Solutions of non-Newtonian flow problems.

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University of Windsor

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SOLUTIONS OF NON-NEWTONIAN FLOW PROBLEMS

by

Edima Okon Oku-Ukpong

A Dissertation
submitted to the Faculty of Graduate Studies and Research
through the Department of Mathematics and Statistics
in partial fulfillment of the requirements for the
degree of Doctor of Philosophy at the
University of Windsor

Windsor, Ontario, Canada
1994
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ABSTRACT

In this dissertation, we have investigated analytically various flow problems of subclasses of viscoelastic (non-Newtonian) fluids of differential type of complexity 2 and 3. Both steady and unsteady fluid flows are considered, and in some cases, the fluids are electrically conducting.

Exact solutions of the equations of the steady, plane, isochoric motion of a second grade fluid and an electrically conducting second grade fluid are obtained employing the von Mises transformations. In the latter case, we explore the possibility of obtaining solutions for fluids of infinite and finite electrical conductivities. We investigate if these fluids can flow along a given family of curves. If this answer is determined in the affirmative, we proceed to obtain the exact integral of the flow along the given family of curves.

Next, we obtain inverse solutions of the equations of the steady, plane flow of an incompressible second grade fluid by assuming a certain form of the stream function in one case, and in another by assuming certain forms of the vorticity function. The exact solutions of these flows are determined.

An attempt is made to investigate the possibility of an incompressible, electrically conducting third grade fluid of infinite and finite magnetic Reynolds numbers admitting a von Kármán-type solution for the equations of a steady, plane flow between two parallel plates, one of which is porous. The lower plate is being stretched by two equal and opposite forces so that the origin is fixed. We find that electrically conducting third grade fluid flow under the aforementioned circumstances is impossible. Approximate solutions, employing the perturbation method, are obtained.
for the flow of an electrically conducting second grade fluid of infinite magnetic
Reynolds number.

Finally, exact solutions are obtained for the equations of an unsteady, plane,
Isochoric motion of an electrically conducting second grade fluid. Inverse method is
employed for which the vorticity distribution is proportional to the stream function
perturbed by a uniform stream. The cases when the electrically conducting fluid
has finite and infinite electrical conductivities are studied.
Dedicated to my Father, posthumously, and to my beloved Mother
ACKNOWLEDGEMENTS

In light of an accomplishment, for the most part, not being an exclusive preserve of an individual, I would like to extend my sincere gratitude to everyone who, either directly or indirectly, contributed to the attainment of this goal.

I am most particularly indebted to my Supervisor, Dr. O. P. Chandna, for his patience and professional guidance which led to the completion of this work. He has often gone beyond the call of supervisory duties, for he has been like a father to me. I lack adequate and appropriate words to express my heartfelt gratitude to him for his friendship.

Worthy of special mention are Drs. R. M. Barron, K. L. Duggal and P. N. Kaloni who have always provided me with support and encouragement, morally and academically. I am deeply appreciative of their kind gestures.

Special thanks are extended to Dr. L. K. Smedick, Dean of the Faculty of Graduate Studies and Research, and Dr. R. J. Caron, Head of the Department of Mathematics and Statistics, for their invaluable help over the years.

I also wish to thank the Secretaries in the Department of Mathematics and Statistics, Mrs. E. M. Bunt, Ms. S. Stephen and Ms. R. Gignac, for their assistance within the department. My gratitude also goes to the staff of the Faculty of Graduate Studies and Research, Mrs. L. Authier, Ms. L. Dajas, Ms. A. Samson, Mrs. V. Smith and Mrs. V. Wilkinson, for their help on issues that required the attention of a faculty staff member.

The assistance rendered by Dr. T. Traynor, Dr. F. Labropulu, Mr. I. Husain and Mr. S. Venkatasubramanian is gratefully acknowledged.
I would like to also express my appreciation to Drs. G. W. Rankin and P. J. Sullivan for accepting to serve, respectively, on the examining committee and as the external examiner.

My mother, Mrs. A. Ukpong, and siblings, Mrs. A. Nkereuwem, Mrs. A. Nduonofit, Mr. O. Oku-Ukpong, Mrs. J. Anwanakak, Mr. N. Oku-Ukpong, Mrs. E. Onumara, Ms. G. Oku-Ukpong, Mr. U. Oku-Ukpong and Mr. B. Oku-Ukpong deserve special mention. The ties of consanguinity that unite us have served as support of leviathan proportion throughout the years of separation from them.

These acknowledgements will be incomplete without the mention of good friends like Ms. S. Cotterell, Mr. J. Okafor, Dr. G. Ogbonna, Mr. O. Folami, Mr. E. Frank, Dr. F. Ikem, Mr. R. Arop, Mr. S. Otum, Ms. J. Quick, Ms. V. Woodward, Mr. F. Zaldana, Mr. W. Bunde, Ms. E. Tribune, Mr. G. Lazurck and Dr. W. Banfield whose varied forms of support have been sources of inspiration to me.

Most importantly, I would like to thank my Heavenly Father for His guidance, strength and love through the years.
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Figure 3.7: Streamline pattern for \( x(y - m_1 x - m_2) = \) constant in unbounded domain.

Figure 3.8: Streamline pattern for \( x(y - m_1 x - m_2) = \) constant for boundary value problem.

Figure 3.9: Streamline pattern for \( y(1 + m_1 c^{m_2}) - m_2 c^{m_2} = \) constant in unbounded domain.

Figure 3.10: Streamline pattern for \( y(1 + m_1 c^{m_2}) - m_2 c^{m_2} = \) constant for boundary value problem.

Figure 3.11: Streamline pattern for \( \theta - Ax^4 - \frac{B}{r} \) ln \( r = \) constant in unbounded domain.
Figure 3.12: Streamlines for $\theta - Blnr = \text{constant}$
in unbounded domain.

Figure 3.13: Streamlines for $\theta - Ar^2 - Blnr = \text{constant}$
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Figure 4.1: Streamline pattern for $y - z^3 = \text{constant}$
in unbounded domain.

Figure 4.2: Streamline pattern for $y - z^3 = \text{constant}$
for boundary value problem.

Figure 4.3: Streamline pattern for $x^2y = \text{constant}$
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Figure 4.4: Streamline pattern for $x^2y = \text{constant}$
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Figure 5.1: Streamline pattern for $y + Ae^{(mz+ky)} = \text{constant}$
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Figure 6.1: Streamline pattern for $Ax^2 + By^2 + Czy + Dx + Ey + 2(A + B)$
= constant in unbounded domain.

Figure 6.2: Streamline pattern for $By^2 + Czy + Dx + Ey + 2B = \text{constant}$
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Figure 6.3: Streamline pattern for $By^2 + Czy + Dx + Ey + 2B = \text{constant}$
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Figure 6.4: Streamline pattern for $By^2 + Dx + Ey + 2B = \text{constant}$
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Figure 6.6: Streamline pattern for $Kc \tau \beta y - (By^2 + Dx + Ey + 2B) = \text{constant}$ in unbounded domain.

Figure 6.7: Streamline pattern for $Kc \tau \beta y - (By^2 + Dx + Ey + 2B) = \text{constant}$ for boundary value problem.

Figure 6.8: Streamline pattern for $A_1 \exp \left[ \lambda_1 Dx + (\lambda_1 E + \frac{1}{\beta B}) y \right] + A_2 \exp \left[ \lambda_2 Dx + (\lambda_2 E + \frac{1}{\beta B}) y \right] - (Dx + Ey) = \text{constant}$ with a stagnation point in unbounded domain.

Figure 6.9: Streamline pattern for $A_1 \exp \left[ \lambda_1 Dx + (\lambda_1 E + \frac{1}{\beta B}) y \right] + A_2 \exp \left[ \lambda_2 Dx + (\lambda_2 E + \frac{1}{\beta B}) y \right] - (Dx + Ey) = \text{constant}$ without a stagnation point in unbounded domain.

Figure 6.10: Streamline pattern for $[B_1 + B_2(Dx + Ey)] \exp \left[ F(Dy - Ez) \right] -(Dx + Ey) = \text{constant}$ with a stagnation point in unbounded domain.

Figure 6.11: Streamline pattern for $[B_1 + B_2(Dx + Ey)] \exp \left[ F(Dy - Ez) \right] -(Dx + Ey) = \text{constant}$ without a stagnation point in unbounded domain.

Figure 6.12: Streamline pattern for $C_1 \cos \left[ m(Dx + Ey) + C_2 \right] \exp \left[ \frac{Dy - Ez}{f(D^2 + E^2)} \right] -(Dx + Ey) = \text{constant}$ in unbounded domain.

Figure 7.1: Problem I and Problem II in dimensional variables.

Figure 7.2: Problem I in dimensionless variables.

Figure 7.3: Problem II in dimensionless variables.
NOMENCLATURE

In dimensional analysis, the fundamental dimensions are those of mass (M), length (L), time (T), and electric charge (Q). In the International System (SI), the units of the aforementioned are, respectively, kilogramme (kg), metre (m), second (s), and Coulomb (C).

The notations used in this work, along with their dimensions, are as follows:

- $A_1 \approx T^{-1}$  \hspace{1cm} \text{first Rivlin-Ericksen tensor}
- $A_2 \approx T^{-2}$  \hspace{1cm} \text{second Rivlin-Ericksen tensor}
- $A_3 \approx T^{-3}$  \hspace{1cm} \text{third Rivlin-Ericksen tensor}
- $\alpha \approx \angle$  \hspace{1cm} \text{angle}
- $\alpha_1, \alpha_2 \approx ML^{-1}$  \hspace{1cm} \text{second order material constants (normal stress moduli)}
- $\bar{\alpha}_1, \bar{\alpha}_2 \approx -$  \hspace{1cm} \text{dimensionless forms of } \alpha_1, \alpha_2
- $\beta_1, \beta_2, \beta_3 \approx ML^{-1}T$  \hspace{1cm} \text{third order material constants}
- $\bar{\beta}_3 \approx -$  \hspace{1cm} \text{dimensionless form of } \beta_3
- $\frac{D}{Dt} \approx T^{-1}$  \hspace{1cm} \text{material derivative}
- $f \approx LT^{-2}$  \hspace{1cm} \text{body force per unit mass}
- $\gamma \approx -$  \hspace{1cm} \text{ratio of normal stress moduli}
- $h' \approx ML^{-1}T^{-2}$  \hspace{1cm} \text{generalized energy function}
- $\mathbf{H}' \approx L^{-1}T^{-1}Q$  \hspace{1cm} \text{magnetic field intensity vector}
- $H_0 \approx L^{-1}T^{-1}Q$  \hspace{1cm} \text{characteristic magnetic field strength}
- $H_1, H_2 \approx L^{-1}T^{-1}Q$  \hspace{1cm} \text{components of magnetic field vector}
- $\check{H}_1, \check{H}_2 \approx L^{-1}T^{-1}Q$  \hspace{1cm} \text{dimensional components of magnetic field vector}
\( I \quad \sim \quad \) unit tensor

\( j \quad \sim \quad L^{-2}T^{-1}Q \quad \) current density

\( L \quad \) L \quad characteristic length

\( \mu \quad \) ML\(^{-1}\)T\(^{-1}\) \quad constant viscosity coefficient

\( \mu^* \quad \) MLQ\(^{-2}\) \quad magnetic permeability

\( p \quad \) ML\(^{-1}\)T\(^{-2}\) \quad fluid pressure

\( \bar{p} \quad \) ML\(^{-1}\)T\(^{-2}\) \quad dimensional fluid pressure

\( \psi \quad \) L\(^2\)T\(^{-1}\) \quad stream function

\( \rho \quad \) ML\(^{-3}\) \quad constant fluid density

\( r \quad \) L \quad radial component of polar coordinates

\( Re \quad \) - \quad (hydrodynamic) Reynolds number

\( R_H \quad \) - \quad magnetic pressure number

\( R_\sigma \quad \) - \quad magnetic Reynolds number

\( \sigma \quad \) M\(^{-1}\)L\(^{-3}\)TQ\(^{2}\) \quad electrical conductivity

\( t \quad \) T \quad time

\( \bar{T} \quad \) ML\(^{-1}\)T\(^{-2}\) \quad stress tensor

\( T_{11}, T_{22} \quad \) ML\(^{-1}\)T\(^{-2}\) \quad normal stress components

\( T_{12} \quad \) ML\(^{-1}\)T\(^{-2}\) \quad shear stress

\( \theta \quad \) - \quad angular component of polar coordinates

\( U \quad \) LT\(^{-1}\) \quad characteristic velocity

\( u, v \quad \) LT\(^{-1}\) \quad components of velocity vector

\( \bar{u}, \bar{v} \quad \) LT\(^{-1}\) \quad dimensional components of velocity vector

\( V \quad \) LT\(^{-2}\) \quad velocity field vector

\( We \quad \) - \quad Weissenberg number

\( \omega \quad \) T\(^{-1}\) \quad vorticity function

\( x, y \quad \) L \quad Cartesian coordinates
$\tilde{z}, \tilde{y} \quad L$ dimensional Cartesian coordinates

These symbols may also appear as dimensionless quantities.
CHAPTER I

INTRODUCTION

1.1 NON-NEWTONIAN FLUIDS.

Fluid Mechanics is the study of the behaviour of fluid-like materials in flow. The words Fluid Mechanics in a book title generally mean something less than that. In many instances, the term is understood to mean the mechanics of Newtonian fluids (viscous and inviscid). Newtonian fluids are those for which the graph of the shear stress versus the shear rate is linear and passes through the origin. Navier-Stokes equations, the equations used to describe the motion of a viscous Newtonian fluid, were first given by Navier (1823) and later by Stokes (1851).

The classical theory of incompressible, viscous fluid is based on the constitutive equation

\[ T \approx -p I \approx + 2\mu D \approx \]  \hspace{1cm} (1.1)

with

\[ \text{tr} D \approx = 0 \] \hspace{1cm} (1.2)

where \( T \approx \) is the symmetric Cauchy stress tensor, \( p \) is the hydrostatic pressure function, \( I \approx \) is the unit tensor, \( \mu \) is the constant fluid viscosity, and \( D \approx \) is the rate of deformation tensor (i.e., the symmetric part of the velocity gradient). Equation (1.2) expresses the assumption of incompressibility of the fluid.
The mechanical behaviour of many real fluids, especially those of low molecular weight, appears to be accurately described by the theory of Navier-Stokes fluids over a wide range of circumstances. However, there are many incompressible fluids of biological and industrial importance whose behaviour cannot be satisfactorily described by means of equations (1.1) and (1.2). These are called non-Newtonian fluids, and they are fluids for which a nonlinear relation exists between the shear stress and the rate of shear, at a given temperature and pressure. Those fluids for which the curve of the shear stress versus the shear rate is linear but does not pass through the origin also fall within the purview of non-Newtonian fluids. These fluids may be divided into three broad groups which will be briefly discussed presently, albeit in reality these classifications, for the most part, are not distinct or sharply defined:

1. Time-independent fluids are those for which the rate of shear at any point is a function of the shear stress at that point;
2. Time-dependent fluids are those for which the relation between the shear stress and the shear rate depends on the time the fluid has been sheared;
3. Viscelastic fluids are those that exhibit characteristics of both elastic solids and fluids, and show partial elastic recovery after deformation.

**Time-independent Non-Newtonian Fluids.**

These are sometimes referred to as non-Newtonian viscous fluids, or as purely viscous fluids. These fluids may be divided into those that appear to exhibit a yield stress below which they do not flow, and those that do not possess a yield stress.

Examples of fluids with a yield stress may be found in certain plastic melts, oil well drilling muds, ores, cement, margarine and shortenings, greases, soap and detergent slurries, and toothpaste. These fluids are often categorized under the Bingham (1922) plastic model. There are some Bingham plastic fluids for which
the curve of shear stress versus shear rate is linear and does not pass through the origin at a given temperature and pressure.

Fluids which do not have a yield stress are subdivided into Pseudo-plastic and Dilatant fluids. Pseudo-plastic fluids are those for which the apparent viscosity is a decreasing function of the shear rate. Shear thinning behaviour, which is characteristic of such fluids, is exemplified by materials such as rubber solutions, polymer solutions and melts, starch suspensions, paints, and biological fluids. Dilatant fluids, on the other hand, are those for which the viscosity function increases as the rate of shear increases. This type of behaviour was originally observed by Reynolds in 1885. Typical examples of dilatant fluids are some aqueous suspensions of titanium dioxide, some corn flour/sugar solutions, starch, potassium silicate, quicksand, and ethylene glycol. Pseudo-plastic and dilatant fluids together constitute what is called Power-law fluids. If the power-law index is less than unity, the fluid exhibits shear thinning behaviour, and for a power-law index greater than unity, the fluid exhibits rheological dilatancy. The constitutive equation for power-law fluids, originally proposed by de Waele in 1923 and Ostwald in 1925 [cf. Tanner (1988)], is similar to that of an incompressible, viscous Newtonian fluid, with the exception that the viscosity is a function of the rate of deformation tensor.

Time-dependent Non-Newtonian Fluids.

These may be divided into Thixotropic and Rheoplectic fluids [cf. Metzner (1956)] depending on whether the shear stress decreases or increases with time at a given shear rate and constant temperature.

Thixotropic fluids are those whose consistency is dependent on the duration of the shear as well as on the shear rate. These substances exhibit a reversible decrease in shear stress with time at a constant shear rate and fixed temperature, resulting in the structure being progressively broken down and also experiencing a decreased
viscosity with time. Fluids exhibiting thixotropy are some solutions or melts of high polymers, printing inks, and many food materials.

Rheoplectic (or antithixotropic) fluids are relatively rare in occurrence. This is a case of gradual formation of structure by shear. Under isothermal conditions, these fluids exhibit a reversible increase in shear stress with time at a constant rate of shear. There is often a critical amount of shear beyond which reformation of structure is not induced and breakdown occurs. Rheopexy is confined to moderate shear rates. Bentonite clay suspensions, gypsum suspensions, vanadium pentoxide suspensions, and certain sols exemplify rheoplectic characteristics.

Viscoelastic Fluids.

The theory of elasticity may account for materials which have a capacity to store mechanical energy without dissipation of the energy. On the other hand, a viscous Newtonian fluid in a nonhydrostatic stress state implies a capacity for dissipating energy, but none for storing it. Therefore, materials which are outside the scope of these two theories are those for which some, but not all, of the work done to deform them, can be recovered. Such materials possess a capacity to both store and dissipate mechanical energy. These materials are said to be viscoelastic, and are characterized by both viscous and elastic properties [cf. Skelland (1967), Christensen (1971)]. Under appropriate circumstances, many materials are viscoelastic. In contrast to purely viscous liquids, viscoelastic fluids flow when subjected to stress, but part of their deformation is gradually recovered upon removal of the stress. Examples of viscoelastic fluids include bitumen, paints, latex, honey, flour dough, liquid metals, ceramics, polymers and polymer melts such as nylon, napalm and similar jellies, and many polymer solutions.

In light of the foregoing brief discussions of the three broad groups which constitute non-Newtonian fluids, it stands to reason that no single constitutive equation
can be obtained in place of equation (1.1) to describe the vast diversity in the physical structures of non-Newtonian fluids. It is for this reason that many rheological models have been proposed and studied, each more general than its predecessors. Among the important references to early theories are the works of Reiner (1946), Oldroyd (1950), Truesdell (1952), Rivlin and Ericksen (1955), Noll (1955), Rivlin (1956), Green and Rivlin (1957), and Coleman and Noll (1961).

Among others, one theory that has gained prominence is the Rivlin-Ericksen (1955) theory of viscoelastic fluids. The fluids that fit this model are called the Rivlin-Ericksen viscoelastic fluids of differential type of complexity \( n \). The stress tensor is a function of the first \( n \) Rivlin-Ericksen tensors \( A_1, A_2, \ldots, A_n \). Under certain invariance considerations, Rivlin and Ericksen derived a constitutive equation which, for incompressible fluids, reduces to

\[
T = -p \mathbb{I} + f(A_1, A_2, \ldots, A_n) \tag{1.3}
\]

where \( T \), \( \mathbb{I} \) and \( p \) have the same meanings as in equation (1.1), \( -p \mathbb{I} \) is the stress due to the constraint of incompressibility, and \( f \) is a tensor-valued, isotropic function of the Rivlin-Ericksen tensors which are given by

\[
A_{n+1} = \frac{DA_n}{Dt} + A_n(\text{grad}V) + (\text{grad}V)^2 A_n \tag{1.4}
\]

where \( A_0 = \mathbb{I} \), \( \frac{DA_n}{Dt} = \frac{D}{Dt} + (V \cdot \text{grad}) \) is the material derivative, \( V \) is the velocity field, and \( n = 0, 1, 2, \ldots \). Coleman and Noll (1960, 1961) developed an approximation scheme giving the same stress equations as those obtained for Rivlin-Ericksen fluids. Following their procedure of recognizing that the essential feature for the construction of various approximate constitutive equations of a incompressible simple fluid is the principle of gradually fading memory, the function \( f \) may be evaluated for different values of \( n \). This principle states that deformations which occurred in the distant past should have less effect on the present value of the stress than deformations which occurred in the recent past. An incompressible simple fluid is one in
which the present state of the stress is determined by the history of the deformation gradient, and that only isochoric motions are admissible [cf. Coleman, Markovitz and Noll (1966)]. For \( n = 1, 2, 3 \), the constitutive equations for incompressible fluids are [cf. Truesdell and Noll (1965)]:

\[
\begin{align*}
T &\approx -pI + \mu A_1 \\
&\approx -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_2^2 \\
&\approx -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_2^2 + \beta_1 A_3 \\
&\approx -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_2^2 + \beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (\text{tr} A_2^2) A_1
\end{align*}
\tag{1.5}
\]

where \( \mu, \alpha_i \ (i = 1, 2) \) and \( \beta_j \ (j = 1, 2, 3) \) are material constants. The constitutive equations represented by equations (1.5) are, respectively, those of a first grade or order (Newtonian) fluid, a viscoelastic second grade fluid, and a viscoelastic third grade fluid.

The mathematical study of non-Newtonian fluid flows is concerned primarily with the partial differential equations arising from known physical conservation laws. These equations are those of the conservation of mass, momentum and energy, as well as the thermodynamic equation of state. Owing to the notoriously nonlinear nature of these governing equations, a general solution is not known. To solve these equations, researchers have recourse to make additional assumptions to ease the complexity of the problems being investigated. Oldroyd (1958) studied the non-Newtonian effects in the steady flow of some viscoelastic liquids. Some unsteady motions of a second grade fluid were studied by Ting (1963), whereas Tanner (1966) investigated the isochoric, creeping motion of the same fluid in a plane. Kaloni (1966) worked on the fluctuating flow past a porous flat plate by a viscoelastic fluid. Kitchens (1967) introduced an integral method for solving boundary layer equations of a second grade fluid. The stability properties of third grade fluids were given by Fosdick and Rajagopal (1980). Rajagopal and Gupta (1981, 1984) obtained exact
solutions of a second grade fluid flow. The flow of a second grade fluid was studied by Kaloni and Siddiqui (1983) using Martin's (1971) approach of introducing a new curvilinear coordinate system in the physical plane. Dorrepaal, Chandna and Labropulu (1992) investigated the flow of a viscoelastic fluid near a point of re-attachment. Some reversed and non-reversed flows of a second grade fluid were obtained by Oku-Ukpong and Chandna (1993). Chandna and Oku-Ukpong (1994) employed the von Mises coordinates to investigate if an incompressible second grade fluid can flow along a given family of curves.
1.2 MAGNETOHYDRODYNAMICS.

Magnetohydrodynamics (MHD) is the branch of continuum mechanics which deals with the motion of an electrically conducting fluid in the presence of a magnetic field. Electric currents induced in the fluid as a result of its motion modify the magnetic field and, at the same time, their flow in the magnetic field produces mechanical forces which influence and change the motion. It is the synthesis of the two classical sciences of fluid mechanics and electromagnetic field theory which defines and characterizes MHD.

The electrical pioneers of the 1830's perceived that MHD might explain certain natural phenomena. Faraday thought that motions of the sea might account for the observed perturbations of the earth's magnetic field in a similar manner as the effect of an electrically conducting fluid influencing the magnetic field. Application of MHD to natural events received belated stimulus when astrophysicists came to a realization of how prevalent throughout the universe are conducting, ionized gases (plasma) and significantly strong magnetic fields. Bigelow in 1889 inferred that there were magnetic fields on the sun. This guess was confirmed by Hale in 1908 and Babcocks about a decade later. Larmor (1919) suggested that the magnetic fields of the sun and other heavenly bodies might be due to the conducting material of the star acting as an armature of a self-exciting dynamo. This theory elicited criticism from Cowling (1934).

Williams (1925) and Hartmann and Lazarus (1927, 1937) performed experiments in the laboratory on the flow of conducting fluids across a magnetic field, and pointed out the influence of the latter on the fluid motion.

Alfvén (1942, 1950), who coined the term magnetohydrodynamics, was the first to mention that the magnetic field changes the fluid motion, and vice versa. He stated that if a highly conducting fluid moves in a magnetic field, the induced currents
will tend, in some sense, to inhibit relative motion of the fluid and field so that the field is convected by the fluid. This is termed the freezing together of the magnetic field and fluid.

MHD has lent itself to application in the following areas, among others [cf. Shercliff (1965)]:

(a) the use of MHD acceleration to shoot plasma into fusion devices or to produce high-energy wind tunnels for simulating hypersonic flight;
(b) the use of MHD to affect the airstream for purposes of thermal protection, braking, propulsion or control;
(c) the use of MHD by acrodynamicists and mechanical engineers to grab fluid in midstream, instead of being confined to pushing at the edges of fluid streams.

Applied mathematicians have recognized the attractiveness of the blending of the two areas of electromagnetism and fluid mechanics. The equations of MHD are the linear electromagnetic and the nonlinear hydrodynamic equations, modified to take account of the interaction between the motion and the magnetic field. As in most electromagnetic problems involving conductors, Maxwell's displacement currents are ignored. Also neglected is the accumulation of electric charge in the equation of continuity of charge.

The aforementioned equations are solved with the assumption of infinite and/or finite electrical conductivity. The equations, being highly nonlinear, a general solution is not known. They, like the governing equations of fluid mechanics, are solved by researchers by making certain simplifying assumptions. Some of these assumptions are based on the geometric relationship that exists between the magnetic field and the velocity field vectors. These special types of flow are classified by researchers as aligned, orthogonal, constantly inclined, variably inclined, transverse, and transverse-aligned flows. These so-called reducible MHD flows [cf. Grad
shall be defined and some of the works that have been performed in each of these flows mentioned.

Aligned Flows.

These are flows for which the magnetic field vector is parallel to the velocity vector everywhere in the flow region. These are the most widely studied reducible MHD flows. Chandrasekhar (1956) studied the stability of the simplest solution of MHD equations, which is the aligned, inviscid, incompressible, electrically conducting fluid flow equations. Smith (1963) applied the method of studying steady, rotational flows of an ideal gas to aligned flows. Complex variable technique was employed by Kingston and Power (1968) to investigate aligned compressible fluid flows. Corresponding to Prim's (1952) substitution principle for classical gas flows, Chandna and Nath (1972) developed a substitution principle for fluids with arbitrary equation of state. Yin (1984) showed that no rotational motions exist for a steady, plane, incompressible, magnetofluiddynamic aligned flow, in the case of non-zero charge density. Finitely conducting aligned MHD confluent flows and Rjabouchinsky's problem applied to MHD aligned flows were investigated by Chandna and Labropulu (1990, 1992).

Orthogonal Flows.

These are flows characterized by the magnetic field vector being orthogonal everywhere to the velocity vector in the flow domain. Ladikov (1962) investigated inviscid, infinitely conducting, orthogonal fluid flows, and obtained two Bernoulli-type equations. The reducibility of certain steady, viscous, incompressible, orthogonal fluid flows to viscous, compressible MHD fluid flows was performed by Power and Walker (1967). Chandna and Nath (1973) studied steady, compressible, orthogonal fluid flow to obtain some flow configurations. Martin's (1971) method was applied by Garg and Chandna (1976) to Hamel's (1916) problem for steady,
plane, viscous, incompressible, orthogonal flow of an electrically conducting fluid of infinite electrical conductivity. Chandna, Barron and Chew (1989) obtained a number of geometric results when they studied the flow of a steady, plane, viscous, incompressible, electrically conducting fluid of finite electrical conductivity, using Martin’s approach of introducing a curvilinear coordinate system in the physical domain.

**Constantly Inclined Flows.**

These are flows for which the magnetic field vector makes a constant, non-zero angle with the velocity field vector in the region of flow. Waterhouse and Kingston (1973) studied plane, inviscid, compressible, constantly inclined flow of an electrically conducting fluid having infinite conductivity. Chandna, Toews and Nath (1975) considered constantly inclined flows of viscous, incompressible, electrically conducting fluids. Chandna and Toews (1977) investigated plane, constantly inclined MHD flows with isometric geometry. Plane, constantly inclined, magnetogasdynamic flows were studied by Chandna and Barron (1981). Barron and Chandna (1981) used hodograph transformations to obtain solutions in constantly inclined MHD flows.

**Varially Inclined Flows.**

These are those flows for which the angle between the velocity field and the magnetic field vectors varies in the domain occupied by the fluid. Since the flow equations are very complicated, not much work has been done for the variably inclined flows. Chandna, Barron and Chew (1982, 1983) employed, respectively, hodograph transformations and Martin’s method to investigate steady, plane flows of incompressible, viscous, variably inclined, electrically conducting fluid having infinite electrical conductivity, to obtain various flow configurations. Chandna, Labropulu and Husain (1991) obtained solutions of variably inclined MHD parallel
flows.

Transverse Flows.

These are those flows having the characteristic that the magnetic field is normal to the plane of flow. Swaminathan, Chandna and Sridhar (1983) studied transverse MHD flows using hodograph transformations. Chandna and Nguyen (1989), using hodograph method, obtained solutions of the equations of steady, plane flows of non-Newtonian electrically conducting fluids of finite and infinite electrical conductivities.

Transverse-aligned Flows.

These flows are defined as those for which the plane projection of the spatial magnetic field on the flow plane is parallel to the planar velocity field everywhere in the flow region. Labropulu and Chandna (1990) applied Martin's method to the study of plane, finitely conducting transverse-aligned MHD confluent flows.
1.3 OUTLINE OF THIS DISSERTATION.

The aim of this dissertation is to obtain analytical solutions of non-Newtonian and non-Newtonian MHD fluid flow problems. The non-Newtonian fluid under consideration is the viscoelastic second grade fluid. The non-Newtonian MHD fluids being investigated are the electrically conducting second grade fluid and, to a very limited extent, the electrically conducting viscoelastic third grade fluid. Both fluids have finite and infinite electrical conductivities, or their dimensionless counterparts. Unsteady and, for the most part, steady fluid flows in a plane are studied. A synopsis of this dissertation is given below.

Chapter II deals with preliminary information which will be put to use in Chapters III and IV. In Section 1, the equations of a plane, isochoric motion of a general electrically conducting third grade fluid are presented. The boundary conditions on the magnetic field are also given. Section 2 gives a few results in differential geometry, which are employed in Section 3 to recast the flow equations in a curvilinear coordinate system developed by Martin (1971).

In Chapter III, the equations of a second grade fluid flow are studied using the von Mises (1927) coordinates. Sections 2, 3, and 4 are devoted to the governing flow equations in Cartesian coordinates, Martin's curvilinear coordinates, and von Mises coordinates. In Section 5, we investigate if second grade fluid can flow along a given family of curves \( \frac{y-f(z)}{g(z)} = \text{constant} \). Having determined an answer in the affirmative so that \( \frac{y-f(z)}{g(z)} = \text{constant} \) are the streamlines, and \( \psi(x, y) = \text{constant} \) on these curves as well, we find a function \( H(\psi) \) such that \( \frac{y-f(z)}{g(z)} = H(\psi) \) with \( H'(\psi) \neq 0 \). The exact integrals [cf. Berker (1963)] of the flow defined by the given streamline pattern in both unbounded and bounded domains are then determined. Several examples are solved corresponding to different continuously differentiable forms of \( f(z) \) and \( g(z) \). Some examples are also given employing polar coordinates.
Also obtained are the stress components at the boundaries in respect of the solutions in bounded domains.

Chapter IV involves obtaining exact solutions of the MHD aligned flow equations of a plane, steady, incompressible, electrically conducting second grade fluid whose electrical conductivity is either finite or infinite. Like in Chapter III, we shall employ von Mises transformations in Section 2 to investigate if the aforementioned fluid can flow along a given family of curves \( \frac{y - g(x)}{t(x)} = \text{constant} \). If the answer to this question has been determined in the affirmative so that the curves are the streamlines, and \( \psi(x,y) = \text{constant} \) on these curves as well, we obtain some function \( \beta(\psi) \) such that \( y - g(x) = t(x)\beta(\psi) \) with \( \beta'(\psi) \neq 0 \). The exact solutions for the permissible streamline patterns are obtained in Section 3 for bounded and unbounded domains. For solutions in a bounded domains, the stress components at the boundaries are determined.

In Chapter V, an inverse method is utilized to find some exact solutions of the equations of a steady, plane, isochoric motion of a second grade fluid. Inverse method involves making certain appropriate hypotheses on the forms of the velocity field and the pressure. These hypotheses, for the most part, are made on the velocity field. Section 2 deals with obtaining the governing equations, including the stress components. In Section 3, the method of hypothesizing on the form of the stream function is formulated and applied to the flow equations. Exact solutions of reversed and non-reversed flows are obtained for unbounded and bounded domains. The components of the stress are determined for the latter.

In Chapter VI, exact solutions of a second grade fluid which undergoes steady, isochoric motion in a plane for chosen vorticity functions are obtained. The flow equations, the stress components, and the method, which consists of choosing some general vorticity function and applying same to the governing equations, are given
in Section 2. In Section 3, the exact solutions corresponding to different specific forms of the general vorticity function are obtained in an unbounded domain and for boundary value problems. The stress components are determined at the boundaries.

Chapter VII deals with an attempt to find out whether or not the equations of a steady, plane, MHD aligned flow of an incompressible, electrically conducting third grade fluid having either finite or infinite magnetic Reynolds number, accepts a von Kármán-type solution for flow between two parallel plates, for given rheological conditions on the material constants. In Section 2, the governing equations are obtained. Also written down are the boundary conditions on the magnetic field intensity at the plates, and the stress components. The lower plate is stretched by two equal and opposite forces in the x-direction so that the origin is fixed. Two problems are proposed to be investigated in which: (i) the upper plate is porous when the lower plate is non-porous; (ii) the upper plate is non-porous and the lower plate is porous. Section 3 gives the von Kármán-type solution which is substituted into the equations, and the boundary conditions for the two problems obtained. In both problems, it is found that third grade MHD fluid flow is impossible under the constitutive restrictions employed. The remainder of the problems are solved for second grade MHD fluid flow by the perturbation method for small hydrodynamic Reynolds number, which is the chosen perturbation parameter. The stress components for the problems are obtained.

Chapter VIII employs an inverse method to find the exact solutions of the equations governing the MHD aligned, unsteady, plane, isochoric motion of an electrically conducting second grade fluid with infinite or finite electrical conductivity. In Section 2, the equations for the unsteady flow and the method are given. The method consists of investigating fluid motion for which the vorticity distribution is proportional to the stream function perturbed by a uniform stream parallel to the
z-axis. Flows of infinite and finite electrical conductivities are studied in Section 3 to obtain steady and unsteady solutions, respectively. The similarities and contrasts to works of a few researchers are given in the conclusion of Section 4.
CHAPTER II

PRELIMINARIES

2.1 GOVERNING EQUATIONS OF NON-NEWTONIAN AND MAGNETOHYDRODYNAMIC NON-NEWTONIAN FLUID FLOWS.

The equations governing the unsteady magnetohydrodynamic (MHD) flow of an electrically conducting third grade fluid, in the absence of thermal effects, are [cf. Dragoș (1975)]:

\[
\begin{align*}
\frac{D}{Dt} \rho V + \text{div}(\rho V) &= 0 \quad \text{(continuity)} \\
\frac{D}{Dt} \rho V + \mu \text{grad} &+ \mu^*(\text{curl} \mathbf{H}) \times \mathbf{H} \quad \text{(linear momentum)} \\
\frac{\partial \mathbf{H}}{\partial t} &= \text{curl}(V \times \mathbf{H}) - \frac{1}{\mu^*} \text{curl}(\text{curl} \mathbf{H}) \quad \text{(diffusion)} \\
\text{div} \mathbf{H} &= 0 \quad \text{(solenoidal)}
\end{align*}
\]

(2.1)

The constitutive equation for the Cauchy stress [cf. Schowalter (1978)] is

\[
T = -p \mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 \mathbf{A}_2^T + \alpha_2 \mathbf{A}_2^2 + \beta_1 \mathbf{A}_3 \\
+ \beta_2 (\mathbf{A}_2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_2^T) + \beta_3 (\text{tr} \mathbf{A}_2^2) \mathbf{A}_3
\]

(2.2)

where

\[
\begin{align*}
\frac{D}{Dt} &= \frac{\partial}{\partial t} + (V \cdot \text{grad}) \\
\mathbf{A}_{n+1} &= \frac{D}{Dt} \mathbf{A}_n + \mathbf{A}_n (\text{grad} V) + (\text{grad} V)^T \mathbf{A}_n \quad (n = 0, 1, 2) \\
\mathbf{A}_0 &= \mathbf{I}
\end{align*}
\]

(2.3)
In system (2.1), \( V \) is the velocity vector field, \( f \) is the body force per unit mass, \( H \) is the magnetic field intensity, \( \rho \) is the fluid density, \( \mu^* \) is the magnetic permeability, \( \sigma \) is the electrical conductivity, and \( \frac{D}{Dt} \) is the material derivative. In equation (2.2), \( p \) is the dynamic pressure function, \( \mathbf{I} \) is the unit tensor, \( A_n \) \( (n = 1, 2, 3) \) [cf. Rivlin and Ericksen (1955)] are the first three Rivlin-Ericksen tensors, \( \mu \) is the constant viscosity coefficient, \( \alpha_1, \alpha_2 \) and \( \beta_1, \beta_2, \beta_3 \) are the second and third order material constants. The solenoidal equation in system (2.1) is an additional equation stipulating the absence of magnetic poles in the flow field.

Fosdick and Rajagopal (1980) found that if the fluid of third grade is to undergo motions which are compatible with the Clausius-Duhem inequality and the assumption that the Helmholtz free energy be a minimum for the fluid in equilibrium under isothermal conditions, the material constants in (2.2) must satisfy

\[
\mu \geq 0, \quad \alpha_1 \geq 0, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3} \quad (2.4)
\]

If, in (2.2) and (2.4), \( \beta_1 = \beta_2 = \beta_3 = 0 \), we have the flow equations for an unsteady second grade electrically conducting MHD fluid. When the variables are time-independent, the flow is steady. When the motion is steady and, in addition \( H = 0 \), we have the governing equations of the flow of a steady, second grade fluid, in which the diffusion and solenoidal equations in system (2.1) are absent. In these cases of second grade fluid flow, the constitutive restrictions are due to Dunn and Fosdick (1974), and are given by

\[
\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0 \quad (2.5)
\]

Fosdick and Rajagopal (1979) showed that the second grade fluid model for which \( \alpha_1 < 0 \) is unstable and exhibits anomalous behaviour unexpected of any fluid of rheological interest. In this dissertation, we shall assume that \( \mu > 0, \alpha_1 > 0, \beta_3 > 0 \) for third grade fluid flow, and \( \mu > 0, \alpha_2 > 0 \) for second grade fluid flow because of positive definite energy.
We now consider an unsteady, planar, isochoric motion of an electrically conducting third grade MHD fluid, with negligible body force. On taking \( \mathbf{V} = (u(x, y, t), v(x, y, t), 0) \), \( \mathbf{H} = (H_1(x, y, t), H_2(x, y, t), 0) \), \( f = 0 \), \( p = p(x, y, t) \) and \( \rho = \text{constant} \), the system (2.1) becomes

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial p}{\partial x} = \mu \nabla^2 u + \alpha_1 \left\{ \nabla^2 \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left[ 2u \frac{\partial^2 u}{\partial x^2} + 2v \frac{\partial^2 u}{\partial x \partial y} + 4 \left( \frac{\partial u}{\partial x} \right)^2 \right] \right. \\
+ \frac{\partial}{\partial y} \left[ (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right] \right\} \\
+ \alpha_2 \left\{ \frac{\partial}{\partial x} \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] \right. \\
+ \frac{\partial}{\partial y} \left[ \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \left( 4 \left( \frac{\partial u}{\partial x} \right)^2 + 4 \left( \frac{\partial u}{\partial y} \right)^2 + 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right) \right] \right\} \\
- \mu^* H_2 \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right)
\]

\[
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + \frac{\partial p}{\partial y} = \mu \nabla^2 v + \alpha_1 \left\{ \nabla^2 \frac{\partial v}{\partial t} + \frac{\partial}{\partial y} \left[ 2u \frac{\partial^2 v}{\partial x \partial y} + 2v \frac{\partial^2 v}{\partial y^2} + 4 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \frac{\partial u}{\partial y} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \right\} \\
+ \frac{\partial}{\partial y} \left[ \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \left( 4 \left( \frac{\partial u}{\partial x} \right)^2 + 4 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right) \right] \right\} \\
+ \alpha_2 \left\{ \frac{\partial}{\partial y} \left[ 4 \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] \right. \\
+ \frac{\partial}{\partial x} \left[ \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \left( 4 \left( \frac{\partial u}{\partial x} \right)^2 + 4 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right) \right] \right\} \\
+ \mu^* H_1 \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right)
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) = \nabla^2 \left[ \frac{1}{\mu^* \sigma} \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) + u H_1 - u H_2 \right]
\]

\[
\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0
\]

(2.6)

This is a system of five equations for the velocity components \( u, v \), the magnetic field components \( H_1, H_2 \), and the pressure \( p \) as functions of \( x, y, t \). \( \nabla^2 \) is the
Laplacian given by

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]

The boundary conditions for the magnetic field \( \mathbf{H} \) along a given surface \( S \) are given by [cf. Dragoș (1975)]

\[ [\mathbf{H}].t = 0, \quad [\mathbf{H}].n = 0 \]  \hspace{1cm} (2.7)

where \( t \) and \( n \) are, respectively, the tangential and normal vectors to the surface \( S \). Equations (2.7) state that the tangential and normal components of \( \mathbf{H} \) are continuous across the surface \( S \), where \([\mathbf{H}] = \mathbf{H}^L - \mathbf{H}^R\), \( \mathbf{H}^L \) and \( \mathbf{H}^R \) being the magnetic field intensities on both sides of the surface.
2.2 A FEW RESULTS FROM DIFFERENTIAL GEOMETRY.

Let

\[ x = x(\phi, \psi), \quad y = y(\phi, \psi) \quad (2.8) \]

define a curvilinear coordinate system in the \( xy \)-plane. Therefore, the squared differential element of arc length is given by

\[ ds^2 = E\phi^2 + 2F\phi\psi + G\psi^2 \quad (2.9) \]

where

\[
E = \left( \frac{\partial x}{\partial \phi} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 \\
F = \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \psi} \\
G = \left( \frac{\partial x}{\partial \psi} \right)^2 + \left( \frac{\partial y}{\partial \psi} \right)^2
\]

Equations (2.8) may be solved to get \( \phi, \psi \) as functions of \( x, y \) so that

\[
\frac{\partial x}{\partial \phi} = J \frac{\partial y}{\partial \psi}, \quad \frac{\partial x}{\partial \psi} = -J \frac{\partial y}{\partial \phi} \\
\frac{\partial y}{\partial \phi} = -J \frac{\partial x}{\partial \psi}, \quad \frac{\partial y}{\partial \psi} = J \frac{\partial x}{\partial \phi}
\]

provided the Jacobian of the transformation

\[
J = \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial y}{\partial \phi} = \pm \sqrt{EG - F^2} = \pm W \quad \text{(say)} \quad (2.12)
\]

is non-zero.

Let \( \alpha(\phi, \psi) \) denote the angle of inclination of the tangent to the coordinate line

\( \psi \) = constant directed in the sense of increasing \( \phi \) with the positive \( x \)-direction.

Using the first of equations (2.10), we obtain [cf. Weatherburn (1939)]

\[
\frac{\partial x}{\partial \phi} = \sqrt{E}\cos \alpha, \quad \frac{\partial y}{\partial \phi} = \sqrt{E}\sin \alpha
\]

Rewriting the first two equations in system (2.10) in the form

\[
\frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \phi} = E \\
\frac{\partial x}{\partial \psi} \frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \psi} \frac{\partial y}{\partial \psi} = F
\]

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and solving for $\frac{\partial y}{\partial \phi}$ and $\frac{\partial y}{\partial \phi}$ yields

\[ J \frac{\partial x}{\partial \phi} = E \frac{\partial y}{\partial \psi} - F \frac{\partial y}{\partial \phi} \]
\[ J \frac{\partial y}{\partial \phi} = F \frac{\partial x}{\partial \psi} - E \frac{\partial x}{\partial \phi} \]

Employing equations (2.13) in the above equations, we get

\[ \frac{\partial x}{\partial \psi} = \frac{1}{\sqrt{E}} (F \cos \alpha - J \sin \alpha) \]
\[ \frac{\partial y}{\partial \psi} = \frac{1}{\sqrt{E}} (J \cos \alpha + F \sin \alpha) \tag{2.14} \]

We evaluate the integrability conditions $\frac{\partial^2 x}{\partial \phi \partial \psi} = \frac{\partial^2 x}{\partial \psi \partial \phi}$, $\frac{\partial^2 y}{\partial \phi \partial \psi} = \frac{\partial^2 y}{\partial \psi \partial \phi}$ to get

\[ (F \sin \alpha + J \cos \alpha) \frac{\partial \alpha}{\partial \phi} - E \sin \alpha \frac{\partial \alpha}{\partial \psi} = \left( \frac{\partial F}{\partial \phi} - \frac{F}{2E} \frac{\partial E}{\partial \phi} - \frac{1}{2} \frac{\partial E}{\partial \psi} \right) \cos \alpha \]
\[ + \left( \frac{J}{2E} \frac{\partial E}{\partial \phi} - \frac{1}{2} \frac{\partial F}{\partial \phi} \right) \sin \alpha \tag{2.15} \]
\[ (J \sin \alpha - F \cos \alpha) \frac{\partial \alpha}{\partial \phi} + E \cos \alpha \frac{\partial \alpha}{\partial \psi} = \left( \frac{\partial F}{\partial \phi} - \frac{F}{2E} \frac{\partial E}{\partial \phi} - \frac{1}{2} \frac{\partial E}{\partial \psi} \right) \sin \alpha \]
\[ + \left( \frac{\partial J}{\partial \phi} - \frac{J}{2E} \frac{\partial E}{\partial \phi} \right) \cos \alpha \]

Equation (2.12) gives

\[ \frac{\partial J}{\partial \phi} = \frac{1}{2J} \left( G \frac{\partial E}{\partial \phi} + E \frac{\partial G}{\partial \phi} - 2F \frac{\partial F}{\partial \phi} \right) \]

Substituting this equation in equations (2.15), and solving for $\frac{\partial \alpha}{\partial \phi}$ and $\frac{\partial \alpha}{\partial \phi}$, we obtain

\[ \frac{\partial \alpha}{\partial \phi} = \frac{1}{2EJ} \left( 2E \frac{\partial F}{\partial \phi} - F \frac{\partial E}{\partial \phi} - E \frac{\partial E}{\partial \psi} \right) \]
\[ \frac{\partial \alpha}{\partial \psi} = \frac{1}{2EJ} \left( E \frac{\partial G}{\partial \phi} - F \frac{\partial E}{\partial \phi} \right) \]

These may be written as

\[ \frac{\partial \alpha}{\partial \phi} = \frac{J}{E} \Gamma_{11}, \quad \frac{\partial \alpha}{\partial \phi} = \frac{J}{E} \Gamma_{12} \tag{2.16} \]
where
\[
\Gamma_{11}^2 = \frac{1}{2W^2} \left( 2E \frac{\partial F}{\partial \phi} - F \frac{\partial E}{\partial \phi} - E \frac{\partial F}{\partial \psi} \right)
\]
\[
\Gamma_{12}^2 = \frac{1}{2W^2} \left( E \frac{\partial G}{\partial \phi} - F \frac{\partial E}{\partial \phi} \right)
\]
are the Christoffel symbols of the second kind. We get, from equations (2.16) and the integrability condition \( \frac{\partial^2 \sigma}{\partial \phi \partial \psi} = \frac{\partial^2 \sigma}{\partial \psi \partial \phi} \),
\[
\frac{\partial}{\partial \psi} \left( \frac{W}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left( \frac{W}{E} \Gamma_{12}^2 \right) = 0
\]
(2.18)

This implies the Gaussian curvature of a plane
\[
K = \frac{1}{W} \left[ \frac{\partial}{\partial \psi} \left( \frac{W}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left( \frac{W}{E} \Gamma_{12}^2 \right) \right]
\]
equals zero. Equation (2.18) is called the Gauss formula.

Conversely, if \( E, F, G \) are given as functions of \( \phi, \psi \) so that the Gauss formula (2.18) is satisfied, then the functions \( x(\phi, \psi), y(\phi, \psi) \) may be obtained in terms of \( E, F, G \), where these satisfy equation (2.9).

The functions \( x(\phi, \psi), y(\phi, \psi), \alpha(\phi, \psi) \) are given by
\[
x = \int \left( \frac{\partial x}{\partial \phi} d\phi + \frac{\partial x}{\partial \psi} d\psi \right) = \int \frac{1}{\sqrt{E}} \left[ E \cos \alpha d\phi + (F \cos \alpha - J \sin \alpha) d\psi \right]
\]
\[
y = \int \left( \frac{\partial y}{\partial \phi} d\phi + \frac{\partial y}{\partial \psi} d\psi \right) = \int \frac{1}{\sqrt{E}} \left[ E \sin \alpha d\phi + (J \cos \alpha + F \sin \alpha) d\psi \right]
\]
\[
\alpha = \int \left( \frac{\partial \alpha}{\partial \phi} d\phi + \frac{\partial \alpha}{\partial \psi} d\psi \right) = \int \frac{J}{E} \left( \Gamma_{11}^2 d\phi + \Gamma_{12}^2 d\psi \right)
\]
(2.19)
(2.20)

From equations (2.19), the complex variable \( z = x + iy \) is given by
\[
z = \int \frac{e^{i\alpha}}{\sqrt{E}} \left[ Ed\phi + (F + iJ) d\psi \right]
\]
(2.21)

where \( \alpha \) is given by equation (2.20) and \( i = \sqrt{-1} \).

We sum up the aforementioned results in the following theorem:
Theorem 2.1. The squared differential element of arc length for a plane

\[ ds^2 = E\phi^2 + 2Fd\phi d\psi + Gd\psi^2 \]

has the functions \( E, F, G \) of \( \phi, \psi \) as coefficients, with a curvilinear coordinate system

\[ x = x(\phi, \psi), \quad y = y(\phi, \psi) \]

if and only if they satisfy the Gauss formula

\[ \frac{\partial}{\partial \psi} \left( \frac{W}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left( \frac{W}{E} \Gamma_{12}^2 \right) = 0 \]

If this condition is satisfied, the functions \( x(\phi, \psi), y(\phi, \psi) \) defining the curvilinear coordinate system are given in complex form by equation (2.21) in terms of \( E, F, G \), where \( \alpha \) is given by equation (2.20) and \( J = \pm W = \pm \sqrt{EG - F^2} \).

Useful identities that are derivable from \( W = \sqrt{EG - F^2} \) are

\[ \frac{\partial}{\partial \phi} \left( \frac{E}{2W^2} \right) = \frac{1}{W^2} (F\Gamma_{11}^2 - E\Gamma_{12}^2) \]

\[ \frac{\partial}{\partial \psi} \left( \frac{E}{2W^2} \right) = \frac{1}{W^2} (F\Gamma_{12}^2 - E\Gamma_{22}^2) \] \hspace{1cm} (2.22)

\[ \frac{\partial}{\partial \phi} \left( \frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left( \frac{E}{W} \right) = \frac{1}{W} (G\Gamma_{11}^2 - 2F\Gamma_{12}^2 + E\Gamma_{22}^2) \] \hspace{1cm} (2.23)

where

\[ \Gamma_{22}^2 = \frac{1}{2W^2} \left( E \frac{\partial G}{\partial \psi} - 2F \frac{\partial F}{\partial \psi} + F \frac{\partial G}{\partial \phi} \right) \] \hspace{1cm} (2.24)
2.3 A METHOD DUE TO MARTIN.

We have, from system (2.6), that the steady, plane, isochoric MHD flow of an electrically conducting second grade fluid is governed by the system:

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\rho \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial p}{\partial x} &= \mu \nabla^2 u + \alpha_1 \left\{ \frac{\partial}{\partial x} \left[ 2u \frac{\partial^2 u}{\partial x^2} + 2v \frac{\partial^2 u}{\partial x \partial y} + 4 \left( \frac{\partial u}{\partial x} \right)^2 \right] \\
+ 2 \frac{\partial v}{\partial x} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left[ \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right] + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \right\} \\
+ \alpha_2 \left\{ \frac{\partial}{\partial x} \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] \right\} - \mu^* H_2 \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) \\
\rho \left( \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + \frac{\partial p}{\partial y} &= \mu \nabla^2 v + \alpha_1 \left\{ \frac{\partial}{\partial y} \left[ \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right] + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \right\} \\
+ 2 \frac{\partial v}{\partial y} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial x} \left[ 2u \frac{\partial^2 v}{\partial x \partial y} + 2v \frac{\partial^2 v}{\partial y^2} + 4 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} \right] \right\} \\
+ \alpha_2 \left\{ \frac{\partial}{\partial y} \left[ 4 \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] \right\} + \mu^* H_1 \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) \\
\frac{1}{\mu^* \sigma} \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) + \nu H_1 - u H_2 &= k \\
\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} &= 0 \\
\end{align*}
\]

(2.25)

of five equations in \( u, v, H_1, H_2, p \) as functions of \( (x, y) \), where \( k \) is an integration constant.

Introducing the vorticity, current density, and the generalized energy functions given, respectively, by [cf. Martin (1971), Kaloni and Siddiqui (1983)]

\[
\begin{align*}
\omega &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\
j &= \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \\
h &= p + \frac{1}{2} \rho (u^2 + v^2) - \alpha_1 (u \nabla^2 u + v \nabla^2 v) \\
&\quad - \left( \frac{3 \alpha_1 + 2 \alpha_2}{4} \right) \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + 4 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right]
\end{align*}
\]

(2.26)
the third order system (2.25) may be replaced by the second order system:

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \quad \text{(continuity)} \\
\frac{\partial h}{\partial x} &= -\mu \frac{\partial \omega}{\partial y} + \rho v \omega - \mu \ast j \cdot H_2 - \alpha_1 v \nabla^2 \omega \\
\frac{\partial h}{\partial y} &= \mu \frac{\partial \omega}{\partial x} - \rho u \omega + \mu \ast j \cdot H_1 + \alpha_1 u \nabla^2 \omega \\
\omega &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{(vorticity)} \quad (2.27) \\
j &= \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \quad \text{(current density)} \\
\frac{j}{\mu \ast \sigma} &= u H_2 - v H_1 + k \quad \text{(diffusion)} \\
\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} &= 0 \quad \text{(solenoidal)}
\end{align*}
\]

of seven equations in seven unknown functions \(u, v, H_1, H_2, \omega, j, h\) of \(x, y\).

We study, in this dissertation, aligned flows for which the magnetic field intensity is parallel to the velocity field everywhere in the flow region, so that

\[
(H_1, H_2) = (fu, fv) \quad (2.28)
\]

where \(f\) is some non-zero function of \(x, y\). The aligned flows to be investigated will be divided into those for which the electrically conducting fluids are of infinite or finite electrical conductivity \(\sigma\). However, in the finitely conducting aligned flows, the current density is a non-zero constant \(j_0\).

**Infinitely Conducting Fluid Flows.** We substitute equation (2.28) into system
(2.27) to obtain the system:

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \quad \text{(continuity)} \\
\frac{\partial h}{\partial x} &= -\mu \frac{\partial \omega}{\partial y} + (\rho \omega - \mu^* f j - \alpha_1 \nabla^2 \omega) v \\
\frac{\partial h}{\partial x} &= -\mu \frac{\partial \omega}{\partial y} - (\rho \omega - \mu^* f j - \alpha_1 \nabla^2 \omega) u \\
\omega &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{(vorticity)} \\
j &= f \omega + v \frac{\partial f}{\partial x} - u \frac{\partial f}{\partial y} \quad \text{(current density)} \\
u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} &= 0 \quad \text{(solenoidal)}
\end{align*}
\]

(2.29)

of six equations in six functions \(u, v, f, \omega, j, h\) of \(x, y\), which constitute the system of equations governing the flow of a steady, plane, incompressible, second grade, infinitely conducting, MHD aligned fluid. The diffusion equation in system (2.27) yields \(k = 0\).

**Finitely Conducting Fluid Flows.** The equations satisfied by the motion of a steady, plane, incompressible, second grade, finitely conducting, MHD aligned fluid are given by system (2.29) when \(j\) is replaced by a constant \(j_0\), since the diffusion equation gives \(j = \mu^* \sigma k \equiv j_0\).

The continuity equation implies the existence of a stream function \(\psi(x, y)\) such that

\[
u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}
\]

(2.30)

We introduce a curvilinear coordinate system \(\phi, \psi\) in the physical plane (xy-plane) in which the curves \(\psi(x, y) = \text{constant}\) are the streamlines and the curves \(\phi(x, y) = \text{constant}\) are kept arbitrary.

We now obtain new forms of the equations of motion, given by system (2.29), based on the results of Section 2.2 in the new variables \(\phi, \psi\).
Continuity Equation. Employing equation (2.30) in the first and third equations of (2.11), we get
\[ \frac{\partial x}{\partial \phi} = Ju, \quad \frac{\partial y}{\partial \phi} = Ju \] (2.31)

Introducing polar coordinates \( q, \theta \) in the hodograph plane (uv-plane), then
\[ u = q \cos \theta, \quad v = q \sin \theta \]

where \( q \) is the fluid speed and \( \theta \) is the direction of flow in the physical plane. These equations are employed in equations (2.31) to obtain
\[ \frac{\partial x}{\partial \phi} = qJ \cos \theta, \quad \frac{\partial y}{\partial \phi} = qJ \sin \theta \] (2.32)

Comparing equations (2.32) with equations (2.13) gives two possibilities:

(i) \( \theta = \alpha, \quad qJ = \sqrt{E}, \quad J > 0 \)

(ii) \( \theta = \alpha + \pi, \quad qJ = -\sqrt{E}, \quad J < 0 \)

In (i) and (ii), the fluid flows along the streamlines toward higher and lower parameter values of \( \phi \), respectively, and \( q \) is given by
\[ q = \frac{\sqrt{E}}{W} \] (2.33)

This is a consequence of the continuity equation. Conversely, equation (2.33) implies the continuity equation since when it holds true, equations (2.13) give
\[ \frac{\partial x}{\partial \phi} = qW \cos \alpha, \quad \frac{\partial y}{\partial \phi} = qW \sin \alpha. \]

Linear Momentum Equations. Introducing the new independent variables \( \phi, \psi \) into the linear momentum equations of system (2.29), we write them in the form
\[ \frac{\partial h}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial h}{\partial \psi} \frac{\partial \phi}{\partial x} = -\mu \left( \frac{\partial \omega}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial \omega}{\partial \psi} \frac{\partial \phi}{\partial x} \right) + \frac{1}{J} \left( \rho \omega - \mu^* f j - \alpha_1 \Delta_2 \omega \right) \frac{\partial y}{\partial \phi} \]
\[ \frac{\partial h}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial h}{\partial \psi} \frac{\partial \phi}{\partial y} = \mu \left( \frac{\partial \omega}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial \omega}{\partial \psi} \frac{\partial \phi}{\partial x} \right) - \frac{1}{J} \left( \rho \omega - \mu^* f j - \alpha_1 \Delta_2 \omega \right) \frac{\partial x}{\partial \phi} \] (2.34)
where use has been made of equations (2.31), and \( \nabla^2 \omega \) in \( x, y \) coordinates becomes the Beltrami's differential parameter of second order (see Appendix A) in the new variables \( \phi, \psi \), and is given by

\[
\Delta_2 \omega = \frac{1}{W} \left[ \frac{\partial}{\partial \phi} \left( \frac{G}{W} \frac{\partial \omega}{\partial \phi} - \frac{F}{W} \frac{\partial \omega}{\partial \psi} \right) + \frac{\partial}{\partial \psi} \left( -\frac{F}{W} \frac{\partial \omega}{\partial \phi} + \frac{E}{W} \frac{\partial \omega}{\partial \psi} \right) \right]
\]  

(2.35)

Employing equations (2.11) in equations (2.34), we get

\[
\frac{\partial h}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial h}{\partial \psi} \frac{\partial y}{\partial \phi} = \mu \left( \frac{\partial \omega}{\partial \phi} \frac{\partial x}{\partial \psi} - \frac{\partial \omega}{\partial \psi} \frac{\partial x}{\partial \phi} \right) + \left( \rho \omega - \mu^* f j - \alpha_1 \Delta_2 \omega \right) \frac{\partial y}{\partial \phi} 
\]  

(2.36)

\[
- \frac{\partial h}{\partial \phi} \frac{\partial x}{\partial \psi} + \frac{\partial h}{\partial \psi} \frac{\partial x}{\partial \phi} = \mu \left( \frac{\partial \omega}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial \omega}{\partial \psi} \frac{\partial y}{\partial \phi} \right) - \left( \rho \omega - \mu^* f j - \alpha_1 \Delta_2 \omega \right) \frac{\partial x}{\partial \phi} 
\]  

(2.37)

We multiply equation (2.36) by \( \frac{\partial y}{\partial \phi} \) and add to it the product of equation (2.37) and \( \frac{\partial y}{\partial \psi} \). We also obtain the product of equation (2.37) and \( \frac{\partial x}{\partial \psi} \). Equations (2.10) and (2.12) are then applied to the resulting equations to obtain the linear momentum equations in \( \phi, \psi \) variables:

\[
\frac{\partial h}{\partial \phi} = \frac{\mu}{J} \left( \frac{F}{\partial \phi} \frac{\partial \omega}{\partial \psi} - \frac{E}{\partial \psi} \frac{\partial \omega}{\partial \phi} \right) 
\]  

(2.38)

Vorticity Equation. The vorticity equation in system (2.29) may be written as

\[
\omega = \frac{\partial v}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial v}{\partial \psi} \frac{\partial \psi}{\partial x} - \frac{\partial u}{\partial \phi} \frac{\partial \phi}{\partial y} - \frac{\partial u}{\partial \psi} \frac{\partial \psi}{\partial y} 
\]  

\[
= \frac{1}{J} \left[ \frac{\partial y}{\partial \phi} \frac{\partial}{\partial \phi} \left( \frac{1}{J} \frac{\partial y}{\partial \phi} \right) - \frac{\partial y}{\partial \psi} \frac{\partial}{\partial \psi} \left( \frac{1}{J} \frac{\partial y}{\partial \psi} \right) + \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \phi} \left( \frac{1}{J} \frac{\partial x}{\partial \phi} \right) - \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \psi} \left( \frac{1}{J} \frac{\partial x}{\partial \psi} \right) \right]
\]  

after employing equations (2.31) and (2.11). Expanding the right hand side of this equation, making use of equations (2.10) and

\[
\frac{\partial x}{\partial \phi} \frac{\partial^2 x}{\partial \phi^2} + \frac{\partial y}{\partial \psi} \frac{\partial^2 y}{\partial \phi^2} = \frac{\partial}{\partial \phi} \left( \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \phi} \frac{\partial x}{\partial \psi} \right) 
\]  

\[
\frac{\partial x}{\partial \phi} \frac{\partial^2 x}{\partial \phi \partial \psi} + \frac{\partial y}{\partial \psi} \frac{\partial^2 y}{\partial \phi \partial \psi} = \frac{\partial}{\partial \psi} \left( \frac{\partial}{\partial \phi} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 
\]  

we get

\[
\omega = \frac{1}{W} \left[ \frac{\partial}{\partial \phi} \left( \frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left( \frac{E}{W} \right) \right] 
\]  

(2.39)

Using equation (2.23), the vorticity equation may also be written as

\[
\omega = \frac{1}{W^2} \left( GT_{11}^2 - 2 FT_{12}^2 + ET_{22}^2 \right) 
\]  

(2.40)
Current Density Equation. In terms of \( \phi, \psi \) variables, the current density is given by

\[
j = f \omega + \frac{1}{J} \left[ \frac{\partial y}{\partial \psi} \left( \frac{\partial f}{\partial \phi} \frac{\partial \psi}{\partial x} + \frac{\partial f}{\partial \psi} \frac{\partial \phi}{\partial x} \right) - \frac{\partial x}{\partial \phi} \left( \frac{\partial f}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial f}{\partial \psi} \frac{\partial \psi}{\partial y} \right) \right]
\]

Similarly, if we use equations (2.11) and (2.10), we get

\[
j = f \omega + \frac{1}{W^2} \left( \rho \frac{\partial f}{\partial \phi} - E \frac{\partial f}{\partial \psi} \right)
\]  \( (2.41) \)

Solenoidal Condition on \( H \). The additional equation stipulating the solenoidal condition on the magnetic field becomes

\[
\frac{1}{J} \left[ \frac{\partial x}{\partial \psi} \left( \frac{\partial f}{\partial \phi} \frac{\partial \psi}{\partial x} + \frac{\partial f}{\partial \psi} \frac{\partial \phi}{\partial x} \right) + \frac{\partial y}{\partial \phi} \left( \frac{\partial f}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial f}{\partial \psi} \frac{\partial \psi}{\partial y} \right) \right] = 0
\]

Substitution of equations (2.11) and (2.12) in the above equation yields

\[
\frac{\partial f}{\partial \phi} = 0
\]  \( (2.42) \)

Summing up the results developed above, we have:

**Theorem 2.2.** When the streamlines, \( \psi(x,y) = \text{constant} \), of a steady, planar, MHD aligned motion of an incompressible, electrically conducting second grade fluid of infinite electrical conductivity are taken as one set of coordinate lines in a curvilinear coordinate system \( \phi, \psi \) in the physical plane, then system (2.29) of six equations for \( u, v, f, \omega, j, h \) as functions of \( x, y \) may be replaced by the
underdetermined system:

\[ \frac{\partial h}{\partial \phi} = \frac{\mu}{J} \left( F \frac{\partial \omega}{\partial \phi} - E \frac{\partial \omega}{\partial \psi} \right) \]  
\[ \begin{aligned}
\frac{\partial h}{\partial \psi} &= \frac{\mu}{J} \left( G \frac{\partial \omega}{\partial \phi} - F \frac{\partial \omega}{\partial \psi} \right) - \left( \rho \omega - \mu^* f j - \alpha_1 \Delta_2 \omega \right) \\
\omega &= \frac{1}{W} \left[ \frac{\partial}{\partial \phi} \left( \frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left( \frac{E}{W} \right) \right] \\
j &= f \omega - \frac{E}{W^2} \frac{\partial f}{\partial \psi} \\
\frac{\partial f}{\partial \phi} &= 0 \\
\frac{\partial}{\partial \psi} \left( \frac{W}{E} \Gamma_{11} \right) - \frac{\partial}{\partial \phi} \left( \frac{W}{E} \Gamma_{12} \right) &= 0
\end{aligned} \]  
\[ (\text{linear momentum}) \]
\[ (\text{vorticity}) \]
\[ (\text{current density}) \]
\[ (\text{solenoidal}) \]
\[ (\text{Gauss}) \]

of six equations for seven functions $E$, $F$, $G$, $f$, $\omega$, $j$, $h$ of $\phi$, $\psi$, where $J$ is positive or negative, respectively, in accordance with $\phi$ increasing or decreasing along the streamlines in the flow direction.

The speed $q$ and the pressure $p$ of the fluid flow are given by equation (2.33) and (see Appendix B)

\[ p = h - \frac{\rho E}{2W^2} + \frac{\alpha_1}{W^2} \left( F \frac{\partial \omega}{\partial \phi} - E \frac{\partial \omega}{\partial \psi} \right) + \frac{3\alpha_1 + 2\alpha_2}{2} \left\{ \omega^2 \right. \\
\left. + \frac{4}{W\sqrt{E}} \left[ \Gamma_{11} \frac{\partial}{\partial \phi} \left( \frac{\sqrt{E}}{W} \right) - \Gamma_{12} \frac{\partial}{\partial \psi} \left( \frac{\sqrt{E}}{W} \right) \right] \right\} \]  
\[ (2.44) \]

We eliminate $h$ from the linear momentum equations with the integrability condition $\frac{\partial h}{\partial \phi} = \frac{\partial^2 h}{\partial \psi \partial \phi}$ to get:

**Theorem 2.3.** When the streamlines, $\psi(x, y) = \text{constant}$, of an MHD aligned, steady, plane, isochoric flow of an electrically conducting second grade of infinite electrical conductivity constitute a set of coordinate curves in a curvilinear coordinate system $\phi$, $\psi$ in the physical plane, then the flow is governed by the underde-
terminated system:

\[ \pm \mu W \Delta_2 \omega - \rho \frac{\partial \omega}{\partial \phi} + \mu^* f \frac{\partial j}{\partial \phi} + \alpha_1 \frac{\partial}{\partial \phi} (\Delta_2 \omega) = 0 \] (integrability)

\[ \omega = \frac{1}{W^2} (GR_{11}^2 - 2FT_{12}^2 + ER_{22}^2) \] (vorticity)

\[ j = f \omega - \frac{E}{W^2} \frac{\partial f}{\partial \phi} \] (current density) \hspace{1cm} (2.45)

\[ \frac{\partial f}{\partial \phi} = 0 \] (solenoidal)

\[ \frac{\partial}{\partial \phi} \left( \frac{W}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left( \frac{W}{E} \Gamma_{12}^2 \right) = 0 \] (Gauss)

of five equations for the six functions \( E, F, G, f, \omega, j \) of \( \phi, \psi \). The upper and lower signs in the integrability equation are used if \( J \) is positive and negative, respectively.

We note that \( f = f(\psi) \), since \( \frac{\partial f}{\partial \phi} = 0 \).

Once a solution \( E, F, G, f, \omega, j \) as functions of \( \phi, \psi \) of the system (2.45) has been established, the energy \( h = h(\phi, \psi) \) is obtained from the linear momentum equations in system (2.43), the speed \( q = q(\phi, \psi) \) and the pressure \( p = p(\phi, \psi) \) are given by equations (2.33) and (2.44).

Furthermore, the \( \phi \psi \)-plane is mapped upon the physical plane by equation (2.21) and upon the hodograph plane by

\[ u + iv = \frac{\sqrt{E}}{f} e^{i\alpha} \]

to provide a complete solution of the flow, where \( i = \sqrt{-1} \) and \( \alpha \) is given by equation (2.20).

It should be noted that if we are considering a finitely conducting fluid flow, the current density \( j \) in the above analysis will be the constant \( j_0 \). Also, if the fluid is non-MHD, we shall set \( H = (H_1(x, y), H_2(x, y), 0) = 0 \) in the foregoing work.
CHAPTER III

SOLUTIONS OF THE EQUATIONS OF A SECOND GRADE FLUID FLOW BY VON MISES COORDINATE TRANSFORMATIONS

3.1 INTRODUCTION.

Exact solutions of the Navier-Stokes equations have been well-documented by Berker (1963), and an excellent review of these solutions has also been given by Wang (1991). When dealing with viscoelastic fluids, exact solutions in the literature are rare because of the presence of higher order nonlinear terms in the governing equations. In the specific case of a second grade viscoelastic fluid, the nonlinearities introduced involve the normal stress moduli $\alpha_1$ and $\alpha_2$. The objective of this chapter is to introduce a direct approach for determining exact solutions of the equations of the steady, isochoric motion of a second grade fluid in a plane.

Berker (1963) has given two definitions of an exact solution (or integral):

(a) a set of the velocity components and the pressure function which satisfy the continuity and the linear momentum equations constitutes an exact solution;

(b) a set of the velocity components and the pressure function satisfying the continuity and the linear momentum equations is an exact solution if these functions
give the solution to a physical problem with fixed or movable boundaries on which
the no-slip condition is satisfied.

The indirect or inverse method of assuming a certain form of the stream func-
tion \textit{a priori} has been extensively used by researchers to find exact solutions.
Markovitz and Coleman (1964), Rajagopal and Gupta (1981, 1984), Kaloni and
Huschilt (1984), and Siddiqui (1990) are some of the works that have utilized this
approach.

A direct method is employed to investigate if second grade fluid can flow along a
given family of curves \( \frac{y - f(x)}{g(x)} = \text{constant} \), where \( f(x) \) and \( g(x) \neq 0 \) are continuously
differentiable functions. Having determined the answer in the affirmative so that
these curves are streamlines, and \( \psi(x, y) = \text{constant} \) on these curves as well, there
exists some function \( H(\psi) \) such that \( y - f(x) = g(x)H(\psi) \) with \( H'(\psi) \neq 0 \). This
method employs the von Mises (1927) coordinates \( z, \psi \) to solve the problem. This
method has been employed by Ames (1965) in nonlinear partial differential equa-
tions, and by Barron (1989) in solving, numerically, an incompressible potential
flow problem.

The exact integral defined by the given streamline pattern in an unbounded do-
main, and when the flow impinges on a fixed or stretching porous or non-porous
plate are found. We also study flows when polar coordinates are used. To accom-
plish the aforementioned objectives, we employ the extension of the work of Martin
(1971) for the Navier-Stokes equations to the second grade fluid flow equations
obtained by Kaloni and Siddiqui (1983). This work deals with solutions in both
unbounded and bounded domains.
3.2 EQUATIONS OF MOTION IN CARTESIAN COORDINATES.

A steady, plane, second grade fluid which undergoes isochoric motion, in dimensional variables, is governed by the system:

\[
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0
\]

\[
\rho (\frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{\partial \tilde{u}}{\partial y}) + \frac{\partial \tilde{p}}{\partial x} = \mu \tilde{V}^2 \tilde{u} + \alpha_1 \left\{ \frac{\partial}{\partial x} \left[ 4(\frac{\partial \tilde{u}}{\partial x})^2 + (\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y})^2 \right] \right\}
\]

\[
+ 2 \frac{\partial \tilde{u}}{\partial x} \left( \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} \right) + \frac{\partial \tilde{v}}{\partial y} \left[ (\frac{\partial \tilde{u}}{\partial x} + \tilde{u} \frac{\partial \tilde{v}}{\partial y}) \left( \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} \right) + 2 \frac{\partial \tilde{u}}{\partial x} \frac{\partial \tilde{u}}{\partial y} \right] \}
\]

\[
\rho (\frac{\partial \tilde{v}}{\partial x} + \tilde{u} \frac{\partial \tilde{v}}{\partial y}) + \frac{\partial \tilde{p}}{\partial y} = \mu \tilde{V}^2 \tilde{v} + \alpha_1 \left\{ \frac{\partial}{\partial y} \left[ 4(\frac{\partial \tilde{v}}{\partial y})^2 + (\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y})^2 \right] \right\}
\]

\[
+ 2 \frac{\partial \tilde{v}}{\partial x} \left( \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} \right) + \frac{\partial \tilde{u}}{\partial y} \left[ (\frac{\partial \tilde{v}}{\partial x} + \tilde{u} \frac{\partial \tilde{v}}{\partial y}) \left( \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} \right) + 2 \frac{\partial \tilde{v}}{\partial x} \frac{\partial \tilde{u}}{\partial y} \right] \}
\]

which is obtained from system (2.6) by letting the variables be time-independent and \( \mathbf{H} = 0 \), and

\[ \tilde{V}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \]

To nondimensionalize system (3.1), we introduce the dimensionless variables

\[
x = \frac{\tilde{x}}{L}, \quad y = \frac{\tilde{y}}{L}, \quad u = \frac{\tilde{u}}{U}, \quad v = \frac{\tilde{v}}{U}, \quad p = \frac{\tilde{p}}{\rho U^2} \quad (3.2)
\]

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System (3.1) becomes

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{Re} \nabla^2 u + \frac{We}{Re} \left\{ \frac{\partial}{\partial x} \left[ 2u \frac{\partial^2 u}{\partial x^2} + 2v \frac{\partial^2 u}{\partial x \partial y} + 4 \left( \frac{\partial u}{\partial x} \right)^2 \right] \right. \\
+ 2 \left( \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial u}{\partial y} \right) \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left[ u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right] \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \\
+ 2 \left( \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right) \right\} + \frac{\gamma We}{Re} \left\{ \frac{\partial}{\partial x} \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] \right. \\
+ 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) \right\} \right) \\
\left. + \frac{\gamma We}{Re} \left\{ \frac{\partial}{\partial y} \left[ 4 \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] \right. \\
+ 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \left( \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \right) \right\} \right\} \\
(3.3)
\]

for the velocity components \(u\), \(v\), and the pressure \(p\) as functions of \(x\), \(y\), where

\(Re = \frac{\rho U L}{\mu}\) is the Reynolds number, \(We = \frac{\alpha U L}{\mu L^2}\) is the Weissenberg number, \(\gamma = \frac{\alpha_2}{\alpha_1}\) is the ratio of the normal stress moduli, \(U\) is the characteristic velocity, \(L\) is the characteristic length.

Introducing the vorticity function \(\omega\) and the energy function \(h\), in dimensionless form, as

\[
\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},
\]

\[
h = p + \frac{1}{2} (u^2 + v^2) - \frac{We}{Re} (u \nabla^2 u + v \nabla^2 v) - \frac{3 + 2\gamma}{4} \frac{We}{Re} \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + 4 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right]
\]

the second order system (3.3) becomes

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{(continuity)}
\]

\[
\frac{\partial h}{\partial x} = -\frac{1}{Re} \frac{\partial \omega}{\partial y} + v \left( \omega - \frac{We}{Re} \nabla^2 \omega \right) \quad \text{(linear momentum)}
\]

\[
\frac{\partial h}{\partial y} = \frac{1}{Re} \frac{\partial \omega}{\partial x} + u \left( \frac{We}{Re} \nabla^2 \omega - \omega \right)
\]

\[
\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{(vorticity)}
\]

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for the four unknown functions $u, v, \omega, h$ of $x, y$. When these functions have been obtained, the pressure $p$ is given by the second of equations (3.4).

In these dimensionless variables, the normal stress components are

\[
T_{11} = -p + \frac{2}{Re} \frac{\partial u}{\partial x} + \frac{We}{Re} \left( \frac{2u \partial^2 u}{\partial x^2} + 2v \frac{\partial^2 u}{\partial x \partial y} + 4(1 + \gamma) \left( \frac{\partial u}{\partial x} \right)^2 \right)
\]
\[
+ \left[ (2 + \gamma) \frac{\partial v}{\partial x} + \gamma \frac{\partial u}{\partial y} \right] \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)
\]

\[
T_{22} = -p + \frac{2}{Re} \frac{\partial v}{\partial y} + \frac{We}{Re} \left( \frac{2u \partial^2 v}{\partial x \partial y} + 2v \frac{\partial^2 v}{\partial y^2} + 4(1 + \gamma) \left( \frac{\partial v}{\partial y} \right)^2 \right)
\]
\[
+ \left[ (2 + \gamma) \frac{\partial u}{\partial y} + \gamma \frac{\partial v}{\partial x} \right] \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)
\]

(3.6)

and the shear stress is

\[
T_{12} = T_{21} = \frac{1}{Re} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \frac{We}{Re} \left[ (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)
\]
\[
+ 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + 2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right]
\]

(3.7)
3.3 FLOW EQUATIONS USING MARTIN’S APPROACH.

The continuity equation in system (3.5) means there exists some stream function \( \psi(x,y) \) so that equations (2.30) hold true.

Martin introduced a curvilinear coordinate system \( \phi, \psi \) in which the curves \( \psi(x,y) = \) constant are the streamlines and the curves \( \phi(x,y) = \) constant are arbitrarily chosen.

In Section 2.3, the equations of motion were recast in these new variables \( \phi, \psi \).

The results for our flow are summed up in the following:

**Theorem 3.1.** When the streamlines, \( \psi(x,y) = \) constant, of a steady, plane, second grade fluid which undergoes isochoric motion are taken as one set of coordinate curves in a curvilinear coordinate system \( \phi, \psi \) in the physical plane, the system (3.5) of four equations for unknowns \( u, v, \omega, h \) as functions of \( x, y \) may be replaced by the underdetermined system:

\[
\frac{\partial h}{\partial \phi} = \frac{1}{ReJ} \left( F \frac{\partial \omega}{\partial \phi} - E \frac{\partial \omega}{\partial \psi} \right) \tag{linear momentum}
\]

\[
\frac{\partial h}{\partial \psi} = \frac{1}{ReJ} \left( G \frac{\partial \omega}{\partial \phi} - F \frac{\partial \omega}{\partial \psi} \right) - \omega + \frac{We}{Re} \Delta_2 \omega \tag{3.8}
\]

\[
\omega = \frac{1}{W} \left[ \frac{\partial}{\partial \psi} \left( \frac{F}{W} \right) - \frac{\partial}{\partial \phi} \left( \frac{E}{W} \right) \right] \tag{vorticity}
\]

\[
\frac{\partial}{\partial \phi} \left( \frac{W}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \psi} \left( \frac{W}{E} \Gamma_{12}^2 \right) = 0 \tag{Gauss}
\]

of four equations for five functions \( E, F, G, \omega, h \) of \( \phi, \psi \), where \( J > 0 \) or \( J < 0 \) accordingly as \( \phi \) increases or decreases along the streamlines in the flow direction, and the Christoffel symbols \( \Gamma_{11}^2, \Gamma_{12}^2 \) are given by equations (2.17).

The speed \( q, \Delta_2 \omega \) and the pressure \( p \) of the motion of the fluid are given by equations (2.33), (2.35) and (2.44), respectively.

Using the integrability condition \( \frac{\partial^2 \omega}{\partial \phi \partial \psi} = \frac{\partial^2 \omega}{\partial \psi \partial \phi} \), the energy \( h \) is eliminated from the linear momentum equations. Along with the vorticity equation and Gauss formula, we obtain the following:
Theorem 3.2. When the streamlines, \( \psi(x,y) = \text{constant} \), of a steady, plane, incompressible, second grade fluid flow constitute a set of coordinate lines in a curvi-linear coordinate system \( \phi, \psi \) in the physical plane, then the flow is governed by the underdetermined system:

\[
W \Delta_2 \omega \mp Re \frac{\partial \omega}{\partial \phi} \pm We \frac{\partial}{\partial \phi} (\Delta_2 \omega) = 0 \quad \text{(integrability)}
\]

\[
\omega = \frac{1}{W^2} (GT_{11}^2 - 2FT_{12}^2 + ET_{22}^2) \quad \text{(vorticity)} \tag{3.9}
\]

\[
\frac{\partial}{\partial \psi} \left( \frac{W}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left( \frac{W}{E} \Gamma_{12}^2 \right) = 0 \quad \text{(Gauss)}
\]

of three equations for \( E, F, G, \omega \) as functions of \( \phi, \psi \). The upper and lower signs in the integrability equation are taken if \( J \) is positive and negative, respectively.
3.4 EQUATIONS IN VON MISES COORDINATES.

The system (3.9) is underdetermined because of the arbitrariness inherent in the choice of the coordinate curves \( \phi(x, y) = \text{constant} \). The system (3.9) can be made determinate in a number of ways. In this dissertation, the system shall be made determinate by setting \( \phi(x, y) = x = \text{constant} \), so that the curvilinear coordinate net is the von Mises (1927) net \( x, \psi \).

We assume that the family of curves

\[
\frac{y - f(x)}{g(x)} = \text{constant}
\]

is a permissible streamline pattern, where \( f(x) \) and \( g(x) \neq 0 \) are continuously differentiable functions. Since \( \psi(x, y) = \text{constant} \) as well along these curves, it follows that there exists some function \( H(\psi) \) such that

\[
y - f(x) = g(x)H(\psi), \quad H'(\psi) \neq 0
\]

where \( H'(\psi) \) is the derivative of \( H(\psi) \) with respect to \( \psi \). This equation gives a unique \( y \) for every \( x \) on each individual streamline. In the von Mises coordinates \( x, \psi \), equations (2.10) and (2.12), which give the coefficients of the squared differential element of arc length and the Jacobian of the transformation, become

\[
E = 1 + \left[ f'(x) + g'(x)H(\psi) \right]^2, \quad F = \left[ f'(x) + g'(x)H(\psi) \right]g(x)H'(\psi) \\
G = g^2(x)H'^2(\psi), \quad J = W = g(x)H'(\psi)
\]

so that \( E = 1 + \frac{F^2}{G} \) when the fluid is assumed to be flowing in the direction of increasing \( x \) along the streamlines. Employing equations (3.12) in system (3.9), we note that the Gauss formula is identically satisfied, and we get:

**Theorem 3.3.** If the curves \( \frac{y - f(x)}{g(x)} = \text{constant} \), \( g(x) \neq 0 \), constitute a streamline pattern for a steady, plane, isochoric motion of a second grade fluid, then the flow must satisfy:

\[
\omega = \frac{1}{g^2(x)H'(\psi)} \left[ g(x)f''(x) - 2f'(x)g'(x) + [g(x)g''(x) - 2g'^2(x)]H(\psi) \right]
\]
\[
+ [1 + f'^2(x)] \left\{ \frac{H''(\psi)}{H^2(\psi)} + 2f'(x)g'(x) \frac{H(\psi)H''(\psi)}{H^2(\psi)} + \frac{g'^2(x)H^2(\psi)H''(\psi)}{H^2(\psi)} \right\} \tag{3.13}
\]

and
\[
g(x)H'(\psi) \frac{\partial^2 \omega}{\partial x^2} - 2[f'(x) + g'(x)H(\psi)] \frac{\partial^2 \omega}{\partial x \partial \psi} + \frac{1}{g(x)H'(\psi)} \left[ 1 + f'^2(x) \right.
\]
\[
+ 2f'(x)g'(x)H(\psi) + g'^2(\psi)H^2(\psi)] \frac{\partial^2 \omega}{\partial \psi^2} + \frac{1}{g(x)H'(\psi)} \left[ 2f'(x)g'(x) - g(x)f''(x) \right.
\]
\[
+ [2g'^2(x) - g(x)g''(x)] H(\psi) - [1 + f'^2(x)] \frac{H''(\psi)}{H^2(\psi)} - 2f'(x)g'(x) \frac{H(\psi)H''(\psi)}{H^2(\psi)}
\]
\[
- g'^2(x) \frac{H^2(\psi)H''(\psi)}{H^2(\psi)} \frac{\partial \omega}{\partial \psi} - Re \frac{\partial \omega}{\partial x} + \frac{W}{g(x)H'(\psi)} \left[ g(x)H'(\psi) \frac{\partial^2 \omega}{\partial x^2} \right.
\]
\[
- 2[f'(x) + g'(x)H(\psi)] \frac{\partial^3 \omega}{\partial x^2 \partial \psi} + \frac{1}{g(x)H'(\psi)} \left[ 1 + f'^2(x) + 2f'(x)g'(x)H(\psi) \right.
\]
\[
+ g'^2(x)H^2(\psi)] \frac{\partial^3 \omega}{\partial x^2 \partial \psi} + \frac{1}{g(x)} \left[ 4f'(x)g'(x) - 3g(x)f''(x) + [4g'^2(x) \right.
\]
\[
- 3g(x)g''(x)] H(\psi) - [1 + f'^2(x)] \frac{H''(\psi)}{H^2(\psi)} - 2f'(x)g'(x) \frac{H(\psi)H''(\psi)}{H^2(\psi)}
\]
\[
- g'^2(x) \frac{H^2(\psi)H''(\psi)}{H^2(\psi)} \frac{\partial \omega}{\partial \psi} + \frac{2}{g^2(x)H'(\psi)} \left[ g(x)f'(x)f''(x) - g'(x)[1 + f'^2(x)] \right.
\]
\[
+ [g(x)g'(x)f''(x) + g(x)f'(x)g''(x) - 2f'(x)g'^2(x)] H(\psi) + g'(x)[g(x)g''(x)
\]
\[
- g'^2(x)] H^2(\psi)] \frac{\partial^2 \omega}{\partial \psi^2} + \frac{1}{g^2(x)} \left[ 3g'(x)f''(x) + 2f'(x)g''(x) - 4f'(x)g'^2(x) \right.
\]
\[
- g'^2(x)f''''(x) + [5g(x)g'(x)g''(x) - 4g'^2(x) - g^2(x)g''(x)] H(\psi) + 2[g'(x)
\]
\[
+ g'(x)f'^2(x) - g(x)f'(x)f''(x)] H''(\psi) \frac{H'(\psi)}{H^2(\psi)} + 2[2f'(x)g'^2(x) - g(x)f'(x)g''(x)
\]
\[
- g(x)g'(x)f''(x)] \frac{H(\psi)H''(\psi)}{H^2(\psi)} + 2g'(x)[g'^2(x)
\]
\[
- g(x)g''(x)] \frac{H^2(\psi)H''(\psi)}{H^2(\psi)} \frac{\partial \omega}{\partial \psi} \right\} = 0 \tag{3.14}
\]

where \(H(\psi)\) is such that \(H'(\psi) \neq 0\).

It is germane to point out that by interchanging the roles of \(x\) and \(y\) in the foregoing analysis, second grade fluid flow along the family of curves
\[
\frac{x - f(y)}{g(y)} = \text{constant} \tag{3.15}
\]
can also be studied. Assuming that the curves (3.17) constitute a streamline pattern, then there exists some function $H(\psi)$ such that

$$x - f(y) = g(y)H(\psi), \quad H'(\psi) \neq 0$$  (3.16)

In this case, (3.16) defines a unique $x$ for every $y$ on each individual streamline so that $x = z(y, \psi)$, and $\phi(x, y) = y = \text{constant}$.

The above approach utilizes rectangular Cartesian coordinates $x, y$. Application of other coordinates is at times more advantageous for the method outlined above. If, for instance, an equation of the form

$$L(x, y) = H(\psi), \quad H'(\psi) \neq 0$$  (3.17)

(where $L$ is a known function of $x, y$) is transformed to polar coordinates, three possibilities exist. The first possibility yields for every $r$ a unique $\theta$ on each individual streamline so that $\theta = \theta(r, \psi)$. The family of curves $\phi(x, y) = \text{constant}$ in system (3.9) is chosen to be the curves $r = \text{constant}$. For this $r, \psi$-net, equations (2.10) and (2.12), with $x = r \cos \theta, y = r \sin \theta$, give

$$E = 1 + r^2 \left(\frac{\partial \theta}{\partial r}\right)^2, \quad F = r^2 \frac{\partial \theta}{\partial r} \frac{\partial \theta}{\partial \psi}$$
$$G = r^2 \left(\frac{\partial \theta}{\partial \psi}\right)^2, \quad J = W = \frac{\partial (x, y)}{\partial (r, \psi)} = r \frac{\partial \theta}{\partial \psi}$$  (3.18)

so that $E = 1 + \frac{F^2}{G}$, where the fluid is assumed to be flowing in the direction of increasing $r$ along the streamlines. The second possibility yields for every $\theta$ a unique $r$ on each individual streamline so that $r = r(\theta, \psi)$. The family of curves $\phi(x, y) = \text{constant}$ in system (3.9) is chosen to be the curves $\theta = \text{constant}$. For this $\theta, \psi$-net, we get

$$E = r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2, \quad F = \frac{\partial r}{\partial \theta} \frac{\partial r}{\partial \psi}$$
$$G = \left(\frac{\partial r}{\partial \psi}\right)^2, \quad J = W = \frac{\partial (x, y)}{\partial (\theta, \psi)} = -r \frac{\partial r}{\partial \psi}$$  (3.19)
which yield $E = r^2 + \frac{F^2}{\sigma}$ if the fluid flows along a streamline in the direction of $\theta$ increasing. The third possibility is one for which the transformed equation is such that neither for every $r$ we have a unique $\theta$ nor for every $\theta$ we have a unique $r$ on each individual streamline. When this possibility is encountered, we proceed as in the case involving the Cartesian coordinates.
3.5 EXACT SOLUTIONS.

An exact solution (or integral) of the equations of motion of a given viscoelastic fluid will be defined for both unbounded and bounded domains:

(i) An exact solution of the equations of a second grade fluid flow, in dimensionless variables, is a set of functions for the velocity components and the fluid pressure if the equations are satisfied by these functions for all values of the Reynolds number $Re$, Weissenberg number $We$, and the ratio of the normal stress moduli $\gamma$;

(ii) An exact integral of a second grade fluid flow, in dimensionless variables, is a set of functions for the velocity components and the pressure if these functions satisfy the flow equations, and the boundary conditions of a realistically imposed physical problem for all values of the Reynolds number $Re$, Weissenberg number $We$, and the ratio of the normal stress moduli $\gamma$.

We now study various flow configurations by employing (3.13) in (3.14) for chosen forms of $f(x)$ and $g(x)$.

Example 1.

We consider that the fluid is flowing along the family of curves $y - m_1 x^2 - m_2 x = \text{constant}$. Since $\psi(x, y) = \text{constant}$ on these curves as well, it follows that there exists some function $H(\psi)$ such that

$$ y - m_1 x^2 - m_2 x = H(\psi), \quad H'(\psi) \neq 0 $$

(3.20)

where $m_1, m_2$ are real constants. Comparing (3.20) with (3.11), we get $f(x) = m_1 x^2 + m_2 x$, $g(x) = 1$. Equations (3.13) and (3.14) become

$$ \omega = \frac{1}{H'(\psi)} \left\{ 2m_1 + [1 + (2m_1 x + m_2)^2] \frac{H''(\psi)}{H'^2(\psi)} \right\} $$

(3.21)
and

\[
H'(\psi) \frac{\partial^2 \omega}{\partial x^2} - (2m_1 x + m_2) \frac{\partial^2 \omega}{\partial x \partial \psi} + \left[1 + (2m_1 x + m_2)^2\right] \frac{1}{H'(\psi)} \frac{\partial^2 \omega}{\partial \psi^2} - \left[2m_1 + \left[1 + (2m_1 x + m_2)^2\right] \frac{H''(\psi)}{H'^2(\psi)} \frac{\partial \omega}{\partial x \partial \psi} - \frac{W e}{H'(\psi)} \frac{\partial^3 \omega}{\partial x \partial \psi^2} - \frac{1}{H'(\psi)} \frac{\partial^2 \omega}{\partial x \partial \psi^2} - \left[6m_1 + \left[1 + (2m_1 x + m_2)^2\right] \frac{H''(\psi)}{H'^2(\psi)} \frac{\partial^2 \omega}{\partial x^2 \partial \psi} + 4m_1 (2m_1 x + m_2) \frac{1}{H'(\psi)} \frac{\partial^2 \omega}{\partial \psi^2} - 4m_1 (2m_1 x + m_2) \frac{H''(\psi)}{H'^2(\psi)} \frac{\partial \omega}{\partial \psi}\right] = 0
\]

(3.22)

Using (3.21) to eliminate \(\omega\) from (3.22) yields

\[
\left\{12m_1^2 \frac{H''(\psi)}{H'^2(\psi)} - 4m_1 \left(\frac{H''(\psi)}{H'^3(\psi)}\right)' + \left[\frac{1}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right)\right]'\right\} - \frac{4m_1}{H'(\psi)} \left\{Re \frac{H''(\psi)}{H'^2(\psi)}\right\} + 12m_1 W e \left(\frac{H''(\psi)}{H'^3(\psi)}\right)' - 2W e \left[\frac{1}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right)'\right] (2m_1 x + m_2)
\]

\[+ 8m_1 W e \left[\frac{1}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right)'\right] (2m_1 x + m_2)^2
\]

\[+ \left[\frac{1}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right)'\right] (2m_1 x + m_2)^3 \equiv \sum_{n=0}^{4} A_n(\psi)(2m_1 x + m_2)^n = 0
\]

(3.23)

Since \(x, \psi\) are independent variables and \((2m_1 x + m_2)^n : n = 0, 1, 2, 3, 4\) is a linearly independent set, it follows that \(A_n(\psi) = 0\) for \(n = 0, 1, 2, 3, 4\). \(A_4(\psi) = 0\) gives

\[
\frac{H''(\psi)}{H'^3(\psi)} = a_1 H + a_2
\]

(3.24)

where \(a_1\) and \(a_2\) are arbitrary constants. \(A_4(\psi) = 0\) is identically satisfied. \(A_n(\psi) = \)
0 for $n = 0, 1, 2$ become

$$
[a_1(3m_1H - 1) + 3m_1a_2]H'(\psi) = 0
$$

$$
a_1[12m_1We - ReH(\psi)] - a_2Re = 0
$$

$$
a_1H'(\psi) = 0 \quad (3.25)
$$

Since $m_1 \neq 0$, these equations give $a_1 = a_2 = 0$.

Thus, the five equations $A_n(\psi) = 0$ for $n = 0, 1, 2, 3, 4$ hold true simultaneously only if $m_1 \neq 0$ and $H''(\psi) = 0$. Therefore, (3.20) takes the form

$$
y - m_1x^2 - m_2x = c_1\psi + c_2 \quad (3.26)
$$

where $c_1 \neq 0$, $c_2$ are arbitrary real constants. We then use (2.30) to find the velocity components $u$, $v$, and employ the latter in system (3.5) to find the vorticity $\omega$ and subsequently find $h$ from the linear momentum equations. The second of equations (3.4) is finally used to find the pressure $p$.

We, thus, have that if $m_1 \neq 0$, $m_2$ are real constants, then a family of curves $y - m_1x^2 - m_2x = \text{constant}$ can form a streamline pattern for system (3.1), and the exact integral for this rotational flow in an unbounded domain is

$$
u = \frac{1}{c_1}, \quad v = \frac{1}{c_1}(2m_1x + m_2), \quad p = p_0 - \frac{1}{2c_1^2}[1 - 4m_1^2(3 + 2\gamma)\frac{We}{Re} + 4m_1y] \quad (3.27)
$$

where $p_0$ is an arbitrary constant.

Imposing the boundary conditions

$$
u(0, y) = u_0, \quad v(0, y) = v_0; \quad u_0 \neq 0 \quad (3.28)
$$

on a plate situated along $x = 0$ so that the fluid occupies the region $x < 0$, we get the solution

$$
u = u_0, \quad v = 2m_1u_0x + v_0, \quad p = p_0 - \frac{u_0^2}{2}[1 - 4m_1^2(3 + 2\gamma)\frac{We}{Re} + 4m_1y] \quad (3.29)
$$
where \( m_1 \neq 0 \) is an arbitrary real number, \( c_1 = \frac{1}{u_0} \), \( m_2 = \frac{u_0}{u_0} \). There is uniform suction \((u_0 > 0)\) or blowing \((u_0 < 0)\) at the plate \( x = 0 \), which is stretching at a uniform rate. If \( u_0 = 0 \) (or \( m_2 = 0 \)), there is no stretching of the plate.

From equations (3.6) and (3.7), the normal stresses \( T_{11}, T_{22} \) and the shear stress \( T_{12} \) at the plate \( x = 0 \) are

\[
T_{11} = -p_0 + \frac{u_0^2}{2} \left[ 1 + \frac{4m_1^2We}{Re} + 4m_1y \right]
\]
\[
T_{22} = -p_0 + \frac{u_0^2}{2} \left[ 1 - \frac{12m_1^2We}{Re} + 4m_1y \right]
\]
\[
T_{12} = \frac{2m_1u_0}{Re}
\]

(3.30)

The streamlines in the unbounded and bounded domains are, respectively, given in Figures 3.1 and 3.2.

**Example 2.**

Assuming the family of curves \( y - e^{mx} - m_1x^2 - m_2x = \) constant constitutes a family of streamlines, we have

\[
y - e^{mx} - m_1x^2 - m_2x = H(\psi), \quad H'(\psi) \neq 0
\]

(3.31)

where \( f(x) = e^{mx} + m_1x^2 + m_2x \), \( g(x) = 1 \) and \( m \neq 0 \), \( m_1 \), \( m_2 \) are real constants.

Substituting \( f(x) = e^{mx} + m_1x^2 + m_2x \), \( g(x) = 1 \) into (3.21) and (3.22), and following the procedure of Example 1, we get \( H''(\psi) = 0 \) with

\[
H'(\psi) = \frac{Re - m^2We}{m} \neq 0
\]

so that

\[
y - e^{mx} - m_1x^2 - m_2x = \left( \frac{Re - m^2We}{m} \right) \psi + c
\]

(3.32)

where \( c \) is an arbitrary constant. The exact integral for this rotational flow in an
unbounded domain is
\[ u = \frac{m}{Re - m^2 We}, \quad v = \frac{m}{Re - m^2 We} \left( m e^{m x} + 2m_1 x + m_2 \right), \]
\[ p = p_0 + \frac{m^2}{2Re(Re - m^2 We)} \left[ 2We(2 + \gamma)(m^2 e^{m x} + 2m_1)^2 - 4m_1 Rey \right] - \left( Re + 2m_1^2 We \right) \]  \hspace{1cm} (3.33)
where \( p_0 \) is an arbitrary constant.

The boundary conditions specified on a plate located at \( x = 0 \) are
\[ u(0, y) = u_0, \quad v(0, y) = v_0; \quad u_0 \neq 0 \]  \hspace{1cm} (3.34)
Thus, \( m, m_2 \) are given by
\[ u_0 We m^2 + m - Re = 0, \quad m_2 = \frac{v_0}{u_0} - m \]  \hspace{1cm} (3.35)
\( u_0 > 0 \) and \( u_0 < 0 \) correspond, respectively, to uniform suction and blowing at the plate. The flow takes place in the region \( x < 0 \).

Therefore, if \( m, m_2 \) are given by (3.35), and \( m_1 \) is an arbitrary real number, then the exact integral of the steady, plane, second grade fluid which undergoes isochoric motion along the streamlines \( y - e^{m x} - m_1 x^2 - m_2 x = \) constant when a plate is situated at \( x = 0 \) is
\[ u = u_0, \quad v = m u_0 \left( e^{m x} - 1 \right) + 2m_1 u_0 x + v_0, \]
\[ p = p_0 + \frac{u_0^2}{2Re} \left[ 2We(2 + \gamma)(m^2 e^{m x} + 2m_1)^2 - 4m_1 Rey - (Re + 2m_1^2 We) \right] \]  \hspace{1cm} (3.36)
when the boundary conditions on the plate are given by (3.34). If \( v_0 = 0 \) (or \( m_2 = -m \)), the velocity profile in (3.36) is attained asymptotically, and may be regarded as the asymptotic suction profile [cf. Schlichting (1968)].

The normal stresses \( T_{11}, T_{22} \) and the shear stress \( T_{12} \) at the plate \( x = 0 \), from (3.6) and (3.7), are
\[ T_{11} = -p_0 + \frac{u_0^2}{2Re} \left[ 4m_1 Rey + (Re + 2m_1^2 We) \right] \]
\[ T_{22} = -p_0 + \frac{u_0^2}{2Re} \left[ 4m_1 Rey + (Re + 2m_1^2 We) - 4(m^2 + 2m_1)We \right] \]  \hspace{1cm} (3.37)
\[ T_{12} = \frac{u_0}{Re} \left[ (m^2 + 2m_1) + m^3 u_0 We \right] \]
The streamline patterns in the unbounded domain and for the boundary value problem are shown in Figures 3.3 and 3.4, respectively.

If $m_1 = 0$ or $m_1 = m_2 = 0$ in the foregoing, the solutions obtained still hold true. In these cases, the families of curves $y - e^{m_1 x} - m_1 x = \text{constant}$ and $y - e^{m_2 x} = \text{constant}$ form streamline patterns for system (3.1).

**Example 3.**

If $y - e^{m_1 x} - e^{n_1} - m_1 x^2 - m_2 x = \text{constant}$ denotes a family of streamlines, then

$$y - e^{m_1 x} - e^{n_1} - m_1 x^2 - m_2 x = H(\psi), \quad H'(\psi) \neq 0$$  \hspace{1cm} (3.38)

where $m \neq 0$, $n \neq 0$, $m_1$ and $m_2$ are real constants. $m$ and $n$ satisfy the relation $mn = -\frac{R_\infty}{W_e}$. $f(x) = e^{m_1 x} + e^{n_1} + m_1 x^2 + m_2 x$, $g(x) = 1$ in (3.21) and (3.22), and proceeding as in Example 1, we get $H''(\psi) = 0$ with

$$H'(\psi) = -(m + n)W_e \neq 0$$

Equation (3.38) takes the form

$$y - e^{m_1 x} - e^{n_1} - m_1 x^2 - m_2 x = -(m + n)W_e \psi + c$$  \hspace{1cm} (3.39)

where $c$ is an arbitrary constant. The exact solution for this rotational flow in an unbounded domain is

$$u = -\frac{1}{(m + n)W_e}, \quad v = -\frac{1}{(m + n)W_e} \left( me^{m_1 x} + ne^{n_1} + 2m_1 x + m_2 \right)$$

$$p = p_0 - \frac{1}{2mn(m + n)^2W_e^2} \left[ mn(1 + 4m_1 y) + 2(2 + \gamma)(m_1 e^{m_1 x} + n_1 e^{n_1} + 2m_1)^2 \right]$$  \hspace{1cm} (3.40)

where $p_0$ is an arbitrary constant.

These flows were obtained by Kaloni and Huschilt (1984) by using the inverse method in the extension of Jeffery's (1915) problems to second grade fluid flow.
The boundary conditions

\[ u(0, y) = u_0, \quad v(0, y) = v_0; \quad u_0 \neq 0 \] (3.41)

imposed on the plate located at \( z = 0 \) give

\[ m + n = -\frac{1}{Weu_0}, \quad m_2 = \frac{v_0}{u_0} - (m + n) \] (3.42)

where the fluid occupies the left half plane \( z < 0 \).

Therefore, the exact integral of a steady, plane, incompressible, second grade fluid flow along the streamlines \( y - e^{m_1x} - e^{m_2x} = \text{constant} \) when the fluid impinges on a porous plate with uniform stretching situated at \( z = 0 \) is

\[ u = u_0, \quad v = v_0 \left( m e^{m_1x} + n e^{m_2x} + 2m_1x - m - n \right) + v_0 \]

\[ p = p_0 - \frac{u_0^2}{2mn} \left[ mn(1 + 4m_1y) + 2(2 + \gamma)(m^2 e^{m_1x} + n^2 e^{m_2x} + 2m_1)^2 \right] \] (3.43)

where \( m_1 \) is an arbitrary constant, \( m, n, m_2 \) are given by (3.42) and \( mn = -\frac{Re}{We} \).

The boundary conditions at the plate are as shown in (3.41).

\[ v_0 = 0 \text{ (or } m_2 = -(m + n)\text{)} \] corresponds to a flow impinging on a non-stretching plate with uniform suction or blowing at \( z = 0 \).

The normal and shear stresses at the plate \( z = 0 \) are

\[ T_{11} = -p_0 + \frac{u_0^2}{2} \left( 1 + 4m_1y \right) \]

\[ T_{22} = -p_0 + \frac{u_0^2}{2mn} \left[ mn(1 + 4m_1y) + 4(m^2 + n^2 + 2m_1)^2 \right] \]

\[ T_{12} = -\frac{u_0^2}{mnWe} \left[ (m^2 + n^2 + 2m_1) - u_0 We(m^3 + n^3) \right] \] (3.44)

Figures 3.5 and 3.6, respectively, give the flow patterns in the unbounded domain and for the boundary value problem.

Example 4.

Let \( x(y - m_1x - m_2) = \text{constant} \) be a family of streamlines so that \( \psi(x, y) = \text{constant} \) along these curves as well. It follows that there exists some function \( H(\psi) \) such that

\[ x(y - m_1x - m_2) = H(\psi), \quad H'(\psi) \neq 0 \] (3.45)
where \( m_1, m_2 \) are real constants. We employ \( f(x) = m_1 x + m_2, g(x) = \frac{1}{x} \) in (3.21) and (3.22), and apply the resulting form of the vorticity equation to that of the integrability equation, to get

\[
\left(1 - \frac{4 W e}{H'(\psi)}\right) \left\{ \frac{6 H^2(\psi) H''(\psi)}{H'^3(\psi)} - 6 H(\psi) \left(\frac{H^2(\psi) H''(\psi)}{H'^3(\psi)}\right)' + \left[ \frac{H^2(\psi)}{H'^3(\psi)} \left(\frac{H^2(\psi) H''(\psi)}{H'^3(\psi)}\right)' \right]' \right\} + 2 \left\{ \text{Re} \frac{H^2(\psi) H''(\psi)}{H'^3(\psi)} \right\}'
\]

\[
+ m_1 \left(1 - \frac{2 W e}{H'(\psi)}\right) \left[ \frac{2 H(\psi) H''(\psi)}{H'^2(\psi)} + \left(\frac{H'(\psi)}{H'^2(\psi)}\right)' \right] + 2 H(\psi) \left(\frac{H(\psi) H''(\psi)}{H'^3(\psi)}\right)'
\]

\[
- \left[ \frac{H(\psi)}{H'(\psi)} \left(\frac{H^2(\psi) H''(\psi)}{H'^3(\psi)}\right)' \right]' - \left\{ \frac{1}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right) \right\}' \right]\}
\]

\[
x^2 + \left\{ \left(1 + m_1^2\right) \left[ \frac{2 H''(\psi)}{H'^3(\psi)} + 2 H(\psi) \left(\frac{H''(\psi)}{H'^3(\psi)}\right)' + \left[ \frac{1}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right) \right]' \right] \right\}'
\]

\[
+ \left[ \frac{H^2(\psi)}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right)' \right]' + 4 m_1^2 \left[ \left(\frac{H(\psi) H''(\psi)}{H'^3(\psi)}\right)' \right]
\]

\[
+ \left[ \frac{H(\psi)}{H'(\psi)} \left(\frac{H(\psi) H''(\psi)}{H'^3(\psi)}\right)' \right]' \right\} x^4 - 2 \left(1 + m_1^2\right) \left\{ \text{Re} \frac{H''(\psi)}{H'^3(\psi)} \right\} \}
\]

\[
+ m_1 \left(1 + \frac{2 W e}{H'(\psi)}\right) \left[ 3 \left(\frac{H''(\psi)}{H'^3(\psi)}\right)' + \left[ \frac{1}{H'(\psi)} \left(\frac{H(\psi) H''(\psi)}{H'^3(\psi)}\right)' \right]' \right]
\]

\[
+ \left[ \frac{H(\psi)}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right)' \right]' \right\} x^6 + (1 + m_1^2)^2 \left\{ \left(1 + \frac{4 W e}{H'(\psi)}\right) \left[ \frac{1}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right)' \right]' \right\} x^8
\]

\[
\equiv \sum_{n=0}^{4} A_{2n}(\psi) x^{2n} = 0 \quad (3.46)
\]

Since \( x, \phi \) are independent variables and \( \{1, x^2, x^4, x^6, x^8\} \) is a linearly independent set, it follows that \( A_{2n}(\psi) = 0 \) for \( n = 0, 1, 2, 3, 4 \). In particular, \( A_8(\psi) = 0 \) gives either

\[
H'(\psi) = -4 W e \quad (3.47)
\]

or

\[
\frac{H''(\psi)}{H'^3(\psi)} = a_1 H(\psi) + a_2 \quad (3.48)
\]
where $a_1$, $a_2$ are arbitrary constants.

When (3.47) is introduced into $A_{2n}(\psi) = 0$ for $n = 0,1,2,3,4$, we note that these are identically satisfied. Taking the case when (3.48) holds true, we have that $A_{2n}(\psi) = 0$ for $n = 0,4$ are identically satisfied, and $A_{2n}(\psi) = 0$ for $n = 1,2,3$ give

\[ Re[a_1 H(\psi) + a_2] - 6m_1 a_1 H'(\psi) + 12Wem_1 a_1 = 0 \]
\[ [3a_1 H(\psi) + a_2] H'(\psi) = 0 \] (3.49)

\[ Re[a_1 H(\psi) + a_2] + 6m_1 a_1 H'(\psi) + 12Wem_1 a_1 = 0 \]

or only if $3a_1 H(\psi) + a_2 = 0$, $m_1 a_1 = 0$
or only if $a_1 = a_2 = 0$.

Therefore, (3.45) takes the form

\[ x(y - m_1 x - m_2) = c_1 \psi + c_2 \] (3.50)

where $c_1 \neq 0$, $c_2$ are arbitrary constants. $c_1$ may also take on the value $-4Wc$.

Summing up, we have that the family of curves $x(y - m_1 x - m_2) = \text{constant}$ can form a streamline pattern for system (3.1), and the exact solution for this rotational flow in an unbounded domain is

\[ u = \frac{x}{c_1}, \quad v = \frac{1}{c_1} (2m_1 x - y + m_2) \]
\[ p = p_0 + \frac{1}{2Re^2} \left[ 4Wc(3 + 2\gamma)(1 + m_1^2) + Re(2m_2 y - y^2 - x^2) \right] \] (3.51)

where $p_0$ is an arbitrary constant. This flow represents an impingement of two constant-vorticity jets with stagnation point $(0,m_2)$.

We impose the boundary conditions

\[ u(0,y) = 0, \quad v(0,y) = v_0 y + v_1; \quad v_0 \neq 0 \] (3.52)

along a plate situated at $z = 0$. The flow is confined to the right half plane $x > 0$.

The boundary conditions imply that $c_1$ and $m_2$ are given by

\[ v_0 = -\frac{1}{c_1}, \quad v_1 = -m_2 v_0 \] (3.53)
Thus, if \( m_1 \) is an arbitrary real constant, and \( m_2 \) is given by (3.53), then the exact integral of the steady, plane, incompressible, second grade fluid along the streamlines \( x(y - m_1 x - m_2) = \text{constant} \), when the flow impinges on a non-porous plate situated at \( x = 0 \), which stretches linearly, is

\[
u = -v_0 x, \quad v = v_0 (y - 2m_1 x) + v_1 \\
p = p_0 + \frac{v_0}{2Re} \left\{ 4W e v_0 (3 + 2\gamma)(1 + m_1^2) - Re \left[ v_0 (x^2 + y^2) + 2v_1 y \right] \right\} \tag{3.54}
\]

where the specified boundary conditions at the plate are given by (3.52). A similar non-orthogonal stagnation-point flow, when the plate is \( y = 0 \), was studied by Dorrepaal (1986) for the Navier-Stokes equations.

The normal stresses \( T_{11}, T_{22} \) and the shear stress \( T_{12} \) at the plate \( x = 0 \) are

\[
T_{11} = -p_0 + \frac{v_0}{2Re} \left[ Re(v_0 y + 2v_1) + 4W e v_0 (m_1^2 - 1) - 4 \right] \\
T_{22} = -p_0 + \frac{v_0}{2Re} \left[ Re(v_0 y + 2v_1) - 4W e v_0 (3m_1^2 + 1) + 4 \right] \tag{3.55} \\
T_{12} = -\frac{2m_1 v_0}{Re} (1 + We v_0)
\]

For flows in the unbounded and bounded domains, the streamlines are given in Figures 3.7 and 3.8, respectively.

**Example 5.**

Letting \( y(1 + m_1 e^{m_2}) - m_2 e^{m_2} = \text{constant} \) denote a streamline pattern, we have

\[
y(1 + m_1 e^{m_2}) - m_2 e^{m_2} = H(\psi), \quad H'(\psi) \neq 0 \tag{3.56}
\]

where \( m \neq 0, m_1 \neq 0 \) and \( m_2 \) are real numbers. Taking \( f(x) = \frac{m_2 e^{m_2}}{1 + m_1 e^{m_2}} \), \( g(x) = \frac{1}{1 + m_1 e^{m_2}} \), and employing the method of Example 4, we obtain \( H''(\psi) = 0 \) with

\[
H'(\psi) = \frac{Re - m^2 We}{m} \neq 0.
\]

Equation (3.56) becomes

\[
y(1 + m_1 e^{m_2}) - m_2 e^{m_2} = \left( \frac{Re - m^2 We}{m} \right) \psi + c \tag{3.57}
\]
The exact solution for this rotational flow in an unbounded domain is

\[
u = \frac{m}{Re - m^2 We} \left(1 + m_1 e^{m_2 x}\right), \quad v = \frac{m^2}{Re - m^2 We} \left(m_2 - m_1 y\right) e^{m_2 x}
\]

\[
p = p_0 + \frac{m^2}{2Re(Re - m^2 We)^2} \left\{ \left[ 2m^2 We \left(7m_1^2 + 2m^2(m_2 - m_1 y)^2\right) \right] - m_1^2 Re \right\} e^{m_2 x} - Re \right\}
\]

(3.58)

Kaloni and Huschilt (1984) obtained these flows when they studied Riabouchinsky (1924)-type flows using the inverse method.

Specifying the boundary conditions

\[
u(0, y) = u_0, \quad v(0, y) = v_0 y + v_1; \quad v_0 \neq 0
\]

(3.59)

at the plate \(x = 0\), we get

\[
u_0 = \frac{m(1 + m_1)}{Re - m^2 We}, \quad m = \frac{v_0(1 + m_1)}{m_1 u_0}, \quad m_2 = \frac{m_1 v_1}{v_0}
\]

(3.60)

where the flow occurs in the region \(x < 0\).

Let \(m \neq 0, m_2\) (given by (3.63)), and arbitrary \(m_1 \neq 0, -1\) be real constants. Then the exact solution of the steady, plane, incompressible, second grade fluid flow along the streamlines \(y(1 + m_1 e^{m_2 x}) - m_2 e^{m_2 x} = \text{constant}\) when the fluid impinges on a stretching porous plate located at \(x = 0\) is

\[
u = \frac{u_0}{1 + m_1} \left(1 + m_1 e^{m_2 x}\right), \quad v = (v_0 y + v_1) e^{m_2 x}
\]

\[
p = p_0 + \frac{1}{2m^2 Re} \left\{ \left[ 2m^2 We \left(7v_0^2 + 2m^2(v_0 y + v_1)^2\right) \right] - Re v_0^2 \right\} e^{m_2 x} - \frac{Re v_0^2}{m_1^2}
\]

(3.61)

where the boundary conditions at the plate are given by (3.59). The streamlines for the flows in an unbounded domain and for the boundary value problem are, respectively, Figures 3.9 and 3.10.
In the particular case of $m_1 = -1$, the boundary conditions at the non-porous plate $x = 0$ are

$$ u(0, y) = 0, \; v(0, y) = v_0 y + v_1; \; \; v_0 \neq 0 \quad (3.62) $$

and the exact integral for this case is

$$ u = \frac{v_0}{m} (1 - e^{mx}), \; v = (v_0 y + v_1) e^{mx} $$

$$ p = p_0 + \frac{1}{2m^2 Re} \left\{ 2m^2 W e \left[ 7v_0^2 + 2m^2 (v_0 y + v_1)^2 \right] + 2\gamma m^2 W e \left[ 4v_0^2 + m^2 (v_0 y + v_1)^2 \right] - Re v_0^2 \right\} e^{2mx} - Re v_0^2 \quad (3.63) $$

where $m_2 = \frac{v_0}{v_0}$ and $m \neq 0$ is given by

$$ m^2 (1 + W e v_0) = Re v_0. $$

For solution (3.61) in which $m_1 \neq 0, -1$, the normal and shear stresses at the plate $x = 0$ are

$$ T_{11} = -p_0 + \frac{1}{2m^2 Re} \left\{ \frac{Re v_0^2}{m_1^2} (1 + m_1^2) - 4m^2 v_0 - 2m^2 W e v_0 (3v_0 + 2mu_0) \right\} $$

$$ T_{22} = -p_0 + \frac{1}{2m^2 Re} \left\{ \frac{Re v_0^2}{m_1^2} (1 + m_1^2) + 4m^2 v_0 - 2m^2 W e \left[ v_0 (3v_0 - 2mu_0) \right] + 2m^2 (v_0 y + v_1)^2 \right\} $$

$$ T_{12} = \frac{m}{Re} \left[ 1 + W e (3v_0 + mu_0) \right] (v_0 y + v_1) \quad (3.64) $$

and for solution (3.63) in which $m_1 = -1$, the corresponding stress components are

$$ T_{11} = -p_0 + \frac{v_0}{m^2 Re} \left[ v_0 (Re - 3m^2 W e) - 2m^2 \right] $$

$$ T_{22} = -p_0 + \frac{1}{m^2 Re} \left[ v_0^2 (Re - 3m^2 W e) + 2m^2 v_0 - 2m^4 W e (v_0 y + v_1)^2 \right] $$

$$ T_{12} = \frac{m}{Re} (1 + 3W e v_0) (v_0 y + v_1) \quad (3.65) $$
Example 6.

We wish to show that second grade fluid cannot flow along the family of curves \( y - x^3 = \text{constant} \). We prove this result by assuming the contrary and arriving at a contradiction. We assume that the fluid is flowing along these curves. In keeping with this assumption, we have that \( \psi(x, y) = \text{constant on these curves as well, and so there exists some function } H(\psi) \text{ such that } \)

\[
y - x^3 = H(\psi), \quad H'(\psi) \neq 0
\]

(3.66)

By making appropriate identification with (3.11), we find that \( f(x) = x^3, g(x) = 1 \). Employing these in (3.13), and subsequently substituting the resulting expression in (3.14), we obtain

\[
\left\{ \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H^{13}(\psi)} \right) \right]' - \frac{6Re}{H'(\psi)} \frac{12We}{H'(\psi)} \left( \frac{H''(\psi)}{H^{13}(\psi)} \right)' + 12 \left\{ \frac{30We}{H'(\psi)} \frac{H''(\psi)}{H^{13}(\psi)} \right\} + \frac{180}{H^{13}(\psi)} \right\} x^2 + 36 \left\{ \frac{2Re}{H'(\psi)} \left( \frac{H''(\psi)}{H^{13}(\psi)} \right)' \right\} x^4
\]

\[
- \frac{Re}{H^{13}(\psi)} \right\} x^3 + 18 \left\{ \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H^{13}(\psi)} \right) \right]' - \frac{90We}{H'(\psi)} \left( \frac{H''(\psi)}{H^{13}(\psi)} \right)' \right\} x^4
\]

\[
- 324 \left\{ \left( \frac{H''(\psi)}{H^{13}(\psi)} \right)' \right\} x^5 + 648We \left\{ \frac{1}{H'(\psi)} \left( \frac{1}{H^{13}(\psi)} \right)' \right\} x^7
\]

\[
+ 81 \left\{ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H^{13}(\psi)} \right)' \right\} x^8 \equiv \sum_{n=0}^{8} A_n(\psi)x^n = 0
\]

(3.67)

Since \( x, \psi \) are independent variables and \( \{x^n : n = 0, 1, ..., 5, 7, 8 \} \) is a linearly independent set, it follows that (3.67) holds true only when the coefficients \( A_n(\psi) \) for \( n = 0, 1, ..., 5, 7, 8 \) vanish identically. Considering that \( A_2(\psi) = 0 \) holds true only if \( H''(\psi) = 0 \), we note that \( A_n(\psi) = 0 \) for \( n = 1, 2, ..., 5, 7, 8 \) are identically satisfied. However, with \( H''(\psi) = 0, A_0(\psi) = 0 \) as well only if \( Re = 0 \). Therefore,
the eight equations $A_n(\psi) = 0$ for $n = 0, 1, ..., 5, 7, 8$ are simultaneously satisfied only if $H''(\psi) = 0$ and $Re = 0$. This represents Stokes flow. Since $Re = 0$ in assumed second grade fluid motion is a contradiction, it follows that second grade fluid cannot flow along the family of curves $y - x^3 = \text{constant}$.

Employing the same procedure as in the above example, it can be shown that $y - x^n = \text{constant}$ when $n = 3, 4, 5, ...$; $y - \ln x = \text{constant}$; $y - \sin x = \text{constant}$; $ye^x = \text{constant}$ exemplify families of curves along which second grade fluid cannot flow.

**Example 7.**

We consider the family of curves

$$A(x^2 + y^2)\frac{y}{z} + \frac{1}{2} B \ln(x^2 + y^2) - \tan^{-1} \frac{y}{z} = \text{constant}$$

which is assumed to define a streamline pattern for system (3.1), where $A$, $B$ are known non-zero constants and $m$ is some real constant to be determined. Thus, there exists some function $H(\psi)$ such that

$$\theta - Arn - Blnr = H(\psi), \quad H'(\psi) \neq 0 \quad (3.68)$$

where polar coordinates $r, \theta$ have been introduced. Since $\theta = \theta(r, \psi)$, it follows that we may make system (3.9) determinate by replacing the arbitrary family of curves $\phi(x, y) = \text{constant}$ with the curves $\tau = \text{constant}$.

For this $r, \psi$-net, the squared differential element of arc length is

$$ds^2 = dr^2 + r^2 d\theta^2 \equiv E dr^2 + 2Fr dr d\psi + G d\psi^2$$

where

$$E = 1 + (Amr^m + B)^2, \quad F = r(Amr^m + B)H'(\psi)$$

$$G = r^2 H''(\psi), \quad J = W = rH'(\psi) \quad (3.69)$$
Here the fluid is assumed to flow along the streamlines in the direction of increasing $r$. Applying (3.69) to system (3.9), we note that the Gauss formula is identically satisfied, and we obtain

$$\omega = \frac{Am^2 r^{m-2}}{H'(\psi)} + \left[1 + \left(\frac{Amr^m + B}{r^2}\right)^2\right] \frac{H''(\psi)}{H'^3(\psi)}$$

(3.70)

and

$$rH'(\psi) \frac{\partial^2 \omega}{\partial r^2} - 2(Amr^m + B) \frac{\partial^2 \omega}{\partial r \partial \psi} + \left[1 + \left(\frac{Amr^m + B}{r^2}\right)^2\right] \frac{\partial^2 \omega}{\partial \psi^2} + H'(\psi) \frac{\partial \omega}{\partial r} = rH'(\psi) \frac{\partial^3 \omega}{\partial r \partial \psi^2} + \frac{2B}{r^2 H'(\psi)} - \left[1 + \left(\frac{Amr^m + B}{r^2}\right)^2\right] \frac{H''(\psi)}{H'^3(\psi)} \frac{\partial^2 \omega}{\partial r \partial \psi}$$

$$+ \left[\frac{Am(2 - 3m)r^{m-2}}{H'(\psi)} + \frac{2B}{r^2 H'(\psi)} - \left[1 + \left(\frac{Amr^m + B}{r^2}\right)^2\right] \frac{H''(\psi)}{H'^3(\psi)} \right] \frac{\partial^2 \omega}{\partial \psi^2} - \frac{1}{r^2} \frac{\partial \omega}{\partial r}$$

$$+ \left[2 \left[1 + \left(\frac{Amr^m + B}{r^3}\right)^2\right] \frac{H''(\psi)}{H'^3(\psi)} - 2Am^2 r^{m-3} (Amr^m + B) \frac{H''(\psi)}{H'^3(\psi)} \right] \frac{\partial \omega}{\partial \psi} \right\} = 0$$

(3.71)

We eliminate $\omega$ between (3.70) and (3.71) to get

$$\left(1 + B^2\right) \left\{ 4 \frac{H''(\psi)}{H'^2(\psi)} + 2Re \frac{H''(\psi)}{H'^3(\psi)} + 4B \left(\frac{H''(\psi)}{H'^3(\psi)}\right)' \right\}$$

$$+ \left(1 + B^2\right) \left[ \frac{1}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right)' \right]^r \left(1 + B^2\right) \left[ \frac{1}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right)' \right]_{r^3} - 4\left(1 + B^2\right) \frac{We}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right)$$

$$+ 4B \left(\frac{H''(\psi)}{H'^3(\psi)}\right)' + (1 + B^2) \left[ \frac{1}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right)' \right]_{r^5}$$

$$+ Am \left\{ (m - 2) \left[ m(m - 2) - \frac{mRe}{H'(\psi)} + 4B(m - 1) \frac{H''(\psi)}{H'^2(\psi)} \right] - 2BRe \frac{H''(\psi)}{H'^3(\psi)} \right\} r^{m-3}$$

$$- 2(1 + B^2) \left[ \frac{H''(\psi)}{H'^3(\psi)} \right]' + 4B(1 + B^2) \left[ \frac{1}{H'(\psi)} \left(\frac{H''(\psi)}{H'^3(\psi)}\right)' \right]_{r^m}$$
\begin{align*}
&+ Am(m-4) \frac{W e}{H'(\psi)} \left\{ (m-2) \left[ m(m-2) + 4B(m-1) \frac{H''(\psi)}{H'^{2}(\psi)} \right] \\
&- 2(1+3B^2) \frac{H''(\psi)}{H'^{2}(\psi)} \right\} r^{m-5} \\
&+ A^2 m^2 \left\{ (m-2)(5m-2) \frac{H''(\psi)}{H'^{2}(\psi)} - 2(m-1) \left[ \text{Re} \frac{H''(\psi)}{H'^{2}(\psi)} \right] \\
&+ 6B \frac{H''(\psi)}{H'^{2}(\psi)} \right\} + 2(1+3B^2) \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H'^{2}(\psi)} \right) \right] ^{'} r^{2m-3} \\
&+ 2A^2 m^2 (m-2) \frac{W e}{H'(\psi)} \left\{ (m-2)(7m+2) \frac{H''(\psi)}{H'^{2}(\psi)} \right\} r^{2m-5} \end{align*}

\begin{align*}
&- 12B(m-1) \frac{H'(\psi)}{H'^{2}(\psi)} + 2(1+3B^2) \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H'^{2}(\psi)} \right) \right\} r^{2m-5} \\
&- 2A^2 m^3 \left\{ (3m-2) \frac{H''(\psi)}{H'^{2}(\psi)} - 2B \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H'^{2}(\psi)} \right) \right\} r^{3m-3} \\
&- 4m^3 \frac{W e}{H'(\psi)} \left\{ (4m-1)^2 \frac{H''(\psi)}{H'^{2}(\psi)} - B(3m-4) \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H'^{2}(\psi)} \right) \right\} r^{3m-5} \end{align*}

\begin{align*}
+ A^4 m^4 \left\{ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H'^{2}(\psi)} \right) \right\} r^{4m-3} \\
+ 4A^4 m^4 (m-1) \frac{W e}{H'(\psi)} \left\{ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H'^{2}(\psi)} \right) \right\} r^{4m-5} \end{align*}

\begin{equation}
\equiv K_1(\psi)r^{-3} + K_2(\psi)r^{-5} + K_3(\psi)r^{m-3} + K_4(\psi)r^{m-5} + K_5(\psi)r^{2m-3} + K_6(\psi)r^{2m-5} + K_7(\psi)r^{3m-3} + K_8(\psi)r^{3m-5} + K_9(\psi)r^{4m-3} + K_{10}(\psi)r^{4m-5} = 0 \tag{3.72}
\end{equation}

Since \(\tau, \psi\) are independent variables, it follows that \(K_n(\psi) = 0; n = 1, 2, ..., 10\) for real values of \(m \neq 0, \pm \frac{1}{2}, \pm \frac{2}{3}, \pm 1, \pm 2\) in which case \(\{r^{-3}, r^{-5}, r^{m-3}, r^{m-5}, r^{2m-3}, r^{2m-5}, r^{3m-3}, r^{3m-5}, r^{4m-3}, r^{4m-5}\}\) is a linearly independent set. The ten equations \(K_n(\psi) = 0; n = 1, 2, ..., 10\) hold true simultaneously only if

\begin{equation}
H(\psi) = \frac{\text{Re}}{2} \psi + c; \quad m = 4
\end{equation}
where $c$ is an arbitrary constant.

Equation (3.68) becomes

$$
\theta = Ar^4 + B \ln r + \frac{Re}{2} \psi + c
$$

(3.73)

We use (3.73) in

$$
u = \frac{\partial \psi}{\partial y} = \frac{1}{J} \frac{\partial x}{\partial r} = \frac{1}{rH'(\psi)} \left[ \cos \theta - r \sin \theta \frac{\partial \theta}{\partial r} \right]$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{1}{J} \frac{\partial y}{\partial r} = \frac{1}{rH'(\psi)} \left[ \sin \theta + r \cos \theta \frac{\partial \theta}{\partial r} \right]
$$

to obtain the velocity field, where $J = r \frac{\partial \theta}{\partial \psi}$ (see (3.18)). Following the established approach, we get: The family of curves $A(x^2 + y^2)^2 + \frac{1}{2} B \ln(x^2 + y^2) - \tan^{-1} \frac{y}{x}$ = constant can define a streamline pattern for a steady, plane, incompressible, second grade fluid flow, and the exact integral for this rotational flow is

$$u = \frac{2}{Re} \left[ \frac{x - By}{x^2 + y^2} - 4Ay(x^2 + y^2) \right], \quad v = \frac{2}{Re} \left[ \frac{Bx + y}{x^2 + y^2} + 4Ax(x^2 + y^2) \right]
$$

$$p = p_0 + \frac{2}{3Re^3} \left\{ Re \left[ 16A^2(x^2 + y^2)^3 + 24AB(x^2 + y^2) - \frac{3(1 + B^2)}{x^2 + y^2} \right] + 12We \left[ 80A^2(x^2 + y^2)^2 + \frac{3(1 + B^2)}{(x^2 + y^2)^2} - 16AB \ln(x^2 + y^2) + 32.1 \tan^{-1} \frac{y}{x} \right. \right.
$$

$$- 8AB \right\} + 24\gamma We \left[ 16A^2(x^2 + y^2)^2 + \frac{(1 + B^2)}{(x^2 + y^2)^2} - 8AB \right]$$

(3.74)

where $p_0$ is an arbitrary constant, and $x = r \cos \theta, y = r \sin \theta$. For this flow in an unbounded domain, the streamline pattern is shown in Figure 3.11.

We shall consider separately the cases when

$$m = 0, \quad \pm \frac{1}{2}, \quad \pm \frac{2}{3}, \quad \pm 1, \quad \pm 2.$$
\( m = 0 \).

Equation (3.72) becomes

\[
\left\{ 2 \left[ 2 - \frac{Re}{H'(\psi)} \right] \frac{H''(\psi)}{H''(\psi)} + 4B \left( \frac{H''(\psi)}{H''(\psi)} \right)' + (1 + B^2) \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H''(\psi)} \right)' \right]' \right\} r^{-3} \\
- \frac{4We}{H'(\psi)} \left\{ 4 \frac{H''(\psi)}{H''(\psi)} + 4B \left( \frac{H''(\psi)}{H''(\psi)} \right)' + (1 + B^2) \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H''(\psi)} \right)' \right]' \right\} r^{-5} = 0
\]

(3.75)

since \((1 + B^2) \neq 0\).

Since \( r, \psi \) are independent variables and \( \{ r^{-3}, r^{-5} \} \) is a linearly independent set, it follows that the coefficients of the respective powers of \( r \) will be zero. Since \( Re \neq 0 \), we get, from these two equations, that \( H''(\psi) = 0 \) or \( H(\psi) = a_1 \psi + a_2 \), where \( a_1 \neq 0 \) and \( a_2 \) are arbitrary real constants.

Thus, equation (3.68) becomes

\[
\theta - B \ln r = a_1 \psi + a_2
\]

(3.76)

Following the procedure of the first part of the problem, the exact integral of this flow is

\[
u = \frac{x - By}{a_1(x^2 + y^2)}, \quad u = \frac{Bx + y}{a_1(x^2 + y^2)}
\]

\[
p = p_0 + \frac{(1 + B^2)}{2a_1^2 Re(x^2 + y^2)} \left[ \frac{4(3 + 2\gamma) We}{x^2 + y^2} - Re \right]
\]

(3.77)

where \( p_0 \) is an arbitrary constant and \( x = r \cos \theta, y = r \sin \theta \). The streamlines for this flow in unbounded domain are as shown in Figure 3.12.
\(m = 2\).

In this case, equation (3.72) takes the form

\[
-4(1 + B^2) \frac{We}{H'(\psi)} \left\{ 4 \frac{H''''(\psi)}{H'^3(\psi)} + 4B \left( \frac{H''''(\psi)}{H'^3(\psi)} \right)' + (1 + B^2) \left[ \frac{1}{H'(\psi)} \left( \frac{H''''(\psi)}{H'^3(\psi)} \right)' \right] \right\} r^{-5}
\]

\[+ (1 + B^2) \left\{ 2 \left[ 2 + \frac{Re}{H'(\psi)} \right] \frac{H''''(\psi)}{H'^2(\psi)} + 4B \left( \frac{H''''(\psi)}{H'^3(\psi)} \right)' \right\} r^{-3}
\]

\[+ \left( 1 + B^2 \right) - 16AB \frac{We}{H'(\psi)} \left[ \frac{1}{H'(\psi)} \left( \frac{H''''(\psi)}{H'^3(\psi)} \right)' \right] \right\} r^{-1}
\]

\[+ 8AB(1 + B^2) \left\{ \left[ \frac{1}{H'(\psi)} \left( \frac{H''''(\psi)}{H'^3(\psi)} \right)' \right] \right\} r^{-1} - 8 \left\{ \frac{Re}{H'^3(\psi)} \right\}
\]

\[+ 2 \left[ 3B + 8A^3 \frac{We}{H'(\psi)} \left( \frac{H''''(\psi)}{H'^3(\psi)} \right)' \right] - (1 + 3B^2)
\]

\[+ 8A^3 \frac{We}{H'(\psi)} \left[ \frac{1}{H'(\psi)} \left( \frac{H''''(\psi)}{H'^3(\psi)} \right)' \right] \right\} r + 64A^4 \frac{We}{H'(\psi)} \left\{ \left[ \frac{1}{H'(\psi)} \left( \frac{H''''(\psi)}{H'^3(\psi)} \right)' \right] \right\} r^3
\]

\[+ 16A^4 \left\{ \left[ \frac{1}{H'(\psi)} \left( \frac{H''''(\psi)}{H'^3(\psi)} \right)' \right] \right\} r^5 \equiv \sum_{n=-3}^{2} K_{2n+1}(\psi)r^{2n+1} = 0
\]

(3.78)

Since \(r, \psi\) are independent variables and \(\{r^{-5}, r^{-3}, r^{-1}, r, r^3, r^5\}\) is a linearly independent set, it implies that \(K_{2n+1}(\psi) = 0\) for \(n = -3, -2, ..., 2\). \(K_3(\psi) = 0, K_{-3}(\psi) = 0\) and \(K_{-5}(\psi) = 0\) give \(H''(\psi) = 0\) or \(H(\psi) = a_1\psi + a_2\), where \(a_1 \neq 0\) and \(a_2\) are arbitrary real constants. \(K_5(\psi) = 0, K_1(\psi) = 0\) and \(K_{-1}(\psi) = 0\) are identically satisfied.

Equation (3.68), therefore, becomes

\[
\theta - Ar^2 - Blnr = a_1\psi + a_2
\]

(3.79)
Following the established procedure, the exact solution for this flow is

\[
    u = \frac{1}{a_1} \left[ \frac{z - By}{x^2 + y^2} - 2Ay \right], \quad v = \frac{1}{a_1} \left[ \frac{Bx + y}{x^2 + y^2} + 2Ax \right]
\]

\[
    p = p_0 + \frac{1}{a_1^2 \text{Re}} \left\{ \text{Re} \left[ 2AB \ln(x^2 + y^2) - 4A \tan^{-1} \frac{y}{x} - \frac{(1 + B^2)}{x^2 + y^2} - 4AB \right] \frac{2(1 + B^2)(3 + 2\gamma) \text{We}}{(x^2 + y^2)^2} \right\}
\]

where \( p_0 \) is an arbitrary constant and \( x = r \cos \theta, \quad y = r \sin \theta \). For this flow, the streamlines are shown in Figure 3.13.

\( m = 1 \).

Equation (3.72) becomes

\[
-4(1 + B^2) \frac{\text{We}}{H'(\psi)} \left\{ 4 \frac{H''''(\psi)}{H''(\psi)} + 4B \left( \frac{H''(\psi)}{H''(\psi)} \right)' + (1 + B^2) \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H''(\psi)} \right)' \right] \right\} r^{-5}
\]

\[
- \frac{3 \text{We}}{H'(\psi)} \left\{ 1 + 2(1 + 3B^2) \left( \frac{H''(\psi)}{H''(\psi)} \right)' + 4B(1 + B^2) \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H''(\psi)} \right)' \right] \right\} r^{-4}
\]

\[
+ \left\{ 2 \left[ 2(1 + B^2) + (1 + B^2) \frac{\text{Re}}{H'(\psi)} + 9A^2 \frac{\text{We}}{H'(\psi)} \right] \frac{H''(\psi)}{H''(\psi)} + 4B(1 + B^2) \left( \frac{H''(\psi)}{H''(\psi)} \right)' \right\} r^{-3} + A \left\{ 1 + \frac{\text{Re}}{H'(\psi)} \right\}
\]

\[
+ 2B \frac{H''(\psi)}{H''(\psi)} + 2(1 + 3B^2) \left( \frac{H''(\psi)}{H''(\psi)} \right) \right\}
\]

\[
+ 4B \left[ (1 + B^2) - A^2 \frac{\text{We}}{H'(\psi)} \right] \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H''(\psi)} \right)' \right] r^{-2} - A^2 \left\{ \frac{3 H''(\psi)}{H'(\psi)} \right\}
\]

\[-2(1 + 3B^2) \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H''(\psi)} \right)' \right] r^{-1} - 2A^3 \left( \frac{H''(\psi)}{H''(\psi)} \right) \right\} r
\]

\[
- 2B \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H''(\psi)} \right)' \right] + A^4 \left\{ \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H''(\psi)} \right)' \right] \right\}
\]

\[
= \sum_{n=-5}^{1} K_n(\psi) r^n = 0
\]
Since \( r, \psi \) are independent variables and \( \{ r^{-5}, r^{-4}, r^{-3}, r^{-2}, r^{-1}, 1, r \} \) is a linearly independent set, it follows that \( K_n(\psi) = 0 \) for \( n = -5, -4, \ldots, 1 \). \( K_1(\psi) = 0 \) and \( K_{-1}(\psi) = 0 \) give \( H''(\psi) = 0 \) or \( H(\psi) = a_1 \psi + a_2 \), where \( a_1 \neq 0 \) and \( a_2 \) are real arbitrary constants. \( K_{-5}(\psi) = 0 \), \( K_{-3}(\psi) = 0 \) and \( K_0(\psi) = 0 \) are identically satisfied. \( K_{-4}(\psi) = 0 \) and \( K_{-2}(\psi) = 0 \) give

\[
\frac{Wc}{H'(\psi)} = 0 \quad \text{and} \quad 1 + \frac{Re}{H'(\psi)} = 0.
\]

We, thus, conclude that second grade fluid cannot flow along the family of curves \( \theta - Ar - Blnr = \text{constant} \) since \( Wc = 0 \), but Newtonian fluids can.

Proceeding as in the case of \( m = 1 \), we find that Newtonian fluids can flow along the families of curves

\[
\theta - Ar^m - Blnr = \text{constant},
\]

but second grade fluid cannot flow along them when

\[
m = \pm \frac{1}{2}, \pm \frac{2}{3}, \pm 1, \pm 2
\]

since \( Wc = 0 \).

**Example 8.**

We consider the family of curves \( \sqrt{x^2 + y^2} - a \tan^{-1} \frac{y}{x} = \text{constant} \), where \( a \) is a non-zero known constant. We wish to show that second grade fluid cannot flow along these curves, by assuming the contrary to arrive at a contradiction.

Assuming that the fluid flows along this family of curves, and introducing polar coordinates, we obtain \( r = r(\theta, \psi) \) given by

\[
r = a\theta + H(\psi), H'(\psi) \neq 0
\]

(3.82)

where \( H \) is some function of \( \psi \). In this \( \theta, \psi \)-net, following the procedure of Example 7, we obtain

\[
E = a^2 + [a\theta + H(\psi)]^2, \quad F = aH'(\psi)
\]

(3.83)

\[
G = H'^2(\psi), \quad J = W = -[a\theta + H(\psi)]H'(\psi)
\]

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where the fluid is assumed to flow along the streamlines in the direction of $\theta$ increasing. Employing (3.83) in system (3.9), it is observed that the Gauss formula is identically satisfied, and the other two equations become

$$\omega = \frac{a}{[a \theta + H(\psi)]^2} \frac{H''(\psi)}{H'^3(\psi)} + \frac{H''(\psi)}{H'^3(\psi)} - \frac{\iota}{[a \theta + H(\psi)] H'(\psi)} \tag{3.84}$$

and

$$- \frac{H'(\psi)}{a \theta + H(\psi)} \frac{\partial^2 \omega}{\partial \theta^2} + \frac{2a}{a \theta + H(\psi)} \frac{\partial^2 \omega}{\partial \theta \partial \psi} - \frac{a^2}{[a \theta + H(\psi)] H'(\psi)} + \frac{[a \theta + H(\psi)]}{H'(\psi)} \frac{\partial^2 \omega}{\partial \psi^2}$$

$$+ \left[ \frac{a^2}{[a \theta + H(\psi)]^2} \frac{H''(\psi)}{H'^3(\psi)} + [a \theta + H(\psi)] \frac{H''(\psi)}{H'^2(\psi)} + \frac{a^2}{[a \theta + H(\psi)]^2} - 1 \right] \frac{\partial \omega}{\partial \psi} + Re \frac{\partial \omega}{\partial \theta}$$

$$+ \text{We} \left\{ - \frac{1}{[a \theta + H(\psi)]^2} \frac{\partial^3 \omega}{\partial \theta^3} - \frac{2a}{[a \theta + H(\psi)]^2} \frac{\partial^3 \omega}{\partial \theta^2 \partial \psi} + \left[ \frac{1}{H'^2(\psi)} \right] \frac{\partial^3 \omega}{\partial \theta^3} \right. $$

$$+ \left[ \frac{a^2}{[a \theta + H(\psi)]^2} \frac{H''(\psi)}{H'^3(\psi)} \right] \frac{\partial^3 \omega}{\partial \theta \partial \psi^2} - \frac{2a}{[a \theta + H(\psi)]^3} \frac{\partial^3 \omega}{\partial \theta^2 \partial \psi} + \left[ \frac{3a^2}{[a \theta + H(\psi)]^3} \frac{H''(\psi)}{H'^3(\psi)} \right] \frac{\partial^3 \omega}{\partial \theta^3 \partial \psi}$$

$$+ \left[ \frac{1}{[a \theta + H(\psi)] H'(\psi)} - \frac{a^2}{[a \theta + H(\psi)]^2} \frac{H''(\psi)}{H'^3(\psi)} + \frac{H''(\psi)}{H'^3(\psi)} \right] \frac{\partial^3 \omega}{\partial \theta \partial \psi^2}$$

$$- \frac{2a^3}{[a \theta + H(\psi)]^3} \frac{H'^2(\psi)}{H^4(\psi)} + \left[ \frac{3a^3}{[a \theta + H(\psi)] H(\psi)} + \frac{2a^3}{[a \theta + H(\psi)]^3} \frac{H''(\psi)}{H'^4(\psi)} \right]$$

$$- \frac{a}{[a \theta + H(\psi)]^2} \frac{\partial \omega}{\partial \psi} \right\} = 0 \tag{3.85}$$
Substitution of (3.84) in (3.85) yields

\[-12a^5 \text{We} \left\{ \frac{H''(\psi)}{H^3(\psi)} \right\} + 5a^3 \frac{\text{We}}{H'(\psi)} \left\{ 1 + \left( \frac{H''(\psi)}{H^3(\psi)} \right)' \right\} [a\theta + H(\psi)]
\]

\[-2a^3 \left\{ a \left( \frac{H''(\psi)}{H^3(\psi)} \right)' + 6 \text{We} \frac{H''(\psi)}{H^3(\psi)} + 2a^2 \frac{\text{We}}{H'(\psi)} \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H^3(\psi)} \right)' \right]' \right\} [a\theta + H(\psi)]^2
\]

\[+ a^3 \left\{ a \left( \frac{H''(\psi)}{H^3(\psi)} \right)' + \frac{9 \text{We}}{H'(\psi)} \left( \frac{H''(\psi)}{H^3(\psi)} \right)' \right\} [a\theta + H(\psi)]^3
\]

\[+ a \left\{ 2a^2 \text{Re} \frac{H''(\psi)}{H^3(\psi)} - 3a \frac{H''(\psi)}{H^2(\psi)} + 2 \text{We} \frac{H''(\psi)}{H^3(\psi)} - a^3 \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H^3(\psi)} \right)' \right]' \right\}
\]

\[- 4a^2 \frac{\text{We}}{H'(\psi)} \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H^3(\psi)} \right)' \right]' \right\} [a\theta + H(\psi)]^4 + \left\{ 1 - \frac{\text{Re}}{H'(\psi)} \right\}
\]

\[+ 3a^2 \left( \frac{H''(\psi)}{H^3(\psi)} \right)' - 2a \frac{\text{We}}{H'(\psi)} \left( \frac{H''(\psi)}{H^3(\psi)} \right)' \right\} [a\theta + H(\psi)]^5 + \left\{ \frac{H''(\psi)}{H^2(\psi)} \right\]

\[+ 3a^2 \left( \frac{H''(\psi)}{H^3(\psi)} \right)' - 2a \frac{\text{We}}{H'(\psi)} \left( \frac{H''(\psi)}{H^3(\psi)} \right)' \right\} [a\theta + H(\psi)]^5 + \left\{ \frac{H''(\psi)}{H^2(\psi)} \right\]

\[- 2a^2 \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H^3(\psi)} \right)' \right]' \right\} [a\theta + H(\psi)]^6 - 2 \left\{ \frac{H''(\psi)}{H^3(\psi)} \right\}' \right\} [a\theta + H(\psi)]^7 \tag{3.86}
\]

\[- \left\{ \left[ \frac{1}{H'(\psi)} \left( \frac{H''(\psi)}{H^3(\psi)} \right)' \right]' \right\} [a\theta + H(\psi)]^8 = 0
\]

Since \(\theta, \psi\) are independent variables and \(\{\theta^n : n = 0, 1, \ldots, 8\}\) is a linearly independent set, we have that all their coefficients (functions of \(H(\psi)\) and its derivatives) will vanish. Since requiring the coefficients of \(\theta^8, \theta^7, \text{ and } \theta^6\) to vanish simultaneously holds true only if \(H''(\psi) = 0\), it follows that the nine coefficients of the various powers of \(\theta\) in (3.86) vanish simultaneously only if \(H''(\psi) = 0\) and

\[5a^3 \frac{\text{We}}{H'(\psi)} [a\theta + H(\psi)] + \left[ 1 - \frac{\text{Re}}{H'(\psi)} \right] [a\theta + H(\psi)]^5 = 0 \tag{3.87}
\]

The coefficients of the six powers of \(\theta\) in (3.87) are identically zero, and so we have \(\text{We} = 0\).

We therefore have that second grade fluid cannot flow along the family of curves \(\sqrt{x^2 + y^2} = \tan^{-1} \frac{z}{x} = \text{constant}\), but Newtonian fluid can flow along these curves.
Streamlines for $y - m_1 x^2 - m_2 x = \text{constant}$ in unbounded domain when $m_1 = m_2 = 1$.

Figure 3.1
Streamlines for $y - m_1 x^2 - m_2 x = \text{constant}$ for boundary value problem when $m_1 = m_2 = 1$.  

Figure 3.2
Streamlines for $y - e^{mx} - m_1 x^2 - m_2 x = \text{constant}$ in unbounded domain when

$m = m_1 = m_2 = 1.$

Figure 3.3
Streamlines for $y - e^{mx} - m_1 x^2 - m_2 x = \text{constant}$ for boundary value problem

when $m = m_1 = m_2 = 1$.

Figure 3.4
Streamlines for $y - e^{mx} - e^{nx} - m_1 x^2 - m_2 x = \text{constant}$ in unbounded domain

when $m = 0.67332$, $n = -2.67332$, $m_1 = m_2 = 1$.

Figure 3.5
Streamlines for \( y - e^{mx} - e^{nz} - m_1x^2 - m_2z = \text{constant} \) for boundary value problem when \( m = 0.67332, n = -2.67332, m_1 = m_2 = 1. \)

Figure 3.6
Streamlines for $x(y - m_1 x - m_2) = \text{constant}$ in unbounded domain when \[ m_1 = m_2 = 1. \]

Figure 3.7
Streamlines for $z(y - m_1 x - m_2) = \text{constant}$ for boundary value problem when $m_1 = m_2 = 1$.

Figure 3.8
Streamlines for $y(1 + m_1 e^{mx}) - m_2 e^{mx} = \text{constant in unbounded domain when}$

$m = m_1 = m_2 = 1.$

Figure 3.9
Streamlines for \( y(1 + m_1 e^{mx}) - m_2 e^{mx} = \) constant for boundary value problem

when \( m = m_1 = m_2 = 1 \).

Figure 3.10
Streamlines for $\theta - A r^4 - B \ln r = \text{constant}$ in unbounded domain when

$A = B = 1$.

Figure 3.11
Streamlines for $\theta - B\ln r = \text{constant}$ in unbounded domain when $B = 1$.

Figure 3.12
Streamlines for $\theta - Ar^2 - Blnr = \text{constant in unbounded domain when}$

$A = B = 1.$

Figure 3.13
CHAPTER IV

SOLUTIONS OF SECOND GRADE MAGNETOHYDRODYNAMIC ALIGNED FLOW BY VON MISES COORDINATES

4.1 INTRODUCTION.

The first objective of this chapter is to investigate if an electrically conducting second grade fluid can flow along a given family of curves \( \frac{v - \beta(x)}{\beta'(x)} = \text{constant} \). If the answer to this question has been determined in the affirmative so that these curves are streamlines, and \( \psi(x, y) = \text{constant} \) on these curves as well, we find a function \( \beta(\psi) \) such that \( \frac{v - \beta(x)}{\beta'(x)} = \beta(\psi) \) with \( \beta'(\psi) \neq 0 \). The exact integral of the flow defined by the given streamline pattern in both unbounded and bounded domains are then determined, which constitutes the second objective of this chapter.

The search for exact solutions of problems arising from definite physical flows faces great difficulties due to the nonlinearity of the governing equations. Five functions \( u(x, y), v(x, y), H_1(x, y), H_2(x, y) \) and \( p(x, y) \) that satisfy the continuity, linear momentum, diffusion and solenoidal (condition on the magnetic field) equations constitute an exact solution if these give the solution to a problem with or
without boundaries. The boundaries may be fixed or movable surfaces on which the no-slip condition is satisfied.

To accomplish the aforementioned objectives, we employ an equivalent alternative formulation of the flow equations developed by Martin (1971), and present a direct approach for the determination of solutions.
4.2 GOVERNING EQUATIONS IN CARTESIAN, MARTIN'S AND
VON MISSES COORDINATES.

The equations governing the steady, planar, isochoric motion of an electrically
cconducting second grade fluid are given by the system (2.25).

We shall study aligned flows for which the magnetic field is everywhere parallel
to the velocity field in the region of flow, so that equation (2.28) holds true. These
aligned flows will be those for which the fluid is of either infinite or finite electrical
conductivity \( \sigma \).

If we introduce the vorticity, current density and energy, given by system (2.26),
and (2.28) into system (2.25), we get for:

Infinitely Conducting Flows.

The flow of a steady, planar, MHD aligned, isochoric motion of an electrically
cconducting second grade fluid of infinite electrical conductivity is satisfied by the
system (2.29), which is a system of six equations in six unknowns \( u, v, f, \omega, j, h \)
as functions of \( x, y \), where the diffusion equation gives \( k = 0 \).

Finitely Conducting Flows.

The equations governing the aligned, steady, plane flow of an incompressible,
electrically conducting second grade fluid having finite electrical conductivity are
given by system (2.29) when the current density \( j \) is replaced by a constant \( j_0 \), since
the diffusion equation gives \( j = \mu^* \sigma k \equiv j_0 \).

We introduce curvilinear coordinates \( \phi, \psi \) in the physical plane, with \( \phi(x,y) = 
\) constant and \( \psi(x,y) = \) constant being two families of curves, where \( \psi(x,y) \) is the
stream function satisfying equations (2.30), and \( \phi = \phi(x,y) \) is an arbitrary function
of \( x, y \).

Employing the results of Section 2.2 and the method of Section 2.3 in the
system (2.29), we get:
Infinitely Conducting Flows.

The plane, steady, aligned, isochoric flow of an electrically conducting second grade fluid having infinite electrical conductivity is satisfied by the system (2.43), which is an underdetermined system of six equations for seven functions $E$, $F$, $G$, $\omega$, $j$, $h$ of $\phi$, $\psi$. $J > 0$ or $J < 0$ accordingly as $\phi$ increases or decreases along the streamline in the direction of flow, where $\Delta_2 \omega$, given by equation (2.35) is the Beltrami's differential parameter of second order [cf. Martin (1971)].

The energy $h$ is eliminated from the linear momentum equations with the aid of the integrability equation \( \frac{\partial^2 h}{\partial \phi \partial \psi} = \frac{\partial^2 h}{\partial \psi \partial \phi} \). We, therefore, get Theorem 2.3, the equations of which are satisfied by the flow.

Having obtained the solution $E$, $F$, $G$, $f$, $\omega$, $j$ as functions of $\phi$, $\psi$ for system (2.45), the energy $h = h(\phi, \psi)$ is obtained from the linear momentum equations in system (2.43). The speed $q = q(\phi, \psi)$ and the pressure $p = p(\phi, \psi)$ are given by equations (2.33) and (2.44), respectively.

Finitely Conducting Flows.

The flow when one set of the coordinates curves in a curvilinear coordinate system $\phi$, $\psi$ are the streamlines $\psi(x, y) = \text{constant}$ of a steady, plane, MHD aligned motion of an incompressible, electrically conducting second grade of finite electrical conductivity is governed by system (2.45), provided the current density $j = j_0 = f\omega - \frac{E \psi}{\psi} f'(\psi)$, where $j_0$ is a non-zero real constant, and the term $\mu^* f \frac{\partial j}{\partial \phi} = 0$ in the integrability equation.

The system (2.45) is underdetermined because of the arbitrariness inherent in the choice of the coordinate curves $\phi(x, y) = \text{constant}$. This system can be made determinate, for instance, by setting $\phi(x, y) = x = \text{constant}$ so that the von Mises (1927) coordinates $x$, $\psi$ is the curvilinear net.

We investigate the first objective of this chapter by assuming that a given family
of curves \( \frac{y-g(x)}{l(x)} = \text{constant} \) is a permissible family of streamlines, where \( g(x), \ l(x) \neq 0 \) are continuously differentiable functions. This implies that \( \psi(x,y) = \text{constant} \) as well along these curves, and so there exists some function \( \beta(\psi) \) such that

\[
\frac{y-g(x)}{l(x)} = \beta(\psi), \quad \beta'(\psi) \neq 0
\]  

(4.1)

where \( \beta'(\psi) \) is the derivative of the unknown function \( \beta(\psi) \). Equation (4.1) gives for every \( x \), a unique \( y \) on each individual streamline so that \( y = y(x, \psi) \).

For this von Mises coordinates \( x, \psi \), and flow with streamlines given by equation (4.1), the squared differential element of arc length is

\[
ds^2 = dx^2 + dy^2 = E dx^2 + 2F dx d\psi + G d\psi^2
\]

where

\[
E = 1 + [g'(x) + l'(x)\beta(\psi)]^2, \quad F = [g'(x) + l'(x)\beta(\psi)]l(x)\beta'(\psi)
\]

\[
G = l^2(x)\beta'^2(\psi), \quad J = W = l(x)\beta'(\psi)
\]

(4.2)

when the fluid is assumed to flow along the streamlines in the direction of \( z \) increasing.

Infinitely Conducting Flows.

When equations (4.2) are introduced into the system (2.45), the Gauss formula is identically satisfied, and we obtain:

**Theorem 4.1.** Let the family of curves \( \frac{y-g(x)}{l(x)} = \text{constant} \), \( l(x) \neq 0 \) form a streamline pattern for the aligned, steady, plane flow of an incompressible, electrically conducting second grade fluid of infinite electrical conductivity. Then

\[
\omega = \frac{1}{l^2(x)\beta'(\psi)} \left\{ l(x)g''(x) - 2g'(x)l'(x) + [l(x)l''(x) - 2l'^2(x)]\beta(\psi) \right. \\
+ \left[ 1 + g'^2(x) \right] \frac{\beta''(\psi)}{\beta'^2(\psi)} + 2g'(x)l'(x)\frac{\beta(\psi)\beta''(\psi)}{\beta'^2(\psi)} + l'^2(x)\frac{\beta'(\psi)\beta''(\psi)}{\beta'^2(\psi)} \right\}
\]

(4.3)
\[ j = f(\psi) \omega - \frac{1 + [g'(x) + l'(x)\beta(\psi)]^2}{l^2(x)\beta'^2(\psi)} f'(\psi) \]  

(4.4)

and

\[
\mu \left\{ l(x) \beta'(\psi) \frac{\partial^2 \omega}{\partial x^2} - 2 \left[ g'(x) + l'(x) \beta(\psi) \right] \frac{\partial^2 \omega}{\partial x \partial \psi} + \frac{1}{l(x) \beta'(\psi)} \left[ 1 + g'^2(x) \right] \right. \\
+ 2g'(x)l'(x)\beta(\psi) + l^2(x)\beta^2(\psi) \left[ \frac{\partial^2 \omega}{\partial \psi^2} - \frac{1}{l(x)} \right] \left[ l(x) g''(x) - 2g'(x)l'(x) \right] \\
+ \left[ l(x) l''(x) - 2l^2(x) \right] \beta(\psi) + \left[ 1 + g'^2(x) \right] \frac{\beta''(\psi)}{\beta'^2(\psi)} + 2g'(x)l'(x) \frac{\beta(\psi) \beta''(\psi)}{\beta'^2(\psi)} \\
+ l^2(x) \frac{\beta^2(\psi) \beta''(\psi)}{\beta'^2(\psi)} \right\} + \left[ \mu^* f'(\psi) - \rho \right] \frac{\partial \omega}{\partial x} \\
+ 2 \mu^* f(\psi) f'(\psi) \left\{ l'(x) \left[ 1 + g'^2(x) \right] - l(x) g'(x) g''(x) + \left[ g'(x) \left[ 2l^2(x) - l(x) l''(x) \right] \right. \\
- l(x) l'(x) g''(x) \right] \beta(\psi) + l'(x) \left[ l'^2(x) - l(x) l''(x) \right] \beta^2(\psi) \} \\
\]

\[
+ \frac{\alpha_1}{l(x) \beta'(\psi)} \left\{ l(x) \beta'(\psi) \frac{\partial^3 \omega}{\partial x^3} - 2 \left[ g'(x) + l'(x) \beta(\psi) \right] \frac{\partial^3 \omega}{\partial x \partial \psi} + \frac{1}{l(x) \beta'(\psi)} \left[ 1 + g'^2(x) \right] \right. \\
+ g'^2(x) + 2g'(x)l'(x)\beta(\psi) + l^2(x)\beta^2(\psi) \right\} \frac{\partial^3 \omega}{\partial \psi^3} + \frac{1}{l(x) \beta'(\psi)} \frac{\partial^3 \omega}{\partial \psi \partial x} \left[ 4g'(x) l''(x) \\
- 3l(x) g''(x) + \left[ 4l'^2(x) - 3l(x) l''(x) \right] \beta(\psi) - \left[ 1 + g'^2(x) \right] \frac{\beta''(\psi)}{\beta'^2(\psi)} \\
- 2g'(x)l'(x) \frac{\beta(\psi) \beta''(\psi)}{\beta'^2(\psi)} - l^2(x) \frac{\beta^2(\psi) \beta''(\psi)}{\beta'^2(\psi)} \right\} \frac{\partial^3 \omega}{\partial \psi \partial x} + \frac{2}{l^2(x) \beta'(\psi)} \left[ l(x) g'(x)g''(x) \\
- l'(x) \left[ 1 + g'^2(x) \right] + \left[ l(x) l'(x) g''(x) + l(x) g'(x) l''(x) - 2g'(x) l'^2(x) \right] \beta(\psi) \right. \\
+ \left[ l'(x) \left[ l(x) l''(x) - l'^2(x) \right] \beta^2(\psi) \right\} \frac{\partial^2 \omega}{\partial x^2} + \frac{1}{l(x)} \left[ 2g'(x) \left[ l(x) l''(x) - 2l'^2(x) \right] \\
+ l(x) \left[ 3l'(x) g''(x) - l(x) g'''(x) \right] + \left[ 5l'(x) l''(x) - l^2(x) l'''(x) \right. \\
- 4l^3(x) \right] \beta(\psi) + 2 \left[ l'(x) + l'(x) g'^2(x) - l(x) g'(x) g''(x) \right] \frac{\beta''(\psi)}{\beta'^2(\psi)} \\
+ 2 \left[ 2g'(x) l'^2(x) - l(x) g'(x) l''(x) - l(x) l'(x) g''(x) \right] \frac{\beta(\psi) \beta''(\psi)}{\beta'^2(\psi)} \\
+ 2 \left[ l'^2(x) - l(x) l''(x) \right] \frac{\beta^2(\psi) \beta''(\psi)}{\beta'^2(\psi)} \right\} \frac{\partial \omega}{\partial \psi} = 0
\]  

are the equations governing the flow.
Finitely Conducting Flows.

In the case of an aligned flow of a finitely conducting second grade fluid, equations (4.3) through (4.5) hold true except that \( j = j_0 \) (non-zero real constant) in (4.4), and in (4.5) we have

\[
\mu^* f^2(\psi) \frac{\partial \omega}{\partial \psi} + \frac{2\mu^* f(\psi) f'(\psi)}{l^2(x) \beta^2(\psi)} \left\{ \frac{l'(x)}{1 + g^2(x)} - l(x) g'(x) g''(x) \right. \\
+ \left[ g'(x) \left[ 2l'^2(x) - l(x) l''(x) \right] + l(x) l'(x) g''(x) \right] \beta(\psi) \\
+ l'(x) \left[ l^2(x) - l(x) l''(x) \right] \beta^2(\psi) \right\} = 0
\]  

(4.6)

From the foregoing, if the streamlines are

\[
\frac{x - g(y)}{l(y)} = \text{constant},
\]

then there exists some function \( \beta(\psi) \) such that \( x - g(y) = l(y) \beta(\psi) \), \( \beta'(\psi) \neq 0 \) so that \( x = x(y, \psi) \).
4.3 EXACT SOLUTIONS.

We now explore some examples with a view to obtaining solutions for the aligned flows of infinitely and finitely conducting second grade fluid.

Example 1.

We assume an answer in the affirmative to the question of whether or not the family of curves \( y - e^{mz} - m_1 x^2 - m_2 x = \text{constant} \) can be streamlines for system (2.25). Then there exists some function \( \beta(\psi) \) such that

\[
y - e^{mz} - m_1 x^2 - m_2 x = \beta(\psi), \quad \beta'(\psi) \neq 0 \tag{4.7}
\]

where \( m \neq 0, m_1 \) and \( m_2 \) are real constants.

Infinitely Conducting Flows.

Applying (4.7) to equations (4.3) to (4.5), we get

\[
\omega = \frac{m^2 e^{mz} - 2m_1}{\beta'(\psi)} + \left[ 1 + (me^{mz} + 2m_1x + m_2)^2 \right] \frac{\beta''(\psi)}{\beta'^2(\psi)} \tag{4.8}
\]

\[
j = f(\psi) \omega - \left[ 1 + (me^{mz} + 2m_1x + m_2)^2 \right] \frac{f'(\psi)}{\beta'^2(\psi)} \tag{4.9}
\]

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and

\[
\begin{align*}
&\mu \left\{ \beta'(\psi) \frac{\partial^2 \omega}{\partial x^2} - 2(me_{mz} + 2m_1x + m_2) \frac{\partial^2 \omega}{\partial x \partial \psi} \\
&+ \left[ 1 + (me_{mz} + 2m_1x + m_2)^2 \right] \frac{1}{\beta'(\psi)} \frac{\partial^2 \omega}{\partial \psi^2} - \left( m^2e_{mz} + 2m_1 \right) \\
&+ \left[ 1 + (me_{mz} + 2m_1x + m_2)^2 \right] \frac{\beta''(\psi)}{\beta'^2(\psi)} \frac{\partial \omega}{\partial \psi} \right\} + \left[ \mu^* f^2(\psi) - \rho \right] \frac{\partial \omega}{\partial x} \\
&- 2\mu^* (m^2e_{mz} + 2m_1)(me_{mz} + 2m_1x + m_2) \frac{\partial^3 \omega}{\partial x^2 \partial \psi} + \left[ 1 + (me_{mz} + 2m_1x + m_2)^2 \right] \frac{1}{\beta'(\psi)} \frac{\partial^3 \omega}{\partial x \partial \psi^2} \\
&- 2(me_{mz} + 2m_1x + m_2) \frac{\partial^3 \omega}{\partial x^2 \partial \psi} \right\} + \left[ 1 + (me_{mz} + 2m_1x + m_2)^2 \right] \frac{1}{\beta'(\psi)} \frac{\partial^3 \omega}{\partial x \partial \psi^2} \\
&- \left[ 3(m^2e_{mz} + 2m_1) + \left[ 1 + (me_{mz} + 2m_1x + m_2)^2 \right] \frac{\beta''(\psi)}{\beta'^2(\psi)} \frac{\partial^2 \omega}{\partial x \partial \psi} \\
&+ 2(m^2e_{mz} + 2m_1)(me_{mz} + 2m_1x + m_2) \frac{1}{\beta'(\psi)} \frac{\partial^2 \omega}{\partial \psi^2} \right\} = 0
\end{align*}
\]

Employing (4.8) in (4.10), we obtain

\[
\begin{align*}
\mu \left[ \frac{1}{\beta'(\psi)} \left( \frac{\beta''(\psi)}{\beta'^3(\psi)} \right) \right]' + m^3 \left[ \mu m + \alpha_1 m^2 + \mu^* f^2(\psi) - \rho - \frac{2\alpha_1}{\beta'(\psi)} \left( \frac{\beta''(\psi)}{\beta'^3(\psi)} \right)' \right] e_{mz} \\
- 2\mu \left( \frac{\beta''(\psi)}{\beta'^3(\psi)} \right)' (m^2e_{mz} + 2m_1) + 10\alpha_1 m^3 \frac{\beta''(\psi)}{\beta'^3(\psi)} e_{mz} (m^2e_{mz} + 2m_1) \\
+ 4m^3 \left[ \mu + \frac{\alpha_1 m^2}{\beta'(\psi)} \frac{\beta''(\psi)}{\beta'^3(\psi)} e_{mz} (me_{mz} + 2m_1x + m_2) + 3\mu \frac{\beta''(\psi)}{\beta'^3(\psi)} (m^2e_{mz} + 2m_1)^2 \\
+ 2\mu \left[ \frac{1}{\beta'(\psi)} \left( \frac{\beta''(\psi)}{\beta'^3(\psi)} \right) \right]' (me_{mz} + 2m_1x + m_2)^2 \\
- 6\alpha_1 m^2 \frac{\beta''(\psi)}{\beta'(\psi)} \left( \frac{\beta''(\psi)}{\beta'^3(\psi)} \right)' e_{mz} (me_{mz} + 2m_1x + m_2)^2 \\
+ \frac{2}{\beta'(\psi)} \left[ \mu^* f^2(\psi) - \rho \right] \frac{\beta''(\psi)}{\beta'^3(\psi)} - \frac{\mu f(\psi)f'(\psi)}{\beta'(\psi)} \\
+ 2\alpha_1 \left( \frac{1}{\beta'(\psi)} \right)' \left( m^2e_{mz} + 2m_1 \right)(me_{mz} + 2m_1x + m_2) \\
- \frac{12\alpha_1}{\beta'(\psi)} \left( \frac{\beta''(\psi)}{\beta'^3(\psi)} \right)' (m^2e_{mz} + 2m_1)^2 (me_{mz} + 2m_1x + m_2)
\right] = 0
\end{align*}
\]

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\[-6\mu \left( \frac{\beta''(\psi)}{\beta'(\psi)} \right)'(m^2e^{mx} + 2m_1)(me^{mx} + 2m_1x + m_2)^2
+ \frac{4\alpha_1}{\beta'(\psi)} \left[ \frac{1}{\beta'(\psi)} \left( \frac{\beta''(\psi)}{\beta'(\psi)} \right)' \right]'(m^2e^{mx} + 2m_1)(me^{mx} + 2m_1x + m_2)^3 \]
\[+ \mu \left[ \frac{1}{\beta'(\psi)} \left( \frac{\beta''(\psi)}{\beta'(\psi)} \right)' \right]'(me^{mx} + 2m_1x + m_2)^4 = 0 \quad (4.11)\]

Since \(x, \psi\) are independent variables and \(\{1, x, x^2, x^3, x^4, e^{mx}, e^{2mx}, e^{3mx}, e^{4mx}, xe^{mx}, xe^{2mx}, xe^{3mx}, xe^{4mx}, x^2e^{mx}, x^2e^{2mx}, x^2e^{3mx}, x^3e^{mx}\}\) is a linearly independent set, it follows that the respective coefficients of these functions of \(x\) will equate to zero. The coefficient of \(x^4\) gives
\[\left[ \frac{1}{\beta'(\psi)} \left( \frac{\beta''(\psi)}{\beta'(\psi)} \right)' \right]' = 0\]

Substituting this in (4.11) and extracting the coefficient of \(x^2\), we get
\[\left( \frac{\beta''(\psi)}{\beta'(\psi)} \right)' = 0\]

This, in turn, is employed in the resulting equation, and the coefficients of \(x\) and unity yield, respectively
\[\left[ \mu^* f^2(\psi) - \rho \right] \frac{\beta''(\psi)}{\beta'(\psi)} - \mu^* f(\psi)f'(\psi) = 0\]
\[m_2 \left\{ \left[ \mu^* f^2(\psi) - \rho \right] \frac{\beta''(\psi)}{\beta'(\psi)} - \mu^* f(\psi)f'(\psi) \right\} + \mu m_1 \beta''(\psi) = 0 \quad (4.12)\]

Equations (4.12) hold true simultaneously only if
\[\beta''(\psi) = 0, \quad f(\psi)f'(\psi) = 0\]

The equation (4.11) in its final form is
\[m^3 \left[ \mu m + \frac{\alpha_1 m^2 + \mu^* f_0^2 - \rho}{\beta'(\psi)} \right] e^{mx} = 0\]
or
\[\beta'(\psi) = \frac{\rho - \mu^* f_0^2 - \alpha_1 m^2}{\mu m} \neq 0\]

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Equations (4.7) and (2.28) become

\[
y - c^{mz} - m_1 x^2 - m_2 x = \left( \frac{\rho - \mu^* f_0^2 - \alpha_1 m^2}{\mu m} \right) \psi + c \tag{4.13}
\]

\[
(H_1, H_2) = (f_0 u, f_0 v) \tag{4.14}
\]

where \( c \) and \( f(\psi) = f_0 \neq 0 \) are real constants. The exact solution for this rotational flow in an unbounded domain is

\[
u = \frac{\mu m}{\rho - \mu^* f_0^2 - \alpha_1 m^2}, \quad v = \frac{\mu m}{\rho - \mu^* f_0^2 - \alpha_1 m^2} \left( m c^{mz} + 2m_1 x + m_2 \right), \quad H_1 = f_0 u
\]

\[
H_2 = f_0 v, \quad p = p_0 + \frac{\mu^2 m^2}{2(\rho - \mu^* f_0^2 - \alpha_1 m^2)} \left\{ 2(2\alpha_1 + \alpha_2) \left( m c^{mz} + 2m_1 + m_2 \right)^2 - \mu^* f_0^2 \left( m c^{mz} + 2m_1 x + m_2 \right)^2 + 4m_1 (\mu^* f_0^2 - \rho) y - \left[ \rho + \alpha_1 \left( 4m_1^2 + m_2^2 \right) \right] \right\}
\]

(4.15)

where \( p_0 \) is an arbitrary constant.

We impose the boundary conditions

\[
u(0, y) = u_0, \quad v(0, y) = v_0; \quad u_0 \neq 0 \tag{4.16}
\]

\[
[H].t = 0, \quad [H].n = 0 \quad \text{at} \quad z = 0 \tag{4.17}
\]

on a plate situated along \( z = 0 \) so that the fluid occupies \( z < 0 \), where \( u_0 \) is the constant suction (or blowing) speed and \( v_0 \) is the constant stretching speed of the plate. Equations (4.17) state that the tangential and normal components of the magnetic field \( H \) are continuous across the plate \( z = 0 \) where \( [H] = H^L - H^R \) or \( [H] = H^R - H^L \), \( H^L \) and \( H^R \) being the magnetic field intensities in the flow region and on the right of the plate, respectively [cf. Dragos (1975)].

Equations (4.16) imply that \( m, m_2 \) are given by

\[
u_0 \alpha_1 m^2 + \mu m + u_0 (\mu^* f_0^2 - \rho) = 0, \quad m_2 = \frac{v_0}{u_0} - m \tag{4.18}
\]
where \( u_0 > 0 \) and \( u_0 < 0 \) denote, respectively, uniform suction and blowing at the plate. Let \( H^R = (A, B) \), where \( A \neq 0 \) and \( B \) are constants. Since \( H^L = (f_0 u, f_0 v) \), we have, from (4.17), that

\[
A = f_0 u_0, \quad B = f_0 v_0.
\]

Thus, the exact solution of this boundary value problem is

\[
u = u_0, \quad v = u_0 \left( m e^{m x} + 2m_1 x - m \right) + v_0, \quad H_1 = f_0 u, \quad H_2 = f_0 v
\]

\[
p = p_0 + \frac{1}{2} \left\{ 2u_0^2 \left( 2\alpha_1 + \alpha_2 \right) \left( m^2 e^{m x} + 2m_1 \right)^2 - \mu^* f_0^2 \left[ u_0 \left( m e^{m x} + 2m_1 x - m \right) + v_0 \right]^2 
\]

\[
+ 4m_1 \left( \mu^* f_0^2 - \rho \right) y - \left[ u_0^2 \left( \rho + 4\alpha_1 m_1^2 \right) + \alpha_1 m_1^2 \left( v_0 - m u_0 \right)^2 \right] \right\}
\]

(4.19)

where \( m_1 \) is an arbitrary constant, \( m_2 = \frac{v_0}{u_0} - m \), \( f_0 = \frac{A}{u_0} \), \( v_0 = \frac{B u_0}{A} \), and \( m \) is given by

\[
u_0^2 \alpha_3 m^2 + u_0 \mu m + \mu^* A^2 - u_0^2 \rho = 0
\]

In the particular case of \( m_2 = -m \), the velocity boundary conditions on the plate

\[z = 0 \text{ arc}
\]

\[
u(0, y) = u_0, \quad v(0, y) = 0; \quad u_0 \neq 0
\]

(4.20)

In this case, \( H^R = (A, 0) \), where \( A \neq 0 \) is a constant. From \( H^L = (f_0 u, f_0 v) \), the boundary conditions (4.17) give \( A = f_0 u_0 \). The exact solution in this case of \( m_2 = -m \) is obtained from solution (4.19) by setting \( v_0 = 0 \) (or \( B = 0 \)).

The normal stress components \( T_{11}, T_{22} \) and the shear stress \( T_{12} \) are

\[
T_{11} = -p + 2\mu \frac{\partial u}{\partial x} + 2\alpha_1 \left[ u \frac{\partial^2 u}{\partial x^2} + v \frac{\partial^2 u}{\partial x \partial y} + 2 \left( \frac{\partial u}{\partial x} \right)^2 + \frac{\partial v}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) \right]
\]

\[
+ \alpha_2 \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right]
\]

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\[ T_{22} = -p + 2\mu \frac{\partial v}{\partial y} + 2\alpha_1 \left[ u \frac{\partial^2 v}{\partial x \partial y} + v \frac{\partial^2 v}{\partial y^2} + 2 \left( \frac{\partial v}{\partial y} \right)^2 + \frac{\partial u}{\partial y} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \\
+ \alpha_2 \left[ 4 \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] \\
T_{12} = T_{21} = \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \alpha_1 \left[ \left( u \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial y} \right) \right] + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + 2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right] \\
(4.21) \]

The values of these stress components on the plate \( x = 0 \) for the solution (4.19) are given by

\[ T_{11} = -p_0 + \frac{1}{2} \left[ \nu_0^2 \mu^* \frac{f_0^2}{f_0^2} + \nu_0^2 (\rho + 4\alpha_1 m_1^2) - m^2 (\nu_0 - m u_0)^2 + 4m_1 (\rho - \mu^* f_0^2) y \right] \]
\[ T_{22} = -p_0 + \frac{1}{2} \left[ \nu_0^2 \mu^* f_0^2 + \nu_0^2 (\rho + 4\alpha_1 m_1^2) - 4\nu_0^2 \alpha_1 (m^2 + 2m_1)^2 \right] \\
- \alpha_1 m^2 (\nu_0 - m u_0)^2 + 4m_1 (\rho - \mu^* f_0^2) y \right] \\
T_{12} = u_0 \left[ \mu (m^2 + 2m_1) + u_0 \alpha_1 m^3 \right] \\
(4.22) \]

The streamlines for flows in unbounded and bounded domains are as shown in Figures 3.3 and 3.4.

Finitely Conducting Flows.

Equation (4.6) is identically satisfied only if

\[ m^3 f(\psi)e^{mz} + \frac{2}{\beta'(\psi)} \left[ \beta''(\psi) - f'(\psi) \right] (m^3 e^{mz} + 2m_1) (me^{mz} + 2m_1x + m_2) = 0 \]

(4.23)

Since \( x, \psi \) are independent variables and \( \{1, z, e^{mz}, ze^{mz}, e^{2mz}\} \) is a linearly independent set, it follows that each coefficient of these functions of \( z \) will be zero.

The coefficient of unity and \( e^{mz} \), respectively, give

\[ \frac{\beta''(\psi)}{\beta'(\psi)} - f'(\psi) = 0 \]
\[ \frac{\beta''(\psi)}{\beta'(\psi)} - f'(\psi) + f(\psi) = 0 \]

(4.24)

From (4.24), we have \( f(\psi) = 0 \) and \( \beta''(\psi) = 0 \), which is the result for non-MHD flow. The exact solution associated with this flow was studied in Example 2 of
Section 3.5. Therefore, second grade, incompressible, finitely conducting MHD aligned fluid flow in a plane along the family of curves

\[ y = c^{m_2} - m_1 x^2 - m_2 x = \text{constant} \]

is impossible.

Example 2.

Assuming the family of curves \( x(y - m_1 x - m_2) = \text{constant} \) constitutes a streamline pattern, (4.1) becomes

\[ x(y - m_1 x - m_2) = \beta(\psi), \quad \beta'(\psi) \neq 0 \tag{4.25} \]

where \( m_1 \) and \( m_2 \) are real constants, and

\[ g(x) = m_1 x + m_2, \quad l(x) = \frac{1}{x}. \]

Infinitely Conducting Flows.

Employing (4.25) in (4.3) through (4.5), we obtain

\[ \omega = \frac{1}{x^2} \frac{\beta^2(\psi)\beta''(\psi)}{\beta'^3(\psi)} + 2m_1 \left[ \frac{1}{\beta'(\psi)} - \frac{\beta(\psi)\beta''(\psi)}{\beta'^3(\psi)} \right] + (1 + m_1^2)x^2 \frac{\beta''(\psi)}{\beta'^4(\psi)} \tag{4.26} \]

\[ j = f(\psi)\omega - \left( x^2 + \left[ m_1 x - \frac{1}{x} \beta(\psi) \right]^2 \right) \frac{f'(\psi)}{\beta'^2(\psi)} \tag{4.27} \]

and

\[
\begin{align*}
\mu & \left[ \frac{1}{x} \beta'(\psi) \frac{\partial^2 \omega}{\partial x^2} + 2 \left[ \frac{1}{x^2} \beta(\psi) - m_1 \right] \frac{\partial^2 \omega}{\partial x \partial \psi} + \left[ \frac{1}{x^3} \beta'^2(\psi) - \frac{2m_1}{x} \beta(\psi) \right] \\
& + (1 + m_1^2) \right] \frac{1}{\beta'(\psi)} \frac{\partial^2 \omega}{\partial \psi^2} - \left[ \frac{1}{x^3} \beta^2(\psi) \beta''(\psi) \right] + 2m_1 \frac{\partial \omega}{\partial x} \left[ 1 - \frac{\beta(\psi)\beta''(\psi)}{\beta'^2(\psi)} \right] \\
& + (1 + m_1^2) \frac{x^2 \beta''(\psi)}{\beta'^2(\psi)} \frac{\partial \omega}{\partial \psi} \right] + \left[ \mu^* f^2(\psi) - \rho \right] \frac{\partial \omega}{\partial x} \\
& + 2 \mu^* \frac{f(\psi)f'(\psi)}{\beta'^2(\psi)} \left[ \frac{1}{x^3} \beta^2(\psi) - (1 + m_1^2)x \right] + \frac{\alpha_1}{\beta'(\psi)} \left[ \beta'(\psi) \frac{\partial^2 \omega}{\partial x^3} \\
& + 2 \left[ \frac{1}{x} \beta(\psi) - m_1 x \right] \frac{\partial^2 \omega}{\partial x^2 \partial \psi} + \left[ \frac{1}{x^2} \beta^2(\psi) - 2m_1 \beta(\psi) + (1 + m_1^2)x^2 \right] \frac{1}{\beta'(\psi)} \frac{\partial^2 \omega}{\partial x \partial \psi} \right]
\end{align*}
\]
\[
- \left[ \frac{1}{x^2} \left[ 2\beta'(\psi) + \frac{\beta^2(\psi)\beta''(\psi)}{\beta'(\psi)} \right] \right] + 2m_1 \left[ 2 - \frac{\beta(\psi)\beta''(\psi)}{\beta'(\psi)} \right] + (1 + m_1^2)x^2 \frac{\beta''(\psi)}{\beta'(\psi)} \frac{\partial^2 \omega}{\partial x \partial \psi} \\
- 2 \left[ \frac{1}{x^3} \beta^2(\psi) - (1 + m_1^2)x \right] \frac{1}{\beta'(\psi)} \frac{\partial^2 \omega}{\partial \psi^2} + 2 \left[ \frac{1}{x^3} \frac{\beta(\psi)\beta''(\psi)}{\beta'(\psi)} \right] \\
- (1 + m_1^2)x \frac{\beta''(\psi)}{\beta'(\psi)} \frac{\partial \omega}{\partial \psi} \right\} = 0 
\]

(4.28)

Substituting (4.26) in (4.28) yields

\[
\left[ \mu - \frac{4\alpha_1}{\beta'(\psi)} \right] \left\{ 12 \frac{\beta^2(\psi)\beta''(\psi)}{\beta'(\psi)} - 6 \frac{(\beta^2(\psi)\beta''(\psi))'}{\beta'(\psi)} + \left[ \frac{\beta^2(\psi)}{\beta'(\psi)} \left( \frac{\beta^2(\psi)\beta''(\psi)}{\beta'(\psi)} \right) \right] \right\} \\
+ 2 \left\{ \rho - \mu^* f(\psi) \right\} \frac{\beta^2(\psi)\beta''(\psi)}{\beta'(\psi)} + \mu^* f(\psi) f'(\psi) \frac{\beta(\psi)\beta'(\psi)}{\beta'(\psi)} \\
+ m_1 \left[ \mu - \frac{2\alpha_1}{\beta'(\psi)} \right] \left( \frac{\beta^2(\psi)\beta''(\psi)}{\beta'(\psi)} \right) \frac{\partial^2 \omega}{\partial \psi^2} + \mu \left( 2(1 + m_1^2) \frac{(\beta(\psi)\beta''(\psi))'}{\beta'(\psi)} \right) \\
+ (1 + m_1^2) \left[ \frac{\beta^2(\psi)}{\beta'(\psi)} \left( \frac{\beta'(\psi)}{\beta'(\psi)} \right) \right] \frac{\partial^2 \omega}{\partial \psi^2} + \mu \left( 2(1 + m_1^2) \left[ \frac{\beta(\psi)\beta''(\psi)}{\beta'(\psi)} \right] \right) \\
+ 4m_1^2 \left[ \frac{\beta(\psi)}{\beta'(\psi)} \left( \frac{\beta'(\psi)}{\beta'(\psi)} \right) \right] \frac{\partial^2 \omega}{\partial \psi^2} \\
+ m_1 \left[ \mu + \frac{2\alpha_1}{\beta'(\psi)} \right] \left( \frac{\beta''(\psi)}{\beta'(\psi)} \right) \frac{\partial^2 \omega}{\partial \psi^2} + \mu \left[ \frac{\beta(\psi)}{\beta'(\psi)} \left( \frac{\beta'(\psi)}{\beta'(\psi)} \right) \right] \frac{\partial^2 \omega}{\partial \psi^2} \\
+ \left[ \frac{1}{\beta'(\psi)} \left( \frac{\beta'(\psi)}{\beta'(\psi)} \right) \right] \frac{\partial^2 \omega}{\partial \psi^2} + (1 + m_1^2) \left[ \mu + \frac{4\alpha_1}{\beta'(\psi)} \right] \left[ \frac{1}{\beta'(\psi)} \left( \frac{\beta'(\psi)}{\beta'(\psi)} \right) \right] \frac{\partial^2 \omega}{\partial \psi^2} \\
\equiv \sum_{n=0}^{4} A_{2n}(\psi)x^{2n} = 0 
\]

(4.29)

The fact that \( x, \psi \) are independent variables and \( \{1, x^2, x^4, x^6, x^8\} \) is a linearly independent set implies \( A_{2n}(\psi) = 0 \) for \( n = 0, 1, 2, 3, 4 \). \( A_8(\psi) = 0 \) gives

\[
\beta'(\psi) = -\frac{4\alpha_1}{\mu} 
\]

(4.30)
or
\[ \frac{\beta''(\psi)}{\beta^2(\psi)} = a_1 \beta(\psi) + a_2 \] (4.31)

where \(a_1\) and \(a_2\) are arbitrary constants.

When equation (4.30) is introduced into \(A_{2n}(\psi) = 0\) for \(n = 0, 1, 2, 3, 4\), we note that these are identically satisfied only if \(f(\psi)f'(\psi) = 0\) or \(f(\psi) = f_0\), a non-zero real constant. Taking the case when equation (4.31) holds true, we observe that \(A_{2n}(\psi) = 0\) for \(n = 0, 1, 2, 3, 4\) are satisfied simultaneously only if
\[
[r - \mu^* f^2(\psi)] [a_1 \beta(\psi) + a_2] \beta''(\psi) + \mu^* f(\psi)f'(\psi) + 6m_2 a_1 [\mu \beta'(\psi) - 2\alpha_1] \beta'(\psi) = 0
\]
\[
[a_1 \beta(\psi) + a_2] \beta'(\psi) = 0
\]
\[
[r - \mu^* f^2(\psi)] [a_1 \beta(\psi) + a_2] \beta''(\psi) + \mu^* f(\psi)f'(\psi) + 6m_1 a_1 [\mu \beta'(\psi) + 2\alpha_1] \beta'(\psi) = 0
\]
(4.32)
or only if \(a_1 \beta(\psi) + a_2 = 0\), \(m_1 a_1 = 0\)
or only if \(a_1 = a_2 = 0\).

This result implies, from equations (4.31) and (4.32), that \(\beta''(\psi) = 0\) and \(f(\psi)f'(\psi) = 0\).

Therefore, (4.25) takes the form
\[ x(y - m_1 x - m_2) = c_1 \psi + c_2 \] (4.33)

where \(c_1 \neq 0\) and \(c_2\) are real constants. \(c_1 = -\frac{4m_1}{\mu}\) if possibility (4.30) applies. The rotational flow described above in an unbounded domain has the exact integral
\[ u = \frac{x}{c_1}, \quad v = \frac{1}{c_1} (2m_1 x - y + m_2), \quad H_1 = f_0 u, \quad H_2 = f_0 v \]
\[ p = p_0 + \frac{1}{2c_1^2} \left[ r(2m_1 y - y^2 - x^2) + 4m_1 \mu^* f_0^2 x(y - m_1 x - m_2) - \rho m_2^2 \right]
+ 4(3\alpha_1 + 2\alpha_2)(1 + m_1^2) \] (4.34)

where \(p_0\) and \(f(\psi) = f_0 \neq 0\) are arbitrary real constants. Equation (4.33) represents an impingement of two constant-vorticity oblique flows with stagnation point at \((0, m_2)\).
The boundary conditions specified on a plate located at \( z = 0 \) are

\[
  u(0, y) = 0, \quad v(0, y) = v_0 y + v_1; \quad v_0 \neq 0
\]

\[
  H^L(0, y) = H^R(0, y)
\]

where \( H^L(x, y) = (f_0 u, f_0 v) \) and \( H^R(x, y) = (A v_0 x, A (v_0 y + v_1)) \) for \( A \neq 0 \).

These give

\[
  c_1 = -\frac{1}{v_0}, \quad m_2 = -\frac{v_1}{v_0}, \quad f_0 = A
\]

Therefore, the exact integral of the equations of a steady, plane, incompressible, electrically conducting second grade MHD aligned fluid having infinite conductivity along the streamlines \( x(y - m_1 x - m_2) = \text{constant} \) when the fluid impinges on a non-porous, stretching plate situated at \( z = 0 \) is

\[
  u = -v_0 x, \quad v = v_0 (y - 2m_1 x) + v_1, \quad H_1 = Au, \quad H_2 = Av
\]

\[
  p = p_0 + \frac{1}{2} \left\{ 4v_0^2 (3\alpha_1 + 2\alpha_2) (1 + m_1^2) - \rho v_1^2 - \rho v_0 \left[ v_0 (x^2 + y^2) + 2v_1 y \right] \\
  + 4m_1 v_0 \mu A x \left[ v_0 (y - m_1 x) + v_1 \right] \right\}
\]

(4.36)

where \( m_1 \) is an arbitrary real constant.

At the plate \( z = 0 \), the normal and shear stresses are

\[
  T_{11} = -p_0 + \frac{1}{2} \left[ \rho v_1^2 - 4\mu v_0 + \rho v_0 y (v_0 y + 2v_1) + 4\alpha_1 v_0^2 (m_1^2 - 1) \right]
\]

\[
  T_{22} = -p_0 + \frac{1}{2} \left[ \rho v_1^2 + 4\mu v_0 + \rho v_0 y (v_0 y + 2v_1) - 4\alpha_1 v_0^2 (3m_1^2 + 1) \right]
\]

(4.37)

\[
  T_{12} = -2m_1 v_0 (\mu + 2\alpha_1 v_0)
\]

The flow pattern in the unbounded domain and for the boundary value problem are given in Figures 3.7 and 3.8.

Finitely Conducting Flows.

Equation (4.6), the condition for the existence of aligned flows of a finitely conducting fluid, on being multiplied through by \( x^2 \), becomes

\[
  \left[ (1 + m_1^2) x^4 + \beta^2(\psi) \right] \left[ \frac{f(\psi)\beta''(\psi)}{\beta'(\psi)} - f'(\psi) \right] = 0
\]

(4.38)
The fact that \( z, \psi \) are independent variables and \( \{1, x^4\} \) is a linearly independent set implies that the coefficients of unity and \( x^4 \) will equate to zero. Since \( \beta'(\psi) \neq 0 \), \( (1 + m_2^2) \neq 0 \), we have

\[
\frac{\beta''(\psi)}{\beta'(\psi)} - \frac{f'(\psi)}{f(\psi)} = 0
\]

which gives \( f(\psi) = b\beta'(\psi) \), where \( b \) is an arbitrary constant. Equations (4.32) for the infinitely conducting flow hold true for the finitely conducting case, provided

\[
f(\psi)f'(\psi) - f^2(\psi)\left[a_1\beta(\psi) + a_2\right]\beta''(\psi) = 0
\]

The remainder of the equations may be solved to obtain \( a_1 = a_2 = 0 \) or \( \beta''(\psi) = 0 \), which gives \( f(\psi) = bc \psi \) when \( \beta'(\psi) = c_1\psi + c_2 \), where \( c_1 \neq 0 \), \( c_2 \) are real constants. From (4.26) and (4.27), with \( j = j_0 \) (constant current density), we get \( b = \frac{j_0}{2m_1} \).

If \( f_0 = \frac{j_0}{2m_1} \) is substituted into solution (4.34), we obtain the exact integral for finitely conducting MHD aligned flow along the streamlines \( x(y - m_1x - m_2) = \) constant in unbounded domain. This value of \( f_0 \) also applies to the results obtained when the fluid impinges on a plate situated at \( x = 0 \).

**Example 3.**

Letting \( y - m_1x^2 - m_2x = \) constant be a family of streamlines, (4.1) takes the form

\[
y - m_1x^2 - m_2x = \beta(\psi), \quad \beta'(\psi) \neq 0
\]

(4.39)

where \( m_1 \neq 0 \), \( m_2 \) are real constants.

**Infinitely Conducting Flows.**

Substituting \( g(x) = m_1x^2 + m_2x, l(x) = 1 \) into (4.3) to (4.5), and following the procedure of Example 1, we obtain \( \beta''(\psi) = 0 \) and \( f(\psi) = f_0 \), so that

\[
y - m_1x^2 - m_2x = c_1\psi + c_2
\]

(4.40)
where \(f_0 \neq 0, c_1 \neq 0, c_2\) are real constants. The exact solution for system (2.25) in unbounded domain is
\[
\begin{align*}
  u &= \frac{1}{c_1}, \quad v = \frac{1}{c_1} (2m_1 x + m_2), \quad H_1 = f_0 u, \quad H_2 = f_0 v \\
  p &= p_0 + \frac{1}{2c_1^2} \left[ 4m_1^2 (3\alpha_1 + 2\alpha_2) - (\rho + m_2^2 \mu^2 f_0^2) - 4m_1 \rho y \right. \\
  &\quad \left. + 4m_1 \mu^2 f_0^2 (y - m_1 x^2 - m_2 x) \right]
\end{align*}
\]
where \(p_0\) and \(f_0\) are real constants.

The boundary conditions at a plate located at \(x = 0\) are
\[
\begin{align*}
  u(0, y) = u_0, \quad v(0, y) = v_0; \quad u_0 \neq 0 \\
  \mathbf{H}^L(0, y) = \mathbf{H}^R(0, y)
\end{align*}
\]
where \(\mathbf{H}^L(x, y) = (H_1, H_2) = (f_0 u, f_0 v)\) and \(\mathbf{H}^R(x, y) = (A, B)\) for real constants \(A \neq 0\) and \(B\). We get, from (4.42),
\[
\begin{align*}
  c_1 &= \frac{1}{u_0}, \quad m_2 = \frac{v_0}{u_0} = \frac{B}{A}, \quad f_0 = \frac{A}{u_0}
\end{align*}
\]
Therefore, the exact solution for this infinitely conducting rotational flow impinging on a plate \(x = 0\) with boundary conditions (4.42) is
\[
\begin{align*}
  u &= u_0, \quad v = 2m_1 u_0 x + v_0, \quad H_1 = A, \quad H_2 = \frac{A}{u_0} y \\
  p &= p_0 + \frac{1}{2} \left[ 4m_1^2 u_0^2 (3\alpha_1 + 2\alpha_2) - (\rho u_0^2 + \mu^2 B^2) - 4m_1 u_0^2 \rho y \right. \\
  &\quad \left. + 4m_1 \mu^2 A (Ay - A m_1 x^2 - Bx) \right]
\end{align*}
\]
where \(m_1 \neq 0\) is an arbitrary real constant, and represents uniform suction \((u_0 > 0)\) or blowing \((u_0 < 0)\) at the plate \(x = 0\), which is stretching uniformly. If \(v_0 = 0\) (or \(m_2 = B = 0\)), the plate is fixed. For flows in unbounded and bounded domains, the streamline patterns are as shown in Figures 3.1 and 3.2.

The components of the normal stress \(T_{11}, T_{22}\) and the shear stress \(T_{12}\) at the plate \(x = 0\) are
\[
\begin{align*}
  T_{11} &= -p_0 + \frac{1}{2} \left[ (\rho u_0^2 + \mu^2 B^2 + 4m_1^2 u_0^2 \alpha_1) + 4m_1 (u_0^2 \rho - \mu^2 A^2) y \right] \\
  T_{22} &= -p_0 + \frac{1}{2} \left[ (\rho u_0^2 + \mu^2 B^2 - 12m_1^2 u_0^2 \alpha_1) + 4m_1 (u_0^2 \rho - \mu^2 A^2) y \right] \\
  T_{12} &= 2m_1 u_0 \rho
\end{align*}
\]
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We have, from equation (4.6),
\[
(2m_1 x + m_2) \left[ \frac{f(\psi) \beta''(\psi)}{\beta'(\psi)} - f'(\psi) \right] = 0
\]  
(4.45)
which gives \( f(\psi) = a \beta'(\psi) \), where \( a \) is an arbitrary constant. Employing equation (4.3) in equations (4.4) and (4.5) with \( j = j_0 \) (non-zero real constant), we get 
\[ \beta''(\psi) = 0 \text{ and } f(\psi) = ac, \]  
where \( \beta(\psi) = c_1 \psi + c_2 \) with \( c_1 \neq 0, c_2 \) being real constants. From equation (4.4), we get \( j_0 = 2m_1 a \).

Thus, if \( f_0 \) is replaced by \( \frac{j_0}{2m_1} \) in solution (4.41), we obtain the exact solution for finitely conducting flows along the streamlines \( y - m_1 x^2 - m_2 x = \text{constant} \).

Example 4.

Taking \( y - e^{m_1 x} - e^{n_2 x} - m_1 x^2 - m_2 x = \text{constant} \) as the streamlines, we have \( y - e^{m_1 x} - e^{n_2 x} - m_1 x^2 - m_2 x = \beta(\psi), \beta'(\psi) \neq 0 \)  
(4.46)
where \( m \neq 0, n \neq 0, m_1 \) and \( m_2 \) are real constants.

Infinitely Conducting Flows.

Proceeding as in Example 1, we get \( \beta''(\psi) = 0 \) with \( \beta'(\psi) = -\frac{\alpha_1 (m+n)}{\mu} \neq 0 \), where \( f(\psi) = f_0, mn = \frac{\mu^2 f_0^2 - \rho}{\alpha_1} \), \( f_0 \) being a non-zero real constant. Equation (4.46) takes the form
\[
y - e^{m_1 x} - e^{n_2 x} - m_1 x^2 - m_2 x = -\frac{\alpha_1 (m+n)}{\mu} \psi + a
\]  
(4.47)
where \( a \) is an arbitrary constant. Thus, the exact integral of the equations of the isochoric, aligned flow of a steady, plane, electrically conducting second grade fluid of infinite electrical conductivity in unbounded domain is
\[
u = \frac{\mu}{\alpha_1 (m+n)}, \quad v = u(m e^{m_1 x} + n e^{n_2 x} + 2m_1 x + m_2), \quad H_1 = f_0 u, \quad H_2 = f_0 v
\]
\[
p = p_0 + \frac{u^2}{2} \left[ 4\alpha_1 mnm_1 y + 2(2\alpha_1 + \alpha_2)(m^2 e^{m_1 x} + n^2 e^{n_2 x} + 2m_1)^2 \\
- \mu^2 f_0^2 (me^{m_1 x} + ne^{n_2 x} + 2m_1 x + m_2)^2 - \rho \right]
\]  
(4.48)
where \( p_0 \) is an arbitrary constant.

We specify the boundary conditions

\[
\begin{align*}
  u(0, y) &= u_0, \quad v(0, y) = v_0; \quad u_0 \neq 0, \\
  H^L(0, y) &= H^R(0, y) 
\end{align*}
\]

(4.49)

at the plate located at \( x = 0 \). Here \( H^L(x, y) = (f_0 u, f_0 v) \) and \( H^R(x, y) = (A, B) \), where \( A \neq 0, B \) are real constants. The boundary conditions (4.49) give

\[
\begin{align*}
  m + n &= -\frac{\mu}{u_0 \alpha_1}, \quad m_2 = \frac{v_0}{u_0} - (m + n), \quad v_0 = \frac{B}{A}, \quad f_0 = \frac{A}{u_0}
\end{align*}
\]

With these, the exact integral of this flow on the left of the plate \( x = 0 \) is

\[
\begin{align*}
  u &= u_0, \quad v = u_0 (me^{m_2} + ne^{n_2} + 2m_1 x - m - n) + v_0, \quad H_1 = A, \quad H_2 = \frac{A}{u_0} v \\
  p &= p_0 + \frac{1}{2} \left\{ 4\alpha_1 mn m_1 u_0 y + 2u_0^2 (2\alpha_1 + \alpha_2) (m^2 e^{m_2} + n^2 e^{n_2} + 2m_1) \right\} \\
  &\quad - \mu^* \left[ A (me^{m_2} + ne^{n_2} + 2m_1 x - m - n) + B \right] - \rho u_0^2 
\end{align*}
\]

(4.50)

where \( m_1 \) is an arbitrary real constant.

\( v_0 = 0 \) (or \( m_2 = -(m + n) \)) represents a flow impinging on a non-stretching plate at \( x = 0 \) with uniform suction or blowing. The streamlines for flows in the unbounded domain and for the boundary value problem are given in Figures 3.5 and 3.6.

The normal and shear stress components at the plate \( x = 0 \) corresponding to solution (4.50) are

\[
\begin{align*}
  T_{11} &= -p_0 + \frac{1}{2} \left[ \rho u_0^2 + \mu^* B^2 - 4\alpha_1 mn m_1 u_0^2 y \right] \\
  T_{22} &= -p_0 + \frac{1}{2} \left[ \rho u_0^2 + \mu^* B^2 - 4\alpha_1 u_0^2 (m^2 + n^2 + 2m_1)^2 - 4\alpha_1 mn m_1 u_0^2 y \right] \\
  T_{12} &= u_0 \left[ \mu (m^2 + n^2 + 2m_1) + \alpha_1 u_0 (m^3 + n^3) \right]
\end{align*}
\]

(4.51)
Finely Conducting Flows.

Equation (4.6) gives

\[
\frac{2}{\beta'(\psi)} \left[ \frac{f(\psi)\beta''(\psi)}{\beta'(\psi)} - f'(\psi) \right] \left( m_2 e^{m_2 x} + n_2 e^{n_2 x} + 2m_1 \right) \left( m_1 e^{m_1 x} + n_1 e^{n_1 x} + 2m_1 x + m_2 \right) f(\psi) \left( m_2 e^{m_2 x} + n_3 e^{n_3 x} \right) = 0
\]

(4.52)

The fact that \( z, \psi \) are independent variables and \( \{1, z, e^{m_2 x}, e^{n_2 x}, x e^{m_2 x}, x e^{n_2 x}, e^{2m_2 x}, e^{(m+n)z} \} \) is a linearly independent set, we have that the coefficients of these functions of \( z \) must equate to zero. The coefficients of unity and \( e^{m_2 z} \) give

\[
\frac{m_1 m_2}{\beta'(\psi)} \left[ \frac{f(\psi)\beta''(\psi)}{\beta'(\psi)} - f'(\psi) \right] = 0
\]

(4.53)

Equations (4.53) together give \( f(\psi) = 0 \), which implies that finitely conducting MHD aligned second grade fluid flow along the curves \( y - e^{m_2 x} - e^{n_2 x} - m_1 x^2 - m_2 x = \) constant is not possible.

Example 5.

Letting \( y - x^3 = \) constant denote a streamline pattern, we have

\[
y - x^3 = \beta(\psi), \quad \beta'(\psi) \neq 0
\]

(4.54)

Infinitely Conducting Flows.

Taking \( g(z) = z^3, l(z) = 1 \), and employing the method of Example 1, we obtain

\( f(\psi) = \sqrt{\frac{\rho}{\mu^2}} \) and \( \beta''(\psi) = 0 \) with \( \beta(\psi) = c_1 \psi + c_2 \), where \( c_1 \neq 0, c_2 \) are arbitrary constants. Equation (4.54) becomes

\[
y - x^3 = c_1 \psi + c_2
\]

(4.55)

and the exact solution for this rotational flow in an unbounded domain is

\[
u = \frac{1}{c_1}, \quad v = \frac{3x^2}{c_1}, \quad H_1 = \sqrt{\frac{\rho}{\mu^2}} u, \quad H_2 = \sqrt{\frac{\rho}{\mu^2}} v
\]

\[
p = p_0 + \frac{1}{2c_1^2} \left[ 12 \mu c_1 y + 72 (2\alpha_1 + \alpha_2) x^2 - 9 \rho x^4 - \rho \right]
\]

(4.56)
where \( p_0 \) is an arbitrary constant. Unlike the flow of a second grade non-MHD fluid flow along the curves \( y - x^3 = \) constant, we find that the above infinitely conducting flow is possible.

The boundary conditions specified on the plate situated at \( x = 0 \) are:
\[
\begin{align*}
u(0, y) &= u_0, & \psi(0, y) &= 0; & u_0 \neq 0 \\
H^L(0, y) &= H^R(0, y)
\end{align*}
\]

(4.57)

where \( H^L(x, y) = (H_1, H_2) \) and \( H^R(x, y) = (\sqrt{\frac{\rho}{\mu^*}}u, x) \). The fluid occupies the left half plane \( x < 0 \). These imply that \( c_1 = \frac{1}{u_0} \). Therefore, the exact solution for this infinitely conducting flow which impinges on a plate \( x = 0 \) is:
\[
\begin{align*}
u &= u_0, & \psi &= 3u_0 x^2, & H_1 &= \sqrt{\frac{\rho}{\mu^*}}u_0, & H_2 &= \sqrt{\frac{\rho}{\mu^*}}v \\
p &= p_0 + \frac{u_0}{2} \left\{ 12\mu y + u_0 \left[ 72(2\alpha_1 + \alpha_2)x^2 - 9\rho x^4 - \rho \right] \right\}
\end{align*}
\]

(4.58)

The normal and the shear stresses for this flow at the plate \( x = 0 \) are:
\[
\begin{align*}
T_{11} &= -p_0 + \frac{u_0}{2} (\rho u_0 - 12\mu y) \\
T_{22} &= -p_0 + \frac{u_0}{2} (\rho u_0 - 12\mu y) \\
T_{12} &= 6\alpha_1 u_0^2
\end{align*}
\]

(4.59)

Figures 4.1 and 4.2, respectively, give the flow patterns in the unbounded and bounded domains.

**Finitely Conducting Flows.**

Following the procedure of Example 4, we find that \( f(\psi) = 0 \). This means that second grade finitely conducting MHD aligned flow along the curves \( y - x^3 = \) constant is not possible. This is the case for a second grade non-MHD fluid flow along these curves.

**Example 6.**

Let \( x^2 y = \) constant be a family of streamlines. It follows that
\[
x^2 y = \beta(\psi), \quad \beta'(\psi) \neq 0
\]

(4.60)
Infinitely Conducting Flows.

In this case, \(g(x) = 0, l(x) = x^{-2}\). Proceeding as in Example 2, we get \(f(\psi) = \sqrt{\frac{\rho}{\mu^*}}\) and \(\beta''(\psi) = 0\). Therefore, equation (4.60) takes the form

\[x^2y = c_1\psi + c_2\] (4.61)

where \(c_1 \neq 0, c_2\) are real arbitrary constants. The exact integral for the rotational, second grade, infinitely conducting MHD aligned fluid flow along the curves \(x^2y = \) constant in an unbounded domain is

\[u = \frac{x^2}{c_1}, \quad v = -\frac{2xy}{c_1}, \quad H_1 = \sqrt{\frac{\rho}{\mu^*}}u, \quad H_2 = \sqrt{\frac{\rho}{\mu^*}}v\]

\[p = p_0 + \frac{1}{2c_1^2} \left[ 4\mu c_1 x - \rho x^2(x^2 + y^2) + 4\alpha_1(13x^2 + 3y^2) + 8\alpha_2(4x^2 + y^2) \right] \]

where \(p_0\) is an arbitrary constant of integration.

We prescribe the boundary conditions

\[u(0, y) = v(0, y) = 0\] (4.63)

\[\mathbf{H}^L(0, y) = \mathbf{H}^R(0, y)\]

on a plate located at \(x = 0\), where \(\mathbf{H}^L(x, y) = (H_1, H_2)\) and \(\mathbf{H}^R(x, y) = (Ax^2, -2Ax y)\). The flow is confined to the region \(x < 0\). Thus, the exact integral for this flow impinging orthogonally on a rigid non-porous plate \(z = 0\) is (4.62), where \(c_1 \neq 0\) is left arbitrary. The streamline patterns in the unbounded domain and for the boundary value problem are shown in Figures 4.3 and 4.4, respectively.

The normal components \(T_{11}, T_{22}\) of the stress tensor, and the shear stress at the plate \(z = 0\) are

\[T_{11} = -p_0 + \frac{2\alpha_1}{c_1^2} y^2\]

\[T_{22} = -p_0 - \frac{6\alpha_1}{c_1^2} y^2\] (4.64)

\[T_{12} = -\frac{2\mu}{c_1} y\]
Finitely Conducting Flows.

Employing \( g(x) = 0, \ l(x) = x^{-2} \) of Example 1, we obtain \( f(\psi) = 0 \). We, therefore, conclude that second grade, finitely conducting flow along the curves \( x^2 y = \text{constant} \) is impossible.

Example 7.

We assume that the family of curves \( y(1 + m_1 e^{m_2}) - m_2 e^{m_2} = \text{constant} \) constitutes a permissible streamline pattern. Then

\[
y(1 + m_1 e^{m_2}) - m_2 e^{m_2} = \beta(\psi), \quad \beta'(\psi) \neq 0
\]  

(4.65)

where \( m \neq 0, m_1, m_2 \) are real constants.

Infinitely Conducting Flows.

Substituting \( g(x) = \frac{m_2 e^{m_2}}{1 + m_1 e^{m_2}}, \ l(x) = \frac{1}{1 + m_1 e^{m_2}} \) into equations (4.3) through (4.5), and following the procedure of Example 2, we obtain \( f(\psi) = f_0 \) and \( \beta''(\psi) = 0 \) with \( \beta'(\psi) = \frac{\rho - \mu^* f_0^2 - \alpha_1 m_2^2}{\mu m} \neq 0 \) so that

\[
y(1 + m_1 e^{m_2}) - m_2 e^{m_2} = \left( \frac{\rho - \mu^* f_0^2 - \alpha_1 m_2^2}{\mu m} \right) \psi + c
\]  

(4.66)

where \( f_0 \neq 0, c \) are real constants. The exact solution for this rotational, infinitely conducting fluid flow, in an unbounded domain, is

\[
u = \frac{\mu m}{\rho - \mu^* f_0^2 - \alpha_1 m_2} (1 + m_1 e^{m_2}), \quad \mu m_2 \]

\[
H_1 = f_0 u, \quad H_2 = f_0 v, \quad p = p_0 + \frac{\mu^2 m^2}{2(\rho - \mu^* f_0^2 - \alpha_1 m_2)} \left\{ 2m_1^2 (2\alpha_1 + \alpha_2) \left[ 2m_2 (2\alpha_1 + \alpha_2) \left[ 4m_1^2 + m_2 (m_1 - m_2)^2 \right] - m_1 (\rho m_2 (2\alpha_1 + \alpha_2)) \right] e^{2m_2} - \rho \right\}
\]  

(4.67)

where \( p_0 \) is an arbitrary constant.

Imposing the boundary conditions

\[
u(0, y) = u_0, \quad v(0, y) = v_0 y + v_1; \quad v_0 \neq 0
\]  

(4.68)

\[
H^L(0, y) = H^R(0, y)
\]
on a plate situated at \( z = 0 \), where \( \mathbf{H}^L(x, y) = (f_0 u, f_0 v) \) and \( \mathbf{H}^R(x, y) = (-v_0 x, v_0 y + v_1) \), we get

\[
m_1 = -1, \quad m_2 = \frac{v_1}{v_0}, \quad u_0 = 0, \quad m_2 = \frac{v_0 (\rho - \mu^* f_0^2)}{\mu + \alpha_1 v_0}
\]

We, therefore, have that the flow of an infinitely conducting second grade fluid along the streamlines \( y(1 + m_1 e^{m_2 x}) - m_2 e^{m_2 x} = \text{constant} \) when the fluid impinges on a plate \( z = 0 \) is

\[
u = \frac{v_0}{m} (1 - e^{m_2 x}), \quad v = (v_0 y + v_1) e^{m_2 x}, \quad H_1 = f_0 u, \quad H_2 = f_0 v
\]

\[
p = p_0 + \frac{1}{2m^2} \left\{ 2m^2 (2\alpha_1 + \alpha_2) \left[ 4v_0^2 + m^2 (v_0 y + v_1)^2 \right] + \mu^* f_0^2 \left[ m^2 (v_0 y + v_1)^2 - v_0^2 \right] - (\rho + 2\alpha_1 m^2) e^{2m_2 x} - \rho \right\}
\]

where \( f_0 \neq 0 \) is kept arbitrary and \( m_1 = -1 \).

It should be noted that plane, second grade non-MHD fluid flow admits a solution for \( m_1 \neq 0, -1 \). The streamlines for the flows in an unbounded domain and for the boundary value problem are, respectively, in Figures 3.9 and 3.10.

The normal and shear stresses at the plate \( z = 0 \) are

\[
T_{11} = -p_0 + \frac{1}{2m^2} \left\{ 2(\rho - 2\mu m^2 v_0) + 2\alpha_1 m^2 (1 - 4v_0^2) + \mu^* f_0^2 \left[ v_0^2 - m^2 (v_0 y + v_1)^2 \right] \right\}
\]

\[
T_{22} = -p_0 + \frac{1}{2m^2} \left\{ 2(\rho + 2\mu m^2 v_0) + 2\alpha_1 m^2 \left[ 1 - 4v_0^2 - 2m^2 (v_0 y + v_1)^2 \right] + \mu^* f_0^2 \left[ v_0^2 - m^2 (v_0 y + v_1)^2 \right] \right\}
\]

\[
T_{12} = m(\mu + 3\alpha_1 v_0) (v_0 y + v_1)
\]

(4.70)

Finitely Conducting Flows.

Following the procedure of Example 6, we find that \( f(\psi) = 0 \), implying that second grade, finitely conducting MHD aligned flow along the curves \( y(1 + m_1 e^{m_2 x}) - m_2 e^{m_2 x} = \text{constant} \) is impossible.
Streamlines for $y - x^3 = \text{constant}$ in unbounded domain.

Figure 4.1
Streamlines for $y - x^3 = \text{constant}$ for boundary value problem.

Figure 4.2
Streamlines for $x^2y = \text{constant}$ in unbounded domain.

Figure 4.3
Streamlines for $x^2y = \text{constant}$ for boundary value problem.

Figure 4.4
CHAPTER V

REVERSED AND
NON-REVERSED FLOWS OF
A SECOND GRADE FLUID

5.1 INTRODUCTION.

Two-dimensional Navier-Stokes equations were studied by Kovasznay (1948) by assuming the stream function to be of the form $\psi(x, y) = y + f(x)\sin 2\pi y$. He obtained a laminar flow behind a two-dimensional grid and a flow of alternating vortices superposed on a main flow perpendicular to their plane.

We wish to study steady, plane, incompressible, second grade fluid flow by assuming that $\psi(x, y) = y + f(x)e^{ky}$, where $k$ is a real constant, and $f(x)$ is a function to be determined. When the Weissenberg number is zero, some of the results of Lin and Tobak (1986) and Hui (1987) are realized. Reversed and non-reversed flows are obtained.

The method employed in this study is the inverse method [cf. Neményi (1951)]. It consists of making certain hypotheses a priori on the form of the velocity field and the pressure, without making any on the boundaries of the domain occupied by the fluid. These hypotheses are often made on the velocity field and rarely on the pressure. This method has been extensively used by researchers such as Taylor (1923), Riabouchinsky (1924), and Siddiqui and Kaloni (1986), to mention a few.
5.2 FLOW EQUATIONS AND METHOD.

The basic equations governing the steady, isochoric, planar motion of a second
grade fluid, in nondimensional variables, are given by system (3.3).

The normal and shear stress components are given by equations (3.6) and (3.7).

Introducing the vorticity function \( \omega \) and the energy function \( h \) given by system
(3.4), the system (3.3) becomes system (3.5) for the functions \( u, v, \omega \) and \( h \) of \( x, y \).

Applying the integrability condition \( \frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x} \) to the linear momentum equa-
tions in system (3.4), we get the compatibility equation

\[
\frac{u}{Re} \frac{\partial \omega}{\partial x} + \frac{v}{Re} \frac{\partial \omega}{\partial y} = \frac{1}{Re} \nabla^2 \omega + \frac{W \varepsilon}{Re} \left[ u \frac{\partial}{\partial x} (\nabla^2 \omega) + v \frac{\partial}{\partial y} (\nabla^2 \omega) \right]
\] (5.1)

The continuity equation implies the existence of some stream function \( \psi(x, y) \)
such that equations (2.30) hold true.

Employing (5.2) in the vorticity equation and the compatibility equation (5.1),
we get \( \omega = -\nabla^2 \psi \) and the vorticity transport equation

\[
\nabla^4 \psi + \frac{W \varepsilon}{Re} \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, y)} - \frac{W \varepsilon}{Re} \frac{\partial (\psi, \nabla^4 \psi)}{\partial (x, y)} = 0
\] (5.3)

The general solution of the inherently nonlinear set of equations of the second
grade fluid flow is not known. Some exact solutions can be found using the inverse
method by assuming the stream function in the form

\[
\psi(x, y) = y + f(x)e^{ky}
\] (5.4)

where \( k \) is a real constant, and \( f(x) \neq 0 \) is some function to be determined such
that the vorticity function is proportional to the stream function perturbed by a
uniform stream parallel to the \( z \)-axis, in the cases when the flows are rotational.
Employing (5.4) in (5.3), we obtain
\[
\begin{align*}
\left\{ W e f''(x) + f^i v(x) + (2k^2 W e - Re)f'''(x) + 2k^2 f''(x) + k^2 W e f'(x) + k^4 f(x) \right\}e^{k y} + k \left\{ W e [f(x)f^i v(x) - f'(x)f^i v(x)] \right\} e^{2k y} &= 0 \\
+ (2k^2 W e - Re) [f(x)f'''(x) - f'(x)f''(x)] \right\} e^{2k y} &= 0
\end{align*}
\] (5.5)

Since \( z, y \) are independent variables and \( \{ e^{k y}, e^{2k y} \} \) is a linearly independent set, we have that \( f(x) \) in (5.4) must satisfy the system of two equations:
\[
\begin{align*}
W e f''(x) + f^i v(x) + (2k^2 W e - Re)f'''(x) + 2k^2 f''(x) \\
+ k^2 (k^2 W e - Re)f'(x) + k^4 f(x) &= 0
\end{align*}
\] (5.6)

\[
W e [f(x)f^i v(x) - f'(x)f^i v(x)] + (2k^2 W e - Re) [f(x)f'''(x) - f'(x)f''(x)] = 0
\] (5.7)

Dividing (5.7) by \( f^2(x) \) and integrating with respect to \( x \) yields the modified form of (5.7):
\[
W e f^i v(x) + (2k^2 W e - Re)f''(x) + Ef(x) = 0
\] (5.8)

where \( E \) is an arbitrary constant of integration.
5.3 EXACT SOLUTIONS.

We shall obtain flows corresponding to \( k = 0 \) and \( k \neq 0 \) by finding \( f(x) \) which satisfies (5.6) and (5.8).

\( k = 0. \)

Equations (5.6) and (5.8) take the forms

\[
Wef^v(x) + f^{iv}(x) - Ref'''(x) = 0 \tag{5.9}
\]

\[
Wef^{iv}(x) - Ref''(x) + Ef(x) = 0 \tag{5.10}
\]

The general solution of (5.9) is

\[
f(x) = A_1 e^{m_1 x} + A_2 e^{m_2 x} + A_3 x^2 + A_4 x + A_5 \tag{5.11}
\]

where \( A_i \) for \( i = 1,2,3,4,5 \) are arbitrary constants, and

\[
m_{1,2} = \frac{-1 \pm \sqrt{1 + 4ReWe}}{2We} \tag{5.12}
\]

Substituting (5.11) in (5.10), we obtain

\[
(Wem_1^4 - Rem_1^2 + E)A_1 e^{m_1 x} + (Wem_2^4 - Rem_2^2 + E)A_2 e^{m_2 x}
\]

\[
EA_3 x^2 + EA_4 x + (EA_5 - 2ReA_3) = 0 \tag{5.13}
\]

Equations (5.9) through (5.13) must be simultaneously satisfied, and a non-trivial solution for \( f(x) \) is given by (5.11) only if one of the underlisted conditions holds true:

(i) \( E = A_1 = A_2 = A_3 = 0 \)

(ii) \( E = m_1^2, \ A_2 = A_3 = A_4 = A_5 = 0 \)

(iii) \( E = m_2^2, \ A_1 = A_3 = A_4 = A_5 = 0. \)

In case (i), the stream function (5.4) becomes

\[
\psi(x,y) = y + A_4 x + A_5 \tag{5.14}
\]
which represents a linear irrotational flow. The exact solution for this flow is

\[ u = 1, \quad v = -A_4, \quad p = p_0 - \frac{1}{2}(1 + A_4^2) \]  

(5.15)

where \( p_0 \) is an arbitrary constant of integration.

Cases (ii) and (iii) may be combined to get the stream function

\[ \psi(x, y) = y + A_i e^{m_i x}; \quad i = 1, 2 \]  

(5.16)

where \( m_1, m_2 \) are given by (5.12). These rotational flows, whose vorticities are proportional to their stream functions superposed on a uniform stream, have the exact integrals

\[ u = 1, \quad v = -A_i m_i e^{m_i x}, \quad p = p_0 - \frac{1}{2} + \frac{1}{2 + \gamma} \frac{W e}{Re} A_i^2 m_i^2 e^{2m_i x} \]  

(5.17)

where \( W e m_i^2 + m_i - Re = 0 \), and \( p_0 \) is an arbitrary integration constant. These solutions represent a uniform flow parallel to the \( z \)-axis perturbed by a term which is exponential in \( x \). They may also represent flows above a plate situated along \( y = 0 \) with suction or blowing in accordance with \( A_i m_i \) being positive or negative, respectively.

From equations (3.6) and (3.7), the normal and shear stresses at the plate \( y = 0 \) are

\[ T_{11} = -p_0 + \frac{1}{2} \]
\[ T_{22} = -p_0 + \frac{1}{2} - \frac{2W e}{Re} A_i^2 m_i^2 e^{2m_i x} \]  

(5.18)
\[ T_{12} = -A_i m_i e^{m_i x} \]

where use has been made of \( W e m_i^2 + m_i - Re = 0 \).

\( k \neq 0 \).

It is obvious that neither of equations (5.6) nor (5.8) can be solved for all values of \( k, Re \) and \( We \). We shall, therefore, solve (5.6), whose auxiliary equation is

\[ (\beta^2 + k^2) \left[ We\beta^2 + \beta^2 + (k^2 W e - Re)\beta + k^2 \right] = 0, \]  

(5.19)
for chosen values of \( k, \text{Re} \) and \( \text{We} \). The cubic equation in the bracket of (5.19) will be solved by Maple for assigned values of \( k, \text{Re} \) and \( \text{We} \). We now consider three possibilities:

(I) when all the roots of the cubic equation (in the bracket) are real and distinct;

(II) when the roots of the cubic equation consists of one real root and two complex conjugate roots;

(III) when the roots of the cubic equation are real and one root is repeated.

We study three examples to exemplify these three possibilities.

**Example 1:** \( k = \pm 0.05, \text{Re} = 0.2, \text{We} = 0.4 \).

By these choices for the values of \( k, \text{Re} \) and \( \text{We} \), the cubic equation in (5.19) to be solved is

\[
800\beta^3 + 2000\beta^2 - 398\beta + 5 = 0
\]

Employing Maple, the roots of (5.20) are

\[
\beta_1 = 0.013480990, \quad \beta_2 = 0.172593727, \quad \beta_3 = -2.686080364
\]

Therefore, the general solution of (5.6) with the aforementioned assigned values of \( k, \text{Re} \) and \( \text{We} \) is

\[
f(x) = B_1 e^{\beta_1 x} + B_2 e^{\beta_2 x} + B_3 e^{\beta_3 x} + B_4 \cos \frac{1}{20} x + B_5 \sin \frac{1}{20} x
\]

(5.22)

where \( B_n \) for \( n = 1, 2, 3, 4, 5 \) are arbitrary constants. Since this \( f(x) \) must satisfy (5.8) as well, for the chosen values of \( k, \text{Re} \) and \( \text{We} \), we apply (5.22) to (5.8) to obtain

\[
\left[ \frac{2}{5} \beta_1^4 - \frac{99}{200} \beta_1^2 + E \right] B_1 e^{\beta_1 x} + \left[ \frac{2}{5} \beta_2^4 - \frac{99}{200} \beta_2^2 + E \right] B_2 e^{\beta_2 x} + \left[ \frac{2}{5} \beta_3^4 - \frac{99}{200} \beta_3^2 + E \right] B_3 e^{\beta_3 x} + \left[ \frac{1}{20} x + \frac{1}{20} x \right] = 0
\]

Equations (5.20) to (5.23) hold true simultaneously, and a non-trivial solution for \( f(x) \) is given by (5.22) only if one of the following holds true:
(a) $E = -0.00124, B_1 = B_2 = B_3 = 0$
(b) $E = 0.0000898, B_2 = B_3 = B_4 = B_5 = 0$
(c) $E = 0.0143913, B_1 = B_3 = B_4 = B_5 = 0$
(d) $E = -17.251209, B_1 = B_2 = B_4 = B_5 = 0$.

In Case (a), the stream function (5.4) yields

$$
\psi(x, y) = \begin{cases} 
  y + (B_4 \cos \frac{1}{20} x + B_5 \sin \frac{1}{20} x)e^{0.05y} & ; k = 0.05 \\
  y + (B'_4 \cos \frac{1}{20} x + B'_5 \sin \frac{1}{20} x)e^{-0.05y} & ; k = -0.05
\end{cases}
$$

(5.24)

which represent irrotational flows. The exact solutions for these flows are:

$$
\begin{align*}
  u &= 1 + \frac{1}{20}(B_4 \cos \frac{1}{20} x + B_5 \sin \frac{1}{20} x)e^{0.05y} \\
  v &= \frac{1}{20}(B_4 \sin \frac{1}{20} x - B_5 \cos \frac{1}{20} x)e^{0.05y}, \quad p = p_0 - \frac{1}{2}
\end{align*}
$$

(5.25)

$$
\begin{align*}
  + \frac{1}{40000} \left[ (2\gamma - 47)(B_4'^2 + B_5'^2)e^{0.05y} - 2000(B_4 \cos \frac{1}{20} x + B_5 \sin \frac{1}{20} x) \right] e^{0.05y}
\end{align*}
$$

when $k = 0.05$, and

$$
\begin{align*}
  u &= 1 - \frac{1}{20}(B_4' \cos \frac{1}{20} x + B_5' \sin \frac{1}{20} x)e^{-0.05y} \\
  v &= \frac{1}{20}(B_4' \sin \frac{1}{20} x - B_5' \cos \frac{1}{20} x)e^{-0.05y}, \quad p = p_0 - \frac{1}{2}
\end{align*}
$$

(5.26)

$$
\begin{align*}
  + \frac{1}{40000} \left[ (2\gamma - 47)(B_4'^2 + B_5'^2)e^{-0.05y} + 2000(B_4' \cos \frac{1}{20} x + B_5' \sin \frac{1}{20} x) \right] e^{-0.05y}
\end{align*}
$$

when $k = -0.05$, where $p_0$ is an arbitrary constant of integration.

Cases (b) through (d) may be combined to get the stream function

$$
\psi(x, y) = \begin{cases} 
  y + B_n e^{(\beta_n x + 0.05y)} & ; k = 0.05 \\
  y + B'_n e^{(\beta_n x - 0.05y)} & ; k = -0.05
\end{cases}
$$

(5.27)

where $\beta_n$ for $n = 1, 2, 3$ are given by (5.21). These are rotational flows with exact solutions

$$
\begin{align*}
  u &= 1 + \frac{1}{20}B_n e^{(\beta_n x + 0.05y)}, \quad v = -B_n \beta_n e^{(\beta_n x + 0.05y)} \\
  p &= p_0 - \frac{1}{2} + 2(2 + \gamma)(\beta_n^2 + \frac{1}{400})^2 B_n^2 e^{2(\beta_n x + 0.05y)}, \quad n = 1, 2, 3
\end{align*}
$$

(5.28)
when \( k = 0.05 \), and
\[
\begin{align*}
    u &= 1 - \frac{1}{20} B'_n e^{(\beta_n z + 0.05y)}, \quad v = -B'_n \beta_n e^{(\beta_n z + 0.05y)} \\
p &= p_0 - \frac{1}{2} + 2(2 + \gamma)(\beta_n^2 + \frac{1}{400})^2 B'^2_n e^{2(\beta_n z + 0.05y)}; \quad n = 1, 2, 3
\end{align*}
\]  
(5.29)

when \( k = -0.05 \), where \( p_0 \) is an arbitrary integration constant.

The solutions (5.28) and (5.29) may represent flows to the left of a plate situated along \( z = 0 \) with suction or blowing when
\[
y \geq 20 \ln \left( -\frac{20}{B_n} \right) \quad \text{and} \quad y \leq -20 \ln \left( \frac{20}{B'_n} \right); \quad n = 1, 2, 3
\]  
(5.30)

provided each \( B_n < 0 \) and \( B'_n > 0 \), respectively.

The solution (5.28) may also represent a flow above a plate located along \( y = 0 \) with suction or blowing according as
\[
B_n \beta_n \geq 0
\]  
(5.31)

to the left of the \( y \)-axis when \( n = 1, 2 \), and to the right of the \( y \)-axis when \( n = 3 \). \( B_1 > 0, B_2 > 0, B_3 < 0 \) for suction and \( B_1 < 0, B_2 < 0, B_3 > 0 \) for blowing at the plate. The above results hold true for solution (5.34) with \( B_n \) being replaced by \( B'_n \).

The normal and shear stresses at the plates \( z = 0 \) and \( y = 0 \) when \( k = 0.05 \), from equations (3.6) and (3.7), are given by
\[
\begin{align*}
    T_{11} &= -p_0 + \frac{1}{2} + \frac{B_n}{100} \left[ 10\beta_n (5 + 2\beta_n) - B_n \left( \beta_n^2 + \frac{1}{400} \right) e^{0.05y} \right] e^{0.05y} \\
    T_{22} &= -p_0 + \frac{1}{2} - \frac{B_n \beta_n}{10} \left[ (5 + 2\beta_n) + 40B_n \beta_n \left( \beta_n^2 + \frac{1}{400} \right) e^{0.05y} \right] e^{0.05y} \\
    T_{12} &= \frac{B_n}{5} \left[ 5(5 + 2\beta_n) \left( \frac{1}{400} - \beta_n^2 \right) + B_n \beta_n \left( \beta_n^2 + \frac{1}{400} \right) e^{0.05y} \right] e^{0.05y}
\end{align*}
\]  
(5.32)

and
\[
\begin{align*}
    T_{11} &= -p_0 + \frac{1}{2} + \frac{B_n}{100} \left[ 10\beta_n (5 + 2\beta_n) - B_n \left( \beta_n^2 + \frac{1}{400} \right) e^{\beta_n x} \right] e^{\beta_n x} \\
    T_{22} &= -p_0 + \frac{1}{2} + \frac{B_n \beta_n}{10} \left[ (5 + 2\beta_n) + 40B_n \beta_n \left( \beta_n^2 + \frac{1}{400} \right) e^{\beta_n x} \right] e^{\beta_n x} \\
    T_{12} &= \frac{B_n}{5} \left[ 5(5 + 2\beta_n) \left( \frac{1}{400} - \beta_n^2 \right) + B_n \beta_n \left( \beta_n^2 + \frac{1}{400} \right) e^{\beta_n x} \right] e^{\beta_n x}
\end{align*}
\]  
(5.33)
where $\beta_n$ for $n = 1, 2, 3$ are given by (5.21). The normal and shear stresses at the plates $z = 0$ and $y = 0$ when $k = -0.05$ can, similarly, be obtained.

**Example 2: $k = \pm 4$, $Re = 8$, $We = 0.6$.**

The resulting cubic equation, from the auxiliary equation (5.19), for these values of $k$, $Re$ and $We$ is

$$3\beta^3 + 5\beta^2 + 8\beta + 80 = 0$$

(5.34)

The roots of (5.34), using Maple, are

$$\lambda_1 = -3.303269906, \quad \lambda_{2,3} = 0.8183016194 \pm 2.720880622i$$

where $i = \sqrt{-1}$. The general solution of (5.6) is

$$f(x) = C_1 e^{\lambda_1 x} + e^{\lambda_2 x} \left( C_2 \cos \lambda_3 x + C_3 \sin \lambda_3 x \right) + C_4 \cos 4x + C_5 \sin 4x$$

(5.35)

where $C_n$ for $n = 1, 2, 3, 4, 5$ are arbitrary constants and

$$\lambda_1 = -3.303269906, \quad \lambda_2 = 0.8183016194, \quad \lambda_3 = 2.720880622$$

(5.36)

We substitute (5.35) in (5.8) to get

$$\left[ 3\lambda_1^4 + 56\lambda_1^2 + 5E \right] C_1 e^{\lambda_1 x} + \left[ C_2 (3\lambda_2^4 - 18\lambda_2^2\lambda_3^2 + 3\lambda_3^4 - 56\lambda_2^2 - 56\lambda_3^2 + 5E) \right. \\
+ 4C_3 \lambda_2 \lambda_3 (3\lambda_2^2 - 3\lambda_3^2 + 28)e^{\lambda_2 x} \cos \lambda_3 x + \left. \left[ 4C_4 \lambda_2 \lambda_3 (3\lambda_2^2 - 3\lambda_3^2 - 28) \\
+ C_5 (3\lambda_2^4 - 18\lambda_2^2\lambda_3^2 + 3\lambda_3^4 - 56\lambda_2^2 - 56\lambda_3^2 + 5E) \right] e^{\lambda_2 x} \sin \lambda_3 x \\
+ (5E - 128)(C_4 \cos 4x + C_5 \sin 4x) = 0$$

(5.37)

For non-trivial $f(x)$, given by equation (5.35), we have that equations (5.34) to (5.37) must be simultaneously satisfied only if one of the underlisted holds true:

1. $E = 25.6, \ C_1 = C_2 = C_3 = 0$
2. $E = -193.64964, \ C_2 = C_3 = C_4 = C_5 = 0$. 

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The stream function for Case (1) is
\[
\psi(x, y) = \begin{cases} 
  y + (C_4 \cos 4x + C_5 \sin 4x)e^{4y} ; & k = 4 \\
  y + (C'_4 \cos 4x + C'_5 \sin 4x)e^{-4y} ; & k = -4
\end{cases}
\] (5.38)

The exact integral for these irrotational flows are
\[
u = 1 + 4(C_4 \cos 4x + C_5 \sin 4x)e^{4y}, \quad v = 4(C_4 \sin 4x - C_5 \cos 4x)e^{4y}
\]
\[
p = p_0 - \frac{1}{2} + \frac{4}{5} \left[ 2(67 + 48\gamma)(C_4^2 + C_5^2)e^{4y} - 5(C_4 \cos 4x + C_5 \sin 4x) \right] e^{4y}
\] (5.39)

when \( k = 4 \), and
\[
u = 1 - 4(C'_4 \cos 4x + C'_5 \sin 4x)e^{-4y}, \quad v = 4(C'_4 \sin 4x - C'_5 \cos 4x)e^{-4y}
\]
\[
p = p_0 - \frac{1}{2} + \frac{4}{5} \left[ 2(67 + 48\gamma)(C'_4^2 + C'_5^2)e^{-4y} + 5(C'_4 \cos 4x + C'_5 \sin 4x) \right] e^{-4y}
\] (5.40)

when \( k = -4 \), where \( p_0 \) is an arbitrary constant of integration.

In Case (2), the stream function (5.4) becomes
\[
\psi(x, y) = \begin{cases} 
  y + C_1 e^{(\lambda_1 x + 4y)} ; & k = 4 \\
  y + C'_1 e^{(\lambda_1 x - 4y)} ; & k = -4
\end{cases}
\] (5.41)

where \( \lambda_1 = -3.303269906 \). These rotational flows have exact integrals
\[
u = 1 + 4C_1 e^{(\lambda_1 x + 4y)}, \quad v = -C_1 \lambda_1 e^{(\lambda_1 x + 4y)}
\]
\[
p = p_0 - \frac{1}{2} + \frac{3}{40}(2 + \gamma)(\lambda_1^2 + 16) C_1^2 e^{2(\lambda_1 x + 4y)}
\] (5.42)

when \( k = 4 \), and
\[
u = 1 - 4C'_1 e^{(\lambda_1 x - 4y)}, \quad v = -C'_1 \lambda_1 e^{(\lambda_1 x - 4y)}
\]
\[
p = p_0 - \frac{1}{2} + \frac{3}{40}(2 + \gamma)(\lambda_1^2 + 16) C'_1^2 e^{2(\lambda_1 x - 4y)}
\] (5.43)

when \( k = -4 \), where \( p_0 \) is an arbitrary integration constant.

Solutions (5.42) and (5.43) may represent flows to the left of a plate situated along \( x = 0 \) with suction or blowing, respectively, in accordance with
\[
y \geq \frac{1}{4} \ln \left( -\frac{1}{4C_1} \right) \quad \text{and} \quad y \geq -\frac{1}{4} \ln \left( \frac{1}{4C'_1} \right)
\] (5.44)
provided \( C_1 < 0 \) and \( C'_1 > 0 \).

The aforementioned solutions may also represent flows above a plate located along \( y = 0 \) with suction or blowing when

\[
C_1 \lambda_1 \geq 0 \quad \text{and} \quad C'_1 \lambda_1 \geq 0
\]

to the right of the \( y \)-axis, where \( C_1 < 0 \) and \( C'_1 < 0 \) for suction, and \( C_1 > 0 \) and \( C'_1 > 0 \) for blowing at the plate.

From equations (3.6) and (3.7), the normal stress components and the shear stress at the plates \( z = 0 \) and \( y = 0 \) when \( k = 4 \) are

\[
T_{11} = -\rho_0 + \frac{1}{2} + \frac{C_1}{5} \left[ \lambda_1 (5 + 3 \lambda_1) - 12 C_1 (\lambda_1^2 + 16) e^{4y} \right] e^{4y}
\]
\[
T_{22} = -\rho_0 + \frac{1}{2} - \frac{C_1 \lambda_1}{20} \left[ 4 (5 + 3 \lambda_1) + 3 C_1 \lambda_1 (\lambda_1^2 + 16) e^{4y} \right] e^{4y}
\]
\[
T_{12} = \frac{C_1}{40} \left[ (5 + 3 \lambda_1) (16 - \lambda_1^2) + 24 C_1 \lambda_1 (\lambda_1^2 + 16) e^{4y} \right] e^{4y}
\]

and

\[
T_{11} = -\rho_0 + \frac{1}{2} + \frac{C_1}{5} \left[ \lambda_1 (5 + 3 \lambda_1) - 12 C_1 (\lambda_1^2 + 16) e^{4iz} \right] e^{4iz}
\]
\[
T_{22} = -\rho_0 + \frac{1}{2} - \frac{C_1 \lambda_1}{20} \left[ 4 (5 + 3 \lambda_1) + 3 C_1 \lambda_1 (\lambda_1^2 + 16) e^{4iz} \right] e^{4iz}
\]
\[
T_{12} = \frac{C_1}{40} \left[ (5 + 3 \lambda_1) (16 - \lambda_1^2) + 24 C_1 \lambda_1 (\lambda_1^2 + 16) e^{4iz} \right] e^{4iz}
\]

where \( \lambda_1 = -3.303269905 \). In a similar manner, the corresponding forms of (5.46) and (5.47) may be obtained for \( k = -4 \).

**Example 3:** \( k = \pm \frac{1}{\sqrt{3}} \), \( Re = \frac{49}{35} \), \( We = \frac{1}{3} \).

The cubic equation, from (5.19), resulting from these values of \( k \), \( Re \) and \( We \) is

\[
4 \beta^3 + 12 \beta^2 - 15 \beta + 4 = 0
\]

and its solution is

\[
\delta_1 = -4, \quad \delta_2 = \delta_3 = \frac{1}{2}
\]
For the chosen values of $k$, $Re$ and $Wc$, the general solution of (5.6) is

$$f(x) = D_1 e^{-4x} + (D_2 + D_3 x) e^{\frac{x}{2}} + D_4 \cos \frac{1}{\sqrt{3}} x + D_5 \sin \frac{1}{\sqrt{3}} x$$

(5.50)

where $D_n$ for $n = 1, 2, 3, 4, 5$ are arbitrary constants.

Since (5.8) has to also be satisfied, we employ (5.50) in (5.8) to get

$$
(9E + 604) D_1 e^{-4x} + \frac{1}{8} \left[ (72E - 19)(D_2 + D_3 x) - 61D_3 \right] e^{\frac{x}{2}} \\
+ \frac{1}{48} (432E + 139)(D_4 \cos \frac{1}{\sqrt{3}} x + D_5 \sin \frac{1}{\sqrt{3}} x) = 0
$$

(5.51)

The above equations are collectively satisfied only if one of the following holds true:

(I) $E = -0.3217592$, $D_1 = D_2 = D_3 = 0$

(II) $E = -67.111111$, $D_2 = D_3 = D_4 = D_5 = 0$

(III) $E = 0.2638888$, $D_1 = D_3 = D_4 = D_5 = 0$.

The stream function (5.4), for Case (I), becomes

$$
\psi(x,y) = \left\{ \begin{array}{l}
y + (D_4 \cos \frac{1}{\sqrt{3}} x + D_5 \sin \frac{1}{\sqrt{3}} x) e^{\sqrt{3}y}, \quad k = \frac{1}{\sqrt{3}} \\
y + (D_4' \cos \frac{1}{\sqrt{3}} x + D_5' \sin \frac{1}{\sqrt{3}} x) e^{-\sqrt{3}y}, \quad k = -\frac{1}{\sqrt{3}}
\end{array} \right.
$$

(5.52)

These represent irrotational flows with exact solutions

$$
u = 1 + \frac{1}{\sqrt{3}} (D_4 \cos \frac{1}{\sqrt{3}} x + D_5 \sin \frac{1}{\sqrt{3}} x) e^{\sqrt{3}y}, \quad v = \frac{1}{\sqrt{3}} (D_4 \sin \frac{1}{\sqrt{3}} x - D_5 \cos \frac{1}{\sqrt{3}} x) e^{\sqrt{3}y},
$$

$$
u = \frac{1}{\sqrt{3}} (D_4' \cos \frac{1}{\sqrt{3}} x + D_5' \sin \frac{1}{\sqrt{3}} x) e^{\sqrt{3}y}, \quad v = \frac{1}{\sqrt{3}} (D_4' \sin \frac{1}{\sqrt{3}} x - D_5' \cos \frac{1}{\sqrt{3}} x) e^{\sqrt{3}y}
$$

(5.53)

$$
u = 1 - \frac{1}{\sqrt{3}} (D_4' \cos \frac{1}{\sqrt{3}} x + D_5' \sin \frac{1}{\sqrt{3}} x) e^{-\sqrt{3}y}, \quad v = \frac{1}{\sqrt{3}} (D_4' \sin \frac{1}{\sqrt{3}} x - D_5' \cos \frac{1}{\sqrt{3}} x) e^{-\sqrt{3}y},
$$

$$
u = \frac{1}{\sqrt{3}} (D_4' \cos \frac{1}{\sqrt{3}} x + D_5' \sin \frac{1}{\sqrt{3}} x) e^{-\sqrt{3}y}, \quad v = \frac{1}{\sqrt{3}} (D_4' \sin \frac{1}{\sqrt{3}} x - D_5' \cos \frac{1}{\sqrt{3}} x) e^{-\sqrt{3}y}
$$

(5.54)
when \( k = -\frac{1}{\sqrt{3}} \), where \( p_0 \) is an arbitrary constant of integration.

We combine Cases (II) and (III) to obtain the stream function

\[
\psi(x, y) = \begin{cases} 
  y + D_n e^{(\delta_n x + \frac{1}{\sqrt{3}} y)} ; & k = \frac{1}{\sqrt{3}} \\
  y + D'_n e^{(\delta_n x - \frac{1}{\sqrt{3}} y)} ; & k = -\frac{1}{\sqrt{3}}
\end{cases}
\]

(5.55)

where \( n = 1, 2 \) and \( \delta_1 = -4, \delta_2 = \frac{1}{2} \). These rotational flows have exact solutions

\[
u = 1 + \frac{1}{\sqrt{3}} D_n e^{(\delta_n x + \frac{1}{\sqrt{3}} y)}, \quad v = -D_n \delta_n e^{(\delta_n x + \frac{1}{\sqrt{3}} y)} \\
p = p_0 - \frac{1}{2} + \frac{12}{49} (2 + \gamma) (\delta_n^2 + \frac{1}{3})^2 D_n^2 e^{2(\delta_n x + \frac{1}{\sqrt{3}} y)}
\]

(5.56)

when \( k = \frac{1}{\sqrt{3}} \), and

\[
u = 1 - \frac{1}{\sqrt{3}} D'_n e^{(\delta_n x - \frac{1}{\sqrt{3}} y)}, \quad v = -D'_n \delta_n e^{(\delta_n x - \frac{1}{\sqrt{3}} y)} \\
p = p_0 - \frac{1}{2} + \frac{12}{49} (2 + \gamma) (\delta_n^2 + \frac{1}{3})^2 D'_n^2 e^{2(\delta_n x - \frac{1}{\sqrt{3}} y)}
\]

(5.57)

when \( k = -\frac{1}{\sqrt{3}} \), with \( p_0 \) being an arbitrary integration constant.

The above solutions may be used to describe flows to the left of a plate along \( z = 0 \) with suction and blowing when

\[
y \geq \sqrt{3} \ln \left(-\frac{\sqrt{3}}{D_n}\right) \quad \text{and} \quad y \geq -\sqrt{3} \ln \left(\frac{\sqrt{3}}{D'_n}\right); \quad n = 1, 2
\]

(5.58)

where \( D_n < 0 \) and \( D'_n > 0 \).

Solution (5.57) may also describe a flow above a plate along \( y = 0 \) with suction or blowing according as

\[
D'_n \delta_n \geq 0
\]

(5.59)

to the right of the \( y \)-axis when \( n = 1 \), and to the left of the \( y \)-axis when \( n = 2 \).

\( D'_1 < 0, \ D'_2 > 0 \) for suction and for blowing at the plate, \( D'_1 > 0, \ D'_2 < 0 \). The result (5.59) holds true for solution (5.56) when \( D'_n \) is replaced by \( D_n \).
At the plates \( x = 0 \) and \( y = 0 \), the normal and shear stress components when \( k = -\frac{1}{\sqrt{3}} \) are

\[
T_{11} = -p_0 + \frac{1}{2} - \frac{8}{49} D'_n \left[ \sqrt{3} \delta_n (3 + \delta_n) + \left( \delta_n^2 + \frac{1}{3} \right) D'_n e^{-\frac{1}{\sqrt{3}} y} \right] e^{-\frac{1}{\sqrt{3}} y}
\]

\[
T_{22} = -p_0 + \frac{1}{2} + \frac{8}{49} D'_n \delta_n \left[ \sqrt{3} (3 + \delta_n) - 3 \delta_n (\delta_n^2 + \frac{1}{3}) D'_n e^{-\frac{1}{\sqrt{3}} y} \right] e^{-\frac{1}{\sqrt{3}} y}
\]

\[
T_{12} = \frac{12}{49 \sqrt{3}} D'_n \left[ \sqrt{3} (3 + \delta_n) \left( \frac{1}{3} - \delta_n^2 \right) - 2 \delta_n (\delta_n^2 + \frac{1}{3}) D'_n e^{-\frac{1}{\sqrt{3}} y} \right] e^{-\frac{1}{\sqrt{3}} y}
\]  \hspace{1cm} (5.60)

and

\[
T_{11} = -p_0 + \frac{1}{2} - \frac{8}{49} D'_n \left[ \sqrt{3} \delta_n (3 + \delta_n) + \left( \delta_n^2 + \frac{1}{3} \right) D'_n e^{\delta_n x} \right] e^{\delta_n x}
\]

\[
T_{22} = -p_0 + \frac{1}{2} + \frac{8}{49} D'_n \delta_n \left[ \sqrt{3} (3 + \delta_n) - 3 \delta_n (\delta_n^2 + \frac{1}{3}) D'_n e^{\delta_n x} \right] e^{\delta_n x}
\]

\[
T_{12} = \frac{12}{49 \sqrt{3}} D'_n \left[ \sqrt{3} (3 + \delta_n) \left( \frac{1}{3} - \delta_n^2 \right) - 2 \delta_n (\delta_n^2 + \frac{1}{3}) D'_n e^{\delta_n x} \right] e^{\delta_n x}
\]  \hspace{1cm} (5.61)

where \( \delta_1 = -4, \delta_2 = \frac{1}{2} \). Corresponding results for \( k = \frac{1}{\sqrt{3}} \), if desired, may be similarly obtained.

Reversed flows are obtained in the rotational cases only when \( k \) is negative. When \( k \) is non-negative, non-reversed flows result. When \( We \equiv 0 \), some of the results of Lin and Tobak (1986) and Hui (1987) are recovered.

Figures 5.1 and 5.2 illustrate, respectively, reversed and non-reversed flows when \( k \neq 0 \). The non-reversed flows obtained when \( k = 0 \) is shown in Figure 5.3.
Streamlines for $y + Ae^{(mz + ky)} = \text{constant}$ when $A = 80, m = 0.1725994, k = -0.05$.

Figure 5.1
Streamlines for $y + Ae^{\left(mx + ky\right)} = \text{constant}$ when $A = 80$, $m = 0.1725994$, $k = 0.05$.

Figure 5.2
Streamlines for $y + Ae^{(mx+ky)} = \text{constant}$ when $A = 80$, $m = 0.198039$, $k = 0$.

Figure 5.3
CHAPTER VI

EXACT SOLUTIONS
OF A SECOND GRADE
FLUID FLOW FOR CHOSEN
VORTICITY FUNCTIONS

6.1 INTRODUCTION.

Wang (1991) has given an excellent review of the exact solutions of the Navier-
Stokes equations obtained within the last century or more. These known solutions
of viscous, incompressible Newtonian fluids may be classified into three types:
(i) flows for which the nonlinear inertia terms in the linear momentum equations
vanish identically. Parallel flows and flows with uniform suction are examples of
these flows;
(ii) flows with similarity properties such that the flow equations reduce to a set of
ordinary differential equations. Stagnation point flow is an example of such flows;
(iii) flows for which the vorticity function is so chosen that the governing equation, in
terms of the stream function, reduces to a linear equation. Taylor (1923), Kampé de
Fériet (1930), Kovasznay (1948), Wang (1966) and Lin and Tobak (1986) employed
this approach, taking $\nabla^2 \psi = K \psi$, $\nabla^2 \psi = f(\psi)$, $\nabla^2 \psi = y + (K^2 - 4\pi^2) \psi$, $\nabla^2 \psi = A \psi + By$ and $\nabla^2 \psi = K(\psi - Ry)$, respectively.
In this chapter, we study the steady, planar, isochoric motion of a second grade fluid when the vorticity function $\omega = -\nabla^2 \psi$ is given by $\nabla^2 \psi = \psi + Ax^2 + By^2 + Cxy + Dz + Ey$, where $A$, $B$, $C$, $D$, $E$ are real constants. Cases when all the constants may be non-zero, $A = 0$, $A = C = 0$, and $A = B = C = 0$ will be considered. Some of the flows obtained are Beltrami flows. These are flows for which $\text{curl}(\omega \times V) = 0$, where $\omega$ and $V$ are, respectively, the vorticity and velocity vectors. These are such that $\omega \neq 0$ and $\omega \times V \neq 0$. This problem was investigated by Chandna and Oku-Ukpong (1994) for the Navier-Stokes equations.
6.2 BASIC EQUATIONS AND METHOD.

Second grade fluid which undergoes steady, isochoric motion in a plane, in the absence of external forces and in dimensionless variables, is governed by the system (3.3) of three equations for the velocity components $u, v$ and the pressure $p$ as functions of $z, y$.

We introduce the vorticity function $\omega(x, y)$ and the generalized energy function $h(x, y)$ given by system (3.4) in system (3.3) to get system (3.5) of four equations in four unknowns $u, v, \omega$ and $h$.

We apply the integrability condition $\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x}$ to the linear momentum equations to obtain the compatibility equation (5.1).

Introducing the stream function $\psi(x, y)$ in equation (5.1) such that equations (2.30) are satisfied, we find that $\psi(x, y)$ must satisfy the vorticity transport equation (5.3), where $\omega = -\nabla^2 \psi$.

We shall study flows for which the vorticity distribution takes the form

$$\nabla^2 \psi = \psi + Ax^2 + By^2 + Cxy + Dz + Ey$$  \hspace{1cm} (6.1)

where $A, B, C, D$ and $E$ are real constants.

Employing (6.1) in the compatibility equation (5.3), we get

$$F(2By + Cz + E) \frac{\partial \psi}{\partial x} - F(2Ax + Cy + D) \frac{\partial \psi}{\partial y} + \psi$$

$$+ Ax^2 + By^2 + Cxy + Dz + Ey + 2(A + B) = 0$$  \hspace{1cm} (6.2)

where $F = Re - We$. The differential equation satisfied by the characteristic curves of (6.2) when $F \neq 0$ is

$$(2Ax + Cy + D) dx + (2By + Cx + E) dy = 0$$

The characteristic curves of (6.2) for $F \neq 0$ are

$$Ax^2 + By^2 + Cxy + Dz + Ey = \text{constant}$$
Employing the canonical coordinates

\[ \xi = Ax^2 + By^2 + Cxy + Dx + Ey \]
\[ \eta = y \]

(6.3)

where \(2Ax + Cy + D \neq 0\), (6.2) may be written as

\[-\mathcal{F}(2Ax + C\eta + D) \frac{\partial \psi}{\partial \eta} + \psi + \xi + 2(A + B) = 0\]

(6.4)
6.3 EXACT SOLUTIONS.

We shall find exact solutions when $F = 0$ (or $Re = We$) and when $F \neq 0$ (or $Re \neq We$).

Example 1.

When $F = 0$, we get the stream function, from (6.2), to be

$$
\psi(x,y) = -[Ax^2 + By^2 + Cxy + Dx + Ey + 2(A + B)]
$$

(6.5)

Using this equation in (2.30) and the linear momentum equations yields the exact solution in unbounded domain:

$$
u = -(2By + Cx + E), \quad v = 2Ax + Cy + D, \quad p = p_0 + \frac{1}{2}\left\{(4AB - C^2)(x^2 + y^2) + 2(2BD - CE)x + 2(2AE - CD)y - (D^2 + E^2) + 4(3 + 2\gamma)[(A - B)^2 + C^2]\right\}
$$

(6.6)

where $p_0$ is an arbitrary constant. The stagnation point for this flow is

$$(x,y) = \left(\frac{2BD - CE}{C^2 - 4AB}, \frac{2AE - CD}{C^2 - 4AB}\right)
$$

(6.7)

where $C^2 \neq 4AB$. The streamlines for this Beltrami flow in an unbounded domain is shown in Figure 6.1.

We now consider three possibilities when $F \neq 0$: $A = 0$, $A = C = 0$, $A = B = C = 0$.

Example 2: $A = 0$.

Equation (6.4) becomes

$$
-F(C\eta + D)\frac{\partial \psi}{\partial \eta} + \psi + \xi + 2B = 0
$$

(6.8)

Equation (6.8) is solved to obtain

$$
\psi = f(\xi)(C\eta + D)^{\nu} - (\xi + 2B)
$$

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where \((C\eta + D) \neq 0\) and \(f\) is an arbitrary function of \(\xi\). Since

\[
\xi = By^2 + Cxz + Dy, \quad \eta = y
\]

we get

\[
\psi = f(\xi)(Cy + D) - (By^2 + Cxz + Dy + 2B)
\]  

(6.9)

Introducing (6.9) into (6.1) with \(A = 0\) yields

\[
\left[\left((2By + Cx + E)^2 + (Cy + D)^2\right)f''(\xi) + 2\left[B + \frac{(2By + Cx + E)}{F(Cy + D)}\right]f'(\xi)\right] + \left[\frac{1 - FC}{F^2(Cy + D)^2} - 1\right]f(\xi) = 0
\]  

(6.10)

Employing \(x = \frac{\xi - y(By + E)}{C\eta + D}\), \(y = \eta\) in (6.10), we obtain

\[
\left\{\frac{F^2}{D^2(D + E^2) + 2CDE\xi + C^2\xi^2}f''(\xi) + 2F\left[D(FBD + E) + C\xi\right]f'(\xi)\right. \\
+\left.(1 - FC - F^2D^2)f(\xi)\right\} + 2F\left[D(BE + CD) + BC\xi\right]f''(\xi) \\
+ 2B(1 + FC)f'(\xi) - FCf(\xi)\right\} \eta + \left\{2F\left[D^2(2B^2 + C^2) + BCDE\right. \\
+ BC^2\xi\left.]f''(\xi) + 2BC(1 + FC)f'(\xi) - FC^2f(\xi)\right\} \eta^2 \\
+ 4F^2CD\left\{(B^2 + C^2)f''(\xi)\right\} \eta^3 + F^2C^2\left\{(B^2 + C^2)f''(\xi)\right\} \eta^4
\]

\[
\equiv \sum_{m=1}^{4} G_m(\xi)\eta^m = 0
\]  

(6.11)

Since \(\xi, \eta\) are independent variables and \(\{1, \eta, \eta^2, \eta^3, \eta^4\}\) is a linearly independent set, it follows that the coefficients \(G_m(\xi) = 0\) for \(m = 0, 1, 2, 3, 4\). From these coefficients, we obtain

\[
f(\xi) = c_1 \xi + c_2
\]  

(6.12)

\[
2Bc_1(1 + FC) - FC(c_1 \xi + c_2) = 0
\]

where \(c_1, c_2\) are arbitrary constants. Equations (6.12) are solved to obtain \(c_1 = c_2 = 0\), which implies \(f(\xi) = 0\).
From (6.9), the stream function is given by

\[ \psi(x,y) = -(By^2 + Cxy + Dx + Ey + 2B) \]  \hspace{1cm} (6.13)

The exact integral of this Beltrami flow in an unbounded domain is

\[ u = -(2By + Cx + E), \quad v = Cy + D, \]

\[ p = p_0 - \frac{1}{2} \left[ C^2(x^2 + y^2) + 2(CE - 2BD)x + 2CDy + (D^2 + E^2) \right] - \frac{4W_e}{Re}(3 + 2\gamma)(B^2 + C^2) \]  \hspace{1cm} (6.14)

where \( p_0 \) is an arbitrary constant.

Equation (6.13) represents an impingement of two constant-vorticity oblique jets with stagnation point

\[ (x, y) = \left( \frac{2BD - CE}{C^2}, -\frac{D}{C} \right) \]  \hspace{1cm} (6.15)

for non-zero value of \( C \). The stagnation point shifts upward as \( C \) gets smaller for fixed values of \( B, C, E \). We remark that when \( B = C = -1, D = E = 0 \), the solution (6.13) reduces to one of the flows in Wang's (1991) paper for Navier-Stokes equations.

Imposing the boundary conditions

\[ u(x, 0) = u_0 x + u_1, \quad v(x, 0) = 0; \quad u_0 \neq 0 \]  \hspace{1cm} (6.16)

on a plate situated along \( y = 0 \) so that the fluid occupies the upper half plane \( y > 0 \), we get the solution

\[ u = u_0 x - 2By + u_1, \quad v = -u_0 y \]

\[ p = p_0 - \frac{1}{2} \left[ u_0^2(x^2 + y^2) + 2u_0u_1x + u_1^2 - \frac{4W_e}{Re}(3 + 2\gamma)(B^2 + u_0^2) \right] \]  \hspace{1cm} (6.17)

where \( B \) is an arbitrary real number, \( C = -u_0, D = 0, E = -u_1 \). The plate \( y = 0 \) on which the fluid impinges stretches linearly and is non-porous. This boundary value problem describes a flow impinging obliquely on a plate at \( y = 0 \). 

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In a study of the Navier-Stokes equations, Dorrepaal (1986) solved a similar non-orthogonal stagnation point flow in the region $y > 0$, in which the dividing streamline $\psi = 0$ impinges on a plate at $z = 0$, by considering the stream function in the form

$$\psi(x, y) = \frac{1}{2} y^2 \cos \alpha + y \sin \alpha.$$ 

The slope of the dividing streamline is $m = -2 \tan \alpha$, where $0 \leq \alpha \leq \frac{\pi}{2}$ is a parameter, Dorrepaal found that the ratio of the slope of the dividing streamline at the wall to that at infinity is constant (3.748513), and is independent of the type of non-orthogonal stagnation-point flow under consideration.

The normal stresses $T_{11}$, $T_{22}$ and the shear stress $T_{12}$ at the plate $y = 0$ are

$$T_{11} = -p_0 + \frac{1}{2} \left\{ u_0^2 x^2 + 2 u_0 u_1 x + u_1^2 + \frac{4 u_0}{Re} \left[ 6 B^2 + (2 + 3 \gamma) u_0^2 \right] \right\}$$

$$T_{22} = -p_0 + \frac{1}{2} \left\{ u_0^2 x^2 + 2 u_0 u_1 x + u_1^2 - \frac{4 u_0}{Re} \left[ 2 B^2 - (2 + 3 \gamma) u_0^2 \right] \right\}$$

$$T_{12} = -\frac{2 B}{Re} (1 + 2 We u_0)$$

(6.26)

using equations (3.6) and (3.7). Figures 6.2 and 6.3 give the streamlines for flows in unbounded and bounded domains, respectively.

**Example 3: $A = C = 0.$**

In this case, (6.4) takes the form

$$-FD \frac{\partial \psi}{\partial \eta} + \psi + \xi + 2B = 0$$

(6.19)

where $D \neq 0$. We solve this equation to get

$$\psi = g(\xi) e^{\gamma B \eta} - (\xi + 2B)$$

where $g$ is an arbitrary function of $\xi$. Using

$$\xi = By^2 + Dz + Ey, \quad \eta = y$$
we get

$$\psi = g(\xi) e^{(B y^2 + D x + E y + 2B) \xi}$$  \hspace{1cm} (6.20)

We substitute (6.20) into (6.1) with $A = C = 0$ to get

$$\left[ D^2 + (2B\eta + E)^2 \right] g''(\xi) + 2 \left[ B + \frac{2B\eta + E}{F_{D}} \right] g'(\xi) + \left[ \frac{1}{F_{D}^2 D^2} - 1 \right] g(\xi) = 0$$

where $y = \eta$, or

$$\left[ F^2 D^4 g''(\xi) + 2 F^2 B D^2 g'(\xi) + (1 - F^2 D^2) g(\xi) \right] + 2 F D g'(\xi)(2B\eta + E) + F^2 D^2 g''(\xi)(2B\eta + E)^2 = 0$$  \hspace{1cm} (6.21)

The fact that $\xi$, $\eta$ are independent variables and \{1, $(2B\eta + E), (2B\eta + E)^2$\} is a linearly independent set implies that the respective coefficients of these functions of $\eta$ will be identically zero. Therefore,

$$g''(\xi) = 0, \quad g'(\xi) = 0, \quad (1 - F^2 D^2) g(\xi) = 0$$  \hspace{1cm} (6.22)

From $(1 - F^2 D^2) g(\xi) = 0$, we get the three possibilities:

$$g(\xi) = 0, \quad F^2 D^2 \neq 1; \quad g(\xi) = 0, \quad F^2 D^2 = 1; \quad g(\xi) \neq 0, \quad F^2 D^2 = 1.$$  

When $g(\xi) \neq 0$, we have $g(\xi) = K$, a non-zero real constant, since $g'(\xi) = 0$.

The stream function (6.28) is given by

$$\psi(x, y) = \begin{cases} 
-(B y^2 + D x + E y + 2B); & g = 0, \quad F^2 D^2 = 1 \text{ or } F^2 D^2 \neq 1 \\
K e^{(B y^2 + D x + E y + 2B)}; & g \neq 0, \quad F^2 D^2 = 1
\end{cases}$$  \hspace{1cm} (6.23)

When the stream function is given by

$$\psi(x, y) = -(B y^2 + D x + E y + 2B); \quad F^2 D^2 = 1 \quad \text{or} \quad F^2 D^2 \neq 1$$  \hspace{1cm} (6.24)

the exact solution for this Beltrami flow of a second grade fluid in an unbounded domain is

$$u = -(2B y + E), \quad v = D$$

$$p = p_0 + \frac{1}{2} \left[ 4BDx - (D^2 + E^2) \right] + \frac{4We}{Re} (3 + 2\gamma) B^2$$  \hspace{1cm} (6.25)
where \(p_0\) is an arbitrary constant.

The boundary conditions specified on a plate located along \(y = 0\) are

\[
u(x, 0) = u_0, \quad v(x, 0) = v_0; \quad v_0 \neq 0
\]  

(6.26)

Therefore, \(D = v_0\), \(E = -u_0\). There is uniform suction (\(v_0 > 0\)) or blowing (\(v_0 < 0\)) at the plate \(y = 0\), which is stretching at a uniform rate, where the fluid occupies the region \(y < 0\).

Thus, the exact integral of this flow impinging on a uniformly stretching porous plate at \(y = 0\) is

\[
u = u_0 - 2By, \quad v = v_0
\]

\[
p = p_0 + \frac{1}{2} \left[4Bv_0x - (u_0^2 + v_0^2) + \frac{4We}{Re} (3 + 2\gamma)B^2 \right]
\]  

(6.27)

where \(B\) is an arbitrary real number.

The normal stress components \(T_{11}\), \(T_{22}\) and the shear stress \(T_{12}\) at the plate \(y = 0\), from equations (3.6) and (3.7), are

\[
T_{11} = -p_0 + \frac{1}{2} \left[ (u_0^2 + v_0^2) - \frac{12We}{Re} B^2 - 4Bv_0x \right]
\]

\[
T_{22} = -p_0 + \frac{1}{2} \left[ (u_0^2 + v_0^2) + \frac{4We}{Re} B^2 - 4Bv_0x \right]
\]  

(6.28)

\[
T_{12} = -\frac{2B}{Re}
\]

The flow patterns, respectively, for flow in unbounded domain and for the boundary value problem are given in Figures 6.4 and 6.5.

The exact solution for the second grade fluid flow in an unbounded domain, when the stream function is given by

\[
\psi(x, y) = Ke^{x\nu} - (By^2 + Dx + E\nu + 2B); \quad \mathcal{F}^2 D^2 = 1
\]  

(6.29)

is

\[
u = \frac{K}{\mathcal{F}D} e^{x\nu} - (2By + E), \quad v = D
\]

\[
p = p_0 + \frac{1}{2} \left[4BDx + \frac{2(2 + \gamma)We}{Re} K(Ke^{x\nu} - 4B)e^{x\nu} + \frac{4(3 + 2\gamma)We}{Re} B^2
\]

\[- (D^2 + E^2) \right]

(6.30)
where \( p_0 \) is an arbitrary constant, and \( \mathcal{F} = Re - We \).

If \( K = \mathcal{F}DE \) in (6.29) and (6.30), the velocity profile in (6.30) may be realized on a plate located along \( y = 0 \) with uniform suction. The velocity profile attains the form

\[
u = E(e^{\frac{K}{Fv_0}y} - 1) - 2By, \quad v = D
\]  

only asymptotically, and so may be regarded as the asymptotic suction profile [cf. Schlichting (1968)]. \( D < 0 \) for blowing and \( D > 0 \) for suction at the plate if the flow is confined to the region \( y < 0 \).

We may impose the boundary conditions

\[
u(x, 0) = u_0, \quad v(x, 0) = v_0; \quad v_0 \neq 0
\]  

on a plate situated along \( y = 0 \). These imply that \( D = v_0, \quad E = \frac{K}{Fv_0} - u_0, \) and there is uniform suction \( (v_0 > 0) \) or blowing \( (v_0 < 0) \) at the uniformly stretching plate for flow in the lower half plane \( y < 0 \). For this second grade fluid flow impinging on a plate \( y = 0 \), the exact solution, when \( B \) is an arbitrary real number, is

\[
u = \frac{K}{Fv_0} (e^{\frac{K}{Fv_0}y} - 1) - 2By + u_0, \quad v = v_0
\]

\[
p = p_0 + \frac{1}{2} \left[ 4Bv_0x + \frac{2We}{Re} K(2 + \gamma)(Ke^{\frac{1}{Fv_0}y} - 4B)e^{\frac{1}{Fv_0}y} + 2(3 + 2\gamma)B^2 \right]
\]

\[- \left( u_0 - \frac{K}{Fv_0} \right)^2 - v_0^2 \right] \right)
\]  

(6.33)

where \( \mathcal{F}^2 v_0^2 = 1 \).

The normal and shear stress components for this flow at the plate \( y = 0 \) are

\[
T_{11} = -p_0 + \frac{1}{2} \left[ (u_0 - \frac{K}{Fv_0})^2 + v_0^2 + \frac{4We}{Re} (K - B)(3B - K) - 4Bv_0x \right]
\]

\[
T_{22} = -p_0 + \frac{1}{2} \left[ (u_0 - \frac{K}{Fv_0})^2 + v_0^2 + \frac{4We}{Re} B^2 - 4Bv_0x \right]
\]  

(6.34)

\[
T_{12} = \frac{K}{F} - \frac{2B}{Re}
\]

For flows in unbounded and bounded domains, the streamline pattern are as shown in Figures 6.6 and 6.7, respectively.
Example 4: \( A = B = C = 0 \).

Equation (6.12) becomes

\[-\mathcal{F}D \frac{\partial \psi}{\partial \eta} + \psi + \xi = 0\]  

(6.35)

where \( D \neq 0 \). The solution of this equation is

\[\psi = l(\xi)e^{\int_{\eta}^{\xi} F} - \xi\]

or

\[\psi = l(\xi)e^{\int_{E}^{\xi}} - (Dx + Ey)\]  

(6.36)

where \( \xi = Dx + Ey, \eta = y \), and \( l \) is an arbitrary function of \( \xi \). We employ (6.36) in (6.1) with \( A = B = C = 0 \) to obtain

\[\mathcal{F}^2D^2(D^2 + E^2)l''(\xi) + 2\mathcal{F}DEl'(\xi) + (1 - \mathcal{F}^2D^2)l(\xi) = 0\]  

(6.37)

The general solution of (6.37) is

\[l(\xi) = \begin{cases} 
A_1e^{\lambda_1 \xi} + A_2e^{\lambda_2 \xi} ; \mathcal{F}^2(D^2 + E^2) - 1 > 0 \quad \text{(a)} \\
(B_1 + B_2 \xi)e^{-\frac{E}{D} \xi} ; \mathcal{F}^2(D^2 + E^2) - 1 = 0 \quad \text{(b)} \\
C_1\cos(m\xi + C_2)e^{\beta \xi} ; \mathcal{F}^2(D^2 + E^2) - 1 < 0 \quad \text{(c)}
\end{cases}\]  

(6.38)

where

\[\lambda_{1,2} = \frac{-E \pm D\sqrt{\mathcal{F}^2(D^2 + E^2) - 1}}{\mathcal{F}D(D^2 + E^2)}, \quad m = \frac{\sqrt{1 - \mathcal{F}^2(D^2 + E^2)}}{\mathcal{F}(D^2 + E^2)}\]  

(6.39)

\[\beta = -\frac{E}{\mathcal{F}D(D^2 + E^2)}\]

and \( A_1, A_2, B_1, B_2, C_1, C_2 \) are arbitrary constants.

We shall study separately the three possibilities corresponding to

\[\mathcal{F}^2(D^2 + E^2) - 1 > 0, \quad \mathcal{F}^2(D^2 + E^2) - 1 = 0, \quad \mathcal{F}^2(D^2 + E^2) - 1 < 0.\]
\[ F^2(D^2 + E^2) - 1 > 0. \]

The stream function, from (6.36) and (6.38a), is
\[
\psi(x, y) = A_1 \exp \left[ \lambda_1 Dx + (\lambda_1 E + \frac{1}{F D}) y \right] + A_2 \exp \left[ \lambda_2 Dx + (\lambda_2 E + \frac{1}{F D}) y \right] - (Dx + Ey) \tag{6.40}
\]
and the exact integral for this second grade fluid flow is
\[
u = (\lambda_1 E + \frac{1}{F D}) A_1 \exp \left[ \lambda_1 Dx + (\lambda_1 E + \frac{1}{F D}) y \right] + (\lambda_2 E + \frac{1}{F D}) A_2 \exp \left[ \lambda_2 Dx + (\lambda_2 E + \frac{1}{F D}) y \right] - E \tag{6.41}
\]
\[
u = -D \left\{ \lambda_1 A_1 \exp \left[ \lambda_1 Dx + (\lambda_1 E + \frac{1}{F D}) y \right] + \lambda_2 A_2 \exp \left[ \lambda_2 Dx + (\lambda_2 E + \frac{1}{F D}) y \right] - 1 \right\}
\]
\[
\nu = p_0 - \frac{1}{2} (D^2 + E^2) + \frac{(2 + \gamma) We}{Re} \left\{ A_1^2 \exp \left[ 2\lambda_1 Dx + 2(\lambda_1 E + \frac{1}{F D}) y \right] + A_2^2 \exp \left[ 2\lambda_2 Dx + 2(\lambda_2 E + \frac{1}{F D}) y \right] \right\} + 2 \left\{ 1 - \frac{1}{F^2 (D^2 + E^2)} \right\}
\]
\[
+ \frac{We}{Re} \left[ \frac{2(5 + 4\gamma)}{F^2 (D^2 + E^2)} + \frac{4(3 + 2\gamma)}{F^4 (D^2 + E^2)^2} \right] A_1 A_2 \exp \left[ \frac{2(Dy - Ex)}{F (D^2 + E^2)} \right]
\]

where \( p_0 \) is an arbitrary constant, \( \lambda_1, \lambda_2 \) are given by (6.39), and \( F = Re - We \).

This flow represents an impingement of an oblique uniform stream with an oblique rotational, divergent jet of infinite speed at infinity, with stagnation point
\[
(x, y) = \frac{F}{2} \left( \frac{D}{\sqrt{F^2 (D^2 + E^2) - 1}} \ln \left( - \frac{A_2}{A_1} \right) + E \ln \left\{ \frac{4A_1 A_2 [1 - F^2 (D^2 + E^2)]}{F^2 (D^2 + E^2)^2} \right\} \right),
\]
\[
- \frac{E}{\sqrt{F^2 (D^2 + E^2) - 1}} \ln \left( - \frac{A_2}{A_1} \right) - D \ln \left\{ \frac{4A_1 A_2 [1 - F^2 (D^2 + E^2)]}{F^2 (D^2 + E^2)^2} \right\}
\]
\tag{6.42}

where \( A_1, A_2 \) are non-zero real constants and, either \( A_1 > 0, A_2 < 0 \) or \( A_1 < 0, A_2 > 0 \). For fixed values of \( F, D, E, \) the stagnation point shifts upward when the absolute value of \( A_2 \) is larger than that of \( A_1 \). This flow may be applied to stagnation point injections behind a strong curved shock [cf. Wang (1966)].
If $A_1$ and $A_2$ are of the same sign, the above phenomenon does not take place, and we have a flow without a stagnation point. The streamlines in an unbounded domain for flows with and without a stagnation point are given, respectively, by Figures 6.8 and 6.9.

\[ \mathcal{F}^2 (D^2 + E^2) - 1 = 0. \]

Using (6.38b) in (6.36), the stream function is

\[ \psi(x, y) = [B_1 + B_2(Dx + Ey)] \exp[\mathcal{F}(Dy - Ex)] - (Dx + Ey) \]  

(6.43)

This flow has the exact solution

\[
\begin{align*}
    u &= \frac{\mathcal{F}D[B_1 + B_2(Dx + Ey)] + EB_2}{\mathcal{F}} \exp[\mathcal{F}(Dy - Ex)] - E \\
    v &= \frac{\mathcal{F}E[B_1 + B_2(Dx + Ey)] - DB_2}{\mathcal{F}} \exp[\mathcal{F}(Dy - Ex)] + D \\
    p &= p_0 - \frac{1}{2\mathcal{F}^2} \left\{ \frac{2WC}{Re} \left[ (2 + \gamma)\mathcal{F}^2[B_1 + B_2(Dx + Ey)]^2 + (7 + 4\gamma)B_2^2 \right] \\
    & \quad - B_2^2 \right\} \exp[2\mathcal{F}(Dy - Ex)]
\end{align*}
\]  

(6.44)

where $p_0$ is an arbitrary constant, and $\mathcal{F} = Re - Wc$.

If $B_2$ is a positive real constant, this flow represents an impingement of an oblique uniform stream with an oblique rotational, divergent jet having infinite speed at infinity, with stagnation point

\[ (x, y) = \frac{1}{D^2 + E^2} \left( \frac{E}{\mathcal{F}} \ln B_2 - \frac{DB_1}{B_2}, -\left[ \frac{D}{\mathcal{F}} \ln B_2 + \frac{EB_1}{B_2} \right] \right) \]  

(6.45)

For fixed values of $\mathcal{F}$ and $D$, the stagnation point shifts upward if $B_1$ and $E$ are of opposite signs, and the absolute value of $B_1$ is larger than $B_2$. This flow may be applied to stagnation point injections behind a straight shock.

If $B_2$ is a negative real number, (6.43) represents an oblique uniform stream which abuts on an oblique rotational, convergent jet. Figures 6.10 and 6.11, respectively, give the flow patterns in an unbounded domain with and without stagnation point.
\[ \mathcal{F}^2(D^2 + E^2) - 1 < 0. \]

From (6.36) and (6.38c), the stream function is given by

\[ \psi(x, y) = C_1 \cos[m(Dx + Ey) + C_2] \exp \left[ \frac{Dy - Ex}{\mathcal{F}(D^2 + E^2)} \right] - (Dx + Ey) \]  \( (6.46) \)

where \( m \) is given by (6.39). The exact integral of this second grade fluid flow is

\[ u = \left\{ \frac{DC_1}{\mathcal{F}(D^2 + E^2)} \cos[m(Dx + Ey) + C_2] \right. \]
\[ - mEC_1 \sin[m(Dx + Ey) + C_2] \left. \right\} \exp \left[ \frac{Dy - Ex}{\mathcal{F}(D^2 + E^2)} \right] - E \]

\[ v = \left\{ \frac{EC_1}{\mathcal{F}(D^2 + E^2)} \cos[m(Dx + Ey) + C_2] \right. \]
\[ + mDC_1 \sin[m(Dx + Ey) + C_2] \left. \right\} \exp \left[ \frac{Dy - Ex}{\mathcal{F}(D^2 + E^2)} \right] + D \]

\[ p = p_0 - \frac{1}{2}(D^2 + E^2) + \frac{C_1^2}{2} \left\{ 1 - \frac{1}{\mathcal{F}^2(D^2 + E^2)} \right. \]
\[ + \frac{We}{Re} \left[ \frac{3(5 + 4\gamma)}{\mathcal{F}^4(D^2 + E^2)^2} \left( 2 + \gamma \cos2[m(Dx + Ey) + C_2] \right) \right] \left. \right\} \exp \left[ \frac{2(Dy - Ex)}{\mathcal{F}(D^2 + E^2)} \right] \]  \( (6.47) \)

where \( p_0 \) is an arbitrary constant, and \( \mathcal{F} = Re - We \).

If \( \mathcal{F} \) and \( C_1 \neq 0 \) are of the same sign, a stagnation point exists in the flow region, and is given by

\[ (x, y) = \left( \frac{\mathcal{F}D[(2n + 1)\frac{\pi}{2} - C_2]}{\sqrt{1 - \mathcal{F}^2(D^2 + E^2)}} \right) + \mathcal{F}E \ln \left[ \frac{C_1 \sqrt{1 - \mathcal{F}^2(D^2 + E^2)}}{\mathcal{F}(D^2 + E^2)} \right], \]

\[ \frac{\mathcal{F}E[(2n + 1)\frac{\pi}{2} - C_2]}{\sqrt{1 - \mathcal{F}^2(D^2 + E^2)}} - \mathcal{F}D \ln \left[ \frac{C_1 \sqrt{1 - \mathcal{F}^2(D^2 + E^2)}}{\mathcal{F}(D^2 + E^2)} \right] \]  \( (6.48) \)

where \( n \) is an integer. The streamline pattern for this flow in an unbounded domain is shown in Figure 6.12.
Streamlines for \( A x^2 + B y^2 + C xy + D x + E y + 2(A + B) = \) constant when

\[ A = B = C = D = E = 1. \]

Figure 6.1
Streamlines for $B y^2 + C x y + D x + E y + 2B = \text{constant}$ in unbounded domain

when $B = C = E = 1, D = 0$.

Figure 6.2
Streamlines for $By^2 + Czy + Dx + Ey + 2B = \text{constant}$ for boundary value problem when $B = C = E = 1, \ D = 0$.

Figure 6.3
Streamlines for $By^2 + Dz + Ey + 2B$ = constant in unbounded domain when $B = D = E = 1$.

Figure 6.4
Streamlines for $By^2 + Dz + Ey + 2B = \text{constant}$ for boundary value problem

when $B = D = E = 1$.

Figure 6.5
Streamlines for $K e^{2 y} - (By^2 + Dz + E_y + 2B) = $ constant in unbounded domain when $F = K = B = D = E = 1.$

Figure 6.6
Streamlines for $K e^{yv} - (By^2 + Dx + Ey + 2B) = \text{constant}$ for boundary value problem when $F = K = B = D = E = 1$.

Figure 6.7
Streamlines for $A_1 \exp \left[ \lambda_1 D x + \left( \lambda_1 E + \frac{1}{F D} \right) y \right] + A_2 \exp \left[ \lambda_2 D x + \left( \lambda_2 E + \frac{1}{F D} \right) y \right] - (D x + E y) = \text{constant}$ when $A_1 = D = E = F = 1$, $\lambda_1 = 0$, $\lambda_2 = A_2 = -1$.

Figure 6.8

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Streamlines for 

\[ A_1 \exp \left[ \lambda_1 Dx + \left( \lambda_1 E + \frac{1}{2F_D} \right)y \right] + A_2 \exp \left[ \lambda_2 Dx + \left( \lambda_2 E + \frac{1}{2F_D} \right)y \right] \]

\[- (Dx + E y) \text{ = constant when } A_1 = 50, A_2 = 60, D = E = 1, F = 2, \]

\[ \lambda_1 = 0.4114378, \lambda_2 = -0.9114378. \]

Figure 6.9
Streamlines for \([B_1 + B_2(Dx + Ey)] \exp[\mathcal{F}(Dy - Ex)] - (Dx + Ey) = \text{constant}\)

when \(B_1 = 0, B_2 = D = E = 1, \mathcal{F} = \frac{1}{\sqrt{2}}\).

Figure 6.10
Streamlines for $[B_1 + B_2(Dx + Ey)] \exp[\mathcal{F}(Dy - Ez)] - (Dx + Ey) = \text{constant}$

when $B_1 = 50$, $B_2 = -60$, $D = E = 1$, $\mathcal{F} = \frac{1}{\sqrt{2}}$.

Figure 6.11
Streamlines for $C_1 \cos \left[ m(Dx+Ey)+C_2 \right] \exp \left[ \frac{-Dy-Ex}{F(D^2+E^2)} \right] - (Dx+Ey) = \text{constant}$

when $C_1 = 5$, $C_2 = 0$, $D = E = 1$, $F = \frac{1}{2}$, $m = \frac{1}{\sqrt{2}}$.

Figure 6.12
CHAPTER VII

VON KÁRMÁN–TYPE
SOLUTIONS OF A THIRD GRADE
MAGNETOHYDRODYNAMIC
ALIGNED FLUID FLOW
BETWEEN PARALLEL PLATES

7.1 INTRODUCTION.

The inherently nonlinear Navier-Stokes equations have no general solutions because of the analytic difficulties associated with solving them. These difficulties are compounded when dealing with fluids of the differential type of order greater than unity. Thus, some of the solutions obtained by researchers are approximated using numerical techniques, analytic techniques, and combinations of both. One of the analytic techniques employed by some researchers is the similarity transformation technique. Similarity solutions exist only for flows which show certain inherent physical symmetries. The aforementioned similarity transforms reduce the governing flow equations to nonlinear ordinary differential equations, since whenever similarity solutions exist, a reduction by at least one in the number of independent variables in the partial differential equations is achieved. These nonlinear ordinary differential equations may be obtained by dimensional analysis or by the method
of stretchings [cf. Hansen (1964), Ames (1965)]. Foremost among the analytic techniques of solving these nonlinear ordinary differential equations is the systematic perturbation (or asymptotic expansion) method in terms of a small or a large parameter or coordinate [cf. Nayfeh (1981)].

In this chapter, we explore the possibility of an MHD aligned, steady, planar, isochoric motion of an electrically conducting third grade fluid between two parallel plates under certain constitutive restrictions, admitting a von Kármán-type solution. The resulting ordinary differential equation will be solved by the perturbation method for small Reynolds number, when one of the plates is porous. A similar problem was studied by Tigoiu (1991).

The transformation

\[ u = \Omega r f'(\xi), \quad v = \Omega r g(\xi), \quad w = -2\sqrt{\Omega} \nu f(\xi) \]

was used by von Kármán (1921) when solving the problem of a flow due to an infinite rotating disc with rotation rate \(\Omega\) in cylindrical coordinates \((r, \theta, z)\), where \(u, v, w\) are the velocity components, \(\xi = z\sqrt{\Omega/\nu}\), \(\nu\) is the kinematic viscosity, \(f\) and \(g\) are functions of \(\xi\) to be determined.
7.2 FLOW EQUATIONS IN NONDIMENSIONAL FORM.

A steady, plane, isochoric motion of an electrically conducting third grade fluid when the body forces are negligible, in dimensional form, is governed by the system:

\[
\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0
\]

\[
\rho \left( \frac{\partial \bar{u}}{\partial \bar{z}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} \right) + \frac{\partial \bar{p}}{\partial \bar{y}} = \mu \bar{\nabla}^2 \bar{u} + \alpha_1 \left\{ \frac{\partial}{\partial \bar{z}} \left[ 2 \bar{u} \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + 2 \bar{v} \frac{\partial^2 \bar{u}}{\partial \bar{x} \partial \bar{y}} + 4 \left( \frac{\partial \bar{u}}{\partial \bar{x}} \right)^2 \right] \right\} \\
+ \frac{\partial}{\partial \bar{y}} \left[ \left( \bar{u} \frac{\partial \bar{v}}{\partial \bar{z}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} \right) \left( \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{z}} + 2 \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} \right) \right] \right\} \\
+ \alpha_2 \left\{ \frac{\partial}{\partial \bar{y}} \left[ 4 \left( \frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 + \left( \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}} \right) \right] \right\} + \beta_3 \left\{ \frac{\partial}{\partial \bar{y}} \left[ 2 \frac{\partial \bar{u}}{\partial \bar{y}} \left( \frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 \right] \right\} \\
+ 4 \left( \frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 + 2 \left( \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 \right] \right\} + \frac{\partial}{\partial \bar{y}} \left[ \left( \frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{y}} \right) \left( \frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 + 4 \left( \frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 \right] \\
+ 2 \left( \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 \right]^2 \right\} - \mu^* \bar{H}_2 \left( \frac{\partial \bar{H}_2}{\partial \bar{z}} - \frac{\partial \bar{H}_1}{\partial \bar{y}} \right)
\]

\[
\rho \left( \frac{\partial \bar{v}}{\partial \bar{z}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} \right) + \frac{\partial \bar{p}}{\partial \bar{y}} = \mu \bar{\nabla}^2 \bar{v} + \alpha_1 \left\{ \frac{\partial}{\partial \bar{z}} \left[ \left( \bar{u} \frac{\partial \bar{v}}{\partial \bar{z}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} \right) \left( \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{z}} + 2 \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} \right) \right] \right\} \\
+ \frac{\partial}{\partial \bar{y}} \left[ \left( \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} + \bar{u} \frac{\partial \bar{v}}{\partial \bar{y}} \right) \left( \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{z}} + 2 \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} \right) \right] \right\} \right. \\
+ \alpha_2 \left\{ \frac{\partial}{\partial \bar{y}} \left[ 4 \left( \frac{\partial \bar{v}}{\partial \bar{y}} \right)^2 + \left( \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}} \right) \right] \right\} + \beta_3 \left\{ \frac{\partial}{\partial \bar{y}} \left[ \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}} \right] \left( \frac{\partial \bar{v}}{\partial \bar{y}} \right)^2 + 4 \left( \frac{\partial \bar{v}}{\partial \bar{y}} \right)^2 \right] \\
+ 4 \left( \frac{\partial \bar{v}}{\partial \bar{y}} \right)^2 + 2 \left( \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 \right] \right\} + \frac{\partial}{\partial \bar{y}} \left[ 2 \left( \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 \right] \right\} \\
+ \mu^* \bar{H}_1 \left( \frac{\partial \bar{H}_1}{\partial \bar{z}} - \frac{\partial \bar{H}_1}{\partial \bar{y}} \right)
\]

\[
\frac{1}{\mu^* \sigma} \left( \frac{\partial \bar{H}_2}{\partial \bar{z}} - \frac{\partial \bar{H}_1}{\partial \bar{y}} \right) + \bar{v} \bar{H}_1 - \bar{u} \bar{H}_2 = \bar{k}
\]

\[
\frac{\partial \bar{H}_1}{\partial \bar{z}} + \frac{\partial \bar{H}_2}{\partial \bar{y}} = 0
\]

(7.1)

for the velocity components \( \bar{u}(\bar{z}, \bar{y}) \), \( \bar{v}(\bar{z}, \bar{y}) \), magnetic field strength components \( H_1(\bar{z}, \bar{y}) \), \( H_2(\bar{z}, \bar{y}) \), and the fluid pressure \( \bar{p}(\bar{z}, \bar{y}) \), where \( \bar{k} \) is an integration constant.

We wish to investigate if an electrically conducting third grade fluid with the constitutive restrictions (2.4) admits a von Kármán-type solution, when it flows
between two parallel plates. The flow is caused by the motion of the lower plate which is situated along the $z$-axis and is being stretched by two equal and opposite forces in the $z$-direction, while keeping the origin fixed. The upper plate is at a distance $h$ from the lower plate. In Problem I, the upper plate is porous whereas the lower plate is non-porous. In Problem II, the upper plate is non-porous and the lower plate is porous. In both cases, $v_w$ is the suction/injection speed at the porous plate.

To nondimensionalize system (7.1), we introduce the dimensionless variables
\[ \bar{v} = -v_w \]

\[ \bar{u} = u_o \bar{x} \]

\[ \bar{v} = -v_w \]

\[ \bar{u} = u_o \bar{x} \]

Figure 7.1
\[ x = \frac{\bar{x}}{h}, \quad y = \frac{\bar{y}}{h}, \quad u = \frac{\bar{u}}{u_0 h}, \quad v = \frac{\bar{v}}{u_0 h}, \quad p = \frac{\bar{p}}{\rho u_0^2 h^2}, \quad H_1 = \frac{\bar{H}_1}{H_0}, \quad H_2 = \frac{\bar{H}_2}{H_0} \]

(7.3)

where \( u_0 \) is a constant having the units of inverse time, and \( H_0 \) is the characteristic magnetic field strength. Employing (7.2) in system (7.1), we get

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\frac{u}{\partial x} + \frac{v}{\partial y} + \frac{\partial p}{\partial x} &= \frac{1}{Re} \nabla^2 u + \bar{\alpha}_1 \left\{ \frac{\partial}{\partial x} \left[ 2u \frac{\partial u}{\partial x} + 2v \frac{\partial^2 u}{\partial x \partial y} + 4(\frac{\partial u}{\partial x})^2 \right] \\
&+ 2\frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left[ (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + 2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right] \right\} \\
&+ \bar{\alpha}_2 \left\{ \frac{\partial}{\partial x} \left[ 4(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})^2 \right] \right\} + \bar{\beta}_3 Re \left\{ \frac{\partial}{\partial x} \left[ 2 \frac{\partial u}{\partial x} \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 \right] \\
&+ 4 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] + \frac{\partial}{\partial y} \left[ \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + 4 \left( \frac{\partial v}{\partial y} \right)^2 \right] \\
&+ 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) \right\} \right\} - RH_2 \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) \\
\frac{u}{\partial x} + \frac{v}{\partial y} + \frac{\partial p}{\partial x} &= \frac{1}{Re} \nabla^2 v + \bar{\alpha}_1 \left\{ \frac{\partial}{\partial x} \left[ (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \right] \\
&+ 2\frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left[ 2u \frac{\partial^2 v}{\partial x \partial y} + 2v \frac{\partial^2 v}{\partial y^2} + 4 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \frac{\partial u}{\partial x} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \right\} \\
&+ \bar{\alpha}_2 \left\{ \frac{\partial}{\partial y} \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] \right\} + \bar{\beta}_3 Re \left\{ \frac{\partial}{\partial x} \left[ (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 \right] \right. \\
&+ \left. 4 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] + \frac{\partial}{\partial y} \left[ 2 \frac{\partial u}{\partial x} \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 \right] + 4 \left( \frac{\partial v}{\partial y} \right)^2 \right] \\
&+ 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) \right\} \right\} + RH_1 \left( \frac{\partial H_1}{\partial x} - \frac{\partial H_1}{\partial y} \right) \\
\frac{1}{R_\sigma} \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) + vH_1 - uH_2 &= k \\
\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} &= 0
\end{align*}
\]

(7.3)

where \( k = \frac{k}{u_0 H_0} \), \( Re = \frac{\rho u_0 h^2}{\mu} = \frac{u_0 h^2}{\nu} \) is the hydrodynamic Reynolds number of the flow and \( R_\sigma = \mu^* \sigma_0 u_0 h^2 = \frac{u_0 h^2}{\nu_H} \) is the magnetic Reynolds number, with \( \nu = \frac{k}{\rho} \).
being the kinematic viscosity and \( \nu_H = \frac{1}{\mu_H} \) the magnetic viscosity or magnetic diffusivity. Despite the similarities in the forms of \( Re \) and \( R_\sigma \), there is a substantial difference between the two quantities. \( Re \) is obtained from dynamic consideration as the ratio of inertial force to viscous force, whereas \( R_\sigma \) is obtained from kinematic consideration as how the magnetic field will be influenced by the fluid motion. \( R_\sigma \) may be regarded as either the ratio of the dimension of the flow field to the characteristic length, or as the ratio of the flow velocity to the characteristic velocity.

\( R_\sigma = \frac{\mu^* H^2}{\rho u_0^2 h} \) is the magnetic pressure number, and is the ratio of the characteristic magnetic pressure, \( \frac{1}{2} \mu^* H_0^2 \), to the characteristic pressure, \( \frac{1}{2} \rho u_0^2 h^2 \). If \( R_\sigma \) is of the order of unity or larger, the fluid flow will be significantly affected by the magnetic field, and if \( R_\sigma \ll 1 \), the terms due to the magnetic field in the linear momentum equations are negligibly small, and the flow will not be noticeably affected by the magnetic field [cf. Pai (1962)]. \( \bar{\alpha}_{1,2} = \frac{\alpha_{1,2}}{\rho h^2} \) and \( \bar{\beta}_3 = \frac{\beta_3}{\rho h^2} \) are the dimensionless forms of \( \alpha_{1,2} \) and \( \beta_3 \).

The boundary conditions on the magnetic field intensity \( H = (H_1, H_2) \) are [cf. Dragos (1975)]

\[
H_a = H_b \quad (7.4)
\]

along the plates, where \( H_a \) and \( H_b \) represent the magnetic field intensities, respectively, above and below each plate.

The normal stress components \( T_{11}, T_{22}, \) and the shear stress component \( T_{12} \) are given by

\[
T_{11} = -p + \frac{2}{Re} \frac{\partial u}{\partial x} + \bar{\alpha}_1 \left[ 2u \frac{\partial^2 u}{\partial x^2} + 2v \frac{\partial^2 u}{\partial x \partial y} + 4 \left( \frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial v}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) \right]
+ \bar{\alpha}_2 \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] + \bar{\beta}_3 Re \left[ \frac{2}{Re} \left( \frac{\partial u}{\partial x} \right)^2 + 4 \left( \frac{\partial u}{\partial y} \right)^2 
+ 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right]
\]
\[ T_{22} = -p + \frac{2}{Re} \frac{\partial v}{\partial y} + \bar{\alpha}_1 \left[ 2u \frac{\partial^2 v}{\partial x \partial y} + 2v \frac{\partial^2 v}{\partial y^2} + 4 \left( \frac{\partial v}{\partial y} \right)^2 \right] \\
+ \bar{\alpha}_2 \left[ 4 \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] + \bar{\beta}_3 \nonumber Re \left\{ 2 \frac{\partial v}{\partial y} \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + 4 \left( \frac{\partial v}{\partial y} \right)^2 \right] \\
+ 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right\} \] \\
(7.5) \nonumber \\
T_{12} = T_{21} = \frac{1}{Re} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \bar{\alpha}_1 \left[ \left( \frac{u}{\partial x} + \frac{v}{\partial y} \right) \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \right] \\
+ 2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \bar{\beta}_3 \nonumber Re \left\{ \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + 4 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] \right\} \]
7.3 METHOD AND SOLUTIONS.

We shall study aligned flows for which the magnetic field is parallel to the velocity field everywhere in the region of flow. Thus, equations (2.28) hold true.

A solution of the von Kármán type, given by

\[ u = xg'(y), \quad v = -g(y) \quad (7.6) \]

will be investigated, where \( g \) is a non-vanishing function of \( y \) to be determined.

Employing equations (2.28) and (7.6) in system (7.3), we observe that the continuity equation is identically satisfied, and the remaining equations become

\[
\frac{\partial p}{\partial x} = -R_H g^2 f \frac{\partial f}{\partial x} + x \left[ \frac{1}{Re} g'' + gg'' - g'^2 + \bar{a}_1 (3g''^2 + 2g'g'' - gg'') \right] \\
+ 2\bar{a}_2 g''^2 + 8\bar{b}_3 \text{Reg}' (g'g'' + 3g'^2) - R_H fg (fg'' + g' \frac{\partial f}{\partial y}) \\
+ 6\bar{b}_3 \text{Re} x^2 g''^2 g'''
\]

\[
\frac{\partial p}{\partial y} = -\left[ \frac{1}{Re} g'' + gg' - \bar{a}_1 (gg'' + 13g'g'') - 8\bar{a}_2 g'g'' + 40\bar{b}_3 \text{Reg}'^2 g'' \right] \\
- R_H x fg g' \frac{\partial f}{\partial x} + x^2 \left[ 2(2\bar{a}_1 + \bar{a}_2) g'' g'' + 2\bar{b}_3 \text{Reg}' (g'^2 - 4g'g'') \right] \\
- R_H fg' (fg'' + g' \frac{\partial f}{\partial y}) \\
k + \frac{1}{Re} \left[ \frac{g}{x} \frac{\partial f}{\partial x} + x (fg'' + g' \frac{\partial f}{\partial y}) \right] = 0
\]

\[ g \frac{\partial f}{\partial y} - xg' \frac{\partial f}{\partial x} = 0 \]

(7.7)

The general solution of the last equation in system (7.7) is \( f = f(xg) \). For the special case of \( f = f_0 \), a real constant, system (7.7) reduces to

\[
\frac{\partial p}{\partial x} = x \left[ \frac{1}{Re} g'' + gg'' - g'^2 + \bar{a}_1 (3g''^2 + 2g'g'' - gg'') \right] \\
+ 8\bar{b}_3 \text{Reg}' (g'g'' + 3g'^2) - R_H f_0^2 gg'' + 6\bar{b}_3 \text{Re} x^2 g''^2 g'''
\]

\[
\frac{\partial p}{\partial y} = -\left[ \frac{1}{Re} g'' + gg' - \bar{a}_1 (gg'' + 13g'g'') - 8\bar{a}_2 g'g'' + 40\bar{b}_3 \text{Reg}'^2 g'' \right] \\
+ x^2 \left[ 2(2\bar{a}_1 + \bar{a}_2) g'' g'' + 2\bar{b}_3 \text{Reg}' (g'^2 - 4g'g'') \right] - R_H f_0^2 g'g'' \\
k + \frac{f_0}{Re} xg'' = 0
\]

(7.8)
where the equation showing that the magnetic field is solenoidal is identically satisfied.

The integrability condition \( \frac{\partial^2 \rho}{\partial x \partial y} = \frac{\partial^2 \rho}{\partial y \partial z} \) is applied to the linear momentum equations in system (7.8) to get the equations

\[
g^{iv} + \text{Re}(1 - R_H f_0^2)(gg'''' - g'g'') + \bar{\alpha}_1 \text{Re}(g'g^{iv} - gg^v)
+ 4\bar{\beta}_3 \text{Re}^2(2g'^2 g^{iv} + 20g'g''g''' + 5g''') + 6\bar{\beta}_3 \text{Re}^2 x^2 g''(g''g^{iv} + 2g''''') = 0
\]

(7.9)

\[k + \frac{f_0}{R_e} x g'' = 0 \]

(7.10)

The aligned flow discussed in the foregoing will be divided into those for which the fluid has infinite and finite magnetic Reynolds number \( R_e \) (or infinite and finite electrical conductivities \( \sigma \)).

**Infinite Magnetic Reynolds Number.**

When the electrical conductivity \( \sigma \to \infty \), the magnetic Reynolds number, \( R_e = \mu^* \sigma u_0 h^2 \), tends to infinity. Thus, the diffusion equation (7.10) gives \( k = 0 \) since \( \frac{1}{R_e} \to 0 \).

Equating the coefficients of unity and \( x^2 \) to zero, we obtain

\[
g^{iv} + \text{Re}(1 - R_H f_0^2)(gg'''' - g'g'') + \bar{\alpha}_1 \text{Re}(g'g^{iv} - gg^v)
+ 4\bar{\beta}_3 \text{Re}^2(2g'^2 g^{iv} + 20g'g''g''' + 5g''') = 0
\]

(7.11)

\[
\bar{\beta}_3 g''(g''g^{iv} + 2g''''') = 0
\]

(7.12)

since \( \text{Re} \neq 0 \).

Equations (7.11) and (7.12) will be solved for **Problem I** and **Problem II**.
Problem 1.

This problem consists of a porous upper plate situated along \( y = 1 \) and a non-porous plate located along \( y = 0 \) which is being stretched by two equal and opposite forces so that the origin is fixed. The motion of the fluid is caused by the stretching of the lower plate.

The boundary conditions on the velocity components for this problem are

\[
  u(x, 0) = x, \quad v(x, 0) = u(x, 1) = 0, \quad v(x, 1) = -v_0
\]

or

\[
g'(0) = 1, \quad g(0) = g'(1) = 0, \quad g(1) = v_0 \quad (7.13)
\]

where \(-v_0 = -\frac{x_m}{u_0h}\) is the injection speed at the upper plate \( y = 1 \).

The solution of (7.12) with \( \bar{\beta}_3 \neq 0 \), subject to the boundary conditions (7.13), does not satisfy (7.11). Therefore, there may exist a solution to the problem only if
\( \bar{\beta}_3 \) (or \( \beta_3 \)) vanishes for the chosen form of \( f \). Thus, we conclude that the aligned flow of an electrically conducting third grade fluid having infinite magnetic Reynolds number, with the rheological restrictions (2.4) on the material constants, cannot accept a von Kármán-type solution for the flow between parallel plates described above when \( f = f_0 \), a real constant.

We explore the possibility of a solution for the MHD aligned, isochoric flow of a second grade fluid of infinite magnetic Reynolds number under the constitutive restrictions (2.5).

With \( \bar{\beta}_3 \equiv 0 \), (7.11) takes the form

\[
g^{iv} + Re(1 - R_H f_0^2)(gg'' - g'g'') + \bar{\alpha}_1 Re(g'g^{iv} - gg'') = 0 \tag{7.14}
\]

subject to

\[
g(0) = 0, \quad g'(0) = 1, \quad g(1) = v_0, \quad g'(1) = 0 \tag{7.15}
\]

The nonlinear equation (7.14) may be solved approximately by the perturbation method. For small values of the perturbation parameter \( Re \), we expand \( g(y) \) in the form

\[
g(y) = g_0(y) + Reg_1(y) + Re^2 g_2(y) + \ldots + Re^n g_n(y) + \ldots \tag{7.16}
\]

which is employed in (7.14) and (7.15) to obtain

\[
\begin{align*}
&\left[ g^{iv}_0 + Reg^{iv}_1 + Re^2 g^{iv}_2 + \ldots + Re^n g^{iv}_n + \ldots \right] + Re(1 - R_H f_0^2) \left[ (g_0 + Reg_1 \\
&+ Re^2 g_2 + \ldots + Re^n g_n + \ldots) (g''_0 + Reg''_1 + Re^2 g''_2 + \ldots + Re^n g''_n + \ldots) - (g'_0 + Reg'_1 + Re^2 g'_2 + \ldots + Re^n g'_n + \ldots) (g''_0 + Reg''_1 + Re^2 g''_2 + \ldots + Re^n g''_n + \ldots) \right] \\
&+ \bar{\alpha}_1 Re \left[ (g^{iv}_0 + Reg^{iv}_1 + Re^2 g^{iv}_2 + \ldots + Re^n g^{iv}_n + \ldots) (g''_0 + Reg''_1 + Re^2 g''_2 + \ldots + Re^n g''_n + \ldots) - (g_0 + Reg_1 + Re^2 g_2 + \ldots + Re^n g_n + \ldots) (g^{iv}_0 + Reg^{iv}_1 + Re^2 g^{iv}_2 + \ldots + Re^n g^{iv}_n + \ldots) \right] = 0
\end{align*}
\]
subject to

\[ g_0(0) + \text{Reg}_1(0) + \text{Re}^2 g_2(0) + \ldots + \text{Re}^n g_n(0) + \ldots = 0 \]

\[ g_0'(0) + \text{Reg}_1'(0) + \text{Re}^2 g_2'(0) + \ldots + \text{Re}^n g_n'(0) + \ldots = 1 \]

\[ g_0(1) + \text{Reg}_1(1) + \text{Re}^2 g_2(1) + \ldots + \text{Re}^n g_n(1) + \ldots = v_0 \]

\[ g_0'(1) + \text{Reg}_1'(1) + \text{Re}^2 g_2'(1) + \ldots + \text{Re}^n g_n'(1) + \ldots = 0 \]

Retaining only terms up to and including \( O(\text{Re}^2) \) in the above expansion, we need to solve the following fourth order ordinary differential equations:

\[ g_0^{iv} = 0 \quad (7.17) \]

\[ g_1^{iv} = (R_H f_0^2 - 1)(g_0 g_0'' - g_0' g_0'') \quad (7.18) \]

\[ g_2^{iv} = (R_H f_0^2 - 1)(g_0 g_1'' - g_0' g_1' - g_0'' g_1' + g_0'' g_1) + \alpha_1 (g_0 g_1'' - g_0' g_1'') \quad (7.19) \]

subject to the respective boundary conditions:

\[ g_0(0) = 0, \quad g_0'(0) = 1, \quad g_0(1) = v_0, \quad g_0'(1) = 0 \quad (7.20) \]

\[ g_1(0) = g_1'(0) = g_1(1) = g_1'(1) = 0 \quad (7.21) \]

\[ g_2(0) = g_2'(0) = g_2(1) = g_2'(1) = 0 \quad (7.22) \]

From (7.17) and (7.20), we get the zeroth order solution:

\[ g_0(y) = y \left[ 1 + (3v_0 - 2)y + (1 - 2v_0)y^2 \right] \quad (7.23) \]

Applying (7.23) to (7.18) yields

\[ g_1^{iv} = 2(1 - R_H f_0^2) \left[ 3v_0 - 2 + 2(3v_0 - 2)^2 y + 6(3v_0 - 2)(1 - 2v_0)y^2 + 6(2v_0 - 1)^2 y^3 \right] \quad (7.24) \]
which is solved subject to the boundary conditions (7.21) to obtain the first order solution:

\[
g_1(y) = \frac{(1 - R_H f_0^2)}{420} y^2 \left[ 3(32v_0^2 - 11v_0 - 6) + 2(27 + 11v_0 - 81v_0^2) y 
+ 35(3v_0 - 2)y^2 + 14(3v_0 - 2)^2 y^3 + 14(3v_0 - 2)(1 - 2v_0)y^4 + 2(2v_0 - 1)^2 y^5 \right] 
\]

(7.25)

Substituting (7.23) and (7.25) into (7.19), we obtain

\[
g_2^{iv} = (1 - R_H f_0^2) [A_0 + A_1 y + A_2 y^2 + A_3 y^3 + A_4 y^4 + A_5 y^5 + A_6 y^6 + A_7 y^7] 
\]

(7.26)

subject to the boundary conditions (7.22), where

\[
A_0 = \frac{(1 - R_H f_0^2)}{70} (32v_0^2 - 11v_0 - 6) + 2\bar{\alpha}_1 (2 - 3v_0)
\]

\[
A_1 = 2(3v_0 - 2) \left[ \frac{(1 - R_H f_0^2)}{35} (32v_0^2 - 11v_0 - 6) + 2\bar{\alpha}_1 (2 - 3v_0) \right]
\]

\[
A_2 = -2 \left[ \frac{(1 - R_H f_0^2)}{35} (339v_0^2 - 276v_0^2 - 8v_0 + 28) + \bar{\alpha}_1 (3v_0 - 2)(18v_0^2 - 18v_0 + 5) \right]
\]

\[
A_3 = 2 \left[ \frac{(1 - R_H f_0^2)}{105} (972v_0^3 - 933v_0^2 + 162v_0 + 22) + 4\bar{\alpha}_1 (2v_0 - 1)(3v_0 - 1)^2 \right]
\]

\[
A_4 = \frac{(1 - R_H f_0^2)}{6} (2 - 3v_0)(18v_0^2 - 30v_0 + 11)
\]

\[
A_5 = \frac{4(1 - R_H f_0^2)}{5} (2v_0 - 1)(9v_0^2 - 18v_0 + 7)
\]

\[
A_6 = \frac{8(1 - R_H f_0^2)}{5} (2 - 3v_0)(2v_0 - 1)^2
\]

\[
A_7 = \frac{24(1 - R_H f_0^2)}{35} (1 - 2v_0)^3
\]

(7.27)

Equation (7.26), with the associated boundary conditions, is solved to obtain the
second order solution:

\[ g_2(y) = \left(1 - R_H f_0^2\right) \frac{y^2}{2} \left[ \frac{1}{12} A_0 + \frac{1}{30} A_1 + \frac{1}{60} A_2 + \frac{1}{105} A_3 + \frac{1}{165} A_4 + \frac{1}{252} A_5 \right. \\
+ \frac{1}{360} A_6 + \frac{1}{495} A_7 - \left(\frac{1}{6} A_0 + \frac{1}{20} A_1 + \frac{1}{45} A_2 + \frac{1}{84} A_3 + \frac{1}{140} A_4 + \frac{1}{216} A_5 \right. \\
+ \frac{1}{315} A_6 + \frac{1}{440} A_7 \right] y + \frac{1}{12} A_0 y^2 + \frac{1}{60} A_1 y^3 + \frac{1}{180} A_2 y^4 + \frac{1}{420} A_3 y^5 \\
+ \frac{1}{840} A_4 y^6 + \frac{1}{1512} A_5 y^7 + \frac{1}{2520} A_6 y^8 + \frac{1}{3960} A_7 y^9 \]

(7.28)

Equations (7.23), (7.25) and (7.28) are employed in (7.16), which in turn is substituted into (7.6) and (2.28) to obtain the velocity and magnetic field components:

\[ u = x[g_0'(y) + Reg_1'(y) + Re^2g_2'(y)] + O(Re^3), \quad H_1 = f_0 u \]

\[ v = -[g_0(y) + Reg_1(y) + Re^2g_2(y)] + O(Re^3), \quad H_2 = f_0 v \]

(7.29)

since \( f = f_0 \), a real constant.

Using the fact that \( \bar{\beta}_0 \equiv 0 \), the linear momentum equations in system (7.8) become

\[ \frac{\partial p}{\partial x} = x \left[ \frac{1}{Re} g'''' + (1 - R_H f_0^2) g g'' - g^2 + \bar{\alpha}_1 (3g''^2 + 2g' g''' - gg''') \right. \\
+ 2\bar{\alpha}_2 g''^2 \right] \]

\[ \frac{\partial p}{\partial y} = - \left[ \frac{1}{Re} g'' + g g' - \bar{\alpha}_1 (gg''' + 13g' g'') - 8\bar{\alpha}_2 g' g'' \right. \\
+ x^2 \left[ 2(2\bar{\alpha}_1 + \bar{\alpha}_2) g'' g''' - R_H f_0^2 g' g'' \right] \]

(7.30)

Integrating these equations and using (7.14), we obtain the fluid pressure

\[ p = p_0 + \frac{1}{Re} g' - \frac{1}{2} g^2 + \bar{\alpha}_1 (gg'' + 6g'^2) + 4\bar{\alpha}_2 g'^2 + \frac{1}{2} x^2 \left[ \frac{1}{Re} g'' \right. \\
+ (1 - R_H f_0^2) gg'' - g^2 + \bar{\alpha}_1 (3g''^2 + 2g' g''' - gg''') + 2\bar{\alpha}_2 g''^2 \]

(7.31)

where \( p_0 \) is an arbitrary constant.
Substitution of (7.6) and (7.31) into (7.5) with $\bar{\beta}_3 \equiv 0$, we get

\[
T_{11} = -p_0 + \frac{3}{Re} g' + \frac{1}{2} g^2 - \bar{\alpha}_1 (3g g'' + 2g'^2) + \frac{1}{2} x^2 \left[ g''^2 + (R_H f_0^2 - 1) gg'' - \frac{1}{Re} g'''' + \bar{\alpha}_1 (gg'' - 2g'g''' - 3g''^2) \right]
\]

\[
T_{22} = -p_0 - \frac{1}{Re} g' + \frac{1}{2} g^2 + \bar{\alpha}_1 (gg'' - 2g'^2) + \frac{1}{2} x^2 \left[ g''^2 + (R_H f_0^2 - 1) gg'' - \frac{1}{Re} g'''' + \bar{\alpha}_1 (gg'' - 2g'g''' + g''^2) \right]
\]

\[
T_{12} = x \left[ \frac{1}{Re} g'' + \bar{\alpha}_1 (3g' g'' - gg''') \right]
\]

The stress components may be evaluated at the plates $y = 0$ and $y = 1$ using

\[
g(y) = g_0(y) + Re g_1(y) + Re^2 g_2(y) + O(Re^3).
\]

The boundary conditions for the magnetic field strength at the plate $y = 1$ are

\[
H^a(x, 1) = H^b(x, 1)
\]

where $H^a(x, y) = (0, A^a)$ and $H^b(x, y) = (f_0 u, f_0 v) = (f_0 x g'(y), -f_0 g(y))$, with $A^*$ being a non-zero real constant. Thus, $f_0 = -\frac{A^*}{v_0}$ since $g(1) = v_0$, $g'(1) = 0$. This value of $f_0$ is substituted into the results obtained above.
Problem II.

\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (6,0) node[right] {$x$};
\draw[->] (0,0) -- (0,3) node[above] {$y$};
\draw (0,1) -- (5,1) node[right] {$1$};
\draw (0,0) -- (0,-1) node[below] {$v = -v_0$};
\draw (5,0) -- (5,-1) node[below] {$u = x$};
\end{tikzpicture}
\end{center}

Figure 7.3

In this problem, the upper plate is fixed and non-porous, whereas the lower plate is porous and is being stretched by equal and opposite forces in such a way that the origin is fixed. As in Problem I, the flow is caused by the motion of the lower plate.

The velocity boundary conditions for this problem are

\[ u(x, 0) = x, \quad v(x, 0) = -v_0, \quad u(x, 1) = v(x, 1) = 0 \]

or

\[ g'(0) = 1, \quad g(0) = v_0, \quad g'(1) = g(1) = 0 \quad (7.34) \]

where \(-v_0 = -\frac{v_0}{v_0 h}\) is the suction speed at \(y = 0\).

Following the procedure of Problem I, the fourth order ordinary differential equations, along with their boundary conditions, we must solve are:
Zeroth order problem.

\[ g_0^{iv} = 0; \]
\[ g_0(0) = v_0, \quad g_0'(0) = 1, \quad g_0(1) = g_0'(1) = 0 \] (7.35)

First order problem.

\[ g_1^{iv} = \left( R_H f_0^2 - 1 \right) \left( g_0' g_0''' - g_0 g_0'' \right); \]
\[ g_1(0) = g_1'(0) = g_1(1) = g_1'(1) = 0 \] (7.36)

Second order problem.

\[ g_2^{iv} = \left( R_H f_0^2 - 1 \right) \left( g_0 g_2''' - g_0' g_2'' - g_0'' g_1' + g_0'' g_1 \right) + \alpha \left( g_0 g_2' - g_0' g_2 \right); \] (7.37)
\[ g_2(0) = g_2'(0) = g_2(1) = g_2'(1) = 0 \]

where

\[ g(y) = g_0(y) + \text{Reg}_1(y) + \text{Re}^2 g_2(y) + O(\text{Re}^3) \] (7.38)

The solution of (7.35) is

\[ g_0(y) = v_0 + y - (3v_0 + 2)y^2 + (2v_0 + 1)y^3 \] (7.39)

Substitution of (7.39) into (7.36) yields the differential equation

\[ g_1^{iv} = 4 \left( R_H f_0^2 - 1 \right) \left[ \left( 3v_0^2 + 3v_0 + 1 \right) - (3v_0 + 2)^2 y \right. \]
\[ \left. + 3(3v_0 + 2)(2v_0 + 1)y^2 - 3(2v_0 + 1)^2 y^3 \right] \] (7.40)

which is solved along with its boundary conditions to obtain

\[ g_1(y) = \frac{(1 - R_H f_0^2)}{210} y^2 \left[ -3(19v_0^2 + 12v_0 + 3) + (129v_0^2 + 94v_0 + 27) y \right. \]
\[ \left. - 35(3v_0^2 + 3v_0 + 1)y^2 + 7(3v_0 + 2)^2 y^3 - 7(3v_0 + 2)(2v_0 + 1)y^4 + 3(2v_0 + 1)^2 y^5 \right] \] (7.41)

Using (7.39) and (7.41) in (7.37) gives

\[ g_2^{iv} = \left( 1 - R_H f_0^2 \right) \left[ B_0 + B_1 y + B_2 y^2 + B_3 y^3 + B_4 y^4 + B_5 y^5 + B_6 y^6 + B_7 y^7 \right] \]
\[ g_2(0) = g_2'(0) = g_2(1) = g_2'(1) = 0 \] (7.42)
where

\[ B_0 = \frac{(R_H f_0^2 - 1)}{35} (129v_0^3 + 113v_0 + 39v_0 + 3) + 4\bar{a}_1 (v_0 + 1)(3v_0 + 1)^2 \]

\[ B_1 = 4 \left[ \frac{(1 - R_H f_0^2)}{35} (12v_0^3 + 179v_0 + 65v_0 + 6) - 2\bar{a}_1 (3v_0 + 2)(3v_0 + 1)^2 \right] \]

\[ B_2 = 2 \left[ \frac{(R_H f_0^2 - 1)}{35} (816v_0^3 + 984v_0 + 358v_0 + 28) + 2\bar{a}_1 (3v_0 + 1)(27v_0^2 + 24v_0 + 5) \right] \]

\[ B_3 = 4 \left[ \frac{(1 - R_H f_0^2)}{105} (1719v_0^3 + 1896v_0 + 549v_0 + 11) - 2\bar{a}_1 (2v_0 + 1)(3v_0 + 1)^2 \right] \]

\[ B_4 = \frac{(R_H f_0^2 - 1)}{3} (99v_0^3 + 63v_0^2 - 15v_0 - 11) \]

\[ B_5 = \frac{4(R_H f_0^2 - 1)}{5} (2v_0 + 1)(9v_0^2 + 18v_0 + 7) \]

\[ B_6 = \frac{8(1 - R_H f_0^2)}{5} (3v_0 + 2)(2v_0 + 1)^2 \]

\[ B_7 = \frac{24(R_H f_0^2 - 1)}{35} (2v_0 + 1)^3 \]  \hspace{1cm} (7.43)

The solution of (7.42) is

\[ g_2(y) = \frac{(1 - R_H f_0^2)}{2} y^2 \left[ \left( \frac{1}{12} B_0 + \frac{1}{30} B_1 + \frac{1}{60} B_2 + \frac{1}{105} B_3 + \frac{1}{160} B_4 + \frac{1}{252} B_5 \right. \right. \]

\[ + \frac{1}{360} B_6 + \frac{1}{495} B_7 ) - \left( \frac{1}{6} B_0 + \frac{1}{20} B_1 + \frac{1}{45} B_2 + \frac{1}{84} B_3 + \frac{1}{140} B_4 + \frac{1}{216} B_5 \right. \]

\[ + \frac{1}{315} B_6 + \frac{1}{440} B_7 ) y + \frac{1}{12} B_0 y^2 + \frac{1}{60} B_1 y^3 + \frac{1}{180} B_2 y^4 + \frac{1}{420} B_3 y^5 \]

\[ + \frac{1}{840} B_4 y^6 + \frac{1}{1512} B_5 y^7 + \frac{1}{2520} B_6 y^8 + \frac{1}{3960} B_7 y^9 \right] \]  \hspace{1cm} (7.44)

The components of the velocity and magnetic fields, and the fluid pressure are given by (7.29) and (7.31), where \( g(y) \) is as shown in (7.38). In this case, \( g_0(y) \), \( g_1(y) \) and \( g_2(y) \) are given, respectively, by (7.39), (7.41) and (7.44). The normal and shear stresses at the plates \( y = 0 \) and \( y = 1 \) may be evaluated from equations (7.32).
The specified boundary conditions at the plate \( y = 0 \) for the magnetic field strength are

\[
\mathbf{H}^a(x, 0) = \mathbf{H}^b(x, 0)
\]

(7.45)

where \( \mathbf{H}^a(x, y) = (f_0 x g'(y), -f_0 g(y)) \) and \( \mathbf{H}^b(x, y) = (B^* x, -B^* y) \). Therefore, \( f_0 = B^* = 0 \) since \( g(0) = v_0 \), \( g'(0) = 1 \), which implies that there is no magnetic field in the flow region. Thus, \( R_H \equiv 0 \) in the above analysis, and the approximate solution obtained is that of a second grade fluid flow between two parallel plates. It should be noted that the aforementioned result is obtained for the constant form of \( f \) chosen.

**Finite Magnetic Reynolds Number.**

The diffusion equation \( x g''(y) = -\frac{k B x}{f_0} \) in system (7.8) implies that both \( x \) and \( g''(y) \) must be constant. This is impossible since \( x \) takes on different values in the flow region. Also, if we let \( k = g''(y) = 0 \), then \( g(y) = c_1 y + c_2 \), where \( c_1 \) and \( c_2 \) are arbitrary constants. It is found that this form of \( g(y) \) does not satisfy the boundary conditions in both Problem I and Problem II. We, therefore, conclude that the MHD aligned, steady, plane, isochoric motion of a finitely conducting second grade fluid between two parallel plates is not possible when \( f \) is a real constant.

It has been shown with two examples (Problem I and Problem II) that the aligned flow of an electrically conducting third grade fluid, with the conditions (2.4) on the material constants cannot accept a solution of the von Kármán type solution for flow between two parallel plates when \( f = f_0 \), a real constant. A solution is possible for the aligned flow of an infinitely conducting second grade fluid between parallel plates with the rheological restrictions (2.5) for Problem I and for ordinary second grade fluid flow for Problem II when \( f \) is a real constant.
CHAPTER VIII

UNSTEADY SECOND
GRADE MAGNETOHYDRODYNAMIC
ALIGNED FLUID FLOW

8.1 INTRODUCTION.

Exact solutions of the equations of motion of non-Newtonian fluids are difficult to obtain because of the nonlinearities of higher order that have been introduced over and above those of the Navier-Stokes equations, which are of second order. The third order equations of the second grade fluid flows will, in general, require an additional boundary and/or initial condition in addition to those required for solving the Navier-Stokes equations. The necessity for this extra condition may be avoided by the application of the inverse method.

The aim of this work is to find some exact solutions of the equations of an unsteady, plane, second grade, electrically conducting, magnetohydrodynamic (MHD) aligned fluid which undergoes isochoric motion. The interest in MHD fluid flows stems from the fact that liquid metals that occur in nature and industry are electrically conducting. These fluids, for the most part, are of finite electrical conductivity. Also considered is an MHD flow of an electrically conducting fluid of infinite electrical conductivity. These two types of fluid flow are attractive both from a mathematical as well as a physical standpoint.
In applying the inverse method to our problem, the vorticity distribution will be assumed to be proportional to the stream function perturbed by a uniform stream. Kovasznay (1948) studied the steady Navier-Stokes equations by making a similar assumption and found the motion behind a two-dimensional grid. More recently, Lin and Tobak (1986) obtained more results by studying a similar flow. Hui (1987) extended this work to unsteady flows. Benharbit and Siddiqui (1992) investigated both steady and unsteady flows for the equations of a second grade fluid. In this work, the effects of the presence of a magnetic field on the unsteady, isochoptic motion of an electrically conducting second grade fluid is studied. Similarities and contrasts to the above works will be identified. The contrasts arise because of the effect of the magnetic field on the motion of the fluid.
8.2 FLOW EQUATIONS AND METHOD.

By setting $\beta_3 \equiv 0$ in system (2.6), we have that an unsteady, plane, isochoric motion of an electrically conducting second grade MHD fluid, in the absence of body forces, is governed by the system:

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} & = 0 \\
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial p}{\partial x} & = \mu \nabla^2 u + \alpha_1 \left\{ \nabla^2 \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left[ 2u \frac{\partial^2 u}{\partial x^2} + 2v \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial u}{\partial x} \right] \\ & + 2 \frac{\partial v}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left[ (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + 2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right] \right\} \\
& + \alpha_2 \left\{ \frac{\partial}{\partial x} \left[ 4 \frac{\partial u}{\partial x} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \right\} - \mu^* H_2 \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) \\
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + \frac{\partial p}{\partial y} & = \mu \nabla^2 v + \alpha_1 \left\{ \frac{\partial}{\partial x} \left[ (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \right] \\ & + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + 2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right] + \frac{\partial}{\partial y} \left[ 2u \frac{\partial^2 v}{\partial x \partial y} + 2v \frac{\partial^2 v}{\partial y^2} + 4 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \frac{\partial u}{\partial y} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \right\} \\
& + \alpha_2 \left\{ \frac{\partial}{\partial y} \left[ 4 \frac{\partial v}{\partial y} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \right\} + \mu^* H_1 \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) \\
\frac{\partial}{\partial t} \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) & = \nabla^2 \left[ \frac{1}{\mu^* \sigma} \left( \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) + v H_1 - u H_2 \right] \\
\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} & = 0
\end{align*}
\] 

(8.1)

of five equations in the unknown functions $u, v, H_1, H_2$ and $p$ of $x, y, t$.

By introducing the vorticity $\omega$, the current density $j$ and the generalized energy
\( h \), given by system (2.26), system (8.1) takes the form

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \quad \text{\text{(continuity)}}
\]

\[
\frac{\partial h}{\partial x} = -\mu \frac{\partial \omega}{\partial y} + \rho (u \omega - \frac{\partial u}{\partial t}) - \mu^* j H_2 - \alpha_1 \left( \frac{\partial^2 \omega}{\partial y \partial t} + v \nabla^2 \omega \right) \quad \text{\text{(linear momentum)}}
\]

\[
\frac{\partial h}{\partial y} = \mu \frac{\partial \omega}{\partial x} - \rho (u \omega + \frac{\partial v}{\partial t}) + \mu^* j H_1 + \alpha_1 \left( \frac{\partial^2 \omega}{\partial x \partial t} + u \nabla^2 \omega \right)
\]

\[
\frac{\partial j}{\partial t} = \nabla^2 \left( \frac{j}{\mu^* \sigma} + v H_1 - u H_2 \right) \quad \text{\text{(diffusion)}}
\]

\[
\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 \quad \text{\text{(solenoidal)}}
\]

(8.2)

which, along with the first two equations in system (2.26), gives a system of seven equations for the functions \( u, v, H_1, H_2, \omega, j \) and \( h \) of \( x, y, t \).

The continuity equation in system (8.2) implies the existence of a stream function \( \psi(x, y, t) \) such that equations (2.30) are satisfied.

We shall study aligned flows for which the magnetic field in the region of flow is everywhere parallel to the velocity field. Therefore, equation (2.28) holds true, where \( f = f(x, y, t) \neq 0 \) is an arbitrary function. We have, from equations (2.30) and (2.28), that

\[
H_1 = f \frac{\partial \psi}{\partial y}, \quad H_2 = -f \frac{\partial \psi}{\partial x}
\]

(8.3)

Substituting (2.30) and (8.3) in the expressions for the vorticity, the current density, and the linear momentum equations in system (8.2), we get

\[
\omega = -\nabla^2 \psi, \quad j = -(f \nabla^2 \psi + \frac{\partial f \psi}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial f \psi}{\partial y} \frac{\partial \psi}{\partial y})
\]

(8.4)
and
\[
\frac{\partial h}{\partial x} = \mu \frac{\partial}{\partial y} (\nabla^2 \psi) + \rho (\nabla^2 \psi \frac{\partial \psi}{\partial x} - \frac{\partial^2 \psi}{\partial y \partial t}) - \mu \frac{\partial f}{\partial x} \left( f \nabla^2 \psi + \frac{\partial f}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial \psi}{\partial y} \right) \\
+ \alpha_1 \left[ \frac{\partial^2}{\partial y \partial t} (\nabla^2 \psi) - \nabla^4 \psi \frac{\partial \psi}{\partial x} \right]
\]
\[
\frac{\partial h}{\partial y} = -\frac{\partial}{\partial x} (\nabla^2 \psi) + \rho (\nabla^2 \psi \frac{\partial \psi}{\partial y} + \frac{\partial^2 \psi}{\partial x \partial t}) - \mu \frac{\partial f}{\partial y} \left( f \nabla^2 \psi + \frac{\partial f}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial \psi}{\partial y} \right) \\
- \alpha_1 \left[ \frac{\partial^2}{\partial x \partial t} (\nabla^2 \psi) + \nabla^4 \psi \frac{\partial \psi}{\partial y} \right]
\]
\]
(8.5)

If we use the integrability condition \( \frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x} \) in equations (8.5) and employ equations (2.30), (8.3) and (8.4) in the diffusion equation and the solenoidal equation for the magnetic field in system (8.2), we obtain
\[
\mu \nabla^4 \psi - \rho \frac{\partial}{\partial t} (\nabla^2 \psi) + (\rho - \mu \nabla^2 f) \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, y)} + \alpha_1 \left[ \frac{\partial}{\partial t} (\nabla^4 \psi) - \frac{\partial (\psi, \nabla^4 \psi)}{\partial (x, y)} \right]
\]
\[
= \mu \left[ (2f \nabla^2 \psi + \frac{\partial f}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial \psi}{\partial y}) \frac{\partial (\psi, f)}{\partial (x, y)} + f \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} \right] \\
+ \frac{\partial f}{\partial x} \frac{\partial (\psi, \frac{\partial \psi}{\partial x})}{\partial (x, y)} + \frac{\partial f}{\partial y} \frac{\partial (\psi, \frac{\partial \psi}{\partial y})}{\partial (x, y)}
\]
\[
= \mu \nabla^4 \psi + 3 \left[ \frac{\partial f}{\partial x} \frac{\partial \psi}{\partial x} (\nabla^2 \psi) + \frac{\partial f}{\partial y} \frac{\partial \psi}{\partial y} (\nabla^2 \psi) \right] + \nabla^2 f \nabla^2 \psi + 2 \left( \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 \psi}{\partial x^2} \\
+ 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} + \frac{\partial^2 f}{\partial y^2} \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial y} \right) \frac{\nabla^2 f}{\partial x \partial y} + \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} (\nabla^2 f)
\]
\[
= \mu \nabla^4 \psi + \nabla^2 f \frac{\partial \psi}{\partial t} + \frac{\partial f}{\partial x} \frac{\partial^2 \psi}{\partial x \partial t} + \frac{\partial f}{\partial y} \frac{\partial^2 \psi}{\partial y \partial t} + \frac{\partial \psi}{\partial x} \frac{\partial^2 f}{\partial x \partial t} + \frac{\partial \psi}{\partial y} \frac{\partial^2 f}{\partial y \partial t}
\]
\[
\frac{\partial \psi}{\partial (x, y)} = 0
\]
(8.6)

The general solution of the last of equations (8.6) is \( f = f(\psi) \). We wish to solve equations (8.6) for the special case of \( f = f(t) \). Employing this form of \( f \) in system (8.6) yields
\[
\mu \nabla^4 \psi - \rho \frac{\partial}{\partial t} (\nabla^2 \psi) + (\rho - \mu \nabla^2 f) \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, y)} + \alpha_1 \left[ \frac{\partial}{\partial t} (\nabla^4 \psi) - \frac{\partial (\psi, \nabla^4 \psi)}{\partial (x, y)} \right] = 0
\]
(8.7)
\[ f \nabla^4 \psi = \mu^* \sigma \left[ f \frac{\partial}{\partial t} (\nabla^2 \psi) + f' \nabla^2 \psi \right] \]  

(8.8)

where \( f' \) is the derivative of \( f \) with respect to \( t \).

Equations (8.7) and (8.8) hold true for an unsteady, plane, MHD aligned motion of an incompressible, second grade fluid of finite electrical conductivity \( \sigma \). For infinitely conducting fluid flows, (8.8) takes the form

\[ f \frac{\partial}{\partial t} (\nabla^2 \psi) + f' \nabla^2 \psi = 0 \]  

(8.9)

We shall investigate fluid motion for which the vorticity distribution is proportional to the stream function perturbed by a uniform stream parallel to the \( x \)-axis. This is given by

\[ \nabla^2 \psi = K (\psi - Uy) \]  

(8.10)

where \( K \neq 0 \) and \( U \) are real constants. The special case of \( K = 0 \) corresponds to an irrotational flow.
8.3 EXACT SOLUTIONS.

The aligned flows will be divided into those for which the fluid has infinite and finite electrical conductivities.

**Infinite Electrical Conductivity.**

Employing equation (8.10) in equations (8.7) and (8.9), we get

\[
(\alpha_1 K - \rho) \frac{\partial \psi}{\partial t} + U (\mu^* f^2 - \rho + \alpha_1 K) \frac{\partial \psi}{\partial x} + \mu K (\psi - U y) = 0
\]

(8.11)

\[
f \frac{\partial \psi}{\partial t} + f' (\psi - U y) = 0
\]

(8.12)

\[
\nabla^2 \psi = K (\psi - U y)
\]

(8.13)

By solving equation (8.12) and requiring that its solution satisfy equation (8.11), we find that \( f \) must be a real non-zero constant \( f_0 \). Thus, system (8.11) through (8.13) becomes

\[
(\alpha_1 K - \rho) \frac{\partial \psi}{\partial t} + U (\mu^* f_0^2 - \rho + \alpha_1 K) \frac{\partial \psi}{\partial x} + \mu K (\psi - U y) = 0
\]

\[
\frac{\partial \psi}{\partial t} = 0
\]

(8.14)

\[
\nabla^2 \psi = K (\psi - U y)
\]

Letting \( \Psi = \psi - U y \) and using \( \Psi_t = 0, \Psi = \Psi(x, y) \), system (8.14) becomes

\[
U (\mu^* f_0^2 - \rho + \alpha_1 K) \frac{\partial \Psi}{\partial x} + \mu K \Psi = 0
\]

(8.15)

\[
\nabla^2 \Psi = K \Psi
\]

(8.16)

\( U = 0 \) or \( \mu^* f_0^2 - \rho + \alpha_1 K = 0 \) leads to the trivial and irrotational solution \( \Psi = 0 \).

For \( U (\mu^* f_0^2 - \rho + \alpha_1 K) \neq 0 \), the solution to equation (8.15) is

\[
\Psi(x, y) = M(y) e^{-\delta x}
\]

(8.17)

where

\[
\delta = \frac{\mu K}{U (\mu^* f_0^2 - \rho + \alpha_1 K)}
\]

(8.18)
Substitution of equation (8.17) into equation (8.16) yields

\[ M''(y) + (\delta^2 - K)M(y) = 0 \]  
(8.19)

The solution of (8.19) is combined with (8.17) and \( \Psi = \psi - Uy \) to obtain

\[
\psi(x, y) = \begin{cases} 
  Uy + R_1e^{(\xi y - \delta x)} + R_2e^{-(\xi y + \delta x)} ; \; \delta^2 - K = -\xi^2 < 0 & \text{(i)} \\
  Uy + (S_1y + S_2)e^{-\delta x} ; \; \delta^2 - K = 0 & \text{(ii)} \\
  Uy + T_1e^{-\delta x} \cos((\xi y + T_2) ; \; \delta^2 - K = \xi^2 > 0 & \text{(iii)}
\end{cases}
\]  
(8.20)

where \( R_1, R_2, S_1, S_2, T_1 \) and \( T_2 \) are arbitrary constants.

Employing (8.20) in (2.30), (8.3) and (8.5) with \( f = f_0 \), a non-zero real constant, we obtain, respectively, the velocity components, magnetic field strength components, and the linear momentum equations. These linear momentum equations may be solved to obtain the energy \( h \) which is substituted into the third of equations (2.26) to obtain the fluid pressure.

The exact integral when the stream function is (8.20i) is given by

\[
u = U + \xi[R_1e^{(\xi y - \delta x)} - R_2e^{-(\xi y + \delta x)}], \quad v = \delta[R_1e^{(\xi y - \delta x)} + R_2e^{-(\xi y + \delta x)}]
\]

\[
H_1 = f_0 u, \quad H_2 = f_0 v, \quad p = p_0 - \frac{1}{2\rho}U^2 - U \mu^* f_0^2 \xi[R_1e^{(\xi y - \delta x)} - R_2e^{-(\xi y + \delta x)}]
\]

\[
+ \frac{1}{2}(\delta^2 + \xi^2)\left[2(2\alpha_1 + \alpha_2)(\delta^2 + \xi^2) - \mu^* f_0^2 \right] \left[R_1e^{2(\xi y - \delta x)} + R_2e^{-2(\xi y + \delta x)} \right]
\]

\[
+ \left[2\rho\delta^2 - \mu^* f_0^2 (\delta^2 + \xi^2) + 4\alpha_1 \delta^2(\delta^2 - 5\xi^2) + 2\alpha_2 (\delta^4 - 6\delta^2\xi^2 + \xi^4) \right] R_1 R_2 e^{-2\delta x}
\]  
(8.21)

where \( p_0 \) is an arbitrary constant, \( \delta \) is given by equation (8.18), and

\[
\xi^2 = K \left[1 - \frac{\mu^2 K}{U^2(\mu^* f_0^2 - \rho + \alpha_1 K)^2} \right]
\]

The stagnation point for this flow is

\[
(x, y) = \left(-\frac{1}{2\delta} \ln \left[-\frac{U^2}{4\xi^2 R_1 R_2}\right], \frac{1}{2\xi} \ln \left[-\frac{R_2}{R_1}\right]\right); \; \delta^2 - K = -\xi^2 < 0
\]  
(8.22)
where \( U, \delta, \xi, R_1 \) and \( R_2 \) are non-zero real constants and, either \( R_1 > 0, R_2 < 0 \)
or \( R_1 < 0, R_2 > 0 \). If \( R_1 \) and \( R_2 \) are of the same sign, there is no stagnation point in the flow region.

The stream function (8.20ii) gives the exact solution

\[
\begin{align*}
    u &= U + S_1 e^{-\delta x}, \quad v = \delta (S_1 y + S_2) e^{-\delta x}, \quad H_1 = f_0 u, \quad H_2 = f_0 v \\
    p &= p_0 - \frac{1}{2} \rho U^2 - U \mu^* f_0^2 S_1 e^{-\delta x} + \frac{1}{2} \left[ 2 \delta^2 (7 \alpha_1 + 4 \alpha_2) - \rho \right] S_1^2 \\
    &\quad + \delta^2 [2 \delta^2 (2 \alpha_1 + \alpha_2) - \mu^* f_0^2] (S_1 y + S_2)^2 \right] e^{-2\delta x} \quad (8.23)
\end{align*}
\]

where \( p_0 \) is an arbitrary constant, and \( \delta \) is given by (8.18).

If \( U, \delta \) and \( S_1 \) are non-vanishing real numbers, this flow represents an impingement of a uniform stream with a rotational divergent jet, with stagnation point

\[
(x, y) = \left( -\frac{1}{\delta} \ln \left[ \frac{U}{S_1} \right], -\frac{S_2}{S_1} \right); \quad \delta^2 - K = 0 \quad (8.24)
\]

where \( U \) and \( S_1 \) are of opposite signs. There is no stagnation point in the flow field if \( U \) and \( S_1 \) are of the same sign. In this case, the flow represents a uniform stream which abuts on a rotational, convergent flow.

For the stream function (8.20iii), the exact integral takes the form

\[
\begin{align*}
    u &= U - \xi T_1 e^{-\delta x} \sin(\xi y + T_2), \quad v = \delta T_1 e^{-\delta x} \cos(\xi y + T_2), \quad H_1 = f_0 u, \quad H_2 = f_0 v \\
    p &= p_0 - \frac{1}{2} \rho U^2 + U \mu^* f_0^2 \xi T_1 e^{-\delta x} \sin(\xi y + T_2) + \frac{1}{4} \left[ \mu^* f_0^2 (\xi^2 - \delta^2) - 2 \rho \xi^2 \\
    &\quad + 4 \alpha_1 \delta^2 (5 \xi^2) + 2 \alpha_2 (\delta^4 + 6 \delta^2 \xi^2 + \xi^4) \right] + (\xi^2 - \delta^2) [\mu^* f_0^2 \\
    &\quad + 2(2 \alpha_1 + \alpha_2) (\xi^2 - \delta^2)] \cos 2(\xi y + T_2) \right] T_1^2 e^{-2\delta x} \quad (8.25)
\end{align*}
\]

where \( p_0 \) is an arbitrary constant, \( \delta \) is given by equation (8.22), and

\[
\xi^2 = K \left[ \frac{\mu^2 K}{U^2 (\mu^* f_0^2 - \rho + \alpha_1 K)^2} - 1 \right]
\]

For non-zero \( \delta, \xi, U \) and \( T_1 \), there exists a stagnation point, given by

\[
(x, y) = \left( \frac{1}{\delta} \ln \left[ \frac{(-1)^n \xi T_1}{U} \right], \frac{1}{\xi} \left[ (2n + 1) \frac{\pi}{2} - T_2 \right] \right); \quad \delta^2 - K = \xi^2 > 0 \quad (8.26)
\]

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where \( n \) is an integer such that \((-1)^n \zeta T_1\) is of the same sign as \( U\).

Solution (8.20i) represents a uniform flow parallel to the \( x \)-axis, in the region \( x > 0 \), perturbed by a part which decays and grows exponentially as \( x \) increases if \( \delta > 0 \) and \( \delta < 0 \), respectively. The reverse holds true for the region \( x < 0 \), and the flow is exponential in \( y \) in both cases. The solution (8.20ii) can be used to describe, in \( x > 0 \), a uniform flow plus a perturbation which is neither exponential nor periodic in \( y \), and decays and grows as \( x \) increases in the same manner as in solution (8.20i). Solution (8.20iii) can also be used to describe a flow in \( x < 0 \). Solution (8.20iii) can be used in the region \( x > 0 \) to represent a uniform flow with a perturbation part which is periodic in \( y \) and decays and grows exponentially as \( x \) increases, respectively, when \( \delta > 0 \) and \( \delta < 0 \). A similar description can be given for a flow in \( x < 0 \).

It should be noted that when \( f_0 = \alpha_1 = 0 \), the aforementioned results reduce to those obtained by Lin and Tobak (1986) and Hui (1987) for Newtonian fluid flow. If \( f_0 = 0, \alpha_1 \neq 0 \), then the steady-state solutions of the equations of motion of a second grade fluid, obtained by Benharbit and Siddiqui (1992), are realized. (8.24iii) is a more general form of the result obtained by Kovasznay (1948) for \( f_0 = \alpha_1 = 0 \).

We observe, from system (8.14), that for infinitely conducting MHD aligned second grade fluid flows, only steady solutions can be obtained.

**Finite Electrical Conductivity.**

We substitute equation (8.10) in equations (8.7) and (8.8) to get

\[
(\alpha_1 K - \rho) \frac{\partial \psi}{\partial t} + U(\mu^* f^2 - \rho + \alpha_1 K) \frac{\partial \psi}{\partial x} + \mu K (\psi - U y) = 0
\]

(8.27)

\[
\mu^* \sigma f \frac{\partial \psi}{\partial t} + (\mu^* \sigma f' - K f) (\psi - U y) = 0
\]

(8.28)

\[
\nabla^2 \psi = K (\psi - U y)
\]

(8.29)

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For case in integrating these equations, we assume that \( f \) is a non-zero real constant \( f_0 \). If we let \( \Psi = \psi - Uy \), then equations (8.27) through (8.29) become

\[
(\alpha_1 K - \rho) \frac{\partial \Psi}{\partial t} + U (\mu^* f_0^2 - \rho + \alpha_1 K) \frac{\partial \Psi}{\partial x} + \mu K \Psi = 0
\]  
(8.30)

\[
\mu^* \sigma \frac{\partial \Psi}{\partial t} - K \Psi = 0
\]  
(8.31)

\[
\nabla^2 \Psi - K \Psi = 0
\]  
(8.32)

The only steady-state solution to equations (8.30) to (8.32) is the uniform, irrotational flow \( \Psi = 0 \) when \( f = f_0 \), a real non-zero constant.

The remainder of this subsection shall be devoted to unsteady, aligned, plane, isochoric motion of a finitely conducting second grade fluid.

**Case I.**

The general solution to equation (8.31) is

\[
\Psi(x, y, t) = F(x, y) \exp \left( \frac{Kt}{\mu^* \sigma} \right)
\]  
(8.33)

where \( F \) is an arbitrary function of \( x, y \). Letting \( U(\mu^* f_0^2 - \rho + \alpha_1 K) \neq 0 \) and \( \mu \mu^* \sigma - \rho + \alpha_1 K \neq 0 \), we employ equation (8.33) in equation (8.30) to obtain

\[
\frac{\partial F}{\partial x} + \lambda F = 0
\]  
(8.34)

where

\[
\lambda = \frac{K(\mu \mu^* \sigma - \rho + \alpha_1 K)}{U \mu^* \sigma (\mu^* f_0^2 - \rho + \alpha_1 K)}
\]  
(8.35)

The solution of equation (8.34) is

\[
F(x, y) = f^*(y) e^{-\lambda x}
\]  
(8.36)

which is substituted into equations (8.33) and (8.32) to get

\[
f^{**}(y) + (\lambda^2 - K) f^*(y) = 0
\]  
(8.37)
The solution of equation (8.37), combined with equations (8.36), (8.33) and \( \Psi = \psi - U y \), yields

\[
\psi(x, y, t) = \begin{cases} 
U y + A_1 \exp \left[ \beta y - \lambda x + \frac{(\lambda^2 + \beta^2) t}{\mu^* \sigma} \right] \\
+ A_2 \exp \left[ -(\beta y + \lambda x) + \frac{(\lambda^2 + \beta^2) t}{\mu^* \sigma} \right] ; \lambda^2 - K = -\beta^2 \quad (a) \\
U y + (B_1 y + B_2) \exp \left[ -\lambda x + \frac{\lambda^2 t}{\mu^* \sigma} \right] ; \lambda^2 - K = 0 \quad (b) \\
U y + C_1 \cos(\beta y + C_2) \exp \left[ -\lambda x + \frac{(\lambda^2 - \beta^2) t}{\mu^* \sigma} \right] ; \lambda^2 - K = \beta^2 \quad (c)
\end{cases}
\]

(8.38)

where \( A_1, A_2, B_1, B_2, C_1 \) and \( C_2 \) are arbitrary constants.

The stream function given by equation (8.38a) gives the exact solution

\[
\begin{align*}
\psi &= U + \beta \left\{ A_1 \exp \left[ \beta y - \lambda x + \frac{(\lambda^2 + \beta^2) t}{\mu^* \sigma} \right] - A_2 \exp \left[ -(\beta y + \lambda x) + \frac{(\lambda^2 + \beta^2) t}{\mu^* \sigma} \right] \right\} \\
u &= \lambda \left\{ A_1 \exp \left[ \beta y - \lambda x + \frac{(\lambda^2 + \beta^2) t}{\mu^* \sigma} \right] + A_2 \exp \left[ -(\beta y + \lambda x) + \frac{(\lambda^2 + \beta^2) t}{\mu^* \sigma} \right] \right\} \\
H_1 &= f_0 u, \quad H_2 = f_0 v, \quad p = p_0(t) - \frac{1}{2} \rho U^2 + U \mu^* f_0^2 \beta \left\{ A_2 \exp \left[ -(\beta y + \lambda x) + \frac{(\lambda^2 + \beta^2) t}{\mu^* \sigma} \right] \right\} + \frac{1}{2} (2\sigma + \alpha_2) (\lambda^2 + \beta^2) \left[ 2(2\alpha_2 + \alpha_1 - \frac{\lambda^2}{\mu^* \sigma}) \right] \\
+ \frac{2(\lambda^2 + \beta^2) t}{\mu^* \sigma} \left\{ A_1 \exp \left[ 2(\beta y - \lambda x) + \frac{(\lambda^2 + \beta^2) t}{\mu^* \sigma} \right] + A_2 \exp \left[ -(2\beta y + \lambda x) + \frac{(\lambda^2 + \beta^2) t}{\mu^* \sigma} \right] \right\} + [2\rho \beta^2 - \mu^* f_0^2 (\lambda^2 + \beta^2) + 4\alpha_1 \lambda^2 (\lambda^2 - 5\beta^2) + 2\alpha_2 (\lambda^4 - 6\lambda^2 \beta^2 + \beta^4)] A_1 A_2 \exp \left[ -2\lambda x + \frac{2(\lambda^2 + \beta^2) t}{\mu^* \sigma} \right]
\end{align*}
\]

(8.39)

where \( p_0(t) \) is an arbitrary function of \( t \), \( \lambda \) is given by equation (8.35), and

\[
\beta^2 = K \left[ 1 - \frac{K (\mu^* \sigma - \rho + \alpha_1 K)^2}{U^2 \mu^* \sigma^2 (\mu^* f_0^2 - \rho + \alpha_1 K)^2} \right]
\]

If \( \beta, \lambda, U \), \( A_1 \) and \( A_2 \) are non-zero real constants, then a stagnation point exists in the flow field if \( A_1 \) and \( A_2 \) are of opposite signs. This stagnation point, at any
time $t$, is given by
\begin{equation}
(x, y) = \left( \frac{(\lambda^2 + \beta^2)t}{\mu^*\sigma\lambda} - \frac{1}{2\lambda} \ln \left[ -\frac{U^2}{4\beta^2 A_1 A_2} \right], \frac{1}{2\beta} \ln \left[ -\frac{A_2}{A_1} \right] \right); \quad \lambda^2 - K = -\beta^2 < 0
\end{equation}
(8.40)

There exists no stagnation point if $A_1$ and $A_2$ are of the same sign.

The exact integral associated with the stream function (8.38b) is
\begin{align*}
u &= U + B_1 \exp \left( -\lambda x + \frac{\lambda^2 t}{\mu^*\sigma} \right), \quad v = \lambda (B_1 y + B_2) \exp \left( -\lambda x + \frac{\lambda^2 t}{\mu^*\sigma} \right), \quad H_1 = f_0 u \\
H_2 &= f_0 v, \quad p = p_0(t) - \frac{1}{2}\rho U^2 - U \mu^* f_0^2 B_1 \exp \left( -\lambda x + \frac{\lambda^2 t}{\mu^*\sigma} \right) + \frac{1}{2} \left[ 2\lambda^2 (7\alpha_1 + 4\alpha_2) - \rho \right] B_1^2 + \lambda^2 \left[ 2\lambda^2 (2\alpha_1 + \alpha_2) - \mu^* f_0^2 \right] (B_1 y + B_2)^2 \exp \left( -2\lambda x + \frac{2\lambda^2 t}{\mu^*\sigma} \right)
\end{align*}
(8.41)

where $p_0(t)$ is an arbitrary function of $t$, and $\lambda$ is given by equation (8.35).

At any time $t$, this flow has the stagnation point
\begin{equation}
(x, y) = \left( \frac{\lambda^2 t}{\mu^*\sigma} - \frac{1}{\lambda} \ln \left[ -\frac{U}{B_1} \right], -\frac{B_2}{B_1} \right); \quad \lambda^2 - K = 0
\end{equation}
(8.42)

provided $U$ and $B_1$ are non-zero constants having opposite signs. It should be noted that $U$ and $B_1$ having the same sign results in a flow with no stagnation point.

The exact solution, when the stream function is (8.38c), is given by
\begin{align*}
u &= U - \beta C_1 \sin (\beta y + C_2) \exp \left[ -\lambda x + \frac{(\lambda^2 - \beta^2)t}{\mu^*\sigma} \right] \\
v &= \lambda C_1 \cos (\beta y + C_2) \exp \left[ -\lambda x + \frac{(\lambda^2 - \beta^2)t}{\mu^*\sigma} \right], \quad H_1 = f_0 u, \quad H_2 = f_0 v \\
p &= p_0(t) - \frac{1}{2}\rho U^2 + U \mu^* f_0^2 \beta C_1 \sin (\beta y + C_2) \exp \left[ -\lambda x + \frac{(\lambda^2 - \beta^2)t}{\mu^*\sigma} \right] \\
&\quad + \frac{1}{4} \left[ \mu^* f_0^2 (\beta^2 - \lambda^2) - 2\rho \beta^2 + 4\alpha_1 \lambda^2 (\lambda^2 + 5\beta^2) + 2\alpha_2 (\lambda^4 + 6\lambda^2 \beta^2 + \beta^4) \right] \\
&\quad + (\beta^2 - \lambda^2) \left[ \mu^* f_0^2 + 2(2\alpha_1 + \alpha_2) (\beta^2 - \lambda^2) \right] \cos 2(\beta y + C_2) \exp \left[ -2\lambda x + \frac{2(\lambda^2 - \beta^2)t}{\mu^*\sigma} \right]
\end{align*}
(8.43)
where \( p_0(t) \) is an arbitrary function of \( t \), \( \lambda \) is given by equation (8.35), and

\[
\beta^2 = K \left[ \frac{K(\mu \mu^* \sigma - \rho + \alpha_1 K)^2}{U^2 \mu^2 \sigma^2 (\mu^* f_0^2 - \rho + \alpha_1 K)^2} \right] - 1
\]

If \( \beta, \lambda, U \) and \( C_1 \) are non-vanishing real constants, the stagnation point for this flow is

\[
(x, y) = \left( \frac{(\lambda^2 - \beta^2) t}{\mu^* \sigma \lambda} + \frac{1}{\lambda} \ln \left[ \frac{(-1)^n \beta C_1}{U} \right] \frac{1}{\beta} \left( (2n + 1) \frac{\pi}{2} - C_2 \right) \right) ; \quad \lambda^2 - K = \beta^2 > 0
\]

at any time \( t \), where \( n \) is an integer and \( (-1)^n \beta C_1 \) and \( U \) are of the same sign.

Solution: (8.38a) to (8.38c) are interpreted in the same manner as solutions (8.20i) to (8.20iii), respectively, of the steady-state equations of infinitely conducting fluid flows. In addition, solutions (8.38a) and (8.38b) grow exponentially in time at a rate equal to \( \frac{\lambda^2 + \beta^2}{\mu^* \sigma} \) and \( \frac{\lambda^2 - \beta^2}{\mu^* \sigma} \). Solution (8.38c) grows and decays exponentially in time according as \( \frac{\lambda^2 - \beta^2}{\mu^* \sigma} \) is positive and negative, respectively. In the case of exponential growth, the solutions can be physically meaningful only in a finite time interval.

The subcase when \( (\mu \mu^* \sigma - \rho + \alpha_1 K) = 0, U(\mu^* f_0^2 - \rho + \alpha_1 K) \neq 0 \) yields the stream function

\[
\psi(y, t) = U y + \left( A e^{\sqrt{K} y} + B e^{-\sqrt{K} y} \right) \exp \left( \frac{K t}{\mu^* \sigma} \right)
\]

(8.45)

which is independent of \( x \), and gives the exact integral

\[
u = U + \sqrt{K} \left( A e^{\sqrt{K} y} - B e^{-\sqrt{K} y} \right) \exp \left( \frac{K t}{\mu^* \sigma} \right), \quad \nu = 0, \quad H_1 = f_0 \nu, \quad H_2 = f_0 \nu
\]

\[
p = p_0(t) - \frac{1}{2} \alpha_1 K U^2 + \frac{1}{2} \left( 3\alpha_1 + 2\alpha_2 \right) K^2 \left( A e^{\sqrt{K} y} - B e^{-\sqrt{K} y} \right)^2 \exp \left( \frac{2K t}{\mu^* \sigma} \right) - \frac{1}{2} \mu^* f_0^2 \left[ U + \sqrt{K} \left( A e^{\sqrt{K} y} - B e^{-\sqrt{K} y} \right) \exp \left( \frac{K t}{\mu^* \sigma} \right) \right]^2
\]

(8.46)

where \( p_0(t) \) is an arbitrary function of \( t \).
Case II.

We rewrite equation (8.30) in the form

$$\frac{\partial \Psi}{\partial t} + \gamma \frac{\partial \Psi}{\partial x} + \frac{\mu K}{\alpha_1 K - \rho} \Psi = 0$$  \hspace{1cm} (8.47)

where

$$\gamma = \frac{U(\mu^2 f_0^2 - \rho + \alpha_1 K)}{\alpha_1 K - \rho}$$  \hspace{1cm} (8.48)

Integrating equation (8.47) with respect to \( t \) we get

$$\Psi(x, y, t) = \exp\left(\frac{\mu K t}{\rho - \alpha_1 K}\right)\exp\left(-\gamma t \frac{\partial}{\partial x}\right)\{G(x, y)\}$$

$$= \exp\left(\frac{\mu K t}{\rho - \alpha_1 K}\right)[G(x, y) - \gamma t \frac{\partial}{\partial x}(G(x, y)) + \frac{(\gamma t)^2}{2!} \frac{\partial^2}{\partial x^2}(G(x, y)) - ...]$$

$$= \exp\left(\frac{\mu K t}{\rho - \alpha_1 K}\right)G(x - \gamma t, y)$$

where \( G \) is an arbitrary function of its arguments. If \( X = x - \gamma t \), we get the solution of (8.47) to be

$$\Psi(X, y, t) = G(X, y)\exp\left(\frac{\mu K t}{\rho - \alpha_1 K}\right)$$  \hspace{1cm} (8.49)

which is substituted into equation (8.32) to obtain

$$\frac{\partial^2 G}{\partial X^2} + \frac{\partial^2 G}{\partial y^2} = KG$$  \hspace{1cm} (8.50)

Plane wave solutions of the Helmholtz equation (8.50) exist in the form

$$G(X, y) = g(\zeta)$$  \hspace{1cm} (8.51)

which when employed in equation (8.50) yields

$$g''(\zeta) - KG(\zeta) = 0$$  \hspace{1cm} (8.52)

where \( \zeta = X\cos\theta + y\sin\theta, -\pi \leq \theta < \pi \). A combination of the solution of equation (8.52) with equations (8.51), (8.49), and \( X = x - \gamma t \) gives the stream functions:

\[ K = -\eta^2 < 0 \]

$$\Psi(x, y, t) = A_1(\theta)\cos\eta[(x - \gamma t)\cos\theta + y\sin\theta + A_2(\theta)]\exp\left(-\frac{\mu\eta^2 t}{\rho + \alpha_1 \eta^2}\right)$$  \hspace{1cm} (8.53)
\[ K = \eta^2 > 0 \]

\[
\Psi(x, y, t) = B_1(\theta) \exp \left\{ \eta \left[ (x - \gamma t) \cos \theta + y \sin \theta + \frac{\mu \eta t}{\rho - \alpha_1 \eta^2} \right] \right\} + B_2(\theta) \exp \left\{ - \eta \left[ (x - \gamma t) \cos \theta + y \sin \theta - \frac{\mu \eta t}{\rho - \alpha_1 \eta^2} \right] \right\} 
\]

where \( A_1(\theta), A_2(\theta), B_1(\theta), B_2(\theta) \) are real constants dependent on the parameter \( \theta \) such that \(-\pi \leq \theta < \pi\), and

\[
\gamma = \begin{cases} 
\frac{U (\alpha_1 \eta^2 + \rho - \mu^* f_\delta^2)}{\alpha_1 \eta^2 + \rho} & K = -\eta^2 < 0 \\
\frac{U (\alpha_1 \eta^2 - \rho + \mu^* f_\delta^2)}{\alpha_1 \eta^2 - \rho} & K = \eta^2 > 0 
\end{cases} 
\]

But equations (8.53) and (8.54) must satisfy equation (8.31) as well. Substitution of equations (8.53) and (8.54) into equation (8.31) give

\[
\eta A_1(\theta) \left\{ \mu^* \sigma \gamma \cos \theta \sin \eta \left[ (x - \gamma t) \cos \theta + y \sin \theta + A_2(\theta) \right] \right\} + \frac{\eta (\alpha_1 \eta^2 + \rho - \mu \mu^* \sigma)}{\alpha_1 \eta^2 + \rho} \cos \eta \left[ (x - \gamma t) \cos \theta + y \sin \theta + A_2(\theta) \right] \exp \left( - \frac{\mu \eta^2 t}{\rho + \alpha_1 \eta^2} \right) = 0 
\]

(8.56)

\[
\eta B_1(\theta) \left\{ \frac{\eta (\alpha_1 \eta^2 - \rho + \mu \mu^* \sigma)}{\rho - \alpha_1 \eta^2} \right\} \exp \left\{ \eta \left[ (x - \gamma t) \cos \theta + y \sin \theta - \frac{\mu \eta t}{\rho - \alpha_1 \eta^2} \right] \right\} + \eta B_2(\theta) \left\{ \frac{\eta (\alpha_1 \eta^2 - \rho + \mu \mu^* \sigma)}{\rho - \alpha_1 \eta^2} + \mu^* \sigma \gamma \cos \theta \exp \left\{ - \eta \left[ (x - \gamma t) \cos \theta - \frac{\mu \eta t}{\rho - \alpha_1 \eta^2} \right] \right\} \right\} = 0 
\]

(8.57)

For non-trivial \( \Psi \), and for equations (8.56) and (8.57) to be identically satisfied, we set

\[
\gamma = 0, \quad \alpha_1 \eta^2 \pm \rho \mp \mu^* \sigma = 0 
\]

since \( \eta, A_1(\theta) \) and \( B_1(\theta) \) cannot vanish. \( \gamma = 0 \) implies \( U = 0 \), and so the forms of the stream function \( \Psi = \psi - Uy = \psi \) become:

\[
K = -\eta^2 < 0 
\]

\[
\psi(x, y, t) = A_1(\theta) \cos \eta \left[ x \cos \theta + y \sin \theta + A_2(\theta) \right] \exp \left( - \frac{\mu \eta^2 t}{\rho + \alpha_1 \eta^2} \right) 
\]

(8.58)
$K = \eta^2 > 0$

\[
\psi(x, y, t) = B_1(\theta)\exp\left[ \eta \left( x\cos\theta + y\sin\theta + \frac{\mu\eta t}{\rho - \alpha_1\eta^2} \right) \right] + B_2(\theta)\exp\left[ -\eta \left( x\cos\theta + y\sin\theta - \frac{\mu\eta t}{\rho - \alpha_1\eta^2} \right) \right]
\]

(8.59)

For the stream function (8.58), the associated exact solution is

\[
u = -\eta A_1(\theta)\sin\theta\sin\eta [x\cos\theta + y\sin\theta + A_2(\theta)]\exp\left(-\frac{\mu\eta^2 t}{\rho + \alpha_1\eta^2}\right)
\]

\[
v = \eta A_1(\theta)\cos\theta\sin\eta \left[ x\cos\theta + y\sin\theta + A_2(\theta) \right]\exp\left(-\frac{\mu\eta^2 t}{\rho + \alpha_1\eta^2}\right), \quad H_1 = f_0u
\]

\[
H_2 = f_0v, \quad p = p_0(t) + \frac{1}{4}\eta^2 A_1^2(\theta)\left[ \eta^2(\alpha_1 + 2\alpha_2) - \rho + 2\alpha_1\eta^3 (x\cos\theta + y\sin\theta)
\right.
\]

\[+ \left[ \mu^* f_0^2 + \eta^2 (5\alpha_1 + 2\alpha_2) \right]\cos 2\eta \left[ x\cos\theta + y\sin\theta + A_2(\theta) \right] - 2\alpha_1\eta^2 \sin 2\eta \left[ x\cos\theta
\]

\[+ y\sin\theta + A_2(\theta) \right] \exp\left(-\frac{\mu\eta^2 t}{\rho + \alpha_1\eta^2}\right)
\]

(8.60)

where $p_0(t)$ is an arbitrary function of $t$, $\eta^2 = -K$, and $-\pi \leq \theta < \pi$.

The exact integral when the stream function is (8.59) is

\[
u = \eta \sin\theta \left\{ B_1(\theta)\exp\left[ \eta \left( x\cos\theta + y\sin\theta + \frac{\mu\eta t}{\rho - \alpha_1\eta^2} \right) \right] - B_2(\theta)\exp\left[ -\eta \left( x\cos\theta + y\sin\theta - \frac{\mu\eta t}{\rho - \alpha_1\eta^2} \right) \right] \right\}, \quad v = \eta \cos\theta \left\{ B_1(\theta)\exp\left[ -\eta \left( x\cos\theta + y\sin\theta - \frac{\mu\eta t}{\rho - \alpha_1\eta^2} \right) \right] - B_2(\theta)\exp\left[ \eta \left( x\cos\theta + y\sin\theta + \frac{\mu\eta t}{\rho - \alpha_1\eta^2} \right) \right] \right\}, \quad H_1 = f_0u, \quad H_2 = f_0v, \quad p = p_0(t)
\]

\[+ \frac{1}{2}\eta^2 \left[ 2\eta^2 (2\alpha_1 + 2\alpha_2) - \mu^* f_0^2 \right] \left\{ B_1^2(\theta)\exp\left[ 2\eta \left( x\cos\theta + y\sin\theta + \frac{\mu\eta t}{\rho - \alpha_1\eta^2} \right) \right] + B_2^2(\theta)\exp\left[ -2\eta \left( x\cos\theta + y\sin\theta - \frac{\mu\eta t}{\rho - \alpha_1\eta^2} \right) \right] \right\} + \eta^2 \left[ (\alpha_1 + 2\alpha_2)\eta^2
\]

\[+ \rho \right\} B_1(\theta)B_2(\theta)\exp\left[ \frac{2\mu\eta^2 t}{\rho - \alpha_1\eta^2} \right]
\]

(8.61)

where $p_0(t)$ is an arbitrary function of $t$, $\eta^2 = K$, and $-\pi \leq \theta < \pi$.  

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Flow (8.58) represents a plane wave that is periodic in \( x \) and \( y \), and decays exponentially in time at a rate that equals \( \frac{\mu n^2}{\rho + \alpha_1 \eta^2} \). Plane wave solution (8.59) is exponential in \( x \), \( y \), and \( t \). It decays or grows exponentially in time if \( \frac{\mu n^2}{\rho - \alpha_1 \eta^2} < 0 \) or \( \frac{\mu n^2}{\rho - \alpha_1 \eta^2} > 0 \). For a physically relevant exponentially growing solution, the time interval over which the flow takes place must be finite.

It is noteworthy to mention that the aforementioned flows are possible if the vorticity distribution is proportional to the stream function only. We observe that these are compatible with those obtained by Hui (1987) and Benharbit and Siddiqui (1992) when \( U = \alpha_1 = f_0 = 0 \) and \( U = f_0 = 0 \), respectively.

**Case III.**

Rewriting equation (8.30) in the form

\[
\frac{\partial \Psi}{\partial x} + \frac{1}{\gamma} \frac{\partial \Psi}{\partial t} + \nu \Psi = 0
\]  
(8.62)

we integrate it with respect to \( x \) to obtain

\[
\Psi(x, y, t) = e^{-\nu z} H^*_0(t, y)
\]

where \( \gamma \) is given by equation (8.48),

\[
\nu = \frac{\mu K}{U(\mu f_0^2 - \rho + \alpha_1 K)}
\]  
(8.63)

and \( m = -\left(\frac{1}{\gamma} \frac{\partial}{\partial t} + \nu\right) \) is the root of the auxiliary equation. Therefore,

\[
\Psi(x, y, t) = e^{-\nu z} \exp\left(-\frac{x}{\gamma} \frac{\partial}{\partial t}\right) \{H^*_0(t, y)\}
\]

\[
= e^{-\nu z} \left[H^*_0(t, y) - \frac{x}{\gamma} \frac{\partial}{\partial t}(H^*_0(t, y)) + \frac{1}{2!} \left(\frac{x}{\gamma}\right)^2 \frac{\partial^2}{\partial t^2}(H^*_0(t, y)) - \ldots\right]
\]

\[
= e^{-\nu z} H^*_0(t - \frac{x}{\gamma}, y) \equiv e^{-\nu z} H^*(x - \gamma t, y)
\]

Letting \( X = x - \gamma t \), the solution of equation (8.62) is

\[
\Psi(X, y, t) = H^*(X, y) \exp\left[-\nu(X + \gamma t)\right]
\]  
(8.64)
In the regions $\sin^2 \theta_0 > \sin^2 \theta$
In the regions $\sin^2 \theta_0 \leq \sin^2 \theta$

where $\theta_0 = \sin^{-1}(\eta/\nu)$, $\eta^2 < \nu^2$

which is employed in equation (8.32) to get

$$\frac{\partial^2 H^*}{\partial X^2} + \frac{\partial^2 H^*}{\partial y^2} - 2\nu \frac{\partial H^*}{\partial X} + (\nu^2 - K)H^* = 0$$  \hspace{1cm} (8.65)

There exist plane wave solutions to equation (8.65) in the form

$$H^*(X, y) = h^*(\zeta)$$  \hspace{1cm} (8.66)

which is substituted into equation (8.65) to obtain

$$h^{*''}(\zeta) - 2\nu \cos \theta h^{*'}(\zeta) + (\nu^2 - K)h^*(\zeta) = 0$$  \hspace{1cm} (8.67)

where $\zeta = X \cos \theta + y \sin \theta$. The auxiliary equation of equation (8.67) has the roots

$$m^* = \nu \cos \theta \pm \sqrt{K - \nu^2 \sin^2 \theta}$$  \hspace{1cm} (8.68)
From equations (8.68), \( X = x - \gamma t \), (8.66) and (8.64), the stream functions are given by:

\[ K = -\eta^2 < 0; \quad -\pi \leq \theta < \pi \]

\[ \Psi(x,y,t) = A_3(\theta) \cos\left\{ \sqrt{\eta^2 + \nu^2 - \eta^2}\left[(x - \gamma t)\cos\theta + y\sin\theta\right] \right. \\
+ A_4(\theta)\left\} \exp\left\{ \nu\cos\theta[(x - \gamma t)\cos\theta + y\sin\theta] - \nu x \right\} \tag{8.69} \]

\[ K = \eta^2 \geq \nu^2; \quad -\pi \leq \theta < \pi \quad \text{and} \]

\[ K = \eta^2 < \nu^2; \quad |\theta| \leq \theta_0 \quad \text{or} \quad \pi - \theta_0 < |\theta| \leq \pi \]

\[ \Psi(x,y,t) = \left[ B_3(\theta) \exp\left\{ \sqrt{\eta^2 - \nu^2 - \eta^2}\left[(x - \gamma t)\cos\theta + y\sin\theta\right] \right. \\
+ B_4(\theta) \left\} \exp\left\{ \nu\cos\theta[(x - \gamma t)\cos\theta + y\sin\theta] - \nu x \right\} \right. \tag{8.70} \]

\[ K = \eta^2 = \nu^2; \quad \theta = \pm \frac{\pi}{2} \]

\[ \Psi(x,y) = (C_3 y + C_4) e^{-\nu x} \tag{8.71} \]

\[ K = \eta^2 < \nu^2; \quad \theta_0 < |\theta| \leq \pi - \theta_0 \]

\[ \Psi(x,y,t) = D_3(\theta) \cos\left\{ \sqrt{\nu^2 + \eta^2 - \eta^2}\left[(x - \gamma t)\cos\theta + y\sin\theta\right] \right. \\
+ D_4(\theta)\right\} \exp\left\{ \nu\cos\theta[(x - \gamma t)\cos\theta + y\sin\theta] - \nu x \right\} \tag{8.72} \]

where \( A_3(\theta), A_4(\theta), B_3(\theta), B_4(\theta), D_3(\theta), D_4(\theta) \) are arbitrary real constants dependent on the parameter \( \theta \), \( C_3, C_4 \) are arbitrary real constants, \( \theta_0 = \sin^{-1} \frac{\eta}{\nu} \), \( \gamma \) and \( \nu \) are given by (8.48) and (8.63).

Equations (8.69) through (8.72) must satisfy equation (8.31). Employing the former equations in the latter yield:
\[ K = -\eta^2 < 0; \quad -\pi \leq \theta < \pi \]

\[
A_3(\theta) \left[ (\eta^2 - \mu \sigma \nu \gamma \cos^2 \theta) \cos \left\{ \sqrt{\eta^2 + \nu^2 \sin^2 \theta} [(x - \gamma t)\cos \theta + y \sin \theta] + A_4(\theta) \right\} + 
\mu \sigma \gamma \cos \theta \sqrt{\eta^2 + \nu^2 \sin^2 \theta} \sin \left\{ \sqrt{\eta^2 + \nu^2 \sin^2 \theta} [(x - \gamma t)\cos \theta + y \sin \theta] + A_4(\theta) \right\} \right] \exp \left\{ \nu \cos \theta [(x - \gamma t)\cos \theta + y \sin \theta] - \nu x \right\} = 0
\]  
(8.73)

\[
K = \eta^2 \geq \nu^2; \quad -\pi \leq \theta < \pi \quad \text{and} \quad K = \eta^2 < \nu^2; \quad |\theta| \leq \theta_0 \quad \text{or} \quad \pi - \theta_0 < |\theta| < \pi
\]

\[
\left[ B_3(\theta) \left[ \eta^2 + \mu \sigma \gamma \cos \theta (\nu \cos \theta + \sqrt{\eta^2 - \nu^2 \sin^2 \theta}) \right] \exp \left\{ \sqrt{\eta^2 - \nu^2 \sin^2 \theta} [(x - \gamma t)\cos \theta + y \sin \theta] \right\} + B_4(\theta) \left[ \eta^2 + \mu \sigma \gamma \cos \theta (\nu \cos \theta - \sqrt{\eta^2 - \nu^2 \sin^2 \theta}) \right] \exp \left\{ -\sqrt{\eta^2 - \nu^2 \sin^2 \theta} [(x - \gamma t)\cos \theta + y \sin \theta] \right\} \right] \exp \left\{ \nu \cos \theta [(x - \gamma t)\cos \theta + y \sin \theta] - \nu x \right\} = 0
\]  
(8.74)

\[
K = \eta^2 = \nu^2; \quad \theta = \pm \frac{\pi}{2}
\]

\[
\eta^2 (C_3 y + C_4) e^{-\nu x} = 0
\]  
(8.75)

\[
K = \eta^2 < \nu^2; \quad \theta \leq |\theta| < \pi - \theta_0
\]

\[
D_3(\theta) \left[ (\eta^2 + \mu \sigma \gamma \nu \cos^2 \theta) \cos \left\{ \sqrt{\nu^2 \sin^2 \theta - \eta^2} [(x - \gamma t)\cos \theta + y \sin \theta] + D_4(\theta) \right\} - 
\mu \sigma \gamma \cos \theta \sqrt{\nu^2 \sin^2 \theta - \eta^2} \sin \left\{ \sqrt{\nu^2 \sin^2 \theta - \eta^2} [(x - \gamma t)\cos \theta + y \sin \theta] \right\} + D_4(\theta) \right\} \exp \left\{ \nu \cos \theta [(x - \gamma t)\cos \theta + y \sin \theta] - \nu x \right\} = 0
\]  
(8.76)

Since \( \eta, \nu \) and \( \gamma \) are non-zero real numbers, equations (8.73) to (8.76) are identically satisfied only if the coefficients \( A_3(\theta), B_3(\theta), B_4(\theta), C_3, C_4 \) and \( D_3(\theta) \) vanish. This case, therefore, yields the uninteresting uniform, irrotational flow given by \( \Psi = 0 \) or \( \psi = U y \). This result is in sharp contrast with those of Hui (1987) and Benharbit and Siddiqui (1992). This is a consequence of the presence of the magnetic field in the flow domain, as will be noted in the conclusion below.
8.4 CONCLUSION.

In attempting to find exact solutions of the equations of MHD aligned, plane, unsteady motion of an incompressible, electrically conducting, second grade, fluid, we found that steady and unsteady solutions were obtained, respectively, for fluids of infinite and finite electrical conductivities when \( f = f_0 \), a real constant.

The solutions (8.20) obtained when studying the motion of the infinitely conducting fluid are compatible with those of Lin and Tobak (1986) and Hui (1987) for the Navier-Stokes equations when \( f_0 = \alpha_1 = 0 \), and those of Benharbit and Siddiqui (1992) when \( f_0 = 0 \). Solution (8.20iii) when \( f_0 = \alpha_1 = 0 \) is a more general form of the result obtained by Kovasznay (1948).

The study of the unsteady motion of a second grade fluid of finite electrical conductivity was divided into three classes:

(I) The first class consists in solving equation (8.31), and requiring the result to satisfy equations (8.30) and (8.32). The resulting stream functions (8.38) do not have any parallel in any of the works of Hui, and Benharbit and Siddiqui. This is because of the introduction of (8.31) which arose by virtue of the presence of the magnetic field in the flow domain;

(II) The second class involves first integrating (8.30), in the form (8.47), with respect to the time variable \( t \). This solution must satisfy equations (8.32) and (8.31). For the latter equation to be satisfied, \( U \) must be zero. Solutions (8.57) are compatible with those of Hui for \( f_0 = \alpha_1 = 0 \) and those of Benharbit and Siddiqui for \( f_0 = 0 \), when \( U = 0 \);

(III) The third class is obtained by first integrating equation (8.30), in the form equation (8.62), with respect to the spatial variable \( z \), the result of which must also satisfy equations (8.32) and (8.31). For equations (8.69) through (8.72) to satisfy equation (8.31), it was found that the coefficients \( A_3(\theta), B_3(\theta), B_4(\theta), C_3, C_4 \) and
$D_3(\theta)$ must vanish, resulting in the irrotational flow $\psi = \hat{U}y$. This result is different from those of Hui for $f_0 = \alpha_1 = 0$ and Benharbit and Siddiqui for $f_0 = 0$ because of equation (8.31), which has no counterpart in the works of the aforementioned. The presence of the magnetic field in the flow region gave rise to equation (8.31).
APPENDIX A: DERIVATION OF $\nabla^2 \omega$ IN TERMS OF THE VARIABLES $\phi$, $\psi$.

\[ \nabla^2 \omega = \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \]

\[ = \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \left[ \frac{\partial \phi}{\partial x} \frac{\partial \omega}{\partial \phi} + \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial \psi} \right] + \frac{\partial \psi}{\partial x} \frac{\partial}{\partial \psi} \left[ \frac{\partial \phi}{\partial x} \frac{\partial \omega}{\partial \phi} + \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial \psi} \right] \]

\[ = \frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi} \left[ \frac{\partial \phi}{\partial y} \frac{\partial \omega}{\partial \phi} + \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial \psi} \right] + \frac{\partial \psi}{\partial y} \frac{\partial}{\partial \psi} \left[ \frac{\partial \phi}{\partial y} \frac{\partial \omega}{\partial \phi} + \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial \psi} \right] \]

\[ = \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right] \frac{\partial^2 \omega}{\partial \phi^2} + 2 \left[ \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial \phi} \right] \frac{\partial^2 \omega}{\partial \phi \partial \psi} + \left[ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 \right] \frac{\partial^2 \omega}{\partial \psi^2} + 2 \left[ \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial \psi} + \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial \psi} \right] \frac{\partial^2 \omega}{\partial \phi \partial \psi} \]

Application of equations (2.11) and (2.12) to (A.1) gives

\[ \nabla^2 \omega = \frac{1}{W} \left[ \frac{1}{W} \left( \frac{\partial x}{\partial \psi} \right)^2 + \frac{\partial y}{\partial \psi} \right] \frac{\partial^2 \omega}{\partial \phi^2} - \frac{2}{W} \left[ \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \psi} \right] \frac{\partial^2 \omega}{\partial \phi \partial \psi} \]

\[ + \frac{1}{W} \left[ \left( \frac{\partial x}{\partial \phi} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 \right] \frac{\partial^2 \omega}{\partial \phi^2} + \left[ \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \phi} (\frac{1}{W} \frac{\partial x}{\partial \phi}) + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \phi} (\frac{1}{W} \frac{\partial y}{\partial \phi}) \right] \frac{\partial \omega}{\partial \phi} \]

\[ - \frac{\partial x}{\partial \phi} \frac{\partial (1 \frac{x}{W})}{\partial \phi} - \frac{\partial y}{\partial \phi} \frac{\partial (1 \frac{y}{W})}{\partial \phi} \frac{1}{W} \frac{\partial \omega}{\partial \phi} + \left[ \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \phi} (\frac{1}{W} \frac{\partial x}{\partial \phi}) + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \phi} (\frac{1}{W} \frac{\partial y}{\partial \phi}) \right] \frac{\partial \omega}{\partial \phi} \]

From (2.10), we have

\[ \frac{\partial}{\partial \phi} \left( \frac{G}{W} \right) - \frac{\partial}{\partial \psi} \left( \frac{F}{W} \right) = \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \phi} \left( \frac{1 \frac{x}{W}}{\partial \phi} \right) + \frac{\partial y}{\partial \phi} \frac{\partial}{\partial \phi} \left( \frac{1 \frac{y}{W}}{\partial \phi} \right) - \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \phi} \left( \frac{1 \frac{x}{W}}{\partial \phi} \right) \]

\[ - \frac{\partial y}{\partial \phi} \frac{\partial}{\partial \phi} \left( \frac{1 \frac{y}{W}}{\partial \phi} \right) \]

\[ \frac{\partial}{\partial \phi} \left( \frac{E}{W} \right) - \frac{\partial}{\partial \psi} \left( \frac{F}{W} \right) = \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \phi} \left( \frac{1 \frac{x}{W}}{\partial \phi} \right) + \frac{\partial y}{\partial \phi} \frac{\partial}{\partial \phi} \left( \frac{1 \frac{y}{W}}{\partial \phi} \right) - \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \phi} \left( \frac{1 \frac{x}{W}}{\partial \phi} \right) \]

\[ - \frac{\partial y}{\partial \phi} \frac{\partial}{\partial \phi} \left( \frac{1 \frac{y}{W}}{\partial \phi} \right) \]
In the new variables \( \phi, \psi \), \( \nabla^2 \omega \) becomes \( \Delta_2 \omega \), the Beltrami's differential operator of second order. Substituting equations (2.10) and (A.2) in (A.1), we obtain

\[
\Delta_2 \omega = \frac{1}{W} \left\{ \frac{G}{W} \frac{\partial^2 \omega}{\partial \phi^2} - \frac{2F}{W} \frac{\partial^2 \omega}{\partial \phi \partial \psi} + \frac{E}{W} \frac{\partial^2 \omega}{\partial \psi^2} + \left[ \frac{\partial}{\partial \phi} \left( \frac{G}{W} \right) - \frac{\partial}{\partial \psi} \left( \frac{F}{W} \right) \right] \frac{\partial \omega}{\partial \phi} \\
+ \left[ - \frac{\partial}{\partial \phi} \left( \frac{F}{W} \right) + \frac{\partial}{\partial \psi} \left( \frac{E}{W} \right) \right] \frac{\partial \omega}{\partial \psi} \right\}
\]

or

\[
\Delta_2 \omega = \frac{1}{W} \left[ \frac{\partial}{\partial \phi} \left( \frac{G}{W} \frac{\partial \omega}{\partial \phi} - \frac{F}{W} \frac{\partial \omega}{\partial \psi} \right) + \frac{\partial}{\partial \psi} \left( - \frac{F}{W} \frac{\partial \omega}{\partial \phi} + \frac{E}{W} \frac{\partial \omega}{\partial \psi} \right) \right] \quad (A.3)
\]

in the new variables \( \phi, \psi \).
APPENDIX B: DERIVATION OF THE ENERGY $\mathcal{H}$ IN TERMS OF THE VARIABLES $\phi$, $\psi$.

The energy in the $x$, $y$ coordinates is given by

\[
\mathcal{H} = p + \frac{1}{2} \rho (u^2 + v^2) - \alpha_1 (u \nabla^2 u + v \nabla^2 v) - \left( \frac{3\alpha_1 + 2\alpha_2}{2} \right) \left[ 2 \left( \frac{\partial u}{\partial x} \right)^2 + 2 \left( \frac{\partial v}{\partial y} \right)^2 \right] + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2
\]

(B.1)

From (2.33), we get

\[
q^2 = u^2 + v^2 = \frac{E}{W^2}
\]

(B.2)

Using the continuity and vorticity equations, we have that

\[
u \nabla^2 u + v \nabla^2 v = v \frac{\partial \omega}{\partial x} - u \frac{\partial \omega}{\partial y} = \frac{\partial y}{\partial \phi} \left[ \frac{\partial \phi}{\partial x} \frac{\partial \omega}{\partial \phi} + \frac{\partial \phi}{\partial x} \frac{\partial \omega}{\partial \psi} \right] - \frac{\partial x}{\partial \phi} \left[ \frac{\partial \phi}{\partial y} \frac{\partial \omega}{\partial \phi} + \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial \psi} \right] = \frac{1}{W^2} \left\{ \frac{\partial x}{\partial \phi} \frac{\partial \omega}{\partial \psi} + \frac{\partial y}{\partial \phi} \frac{\partial \omega}{\partial \psi} \right\} \frac{\partial \omega}{\partial \phi} - \left[ \left( \frac{\partial x}{\partial \phi} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 \right] \frac{\partial \omega}{\partial \psi} = - \frac{1}{W^2} (E \frac{\partial \omega}{\partial \phi} - E \frac{\partial \omega}{\partial \psi})
\]

(B.3)

where use has been made of equations (2.31), (2.11) and (2.10). We also employ the continuity and vorticity equations to get

\[
2 \left( \frac{\partial u}{\partial x} \right)^2 + 2 \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 = 2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 - 4 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + 4 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + 4 \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + 4 \left( \frac{\partial v}{\partial \phi} \frac{\partial u}{\partial \phi} - \frac{\partial u}{\partial \phi} \frac{\partial v}{\partial \phi} \right)
\]

\[
\omega^2 + 4 \left( \frac{\partial v}{\partial \phi} \frac{\partial u}{\partial \phi} - \frac{\partial u}{\partial \phi} \frac{\partial v}{\partial \phi} \right)
\]

Employing

\[
\frac{\partial (v, u)}{\partial (x, y)} = \frac{\partial (\phi, \psi)}{\partial (x, y)} \frac{\partial (v, u)}{\partial (\phi, \psi)} = \frac{1}{J} \frac{\partial (v, u)}{\partial (\phi, \psi)}
\]

Employing

\[
u + iv = \frac{\sqrt{E}}{J} e^{i\alpha}
\]

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and (2.16), we obtain

\[
2\left(\frac{\partial \nu}{\partial x}\right)^2 + 2\left(\frac{\partial \nu}{\partial y}\right)^2 + \left(\frac{\partial \nu}{\partial x} + \frac{\partial \nu}{\partial y}\right)^2 = \omega^2 + \frac{4\sqrt{E}}{JW} \left[ \frac{\partial \alpha}{\partial \phi} \frac{\partial}{\partial \psi} \left(\frac{\sqrt{E}}{W}\right) - \frac{\partial \alpha}{\partial \psi} \frac{\partial}{\partial \phi} \left(\frac{\sqrt{E}}{W}\right) \right]
\]

\[
= \omega^2 + \frac{4}{W\sqrt{E}} \left[ \Gamma_{11}^2 \frac{\partial}{\partial \psi} \left(\frac{\sqrt{E}}{W}\right) - \Gamma_{12}^2 \frac{\partial}{\partial \phi} \left(\frac{\sqrt{E}}{W}\right) \right]
\]

(B.4)

Substitution of equations (B.2) through (B.4) in the energy equation (B.1) yields

\[
h = p + \frac{pE}{2W^2} - \frac{\alpha_1}{W^2} \left( F \frac{\partial \omega}{\partial \phi} - E \frac{\partial \omega}{\partial \psi} \right) - \left( \frac{3\alpha_1 + 2\alpha_2}{2} \right) \left\{ \omega^2 + \frac{4}{W\sqrt{E}} \left[ \Gamma_{12}^2 \frac{\partial}{\partial \phi} \left(\frac{\sqrt{E}}{W}\right) \right. \right.
\]

\[
- \left. \Gamma_{11}^2 \frac{\partial}{\partial \psi} \left(\frac{\sqrt{E}}{W}\right) \right\}
\]

(B.5)

in the new variables \( \phi, \psi \), where \( \Gamma_{11}^2 \) and \( \Gamma_{12}^2 \) are Christoffel symbols of the second kind.
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CONFERENCE PROCEEDINGS:


CONFERENCE PRESENTATIONS:
