisymmetric Soil
        Element
        = 1  Exact Integration is Carried
        = 0  Approximate Integration is Carried
E     =  Modulus of Elasticity
ν     =  Poisson's Ratio
EN    =  New Modulus of Elasticity
σn    =  Normal Pressure on the Wall
τ     =  Shear Stress on the Wall
ϕ     =  Angle of Internal Friction
Figure (3.15) Flow Chart
CHAPTER IV

EXPERIMENTAL INVESTIGATION

4.1 General

In order to conduct the analysis outlined in Chapter III, it is essential to determine the properties of the soil such as the modulus of elasticity, bulk density and the shear-deformation relationship. Also a model of grain storage bin must be built and tested in order to compare the analytical results with experimentally obtained data. Therefore the experimental program undertaken is as follows:

a) A triaxial test is conducted to determine the hyperbolic constants needed for the stress-strain relationship of the sand used in the bin model tests.

b) A shear box test is conducted to define the load-deformation relationship for the interface between the sand used and the galvanized steel.

c) A model bin is loaded with the sand tested in steps a and b in order to measure the pressure of the sand along the bin height.

4.2 Description of the Sand Used in Tests

The sand used in this experimental investigation is a clean, washed, dry sand from Lake Erie. This sand is passed through sieve #40 and retained on sieve #60; the size of the particles is between 0.42 mm and 0.25 mm. The bulk density for the sand is found to be 91.6pcf.

4.3 Triaxial Test

The mechanical properties and the shear strength of
the soil is obtained from the triaxial test. Triaxial tests are carried out with a constant confining pressure ($\sigma_3$) and increasing the principal stress ($\sigma_1$) to failure. In order to obtain consistent results from the triaxial tests, considerations are given to reduce the errors that could have a bearing on the test results. The possible errors are as follows:

a) Friction between the end plates and the sand sample;

b) Diameter to height ratio;

c) Friction between the loading ram and the triaxial cell;

d) Expansion of the cell on application of the cell pressure;

e) Deformation of the sample during testing;

f) Stiffness effect of the membrane on the sample stiffness;

g) The penetration of the membrane into the sample on application of the confining pressure.

It is necessary in triaxial tests to eliminate the end friction between the sample and the end plates to assure a uniform axial strain (23,79), which leads to a shear strength mobilization through the whole sample height. In order to reduce the end friction, two aluminum end plates with a smooth surface covered by a lubricated membrane are used (40).

The triaxial test results become more consistent when a sample of height to diameter ratio of two is used (40).
Hence 8 in. high by 4 in. diameter samples are used in the triaxial tests.

The friction between the loading ram and the triaxial cell can be eliminated using a rotating ram or by measuring the load inside the triaxial cell (40). The torque applied to the loading ram in the former method could create some torsion to the sample. This torque has a serious effect on the test results. The latter method is complicated and inaccurate (40). In the present tests, the loading ram is lubricated and the load is measured outside the triaxial cell using a proving ring.

The volume change measurements in a triaxial test should be corrected for the cell expansion. In the present work, the triaxial cell shown in Figure (4.8) is made of plexiglass and reinforced both horizontally and vertically. Since the confined pressures used in the present tests are relatively small, it is felt that the expansion of the pressure cell can be neglected.

The membrane correction to be applied to the deviator stress and the volume change will be discussed in more details later in this chapter.

4.3.1 Preparation of Samples

The behaviour of sand during shear tends to be erratic due to the variation of the density and packing from test-to-test. A sand deposition device has been used by Walker (92) and Hasnain (40) in order to reproduce approximately the same density and packing. Since the variation of the density
along the diameter and the height is not considered in the present study, a funnel raining device is used for both the triaxial and model tests and the sand falls a total of 36 in. through a 1 in. diameter opening at the base of the funnel shown in Figures (4.1) and (4.2). Incremental sand layers of 3 in. thick are used to fill both the triaxial form and the bin model. After each layer, the funnel containing the sand is raised 3 inches distance to maintain a drop of 36 inches for the sand. In each triaxial test, the sample is prepared using the three piece form as shown in Figure (4.3). Suction between the membrane and the form is provided to ensure a constant sample diameter as shown in Figures (4.4) and (4.5). After the form is filled with sand, the sides and the top of the form are cleaned and excess sand is removed slowly and gently.

The top plate along with a greased rubber membrane is placed in position. It is then given a small rotation in either direction to ensure a perfect seating. It should be noted that the membrane overlap is greased with silicon grease and secured to the top and bottom plates by o-rings to insure sealing. Suction is then applied to the sand sample and then the suction valve is closed. The form is then removed and the sample remained upright as shown in Figure (4.5). This suction is released only after a confining pressure is applied to the sample.

The diameter of the sample is measured by a micrometer at five different places, including the mid-height and the ends of the sample. At every position, a set of three
Figure (4.1) Raining Sand Device for Triaxial Test.
Figure (4.2) Raining Sand Device for Bin Model Test.
Figure (4.3) Triaxial Sample Form.
Figure (4.4) Triaxial Form Connected to Suction Valve.
Figure (4.5) Vacuumed Triaxial Sample.
diameter readings are taken in different directions.

4.3.2 Triaxial Compression Test Procedure

After the sample is prepared as described in the above section, it was transferred to the triaxial machine and the triaxial cell is assembled as shown in Figures (4.6) and (4.8). The cell is filled carefully with distilled water without entrapping any air bubbles. The base of the machine is raised until the loading ram comes into contact with the top plate. The cell pressure is then applied and the suction valve opened to the air in order to establish atmospheric pressure within the sample. A strain rate of about 0.016 in. per minute is used. The volume change measurements are made by recording the amount of air entering or leaving the sample with the help of the volume change measuring device shown in Figures (4.7) and (4.9). The details of the experimental set up is shown in Figures (4.8) and (4.9).

4.3.3 Membrane Corrections

The deviator stress can be corrected to account for the membrane stiffness using either Duncan's (22) equation or Bishop's (7) equation. This correction according to Bishop (7) for the 4 in. diameter sample is found to be 0.25 psi for a membrane thickness of about 0.012 in.

The correction for the volumetric changes due to the membrane penetration is given by Frydman (30). For confining pressures of 4 psi, 8 psi and 12 psi, the volume change corrections are 0.0094 in$^3$, 0.016 in$^2$ and 0.019 in$^3$, respectively.
Figure (4.6) Triaxial Machine.
Figure (4.7) Volume Change Measuring Device.
Figure (4.8) Triaxial Test Set Up.
4.3.4 Analysis of Triaxial Test Results

The test results for the triaxial tests are presented in Tables (1) through (9) in Appendix (C). The values of the hyperbolic parameters used in Equation (3.18) are determined as follows:

a) The test data is inspected carefully to eliminate experimental errors and outliers and smooth curves are drawn;

b) A graph of transformed stress versus strain is plotted as shown in Figure (4.10), and then best fit line for each set of curves is drawn. The initial modulus $E_i$ and the ultimate strength $(\sigma_1 - \sigma_3)_{ult}$ are the reciprocals of the intercepts and the slopes of these lines, respectively. The values of $E_i/P_a$ with the corresponding values of $\sigma_3/P_a$ are plotted on a logarithmic scale as shown in Figure (4.11). The values of the modulus number, $k_h$, and the modulus exponent, $n$, are the intercept and the slope of the best fit line as shown in Figure (4.11).

The modulus number $k_h$ is 290.4, the modulus exponent $n$ is 0.745, the failure ratio $R_f$ is 0.86 and the internal angle of friction is $38^\circ$.

In order to check the accuracy of the hyperbolic representation of the stress-strain curves, the obtained values of $k_h$, $n$, $R_f$, and $\phi$ are substituted into Equation (3.12) and the deviatoric stresses for different assumed axial strains are calculated. The assumed strains together with the calculated stresses are given in Table (10) in Appendix (C) and are plotted in Figure (4.12). It is clear
Figure (4.10) Experimental Stress-Strain Relationship.
Triaxial Test
\[ k_h = 290.4 \]
\[ n = 0.745 \]

Figure (4.11) Experimental Hyperbolic Parameters for the Tangent Modulus.
Figure (4.12) Predicted and Experimental Stress-Strain Curves.
that the hyperbolic function is reasonably representative of the stress-strain relationship.

d) Equation (3.21) is used to determine the value of Poisson's ratio under high overburden pressure. A trial is made here to use this equation under the small pressures involved in the grain storage problem. It is found that the value of Poisson's ratio cannot be predicted correctly using this equation under small values of overburden pressures. The relation between the axial strain and the lateral strain is plotted in Figure (4.13) for different confining pressure. It is found that this relation is approximately a straight line and it does not change considerably with the change of the confining pressure. As a conclusion from the above discussion, Poisson's ratio is assumed to be a constant 0.4 in the present analysis.

4.4 Shear Box Test

The one-dimensional element developed in Chapter III is employed to represent the interface between the grain material and the bin wall. The properties assigned to the interface are determined from the results of direct shear tests on composite specimens consisting partly of soil and partly of the structural galvanized steel.

4.4.1 Preparation of Samples

Six-by-six cm. (2.36 x 2.36 in.) samples are used in the direct shear box tests. The samples are prepared by depositing the sand using the previously described funnel with an opening of 1 in. diameter and 36 in. height. Care
Figure (4.13) Variation of Lateral Strains with Axial Strains.
is taken to produce a uniform thickness. The top of the sample was cleaned before the top plate is adjusted in position.

4.4.2 Experimental Results of Direct Shear Box Tests

The direct shear box tests are carried out as explained in reference (61), a reading is taken every 15 seconds for the first 2 minutes. The values of the normal stresses used in the tests are 2.27, 5.49, and 8.7 psi, respectively.

The value of the friction angle $\phi'$ for a particular test is determined from the maximum ratio of $\tau_s/\sigma_n$, expressed as:

$$\tan \phi' = \frac{\tau_s}{\sigma_n}$$  \hspace{1cm} (4.1)

where $\tau_s$ is the shear resistance and $\sigma_n$ is the normal stress. The experimental results are given in Tables (11) through (13) in Appendix (C).

4.4.3 The Hyperbolic Parameters From the Laboratory Tests

The same procedure explained in section (4.3.5) is applied for the determination of the modulus parameters for the friction between the sand used and the galvanized steel. The value of $K_{\text{si}}$ and $\tau_{\text{ult.}}$ are obtained as shown in Figure (4.14). The values of $K_I$ and $n_s$ are obtained as shown in Figure (4.15). The values of $K_I$, $n_s$ and $R_{fs}$ as calculated from Figures (4.14) and (4.15) are $0.787 \times 10^5$, 0.846 and 0.903, respectively. Table (14) includes the predicted deformations versus the corresponding stresses for different values of normal stresses. Figure (4.16) shows both the experimental and the predicted load-deformation relationship.
Figure (4.14) Experimental Shear Stress-Deformation Relationship.
Figure (4.15) Experimental Hyperbolic Parameters for Shear Stiffness.

Figure (4.16) Experimental and Predicted Stress-Deformation Curves.
using the above parameters for the three stresses chosen. These figures indicate good agreement between the predicted and the experimental values.

4.5 Bin Model Test

Tests are carried out on a cylindrical steel bin of 3½ ft. height by 3 ft. in diameter and 1/16 in. wall thickness. The size of the bin is chosen so that the height to the diameter ratio is less than 1.5 and thus it can be classified as shallow bin, furthermore this size made is suitable for manual filling and emptying. The pressure on the wall is measured at five different locations, at 9, 12, 18, 24 and 30 in. from the bottom of the bin as shown in Figure (4.17). The sand pressure along the wall height is measured using pressure cells of diameter 3.5 in. Each of these pressure cells is screwed to an opening of 3.5 in. diameter in the wall at staggered locations as shown in Figure (4.18). The bottom of the bin was a flat base welded to the bin wall along the perimeter. A one-inch diameter opening on the bottom of the bin facilitated the emptying of the bin at the end of each experiment.

The pressures in the bin model test are expected to be relatively small. Accordingly, very sensitive pressure cells with very small deformations were developed (38). The details of these pressure cells is given in Figures (4.19) through (4.21).

The pressure cells are machined from 6061-T6 aluminum in the form of a circular membrane of thickness 0.012 in.
Figure (4.17) Calibration of Pressure Cells.
Figure (4.18) Location of Pressure Cells.
with a ring beam of 0.25 x 0.3 in$^2$ cross section as shown in Figure (4.20). The diameter of the cell is 3.5 in., while the diameter of the membrane is 3 in. There is a smooth transfer from the membrane thickness to the edge ring beam through a transition curve of radius $\frac{1}{4}$ in.

Four BAE-13-125 BL3-350S electrical strain gauges of gauge factor 2.05 are installed along a diameter on the membrane. These gauges are connected in a full bridge circuit in such a way to be temperature compensating and very sensitive as shown in Figure (4.20). In order to protect the strain gauges and the leading wires, a cover plate is machined from aluminum as shown in Figure (4.21). This cover has an outer diameter of 4 in. and inner diameter of 3.5 in., the edge beam thickness is 0.3 in. while cover plate is 0.125 in. thick. A $\frac{1}{4}$ in. hole is drilled at the centre of the cover plate to pass the lead wires.

Each pressure cell is connected to the bin wall and calibrated using water pressure. The height of water versus the readings of the strain indicator are recorded for three different tests for each pressure cell (Tables 15 through 17 in Appendix (C)). The three readings are close enough to each other with no sign of any hysteresis effect. For each pressure cell, the calibration curve is plotted as shown in Figures (C.1) through (C.3) in Appendix (C).

4.5.1 Experimental Procedure

The bin is set up on a platform about two feet above the floor in order to ease emptying of the bin. The funnel shown in Figure (4.2) is adjusted to the centre of the bin
Figure (4.19) Pressure Cells.

Figure (4.20) Full Bridge Pressure Cell.
Figure (4.21) Details of the Pressure Cells.
at a 36 in. height from the bottom. The funnel is opened and the sand falls freely as shown in Figure (4.2). The strain output of each pressure cell is recorded and the height of the funnel is adjusted to the 36 in. height after every 3 in. incremental layer. For each incremental layer, the readings of the strain output are recorded at different time intervals until two successive readings are the same. Three tests are carried out for every pressure level. The results of the experimental measurements are given in Table (18) in Appendix (C).
CHAPTER V

ANALYSIS AND DISCUSSION

5.1 General

The finite element method was used for a computer analysis of shallow cylindrical grain bins as outlined in Chapter III. In this chapter, a comparison between the analytical results and the experimental results are given. It also includes a comparison between the existing theories and the new approach presented in this work. All the different parameters affecting the characteristics of the induced grain pressure as well as the performance of the bin are examined.

The parameters included in the analysis are: the wall stiffness; the interface shear stiffness; the plane strain and axisymmetric solutions and the effect of the vertical curvature of the bin wall in both the load distribution and the stability of the bin.

The configurations of the bin used in the analysis are shown in Figures (5.1.a) and (5.1.b). The bin has the following dimensions and mechanical properties: a diameter of 36 inches; a height of 42 inches; modulus of elasticity of $30 \times 10^6$ psi and Poisson's ratio of 0.3. The tangent modulus for the sand used in this analysis is obtained from Equation (3.18) using the previously described incremental loading and the iteration technique. For the first iteration, the vertical and horizontal stresses at the centroid of each element are assumed and the corresponding tangent modulus is obtained.
Figure (5.1.a) Experimental Bin Model.

Figure (5.1.b) Shallow Bin With Curved Wall.
Interface analysis requires specifying the normal stiffness and the shear stiffness of the interface. The normal stiffness of the interface is assumed to be high, with a value of $0.1 \times 10^7$ psi. The initial value of the shear stiffness is assumed to be small and equal to unity. As the normal pressure builds up, Equation (3.51) is used to calculate the shear stiffness.

The boundary conditions along the axis of the bin together with the floor have to be defined in the analysis of the grain-bin system. The bottom of the steel bin is assumed to be fixed and the soil nodes along the floor are also assumed fixed with no horizontal or vertical movements. The nodes along the bin axis of symmetry are assumed to have a vertical movement with no radial displacement. The analytical results will be discussed in more detail in the next section.

5.2 Verification of the Method of Analysis and the Computer Program

The computer program developed in this work is used to analyze the experimental bin model described in section (4.5). The experimental and the theoretical results are given in Figures (5.2) through (5.4). The average theoretical normal pressure along each element is plotted in solid lines for different $h/D$. The experimental results given in Table (17) (Appendix C) are also plotted in square points. In order to make a comparison between the experimental and the theoretical results, the theoretical and the experimental normal pressures are approximated by broken lines and dashed lines, respectively, as shown in Figures (5.2) through (5.4).
Figure (5.2) Normal Pressure Distribution.

Figure (5.3) Normal Pressure Distribution.
Figure (5.4) Normal Pressure Distribution.

\[ \frac{h}{2R_R} = 0.82 \]

\[ \frac{t_f}{R} = 3.5 \times 10^{-3} \]
The percentage difference between the theoretical and the experimental results obtained for each h/D is plotted in Figure (5.5). From this figure, it is clear that the difference between the experimental and the theoretical results ranges between 20% for the minimum h/D and 32% for the maximum h/D.

The deviation between the theoretical and the experimental results can be attributed to:

(a) The calibration of the pressure cell is one of the main factors which affects the experimentally measured pressure. Peaker (70) observed that, the pressure-strain curve of a pressure cell calibrated by granular material is steeper than the pressure-strain curve of the same pressure cell when calibrated by air or water pressure as shown in Figure (5.6). Peaker found that for the same strain, the sand pressure is 20% higher than the air pressure, which is attributed to the particles arrangement and size. If a similar percentage of error is considered in the present measured pressures, the experimental results will become close to the analytically obtained results.

(b) When filling the bin, the sand was allowed to fall freely under its own weight to form a conical shape with least stable particles at its surface. This means that the density of the falling sand changes along the diameter and the height from point to point, as indicated by Ross (78). The density variation contributes to the difference between the theoretical and the experimental results.

From the above discussion, it can be concluded that the
Figure (5.5) Relationship Between the Height of Filling and % Error.
Figure (5.6) Pressure-Strain Curve.
experimental results are in reasonable agreement with the theoretically obtained values, which verify the method of analysis and the computer program developed in Chapter III.

5.3 Effect of Wall Stiffness on the Sand Pressure

The stiffness of the bin wall, which is defined as the ratio of the wall thickness by the bin radius, is one of the major factors influencing the pressure distribution along the bin height. To study the effect of the wall stiffness, the bin described in section (5.1) has been given eight different stiffnesses of, $5.5 \times 10^{-2}$, $3.5 \times 10^{-3}$, $2.6 \times 10^{-3}$, $1.72 \times 10^{-3}$, $1.5 \times 10^{-3}$, $1.25 \times 10^{-3}$, $1.00 \times 10^{-3}$ and $8.33 \times 10^{-4}$. Figures (5.7) through (5.18) show the normal pressure distribution along the wall height for different $t/R$ and $h/D$ values. In each figure, two different normal pressures are presented for different $t/R$.

In order to compare between the different pressure distributions mentioned above, each of these pressures is transformed into an equivalent hydrostatic pressure. For each hydrostatic pressure an equivalent coefficient of lateral pressure, $k_a$, is obtained by equating the area under the assumed hydrostatic pressure distribution to the area under the actual pressure distribution. Therefore, the coefficient $k_a$ is as follows:

$$k_a = \frac{2 \times \text{Area Under the Actual Distribution}}{\gamma \cdot h^2} \quad (5.1)$$

The values of $k_a$ and the corresponding values of $t/R$ and $h/D$ are given in Table (5.1).

Figure (5.19) presents the relationship between $k_a$ and
Figure (5.7) Normal Pressure Distribution.

Figure (5.8) Normal Pressure Distribution.
Figure (5.9) Normal Pressure Distribution.
Figure (5.10) Normal Pressure Distribution.

Figure (5.11) Normal Pressure Distribution.
Figure (5.12) Normal Pressure Distribution.
Figure (5.13) Normal Pressure Distribution.

Figure (5.14) Normal Pressure Distribution.
Figure (5.15) Normal Pressure Distribution.
Figure (5.16) Normal Pressure Distribution.

Figure (5.17) Normal Pressure Distribution.
Figure (5.18) Normal Pressure Distribution.
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<td>1.0 x 10⁻⁴</td>
<td>0.393</td>
<td></td>
<td>1.0 x 10⁻⁴</td>
<td>0.393</td>
</tr>
<tr>
<td></td>
<td>8.33 x 10⁻⁴</td>
<td>0.366</td>
<td></td>
<td>8.33 x 10⁻⁴</td>
<td>0.335</td>
<td></td>
<td>8.33 x 10⁻⁴</td>
<td>0.335</td>
</tr>
</tbody>
</table>
Figure (5.19) Relationship Between Wall Stiffners And the Coefficient of Lateral Pressure, $k_a$.
t/R for four different values of h/D. It can be noted that, the coefficient $k_a$ does not change considerably when the wall is rigid and the sand stresses are close to the at rest state of stress. On the other side, when the wall becomes flexible the state of stresses in the sand become close to the active state. It can also be noted that, the coefficient $k_a$ is changed with the change of the height of the stored sand (h/D), with a difference of about 20% between the two limiting cases.

From Figures (5.7) through (5.18), it is clear that the hydrostatic pressure distribution is an accurate approximation, if the stiffness of the wall, t/R, is greater than or equal to $2.6 \times 10^{-3}$. If the wall stiffness is less than $2.6 \times 10^{-3}$, the lateral pressure is released from the upper part and the hydrostatic distribution is no longer valid.

From the above discussion, it can be concluded that the relative stiffness between the bin wall and the stored material controls the lateral pressure distribution along the bin height.

5.4 Comparison Between Conventional and Computed Pressure Distribution

As previously mentioned in Chapter II, for the conventional design procedure for shallow bins, theories of Janssen, Rankine, Coulomb and modified Coulomb were used. In this section, the lateral pressure distribution for the bin dimensions mentioned in section (5.2) were obtained using the conventional and the finite element methods.

Figure (5.20) shows the lateral pressure distribution for the different classical theories as well as the finite
Figure (5.20) Conventional Versus Finite Element.
element analysis. The finite element analysis is based on
wall stiffnesses of \(3.5 \times 10^{-3}\) and \(8.33 \times 10^{-4}\). The total
force acting on the bin wall at different levels due to each
lateral pressure is given in Table (5.2). The difference
between each of these forces and the experimentally obtained
force has been tabulated in Table (5.2).

From Figure (5.20) and Table (5.2), it can be clearly
stated that:

(a) the experimentally obtained lateral force is twice
as much as the force obtained using modified Coulomb;

(b) the lateral force obtained using the finite element
analysis together with wall stiffness of \(3.5 \times 10^{-3}\) has a dif-
ference of \(-12\%\) to \(25\%\) from the experimentally obtained force;

(c) the forces obtained from the analysis using the
classical theories differ among themselves from \(13\%\) to \(46\%\);

(d) when the wall stiffness is reduced to \(8.33 \times 10^{-4}\)
the finite element force is one-and-one-half times the modi-
fi ed Coulomb force. This indicates that the conventional
methods do not account for the wall flexibility and they only
represent the lower limit of the lateral pressure.

5.5 **Effect of Interface Conditions on Stress Distribution**

Along the **Vertical Wall**

The details of the interface analysis have been pre-
sented in sections (4.4) and (5.2). It is found that the one
dimensional element can be used to represent the different
interface conditions. The greatest advantage is that the use
of the linkage element eliminates the difficulty of computing
special boundary conditions for each state of deformation. The
### Table (5.2)

**Lateral Loads and % Difference**

<table>
<thead>
<tr>
<th>Sand Level Below Top</th>
<th>Modified Coulomb</th>
<th>Janssen</th>
<th>Coulomb</th>
<th>Rankine</th>
<th>Finite Element $t/R=8.33 \times 10^{-4}$</th>
<th>Finite Element $t/R=3.5 \times 10^{-4}$</th>
<th>Exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.24</td>
<td>-58%</td>
<td>0.32</td>
<td>-44%</td>
<td>0.24</td>
<td>-57%</td>
<td>0.33</td>
</tr>
<tr>
<td>12</td>
<td>0.93</td>
<td>-53%</td>
<td>1.16</td>
<td>-42%</td>
<td>0.98</td>
<td>-51%</td>
<td>1.17</td>
</tr>
<tr>
<td>18</td>
<td>2.10</td>
<td>-46%</td>
<td>2.40</td>
<td>-37%</td>
<td>2.25</td>
<td>-41%</td>
<td>2.67</td>
</tr>
<tr>
<td>24</td>
<td>3.60</td>
<td>-42%</td>
<td>4.10</td>
<td>-35%</td>
<td>4.10</td>
<td>-35%</td>
<td>4.79</td>
</tr>
</tbody>
</table>
effect of the nonlinear shear stiffness analysis is demonstrated by comparing the stresses obtained from the nonlinear analysis with the stresses obtained from linear shear stiffness analysis.

Figures (5.21) through (5.23) represent the stress distribution using constant shear stiffness. Whenever the shear stiffness exceeds \( \sigma_n \cdot \tan \beta \), slippage at the interface takes place and the shear stiffness is neglected.

It should be noticed that for each incremental loading the shear stiffness at the top element is neglected. Therefore the assumption of zero shear stiffness for each new incremental load is satisfied. For each value of \( h/D \), the area under the shear stress diagram and the area under the normal stress diagram is calculated and recorded with each corresponding figure.

For the same wall stiffness and \( h/D \) used in the above, constant shear stiffness analysis, a nonlinear shear stiffness analysis is carried out and the results are as shown in Figures (5.24) through (5.26). The areas under the shear stress diagrams and the normal stress diagrams have been also calculated and recorded for the corresponding figure.

Comparing the results obtained from the constant shear stiffness analysis with the corresponding results obtained from the nonlinear stiffness analysis, one will observe that:

a) The shear forces obtained from the nonlinear analysis are less than those obtained from the constant stiffness analysis by approximately 15%.

b) The normal forces obtained from the former method are greater than those obtained from the latter method by...
Figure (5.21) Normal and Vertical Pressures Dist. With Constant Shear Stiffness.

Figure (5.22) Normal and Vertical Pressures Dist. With Constant Shear Stiffness.
Figure (5.23) Normal and Vertical Pressures Dist. With Constant Shear Stiffness.
Figure (5.24) Normal and Vertical Pressure Dist. With Variable Shear Stiffness.

Figure (5.25) Normal and Vertical Pressures Dist. with Variable Shear Stiffness.
Figure (5.26) Normal and Vertical Pressures Dist. With Variable Shear Stiffness.
approximately 20%.

In conclusion, the nonlinear stiffness analysis updates the state of stress between the grains and the bin wall, therefore it should be used for the analysis of the shallow grain bins.

5.6 Comparison Between Plane Strain and Axisymmetric Analysis

The mathematical computation and the numerical integration involved in the axisymmetric shell analysis is lengthy and complicated.

In order to simplify the calculations, the bin with straight wall is replaced by beams on elastic supports and the composite grain-shell system is solved as a plane strain problem. In order to check the accuracy of the plane strain assumption, a comparison between the axisymmetric and plane strain stresses has been carried out.

In Figures (5.27) through (5.32), the axisymmetric lateral pressure in broken lines and the plane strain lateral pressure in solid lines for two t/R values are shown. In order to make a comparison between the axisymmetric and the plane strain analysis, the area under each pressure distribution have been calculated and the results tabulated for different t/R and h/D as shown in Table (5.3).

From the results given in Table (5.3), note that the plane strain analysis overestimates the lateral pressure by 10% to 15%. The percentage difference increases as the wall stiffness decreases. The height-to-diameter ratio has no effect on the difference between the two solutions. From
Figure (5.27) Axisymmetric and Plane Strain Pressure Distribution.

Figure (5.28) Axisymmetric and Plane Strain Pressure Distribution.
Figure (5.29) Axisymmetric and Plane Strain Pressure Distribution.
Figure (5.30) Axisymmetric and Plane Strain Pressure Distribution.

Figure (5.31) Axisymmetric and Plane Strain Pressure Distribution.
Figure (5.32) Axisymmetric and Plane Strain Pressure Distribution.

\[ t/R = 1.25 \times 10^{-3} \]
\[ h/2R = 0.92 \]

- Plane Strain
- Axisymmetric
## Table (5.3)

### Difference Between Axisymmetric and Plane Strain Solutions

<table>
<thead>
<tr>
<th>t/R x 10^{-3}</th>
<th>h/D</th>
<th>Plane Strain Force</th>
<th>Axisymmetric Force</th>
<th>% Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5</td>
<td>0.42</td>
<td>4.1</td>
<td>3.64</td>
<td>12.0</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>5.67</td>
<td>5.04</td>
<td>12.5</td>
</tr>
<tr>
<td></td>
<td>0.58</td>
<td>7.57</td>
<td>6.72</td>
<td>12.6</td>
</tr>
<tr>
<td></td>
<td>0.67</td>
<td>9.6</td>
<td>8.68</td>
<td>11.0</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>11.95</td>
<td>10.62</td>
<td>12.5</td>
</tr>
<tr>
<td></td>
<td>0.83</td>
<td>14.53</td>
<td>12.85</td>
<td>13.0</td>
</tr>
<tr>
<td></td>
<td>0.92</td>
<td>17.19</td>
<td>14.98</td>
<td>15.0</td>
</tr>
<tr>
<td>1.72</td>
<td>0.42</td>
<td>3.724</td>
<td>3.36</td>
<td>11.0</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>5.19</td>
<td>4.7</td>
<td>10.0</td>
</tr>
<tr>
<td></td>
<td>0.58</td>
<td>6.9</td>
<td>6.2</td>
<td>11.0</td>
</tr>
<tr>
<td></td>
<td>0.67</td>
<td>8.74</td>
<td>7.8</td>
<td>11.5</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>10.75</td>
<td>9.68</td>
<td>11.0</td>
</tr>
<tr>
<td></td>
<td>0.83</td>
<td>12.99</td>
<td>11.57</td>
<td>12.3</td>
</tr>
<tr>
<td></td>
<td>0.92</td>
<td>15.27</td>
<td>13.56</td>
<td>13.0</td>
</tr>
<tr>
<td>1.25</td>
<td>0.42</td>
<td>3.45</td>
<td>3.04</td>
<td>13.5</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>4.66</td>
<td>4.03</td>
<td>15.5</td>
</tr>
<tr>
<td></td>
<td>0.58</td>
<td>6.00</td>
<td>5.25</td>
<td>14.0</td>
</tr>
<tr>
<td></td>
<td>0.67</td>
<td>7.43</td>
<td>6.56</td>
<td>13.0</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>8.95</td>
<td>7.9</td>
<td>13.0</td>
</tr>
<tr>
<td></td>
<td>0.83</td>
<td>10.76</td>
<td>9.39</td>
<td>14.5</td>
</tr>
<tr>
<td></td>
<td>0.92</td>
<td>12.47</td>
<td>10.97</td>
<td>14.0</td>
</tr>
</tbody>
</table>
Figures (5.27) through (5.32) it is clear that the pressure
distribution due to the plane strain analysis deviates from
the pressure distribution from axisymmetric analysis at the
bottom of the bin when the wall stiffness is reduced.

In conclusion, the plane strain assumption is a
reasonable approximation for the analysis of shallow bins
with vertical straight wall, within a certain limit of the
wall stiffness.

5.7 Grain Bins with Double Curvature

Shallow bins made of cold formed steel may be subjected
to buckling or excessive deformation depending on the vertical
stiffness. In order to improve the load carrying capacity of
these bins, the wall can be curved towards the grain in order
to make use of the interaction characteristics between the
grain and the wall as shown in Figure (5.7.b). This inter-
action will enhance the buckling criteria of the wall in the
vertical direction, without allowing for large vertical de-
formation. Two bins of the proposed shape are analyzed in
order to illustrate their performance.

One bin has a wall stiffness t/R of $3.5 \times 10^{-3}$, a
vertical radius of curvature of 9.25 in. and a sector angle
of $38^\circ$.

The solid lines in Figures (5.33) through (5.35)
represent the normal pressure distribution along the proposed
bin wall, while the broken lines represent the pressure dis-
tribution along the corresponding bin with a vertical straight
wall of the same t/R. Clearly, the lateral pressure along
the straight wall bin is less than the maximum pressure and
Figure (5.33) Normal Pressure Distribution

Figure (5.34) Normal Pressure Distribution.
Figure (5.35) Normal Pressure Distribution
greater than the minimum pressure along the proposed bin.

The second bin has a wall stiffness of $3.5 \times 10^{-4}$, a vertical radius of curvature of 6 inches and a sector angle of $60^\circ$. Figures (5.36) through (5.38) show the normal pressure distribution along the curved wall and the corresponding straight wall. Note that the lateral pressure along the straight wall is almost tangent to the lower limits of the lateral pressure along the curved wall.

From the above discussion, it can be concluded that the vertical curvature of the proposed bin forces the bin wall to displace towards and away from the soil. When the wall moves towards the soil a state of passive pressure develops while a state of active pressure develops when the wall moves away from the soil.

The buckling stresses in the proposed bin is governed by the elastic supports developed by the grain and the wall stiffness. Fisher (25) and Weingarten (93) analyzed the cylindrical shells under combined axial load and internal pressure. According to their analysis, it is found that the relatively small internal pressure due to the grain has no effect on the increase of buckling stresses and therefore can be neglected. The buckling stresses of axisymmetric cylindrical shell of revolution under axisymmetric axial load was found to be the same as the buckling stresses of a beam on elastic springs (88) with a coefficient of spring reaction equal to $Et/R^2$. The buckling load, $P_{cr}$, for the case of many buckling waves along the length of the shell is:
Figure (5.36) Normal Pressure Distribution.

Figure (5.37) Normal Stress Distribution.
Figure (5.38) Normal Stress Distribution.
\[ P_{cr}^u = 2 \frac{E_t}{R^2} \cdot \frac{E_t^3}{12(1-u^2)} \]  

(5.2)

Meyerhof (65) developed a similar formula for the buckling load of conduits and arches embedded in the soil, this formula is given as:

\[ P_{cr}^s = 2 \cdot E' \cdot \frac{E_t}{1-u^2} \]  

(5.3)

where \( E' \) is the coefficient of sand reaction.

Since there is no rational stability analysis for the proposed bin at the present time, an approximate method can be developed for one curved segment only. A combination between the two methods described by Equations (5.2) and (5.3) can be used.

It is expected that the new shape of the grain bins will be capable of resisting higher shear forces compared to the shear forces resisted by the grain bins with vertical wall. In the same time the new bins will have vertical deformation less than the vertical deformation of the grain bins made of corrugated sheets with vertical walls.
CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

In this thesis, the finite element method is used to study the behaviour of shallow cylindrical grain bins and the theoretical results are verified experimentally. To increase the buckling strength and the vertical stiffness of the bin wall, a new configuration is proposed. This configuration is found to be reasonable and practical.

From the present study, the following conclusions are made:

1) The proposed method of analysis is capable of analyzing shallow grain bins for different wall configurations;

2) The relative stiffness between the bin wall and the stored material controls the value and distribution of the lateral and vertical pressures;

3) None of the conventional methods account for the relative stiffness between the grain and the bin wall. The pressures obtained from the conventional methods differ among themselves from 13% to 46% which indicate inaccurate prediction for the lateral pressures;

4) The hydrostatic pressure distribution is valid only in case of isotropic steel bin if the wall stiffness t/R is greater than $2.6 \times 10^{-3}$. If the wall stiffness is reduced the pressure is always released from the top part of the bin. This can be attributed to the movement of the wall compared to the size of the grain particles;

5) The coefficient of lateral pressure is a function of the height of filling and not constant.
6) The use of interface element to represent the state of contact between the grain and the bin is extremely helpful. The nonlinear shear stiffness analysis is powerful and updating the state of contact at the common interface;

7) Shallow grain bins can be analyzed using the plane strain analysis especially for relatively high wall stiffness;

8) The interaction between the proposed bin with vertical curvature develops an active and passive pressure along the bin height. This interaction increases the buckling strength which can decrease the number of buckling failures;

9) It is expected that this proposed shape will reduce considerably the vertical displacements of the bins made of cold formed steel.

Recommendations for Further Work

1. Triaxial tests and shear box tests should be conducted on actual grains in order to define their properties. The results of such experiments can then be fitted to the computer program to study the behaviour of actual grain.

2. Different bins with different wall configurations such as conical bins and half barrel grain storage can be analyzed using the present procedure.

3. A rational buckling analysis of the proposed bins should be established to justify the practical use of this new shape.

4. The effect of both the centric and eccentric discharge on the proposed bins should be studied.

5. The effect of ground motion due to an earthquake or the impact of running train on the pressure distribution
should be studied.
BIBLIOGRAPHY


69. Palmerton, J.B., "Application of Three-Dimensional Finite Element Analysis," Report to U.S. Army Engineer Waterways Experimental Station, Vicksburg, Miss.


Matrices, Integrals and Derivations of Matrix Equations for Grain and Beam Elements

Some of the important matrices and all the integrals used in the static analysis of axisymmetric and plain strain elements under symmetric loading are given in this appendix.

A.1 Matrices Relating Stresses, Strains and Displacement

Because of the axisymmetric body and the symmetry of the loads about the vertical axis, displacements will be developed only in the radial and vertical directions. Where the matrices for the axisymmetric element are the same as for the plane elements, the \( r \) and \( z \) coordinates and the displacements \( u \) and \( w \), replaces the \( x \) and \( y \) coordinates and displacements \( u \) and \( v \), respectively. Equations (3.23) and (3.24) are written for axisymmetric and plane strain as:

\[
\{u\} = [\phi]\{a\} \tag{A.1}
\]

where,

\[
\{u\} = \begin{bmatrix} u \\ v \end{bmatrix}
\]

\[
[\phi] = \begin{bmatrix}
1 & 0 & x & y & 0 & 0 \\
0 & 0 & 0 & 1 & x & y \\
\end{bmatrix}
\] \tag{A.3}

\[
\{a\}^T = \{ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \} \tag{A.4}
\]
The displacement of the element nodes \((i, j, k)\) are found from Equation (3.5):

\[
\{q\} = [A]\{\alpha\} \quad \text{(A.5)}
\]

where,

\[
\{q\}^T = \{u_i \quad u_j \quad u_k \quad v_i \quad v_j \quad v_k\} \quad \text{(A.6)}
\]

\[
[A] = \begin{bmatrix}
1 & x_i & y_i & 0 & 0 & 0 \\
1 & x_j & y_j & 0 & 0 & 0 \\
1 & x_k & y_k & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & x_i & y_i \\
0 & 0 & 0 & 1 & x_j & y_j \\
0 & 0 & 0 & 1 & x_k & y_k \\
\end{bmatrix} \quad \text{(A.7)}
\]

\((x_i, y_i), (x_j, y_j), (x_k, y_k)\) are the nodal coordinates for the plain strain and axisymmetric elements in the anticlockwise direction. Solve Equation (A.5) for \(\{\alpha\}\), yields

\[
\{\alpha\} = [A]^{-1}\{q\} \quad \text{(A.8)}
\]

where,
\[
\begin{bmatrix}
    x_{j} - x_{k} & x_{k} - x_{j} & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & x_{j} - x_{i} & x_{i} - x_{j} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & x_{j} - x_{i} & x_{i} - x_{j} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & x_{j} - x_{i} & x_{i} - x_{j} & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & x_{j} - x_{i} & x_{i} - x_{j} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{j} - x_{i} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{j} - x_{i} \\
\end{bmatrix}
\]

\[
\left[ A \right]^{-1} = \frac{1}{\| A \|} \begin{bmatrix}
    x_{j} - x_{k} & x_{k} - x_{j} & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & x_{j} - x_{i} & x_{i} - x_{j} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & x_{j} - x_{i} & x_{i} - x_{j} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & x_{j} - x_{i} & x_{i} - x_{j} & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & x_{j} - x_{i} & x_{i} - x_{j} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{j} - x_{i} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{j} - x_{i} \\
\end{bmatrix}
\]
\[ |A| = x_i (y_j - y_k) + x_j (y_k - y_i) + x_k (y_i - y_j) \]  \hspace{1cm} (A.10)

The strains at a point are given by Equation (3.4).

For axisymmetric case, Equations (3.25) can be written as:

\[ \{\varepsilon\} = [B]\{\alpha\} \]  \hspace{1cm} (A.11)

where,

\[ \{\varepsilon\}^T = \{\varepsilon_r \quad \varepsilon_z \quad \varepsilon_\theta \quad \gamma_{rz}\} \]  \hspace{1cm} (A.12)

\[ [B] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{R} & 1 & \frac{N}{R} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]  \hspace{1cm} (A.13)

For plane strain case Equation (3.27) can be written as:

\[ \{\varepsilon\} = [B]\{\alpha\} \]  \hspace{1cm} (A.14)

where,

\[ \{\varepsilon\}^T = \{\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}\} \]  \hspace{1cm} (A.15)

\[ [B] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \]  \hspace{1cm} (A.16)

The stress-strain relations given by Equation (3.26) for axisymmetric element can be represented in the form:

\[ \{\sigma\} = [C]\{\varepsilon\} \]  \hspace{1cm} (A.17)
where,

\[
\{\sigma\}^T = \{ \sigma_x \sigma_y \tau_{xy} \} \tag{A.18}
\]

\[
[C] = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix}
\frac{1-\nu}{\nu} & 1 & 1 & 0 \\
1 & \frac{1-\nu}{\nu} & 1 & 0 \\
1 & 1 & \frac{1-\nu}{\nu} & 0 \\
0 & 0 & 0 & \frac{1-2\nu}{2\nu}
\end{bmatrix}
\tag{A.19}
\]

The stress-strain relations given by Equations (3.28) for plane strain elements can be represented in the form:

\[
\{\sigma\} = [C]\{\varepsilon\} \tag{A.20}
\]

where,

\[
\{\sigma\}^T = \{ \sigma_x \sigma_y \tau_{xy} \} \tag{A.21}
\]

\[
[C] = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & 0 \\
\nu & 1-\nu & 0 \\
0 & 0 & \frac{1-2\nu}{2\nu}
\end{bmatrix}
\tag{A.22}
\]

The matrix \([B]^T[C][B]\) which has been used in Equation (3.6) is given for the axisymmetric element as:
\[
[B]^T[C][B] = \frac{E \nu}{(1+\nu)(1-2\nu)} \begin{bmatrix}
\frac{m_1}{r^2} & \frac{1}{\nu r} & \frac{m_1 z}{r^2} & 0 & 0 & \frac{1}{r^2} \\
\frac{1}{\nu r} & \frac{2}{\nu} & \frac{z}{\nu r} & 0 & 0 & 2 \\
\frac{m_1 z}{r^2} & \frac{z}{\nu r} & \left(\frac{m_1 z}{r^2}+m_2\right) & 0 & m_2 & \frac{z}{r^2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & m_2 & 0 & m_2 & 0 \\
\frac{1}{r} & 2 \frac{z}{r} & 0 & 0 & m_1 & \\
\end{bmatrix}
\]

(A.23)

where,

\[
m_1 = \frac{1-\nu}{\nu} \\
m_2 = \frac{1-2\nu}{2\nu}
\]

while for the plane strain the matrix \([B]^T[C][B]\) is given as:

\[
[B]^T[C][B] = \frac{E \nu}{(1+\nu)(1-2\nu)} \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-\nu & 0 & 0 & 0 & \nu \\
0 & 0 & \frac{1-2\nu}{2\nu} & 0 & \frac{1-2\nu}{2\nu} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1-2\nu}{2\nu} & 0 & \frac{1-2\nu}{2\nu} & 0 \\
0 & \nu & 0 & 0 & 0 & (1-\nu) \\
\end{bmatrix}
\]

(A.24)
A.2 Integration of the Element Stiffness Matrix

Using Equation (3.7), the element stiffness matrix for the axisymmetric element is given as:

\[
[k_e] = 2\pi [A^{-1}]^T \int \{[B]^T[C][B] \cdot r dr dz \} \cdot [A^{-1}] \quad (A.25)
\]

This integration includes six lengthy integrals. These integrals are simplified by using Green's Lemma, and it is given as:

\[
v = \int r dr dz = -\int rz dr
\]

\[
= -\left(\frac{a_1}{2}(r_j^2 - r_i^2) + \frac{b_1}{3}(r_j^3 - r_i^3) + \frac{a_2}{2}(r_k^2 - r_j^2)
\right.
\]

\[
+ \frac{b_2}{3}(r_k^3 - r_j^3) + \frac{a_3}{2}(r_i^2 - r_k^2) + \frac{b_3}{3}(r_i^3 - r_k^3)
\]

\[
(A.26)
\]

\[
\lambda_1 = \int r dr dz = -\int z dr
\]

\[
= -\left(a_1(r_j - r_i) + \frac{b_1}{2}(r_j^2 - r_i^2) + a_2(r_k - r_j)
\right)
\]

\[
+ \frac{b_2}{2}(r_k^2 - r_j^2) + a_3(r_i^2 - r_k) + \frac{b_3}{2}(r_i^3 - r_k^3)
\]

\[
(A.27)
\]

\[
\lambda_2 = \int \frac{1}{r} dr dz = -\int \frac{z}{r} dr
\]

\[
= -\left[a_1 \ln \frac{r_j}{r_i} + b_1 (r_j - r_i) + a_2 \ln \left(\frac{r_k}{r_j}\right)
\right.
\]

\[
+ b_2 (r_k - r_j) + a_3 \ln \left(\frac{r_i}{r_k}\right) + b_3 (r_i - r_k)
\]

\[
(A.28)
\]
\[ \lambda_3 = \iint z \text{d}rdz = -\iint \frac{z^2}{2} \text{d}r \]

\[ = -\left[ \frac{a_1^2}{2} (r_j - r_i) + \frac{a_1b_1}{2} (r_j^2 - r_i^2) + \frac{b_1^2}{6} (r_j^3 - r_i^3) + \frac{a_2}{2} (r_k - r_j) + \frac{a_2b_2}{2} (r_k^2 - r_j^2) + \frac{b_2^2}{6} (r_k^3 - r_j^3) + \frac{a_3}{2} (r_k^2 - r_i^2) + \frac{a_3b_3}{2} (r_k^3 - r_i^3) \right] \quad (A.29) \]

\[ \lambda_4 = \iint \frac{z}{r} \text{d}rdz = -\iiint \frac{z^2}{2x} \text{d}r \quad \lambda_4' \]

\[ = -\left[ \frac{a_1^2}{2} \ln \left( \frac{r_j}{r_i} \right) + \frac{a_1b_1}{4} (r_j^2 - r_i^2) + \frac{b_1^2}{4} (r_j^3 - r_i^3) + \frac{a_2^2}{2} \ln \left( \frac{r_k}{r_j} \right) + \frac{a_2b_2}{4} (r_k^2 - r_j^2) + \frac{b_2^2}{4} (r_k^3 - r_j^3) \right] \quad (A.30) \]

\[ \lambda_5 = \iint \frac{z^2}{x} \text{d}rdz = -\iint \frac{z^3}{3x} \text{d}r \]

\[ = -\left[ \frac{a_1^3}{3} \ln \left( \frac{r_j}{r_i} \right) + \frac{a_1b_1^2}{2} (r_j^2 - r_i^2) + \frac{a_1b_1}{3} (r_j^3 - r_i^3) + \frac{b_1^3}{9} (r_j^3 - r_i^3) + \frac{a_2^3}{3} \ln \left( \frac{r_k}{r_j} \right) + \frac{a_2b_2^2}{2} (r_k^2 - r_j^2) + \frac{a_2b_2}{3} (r_k^3 - r_j^3) + \frac{b_2^3}{9} (r_k^3 - r_j^3) + \frac{a_3^3}{3} \ln \left( \frac{r_i}{r_k} \right) + \frac{a_3b_3^2}{2} (r_i^2 - r_k^2) + \frac{a_3b_3}{3} (r_i^3 - r_k^3) + \frac{b_3^3}{9} (r_i^3 - r_k^3) \right] \quad (A.31) \]

where,

\[ a_1 = z_i - b_1^j r_i \quad (A.32.a) \]

\[ a_2 = z_j - b_2^j r_j \quad (A.32.b) \]
\[ a_3 = z_k b_3 r_k \]  \hspace{1cm} (A.32.c)

\[ b_1 = \frac{z_j - z_i}{r_j - r_i} \]  \hspace{1cm} (A.32.d)

\[ b_2 = \frac{z_k - z_j}{r_k - r_j} \]  \hspace{1cm} (A.32.e)

\[ b_3 = \frac{z_i - z_k}{r_i - r_k} \]  \hspace{1cm} (A.32.f)

When any radius is zero, the corresponding logarithm becomes infinite; however, the limit of such terms as determined using L'Hopital's rule is zero. Whenever nodal radial coordinate is zero, the corresponding logarithmic term should be deleted. Whenever two radial coordinates of an element are equal, apparently a and b constants become infinite. In these cases, the line integration is not required and the constants should be set equal to zero.

Instead of using these lengthy integrals, a simplified expression is valid if the cross section of the element is small compared to its radius of revolution. These simplified expressions are given as:

\[ v = A r \]  \hspace{1cm} (A.33.a)

\[ \lambda_1 = A \]  \hspace{1cm} (A.33.b)

\[ \lambda_2 = \frac{A}{r} \]  \hspace{1cm} (A.33.c)

\[ \lambda_3 = \frac{A z}{r} \]  \hspace{1cm} (A.33.d)

\[ \lambda_4 = \frac{A z^2}{r^2} \]  \hspace{1cm} (A.33.e)
and,
\[
\lambda_5 = \frac{A}{12r} \left[ (z_i + z_j)^2 + (z_j + z_k)^2 + (z_k + z_i)^2 \right] \quad (A.33.f)
\]
where,
\[
2A = r_i (z_j - z_k) + r_j (z_k - z_i) + r_k (z_i - z_j) \quad (A.34.a)
\]
\[
\bar{r} = \frac{1}{3} (r_i + r_j + r_k) \quad (A.34.b)
\]
\[
\bar{z} = \frac{1}{3} (z_i + z_j + z_k) \quad (A.34.c)
\]

The integration of Equation (3.7) per unit thickness of the plane strain element is simply given by:
\[
[k_e] = (\text{area})_e [A^{-1}]^T [B]^T [C][B][A^{-1}] \quad (A.35)
\]
The matrices $[A^{-1}]$ and $[B]^T [C][B]$ are given early in this section.

A.3 Integration of Gravity Load Vector

Apply the variational principles to Equation (3.6), the load vector for the axisymmetric case is:
\[
\begin{bmatrix}
F_{ri} \\
F_{rij} \\
F_{rk} \\
F_{zi} \\
F_{zj} \\
F_{zk}
\end{bmatrix} = -\gamma [A^{-1}]^T \begin{bmatrix}
0 \\
0 \\
0 \\
v \\
\lambda_6 \\
\lambda_7
\end{bmatrix} \quad (A.36)
\]
where,

\[ \lambda_6 = \iint r^2 \, dr \, dz = -\int r^2 z \, dr \]

\[ = -\left[ \frac{a_1}{3} (r_j^3 - r_i^3) + \frac{b_1}{4} (r_j^4 - r_i^4) + \frac{a_2}{3} (r_k^3 - r_j^3) \right. \]

\[ + \left. \frac{b_2}{4} (r_k^4 - r_j^4) + \frac{a_3}{3} (r_i^3 - r_k^3) + \frac{b_3}{4} (r_i^4 - r_k^4) \right] \]

(A.37)

\[ \lambda_7 = \iint z r^2 \, dr \, dz = -\int \frac{r z^2}{2} \, dr \]

\[ = -\left[ \frac{a_1}{4} (r_j^2 - r_i^2) + \frac{a_1 b_1}{3} (r_j^3 - r_i^3) + \frac{b_1^2}{8} (r_j^4 - r_i^4) \right. \]

\[ + \left. \frac{a_2}{4} (r_k^2 - r_j^2) + \frac{a_2 b_2}{3} (r_k^3 - r_j^3) + \frac{b_2^2}{8} (r_k^4 - r_j^4) \right. \]

\[ + \left. \frac{a_3}{4} (r_i^2 - r_k^2) + \frac{a_3 b_3}{3} (r_i^3 - r_k^3) + \frac{b_3^2}{8} (r_i^4 - r_k^4) \right] \]

(A.38)

The simplified form of \( \lambda_6 \) and \( \lambda_7 \) are:

\[ \lambda_6 = \frac{A}{12} [(r_i + r_j)^3 + (r_j + r_k)^3 + (r_k + r_i)^3] \]

(A.38)

\[ \lambda_7 = AZ \]

(A.39)

A.4 Beam Element Stiffness Matrix

The element stiffness matrix of the beam element in its local coordinates is given by:
\[
[k_3] = E_e \begin{bmatrix}
\frac{A}{L} & 0 & \frac{12I}{L^3} \\
0 & \frac{A}{L} & 0 \\
0 & 0 & \frac{-12I}{L^3} \frac{12I}{L^3} \\
0 & 0 & \frac{6I}{L^2} \frac{6I}{L^2} \frac{4I}{L} \\
0 & 0 & \frac{6I}{L^2} \frac{6I}{L^2} \frac{2I}{L} \frac{4I}{L}
\end{bmatrix}
\]

\begin{equation}
(A.40)
\end{equation}

where,

\[
E_e = \frac{E}{1-\nu^2}
\]

And A and I are the cross-sectional area and the moment of inertia, respectively. The transformation matrix may be written as:

\[
[T] = \begin{bmatrix}
cosa & sina & 0 & 0 & 0 & 0 \\
0 & 0 & cosa & sina & 0 \\
-sina & cosa & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -sina & cosa & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\begin{equation}
(A.41)
\end{equation}
Matrices, Integrals and Derivation of Matrix Equations
For Shells of Revolution

The relations given by Equations (3.58) and (3.59), together with the corresponding rotation is represented in matrix form as:

\[ \{u\} = [\phi]\{\alpha\} \]  \hspace{1cm} (B.1)

where,

\[ \{u\}^T = \begin{bmatrix} u_1 & u_2 & \chi \end{bmatrix} \]  \hspace{1cm} (B.2)

\[ \{\alpha\}^T = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \end{bmatrix} \]  \hspace{1cm} (B.3)

\[ [\phi] = \begin{bmatrix}
1 & \xi & \xi^2 & \xi^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \xi & \xi^2 & \xi^3 \\
0 & -\eta'\rho & -2\eta'\xi\rho & -3\eta'\xi^2\rho & 0 & \rho & 2\rho\xi & 3\rho\xi^2 \\
\end{bmatrix} \]  \hspace{1cm} (B.4)

Substituting the coordinates of the element nodal points into Equation (B.1), one gets the nodal displacements \( \{q\} \) in terms of generalized coordinates as:

\[ \{q\} = [A]\{\alpha\} \]  \hspace{1cm} (B.5)
The strain-displacement relation given by Equations (3.62) is written as:

\[ \{ \varepsilon \}_r = [B]\{\alpha\} \tag{B.6} \]

where,

\[ \{ \varepsilon \}_r^T = \{ \varepsilon_s \ \varepsilon_\theta \ \kappa_s \ \kappa_\theta \} \tag{B.7} \]

The stress-strain relationship can be written in matrix form as:

\[ \{ \sigma \} = [C]\{\varepsilon\} \tag{B.8} \]

where,

\[ \{ \sigma \}_r^T = \{ N_s \ N_\theta \ M_s \ M_\theta \} \tag{B.9} \]

The \([C]\) matrix for isotropic elastic shell of revolution is:

\[
[C] = \frac{E_t}{1-v^2} \begin{bmatrix}
1 & v & 0 & 0 \\
v & 1 & 0 & 0 \\
0 & 0 & \frac{t^2}{12} & \frac{vt^2}{12} \\
0 & 0 & \frac{vt^2}{12} & t^2/12
\end{bmatrix}
\]

(B.10)

Matrices \([A]\) and \([B]\) are as follows:
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\cos^2 \beta & \cos \beta & 0 & 0 & 0 & 0 & 0 \\
\cos \beta & \cos^2 \beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos^2 \beta & \cos \beta & 0 & 0 & 0 \\
0 & 0 & \cos \beta & \cos^2 \beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{3 \cos^2 \beta}{L} & \frac{2 \cos^2 \beta}{L} & \frac{3 \cos^2 \beta}{L} & \frac{2 \cos^2 \beta}{L} & \frac{3 \cos^2 \beta}{L} & \frac{2 \cos^2 \beta}{L} & \frac{3 \cos^2 \beta}{L} \\
\frac{1}{27} & \frac{2}{27} & \frac{8}{27} & \frac{1}{3} & \frac{1}{9} & \frac{4}{9} & \frac{2}{3} \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{1}{3} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sin \beta \cos \beta & \sin \beta \cos \beta & \sin \beta \cos \beta & \sin \beta \cos \beta & \sin \beta \cos \beta & \sin \beta \cos \beta & \sin \beta \cos \beta \\
\end{bmatrix}
\]
\[
[B] = \begin{bmatrix}
0 & \rho & 2\xi\rho & 3\xi^2\rho & 0 & \eta'\rho & 2\xi\eta'\rho & 3\xi^2\eta'\rho \\
\frac{\sin\psi}{r} & \frac{\xi}{r} & \frac{\xi^2}{r} & \frac{\sin\psi}{r} & \frac{\cos\psi}{r} & \frac{\xi}{r} & \frac{\xi^2}{r} & \frac{\cos\psi}{r} \\
0 & (1-\eta'^2)\phi & 2\xi(1-\eta'^2)\phi+\Omega & 3\xi\left[\xi(1-\eta'^2)\phi+\Omega\right] & 0 & 2\eta'\phi & 4\xi\eta'\phi-\mu & 3\xi(2\xi\eta'\phi-\mu) \\
0 & \eta'\psi & 2\xi\eta'\psi & 3\xi^2\eta'\psi & 0 & -\psi & -2\xi\psi & -3\xi^2\psi
\end{bmatrix}
\]
Substituting Equations (B.10), (B.11) and (B.12) into Equation (3.7), the element stiffness matrix is written as:

$$[k_e] = [A^{-1}]^T \int [B]^T [C] [B] d\nu [A^{-1}]$$  \hspace{1cm} \text{(B.13)}

Equation (B.13) is integrated with respect to the normalized coordinates $\xi$, as:

$$[k_e] = 2\pi [A^{-1}]^T \int_{0}^{1} [B]^T [C] [B] \cdot r \sqrt{1 + \eta} \, d\xi [A^{-1}]$$  \hspace{1cm} \text{(B.14)}

The radius of any circle $r$ is given as:

$$r = r^i + \xi \cdot L \cdot [\sin\psi + \eta \cdot \cos\psi]$$  \hspace{1cm} \text{(B.15)}

where $r^i$ is the radius of the nodal circle, i.e. Direct integration in its closed form for Equation (B.14) is very difficult. The numerical integration technique using Gauss-Legendre Quadratur open formula was adopted in the computer program.

The sign convention for the displacement and rotations and for loads and moments is given in Figure (B.1), while the sign conversion for the output stress-resultants is given in Figure (B.2).
Figure (B.1) Sign Convention for Nodal Displacements and Loads.

Figure (B.2) Sign Convention for Output Stress Resultants.
APPENDIX (C)

Tabulated Experimental Results
## Table 1

<table>
<thead>
<tr>
<th>Axial Strain $e_a \times 10^{-3}$</th>
<th>Area in $^2$</th>
<th>Deviatoric Stress $(\sigma_1 - \sigma_3)$ lb/in$^2$</th>
<th>Volumetric Strain $\Delta V / V \times 10^{-3}$</th>
<th>Corrected Stress $(\sigma_1 - \sigma_3)_c$ lb/in$^2$</th>
<th>Corrected Volumetric Strain $x10^{-3}$</th>
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</thead>
<tbody>
<tr>
<td>2.0</td>
<td>12.46</td>
<td>2.41</td>
<td>- .08</td>
<td>2.15</td>
<td>- .18</td>
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<td>4.81</td>
<td>- .17</td>
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<td>- .27</td>
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<td>- .17</td>
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Confining Pressure = 4 psi  
Original Height = 7.5 in  
Original Diameter = 3.98 in  
Test No. = 1
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<tr>
<th>Axial Strain ( \epsilon_a \times 10^{-3} )</th>
<th>Area in² ( A = \frac{A_0}{1 - \epsilon} )</th>
<th>Deviatoric Stress ( (\sigma_1 - \sigma_3) \text{ lb/in}^2 )</th>
<th>Volumetric Strain ( \frac{\Delta V}{V} \times 10^{-3} )</th>
<th>Corrected Stress ( (\sigma_1 - \sigma_3) \text{ lb/in}^2 )</th>
<th>Corrected Volumetric Strain ( \times 10^{-3} )</th>
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Confining Pressure = 4 psi  
Original Height = 7.687 inches  
Original Diameter = 3.99 inches  
Test No. = 2
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<th>Axial Strain $\varepsilon_a \times 10^{-3}$</th>
<th>Area in $^2$</th>
<th>Deviatoric Stress $(\sigma_1 - \sigma_3)$ lb/in$^2$</th>
<th>Volumetric Strain $\frac{\Delta V}{V} \times 10^{-3}$</th>
<th>Corrected Stress $(\sigma_1 - \sigma_3)_a$ lb/in$^2$</th>
<th>Corrected Volumetric Strain $\times 10^{-3}$</th>
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Confining Pressure = 12 psi; Original Height = 7.875 in.; Original Diameter = 4.047 in.; Test No. 2
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<th>Volumetric Strain, $\frac{\Delta V}{V} \times 10^{-3}$</th>
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Confining Pressure = 12 psi; Original Height = 7.875 in.; Original Diameter = 4.054 in.; Test No. 3
Table 10

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Table 14

Predicted Shear Deformation Versus Shear Stress

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Calibration Reading for Pressure Cell No. 1

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Table 16

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Calibration Readings for Pressure Cell No. 3

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Experimental Pressure Results

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Load-Cell No. 1

P = 6.335(\epsilon - 38.27) \times 10^{-4}

Figure (C.1) Calibration of Load Cell No. 1.
Load–Cell No. 2

\[ P = 7.427(e - 19.09) \times 10^{-4} \]

Figure (C.2) Calibration of Load Cell No. 2.
1947  Born, September 21, in Cairo, Egypt.

1966  Entered the Faculty of Engineering, University of Cairo, Giza, Egypt.

1971  Graduated with a Bachelor of Science Degree (Honours) in Civil Engineering (Structural Division). Appointed as Teaching Assistant in the Department of Mathematics, Faculty of Engineering, University of Cairo.

1971  Entered the Faculty of Science, Ain Shams University, Cairo, Egypt.

1974  Graduated with a Bachelor of Science Degree in Applied Mathematics.

1974  Entered the Faculty of Engineering (Civil Engineering), McMaster University, Hamilton, Ontario, Canada.

1976  Graduated with a Master of Science Degree in Engineering (Structural) from McMaster University, Hamilton, Ontario, Canada.

1976  Entered the Faculty of Engineering (Civil Engineering), University of Windsor, Windsor, Ontario, Canada.

Awarded a Teaching and Research Assistantship for Graduate study at the University of Windsor.

Also Awarded the University of Windsor Scholarship.