Some applications of similarity solutions in fluid-flow problems.

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UMI
Some Applications of Similarity Solutions in
Fluid-Flow Problems

by

Ali El Saheli

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Abstract

In this thesis we apply similarity analysis to find possible similarity solutions for various flow problems with heat transfer. Chapter I introduces two similarity methods, the free parameter method and similarity via separation of variables, and illustrates their basic features by applying both methods to the one-dimensional diffusion equation.

In Chapter II we employ separation of variables to present a unified analysis of the boundary layer equations with heat and mass transfer for viscous fluids and establish conditions under which similarity solutions are possible.

Chapter III deals with the steady three-dimensional boundary layer flow of a second order fluid over a flat plate using the free parameter method. We determine the forms of the free stream velocities $U$ and $W$ for which this method is applicable. It seems that this problem has not been treated before for the same constitutive model.

Finally, Chapter IV considers transport of thermal energy in two-dimensional flows of the second order fluid. We discuss a variety of steady and unsteady problems.

Most of the results in Chapter III and IV appear to be new.
Dedicated to my Mother and Father for their love, care and encouragement.
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Chapter I

Similarity Analysis

1.1. Introduction

The fundamental governing equations of fluid dynamics are nonlinear partial differential equations, hence they are very difficult to solve analytically. To date, no general closed-form solution to these equations for three-dimensional problems has been found. Unlike the case for linear partial differential equations, there is no existing general theory for nonlinear partial differential equations; however many special cases have yielded to appropriate changes of variables, or transformations. These can be classified according to the task they are designed to accomplish. Some transformations linearize the system under consideration, others transform it to one for which a solution is known. Transformations that reduce the system of partial differential equations to a system of ordinary differential equations by exploiting an inherent symmetry of the problem are designated “Similarity transformations”. These are the primary analytical tool in the analysis of boundary layer equations. In this thesis similarity methods will be utilized to study some problems in boundary layer theory.

The physical meaning of similarity will be given in the context of the problems to be discussed later. Mathematically, a similarity transformation of a partial differential equation is defined to be a transformation of independent and dependent variables such that the number of independent variables in the transformed equation is reduced by at least one. Thus, if the original system has two independent variables it will be transformed into a
system of ordinary differential equations.

In engineering problems one has to include the boundary and initial values of the problem in the similarity analysis, and these auxiliary conditions must transform properly, otherwise no similarity solution is possible. As will be demonstrated later, it is possible to end up with a system of ordinary differential equations and auxiliary conditions that are not compatible with each other.

There are four methods by which similarity transformations can be discovered. These are

(a) The free parameter method
(b) Similarity via separation of variables
(c) The group theoretic method
(d) Dimensional analysis

In this thesis we will be concerned with only the first two. Comprehensive treatments of the other two can be found in Hansen [1], Ames [2], Birkhoff [3] and Sedov [4].

The free parameter method and the similarity via separation of variables method differ mostly on what they emphasize: the dependent or independent variables. In the first method the dependent variable is specified at the outset in terms of the similarity variable \( \eta \) without specifying what \( \eta \) is. In the second method the dependent variable is only assumed to be separable while \( \eta \) is initially specified. In our presentation we will follow the treatment by Hansen [1].
1.2 The Free Parameter Method

Let \( u \) be the dependent variable occurring in a partial differential equation. Let \( x_1, x_2, ..., x_n \) and \( y \) be the independent variables such that \( u \) has boundary conditions which depend largely on \( y \). In the free parameter method we assume that \( u \) can be written as

\[
\begin{align*}
    u &= H(x_1, ..., x_n) \cdot F(\eta) \\
    (1a)
\end{align*}
\]

The parameter \( \eta \) is obtained from a transformation of the independent variables

\[
\begin{align*}
    \eta &= \eta(x_1, ..., x_n, y) \\
    (1b)
\end{align*}
\]

such that the boundary conditions on \( u \) can be transformed to constant boundary conditions on \( F(\eta) \).

For instance, if the boundary conditions on \( u \) were

\[
\begin{align*}
    u(x_1, ..., x_n, y_1) &= 0 \\
    (2a)
\end{align*}
\]

or

\[
\begin{align*}
    \lim_{y \to y_2} u(x_1, ..., x_n, y) &= H(x_1, ..., x_n) \\
    (2b)
\end{align*}
\]

then the corresponding boundary conditions for the transformed equation would be

\[
\begin{align*}
    F(\eta_1) &= 0 \\
    (3a)
\end{align*}
\]

or
\[
\lim_{\eta \to \eta_2} F(\eta) = 1 \quad (3b)
\]

where

\[\eta_1 = \eta (x_1, \ldots, x_n, y_1) \quad (4a)\]

\[\eta_2 = \eta (x_1, \ldots, x_n, y_2) \quad (4b)\]

In order to impose proper boundary conditions on the resulting equation in \(\eta\), the values of \(\eta_1\) and \(\eta_2\) in (4) must be constant values. This is the reason why similarity solutions are usually associated with problems with infinite domains. Suppose, for example, that the problem under consideration has the following boundary conditions:

\[u(x, y_1) = c_1\]

\[u(x, y_2) = c_2\]

\(c_1\) and \(c_2\) are constants. As we shall see later \(\eta\) will almost always take the following general form

\[\eta = y \ \phi (x_1, \ldots, x_n)\]

so we assume that

\[\eta = y \ \phi (x)\]

and

\[u(x, y) = F(\eta)\]

For \(\eta_1 = y_1 \ \phi (x)\) and \(\eta_2 = y_2 \ \phi (x)\), the conditions on \(F(\eta)\) would be

\[F(\eta_1) = c_1 \ \text{and} \ F(\eta_2) = c_2\]
It is easily seen that $\eta_1$ and $\eta_2$ will have constant values only if $\phi(x)$ is constant or $y_1$ and $y_2$ have very special values. In any case, we can change origin and have $y_1 \to y_1^1 = 0$ and $y_2 \to y_2^1$ so that

$$\eta_1^1 \to y_1^1 \phi(x) = 0$$

$$\eta_2^1 \to y_2^1 \phi(x).$$

It is most unlikely that a transformation will be found which guarantees that $\eta_2^1$ would also be constant. However, if it happens that $y_2^1 = \infty$ then $\eta_2^1 \to \infty$.

After the similarity transformation has been carried out, the transformed equation is examined to determine the conditions under which it can be reduced to an ordinary differential equation. These conditions occur as restrictions on the form of $H(x_1, \ldots, x_n)$. Satisfaction of these restrictions determines whether or not a similarity solution is possible under the assumed transformation, but it also limits the problem which might be solved.

1.2.1 Example

To illustrate the basic features of the free parameter method we consider the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial y^2}$$

(5)

Equation (5) is a mathematical model of many physical phenomena depending on the physical meaning given to the variables involved. For this problem, let $u$ represent the velocity of a
flow parallel to a plate, define $\alpha$ as the kinematic viscosity of the fluid, let $y$ be the normal distance from the plate, and $t$ be time. Then equation (5) describes the flow of a fluid near a flat plate that is suddenly started and moves in its own plane with velocity $U(t)$, see Rosenhead [5], p. 359.

The initial and boundary conditions for this problem are

\begin{align*}
  t & \leq 0, \ y \geq 0 : u = 0 \\
  t & > 0, \ y = 0 : u = U(t) \\
  \text{As } y \to \infty & : u = 0
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig_a}
\caption{Fig. A}
\end{figure}
First we introduce the following transformation

\[ \eta = \eta(y, t) \quad (6a) \]

\[ u = U(t) \, F(\eta) \quad (6b) \]

The form chosen for \( u \) is necessary to establish constant boundary values for \( F(\eta) \) as follows:

\[ F(\eta) \big|_{y=0} = \frac{u(0, t)}{U} = \frac{U}{U} = 1 \]

\[ \lim_{y \to \infty} F(\eta) = \frac{u(\infty, t)}{U} = \frac{0}{U} = 0 \ . \]

Substituting (6) in (5), we obtain:

\[ F + \frac{U}{U'} \frac{\partial \eta}{\partial t} F' = \alpha \frac{U}{U'} \left( \frac{\partial \eta}{\partial y} \right)^2 F'' + \alpha \frac{U}{U'} \frac{\partial^2 \eta}{\partial y^2} F' \quad (7) \]

Clearly, if the various coefficients of \( F \) and its derivatives are constants, then the transformed equation (7) will be reduced to an ordinary differential equation. Since the order of a differential equation is one of its most important aspects we start with the coefficient of \( F'' \).

For if we start with another coefficient, \( F'' \) might end up with a zero coefficient and, hence, the order will be reduced.

So we write:

\[ \alpha \frac{U}{U'} \left( \frac{\partial \eta}{\partial y} \right)^2 = c_3 \]

Then,
\[ \frac{\partial \eta}{\partial y} = \sqrt{\frac{c_3 U'}{\alpha U}} \]  

(8)

Integrating with respect to ‘\(y\)’, we get

\[ \eta = \sqrt{\frac{c_3 U'}{\alpha U}} y + g(t) \]  

(9)

where \(g(t)\) is an arbitrary function of \(t\).

For the boundary value at \(y = 0\)

\[ \eta(0, t) = g(t) \]

which, in general, is nonconstant. If \(g(t)\) is set equal to zero, then we have

\[ \eta(y, t) = y \left( \frac{c_3 U'}{\alpha U} \right)^{\frac{1}{2}} \]  

(10)

\[ \eta(0, t) = 0 \]

\[ \lim_{y \to \infty} \eta(y, t) = \infty \]

From (10) we obtain

\[ \frac{\partial^2 \eta}{\partial y^2} = 0 \]

\[ \frac{\partial \eta}{\partial t} = \frac{1}{2} \left( \frac{c_3}{\alpha} \right)^{\frac{1}{2}} \left( \frac{U}{U'} \right) \left( \frac{UU'' - U''^2}{U^2} \right) \eta \]
Substitution in equation (7) yields

\[ c_3 F'' + \frac{1}{2} \sqrt{\frac{c_3}{\alpha}} \left( 1 - \frac{UU''}{U''^2} \right) \eta F' - F = 0 \]  \hspace{1cm} (11)

Now we see that the restriction imposed on \( U \) is

\[ 1 - \frac{UU''}{U''^2} = c_4 = \text{constant} \]

which implies that

\[ \frac{U''}{U''} = (1 - c_4) \frac{U'}{U} \]

Upon integration, we get

\[ \ln U'' = (1 - c_4) \ln U + c_5 \]

or,

\[ U'' = c_6 U^{1-c_4} . \] \hspace{1cm} (12)

If \( c_4 = 0 \), then

\[ U = c_7 e^{c_6 t} \]

From the initial condition at \( t = 0 \) we have

\[ u(y, 0) = 0 = U(0) F(\eta) \Rightarrow U(0) = 0 \]

\[ \therefore c_7 = 0 \]

or

\[ U = 0 \]

\[ 9 \]
If $c_4 \neq 0$, (12) yields

$$U^{c_4 - 1} \ dU = c_6 \ dt$$

$$\Rightarrow U^{c_4} = c_5 t + c_8$$

or,

$$U = (c_5 t + c_8)^n$$

where $n = \frac{1}{c_4}$.

From $U(0) = 0$, we have $c_4 = 0$, hence

$$U = c_9 r^n$$

Returning to the similarity variable, we find that, from equation (10)

$$\eta = y \left( \frac{nc_3 c_9}{\alpha c_9 t} \right)^{\frac{1}{2}}$$

(13)

The final differential equation is

$$c_3 F'' + \frac{1}{2} \sqrt{\frac{c_3}{\alpha}} c_4 \eta F' - F = 0$$

(14)

and the boundary conditions are

$$F(0) = 1$$

(15)

$$\lim_{\eta \to \infty} F(\eta) = 0$$

(16)
1.3 Similarity Via Separation Of Variables

Contrary to the free parameter method, when applying separation of variables the similarity variable is initially specified, and the dependent variable is only assumed separable without any attempt to ensure that the boundary values transform properly. It is hoped that the separation procedure, if successful, will lead to functions that are compatible with the boundary conditions.

Consider Equation (5) again

\[
\frac{1}{\alpha} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2}
\]  

(5)

Let \( u \) represent temperature, define \( \alpha \) as the thermal conductivity, let \( t \) be time, and \( y \) be the coordinate normal to a wall. Then Equation (5) describes heat conduction in a semi-infinite solid. See ref. [6], p. 140. The boundary conditions are:

\[
t > 0, y = 0 : u = 0
\]

(17a)

\[
t > 0, y \to \infty : u = 1
\]

(17b)

The initial condition is

\[
t = 0, y \geq 0 : u = 1
\]

(17c)

At first we might want to use classical separation of variables, so we let

\[
u = T(t) Y(y)
\]

(18)

Substitution into Equation (5) yields
\[
\frac{1}{\alpha} \frac{T'}{T} = \frac{Y''}{Y} = \lambda
\]

\[
\therefore T = c_3 e^{\alpha t}.
\]

Employing (19) in (18) and applying (17c) we obtain

\[
1 = c_3 Y
\]

Therefore \( Y \) is always constant. But from (17a) we have \( Y(0) = 0 \) which is impossible.

Rather than abandoning this classical method, we apply a similarity transformation first, then we try separation of variables again. Guided by our results from section 1.5, we introduce the following transformation:

\[
\eta = \frac{y}{2\sqrt{\alpha t}}
\]

\[
\tau = \theta n t
\]

Equation (5) transforms to:

\[
-2\eta \frac{\partial u}{\partial \eta} + 4 \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \eta^2}
\]

Equation (21)

Now, let

\[
u = H(\tau) F(\eta)
\]

then Equation (21) becomes

\[
HF'' + 2\eta HF' = 4H' F.
\]

Dividing by \( HF \), we obtain
\[
\frac{F''}{F} + 2 \eta \frac{F'}{F} = 4 \frac{H'}{H}
\]  

(23)

The left side is a function of \( \eta \) only while the right side is a function of \( \tau \). Thus Equation (23) will separate into two ordinary differential equations as follows:

\[
F'' + 2 \eta F' - \lambda F = 0
\]  

(24a)

\[
\frac{H'}{H} = \frac{\lambda}{4}
\]  

(24b)

where \( \lambda \) is the separation constant.

From (24b),

\[
H(\tau) = c_4 e^{\frac{\lambda}{4} \tau}
\]  

(25a)

Another form for \( H(t) \) is possible if, in (20b), we let \( \tau = t \).

Then, we would have:

\[
H(t) = c_5 t^n
\]

The boundary conditions for (24a) are

\[
F(\infty) = 1
\]

\[
F(0) = 0
\]

Also, from the initial condition we have \( H(0) = 1 \), thus

\[
H(t) = e^{\frac{\lambda}{4} t}
\]
1.4 Concluding Remarks

One advantage of the separation of variables method is that it is an extension of a well-known technique. Another is that in some problems, such as the one treated in the last section, we don't have to assume any specific form for the dependent variable except that it is separable. The free parameter method allows for more flexibility since the similarity variable $\eta$ is not determined at the outset, although this might complicate the calculations involved. In a sense, this method provides a systematic way for determining $\eta$, while in the separation of variables method the form of $\eta$ must be guessed based on previous experience. Sometimes the problem at hand may yield to one method but not to the other, and most problems are best treated using a combination of both methods.

Similarity solutions have always been very important in helping us understand the behaviour of fluids. Even though the solution obtained may not be the solution to a problem of interest, it may provide very important qualitative information.

However, these solutions should be viewed with caution because they may represent physically impossible flows, or they may have a limited range of applicability. For example they may provide a valid representation of a flow over some region of space only up to a critical parameter of the problem, but if this critical value is exceeded the similarity solution may no longer represent a real flow. To obtain a better understanding of these problems one needs to carry out numerical integration of the final differential equations and analyse the results obtained. A number of authors have carried out investigations in this direction ([7], [8], [9] and [10]).
In the following chapters we consider problems of boundary layer flow with heat and mass transfer. The engineering applications of these problems are numerous and varied. For example, in addition to there interest in determining velocity profiles and pressure contours, aerospace and mechanical engineers are interested in finding ways to maintain tolerable surface temperatures on high speed aircrafts and on turbine blades. An effective way of achieving this is the use of porous surfaces through which a coolant is injected into the boundary layer. Suction, on the other hand, prevents flow separation and reduces drag on airfoils.

Convection in a porous medium adjacent to a heated surface appears in a number of geophysical and industrial applications such as petroleum drilling. If the fluids involved in any of the applications exhibit strong non-Newtonian behaviour, then recourse to the theory of non-Newtonian fluids is necessary.
Chapter II

Two-Dimensional Thermal Boundary Layers For Viscous Fluids

2.1 Introduction

When a heated fluid is in motion it carries heat along in the form of internal energy, a mode of energy transport called convection. There is also another kind of heat transfer between the fluid and its surroundings by conduction; the mechanism of heat flow in which energy is transported from a body of higher temperature to a body of lower temperature by the drift of electrons. Heat transfer in fluids is, therefore, usually caused by an interplay between convection and conduction.

In some situations a fluid is caused to move by external forces such as a blower, a pump or gravity. In this case one talks about forced convection. For example, in a liquid-to-liquid exchanger, a pump is used to force the fluid to flow over the tube bundles. In a gas-cooled nuclear reactor, the hot gas from the reactor core is circulated over the tubes of the steam generator by special blowers. In the cooling of air automobile radiator, air is forced to flow over the hot radiator tubes as a result of the motion of the car and the fan.

In the case where no external influences are present, temperature differences can still create motion in the fluid. For example, consider a hot wall placed vertically in a large body of cold fluid at rest as shown in Figure (A).

The fluid near the wall becomes hotter than that away from it, which results in density differences. This, in turn, gives rise to buoyancy forces which causes the hot fluid to move vertically along the plate and a velocity boundary layer to develop. In general, free convection
is caused by motion created by any body force within the system in which the heat transfer takes place. Energy transfer by free convection occurs in many engineering and industrial applications. Heat transfer from a hot radiator to heat a room, refrigeration coils, and electric heating elements are a few examples.

The third mode of energy transport is by radiation, but for the problems considered in this thesis radiative heat transfer will be assumed negligible.

Fig. A
Numerous studies have been made of the thermal boundary layer equations for non-isothermal surfaces. Chambre [11] has examined the structure of the thermal boundary layer with distributed heat sources. Hartnett and Eckert [12] have studied the heat transfer, skin friction, and required coolant flows for transpiration cooling. Sparrow and Gregg have studied the problems of constant-fluid properties [13] and variable-fluid properties [14] in free convection. In another paper [15] Sparrow and Gregg investigated laminar film condensation on a vertical plate using the techniques of boundary layer theory. Sparrow [16] found a series solution for the thermal boundary layer on a non-isothermal surface subjected to non-uniform free stream velocity. Siegel [17] has employed the method of characteristics to obtain solutions to the time dependent free convection equations of momentum and energy placed in integral form. Eichhorn [18] has considered the effect of mass transfer on free convection. Yang [19] has obtained similarity solutions for free convection on vertical plates and cylinders using the free parameter method. Koh, Sparrow and Hartnett [20] have considered the two-phase flow problem in laminar film condensation which arises when induced motions of the vapor are included. Kaviany and Mittal [21] have made an experimental and analytical study of the heat transfer rate from an isothermal plate placed next to a high permeability porous media, and, finally, Prasad in [22] studied fluid-superposed porous layers heated from below.

In this chapter, we present a unified analysis of the laminar boundary layer equations with heat convection and mass transfer to establish conditions under which similarity solutions are possible. We employ the method of similarity via separation of variables and discuss a variety of steady and unsteady problems.
2.2 Basic Equations

The governing equations for the unsteady boundary layer flow are [23]:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial t} = U \frac{\partial U}{\partial x} + g \beta (T - T_\infty) + \frac{\nu}{\partial y^2} \frac{\partial^2 u}{\partial y^2}
\]  

(1)

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}
\]  

(2)

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  

(3)

subject to the boundary conditions:

At \( y = 0 \) : \( u = 0, \quad v = V_w(x, t), \quad T = T_\infty + W(x, t) \)  

(4a)

As \( y \to \infty \) : \( u \to U(x, t), \quad T \to T_\infty \)  

(4b)

Here

\( g \) = acceleration due to gravity

\( \beta \) = coefficient of thermal expansion

\( T_\infty \) = ambient temperature

\( \nu \) = kinematic viscosity

\( \alpha \) = thermal diffusivity

\( V_w \) = velocity normal to wall, suction/blowing distribution

\( W \) = wall temperature variation.

The coordinate system for this problem and the relevant physical quantities are depicted in Figure B.
Equations (1), (2) and (3) may be reduced to two equations in two dependent variables by introducing the stream function \( \psi \), with the equations

\[
\begin{align*}
\frac{\partial \psi}{\partial y} & = u \\
\frac{\partial \psi}{\partial x} & = -v
\end{align*}
\]

The continuity equation (3) is satisfied identically, and Equations (1) and (2) become, respectively,

\[
\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + g \beta (T - T_{\infty}) + \nu \frac{\partial^3 \psi}{\partial y^3}
\]
\[
\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}
\] (7)

The boundary conditions will be discussed later for specific cases.

2.3 The Similarity Transformation

We seek a transformation which will reduce equations (6) and (7) to a system of ordinary differential equations.

We try

\[
\begin{align*}
\xi &= x \\
\eta &= \nu \psi(x, t) \\
\tau &= t
\end{align*}
\] (8)

The various partial derivatives in Equations (6) and (7) transform as follows:

\[
\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial \psi}{\partial \tau} \frac{\partial \tau}{\partial x}
\]

\[
= \frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \eta} \frac{\partial \eta \psi}{\partial x} \eta
\]

\[
\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial \psi}{\partial \tau} \frac{\partial \tau}{\partial y}
\]

\[
= \phi \frac{\partial \phi}{\partial \eta}
\]

\[
\frac{\partial^2 \psi}{\partial y \partial \tau} = \frac{\partial}{\partial \tau} \left( \phi \frac{\partial \psi}{\partial \eta} \right) = \frac{\partial \phi}{\partial \tau} \frac{\partial \psi}{\partial \eta} + \phi
\]
\[
\begin{align*}
\frac{\partial \psi}{\partial \tau} & = \phi \left[ \frac{\partial^2 \psi}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial \tau} + \frac{\partial^2 \psi}{\partial \eta \partial \xi} \frac{\partial \eta}{\partial \tau} + \frac{\partial^2 \psi}{\partial \eta^2} \frac{\partial \eta}{\partial \tau} \right] \\
\frac{\partial \psi}{\partial \eta} & = \frac{\partial \phi}{\partial \tau} \frac{\partial \psi}{\partial \eta} + \phi \left[ \frac{\partial \ln \phi}{\partial \eta} \frac{\partial^2 \psi}{\partial \eta^2} + \frac{\partial^2 \psi}{\partial \eta \partial \tau} \right] \\
\frac{\partial^2 \psi}{\partial \eta^2} & = \frac{\partial}{\partial \eta} \left( \phi \frac{\partial \psi}{\partial \eta} \right) = \phi^2 \frac{\partial^2 \psi}{\partial \eta^2} \\
\frac{\partial^3 \psi}{\partial \eta^3} & = \frac{\partial}{\partial \eta} \left( \phi^2 \frac{\partial^2 \psi}{\partial \eta^2} \right) = \phi^3 \frac{\partial^3 \psi}{\partial \eta^3}
\end{align*}
\]

Similarly,

\[
\begin{align*}
\frac{\partial T}{\partial \tau} & = \frac{\partial \ln \phi}{\partial \tau} \frac{\partial T}{\partial \eta} + \frac{\partial T}{\partial \eta} \\
\frac{\partial T}{\partial \xi} & = \frac{\partial T}{\partial \xi} + \frac{\partial \ln \phi}{\partial \xi} \frac{\partial T}{\partial \eta} \\
\frac{\partial T}{\partial \eta} & = \phi \frac{\partial T}{\partial \eta} \\
\frac{\partial^2 T}{\partial \eta^2} & = \phi^2 \frac{\partial^2 T}{\partial \eta^2}
\end{align*}
\]

Substitution of the above transformed derivatives into Equations (6) and (7) yields
\[
\frac{\partial \phi}{\partial t} \frac{\partial \psi}{\partial \eta} + \phi \left[ \frac{\partial \ln \phi}{\partial t} \frac{\partial \psi}{\partial \eta^2} + \frac{\partial^2 \psi}{\partial \eta \partial \xi} \right] \\
+ \phi \frac{\partial \psi}{\partial \eta} \left[ \frac{\partial \phi}{\partial \xi} \frac{\partial \psi}{\partial \eta} + \phi \left[ \frac{\partial^2 \psi}{\partial \xi \partial \eta} + \frac{\partial^2 \psi}{\partial \eta^2} \frac{\partial \phi}{\partial \xi} \right] \right] \\
- \phi^2 \frac{\partial^2 \psi}{\partial \eta^2} \left[ \frac{\partial \psi}{\partial \xi} + \frac{\partial \ln \phi}{\partial \eta} \frac{\partial \psi}{\partial \eta} \right] = \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial \xi} \\
+ g \beta (T - T_w) + \nu \phi^3 \frac{\partial^3 \psi}{\partial \eta^3}
\]

and

\[
\frac{\partial \ln \phi}{\partial t} \frac{\partial \psi}{\partial \eta} + \frac{\partial T}{\partial \eta} + \phi \frac{\partial \psi}{\partial \eta} \left[ \frac{\partial t}{\partial \xi} + \frac{\partial \ln \phi}{\partial \eta} \frac{\partial T}{\partial \eta} \right] \\
- \phi \frac{\partial T}{\partial \eta} \left[ \frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \eta} \frac{\partial \ln \phi}{\partial \xi} \frac{\partial \psi}{\partial \eta} \right] = \alpha \phi^2 \frac{\partial^2 T}{\partial \eta^2}
\]

Upon simplification, we obtain

\[
\frac{1}{\phi^2} \frac{\partial \ln \phi}{\partial t} \frac{\partial \psi}{\partial \eta} + \frac{1}{\phi^2} \frac{\partial \ln \phi}{\partial t} \frac{\partial^2 \psi}{\partial \eta^2} + \frac{1}{\phi^2} \frac{\partial^2 \psi}{\partial \eta \partial \xi} \\
+ \frac{1}{\phi} \frac{\partial \ln \phi}{\partial \xi} \left( \frac{\partial \psi}{\partial \eta} \right)^2 + \frac{1}{\phi} \frac{\partial \psi}{\partial \eta} \frac{\partial^2 \psi}{\partial \xi^2} \\
- \frac{1}{\phi} \frac{\partial \psi}{\partial \xi} \frac{\partial^2 \psi}{\partial \eta^2} = \frac{1}{\phi^3} \left[ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial \xi} \right] \\
+ \frac{1}{\phi^3} g \beta (T - T_w) + \nu \frac{\partial^3 \psi}{\partial \eta^3}
\]

(9)
and

\[ \frac{1}{\phi^2} \frac{\partial \ln \phi}{\partial \tau} \eta \frac{\partial T}{\partial \eta} + \frac{1}{\phi^2} \frac{\partial T}{\partial \tau} + \frac{1}{\phi} \frac{\partial \psi}{\partial \xi} \frac{\partial T}{\partial \xi} \]

\[ - \frac{1}{\phi} \frac{\partial T}{\partial \eta} \frac{\partial \psi}{\partial \xi} = \alpha \frac{\partial^2 T}{\partial \eta^2} \]

(10)

Now we attempt to solve these equations by the method of separation of variables, so we let

\[ \psi = H(\xi, \tau) F(\eta) \]  

(11a)

\[ T - T_\infty = W(\xi, \tau) K(\eta) \]  

(11b)

Equations (9) and (10) become, respectively,

\[ \frac{1}{\phi^2} \frac{\partial \ln \phi}{\partial \tau} H F' + \frac{1}{\phi^2} \frac{\partial \ln \phi}{\partial \tau} \eta H F'' + \frac{1}{\phi^2} \frac{\partial H}{\partial \tau} F' \]

\[ + \frac{1}{\phi} \frac{\partial \ln \phi}{\partial \xi} H^2 F'^2 + \frac{1}{\phi} \frac{\partial H}{\partial \xi} F'^2 \]

\[ - \frac{1}{\phi} \frac{\partial H}{\partial \xi} H F F'' = \frac{1}{\phi^3} \left[ \frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial \xi} \right] + \frac{1}{\phi^3} \beta \psi W K + \nu H F''' \]

\[ \frac{1}{\phi^2} \frac{\partial \ln \phi}{\partial \tau} \eta W K' + \frac{1}{\phi^2} \frac{\partial W}{\partial \tau} K + \frac{1}{\phi} \frac{\partial W}{\partial \xi} F' K \]

\[ - \frac{1}{\phi} \frac{\partial H}{\partial \xi} F K' = \alpha W K'' \]

or,
\[
\frac{1}{\phi^2} \left( \frac{\partial \eta \Phi}{\partial \tau} + \frac{\partial \eta n H}{\partial \tau} \right) F' + \frac{1}{\phi^2} \frac{\partial \eta n \Phi}{\partial \eta} \eta F'' + \frac{1}{\phi} \left( \frac{\partial \eta n \Phi}{\partial \xi} H + \frac{\partial H}{\partial \xi} \right) F''^2
\]

\[- \frac{1}{\phi} \frac{\partial H}{\partial \xi} F F'' = \frac{1}{\phi^3 \eta} \left( \frac{\partial U}{\partial \tau} + U \frac{\partial \eta}{\partial \xi} \right) + \frac{1}{\phi^3 \eta} g \beta WK + \overline{\nu} F''''\]

(12)

\[
\frac{1}{\phi^2} \frac{\partial \eta n \Phi}{\partial \tau} \eta K' + \frac{1}{\phi^2} \frac{\partial \eta n W}{\partial \tau} K + \frac{H}{\phi} \frac{\partial \eta n W}{\partial \xi} F' K
\]

\[- \frac{1}{\phi} \frac{\partial H}{\partial \xi} F K' = \alpha K''\]

(13)

Equations (12) and (13) will separate into two ordinary differential equations in the similarity variable if the following conditions hold

\[
\frac{1}{\phi^3} \frac{\partial \phi}{\partial \tau} + \frac{1}{\phi^2} \frac{\partial \eta n H}{\partial \tau} = a_1
\]

(14a)

\[
\frac{1}{\phi^3} \frac{\partial \phi}{\partial \tau} = a_2
\]

(14b)

\[
\frac{H}{\phi^2} \frac{\partial \phi}{\partial \xi} + \frac{1}{\phi} \frac{\partial H}{\partial \xi} = a_3
\]

(14c)

\[
\frac{1}{\phi^3} \frac{\partial \phi}{\partial \tau} = 0
\]

(14d)
\[
\frac{1}{\phi^3} \frac{W}{H} = a_5 \tag{14e}
\]

\[
\frac{1}{\phi^2} \frac{\partial \ln W}{\partial \tau} = a_6 \tag{14f}
\]

\[
\frac{H}{\phi} \frac{\partial \ln W}{\partial \xi} = a_7 \tag{14g}
\]

where \( a_i = \text{constant} \ (i = 1, \ldots, 7). \)

### 2.4 Steady Convection

For this case \( \frac{\partial}{\partial \tau} = 0 \), and Equations (14) become

\[
\frac{H}{\phi^2} \frac{\partial \phi}{\partial \xi} + \frac{1}{\phi} \frac{\partial H}{\partial \xi} = a_3 \tag{15a}
\]

\[
\frac{1}{\phi} \frac{\partial H}{\partial \xi} = a_4 \tag{15b}
\]

\[
\frac{1}{\phi^3 H} W = a_5 \tag{15c}
\]

\[
\frac{H}{\phi} \frac{\partial \ln W}{\partial \xi} = a_7 \tag{15d}
\]

Equation (15b) implies

\[
\phi = \frac{1}{a_4} \frac{\partial H}{\partial \xi} , \quad a_4 
eq 0 .
\]

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Differentiation with respect to "\( \xi \)" yields

\[
\frac{\partial \phi}{\partial \xi} = \frac{1}{a_4} \frac{\partial^2 H}{\partial \xi^2}
\]

Using this in (15a), we get

\[
H \frac{\partial^2 H}{\partial \xi^2} \left( \frac{1}{\frac{\partial H}{\partial \xi}} \right)^2 = \frac{a_3}{a_4} - 1,
\]

or

\[
\frac{\partial^2 H}{\partial \xi^2} = \left( \frac{a_3}{a_4} - 1 \right) \frac{1}{H} \frac{\partial H}{\partial \xi}
\]

Integrating once, we obtain

\[
\ln \left( \frac{\partial H}{\partial \xi} \right) = \left( \frac{a_3}{a_4} - 1 \right) \ln H + a_4
\]

which implies

\[
\frac{\partial H}{\partial \xi} = a_9 H^{\frac{a_3}{a_4} - 1}, \quad \text{or}
\]

\[
\frac{\partial H}{\partial \xi} = H^{1 - \frac{a_3}{a_4}} = a_4
\]

Here we have two cases, either \( a_3 = 2a_4 \) or \( a_3 \neq 2a_4 \).

**Case I** If \( a_3 = 2a_4 \), then from (16) we have

\[
\frac{\partial \ln H}{\partial \xi} = a_9
\]
\[ \ln H = a_9 \xi + a_{10} \]
\[ \therefore H = a_{11} e^{a_9 \xi} \quad (17) \]

where \( a_{11} = e^{a_{10}} \), and \( a_{10} \) is a constant of integration.

Now, from (15b) we get,
\[ \phi = \frac{a_9 a_{11}}{a_4} e^{a_9 \xi} \quad (18) \]

**Case II**  
If \( a_3 \neq 2a_4 \), then equation (16) yields
\[ \int_{\frac{a_3}{a_4}}^{1} a_4 \, dH = a_9 d\xi \]

By integration, we get
\[ \frac{2a_4 - a_3}{a_4} \int_{a_4}^{a_3} a_4 \, dH = a_9 \xi + a_{12} \]
\[ \therefore H = \left[ \frac{2a_4 - a_3}{a_4} (a_9 \xi + a_{12}) \right]^{a_1=a_4}_{2a_4-a_3} \quad (19) \]

and, from (15b)
\[ \phi = \frac{a_9}{a_4} \left( a_9 \xi + a_{12} \right) \frac{a_5 - a_4}{2a_4 - a_3} \]  

(20)

### 2.4.1 Wall Temperature Conditions

To find the possible forms \( W \) can take, we note that Equation (15c) and (15d) yield

\[ \frac{\partial W}{\partial \xi} = a_5 a_7 \phi^4. \]  

(21)

For case I, Equations (21) and (18) give

\[ \frac{\partial W}{\partial \xi} = a_5 a_7 \frac{a_9 a_{11}^4}{a_4} e^{a_9 \xi} \]

\[ \therefore W = a_{14} e^{4a_9 \xi} + a_{13} \]  

(22)

where \( a_{14} = \frac{a_5 a_7 a_9 a_{11}^4}{4a_4} \), and \( a_{13} \) is a constant of integration.

For case II, equations (21) and (20) imply

\[ \frac{\partial W}{\partial \xi} = a_5 a_7 \left( \frac{a_9}{a_4} \right)^4 \frac{4(a_9 - a_0)}{2a_4 - a_3} \]  

(23)

Here, again, we have two cases.

Case II', \( a_4 = \frac{3}{2} a_3 \), then Equation (23) becomes
\[
\frac{\partial W}{\partial \xi} = a_3 a_7 \left(\frac{a_9}{a_4}\right)^4 (a_9 \xi + a_{12})^{-1}
\]

\[
\therefore W = a_{14} \ln (a_9 \xi + a_{12})^{-1} + a_{15}
\]

(24)

where \( a_{14} = a_3 a_7 \frac{a_9^3}{a_4^4} \), and \( a_{15} \) is a constant of integration.

Case II", if \( a_4 \neq \frac{3}{2} a_3 \), then for \( s - 1 = \frac{4(a_3 - a_4)}{2a_4 - a_3} \) we have

\[
\frac{\partial W}{\partial \xi} = a_3 a_7 \left(\frac{a_9}{a_4}\right)^4 (a_9 \xi + a_{12})^{s-1}
\]

\[
\therefore W = a_{17} (a_9 \xi + a_{12})^s + a_{16}
\]

(25)

where \( a_{17} = \frac{a_5 a_7 a_9^3}{a_4^4 s} \), and \( a_{16} \) is a constant of integration.

Thus, for this problem to have a similarity solution the wall temperature must take one

of the three forms as in equations (22), (24) and (23).

Yang [19] derived (22) and (25) using the free parameter method, but he overlooked

the fact that from his expression (30) for \( W \) as a linear function of \( x \) varying with any power,

one can not obtain the case when the power is identity.

### 2.4.2 The Free Stream Velocity

In light of Equations (15), with \( \frac{\partial U}{\partial \tau} = 0 \), Equation (12) may be written as
\[
\bar{v} F''' + a_4 F'' - a_3 F' + a_3 g \phi K = \frac{-1}{\phi^3 H} U \frac{\partial U}{\partial \xi} \tag{26}
\]

The left side of Equation (26) is a function of \( \eta \) only whereas the right side in a function of \( \xi \) only. Therefore, we have

\[
\bar{v} F''' + a_4 F'' - a_3 F' + a_3 g \phi K = \lambda \tag{27}
\]

and

\[
\frac{1}{\phi^3 H} U \frac{\partial U}{\partial \xi} = -\lambda \tag{28}
\]

where \( \lambda \) is an arbitrary constant known as the separation constant.

Now, equation (28) implies that

\[
\frac{d}{d\xi} (U^2) = -2\lambda \phi^3 H \\
\therefore U^2 = -2\lambda \int \phi^3 d\xi + a_{18} \tag{29}
\]

For case I, (29) yields

\[
U^2 = -2\lambda \int \frac{a_9 a_{11}^3}{a_4^3} e^{4a_9\xi} d\xi + a_{18} \\
= -\lambda \frac{a_9 a_{11}^4}{2a_4^3} e^{4a_9\xi} + a_{18}
\]

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where the new constant of integration was absorbed into \( a_{18} \).

\[
\therefore \quad U = \left[ -\lambda \frac{a_9^2 a_{11}^4 e^{4a_9 \xi}}{2a_4^3} + a_{18} \right]^\frac{1}{2}
\]  

(30)

For case II, Equation (29) becomes

\[
U^2 = -2\lambda \left( \frac{a_9}{a_4} \right)^3 \left( \frac{2a_4 - a_3}{a_4} \right)^{\frac{a_4}{2a_4 - a_3}} (a_9 \xi + a_{12})^{2r-1} + a_{18}
\]  

(31)

where \( 2r - 1 = \frac{3a_3 - 2a_4}{2a_4 - a_3} \).

Hence,

\[
U^2 = -\frac{\lambda}{r} \frac{a_9^2}{a_4^3} \left( \frac{2a_4 - a_3}{a_4} \right)^{\frac{a_4}{2a_4 - a_3}} (a_9 \xi + a_{12})^{2r} + a_{18}
\]  

(33)

\[
\therefore \quad U = \left[ a_{19} (a_9 \xi + a_{12})^{2r} + a_{18} \right]^\frac{1}{2}
\]  

(34)

where \( a_{19}^2 = -\frac{\lambda}{r} \frac{a_9^2}{a_4^3} \left( \frac{2a_4 - a_3}{a_4} \right)^{\frac{a_4}{2a_4 - a_3}} \)  

(35)

If we let \( a_{18} = 0 \), then we have

\[
U = a_{19} (a_9 \xi + a_{12})^r
\]  

(36)
This result was obtained by Hansen [1].

A logarithmic form is also possible for the free stream velocity. If \( a_3 = 0 \) in Equations (15), then Equation (16) implies

\[
H \, dH = a_9 \, d\xi
\]

\[
\therefore \quad H = \left(2a_9 \xi + a_{20}\right)^{\frac{1}{2}}.
\]

Equation (15b), then, gives

\[
\phi = \frac{a_9}{a_4} \left(2a_9 \xi + a_{20}\right)^{-\frac{1}{2}}
\]

Thus, Equation (29) gives

\[
U^2 = -2\lambda \int \frac{a_9^3}{a_4} \left(2a_9 \xi + a_{20}\right)^{-1} + a_{18}
\]

\[
\therefore \quad U = \left[-\lambda \frac{a_9^2}{a_4} \ln \left(2a_9 \xi + a_{20}\right) + a_{18}\right]^{\frac{1}{2}}
\]

(37)

However, there is another restriction imposed on the form of \( U \) by the boundary conditions. We will see that soon.

The general system to be solved (numerically) is

\[
\overline{V} F''' + a_4 F F'' - a_3 F'^2 + a_5 g \Phi K - \lambda = 0
\]

(38a)
\[ \alpha K'' + a_4 F K' - a_7 F' = 0 \]  
\hspace{1cm} (38b)

The boundary conditions are obtained from

\[ u = \frac{\partial \psi}{\partial y} = \phi H F'(\eta) \]  
\hspace{1cm} (39a)

\[ v = -\frac{\partial \psi}{\partial x} = - \left[ \frac{\partial H}{\partial \xi} F(\eta) + \frac{\partial \eta \phi}{\partial \xi} H \eta F''(\eta) \right] \]  
\hspace{1cm} (39b)

\[ T = T_m + \mathcal{W} K(\eta) \]  
\hspace{1cm} (39c)

At \( \eta = 0, \ y = 0 \)

\[ u = 0 \rightarrow F' = 0 \]

\[ v = V_w \]

If \( V_w = 0 \) then \( F = 0 \). If \( V_w \neq 0 \) we have a porous medium problem and \( V_w \) must be proportional to \( \frac{dH}{d\xi} \).

\[ T = T_m + \mathcal{W} \rightarrow K = 1. \]

As \( \eta \rightarrow \infty, \ y \rightarrow \infty \)

\[ u \rightarrow U \rightarrow F' \rightarrow \frac{U}{\phi H} \]

\[ T \rightarrow T_m \rightarrow K \rightarrow 0. \]

If \( U = 0 \), \( F' \rightarrow 0 \), and we have a free convection problem. In this case \( \lambda = 0 \) in Equations (29) and (38a).
If $U \neq 0$, then the condition

$$F' \rightarrow \frac{U}{\phi H}$$

makes sense, in terms of the possibility of a similarity solution, only if $U$ is proportional to $\phi H$. This means that we have to see whether it is possible to choose the constants in equations (30), (36) and (37) in such a way that these expressions of $U$ are, for each case, compatible with the requirement that

$$U = a_{21} \phi H \quad (40)$$

For then, we would have $F' \rightarrow a_{21}$, $a_{21}$ a constant, otherwise, the boundary condition at $\infty$ will be a variable function of $\xi$, and the resulting system of ordinary differential equation won't be solvable.

For case I, Equations (17), (18), (30), and (40) yield

$$\left[ -\lambda \frac{a_2 a_{11}^2}{2a_4} e^{4a_4 \xi} + a_{18} \right] \frac{1}{2} \frac{a_2 a_{11}^2}{a_4} e^{2a_4 \xi}.$$

On choosing $a_{18} = 0$, $a_{21} = 1$ and $\lambda = -2a_4$, then the above equality holds and

$$U = \frac{a_2 a_{11}^2}{a_4} e^{2a_4 \xi} \quad (41)$$

is acceptable.

For case II, Equations (19), (20), (36), and (40) imply
\[
\left[ \frac{-\lambda (2a_4 - a_3) a_9^2}{a_4^3} \left( \frac{2a_4 - a_3}{a_4} \right)^{a_4 \over 2a_4 - a_3} \right]^{1 \over 2} \left( a_9 \xi + a_{12} \right)^{a_3 \over 2a_4 - a_3}
\]

If we let \( a_{21} = \left[ \frac{2a_4 - a_3}{a_4} \right]^{a_4 \over 2a_4 - a_3} \) and \( \lambda = -a_3 \), then the above relation holds and with those choices Equation (36) is also acceptable.

Finally, for Equation (37), we have

\[
\left[ -\lambda \frac{a_9^2}{a_4^3} \ln \left( 2a_9 \xi + a_{20} + a_{18} \right) \right]^{1 \over 2} \quad a_{21} \frac{a_9}{a_4}
\]

which does not hold, therefore the logarithmic form of the free stream velocity (37) is not possible even from the purely mathematical point of view.

2.4.3 The Transformed Problem

Returning to the similarity variable of and the stream function \( \psi \), we see that they take the following forms:

**Case I:**

\[
\eta(x,y) = \frac{a_9a_{11}}{a_4} ye^{a_y x}
\]

\[
\psi(x,y) = a_{11} e^{a_x} F(\eta)
\]
and the velocity and temperature profiles are

\[ u(x, y) = \frac{a_2 a_{11}^2}{a_4} e^{2a_4 x} F''(\eta) \]

\[ \bar{v}(x, y) = -a_2 a_{11} e^{a_4 x} \left[ F(\eta) + \eta F'(\eta) \right] \]

\[ T(x, y) = T_\infty + a_{14} e^{a_4 x} K(\eta) \quad \text{(for } a_{13} = 0) \]

**Case II:**

\[ \eta(x, y) = \frac{a_2}{a_4} y (a_2 x + a_{12})^{\frac{a_2 - a_4}{2a_4 - a_3}} \]

\[ \psi(x, y) = \left[ \frac{2a_4 - a_3}{a_4} (a_1 x + a_{12}) \right]^{\frac{a_4}{2a_4 - a_3}} \]

\[ u(x, y) = \frac{a_2}{a_4} \left( \frac{2a_4 - a_3}{a_4} \right)^{\frac{a_4}{2a_4 - a_3}} (a_2 x + a_{12})^{\frac{a_3 - a_4}{2a_4 - a_3}} F'(\eta) \]

\[ \bar{v}(x, y) = -a_2 (a_2 x + a_{12})^{\frac{a_2 - a_4}{2a_4 - a_3}} \left[ \left( \frac{2a_4 - a_3}{a_4} \right)^{\frac{a_4}{2a_4 - a_3}} F(\eta) \right. \]

\[ + \left. \left( \frac{a_3 - a_4}{2a_4 - a_3} \right)^{\frac{a_4}{2a_4 - a_3}} \eta F'(\eta) \right] \]

\[ T(x, y) = T_\infty + a_{14} \ln(a_2 x + a_{12}) K(\eta) \quad \text{(case II', } a_{15} = 0) \]

\[ T(x, y) = T_\infty + a_{17} (a_2 x + a_{12})^\gamma K(\eta) \quad \text{(case II'', } a_{16} = 0) \]

The similarity functions \( F, F' \) and \( K \) are obtained from Equations (38) and the associated boundary conditions.
A form of the free stream velocity that occurs in most applications is obtained from (36) by setting $a_{12} = 0$:

$$U = ax^r$$  \hspace{2cm} (42)

This represents flow over a wedge. The exponent $r$ is related to the wedge angle by $\beta = \frac{2r}{1 + r}$ [1]. Figure C depicts $U(x)$ for $r = 0$ and $r = 0$. The physical meaning of similarity can be clarified with the aid of Figure C. Similar solutions have the property that two velocity profiles $u(x, y)$ located at different points on the $x$-coordinate differ by a scale factor, such a factor being a function of $x$. In other words, the shapes of the velocity profiles are geometrically similar.
2.5 Unsteady Convection

In this section we consider the thermal boundary layer on a wall when the wall temperature varies not only with position but with time as well.

2.5.1 Unsteady Forced Convection

Returning to equations (14), (14b) implies

\[ \frac{1}{\phi^3} d\phi = a_2 \, d\tau \]

Upon integration with respect to \( \tau \), we get

\[ -\frac{1}{2\phi^2} = a_2 \, \tau + A_1(\xi) \]

where \( A_1(\xi) \) is an arbitrary function of \( \xi \).

\[ \therefore \quad \phi = (A_1(\xi) - 2a_2 \, \tau)^{-\frac{1}{2}} \quad (43) \]

where a multiplicative factor of \(-2\) has been absorbed into \( A_1(\xi) \).

Using (43) in (14a) and (14b) we obtain
\[
\frac{1}{H} \frac{\partial H}{\partial \tau} = (a_1 - a_2) (A_1(\xi) - 2a_2 \tau)^{-1}
\]

Integration yields

\[
\ln H = -\frac{(a_1 - a_2)}{2a_2} \ln (A_1(\xi) - 2a_2 \tau) + A_2(\xi)
\]

\[
\therefore \quad H = A_3(\xi) (A_1(\xi) - 2a_2 \tau)^{\frac{a_2 - a_1}{2a_2}}
\]

where \( A_3(\xi) = e^{A_{20}} \), and \( c_2 \neq 0 \)

From (14d), we obtain

\[
\frac{\partial \phi}{\partial \xi} = \frac{1}{a_4} \frac{\partial^2 H}{\partial \xi^2}
\]

Also, from (14c) and (14d) we have

\[
\frac{1}{\phi^2} \frac{\partial \phi}{\partial \xi} H = a_3 - a_4, \quad a_3 \neq a_4.
\]

Therefore

\[
\left( \frac{\partial H}{\partial \xi} \right)^{-2} \frac{\partial^2 H}{\partial \xi^2} a_4 H = a_3 - a_4
\]

or

\[
\frac{\partial H}{\partial \xi} = B_2(\tau) \frac{a_1 - a_4}{a_4}, \quad a_4 \neq 0.
\]
\[ H \frac{a_4 - a_3}{a_4} \ dH = B_2(\tau) \ d\xi \]

\[ \frac{a_4}{2a_4 - a_3} H \frac{2a_4 - a_3}{a_4} = B_2(\tau) \xi + B_3(\tau) \]

\[ \therefore H = \left[ \frac{2a_4 - a_3}{a_4} B_2(\tau) \xi + B_3(\tau) \right]^{a_4}_{2a_4 - a_3} \]

\[ \frac{\partial H}{\partial \xi} = B_2(\tau) \left[ \frac{2a_4 - a_3}{a_4} B_2(\tau) \xi + B_3(\tau) \right]^{a_4}_{2a_4 - a_3}. \]

Using this in (14d), we have

\[ \phi = \frac{B_2(\tau)}{a_4} \left[ \frac{2a_4 - a_3}{a_4} B_2(\tau) \xi + B_3(\tau) \right]^{a_4}_{2a_4 - a_3}. \]

On comparing equations (43), (44), (45) and (46), we see that a most convenient solution is obtainable if the following relations hold:

\[ a_3 = a_1 = 0 \]

\[ A_{1(\xi)} = 2a_4 \xi \]

\[ B_2(\tau) = a_4 \]

\[ B_3(\tau) = -2a_2 \tau \]

\( (47) \)
Therefore
\[ \phi(\xi, \tau) = \frac{1}{\sqrt{2}} [a_4 \xi - a_2 \tau]^{-\frac{1}{2}} \]  \hspace{1cm} (48)

\[ H(\xi, \tau) = \sqrt{2} [a_4 \xi - a_2 \tau]^{-\frac{1}{2}} \]  \hspace{1cm} (49)

From (14e), we find that
\[ W(\xi, \tau) = \frac{a_5}{2} [a_4 \xi - a_2 \tau]^{-1} \]  \hspace{1cm} (50)

It is easy to verify that equation (50) is compatible with equations (14f) and (14g).

The governing equations become
\[ \bar{v} F''' + a_4 F F'' - a_2 \eta F'' + a_5 g \beta K = -\frac{1}{\phi^2 H} \left( \frac{\partial U}{\partial \xi} + U \frac{\partial U}{\partial \xi} \right) \]  \hspace{1cm} (51a)

\[ a K'' + (a_4 F - a_2 \eta) K' - (a_7 F' + a_2) K = 0 \]  \hspace{1cm} (51b)

The boundary conditions are determined from
\[ u(x,y,\tau) = \frac{\partial}{\partial y} \psi(x,y,\tau) = \phi(\xi, \tau) H(\xi, \tau) F'(\eta) \]  \hspace{1cm} (51c)

\[ v(x,y,\tau) = -\frac{\partial \psi}{\partial x} (x,y,\tau) = -\left[ \frac{\partial H}{\partial \xi} F(\eta) + \frac{\partial \ln \phi}{\partial \xi} H(\xi, \tau) \eta F'(\eta) \right] \]  \hspace{1cm} (51d)
\[ T(x,y,t) = T_\infty + W(\xi, \tau)K(\eta) \] \hspace{1cm} (51e)

At
\[
\begin{aligned}
  y &= 0 \quad \eta = 0 \\
  u &= 0 \quad F' = 0 \\
  T - T_\infty &= W \quad K = 1 \\
  v &= V_\mu(x,t)
\end{aligned}
\] \hspace{1cm} (51f)

As before, it is seen from (51d) that \( V_\mu(x,t) \) must be proportional to \( \frac{\partial H}{\partial \xi} \).

As
\[
\begin{aligned}
  y &\rightarrow \infty \quad \eta \rightarrow \infty \\
  T &\rightarrow T_\infty \quad K \rightarrow 0 \\
  u &\rightarrow U \quad F' \rightarrow \frac{U}{\phi H}
\end{aligned}
\] \hspace{1cm} (51g)

As with the steady case, we must have
\[ U = a_{22} \phi H \]

or, by employing (48) and (49) in the above identity
\[ U = a_{22} \] \hspace{1cm} (52)

Returning to equation (51a), we see that the left side is a function of \( \eta \) only, while the right side is a function of \( \xi \) and \( \tau \). Therefore

\[ \nabla F'''' + a_4 FF'' - a_2 \eta F'' + a_5 g \beta K = \lambda_1 \]

\[ \frac{1}{\phi^2 H} \left( \frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial \xi} \right) = -\lambda_1 \] \hspace{1cm} (53)
where \( \lambda_1 \) is the separation constant.

But by equation (52), \( U \) must be a constant,

hence

\[ \lambda_1 = 0 \]

The system (51) becomes

\[ \overline{v} F''' + a_4 F F'' - a_2 \eta F'' + a_3 g \beta K = 0 \]  

(54a)

\[ \alpha K'' + (a_4 F - a_2 \eta) K' - (a_1 F' + a_2) K = 0 \]  

(54b)

subject to the boundary conditions (51f, g) with

\[ U = a_{22} \]

The similarity variable and the stream function are given by

\[ \eta(x,y,t) = \frac{y}{\sqrt{2(a_4 x - a_2 t)}} \]  

(55a)

\[ \psi(x,y,t) = \left[ 2(a_4 x - a_2 t) \right]^{\frac{1}{2}} F(\eta) \]  

(55b)

Using (48) and (49) in (51c) - (51e)-, the velocity and temperature profiles can be determined as follows

\[ u(x,y,t) = F'(\eta) \]  

(55c)
\[ v(x, y, t) = \frac{-a_4}{\sqrt{2(a_4 x - a_2 t)}} (F - \eta F') \quad (55d) \]

\[ T(x, y, t) = T_* + \frac{a_5}{2(a_4 x - a_2 t)} K(\eta) \quad (55e) \]

### 2.5.2 Unsteady Free Convection

If \( a_3 = a_4 \) in Equations (14), then \( \phi = \phi(\tau) \).

Therefore, Equation (43) becomes

\[ \phi = [a_{23} - 2a_2 \tau]^{-\frac{1}{2}} \quad (56) \]

Using this in Equation (14a), we have

\[ \frac{1}{H} \frac{\partial H}{\partial \tau} = (a_1 - a_2) [a_{23} - 2a_2 \tau]^{-1} \]

\[ \therefore H = A_4(\xi) [a_{23} - 2a_2 \tau]^{\frac{a_3 - a_1}{2a_2}} \quad (56') \]

Combining this with (14d), we obtain

\[ \phi = \frac{1}{a_4} \frac{\partial H}{\partial \xi} = \frac{A_4'(\xi)}{a_4} [a_{23} - 2a_2 \tau]^{\frac{a_3 - a_1}{2a_2}} \]

Comparing this expression with (56), we find that

\[ A_4'(\xi) = a_4, \text{ and } a_2 = \frac{a_1}{2} \quad . \]

Hence,

\[ A_4(\xi) = a_4 \xi + a_{24} \quad . \]

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Therefore
\[ H = (a_4 \xi + a_{24}) \left[ a_{23} - 2a_2 \tau \right]^{-\frac{1}{2}} \] (57)

The corresponding wall temperature, from (14e), is
\[ W(\xi, \tau) = a_5 (a_4 \xi + a_{24}) \left[ a_{23} - a_2 \tau \right]^{-2} \] (58)

which is compatible with the other two conditions on \( W(\xi, \tau) \), (14f) and (14g).

For this similarity solution, the potential velocity and the separation constant must be zero. Heat convection, in this case, is free convection.

The ordering differential equations are
\[ \bar{\nabla} F''' + a_3 FF'' - a_3 F' + \frac{a_1}{2} \eta F' + a_5 g \beta K = 0 \] (59a)
\[ \alpha K'' + (a_3 F - \frac{a_1}{2} \eta) K' - (a_6 - a_7 F') K = 0 \] (59b)

subject to the boundary conditions
\[ \eta = 0 : F' = 0, K = 1 \]
\[ \eta \to \infty : F' \to 0, k \to 0 \]

Also
\[ \eta = y \left[ a_{23} - 2a_2 \tau \right]^{-\frac{1}{2}} \] (60a)

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\[ \psi = (a_4 x + a_{24}) [a_{23} - 2a_2 \tau]^{-\frac{1}{2}} F(\eta) \quad (60b) \]

\[ u = \frac{(a_4 x + a_{24})}{a_{21} - 2a_2 \tau} F'(\eta) \quad (60c) \]

\[ v = -a_4 (a_{23} - 2a_2 \tau)^{-\frac{1}{2}} F(\eta) \quad (60d) \]

\[ T = T_0 + a_2 \frac{(a_4 x + a_{24})}{(a_{23} - 2a_2 \tau)^2} K(\eta) \quad (60e) \]

### 2.5.3 Asymptotic Solutions

In this case we have \( a_3 = a_4 = 0 \).

From Equation (43) we obtain

\[ \phi = (a_{23} - 2a_2 \tau)^{-\frac{1}{2}} \quad (61) \]

and from \(56'\), we get

\[ H = a_{25} (a_{23} - 2a_2 \tau)^{\frac{a_2 - a_1}{2a_2}} \quad (61a) \]

Therefore

\[ W = a_5 a_{25} (a_{23} - 2a_2 \tau)^{\frac{-2a_2 - a_1}{2a_2}} \quad (61b) \]
This result is in agreement with (14f), but \( a_7 \) must be set equal to zero in order to accommodate Equation (14g).

The differential equations for this problem are

\[
\bar{v} F''' - a_2 \eta F'' - a_1 F' + a_5 g \beta K = \lambda_2
\]  
(62a)

\[
\alpha K'' - a_2 \eta K' - a_6 K = 0
\]  
(62b)

The condition on \( F' \) as \( \eta \to \infty \) requires that

\[
U = a_{25} \phi H = a_{25} (a_{23} - 2a_2 \tau)^{-\frac{1}{2a_2}}
\]  
(63)

which must be constant. Hence, \( a_1 = 0 \) and \( \lambda_2 = 0 \).

Thus, from (61)

\[
H = a_{25}(a_{23} - 2a_2 \tau)^{\frac{1}{2}}
\]  
(64a)

\[
W = a_5 a_{25} (a_{23} - 2a_2 \tau)^{-1}
\]  
(64b)

Equations (62) become

\[
\bar{v} F''' - a_2 \eta F'' + a_5 g \beta K = 0
\]  
(65a)

\[
\alpha K'' - a_2 \eta K' - a_6 K = 0
\]
subject to the usual boundary conditions with $U = a_{25}$.

$$\eta = y(a_{23} - 2a_2t)^{-\frac{1}{2}} \tag{66a}$$

$$\psi = a_{25}(a_{23} - 2a_2t)^{\frac{1}{2}} F(\eta) \tag{66b}$$

$$u = a_{25} F'(\eta) \tag{66c}$$

$$v = 0 \tag{66d}$$

$$T = T_\ast + a_3 a_{25}(a_{23} - 2a_2t)^{-1} K(\eta) \tag{66e}$$

The fact that all derivatives with respect to $\ast$ are zero in the last case implies that it represents an asymptotic solution valid at large distances away from the leading edge. Similar results have been obtained by Yang [19] using the free parameter method.
Chapter III
Boundary Layer Analysis For Second Order Fluids

3.1 Introduction

In many engineering and industrial applications of fluid flow, one often deals with fluids whose behaviour cannot be described, satisfactorily, by the classical theory of incompressible viscous fluids. In this theory, the relation between the stress and the strain-rate tensors is described by the following constitutive equation for an incompressible Newtonian fluid.

$$T_{ij} = -p\delta_{ij} + 2\mu D_{ij}$$  (1)

Here $T_{ij}$ is the symmetric Cauchy stress tensor, $D_{ij}$ is the stretching tensor, $\mu$ is the viscosity of the fluid, and $p$ is the indeterminate pressure.

It is well known that fluids with complex structures, such as drilling muds, certain oils and greases, polymer melts, suspensions, blood and other biological fluids behave in unexpected ways and are not described by (1). These are termed as non-Newtonian or viscoelastic fluids.

Some of the viscoelastic features of such fluids that have been observed are [24].

(a) Shear-rate-dependent viscosity: most of these fluids display "shear thining", that is the viscosity decreases with increasing shear rate.

(b) Normal stress effects: unequal normal stresses in the different directions in steady shear
flow and related simple flows.

(e) Elastic and tensile properties: the resistance of the fluid to stretching.

It has long been recognized that an appropriate generalization of equation (1) must be found, but these non-Newtonian fluids vary greatly among themselves in their physical structure and their responses to the stress. Therefore, many constitutive equations have been proposed as can be seen from the articles by Storer and Green, Walters, Oldroyd, Coleman and Noll in [25], Rivlin, Reiner in [26], Denn [27], Truesdell in [28] and others. An extensive coverage of these constitutive equations can be found in references [24] and [28].

One of the models that have gained considerable support from researchers in the field is the constitutive equation for the second order fluid formulated by R.S. Rivlin and J.L. Ericksen. In this model, the stress tensor $T_{ij}$ is given by [28].

$$T_{ij} = -p \delta_{ij} + \mu A_1 + \mu_1 A_2 + \mu_2 A_1^2$$  \hspace{1cm} (2)$$

where $\mu$ is the coefficient of viscosity, $\mu_1$ the coefficient of viscoelasticity, and $\mu_2$ is the coefficient of cross viscosity. Furthermore, the Rivlin-Ericksen tensors $A_1$ and $A_2$ are defined by

$$A_1 = A_{ij} = (V_{i,j} + V_{j,i})$$  \hspace{1cm} (2a)$$

$$A_2 = \frac{\partial}{\partial t} A_{ij} + V_k A_{ij,k} + A_{im} V_{m,j} + A_{jm} V_{m,i}$$  \hspace{1cm} (2b)$$

$$A_1^2 = A_{ik} A_{kj}$$  \hspace{1cm} (2c)$$
where \( V \) is the velocity vector.

The employment of (2) in the momentum equations results in highly nonlinear equations of motion. As a result, it is very difficult to obtain exact solutions of these equations unless, either there are some simplifying feature to the problem or some simplifying assumptions are made.

In order to study the effects of elasticity, several authors have employed boundary layer type analyses with different idealized constitutive equations. Rivlin [25] has investigated the flow of ordered fluids along a straight, non-circular pipe. He (see Ref. [26]) also has studied torsional flow between parallel discs and between coaxial cones and the helical flow of such fluids. Ting [29] has studied certain non-steady flows, such as channel flows, pipe flows under constant pressure gradient, flow between infinite parallel planes in an incompressible second order fluid. Beard and Walters [30] have solved, numerically, the boundary layer equations for the case of two-dimensional flow near a stagnation point. Sarpkaya and Rainey [31] have studied the same problem with a slightly different approach. Huilgol [32] has considered slow steady flows of second order fluids in which inertia terms are negligible. Rajagopal and Gupta [33, 34] have found certain exact solutions in such fluids in which either the nonlinearities are self cancelling or the equations become linear. Kaloni and Siddiqui [35] have generalized a new method that uses concepts from Differential Geometry to study plane flows of a second order fluid. Siddiqui [36] has used the hodograph transformation method to study the flow of ordered fluids.

The above investigations have dealt mostly with two-dimensional flows. Their results are known not to accurately account for many engineering applications. For example, Hansen
and Herzig [37] have noted that in turbomachine boundary layers, the two-dimensional boundary layer flows tend to become three-dimensional because of the secondary flows generated. Thus, for physically acceptable solutions consideration of the three-dimensional boundary layer equations is much more desirable.

In a recent article, Sacheti and Chandran [38] have studied the steady three-dimensional boundary layer flow of a certain kind of second order fluid over a flat plate. In this chapter a similar study is carried out but using a slightly different constitutive equation.

3.2. Basic Equations And Ordering Analysis

The equations of motion for incompressible flow in the absence of body forces are

linear momentum: \( \frac{\partial T_{ij}}{\partial x_j} = \rho \left( \frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} \right) \) \hspace{1cm} (3a)

continuity: \( \frac{\partial V_i}{\partial x_j} = 0 \) \hspace{1cm} (3b)

If we substitute (1) in (3a) and make use of (2), we obtain

\[ \rho \left( \frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial V_j} \right) = \frac{\partial}{\partial x_j} \left( \rho \delta_{ij} \right) + \mu \frac{\partial^2 V_i}{\partial x_j \partial x_j} \]

\[ + \mu_1 \left[ \frac{\partial}{\partial t} \left( \frac{\partial^2 V_i}{\partial x_j \partial x_j} \right) + \frac{\partial V_k}{\partial x_j} \frac{\partial^2 V_i}{\partial x_j \partial x_k} + \frac{\partial V_k}{\partial x_j} \frac{\partial^2 V_i}{\partial x_j \partial x_j} + \frac{\partial^3 V_i}{\partial x_j \partial x_j \partial x_k} \right] \]

\[ + \frac{\partial V_k}{\partial x_j} \frac{\partial^2 V_i}{\partial x_j \partial x_k} + \frac{\partial V_m}{\partial x_j} \left( \frac{\partial V_i}{\partial x_m} + \frac{\partial V_m}{\partial x_i} \right) \]
\[\begin{align*}
\left( \frac{\partial V_m}{\partial x_j} \right) & \left( \frac{\partial^2 V_i}{\partial x_j \partial x_m} + \frac{\partial^2 V_m}{\partial x_i \partial x_j} \right) + \frac{\partial^2 V_m}{\partial x_i \partial x_j} \\
& \left( \frac{\partial V_i}{\partial x_m} + \frac{\partial V_m}{\partial x_j} \right) \\
& + \frac{\partial V_m}{\partial x_i} + \frac{\partial^2 V_m}{\partial x_j \partial x_k} \left( \frac{\partial V_i}{\partial x_k} + \frac{\partial V_k}{\partial x_i} \right) \\
& + \left( \frac{\partial^2 V_i}{\partial x_k \partial x_j} + \frac{\partial^2 V_k}{\partial x_j \partial x_k} \right) \left( \frac{\partial V_k}{\partial x_i} + \frac{\partial V_l}{\partial x_l} \right)
\end{align*}\]

(4)

where \(x_i = (x, y, z)\) are the usual Cartesian coordinates, and \(V_i = (u, v, w)\) is the velocity field.

Let \(U_0(x, z)\) and \(W_0(x, z)\) be the velocity components of the potential flow in the \(x\) - and \(z\) - directions, respectively, and introduce the nondimensional variables.

\[t = \frac{U_0 t}{L}, \quad \tilde{x}_i = \frac{x_i}{L}, \quad \tilde{P} = \rho \frac{L^2}{\mu^2} P, \quad \tilde{U} = \frac{U}{U_0}, \quad \tilde{W} = \frac{W}{W_0},\]

and \(\tilde{V}_i = \rho \frac{V_i L}{\mu}\).

We now make the usual boundary layer assumptions.

Within the boundary layer, \(u, w, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial z}, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial z}, \frac{\partial p}{\partial x}\) and \(\frac{\partial p}{\partial z}\) are assumed to be \(0(1)\) and \(y\) to be \(0(\delta)\), where \(\delta\) is the boundary layer thickness.

From the equation of continuity, we find that

\[\frac{\partial v}{\partial y} = 0(1),\]

Therefore \(v = 0(\delta)\).
We also have \( \frac{\partial P}{\partial y} = 0(\delta) \), so we can assume that the pressure within the boundary layer can be approximated by the pressure outside. Therefore, form Bernoulli’s equation we have

\[
- \frac{1}{\rho} \frac{\partial P}{\partial x} = U \frac{\partial U}{\partial x} + W \frac{\partial U}{\partial z}
\]  
(5a)

and

\[
- \frac{1}{\rho} \frac{\partial P}{\partial z} = U \frac{\partial W}{\partial x} + W \frac{\partial W}{\partial z}
\]  
(5b)

If we let \( \bar{\mu}_1 = \frac{\mu_1}{\rho L^2} \), \( \bar{\mu}_2 = \frac{\mu_2}{\rho L^2} \), the steady state equations of motion, in dimensionless form, become

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]  
(6a)

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = U \frac{\partial U}{\partial x} + W \frac{\partial U}{\partial z} + \frac{\partial^2 u}{\partial y^2} + \bar{\mu}_1 \left[ u \frac{\partial^2 u}{\partial y^2} + v \frac{\partial^2 u}{\partial y^3} + w \frac{\partial^2 u}{\partial y^2 \partial z} + \frac{u}{\partial y} \left( 3 \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 w}{\partial y \partial z} \right) \right] + \frac{\partial^2 u}{\partial y^2} \left( \frac{\partial u}{\partial x} - 2 \frac{\partial w}{\partial z} \right)^2 + 2 \frac{\partial w}{\partial y} \left( \frac{\partial^2 u}{\partial y^3} + \frac{\partial^2 w}{\partial x \partial y} \right) + \frac{\partial^2 w}{\partial y^2} \left( \frac{\partial u}{\partial z} + 2 \frac{\partial w}{\partial x} \right) \right]
\]

\[
+ \bar{\mu}_2 \left[ \frac{\partial}{\partial y} \left( \frac{\partial y}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial w}{\partial z} - 2 \frac{\partial u}{\partial y} \frac{\partial w}{\partial z} \right] + \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial y} \frac{\partial w}{\partial z} \right) + \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right)^2 \]  
(6b)
\[
\begin{align*}
\frac{u}{\partial x} + \frac{v}{\partial y} + w \frac{\partial w}{\partial z} &= U \frac{\partial W}{\partial x} + W \frac{\partial W}{\partial y} + \frac{\partial^2 w}{\partial y^2} \\
\bar{\mu}_1 \left[ u \frac{\partial^3 w}{\partial x \partial y^2} + u \frac{\partial^3 w}{\partial x y^3} + w \frac{\partial^3 w}{\partial y^2 \partial z} + \frac{\partial w}{\partial y} \left( 3 \frac{\partial^2 w}{\partial y z} - \frac{\partial^2 u}{\partial x \partial y} \right) \right] \\
+ \frac{\partial^2 u}{\partial y^2} \left( \frac{\partial w}{\partial x} + 2 \frac{\partial u}{\partial x} \right) + 2 \frac{\partial u}{\partial y} \left( \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 w}{\partial x \partial y} \right) + \frac{\partial^2 w}{\partial y^2} \left( \frac{\partial w}{\partial x} - 2 \frac{\partial u}{\partial x} \right) \right] \\
+ \bar{\mu}_2 \left[ \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \frac{\partial u}{\partial y} - 2 \frac{\partial u}{\partial x} \frac{\partial w}{\partial y} \right] + \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial w}{\partial y} \right)^2 \right] \\
\end{align*}
\]

subject to the boundary conditions:

At \( y = 0 \) : \( u = v = w = 0 \)

As \( y \to \infty \) : \( u \to U, w \to W \) \hspace{2cm} (6d)

In the above equations, the overbars have been dropped for convenience.

3.3 The Similarity Solution

In order to reduce equations (6) to a set of ordinary differential equations, we introduce

the similarity variable

\[ \eta = y \phi (x, z) , \]  \hspace{2cm} (7a)

and the transformations

\[ u = U(x, z) F' (\eta) \]  \hspace{2cm} (7b)

\[ w = W(x, z) G' (\eta) \]  \hspace{2cm} (7c)
The partial derivatives in (6) transform as follows

\[
\frac{\partial u}{\partial x} = \frac{\partial U}{\partial x} F' + \frac{\partial \eta \Phi}{\partial x} U \eta F''
\]

\[
\frac{\partial w}{\partial z} = \frac{\partial W}{\partial z} G' + \frac{\partial \eta \Phi}{\partial z} W \eta G''
\]

\[
\frac{\partial u}{\partial y} = \phi U F''
\]

\[
\frac{\partial u}{\partial z} = \frac{\partial U}{\partial z} + \frac{\partial \eta \Phi}{\partial z} U \eta F''
\]

\[
\frac{\partial^2 u}{\partial y^2} = \phi^2 U F'''
\]

\[
\frac{\partial^3 u}{\partial x \partial y^2} = \left( 2 \phi \frac{\partial \Phi}{\partial x} U + \phi^2 \frac{\partial U}{\partial x} \right) F''' + \phi \frac{\partial \Phi}{\partial x} U \eta F^{iv}
\]

\[
\frac{\partial^3 u}{\partial y^3} = \phi^3 U F^{iv}
\]

\[
\frac{\partial^3 u}{\partial x \partial y^2} = \left( 2 \phi \frac{\partial \Phi}{\partial z} U + \phi^2 \frac{\partial U}{\partial z} \right) F'''' + \phi \frac{\partial \Phi}{\partial z} U \eta F^{iv}
\]

\[
\frac{\partial w}{\partial y} = \phi W G''
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = \left( \frac{\partial \psi}{\partial x} U + \phi \frac{\partial U}{\partial x} \right) F'' + \frac{\partial \psi}{\partial x} W \eta \frac{\partial F}{\partial x} \\
\frac{\partial^2 w}{\partial y \partial z} = \left( \frac{\partial \psi}{\partial z} W + \phi \frac{\partial W}{\partial z} \right) G'' + \frac{\partial \psi}{\partial z} W \eta \frac{\partial G}{\partial z} \\
\frac{\partial w}{\partial z} = \frac{\partial W}{\partial z} G' + \frac{\partial \ln \phi}{\partial z} W \eta \frac{\partial G}{\partial z} \\
\frac{\partial^2 u}{\partial y \partial z} = \left( \frac{\partial \psi}{\partial z} U + \phi \frac{\partial U}{\partial z} \right) F'' + \frac{\partial \psi}{\partial z} U \eta \frac{\partial F}{\partial z} \\
\frac{\partial^2 w}{\partial x \partial y} = \left( \frac{\partial \psi}{\partial x} W + \phi \frac{\partial W}{\partial x} \right) G'' + \frac{\partial \psi}{\partial x} W \eta \frac{\partial G}{\partial x} \\
\frac{\partial^2 w}{\partial y^2} = \phi^2 W G'' \\
\frac{\partial w}{\partial x} = \frac{\partial W}{\partial x} G' + \frac{\partial \ln \phi}{\partial x} W \eta \frac{\partial G}{\partial x} \\

The continuity equation (6a) becomes
\[
\frac{\partial v}{\partial y} = -\left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right)
= -\left( \frac{\partial U}{\partial x} F' + \frac{\partial \ln \phi}{\partial x} U \eta F'' + \frac{\partial W}{\partial z} G' + \frac{\partial \ln \phi}{\partial z} W \eta G'' \right)
\]
Hence,
\[ \nu = -f \left( \frac{\partial U}{\partial x} F' + \frac{\partial \ln \phi}{\partial x} U \eta F'' + \frac{\partial W}{\partial z} G' + \frac{\partial \ln \phi}{\partial z} W \eta G'' \right) \frac{\partial \eta}{\partial \phi} \]

or,
\[ \phi \nu = \left( U \frac{\partial \ln \phi}{\partial x} - \frac{\partial U}{\partial x} \right) F + \left( W \frac{\partial \ln \phi}{\partial z} - \frac{\partial W}{\partial z} \right) G \]
\[ - U \frac{\partial \ln \phi}{\partial x} \eta F' - W \frac{\partial \ln \phi}{\partial z} \eta G' \]

(8)

The x-momentum equation becomes

\[ \frac{1}{\phi^2} \frac{\partial U}{\partial x} (F'^2 - FF' - 1) + \frac{W}{\phi^2} \frac{\partial \ln U}{\partial z} (F' G' - 1) \]
\[ + \frac{U}{\phi^2} \frac{\partial \ln \phi}{\partial x} F F'' + \frac{1}{\phi^2} \left( \frac{W}{\partial z} \frac{\partial \ln \phi}{\partial z} - \frac{\partial W}{\partial z} \right) F'' G - F''' \]
\[ - \mu \left[ \frac{\partial \ln \phi}{\partial x} U \left( 2 F' F''' + 3 F''^2 + 2 \eta F'' F''' \right) \right] \]
\[ + \frac{\partial U}{\partial x} (2 F' F''' - FF'' + F''^2) + \frac{\partial \ln \phi}{\partial x} UF F'' \]
\[ + \frac{\partial \ln \phi}{\partial z} \left( W (GF'' + 2 G' F''') + F'' G' \right) - \frac{\partial W}{\partial z} (GF''') \]
\[ + G'' F'' + 2 G' F''' + \frac{W}{U} \frac{\partial U}{\partial z} \left( F''' G' + 2 F'' G'' + F' G''' \right) \]
\[ + \frac{\partial n \Phi}{\partial x} \frac{W^2}{U} (2G''^2 + 4\eta G'' G^''') + 2 \frac{W}{U} \frac{\partial W}{\partial x} (G''^2 + G' G''') \]

\[- \mu_2 \left[ \frac{W}{U} \frac{\partial U}{\partial z} (2F'' G'' + F' G''') + \frac{W}{U} \frac{\partial W}{\partial x} (G' G''' + G''^2) \right] + \frac{\partial n \Phi}{\partial x} \frac{W^2}{U} (2\eta G'' G''' + G''^2) - \frac{\partial W}{\partial z} (2 F'' F' + F'' F'') \]

\[+ \frac{\partial n \Phi}{\partial z} WF'' G'' + 2 \frac{\partial n \Phi}{\partial x} U (F''^2 + \eta F'' F''') + 2 \frac{\partial U}{\partial x} F''^2 \right] = 0 \tag{9a} \]

and the z-momentum is

\[ \frac{1}{\phi^2} \frac{\partial W}{\partial z} (G''^2 - GG'' - 1) + \frac{U}{\phi^2} \frac{\partial n W}{\partial x} (F' G' - 1) \]

\[+ \frac{W}{\phi^2} \frac{\partial n \Phi}{\partial z} GG'' + \frac{1}{\phi^2} \left( U \frac{\partial n \Phi}{\partial x} - \frac{\partial U}{\partial x} \right) G'' F - G''' \]

\[- \mu_1 \left[ \frac{\partial n \Phi}{\partial z} W(2G'' G''') + 3G''^2 + 2\eta G'' G''' \right] \]

\[+ \frac{\partial W}{\partial z} (2G'' G'''' - GG^{iv} + 3G''^2) + \frac{\partial n \Phi}{\partial z} WGG^{iv} \]

\[+ \frac{\partial n \Phi}{\partial x} U(FG^{iv} + 2F' G'''' + F'' G'' - \frac{\partial W}{\partial z} (FG^{iv}) \]

\[+ F'' G'' + 2F' G'''' + \frac{U}{W} \frac{\partial W}{\partial x} (G'''' F' + 2F'' G'' + G' F''') \]

\[+ \frac{\partial n \Phi}{\partial z} U^2 \frac{\partial}{\partial x} (2F''^2 + 4\eta F'' F''') + 2 \frac{U}{W} \frac{\partial U}{\partial x} (F''^2 + F' F''') \]
\[
\mu_2 \left[ \frac{U}{W} \frac{\partial W}{\partial x} (2F'' G'' + G' F'') + \frac{U}{W} \frac{\partial U}{\partial z} (F' F''' + F''^2) \\
+ \frac{\partial \ln \phi}{\partial z} \frac{U^2}{W} (2 \eta F' F'' + F''^2) - \frac{\partial U}{\partial x} (2G''' F' + F'' G'') \\
+ \frac{\partial \ln \phi}{\partial x} UF'' G'' + 2 \frac{\partial \ln \phi}{\partial z} W(G''^2 + \eta G'' G''') + 2 \frac{\partial W}{\partial z} G''^2 \right] = 0
\]
\tag{9b}

The partial differential equations (9) will transform into a system of ordinary differential equations provided that the coefficients of \(F, G\) and their derivatives are made proportional. In this case, the following must hold:

\[
\phi^2 = c_1 \quad \tag{10a}
\]

\[
\frac{\partial U}{\partial x} = c_2 \quad \tag{10b}
\]

\[
\frac{\partial W}{\partial z} = c_3 \quad \tag{10c}
\]

\[
\frac{W}{U} \frac{\partial U}{\partial z} = c_4 \quad \tag{10d}
\]

\[
\frac{W}{U} \frac{\partial W}{\partial x} = c_5 \quad \tag{10e}
\]

\[
\frac{U}{W} \frac{\partial W}{\partial x} = c_6 \quad \tag{10f}
\]
\[ \frac{U}{W} \frac{\partial U}{\partial z} = c_7 \]  

(10g)

From (10b), (10c), we get

\[ U = c_2 x + a_1(x) \]  

(11a)

\[ W = c_2 z + a_2(x) \]  

(11b)

Also, equations (10d), (10g), (10e) and (10f) yield

\[ U = c_8 W \]  

(11c)

where \( c_8 = \sqrt{\frac{c_7}{c_4}} = \sqrt{\frac{c_6}{c_5}} \).

Therefore, \( a_2(x) = \frac{c_2}{c_8} x \)

and \( a_1(x) = c_3 c_8 z \)

Thus, equations (11) become

\[ U = c_2 x + c_3 c_8 z \]  

(12a)

\[ W = c_2 z + \frac{c_2}{c_8} x \]  

(12b)

The flows described by (12) represent flows near a stagnation point.

Using (10) in (9a), we obtain (for \( c_i = 1 \))
\[ c_2(F'^2 - F F'' - 1) + c_4(F' G' - 1) - c_3 F' G - F'''
- \mu_1 [c_2(2 F'' F''' - F F^iv + 3 F'^2) - c_3(F^iv G + F'' G'' + 2 F''' G')
+ c_4(F''' G' + 2 F'' G'' + F' G''') + 2 c_5(G'^2 + G' G'' )] \\
- \mu_2 [c_4(2 F'' G' + F' G''') + c_5(G' G''' + G'^2)]
- c_3(2 F''' G' + F'' G'') + 2 c_2 F'^2 = 0 \] (13a)

and

\[ c_3(G'^2 - G G'' - 1) + c_6(G' F' - 1) - c_2 G'' F - G'''
- \mu_1 [c_3(2 G' G''' - G G^iv + 3 G'^2) - c_2(G^iv F + G'' F'' + 2 G''' F')
+ c_6(G''' F' + 2 G'' F'' + G' F''') + 2 c_7(G'^2 + G' G''')] \\
- \mu_2 [c_6(2 G'' F' + G' F'') + c_7(F' F''' + F'^2)]
- c_2(2 G''' F' + G'' F'') + 2 c_3 F'^2 = 0 \] (13b)

The boundary conditions are:

At $\eta = 0$ : $F = 0$, $F' = 0$, $G = 0$, $G' = 0$

As $\eta \to \infty$ : $F' \to 1$, $G' \to 1$

Usually a perturbation technique is applied to the similarity functions, or they are approximated by $n^{th}$ degree polynomials before a numerical solution is carried out.

Values for the velocities can, then, be found from

\[ u(x, y, z) = (c_2 x + c_4 c_5 z)F(y) \] (14a)

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\[ w(x, y, z) = (c_3 z + \frac{c_2}{c_8} x) G'(y) \]  \hspace{1cm} (14b)

\[ \nu(y) = -c_2 F(y) - c_3 G(y) \]  \hspace{1cm} (14c)

### 3.4 Concluding Remarks

When \( \bar{\mu}_1 = \bar{\mu}_2 = 0 \), we have the purely viscous case which has been treated by Hansen and Herzig [37]. Equations (13) become

\[ c_2 (F'' - F' F'' - 1) + c_4 (F' G' - 1) - c_3 F'' G - F''' = 0 \]  \hspace{1cm} (15a)

\[ c_3 (G'' - G' G'' - 1) + c_5 (G' F' - 1) - c_2 G'' F - G''' = 0 \]  \hspace{1cm} (15b)

subject to the same boundary conditions.

By comparing equations (13) and (15), it is obvious that a salient feature of the viscoelastic fluid is the increased order of the governing ordinary differential equations. Equations (15) are of order three, whereas equations (13) are of order four. The presence of elasticity greatly alters the boundary layer equations and, hence, the flow from the purely viscous case.
Chapter IV

Two-Dimensional Thermal Boundary Layers For Second Order Fluids

4.1 Introduction

In this chapter we consider transfer of thermal energy in second order fluids, a subject for which both theory and experiments are in short supply. As we have seen in the last chapter, the fluids elasticity has some rather unexpected results on momentum transfer, and, hence, one might expect anomalies for heat transfer as well. Experiments have shown that drag reducing dilute polymer additives can also reduce heat transfer, but the effect is not in complete analogy to the momentum transfer phenomenon. A review of relevant work is given by Showalter [28].

For this problem the Laminar boundary layer equations for heat convection on a vertical plate are [28] (see also ch. 1):

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + \Theta + \frac{\partial^2 u}{\partial y^2}
\]

\[+ \mu_1 \left( \frac{\partial^3 u}{\partial y^2 \partial t} + u \frac{\partial^3 u}{\partial x \partial y^2} + v \frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y^2} \right)\]

\[+ (3 \mu_1 + 2 \mu_2) \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y}\]  

(16a)

\[
\frac{\partial \Theta}{\partial t} + u \frac{\partial \Theta}{\partial x} + v \frac{\partial \Theta}{\partial y} = \sigma \frac{\partial^2 \Theta}{\partial y^2}
\]

(16b)

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

(16c)
where \( \theta \) is a dimensionless temperature difference and \( \sigma \) is the coefficient of diffusivity.

The boundary conditions are

At \( y = 0 : u = v = 0, \ \theta = W \) (The wall temperature) \hspace{1cm} (17a)

As \( y \to \infty : u \to U, \ \theta \to 0 \). \hspace{1cm} (17b)

The solution of (16c) may be written in terms of a stream function \( \psi \) defined as follows:

\[
\begin{align*}
    u &= \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \\
    \text{(18)}
\end{align*}
\]

Employing (18) in equations (16) we obtain

\[
\begin{align*}
    \frac{\partial^2 \psi}{\partial y \partial t} + \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} &= \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} \\
    + \theta + \frac{\partial^3 \psi}{\partial y^3} + \mu_1 \left[ \frac{\partial^4 \psi}{\partial y^4 \partial t} + \frac{\partial \psi}{\partial y} \frac{\partial^4 \psi}{\partial x \partial y^3} \\
    - \frac{\partial \psi}{\partial x} \frac{\partial^4 \psi}{\partial y^4} + \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial^3 \psi}{\partial y^3} \right] + (3 \mu_1 + 2 \mu_2) \frac{\partial^2 \psi}{\partial y^2} \frac{\partial^3 \psi}{\partial x \partial y^2} \\
    \text{(19a)}
\end{align*}
\]

\[
\begin{align*}
    \frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} &= \sigma \frac{\partial^2 \theta}{\partial y^2} \\
    \text{(19b)}
\end{align*}
\]

### 4.1.1 The Similarity Transformation

Now we introduce the similarity transformations

\[
\eta = y \phi(x, t) \hspace{1cm} (20a)
\]
\[ \theta = W(x, t) K(\eta) \]  
\[ \psi = H(x, t) F(\eta) \]  

(20c)  
(20b)

The partial derivatives in equations (19) transform according to:

\[
\frac{\partial \psi}{\partial x} = \frac{\partial H}{\partial x} F + H \frac{\partial F}{\partial x} = \frac{\partial H}{\partial x} F + H \left( \frac{\partial F}{\partial \eta} \frac{\partial \eta}{\partial x} \right) 
\]

\[
= \frac{\partial H}{\partial x} F + \frac{\partial \eta \phi}{\partial x} H \eta F'
\]

\[
\frac{\partial \psi}{\partial y} = H \frac{\partial F}{\partial \eta} \frac{\partial \eta}{\partial y} = \phi HF'
\]

\[
\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y} (\phi HF') = \phi^2 HF''
\]

\[
\frac{\partial^4 \psi}{\partial y^4} = \frac{\partial}{\partial y} (\phi^2 HF'') = \phi^3 HF'''
\]

\[
\frac{\partial^4 \psi}{\partial y^2} = \frac{\partial}{\partial y} (\phi^3 HF''') = \phi^4 HF^iv
\]

\[
\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial x} (\phi HF') = \left( \frac{\partial \phi}{\partial x} H + \phi \frac{\partial H}{\partial x} \right) F' + \frac{\partial \phi}{\partial x} H \eta F''
\]

\[
\frac{\partial^4 \psi}{\partial x \partial y^3} = \frac{\partial}{\partial x} (\phi^3 HF''')
\]
\[
\frac{\partial^4 \Psi}{\partial x \partial y^3} = \left( 3 \phi^2 \frac{\partial \phi}{\partial x} H + \phi^3 \frac{\partial H}{\partial x} \right) F''' + \phi^2 \frac{\partial \phi}{\partial x} H \eta F^{iv}
\]

\[
\frac{\partial^3 \Psi}{\partial x \partial y^2} = \left( 2 \phi \frac{\partial \phi}{\partial x} H + \phi^2 \frac{\partial H}{\partial x} \right) F'' + \phi \frac{\partial \phi}{\partial x} H \eta F'''
\]

\[
\frac{\partial^2 \Psi}{\partial y \partial t} = \frac{\partial}{\partial t} (\phi H F') = \left( \frac{\partial \phi}{\partial t} H + \phi \frac{\partial H}{\partial t} \right) F' + H \frac{\partial \phi}{\partial t} \eta F''
\]

\[
\frac{\partial^4 \Psi}{\partial t^4} = \left( 3 \phi \frac{\partial \phi}{\partial t} H + \phi^3 \frac{\partial H}{\partial t} \right) F'''' + \phi^2 H \frac{\partial \phi}{\partial t} \eta F^{iv}
\]

\[
\frac{\partial \Theta}{\partial x} = \frac{\partial W}{\partial x} K + \frac{\partial \ln \Phi}{\partial x} W \eta K'
\]

\[
\frac{\partial \Theta}{\partial t} = \frac{\partial W}{\partial t} K + W \frac{\partial \ln \Phi}{\partial t} \eta K'
\]

\[
\frac{\partial \Theta}{\partial y} = W \frac{\partial K}{\partial \eta} \frac{\partial \eta}{\partial y} = \phi W K'
\]

\[
\frac{\partial^2 \Theta}{\partial y^2} = \frac{\partial}{\partial y} (\phi W K') = \phi^2 W K''
\]

Substituting the above into equations (19), we have
\[
\left( \frac{\partial \phi}{\partial t} H + \phi \frac{\partial H}{\partial t} \right) F' + \frac{\partial \phi}{\partial t} H \eta F'' + \phi HF' \left[ \left( \frac{\partial \phi}{\partial x} H + \phi \frac{\partial H}{\partial x} \right) F' + \frac{\partial \phi}{\partial x} H \eta F'' \right] - \phi^2 HF'' \left[ \frac{\partial H}{\partial x} F + \frac{\partial \ln \phi}{\partial x} H \eta F' \right] + W K + \phi^3 HF'''' + \bar{\mu}_1 \left\{ \left( 3 \phi \frac{\partial \phi}{\partial t} H + \phi^3 \frac{\partial H}{\partial t} \right) F''' \right. \\
+ \phi^2 \frac{\partial \phi}{\partial t} \eta F''' + \phi F F' \left[ \left( 3 \phi^2 \frac{\partial \phi}{\partial x} H + \phi^3 \frac{\partial H}{\partial x} \right) F''' \right. \\
+ \phi^2 \frac{\partial \phi}{\partial x} H \eta F''' \right. \left. - \phi^4 H F''' \left[ \frac{\partial H}{\partial x} F + \frac{\partial \ln \phi}{\partial x} H \eta F' \right] \right. \\
+ \phi^3 HF'''' \left[ \left( \frac{\partial \phi}{\partial x} H + \phi \frac{\partial H}{\partial x} \right) F' + \frac{\partial \phi}{\partial x} H \eta F'' \right] \right. \\
+ (3 \bar{\mu}_1 + 2 \bar{\mu}_2) \phi^2 HF'' \left. \left[ \left( 2 \phi \frac{\partial \phi}{\partial x} H + \phi^2 \frac{\partial H}{\partial x} \right) F'' + \phi \frac{\partial \phi}{\partial x} H \eta F''' \right] \right. \\
\right.
\]

and

\[
\frac{\partial W}{\partial t} K + W \frac{\partial \ln \phi}{\partial t} \eta K' + \phi HF' \left[ \frac{\partial W}{\partial x} K + W \frac{\partial \ln \phi}{\partial x} \eta K' \right]
\]
\[-\phi W K' \left[ \frac{\partial H}{\partial x} F + \frac{\partial \ln \phi}{\partial x} H \eta F' \right] = \sigma \phi^2 w K'' \]

Upon simplification, we get

\[\frac{1}{\phi^3 H} \left( \frac{\partial \phi}{\partial t} H + \phi \frac{\partial H}{\partial t} \right) F' + \frac{1}{\phi^3} \frac{\partial \phi}{\partial t} \eta F'' \]

\[+ \left( \frac{H}{\phi^2} \frac{\partial \ln \phi}{\partial x} + \frac{1}{\phi} \frac{\partial H}{\partial x} \right) F'' - \frac{1}{\phi} \frac{\partial H}{\partial x} FF'' \]

\[= \frac{1}{\phi^3 H} \left( \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} \right) + \frac{1}{\phi^3} \frac{W K + F'''}{W} \]

\[+ \bar{\mu}_1 \left( \frac{3}{\phi} \frac{\partial \ln \phi}{\partial t} + \frac{\partial \ln H}{\partial t} \right) F''' + \frac{\partial \ln \phi}{\partial t} \eta F^{iv} \]

\[+ 2 \left( \phi \frac{\partial H}{\partial x} + 2 H \frac{\partial \phi}{\partial x} \right) F' F''' - \phi \frac{\partial H}{\partial x} FF^{iv} \]

\[+ (5 \bar{\mu}_1 + 2 \bar{\mu}_2) H \frac{\partial \phi}{\partial x} \eta F'' F''' \]

\[+ (3 \bar{\mu}_1 + 3 \bar{\mu}_2) \left( 2 H \frac{\partial \phi}{\partial x} + \phi \frac{\partial H}{\partial x} \right) F''^2 \]

(21a)

and,

\[\left( \frac{1}{\phi^2 W} \frac{\partial W}{\partial t} + \frac{H}{\phi W} \frac{\partial W}{\partial x} F' \right) K + \frac{1}{\phi^2} \frac{\partial \ln \phi}{\partial t} \eta K' - \frac{1}{\phi} \frac{\partial H}{\partial x} K' = \sigma K'' \]

(21b)

The above partial differential equations will become ordinary differential equations provided the following conditions hold.
\[ \phi = c_0 = 1 \quad \text{(chosen)} \]  \hfill (22a)

\[ \frac{\partial \ln H}{\partial H} = c_1 \]  \hfill (22b)

\[ \frac{\partial H}{\partial x} = c_2 \]  \hfill (22c)

\[ \frac{1}{H} \left( \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} \right) = c_3 \]  \hfill (22d)

\[ \frac{W}{H} = c_4 \]  \hfill (22e)

\[ \frac{\partial \ln W}{\partial t} = c_5 \]  \hfill (22f)

\[ H \frac{\partial \ln W}{\partial x} = c_6 \]  \hfill (22g)

4.2 \hspace{1cm} \textbf{The Steady Case} \hspace{1cm} \left( \frac{\partial}{\partial t} = 0 \right)

From (22c), we get

\[ H = c_2 x + c_7 \]  \hfill (23a)

Using (23a) in (22e), we have
\( W = c_4 (c_2 x + c_7) \)  \hspace{1cm} (23b)

and (22g) is satisfied with \( c_2 = c_6 \).

From (22d), we obtain

\[
U \frac{\partial U}{\partial x} = c_3 (c_2 x + c_7)
\]

which, upon integration, becomes

\[
\frac{U^2}{2} = c_3 \left( \frac{c_2}{2} x^2 + c_7 x + c_8 \right)
\]

or,

\[
U = \left[ 2 c_3 \left( \frac{c_2}{2} x^2 + c_7 x + c_8 \right) \right]^{1/2}
\]

We still have to worry about the restriction imposed on \( U \) by the boundary condition at \( \infty \), that is \( \frac{U}{\phi H} \) must be constant. Thus, we must have

\[
c_7 = c_8 = 0
\]

Therefore,

\[
H = c_2 x \quad \text{(23e)}
\]

\[
W = c_2 c_4 x \quad \text{(23d)}
\]

\[
U = \sqrt{c_2 c_3} x \quad \text{(23e)}
\]

and, consequently,

\[
\frac{u}{\phi H} \rightarrow \frac{U}{\phi H} = \sqrt{\frac{c_3}{c_2}}.
\]
Equations (21) become
\[ c_2 \left( F''^2 - F F'' \right) = c_3 + c_4 K + F'''' \quad (24a) \]
\[ + c_2 \mu_1 \left( 2 F' F'''' - F F''' \right) + c_2 \left( 3 \mu_1 + 2 \mu_2 \right) F''^2 \]

and
\[ c_2 (F' K - FF') = \sigma K'' \quad , \quad (24b) \]

subject to the boundary conditions:

At \( \eta = 0 \), \( y = 0 \)
\[ u = \phi HF' = 0 \implies F' = 0 \]
\[ v = -c_2 F = 0 \implies F = 0 \]
\[ \theta = WK = W - K = 1 \]

As \( \eta \to \infty \), \( y \to \infty \)
\[ u - U = \sqrt{c_2 c_3} x - F' - \sqrt{\frac{c_3}{c_2}} \]
\[ \theta = WK - 0 \implies K \to 0 \]

The above system is solved numerically, and the velocity and temperature distributions are obtained from the following (with \( \eta = y \)):
\[ \psi = c_2 x F(\eta) \quad (26a) \]
\[ u(x,y) = \frac{\partial \Psi}{\partial y} = c_2 x F'(\eta) \]  \hspace{1cm} (26b)

\[ v(x,y) = -\frac{\partial \Psi}{\partial x} = -c_2 F(\eta) \]  \hspace{1cm} (26c)

\[ \theta(x,y) = c_2 c_4 x K(\eta) \]  \hspace{1cm} (26d)

If \( \bar{\mu}_1 = \bar{\mu}_2 = 0 \) in equation (24), we obtain the purely viscous case

\[ c_2 (F'' - FF'') = c_3 + c_4 K + F'''' \]  \hspace{1cm} (27a)

This case is treated in Chapter 2.

As in the previous chapter the most obvious effect of the fluid's elasticity, at this point, is the increased order of the differential equations. For the viscous fluid the order is three. For the viscoelastic fluid the order is four.

Returning to the free stream velocity, it should be noted that the case \( U = 0 \) is also permissible. This is the case of free convection, where \( c_3 = 0 \) and equations (23a,b) are retained. Equations (24) are the same except that \( c_3 = 0 \) now.

Also, the boundary conditions are the same except as \( \eta \to \infty, F' = 0 \).

The velocity and temperature profiles are, then
\[ \psi = (c_2 x + c_7) F(y) \]  \hspace{1cm} (27b)

\[ u(x,y) = (c_2 x + c_7) F(y) \]  \hspace{1cm} (27c)

\[ v(x,y) = -c_2 F(y) \]  \hspace{1cm} (27d)

\[ \theta(x,y) = c_4 (c_2 x + c_7) K(y) \]  \hspace{1cm} (27e)

where \( \eta = y \).

4.3 Unsteady Free Convection

From (22b) we get

\[ \ln H = c_1 t + a_1(x) \]

or

\[ H = A_1(x) e^{c_1 t} \]

where \( A_1(x) = e^{a_1(x)} \) and \( c_1 < 0 \).

Employing that in (22c), we obtain

\[ A_1'(x)e^{c_1 t} = c_2 \]

\[ A_1'(x) = 0 \text{, and } c_2 = 0 \]

Therefore, \( A_1(x) = c_9 = \text{constant} \), and
\[ H = c_9 e^{c_i t} \]  \hspace{1cm} (28a)

Combining (28a) with (22e), we get

\[ W = c_4 c_9 e^{c_i t} \]  \hspace{1cm} (28b)

and Equation (22f) is satisfied if \( c_5 = c_1 \).

In Equation (22g), \( c_6 \) must be zero.

The similarity transformations (20) become

\[
\begin{align*}
\eta &= \gamma \\
\psi &= c_9 e^{c_i t} F(\eta) \\
\theta &= c_4 c_9 e^{c_i t} K(\eta)
\end{align*}
\]

Equations (21) become

\[
c_1 c_9 e^{c_i t} \frac{1}{c_9} F' + c_4 c_9 e^{c_i t} \frac{1}{c_9} K
\]

\[ = F'''' + \mu \left[ c_1 c_9 e^{c_i t} \frac{1}{c_9} \right] F''''
\]

and

\[
c_1 c_4 c_9 e^{c_i t} \frac{1}{c_4 c_9} K = \sigma K''
\]

or, upon simplification,
\[ cF' + c_4 K = (1 + c_1 \bar{\mu}_1) F''' \]  
\hspace{1cm} (30a)

\[ c_1 K = \sigma K'' \]  
\hspace{1cm} (30b)

with boundary conditions:

At \( \eta = 0 \), \( F' = 0 \), \( K = 1 \) \hspace{1cm} (31a)

At \( \eta \to \infty \), \( F' \to 0 \), \( K \to 0 \). \hspace{1cm} (31b)

The solution of (30b) is

\[ K = c_{10} \cos \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) + c_{11} \sin \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) . \]

From the first boundary condition, \( c_{10} = 1 \). Thus,

\[ K = \cos \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) + c_{11} \sin \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) . \]

The other boundary condition at \( \infty \) can not be satisfied, so it seems that this solution is valid only near the plate or within the boundary layer.

Equation (30a) becomes

\[ (1 + c_1 \bar{\mu}_1) F''' - c_1 F' = c_4 \cos \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) = c_4 c_{11} \sin \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) \]

or

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\[(1 + c_1 \bar{\mu}_1) G'' - c_1 G = c_4 \left[ \cos \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) + c_{11} \sin \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) \right] \quad (32)\] 

where \( G = F' \).

The homogeneous solution is

\[G_H = c_{12} \cos \left( \sqrt{\frac{-c_1}{1 + c_1 \bar{\mu}_1}} \eta \right) + c_{13} \sin \left( \sqrt{\frac{c_1}{1 + c_1 \bar{\mu}_1}} \eta \right)\]

A particular integral of (32) is obtained as follows:

\[G_p = c_{14} \cos \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) + c_{15} \sin \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right)\]

\[C_p' = -c_{14} \sqrt{\frac{-c_1}{\sigma}} \sin \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) + c_{15} \sqrt{\frac{-c_1}{\sigma}} \cos \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right)\]

\[G_p'' = \frac{-c_1}{\sigma} c_{14} \cos \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) - \frac{c_1}{\sigma} c_{15} \sin \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right).\]

Substituting into (32), we get

\[-(1 + c_1 \bar{\mu}_1) \frac{c_1}{\sigma} \left[ c_{14} \cos \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) + c_{15} \sin \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) \right] \]

\[-c_1 \left[ c_{14} \cos \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) + c_{15} \sin \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) \right] \]

\[= c_4 \left[ \cos \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) + c_{11} \sin \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) \right]\]
Comparing terms on both sides, we find that

\[(1 + c_1 \vec{\mu}_1) \frac{c_1}{\sigma} c_{14} + c_1 c_{14} = -c_4\]

and

\[(1 + c_1 \vec{\mu}_1) \frac{c_1}{\sigma} c_{15} + c_1 c_{15} = -c_4 c_{11}\]

or

\[c_{14} = \frac{-c_4 \sigma}{c_1 (1 + c_1 \vec{\mu}_1 + \sigma)}, \text{ and}\]

\[c_{15} = \frac{-c_4 c_{11} \sigma}{c_1 (1 + c_1 \vec{\mu}_1 + \sigma)}\]

Therefore,

\[G = c_{12} \cos \left( \sqrt{\frac{-c_1}{1 + c_1 \vec{\mu}_1}} \eta \right) + c_{13} \sin \left( \sqrt{\frac{-c_1}{1 + c_1 \vec{\mu}_1}} \eta \right)\]

\[-\frac{c_4 \sigma}{c_1 (1 + c_1 \vec{\mu}_1 + \sigma)} \left[ \cos \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) + c_{11} \sin \left( \sqrt{\frac{-c_1}{\sigma}} \eta \right) \right]\]

At \(\eta = 0\), \(G = 0\), so

\[c_{12} = \frac{c_4 \sigma}{c_1 (1 + c_1 \vec{\mu}_1 + \sigma)}\]

Again, the condition at \(\infty\) cannot be satisfied, so the constants \(c_{11}\) and \(c_{13}\) will still be unknown.

Now,
\[ F' = \frac{c_4 \sigma}{c_1(1 + c_1 \mu_1 + \sigma)} \left[ \cos \left( \sqrt{\frac{-c_1}{1 + c_1 \mu_1}} \eta \right) \cos \left( \sqrt{\frac{-c_1}{c_1 \sigma}} \right) c_{11} \sin \left( \sqrt{\frac{-c_1}{c_1 \sigma}} \eta \right) \right] + c_{13} \sin \left( \sqrt{\frac{-c_1}{1 + c_1 \mu_1}} \eta \right) \]

Therefore,

\[ F = \frac{-c_4 \sigma}{c_1(1 + c_1 \mu_1 + \sigma)} \left[ \left( \frac{-c_1}{1 + c_1 \mu_1} \right)^{\frac{1}{2}} \sin \left( \sqrt{\frac{-c_1}{1 + c_1 \mu_1}} \eta \right) - \left( \frac{-c_1}{c_1 \sigma} \right)^{\frac{1}{2}} \sin \left( \sqrt{\frac{-c_1}{c_1 \sigma}} \eta \right) \right] + \left( \frac{-c_1}{c_1 \sigma} \right)^{\frac{1}{2}} \cos \left( \sqrt{\frac{-c_1}{c_1 \sigma}} \eta \right) + c_{13} \left( \frac{-c_1}{1 + c_1 \mu_1} \right)^{\frac{1}{2}} \cos \left( \sqrt{\frac{-c_1}{1 + c_1 \mu_1}} \eta \right) \]

The velocities and temperature are obtained from

\[ u = \left[ \frac{c_4 \sigma}{c_1(1 + c_1 \mu_1 + \sigma)} \left[ \cos \left( \sqrt{\frac{-c_1}{1 + c_1 \mu_1}} \eta \right) - \cos \left( \sqrt{\frac{-c_1}{c_1 \sigma}} \eta \right) \right] - c_{11} \sin \left( \sqrt{\frac{-c_1}{c_1 \sigma}} \eta \right) \right] + c_{13} \sin \left( \sqrt{\frac{-c_1}{1 + c_1 \mu_1}} \eta \right) \right] \right] \right] c_{9} e^{c_{11} t} \]

(32a)

\[ v = 0 \]

(32b)

\[ \theta = c_4 c_4 \left[ \cos \left( \sqrt{\frac{-c_1}{c_1 \sigma}} \eta \right) + c_{11} \sin \left( \sqrt{\frac{-c_1}{c_1 \sigma}} \eta \right) \right] e^{c_{11} t} \]

(32c)

The following observations are in order:
First, (32) represents an asymptotic solution that is valid at large distances away from the leading edge where no changes in the $x$-direction are taking place.

Secondly, this solution is not uniformly valid everywhere, it is only valid near the plate for small $y$.

Finally, since $c_1 < 0$ in (32), it is obvious that as $t \to \infty$, $u \to 0$ and $\theta \to 0$ which is to be expected. For, as $t \to \infty$, the temperature of the fluid and the temperature of the wall would become equal ($\theta = 0$). The temperature difference will disappear and, consequently, the buoyancy forces that have been the only cause of the fluid's motion will also disappear, hence $u = 0$ and $v = 0$.

4.4 Steady Two-Dimensional Flow Over A Flat Plate

From (22c), we get

$$H = c_2 x + c_{14}$$

(22d) yields,

$$U \frac{dU}{dx} = c_2 c_3 x + c_{14}$$

or \[ \frac{dU^2}{dx} = 2 c_2 c_3 x + c_{14} \]

Upon integration, we have

$$U^2 = c_2 c_3 x^2 + c_{14} x + c_{15}$$
Then, \[ U = (c_2c_3x^2 + c_{14}x + c_{15})^{\frac{1}{2}} \]

But the boundary condition at infinity implies that

\[ F' \to \frac{U}{\phi H} = \frac{(c_2c_3x^2 + c_{14}x + c_{15})^{\frac{1}{2}}}{c_2x + c_{14}} \]

\[ \therefore c_{14} = c_{15} = 0 \]

Hence, \[ H = c_2x \quad (33a) \]

\[ U = \sqrt{c_2c_3}x \quad (33b) \]

The equation of motion is

\[ \frac{1}{c_2} F'''' - F^2 + FF'' + \bar{\mu}_1 (QF'F'''' - FF'''' + 3F''^2) \]

\[ + 3\bar{\mu}_2F''^2 + \frac{c_3}{c_2} = 0 \quad (34) \]

subject to the boundary conditions

At \( \eta = 0 \), \( F' = F = 0 \quad (34a) \)

At \( \eta \to \infty \), \( F' \to \sqrt{\frac{c_3}{c_2}}. \quad (34b) \)
\[ \begin{align*}
\psi &= c_2 x F(\eta) \\
u &= c_2 x F'(\eta) \\
v &= -c_2 F(\eta)
\end{align*} \]

(35)

The flow described by (33b) corresponds to flow near a stagnation point represented in Figure A.

\[ \text{Fig. A} \]
For $\bar{\mu}_1 = \bar{\mu}_2 = 0$, we obtain
\[
\frac{1}{c^2} F''' - F'^2 + F F'' + \frac{c_3}{c_2} = 0
\] (36)

This case has been studied in some detail ([5], [23]). Again, we see that the equation for the viscoelastic case (34) is one order higher than the purely viscous one (36). It has been reported that ([30], [31]), for viscoelastic fluids whose constitutive equation is very similar to the one we used in our analysis, the effect of the fluid's elasticity is that it increases the velocity of the fluid in the boundary layer and it also increases the stress on the solid boundary. This is true for both two-dimensional and three-dimensional flows (see [28]).
References


<table>
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<tr>
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<tbody>
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