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Statistical analysis of familial correlations.

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STATISTICAL ANALYSIS OF FAMILIAL CORRELATIONS

by

Ijaz Ul Hassan Mian

A Dissertation
submitted to the
Faculty of Graduate Studies and Research
through the Department of Mathematics and Statistics
in partial Fulfillment for the Degree of
Doctor of Philosophy
at the University of Windsor

Windsor, Ontario, Canada

1989
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In the name of God Almighty,
the compassionate, the merciful.

To the patience and love of
my parents, wife and children.
ABSTRACT

Inferences for familial (intraclass and interclass) correlations are considered for unbalanced familial data from multivariate normal populations. The most commonly used familial correlation is that between siblings (without regard to gender), called intraclass or sib-sib correlation coefficient. The expressions for the large sample biases and variances of several point estimators of the intraclass correlation are derived and the sampling properties of these estimators are compared for a wide variety of unbalanced designs. It is recommended that the Karlin's individual estimator should be used for the small number of groups with a severe degree of unbalancedness. However, for a large number of groups, the Karlin's empirical estimator is recommended provided that the true value of intraclass correlation is less than or equal to 0.5. Several procedures for testing that the intraclass correlation is equal to a specified value are derived and compared by extensive Monte Carlo studies. The Neyman's C(q0) (or partial score) and modified F-ANOVA procedures are shown to be consistently more powerful than the other procedures for the said hypotheses.

Estimation procedures, based on the maximum likelihood and the ANOVA methods, for intraclass correlations in multiple samples are discussed. In order to test the homogeneity of the intraclass correlations in multiple samples, several procedures are derived and compared in
terms of their empirical powers. The use of a test based on Fisher's variance stabilizing transformation is recommended for small values of the common intraclass correlation and of the Neyman's C(α) test for moderate values of the common intraclass correlation is recommended.

The maximum likelihood estimation of sibling correlations (brother-brother, sister-sister, and brother-sister correlations) is considered next and it is shown that the estimates of the parameters can be obtained by numerical maximization of a function of fewer parameters.

The expressions for the large sample variances and covariances of the estimators are derived and the procedures to test the significance of these correlations are discussed. The procedures are illustrated by using a published arterial blood pressures dataset from the literature.

Using a linear model approach, procedures to find the maximum likelihood estimates of five familial correlations (mother-brother, mother-sister, brother-brother, sister-sister and brother-sister correlations) and other parameters are developed. The expressions for the asymptotic variances and covariances of the estimators are derived. Procedures for testing the significance of the above familial correlations are also presented and the methodologies are illustrated on the previously mentioned epidemiological data sets.
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CHAPTER 1
INTRODUCTION

1.1 Familial Correlations

The familial correlations (parent-sib and sib-sib) are commonly used to measure the degree of resemblance among family members with respect to clinically important characteristics such as blood pressures and cholesterol level. They provide useful information about the manner in which the characteristics are inherited. In terms of expressing the resemblance between characters which have a different degree of variability, the concept of correlations was introduced by Galton (1885) more than a century ago. This measure of resemblance was called "co-relation" and the first co-relations were those calculated between several anthropometric measures. It was Pearson (1896) who gave the mathematical formulation to this concept and refined this measure to modern correlation coefficient. Both Galton and Pearson considered this measure as a special case of linear regression and the one best known example (see Galton, 1885) of this is regressing the heights of sons on the height of their father. Let X and Y be any two random variables defined on the same probability space. The correlation coefficient of random variables X and Y, denoted by \( \rho_{xy} \), is defined as

\[
\rho_{xy} = \text{Corr}(X,Y) = \frac{E(X - E(X))(Y - E(Y))}{\sqrt{[E(X - E(X))^2 E(Y - E(Y))^2]}}.
\]
These measures are so defined that they are independent of the scale of measurement and linear transformation of random variables, and always satisfy $-1 \leq \rho_{xy} \leq 1$. The correlations have since been used extensively in several areas of research. When these correlations are used in the context of family studies, they are called "familial correlations". In genetic epidemiology, familial correlations are commonly used to measure the heritability of economically important traits in animal and plant breeding experiments. Heritability is a term used by biologists to characterize the resemblance of related individuals, measured by the proportional contribution of genetic factors to the variability of traits. Although the discussion regarding these correlations is used here in a genetic context, the use of these correlations is not limited to assessing the heritability of traits. A comprehensive review of the historical development of the concept of familial correlations is given by Smith (1980a, b) and examples of their uses can be found in Higgins and Keller (1975), Martarell et al. (1978), Roberts et al. (1978), and others.

Let us focus our attention on inference procedures regarding a random variable that can be measured on the elements of a subdivided population. Suppose that the population is stratified such that its elements can be described as belonging to a particular class within a group. In the analysis of familial data, the groups are human
families, the classes are father, mother, sons (brothers) and daughters (sisters), and the random variable can be some quantitative characteristic such as arterial blood pressures. Basically there are two types of familial correlations, called intraclass and interclass correlations. The "intraclass correlation" is a measure of the degree of resemblance between any two elements of a particular class (such as siblings) within each group (family). On the other hand, the "interclass correlation" measures the degree to which any two elements within the same group (family) but from two different classes are alike.

The simplest and most commonly used of these familial correlations is that between siblings (without regard to gender) and is called intraclass (sib-sib) correlation coefficient. A general review of the inference procedures for intraclass correlation is given by Donner (1986). As pointed out by Fisher (1925), "Whenever observations cluster in sets, such that there is interdependence within sets but independence among sets, the intraclass correlation provides a useful measure of the degree to which observations aggregate in sets". Besides its uses in genetics and epidemiology, the intraclass correlation has been used extensively in several other fields of research. In psychology it plays a fundamental role in assessing the reliability of several judges and raters in diverse situations such as criminal traits. In medical research, it may be used in the sensitivity analysis of the effectiveness
of different experimental treatments (Bradley and Schumann, 1967). Another application of the intraclass correlation is in the theory of sampling surveys where it is used to measure the degree of homogeneity of clusters in stratified sampling and it thus plays an important role in designing cost-efficient surveys.

Unlike intraclass correlation, interclass correlation is used mostly in genetics where it plays an important role in estimating the degree of resemblance, with respect to quantitative characters, between a single parent and his/her offspring, and the resemblance of a brother to his sister.

Clearly, because of the wide applications of familial correlations in several areas of research, there has been sufficient motivation for statisticians to develop time and cost efficient techniques for estimating and testing these correlations. However, the familial correlations defining the familial resemblance cannot be estimated from the outcome of controlled designs and varying family sizes prohibits deriving the exact distributions of many of the estimators of familial correlations.

1.2 Outline of Chapters

Chapters 2, 3 and 4 are devoted to the intraclass correlation. In chapter 2, the expressions for the large sample biases and variances of several point estimators are derived. Since the distributions of estimators for intraclass correlation are very much dependent on the
variability in group (family) sizes, the sampling properties of these estimators are investigated in finite samples for a wide variety of unbalanced designs.

The procedures for testing that the intraclass correlation is equal to a specified value have been ignored in the past. However several test procedures, cited in the literature on variance components, can be modified to test hypotheses regarding intraclass correlation. In Chapter 3, several new test procedures for intraclass correlation are derived and compared with known procedures. Since it is difficult to estimate the required sample size that ensure the applicability of large sample theory, a Monte Carlo study was conducted to compare these procedures in terms of their empirical powers. This is the first time that a simulation study was conducted to compare the significance testing procedures for intraclass correlation. Donner and Wells (1986) and Keen (1987) report the results of Monte Carlo simulations for a number of confidence interval estimators. Since procedures of hypotheses testing are closely related to the procedures of confidence intervals, the results of this simulation study will be compared with the results of Donner and Wells (1986) and Keen (1987).

In family studies, the data on siblings may be available or be collected from several multivariate normal populations. Such problems arise in practice when information is collected under different experimental or observational conditions. For example, Hennekens et al.
(1980) investigated the familial aggregation of cholesterol level among children of men with and without a history of myocardial infarction. Chapter 4 deals with the estimation of intraclass correlations and testing the homogeneity of intraclass correlations in multiple samples. Several procedures for testing the homogeneity of intraclass correlations are derived and compared by an extensive Monte Carlo study for a family size distribution that is likely to occur in practice. The use of $C(o)0$ test of Neyman (1959) is recommended for such hypotheses, especially when data on a large number of families are available. Otherwise, a test based on variance stabilizing transformation due to Fisher (1925) is recommended.

The results of Chapters 2, 3 and 4 deal with the situations when the siblings of both sexes are assumed to be similar with respect to the character under consideration. This assumption will often be unrealistic, especially to those epidemiologists and animal breeders who may be interested in measuring the effect of sex differences on the reported values of sibling correlations. The maximum likelihood estimation of three sibling (brother-brother, sister-sister, and brother-sister) correlations and other parameters in the case of homogeneous and non-homogeneous populations of two natural groups (brothers and sisters) of siblings is considered in Chapter 5. Under the assumption that the sibling scores within each family follow a multivariate normal distribution, it is shown that the
maximum likelihood estimates of sibling (intraclass and interclass) correlations and other parameters can be obtained numerically, by maximizing a function of fewer parameters. The expressions for the asymptotic variances and covariances of the maximum likelihood estimators, for both homogeneous and non-homogeneous case, are also derived. The procedures for testing the significance of these correlations are also discussed and illustrated by using the published arterial blood pressure dataset of Miall and Oldham (1955).

The linear regression approach to the problem of estimating familial correlations was introduced by Kempthorne and Tandon (1953) and has been used by Mak and Ng (1981), Muñoz et al. (1986), and Shoukri and Ward (1989) to find the maximum likelihood estimates of familial correlations when offspring within a family are not divided into classes of brothers and sisters. Chapter 6 deals with the maximum likelihood estimation of five familial (mother-brother, mother-sister, brother-brother, sister-sister and brother-sister) correlations for unbalanced data from multivariate normal populations. The linear regression model of Kempthorne and Tandon (1953) is generalized to accommodate the sex differences in siblings. The expressions for asymptotic variances and covariances of the estimators are derived.

Finally Chapter 7 concludes the dissertation with a summary of major findings and recommendations for future
research.

The remainder of this chapter will be devoted, for the sake of completeness, to some of the well known results on convergence of sequences of random variables and the large sample distribution theory of the maximum likelihood estimators.

1.3 Convergence of Sequences of Random Variables

The approximation of a given random variable by another random variable and a given distribution function by another distribution function are of central importance in many statistical applications. Here, several modes of convergence and their properties are presented. Theorems on convergence are presented without proofs which can be found, for example, in Serfling (1980, Ch. 1).

Definition 1.1. Convergence in Probability

Let \( X_1, X_2, \ldots \) and \( X \) denote random variables on a probability space \( \Omega \). The sequence of random variables \( X_n, n = 1, 2, \ldots \), converges in probability to \( X \) if

\[
\lim_{n \to \infty} P[|X_n - X| < \varepsilon] = 1, \text{ for each } \varepsilon > 0.
\]

This is written \( X_n \xrightarrow{P} X, n \to \infty \).

Definition 1.2. Almost Sure Convergence

Consider the random variables \( X_1, X_2, \ldots \) and \( X \) on a probability space \( \Omega \). The sequence of random variables \( X_n, n = 1, 2, \ldots \), converges almost surely (or strongly, with probability 1, etc.) to \( X \) if

\[
\lim_{n \to \infty} X_n = X, \text{ almost surely.}
\]
\[ \lim_{n \to \infty} P\left[ |X_m - X| < \varepsilon, \text{ all } m \geq n \right] = 1, \text{ for each } \varepsilon > 0. \]

This is written \( X_n \xrightarrow{\text{wp}} X, \ n \to \infty. \)

**Definition 1.3. Convergence in Distribution**

Consider the distribution functions \( F_1(.), F_2(.), \ldots, \) and \( F(.). \) Let \( X_1, X_2, \ldots, \) and \( X \) denote random variables (not necessarily on a common probability space) having these distribution functions, respectively. The sequence of random variables \( X_n, n = 1, 2, \ldots, \) converges in distribution (or in law) to \( X \) if

\[ \lim_{n \to \infty} F_n(t) = F(t), \text{ at each continuity point } t \text{ of } F. \]

This is written \( X_n \xrightarrow{D} X. \)

Let us consider a specified function of the sample values and let \( T_n \) denote the evaluation of the function at the first \( n \) sample observations \( X_1, X_2, \ldots, X_n. \) Furthermore, let \( T_n, n = 1, 2, \ldots, \) be the sequence of relevant statistics. We have the following definitions:

**Definition 1.4. Consistent estimator**

A sequence of estimators \( T_n, n = 1, 2, \ldots, \) is a consistent estimator of parameter \( \theta \) if as \( n \to \infty, \)

\[ T_n \xrightarrow{P} \theta, \quad \text{(Weak consistency)} \]

\[ T_n \xrightarrow{\text{wp}} \theta, \quad \text{(Strong consistency)} \]

**Definition 1.5. \( \sqrt{n} \)-Consistent Estimator (Neyman, 1959)**

A sequence of estimators \( T_n, n = 1, 2, \ldots, \) is said to be \( \sqrt{n} \)-consistent estimator of the parameter \( \theta \) if, as \( n \to \infty, \)

the product
\[ \sqrt{n} \left[ |T_n - \theta| \right] \]

remains bounded in probability.

**Definition 1.6. Locally \( \sqrt{n} \)-Consistent Estimator**

(Neyman, 1959)

If there exists a constant \( A_1 \neq 0 \) such that, as \( n \to \infty \), the product

\[ \sqrt{n} \left[ |(T_{in} - \theta_1) - A_1(\xi - \xi_0)| \right] \]

remains bounded in probability for all \( \xi \) and \( \theta \), then we shall say that the estimator \( T_i \) is locally \( \sqrt{n} \)-consistent estimator of the parameter \( \theta \).

A common situation in mathematical statistics is that the statistic of interest is a slight modification of a random variable having a known limit distribution. The following theorem by Slutsky (1925) plays a fundamental role in establishing the distribution of such a statistic. Note that the random variables \( X_1, X_2, \ldots, X_n \) need not be independent to apply this theorem.

**Theorem 1.1. (Slutsky)** Let \( X_n, n = 1,2,\ldots \) and \( Y_n, n = 1,2,\ldots \), be two sequences of random variables such that \( X_n \xrightarrow{D} X \) and \( Y_n \xrightarrow{D} C \), where \( C \) is a finite constant.

Then

(i) \( X_n + Y_n \xrightarrow{D} X + C \)

(ii) \( X_n Y_n \xrightarrow{D} CX \)

(iii) \( \frac{X_n}{Y_n} \xrightarrow{D} \frac{X}{C} \) if \( C \neq 0 \).
Theorem 1.2. Convergence of Transformed Sequence

Let \( X_1, X_2, \ldots, X \) be random p-vectors defined on a probability space \( \Omega \) and let \( g(\cdot) \) be a vector valued function. Suppose \( g(\cdot) \) is continuous, then

\[
\begin{align*}
(1) \quad & X_n \xrightarrow{w} X \rightarrow g(X_n) \xrightarrow{w} g(X) \\
(2) \quad & X_n \xrightarrow{p} X \rightarrow g(X_n) \xrightarrow{p} g(X) \\
(3) \quad & X_n \xrightarrow{D} X \rightarrow g(X_n) \xrightarrow{D} g(X).
\end{align*}
\]

1.4 The Delta Method

The delta method is often used in the development of asymptotic inference for population parameters. It is an effective way of deriving expressions for the large sample biases, variances and covariances of the functions of random variables. This method is based on the Taylor series expansions of the functions about their means. The method is as follows:

Suppose \( X = [X_1, X_2, \ldots, X_m]^T \) is a random vector such that \( E(X) = \mu = [M_1, M_2, \ldots, M_m]^T \) and \( \text{Var}(X_i) \) is of order \( n^{-1} \) (i.e., \( \text{Var}(X_i) \xrightarrow{n \to \infty} 0 \)). Let \( g(X) = g(X_1, X_2, \ldots, X_m) \) be a function with finite mean and variance such that the first and second order partial derivatives of \( g(X) \) exist in the neighbourhood of \( X = \mu \). Furthermore, let

\[
D_i(g) = \left. \frac{\partial g(X)}{\partial X_i} \right|_{X = \mu}, \quad i = 1, 2, \ldots, m, \quad (1.1)
\]
\[
D_{ij}^2(g) = \left. \frac{\partial^2 g(x)}{\partial x_i \partial x_j} \right|_{x=M}, \quad i, j = 1, 2, \ldots, m. \quad (1.2)
\]

Expanding \(g(x)\) by Taylor series expansion about its mean \(g(M)\), dropping all terms of order higher than 2, and then taking the expectation gives

\[
\text{Bias}(g(x)) \approx \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} D_{ij}^2(g) \text{Cov}(X_i, X_j). \quad (1.3)
\]

The variance of \(g(X)\) is similarly obtained by expanding in a Taylor series and retaining only second order terms, which gives

\[
\text{Var}(g(X)) \approx \sum_{i=1}^{m} \sum_{j=1}^{m} D_{ii}(g) D_{jj}(g) \text{Cov}(X_i, X_j). \quad (1.4)
\]

If \(h(X) = h(X_1, X_2, \ldots, X_m)\) is another function whose first order derivatives exist in the neighbourhood of \(X = M\), then

\[
\text{Cov}(g(X), h(X)) \approx \sum_{i=1}^{m} \sum_{j=1}^{m} D_{ih}(g) D_{j}(h) \text{Cov}(X_i, X_j). \quad (1.5)
\]

**Remark.** If \(g(X) = g(X_1, X_2)\) (i.e. \(m = 2\)), the delta method gives

\[
\text{Bias}(g(X_1, X_2)) \approx \frac{1}{2} \left\{ D_{11}^2(g) \text{Var}(X_1) + 2 D_{12}^2(g) \text{Cov}(X_1, X_2) + D_{22}^2(g) \text{Var}(X_2) \right\} \quad (1.6)
\]

and

\[
\text{Var}(g(X_1, X_2)) \approx [D_1(g)]^2 \text{Var}(X_1) + 2 D_1(g) D_2(g) \text{Cov}(X_1, X_2) + [D_2(g)]^2 \text{Var}(X_2). \quad (1.7)
\]
1.5 Distributions of Functions of Normal Random Variables

Definition 1.7. Multivariate Normal Distribution

(Anderson, 1984)

A p-component random vector \( \mathbf{X} = [X_1, X_2, \ldots, X_p]^T \) is said to have a p-dimensional multivariate normal distribution with mean vector \( \mathbf{\mu} = \text{E}[\mathbf{X}] = [\mu_1, \mu_2, \ldots, \mu_p]^T \) and covariance matrix \( \mathbf{\Sigma} = \text{E}[\mathbf{X} - \mathbf{\mu}][\mathbf{X} - \mathbf{\mu}]^T \) if the probability density function of \( \mathbf{X} \) is

\[
    f(\mathbf{x}) = \frac{1}{(2\pi)^{p/2} |\mathbf{\Sigma}|^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \mathbf{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{x} - \mathbf{\mu}) \right\}.
\]

This is written \( \mathbf{X} \sim \mathcal{N}(\mathbf{\mu}, \mathbf{\Sigma}) \). Here covariance matrix \( \mathbf{\Sigma} \) is symmetric and is positive definite.

Theorem 1.3. Distribution of Linear Combination of Normal Random Variables

(Anderson, 1984)

Let p-component random vector \( \mathbf{X} \) be distributed as \( \mathcal{N}(\mathbf{\mu}, \mathbf{\Sigma}) \) and \( \mathbf{C} \) be a non-singular matrix of real coefficients. Then \( \mathbf{Y} = \mathbf{C} \mathbf{X} \) is distributed as \( \mathcal{N}(\mathbf{C} \mathbf{\mu}, \mathbf{C} \mathbf{\Sigma} \mathbf{C}^T) \).

Let the components of \( \mathbf{X} \) be divided into two groups such that \( \mathbf{X} = [\mathbf{X}_1; \mathbf{X}_2]^T \), where \( \mathbf{X}_1 = [X_1, X_2, \ldots, X_q]^T \) and \( \mathbf{X}_2 = [X_{q+1}, X_{q+2}, \ldots, X_p]^T \). Suppose the mean vector \( \mathbf{\mu} \) is also divided such that \( \mathbf{\mu} = [\mathbf{\mu}_1; \mathbf{\mu}_2]^T \), where \( \mathbf{\mu}_1 = [\mu_1, \mu_2, \ldots, \mu_q]^T \) and \( \mathbf{\mu}_2 = [\mu_{q+1}, \mu_{q+2}, \ldots, \mu_p]^T \).

Similarly partition the covariance matrix \( \mathbf{\Sigma} \) of \( \mathbf{X} \) into \( \mathbf{\Sigma}_{11}, \mathbf{\Sigma}_{12}, \mathbf{\Sigma}_{21}, \) and \( \mathbf{\Sigma}_{22} \) such that

\[
    \mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix},
\]

where
\[ v_{11} = E[x_1 - m_1][x_1 - m_1]^T \]
\[ v_{22} = E[x_2 - m_2][x_2 - m_2]^T \]
\[ v_{12} = v_{21}^T = E[x_1 - m_1][x_2 - m_2]^T. \]

We have following theorems regarding the marginal distribution of \( x_1 \) and conditional distribution of \( x_1 \) given \( x_2 = x_2 \):

**Theorem 1.4. The Marginal Distribution of \( x_1 \)**

Let \( p \)-component random vector \( X \) be distributed as 

\[ MN_p(M, \Sigma) \].

The marginal distribution of the random vector

\[ X_1 = [x_1, x_2, \ldots, x_q]^T \]

is \( MN_q(M_1, V_{11}) \).

**Theorem 1.5. The Conditional Distribution of \( x_1 \) given \( x_2 = x_2 \)**

Let \( p \)-component random vector \( X \) be distributed as 

\[ MN_p(M, \Sigma) \].

Then the conditional distribution of the random vector

\[ X_1 = [x_1, x_2, \ldots, x_q]^T \]

given \( x_2 = x_2 \) is 

\[ MN_q(M_1 + V_{12}^{-1} V_{22}^{-1} [x_2 - m_2], V_{11} - V_{12} V_{22}^{-1} V_{21}) \].

**Theorem 1.6. Condition for Independence**

Let \( p \)-component random vector \( X \) be distributed as 

\[ MN_p(M, \Sigma) \].

A necessary and sufficient condition that \( x_1 \) and \( x_2 \) are independently distributed is that \( v_{12} = v_{21}^T = 0 \).

**Theorem 1.7. Distribution of Quadratic Form**

If the \( p \)-component random vector \( X = [x_1, x_2, \ldots, x_p]^T \) is distributed as \( MN_p(M, \Sigma) \), then \( Y = [X - M]^T V^{-1} [X - M] \) is
distributed as $\chi^2_{(p)}$ (i.e., chi-squared distribution with $p$ degrees of freedom).

Another theorem which is often useful in finding the variances and covariances of functions of normally distributed random variables is as follows:

**Theorem 1.8.** If $Y$ and $Z$ are two normally distributed random variables with zero means then

$$\text{Cov}(Y^2, Z^2) = 2 \left[ \text{Cov}(Y, Z) \right]^2. \quad \text{(Searle, 1971)}$$

**Proof:** The moment generating function of the joint distribution of $Y$ and $Z$ (which is bivariate normal) is

$$\mathbb{E}_{Y,Z}(u, v) = \exp \left\{ \frac{1}{2} \left[ u^2 \sigma_y^2 + 2uv \rho_{yz} \sigma_y \sigma_z + v^2 \sigma_z^2 \right] \right\}.$$ 

Clearly,

$$\text{Cov}(Y, Z) = \mathbb{E}(YZ) = \frac{\partial^2 \mathbb{E}_{Y,Z}(u, v)}{\partial u \partial v} \bigg|_{u=0, v=0} = \rho_{yz} \sigma_y \sigma_z$$

$$\text{Var}(Y) = \mathbb{E}(Y^2) = \frac{\partial^2 \mathbb{E}_{Y,Z}(u, v)}{\partial u^2} \bigg|_{u=0, v=0} = \sigma_y^2$$

$$\text{Var}(Z) = \mathbb{E}(Z^2) = \frac{\partial^2 \mathbb{E}_{Y,Z}(u, v)}{\partial v^2} \bigg|_{u=0, v=0} = \sigma_z^2$$

$$\mathbb{E}(Y^2 Z^2) = \frac{\partial^4 \mathbb{E}_{Y,Z}(u, v)}{\partial u^2 \partial v^2} \bigg|_{u=0, v=0} = 2 \rho_{yz}^2 \sigma_y^2 \sigma_z^2 + \sigma_y^2 \sigma_z^2.$$ 

Thus

$$\text{Cov}(Y^2, Z^2) = \mathbb{E}(Y^2 Z^2) - \mathbb{E}(Y^2) \mathbb{E}(Z^2) = 2 \rho_{yz}^2 \sigma_y \sigma_z = 2 \left[ \text{Cov}(Y, Z) \right]^2.$$
The following theorem based on delta method is often used in the development of asymptotic inference for population parameters. Proof of the theorem can be found, for example, in Rao (1973). This theorem is very useful in determining the first non-trivial term in the Edgeworth expansion of the probability density function of a random variable (or random vector) about its mean.

**Theorem 1.9.** Let \( \mathbf{X}_n = [X_{n1}, X_{n2}, \ldots, X_{np}]^T \) be a random vector such that as \( n \to \infty \)

\[
\sqrt{n} \left[ \mathbf{X}_n - \mu \right] \xrightarrow{D} MN_p(0, \Sigma).
\]

Let \( g(\mathbf{X}) = [g_1(\mathbf{X}), g_2(\mathbf{X}), \ldots, g_q(\mathbf{X})]^T \) be a vector valued function of \( \mathbf{X} \), all of whose first order derivatives exist in the neighbourhood of \( \mathbf{X} = \mu \). Then as \( n \to \infty \),

\[
\sqrt{n} \left[ g(\mathbf{X}_n) - g(\mu) \right] \xrightarrow{D} MN_q(0, \nabla \nabla^T)
\]

where \( \nabla \) is \((q \times p)\) matrix of first order derivatives with \((i, j)\)th entry

\[
\nabla_{ij} = \left. \frac{\partial g_i(\mathbf{X})}{\partial x_j} \right|_{\mathbf{X} = \mu}, \quad i = 1, 2, \ldots, q; \quad j = 1, 2, \ldots, p.
\]

1.6 Asymptotic Theory of Maximum Likelihood

The large sample theory of maximum likelihood estimators can be found in Cramér (1946), Rao (1973), or Cox and Hinkley (1974). This section contains only a brief survey of the maximum likelihood theory. Further details can be found in the above mentioned references.
1.6.1 The Method

Let \( x_1, x_2, \ldots, x_n \) be a random sample from the density (or mass) function \( f(x; \theta) \), where \( \theta = [\theta_1, \theta_2, \ldots, \theta_p]^T \) is a parameter vector. The likelihood function of the sample is defined as

\[
L(\theta) = L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i; \theta).
\]

That is, the joint density (or mass) function of the observations is treated as a function of \( \theta \). The maximum likelihood estimator \( \hat{\theta} = [\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p]^T \) of the parameter vector \( \theta = [\theta_1, \theta_2, \ldots, \theta_p]^T \) is that value of \( \theta \) which maximizes the likelihood function \( L(\theta) \). In most statistical applications \( \hat{\theta} \) exists and is unique. Many likelihood functions satisfy regularity conditions (see, e.g., Cox and Hinkley, 1974, Sec. 9.4 for these conditions), so the maximum likelihood estimator \( \hat{\theta} \) may be obtained by solving the system of likelihood equations

\[
\left. \begin{align*}
\frac{\partial L(\theta)}{\partial \theta_i} & = 0, \quad (i = 1, 2, \ldots, p) \\
\hat{\theta} & = \hat{\theta}_i
\end{align*} \right|_{\theta = \theta_i}
\tag{1.8}
\]

and confirming that the solution \( \hat{\theta} \) indeed maximizes \( L(\theta) \).

Since \( L(\theta) \) and logarithm of \( L(\theta) \) have their maxima at the same value of \( \theta \), it is often more convenient for computations to work with logarithm of \( L(\theta) \).

The closed form solutions of the system of likelihood equations (1.8) do not always exist. If this is the case, the maximization of the (log) likelihood function or the
solution of the system of likelihood equations can be
achieved iteratively. Several optimization methods for this
purpose are available in literature and a detailed
discussion of this topic can be found, for example, in
Lawless (1982, Appendix F). The procedure by
Richards (1961) for maximizing a likelihood function of
several parameters is sometimes very helpful in finding the
maximum likelihood estimators of the parameters, especially
when a large number of parameters needs to be estimated.
The procedure is outlined below:

Denote the parameter vector \( \theta \) by \( [\hat{\theta}^{(1)}, \hat{\theta}^{(2)}]^T \), where
\( \hat{\theta}^{(1)} = [\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r]^T \) and \( \hat{\theta}^{(2)} = [\hat{\theta}_{r+1}, \hat{\theta}_{r+2}, \ldots, \hat{\theta}_p]^T \).

Let \( \hat{\theta}^{(2)}(\hat{\theta}^{(1)}) \) be the maximum likelihood estimator of \( \hat{\theta}^{(2)} \)
for a fixed value of \( \hat{\theta}^{(1)} \) and \( \hat{\theta}^{(1)} \) be the maximum likelihood estimator of \( \hat{\theta}^{(1)} \).

(i) Obtain \( \hat{\theta}^{(2)}(\hat{\theta}^{(1)}) = [\hat{\theta}_{r+1}, \hat{\theta}_{r+2}, \ldots, \hat{\theta}_p]^T \) as functions
of \( \hat{\theta}^{(1)} = [\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r]^T \) by solving

\[
\frac{\partial \log L(\theta)}{\partial \theta_i} = 0 \quad \text{or} \quad \frac{\partial \log L(\theta)}{\partial \theta_i} = 0, \quad (i = r+1, r+2, \ldots, p).
\]

(ii) Substitute \( \hat{\theta}^{(2)}(\hat{\theta}^{(1)}) \) for \( \hat{\theta}^{(2)} \) in the likelihood
function \( L(\theta) \) (or \( \log L(\theta) \)) to obtain a modified
likelihood function which is then be a function of
\( \hat{\theta}^{(1)} \) only.

(iii) Using this modified function from (ii), find the
maximum likelihood estimator \( \hat{\theta}^{(1)} = [\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r]^T \)
of \( \hat{\theta}^{(1)} = [\theta_1, \theta_2, \ldots, \theta_r]^T \) by an ordinary maximum
Replace \( \hat{\Theta}^{(1)} = [\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r]^T \) by 
\( \hat{\Theta}^{(1)} = [\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r]^T \) in \( \hat{\Theta}^{(2)}(\hat{\Theta}^{(1)}) \) from (i) to obtain the maximum likelihood estimate of \( \hat{\Theta}^{(2)} \).

Example 1.1. ML Estimation of Weibull Parameters

Consider the maximum likelihood estimation of the parameters of Weibull distribution. The probability density function of a random variable \( X \) having Weibull distribution is

\[
f(x) = \frac{\beta}{\alpha} \left[ \frac{x}{\alpha} \right]^{\beta-1} \exp\left[ -\left( \frac{x}{\alpha} \right)^{\beta} \right], \quad x \geq 0.
\]

Here \( \alpha > 0 \) and \( \beta > 0 \) are the shape and scale parameters of the distribution, respectively. The log-likelihood function of the sample from the above distribution can be written as

\[
l = -\sum_{i=1}^{n} \ln(x_i) + n \ln(\beta) + \beta \sum_{i=1}^{n} \ln(x_i) - \sum_{i=1}^{n} \frac{x_i}{\alpha}^\beta.
\]

Differentiating \( l \) with respect to \( \alpha \) and \( \beta \) and equating the derivatives to zero gives

\[
\frac{\beta}{\alpha} \left[ n - \frac{1}{\alpha^\beta} \sum_{i=1}^{n} (x_i)^\beta \right] = 0,
\]

\[
\frac{n}{\beta} + \frac{1}{\beta} \sum_{i=1}^{n} \ln\left( \frac{x_i}{\alpha} \right) - \frac{1}{\alpha^\beta} \sum_{i=1}^{n} (x_i)^\beta \ln(x_i) = 0.
\]

Thus from equation (1.9), the maximum likelihood estimator of \( \alpha \), as a function of \( \beta \), is given by

\[
\hat{\alpha}(\beta) = \left[ \frac{1}{n} \sum_{i=1}^{n} (x_i)^\beta \right]^{1/\beta}.
\]

Now substituting (1.11) in log-likelihood function \( l \) and
equation (1.10), we have

\[ \hat{L}^* = -\sum_{i=1}^{n} \ln(x_i) + n \ln(\beta) + \beta \sum_{i=1}^{n} \ln(\hat{\alpha}(\beta)x_i) - \sum_{i=1}^{n} \left(\frac{x_i}{\hat{\alpha}(\beta)}\right)^\beta \]

and

\[ \frac{1}{\hat{\beta}} + \frac{1}{n} \sum_{i=1}^{n} \ln(x_i) - \frac{1}{n} \sum_{i=1}^{n} \frac{(x_i)^\beta \ln(x_i)}{\hat{\beta}} = 0, \quad (1.12) \]

respectively, which are functions of a single parameter \( \beta \) only. The maximum likelihood estimate \( \hat{\beta} \) of \( \beta \) can be obtained by maximizing \( \hat{L}^* \) or by solving equation (1.12) analytically for \( \beta \). Once the estimate \( \hat{\beta} \) of \( \beta \) is found, the maximum likelihood estimate of \( \alpha \) can be obtained from (1.11) by substituting \( \hat{\beta} \) for \( \beta \).

### 1.6.2 Asymptotic Distribution of Maximum Likelihood Estimators

Under mild regularity conditions, the maximum likelihood estimators are strongly consistent, asymptotically normal and asymptotically efficient. That is, if

\[ \hat{\theta} = [\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p]^T \]

is a maximum likelihood estimator of

\[ \theta = [\theta_1, \theta_2, \ldots, \theta_p]^T \]

then

\[ \hat{\theta} \xrightarrow{\text{w}} \theta \quad (1.13) \]

and

\[ \hat{\theta} \xrightarrow{\text{D}} N_p(\theta, \Sigma^{-1}(\hat{\theta})), \quad (1.14) \]

where

\[ \Sigma(\bar{\theta}) = [-E \left\{ \frac{\partial^2 \ln(L)}{\partial \theta_i \partial \theta_j} \right\}]_{p \times p}, \quad i, j = 1, 2, \ldots, p \quad (1.15) \]
is called Fisher's (or expected) information matrix and is positive definite.

1.6.3 Invariance Property of the Maximum Likelihood Estimators

If we transform \( \Theta = [\theta_1, \theta_2, \ldots, \theta_p]^T \) to a new parameter vector \( \tilde{\Theta} = [\beta_1, \beta_2, \ldots, \beta_r]^T \) for \( 1 \leq r \leq p \), where \( \beta_i = g_i(\theta_1, \theta_2, \ldots, \theta_p) \) for some real function \( g(\cdot) \), then the maximum likelihood estimator of \( \tilde{\Theta} = [\beta_1, \beta_2, \ldots, \beta_r]^T \) is given by \( \hat{\tilde{\Theta}} = [\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_r]^T \) where \( \hat{\beta}_i = g_i(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p) \). Under mild regularity conditions on the transformation, the consistency and asymptotic normality properties survive under the transformation.

1.6.4 Asymptotically Optimal Tests for Composite Hypotheses

Let \( X_1, X_2, \ldots, X_n \) denote a simple random sample (SRS) from a probability distribution \( f(X; \Theta) \) which depends on a \( p \)-component parameter vector \( \Theta = [\theta_1, \theta_2, \ldots, \theta_p]^T \). This section deals with the problem of setting up asymptotically optimal tests for a single parameter \( \theta_1 \) (say) when other unknown parameters \( \Theta^{(2)} = [\theta_2, \theta_3, \ldots, \theta_p]^T \) exist but are not specified under the null hypothesis. That is, it is desired to test the null hypothesis \( H_0: \theta_1 = \theta_{10} \) where the values of \( \theta_2, \theta_3, \ldots, \theta_p \) are being unspecified. There are many possible ways to construct an optimal test in the above circumstances. The most commonly used procedures are based on the maximum likelihood theory.

Let \( \hat{\theta}_1 \) be the maximum likelihood estimator of \( \theta_1 \) and
$\text{AVC} (\hat{\theta}_1)$ be its asymptotic variance. It is known from (1.14) that the consistent estimator $\hat{\theta}_1$ is asymptotically normally distributed with mean $\theta_1$ and variance $\text{AVC} (\hat{\theta}_1)$; that is

$$\hat{\theta}_1 \xrightarrow{\text{D}} \text{ANC} (\theta_1, \text{AVC} (\hat{\theta}_1)).$$

If we denote the $(i,j)$th element of Fisher’s information matrix $E(\Theta)$ by $e_{ij}$, the asymptotic variance of $\hat{\theta}_1$, obtained by inverting the Fisher’s information (1.15), is given by

$$\text{AVC} (\hat{\theta}_1) = [e_{11} - E_{12} (\Theta) E_{22}^{-1} (\Theta) E_{21} (\Theta)]^{-1}, \quad (1.16)$$

where

$$E_{12} (\Theta) = E_{21} (\Theta) = [e_{12}, e_{13}, \ldots, e_{1p}] \quad (1.17)$$

and

$$E_{22} (\Theta) = \begin{bmatrix} e_{22} & e_{23} & \cdots & e_{2p} \\ e_{32} & e_{33} & \cdots & e_{3p} \\ \cdots & \cdots & \cdots & \cdots \\ e_{p2} & e_{p3} & \cdots & e_{pp} \end{bmatrix}. \quad (1.18)$$

Thus the limiting distribution of

$$\frac{\hat{\theta}_1 - \theta_1}{[\text{AVC} (\hat{\theta}_1)]^{1/2}}$$

is asymptotically normal with zero mean and unit variance. Replacing $\theta_1$ by the specified value $\theta_{10}$ and all the remaining parameters by their maximum likelihood estimators, we may have a consistent estimator $\text{AVC} (\hat{\theta}_1)$ for the asymptotic variance $\text{AVC} (\hat{\theta}_1)$. Therefore under $H_0: \theta_1 = \theta_{10}$, the test statistic
\[ Z_M = \frac{\hat{\theta}_1 - \theta_{10}}{[AV_0(\hat{\theta}_1^2)]^{1/2}} \rightarrow \text{N}(0,1). \] (1.19)

Hence, an asymptotic test of size \( \alpha \) for two sided alternative \( H_1: \theta_1 \neq \theta_{10} \) is to reject \( H_0: \theta_1 = \theta_{10} \) if
\[ |Z_M| > Z_{1-\alpha/2}, \]
where \( Z_{1-\alpha/2} \) is the 100(1-\( \alpha/2 \)) percentile point of the standard normal distribution.

Wald (1943) suggested that all the parameters, including \( \theta_1 \), in \( AV(\hat{\theta}_1^2) \) should be replaced by their maximum likelihood estimators, such that
\[ AV(\hat{\theta}_1^2) = AV(\hat{\theta}_1^2) \bigg| \theta = \hat{\theta} \]
is a consistent estimator of \( AV(\hat{\theta}_1^2) \). Thus, another test statistic which may be used to test the said hypothesis can be obtained by replacing \( AV_0(\hat{\theta}_1^2) \) in (1.19) by \( AV(\hat{\theta}_1^2) \).

A likelihood ratio (LR) test of the hypothesis \( H_0: \theta_1 = \theta_{10} \) can also be constructed. Let \( LC(\theta_{10}, \hat{\theta}_2, \ldots, \hat{\theta}_p) \) be the maximum value of the likelihood function under \( H_0 \) and \( LC(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p) \) be its maximum value under the alternate hypothesis. It is known that under \( H_0: \theta_1 = \theta_{10} \), the likelihood ratio test statistic
\[ \lambda = -2 \ln \left[ \frac{LC(\theta_{10}, \hat{\theta}_2, \ldots, \hat{\theta}_p)}{LC(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p)} \right] \] (1.20)
is asymptotically distributed as \( \chi^2_{(1)} \). Thus for two sided alternatives, the \( \lambda \) should be compared with the upper percentile points of the chi-square distribution with one
degree of freedom. It should be noted that the LR test should not be used for one sided alternatives because the problem of approximating its distribution for one sided alternatives still remains to be solved.

Several other tests for testing \( H_0 : \theta_1 = \theta_{10} \) can be derived using the C(\omega) theory of Neyman (1939) which is based on the asymptotic properties of the partial scores. Moran (1970) has proved that the locally optimal C(\omega) (or partial score) tests are asymptotically equivalent to the likelihood ratio test and the tests using the maximum likelihood estimators. The partial scores are defined as

\[
U_i(\theta) = \frac{\theta \ln(L(\theta))}{\partial \theta_i}, \quad i = 1, 2, \ldots, p
\]

and

\[
\Psi(\theta) = [U_1(\theta), U_2(\theta), \ldots, U_p(\theta)]^T
\]

is called the score vector. It is known that under mild regularity conditions

\[
\Psi(\theta) \xrightarrow{D} N_p(0, \Sigma(\theta)), \quad (1.23)
\]

where \( \Sigma(\theta) \) is Fisher's information matrix as defined by (1.15). Let us denote

\[
\Psi_{1}(\theta) = U_1(\theta) \bigg|_{\theta_1 = \theta_{10}}
\]

\[
\gamma_i(\theta) = U_{i+1}(\theta) \bigg|_{\theta_1 = \theta_{10}}, \quad i = 1, 2, \ldots, p-1
\]
\[ \chi(\Theta) = [\chi_1(\Theta), \chi_2(\Theta), \ldots, \chi_{p-1}(\Theta)]^T. \]

Following Moran (1970), the C(\omega) (or partial score) test statistic is based on

\[ T = T(\Theta) = \chi(\Theta) - [\chi(\Theta)]^T \beta, \tag{1.24} \]

where \( \beta = [\beta_1, \beta_2, \ldots, \beta_{p-1}]^T \) is a vector of regression coefficients of \( \chi(\Theta) \) on \( \chi(\Theta) \) and should be estimated from the second order partial derivatives. These regression coefficients can be obtained by minimizing

\[ E\left[ \chi(\Theta) - [\chi(\Theta)]^T \beta \right]^2. \tag{1.25} \]

Differentiating (1.25) with respect to \( \beta_i \) \( (i = 1, 2, \ldots, p-1) \), equating the derivatives to zero, we have a system of \( (p-1) \) equations given by

\[ -2 E\left[ \chi(\Theta) [\chi(\Theta) - [\chi(\Theta)]^T \hat{\beta}] \right] = 0, \]

where \( \Theta = [0, 0, \ldots, 0]^T \) is a \((p-1) \times 1\) vector of zeros. Equivalently, the above system of equations can be written as

\[ E_{21}(\Theta) - E_{22}(\Theta) \hat{\beta} = 0 \]

which gives

\[ \hat{\beta} = \hat{\beta}(\Theta) = E_{22}^{-1}(\Theta) E_{21}(\Theta), \tag{1.26} \]

where \( E_{21}(\Theta) \) and \( E_{22}(\Theta) \) are as defined by (1.17) and (1.18).
respectively. Thus substituting $\hat{E}$ for $E$ in (1.24), we have

$$T = \psi(\hat{E}) - [\psi(\hat{E})]^T E_{22}(\hat{E})^{-1} E_{21}(\hat{E}).$$

(1.27)

Since $T$ is a linear combination of asymptotically normally distributed random variables, by theorem 1.3, $T$ has a univariate normal distribution with mean $\text{E}(T) = 0$ and variance

$$\text{AVC}(T) = [\text{AVC}(\hat{\theta}_1)]^{-1} = e_{11} - E_{12}(\hat{E}) E^{-1}(\hat{E}) E_{21}(\hat{E}).$$

(1.28)

Notice that $T$ in (1.27) depends on the nuisance parameter vector $\theta_2 = [\theta_2, \theta_3, \ldots, \theta_p]^T$ and it is still not appropriate to use this quantity. Moran (1970) suggested that the nuisance parameters in (1.27) may be replaced by their $\sqrt{n}$-consistent estimators. Let $\hat{\theta}_2^{(2)} = [\hat{\theta}_2, \hat{\theta}_3, \ldots, \hat{\theta}_p]^T$ be a random vector of some $\sqrt{n}$-consistent estimators of parameter vector $\theta_2^{(2)} = [\theta_2, \theta_3, \ldots, \theta_p]^T$. Then replacing $\theta_1$ by $\theta_{10}$ and $\theta_2^{(2)}$ by $\hat{\theta}_2^{(2)}$ in (1.27), $T$ becomes

$$T^* = T(\theta_{10}, \hat{\theta}_2^{(2)}) = \psi(\theta_{10}, \hat{\theta}_2^{(2)})$$

$$- [\psi(\theta_{10}, \hat{\theta}_2^{(2)})]^T E_{22}(\theta_{10}, \hat{\theta}_2^{(2)})^{-1} E_{21}(\theta_{10}, \hat{\theta}_2^{(2)}).$$

(1.29)

Thus the $C(\omega)$ test statistic for testing $H_0: \theta_1 = \theta_{10}$ is given by

$$Z_c = T^* [\text{AVC}(\hat{\theta}_1)]^{1/2}$$

(1.30)

which is asymptotically normally distributed with zero mean.
and unit variance under \( H_0 \). Thus an approximate test of size \( \alpha \) for the two sided alternative \( H_1: \theta_1 \neq \theta_{10} \) is to reject \( H_0 \) if \( |Z_c| > Z_{1-\alpha/2} \), where \( Z_{1-\alpha/2} \) is the \( 100(1-\alpha/2) \) percentile point of the standard normal distribution.

It is known from (1.13) that the maximum likelihood estimators are \( \sqrt{n} \)-consistent. If the maximum likelihood estimator vector \( \hat{\theta}^{(2)} = [\hat{\theta}_2, \hat{\theta}_3, \ldots, \hat{\theta}_p]^T \) of parameter vector \( \theta^{(2)} = [\theta_2, \theta_3, \ldots, \theta_p]^T \) is used in (1.29), then \( T_M \) becomes

\[
T_M = T(\theta_{10}, \hat{\theta}^{(2)}) = \frac{A}{\theta_{10}, \hat{\theta}^{(2)}}
\]

(1.31)
due to the fact that \( \gamma_1(\hat{\theta}) = 0, \ i = 1, 2, \ldots, p-1 \). Thus another appropriate test statistic for testing the hypothesis under consideration is given by

\[
Z_{CM} = T_M \left[ AV_0(\hat{\theta}_1) \right]^{1/2}
\]

(1.32)

which has asymptotically a standard normal distribution.

The procedures for testing a general composite hypothesis \( H_0: \theta^{(1)} = \theta^{(1)}_0 \) where \( \theta^{(1)} = [\theta_1, \theta_2, \ldots, \theta_q]^T \) (\( q < p \)), when \( \theta^{(2)} = [\theta_{q+1}, \theta_{q+2}, \ldots, \theta_p]^T \) is unspecified under the null hypothesis can also be developed along the lines of this section.

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CHAPTER 2

SAMPLING PROPERTIES OF THE ESTIMATORS OF INTRACLASS CORRELATION

2.1 Introduction

The intraclass (sib-sib) correlation, \( \rho \), is a measure of the degree of similarity of individuals within groups (families). It is one of the oldest measures of association which have been proposed in the statistical literature. Pearson and Filon (1898) introduced the concept and proposed a pairwise estimator of the intraclass correlation, known as Pearson product-moment correlation, computed over all pairs of observations within groups.

Estimation of intraclass correlation is essentially the same problem as estimation of the variance components, as noted by Fisher (1925), and this gives an alternate approach to the problem. Fisher (1925) proposed an analysis of variance (ANOVA) estimator of the intraclass correlation based on balanced one-way random effects model. The sampling distribution of Fisher's ANOVA estimator is known when the design is balanced, that is, when the groups contain the same number of observations.

The sampling distributions of Pearson product-moment and Fisher's ANOVA estimators of the intraclass correlation are still unknown for unbalanced designs. Using point estimators of the variance components of one-way random effects model (2.1), Fieller and Smith (1951) proposed a generalization to the estimator of Fisher (1925) for
unbalanced designs and proposed a weighted estimator of the
intraclass correlation in which weights are proportional to
the group sizes. They also pointed out that the Pearson
product-moment estimator tends to give too much weight to
groups of large sizes. For example, a group of size 10
receives 45 times as much weight in the calculation of a
pairwise estimator as a group of size 2 although it clearly
does not provide 45 times as much information. Smith (1957)
provided a derivation of the asymptotic variance of the
weighted estimator of intraclass correlation which was
proposed by Fieller and Smith (1961). He also introduced
the use of weights other than group sizes and briefly
considered the problem of selecting the weights in order to
produce an estimator of the intraclass correlation with
smallest possible asymptotic variance.

Elston (1975) considered the maximum likelihood
estimation of intraclass correlation for balanced designs
and showed that the maximum likelihood estimator of
intraclass correlation is given by the Pearson product
moment estimator. The closed form solution for the maximum
likelihood estimator of intraclass correlation does not
exist when group sizes are unequal. Donner and Koval
(1980a) provide a simple algorithm to find the maximum
likelihood estimate of intraclass correlation for an
unbalanced design. The expression for the asymptotic
variance of the maximum likelihood estimator of intraclass
correlation is given by Donner and Koval (1980b).

Several other estimators of the intraclass correlation
are available in the literature and a general review of the inference procedures is given by Donner (1986). When group sizes are unequal, the sampling properties of many estimators of $\rho$ are still unknown and thus it is unclear which estimator of $\rho$ should be used. It is therefore especially important to have a reasonably efficient method of estimating the intraclass correlation and study the sampling properties of the estimator. It is also desirable that the method should not be unduly laborious.

The comparison among various estimators of the intraclass correlation has been done before by Donner and Koval (1980a) and Keen (1987) by Monte Carlo simulations. Yet, upon consulting the literature, one finds lacking a comprehensive comparison among several other competitors. The main goal of this chapter is to investigate the sampling properties of several point estimators of the intraclass correlation for unbalanced designs. Under the assumption of multivariate normality, the expressions for large sample biases and variances of several estimators of the intraclass correlation are derived. Because of the complexity of these expressions, the sampling distributions of the estimators are still mathematically intractable and thus the investigation is done by using a wide variety of unbalanced designs.

2.2 The Models

The most frequently adopted model, called one-way random effects model, for the estimation of intraclass correlation assumes that the score of the $j$th individual in the $i$th
group is represented by

\[ y_{ij} = \mu + a_i + \varepsilon_{ij}, \quad i = 1,2, \ldots, K; \quad j = 1,2, \ldots, n_i. \]  

(2.1)

where \( K \) is the total number of groups, \( \mu \) is the grand mean over all groups, the group effects \( \{a_i\} \) have mean zero and variance \( \sigma_a^2 \) and the residual effects \( \{\varepsilon_{ij}\} \) have mean zero and variance \( \sigma_e^2 \). Moreover, it is assumed that \( \{a_i\} \) and \( \{\varepsilon_{ij}\} \) are independently distributed. The ANOVA table related to model (2.1) is:

**ANOVA TABLE OF ONE-WAY RANDOM EFFECTS MODEL**

<table>
<thead>
<tr>
<th></th>
<th>D.F</th>
<th>S.S</th>
<th>M.S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Among Groups</td>
<td>K-1</td>
<td>( \sum_{i=1}^{K} n_i (y_{i1} - \bar{y})^2 )</td>
<td>( \frac{\text{SSA}}{K-1} )</td>
</tr>
<tr>
<td>Within Groups</td>
<td>N-K</td>
<td>( \sum_{i=1}^{K} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 )</td>
<td>( \frac{\text{SSW}}{N-K} )</td>
</tr>
<tr>
<td>Total</td>
<td>N-1</td>
<td>( \sum_{i=1}^{K} \sum_{j=1}^{n_i} (y_{ij} - \bar{y})^2 )</td>
<td>( \frac{\text{SST}}{N-K} )</td>
</tr>
</tbody>
</table>

(2.2)

where

\[
\bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}, \quad \bar{y} = \frac{1}{N} \sum_{i=1}^{K} n_i \bar{y}_i \quad \text{and} \quad N = \sum_{i=1}^{K} n_i.
\]

Under model (2.1), the variance of \( y_{ij} \), say \( \sigma^2 \), is given by

\[ \sigma^2 = \sigma_a^2 + \sigma_e^2 \] and the quantity

\[
\rho = \frac{\text{Cov}(y_{ij}, y_{im})}{\text{Var}(y_{ij})} = \frac{\sigma_a^2}{\sigma^2} = \frac{\sigma_a^2}{\sigma_a^2 + \sigma_e^2}, \quad j \neq m
\]

(2.3)

was termed intraclass correlation by Fisher (1958). It represents the proportion of the total variability in \( y_{ij} \)'s explained by the variation among groups. In many genetic
experiments $\sigma_a^2$ and $\sigma_e^2$ represent the genetic and environmental variation components respectively.

Since this model focuses attention on the variance components $\sigma_a^2$ and $\sigma_e^2$, it is also referred to as the components of variance model.

An alternate model, called multivariate normal model, assumes that the observations $y_{ij}$'s in each group are themselves a simple random sample (SRS) from a multivariate normal population. More specifically, if $n_i$ observations in $i$th group are denoted by $\mathbf{y}_i = [y_{i1}, y_{i2}, \ldots, y_{in_i}]^T$, then it is assumed that

$$\mathbf{y}_i \sim MN_{n_i}(\mu_i, \Sigma_i), \quad i = 1, 2, \ldots, K, \quad (2.4)$$

Where

$$\mu_i = E(\mathbf{y}_i) = \mu \mathbf{1}_{n_i} \cdot \mathbf{1},$$

$$\Sigma_i = \text{Cov} (\mathbf{y}_i) = [(1-\rho) \mathbf{I}_{n_i} + \rho \mathbf{J}_{n_i} \mathbf{1}_{n_i}] \sigma^2,$$

$I_p$ is an identity matrix of order $p$ and $J_{pxq}$ is a $(p \times q)$ matrix each element of which is 1. The parameters $\mu$ and $\sigma^2$ are the mean and variance of the individual observation $y_{ij}$, and $\rho$ is the intraclass correlation. Since

$$E(y_{ij} - \mu)(y_{im} - \mu) = \rho E(y_{ij} - \mu)^2,$$

the intraclass correlation $\rho$ is in a sense simple correlation coefficient between any two randomly chosen observations within the same group. Furthermore, it is assumed that the intraclass correlation $\rho$ is constant over all groups and observations from $i$th and $h$th groups are independent for $i \neq h = 1, 2, \ldots, K$, and in particular,
independent of the group sizes \( n_i, i = 1, 2, \ldots, k \). The model (2.4) is also called the common correlation model.

It should be noted that the one-way random effects model (2.1) assumes \( \sigma_a^2 \geq 0 \) and \( \sigma_e^2 \geq 0 \), and thus \( \rho \geq 0 \) (i.e., the intraclass correlation is constrained to be non-negative). On the other hand the determinant of the covariance matrix \( \Sigma_i \), under the multivariate normal model (2.4) is

\[
|\Sigma_i| = [1 + (n_i - 1) \rho] (1 - \rho)^{n_i} \sigma_e^{n_i}.
\] (2.5)

From this equation, the necessary and sufficient condition for \( \Sigma_i \) to be positive definite is

\[
\frac{-1}{n_i - 1} < \rho < 1,
\] (2.6)

which implies that the model (2.4) permits negative values of the intraclass correlation. It seems that the model (2.4) is more general than the model (2.1) because it permits \( \rho < 0 \). But this generalization is of little consequence unless all the \( n_i \)'s are small because considering all possible values of \( n_i \geq 1 \), the condition \( \rho \geq -1/(n_i - 1) \) reduces to \( \rho \geq 0 \). An advantage of using model (2.1) is that the extensive results on the estimation of variance components can be used for making inferences regarding \( \rho \). There is also an interpretational difference in that adoption of model (2.1) implies that inferences are to extended to some larger populations of groups, while the model (2.4) is more appropriate if the inferences are to be restricted solely to groups in the sample (Donner and Koval, 1980a). However, both the models are essentially equivalent if \( \rho \) is non-negative. In the remainder of this thesis, the
distinction between the two models will be made only if need arises.

2.3 Estimators of Intraclass Correlation

Proceeding with the definitions of equation (2.2), the mean of the observations in the ith group is

$$\bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}, \quad i = 1, 2, \ldots, K.$$ 

The expected value of $\bar{y}_i$ is obviously $\mu$ and variance is given by

$$\text{Var}(\bar{y}_i) = \frac{1}{n_i^2} \left[ \sum_{j=1}^{n_i} \text{Var}(y_{ij}) + \sum_{j=m=1}^{n_i} \sum_{j=m}^{n_i} \text{Cov}(y_{ij}, y_{jm}) \right]$$

$$= \frac{1}{n_i^2} \left[ n_i \sigma^2 + n_i(n_i-1)\rho \sigma^2 \right].$$

Thus

$$\text{Var}(\bar{y}_i) = \frac{\sigma^2}{n_i} \left[ 1 + (n_i-1)\rho \right]. \quad (2.7)$$

Since $\bar{y}_i$ is a linear combination of normally distributed random variables, it is also normally distributed with mean $\mu$ and its variance is given by (2.7). Furthermore because of the natural independence among groups, $\text{Cov}(\bar{y}_i, \bar{y}_h) = 0$ for $i \neq h = 1, 2, \ldots, K$.

Define the grand mean of all observations in K groups as

$$\bar{y} = \frac{1}{N} \sum_{i=1}^{K} \sum_{j=1}^{n_i} y_{ij} = \frac{1}{N} \sum_{i=1}^{K} n_i \bar{y}_i.$$ 

Clearly,

$$\text{E}(\bar{y}) = \frac{1}{N} \sum_{i=1}^{K} n_i \text{E}(\bar{y}_i) = \mu$$

and

$$\text{Var}(\bar{y}) = \frac{1}{N^2} \sum_{i=1}^{K} n_i^2 \text{Var}(\bar{y}_i)$$
because \( \text{Cov}(\bar{y}_i, \bar{y}_h) = 0, \ i \neq h = 1, 2, \ldots, K \). Thus

\[
\text{Var}(\bar{y}) = \frac{\sigma^2}{N'} \left[ (1-\rho) + \frac{\rho}{N} \sum_{i=1}^{K} n_i^2 \right].
\]

(2.8)

Since \( \bar{y} \) is also a linear combination of normally distributed random variables, its distribution is normal with mean \( \mu \) and variance as given by (2.8).

Moreover, if we define the mean of the \( K \) group means to be

\[
\bar{y} = \frac{1}{K} \sum_{i=1}^{K} \bar{y}_i,
\]

(2.9)

the mean of \( \bar{y} \) is \( \text{E}(\bar{y}) = \mu \) and its variance is

\[
\text{Var}(\bar{y}) = \frac{1}{K^2} \sum_{i=1}^{K} \text{Var}(\bar{y}_i)
\]

\[= \frac{\sigma^2}{K^2} \sum_{i=1}^{K} \frac{1}{n_i} [1+(n_i-1)\rho].\]

Thus

\[
\text{Var}(\bar{y}) = \frac{\sigma^2}{K} [\rho+(1-\rho) \lambda_1],
\]

(2.10)

where

\[
\lambda_1 = \frac{1}{K} \sum_{i=1}^{K} n_i^{-1}
\]

(2.11)

is the reciprocal of the harmonic mean of group sizes.

The sum of squares among groups, \( \text{SSA} \), is defined as

\[
\text{SSA} = \sum_{i=1}^{K} n_i (\bar{y}_i - \bar{y})^2 = \sum_{i=1}^{K} n_i (\bar{y}_i - \mu)^2 - N(\bar{y} - \mu)^2.
\]

Hence

\[
\text{E}(\text{SSA}) = \sum_{i=1}^{K} n_i \text{Var}(\bar{y}_i) - N \text{Var}(\bar{y})
\]

\[= (K-1)[1+(n_0-1)\rho] \sigma^2
\]

(2.12)

where
\[ n_0 = \frac{1}{K-1} \left[ N - N^{-1} \sum_{i=1}^{K} n_i^2 \right] \]  

(2.13)

and the variance of SSA is given by

\[
\text{Var}(\text{SSA}) = \sum_{i=1}^{K} n_i^2 \text{Var}(\bar{y}_i - \mu)^2 + N^2 \text{Var}(\bar{y} - \mu)^2 - 2N \sum_{i=1}^{K} n_i \text{Cov}((\bar{y}_i - \mu), (\bar{y} - \mu)^2).
\]

By Theorem 1.8,

\[
\text{Cov}(\bar{y}_i - \mu, (\bar{y} - \mu)^2) = 2\left[ \text{Cov}(\bar{y}_i, \bar{y}) \right]^2
\]

\[= 2\left[ \frac{n_i}{N} \text{Var}(\bar{y}_i) \right]^2.\]

Thus

\[
\text{Var}(\text{SSA}) = 2 \sum_{i=1}^{K} n_i^2 \left( \text{Var}(\bar{y}_i) \right)^2 + 2N^2 \left( \text{Var}(\bar{y}) \right)^2 - \frac{4}{N} \sum_{i=1}^{K} n_i^3 \left( \text{Var}(\bar{y}_i) \right)^2
\]

\[= 2\sigma^4 \left[ \frac{1}{K} \rho^2 + (K-1)(1-\rho)[1+(2n_0-1)\rho] \right]. \tag{2.14}\]

where

\[
\frac{1}{K} = \sum_{i=1}^{K} n_i^2 - 2N^{-1} \sum_{i=1}^{K} n_i^3 + N^{-2} \left[ \sum_{i=1}^{K} n_i^2 \right]^2. \tag{2.15}\]

The expression (2.14) is identical to the expression derived by Tukey (1957) for the variance of SSA.

Let \( SSW_i \) be the sum of squares within the \( i \)th group. More specifically

\[
SSW_i = \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 = \sum_{j=1}^{n_i} (y_{ij} - \mu)^2 - n_i (\bar{y}_i - \mu)^2.
\]

Then
$$E(SSW_1) = n_1 \text{Var}(y_{1j}) - n_1 \text{Var}({\bar{y}}_1)$$

$$= n_1 \sigma^2 - (1+(n_1-1)\rho) \sigma^2$$

$$= (n_1-1)(1-\rho)\sigma^2.$$ 

and

$$\text{Var}(SSW_1) = \text{Var} \left[ \sum_{j=1}^{n_1} (y_{1j} - \mu)^2 \right] + n_1^2 \text{Var} [ (\bar{y}_1 - \mu)^2 ]$$

$$- 2n_1 \sum_{j=1}^{n_1} \text{Cov} [ (y_{1j} - \mu)^2, (\bar{y} - \mu)^2 ]$$

$$= 2 \sum_{j=1}^{n_1} \text{Var}(y_{1j})^2 + 2 \sum_{j=m=1}^{n_1} \text{Cov}(y_{1j}, y_{1m})^2$$

$$+ 2n_1^2 \text{Var}(\bar{y}_1)^2 - 4n_1 \sum_{j=1}^{n_1} \text{Cov}(y_{1j}, \bar{y}_1)^2$$

$$= 2n_1 \sigma^4 + 2n_1(n_1-1)\rho^2 \sigma^4 + 2 (1+(n_1-1)\rho)^2 \sigma^4$$

$$- 4 (1+(n_1-1)\rho)^2 \sigma^4$$

$$= 2(n_1-1)(1-\rho)\sigma^4. \quad (2.16)$$

Since sum of squares within groups, SSW, is

$$SSW = \sum_{i=1}^{K} SSW_i$$

$$E(SSW) = \sum_{i=1}^{K} E(SSW_i) = (N-K)(1-\rho)\sigma^2 \quad (2.17)$$

and because of mutual independence among groups,

$$\text{Var}(SSW) = \sum_{i=1}^{K} \text{Var}(SSW_i)$$

$$= 2 (N-K)(1-\rho)\sigma^4. \quad (2.18)$$

Let $\frac{S_y^2}{y}$ be the sample variance of the sample means defined as

$$\frac{S_y^2}{y} = \frac{1}{K-1} \sum_{i=1}^{K} (\bar{y}_i - \bar{y})^2$$

$$= \frac{1}{K-1} \left[ \sum_{i=1}^{K} (\bar{y}_i - \mu)^2 - K (\bar{y} - \mu)^2 \right]. \quad (2.19)$$
Then

\[ E(S^2_y) = \frac{1}{k-1} \left[ \sum_{i=1}^{K} \text{Var}(\bar{y}_i) - K \text{Var}(\bar{y}) \right] \]

\[ = [\rho + (1-\rho)\lambda_1] \sigma^2, \tag{2.20} \]

and, because of mutual independence among groups,

\[ \text{Var}(S^2_y) = \frac{1}{(k-1)^2} \left[ \sum_{i=1}^{K} \text{Var}(\bar{y}_i) + k^2 \text{Var}(\bar{y} - \mu)^2 \right. \]

\[ - 2k \sum_{i=1}^{K} \text{Cov}(\bar{y}_i, \bar{y} - \mu)^2 \right] \]

\[ = \frac{2}{(k-1)^2} \left[ \sum_{i=1}^{K} \text{Var}(\bar{y}_i)^2 + k^2 \text{Var}(\bar{y})^2 \right. \]

\[ - 2k \sum_{i=1}^{K} \text{Cov}(\bar{y}_i, \bar{y})^2 \]. \]

Since

\[ \text{Cov}(\bar{y}_i, \bar{y}) = \frac{1}{k} \sum_{m=1}^{K} \text{Cov}(\bar{y}_i, \bar{y}_m) = \frac{1}{k} \text{Var}(\bar{y}_i) \]

\[ = \frac{\sigma^2}{Kn_i} \left[ 1 + (n_1 - 1)\rho \right], \]

\[ \text{Var}(S^2_y) = \frac{2\sigma^4}{k-1} \left[ \frac{(k-2)(1-\rho)^2}{k} U + [\rho + (1-\rho)\lambda_1]^2 \right], \tag{2.21} \]

where

\[ U = \frac{k}{k-1} \left( \lambda_2 - \lambda_1^2 \right) \tag{2.22} \]

is the sample variance of \( n_i \)'s and

\[ \lambda_2 = \frac{1}{K} \sum_{i=1}^{K} n_i^{-1}. \tag{2.23} \]

We also know that the mean \( \bar{y}_i \) of the \( i \)-th sample is stochastically independent of SSW, so that SSA and \( S^2_y \) are both stochastically independent of SSW.

2.3.1 Maximum Likelihood Estimator

Under the multivariate normal model (2.4), the log-likelihood of the sample can be written as
\[ l = - \frac{N}{2} \ln(2\pi\sigma^2) - \frac{N-K}{2} \ln(1-\rho) - \frac{1}{2} \sum_{i=1}^{K} \ln[1+(n_i-1)\rho] \]

\[ - \frac{1}{2\sigma^2} \left\{ \frac{SSW}{(1-\rho)} + \sum_{i=1}^{K} \alpha_i(\rho)(\bar{y}_i - \mu)^2 \right\}. \tag{2.24} \]

where

\[ \alpha_i(\rho) = \frac{n_i}{n_i(1+(n_i-1)\rho)}, \quad i = 1, 2, 3, \ldots, K. \]

By taking the derivatives of \( l \) with respect to \( \mu \) and \( \sigma^2 \), equating them to zero and solving for these parameters, one can easily show that the maximum likelihood estimators of \( \mu \) and \( \sigma^2 \), as functions of \( \rho \), are given by

\[ \hat{\mu}(\rho) = \frac{\sum_{i=1}^{K} \alpha_i(\rho) \bar{y}_i}{\sum_{i=1}^{K} \alpha_i(\rho)} \tag{2.25} \]

\[ \hat{\sigma}^2(\rho) = \frac{1}{N} \left\{ \frac{SSW}{(1-\rho)} + \sum_{i=1}^{K} \alpha_i(\rho)(\bar{y}_i - \hat{\mu}(\rho))^2 \right\}. \tag{2.26} \]

Following the procedure of Richards (1961) for maximizing a function of several variables and substituting (2.25) and (2.26) in (2.24) for \( \mu \) and \( \sigma^2 \), respectively, Donner and Koval (1980a) have shown that the maximum likelihood estimator of \( \rho \), say \( \hat{\rho}_M \), can be obtained by maximizing a single variable function \( l^* \), given by

\[ l^* = - \frac{N}{2} \left[ 1 + \ln(2\pi) + \ln(\hat{\sigma}^2(\rho)) \right] - \frac{N-K}{2} \ln(1-\rho) \]

\[ - \frac{1}{2} \sum_{i=1}^{K} \ln[1+(n_i-1)\rho], \tag{2.27} \]

in the interval \( \left[ \max\left\{ \frac{1}{n_i-1} \right\}, 1 \right] \). The large sample variance of \( \hat{\rho}_M \) can be obtained by inverting the Fisher's information matrix corresponding to parameters \( \mu \), \( \sigma^2 \) and \( \rho \). Donner and
Koval (1980b) gave the large sample variance of $\hat{\rho}_M$ as

$$
\text{Var}(\hat{\rho}_M) \approx 2(1-\rho)^2 \left\{ \sum_{i=1}^{K} \frac{a_i(\rho)(n_i-1)[1+(n_i-1)\rho^2]}{[1+(n_i-1)\rho]} \right\}^{-1} - \frac{\rho^2}{N} \left[ \sum_{i=1}^{K} a_i(\rho)(n_i-1) \right]^{-1} \quad (2.28)
$$

2.3.2 ANOVA Estimator

The most commonly used estimator of $\rho$, named by Fisher (1958), the ANOVA estimator is derived by replacing $\sigma_a^2$ and $\sigma_e^2$ in (2.3) by their unbiased estimators under variance components model (2.1). Since

$$
\text{ECMSAO} = \frac{\text{ECSSAO}}{K-1} = \sigma_e^2 + n_0 \sigma_a^2 = [1+(n_0-1)\rho] \sigma^2
$$

$$
\text{ECMSWO} = \frac{\text{ECSSWO}}{N-K} = \sigma_e^2 = (1-\rho) \sigma^2, \quad (2.29)
$$

we have

$$
\hat{\mu} = \bar{y},
$$

$$
\hat{\sigma}^2 = \frac{1}{n_0} [\text{MSA} + (n_0-1)\text{MSW}]
$$

and

$$
\hat{\rho}_A = \frac{\text{MSA - MSW}}{\text{MSA} + (n_0-1)\text{MSW}} = \frac{F_A - 1}{F_A + (n_0-1)} \quad (2.30)
$$

where

$$
F_A = \frac{\text{MSA}}{\text{MSW}} = \frac{1 + (n_0-1)\hat{\rho}_A}{1 - \hat{\rho}_A} \quad (2.31)
$$

is the usual analysis of variance ratio statistic for testing the equality of group means. This statistic can also be used for testing the significance of intraclass correlation.

The ANOVA estimator $\hat{\rho}_A$ is consistent but not an unbiased
estimator of $\rho$. Under the assumption of multivariate normality, the large sample bias and variance of $\hat{\rho}_A$ will be derived by the delta method as described in section 1.4. Under the assumption of normality of observations, the MSA and MSW are asymptotically normally distributed random variables.

Let

$$\hat{X} = [\text{MSA}, \text{MSW}]^T$$

and

$$\hat{\xi} = [\text{EC(MSA)}, \text{EC(MSW)}]^T,$$

where EC(MSA) and EC(MSW) are given by (2.29). Since Cov(MSA, MSW) = 0, the large sample bias and variance of ANOVA estimator $\hat{\rho}_A$ derived by delta method, from (1.6) and (1.7) respectively, are given by

$$\text{Bias}(\hat{\rho}_A) \approx \frac{1}{2} \left[ D_{11}(\hat{\rho}_A) \text{Var(MSA)} + D_{22}(\hat{\rho}_A) \text{Var(MSW)} \right]$$

and

$$\text{Var}(\hat{\rho}_A) \approx \left[ [D_1(\hat{\rho}_A)]^2 \text{Var(MSA)} + [D_2(\hat{\rho}_A)]^2 \text{Var(MSW)} \right],$$

where $D_1$ and $D_2$ ($i = 1, 2$) are as defined by (1.1) and (1.2), respectively. Taking the first and second derivatives of $\hat{\rho}_A$ with respect to MSA and MSW and then evaluating them at the mean vector $\bar{x}$, we find

$$D_1(\hat{\rho}_A) = \frac{1-\rho}{n_0 \sigma^2}, \quad D_2(\hat{\rho}_A) = \frac{-(1+(n_0-1)\rho)}{n_0 \sigma^2},$$

$$D_{11}(\hat{\rho}_A) = \frac{-2(1-\rho)}{n_0^2 \sigma^4}, \quad D_{22}(\hat{\rho}_A) = \frac{2(n_0-1)(1+(n_0-1)\rho)}{n_0^2 \sigma^4}.$$

Furthermore, the variances of MSA and MSW are
\[
\text{Var}(\text{MSA}) = \frac{1}{(K-1)^2} \text{Var}(\text{SSA})
\]
\[
= \frac{2\sigma^4}{(K-1)^2} \left[ \frac{n_0}{K} \rho^2 + (K-1)(1-\rho)(1+(2n_0-1)\rho) \right], \quad (2.34)
\]

and
\[
\text{Var}(\text{MSW}) = \frac{1}{(N-K)^2} \text{Var}(\text{SSW}) = \frac{2(1-\rho)^2\sigma^4}{(N-K)}, \quad (2.35)
\]
respectively. Thus, using the delta method, the large sample bias and variance of \( \hat{\rho}_A \) are given, respectively, by
\[
\text{Bias}(\hat{\rho}_A) \equiv \frac{2(1-\rho)^2}{n_0^2} \left\{ \frac{(n_0-1)(1+(n_0-1)\rho)}{N-K} - \frac{[1+(2n_0-1)\rho]}{K-1} \right\}
\]
\[
- \frac{\frac{1}{n_0} \rho^2}{(K-1)^2(1-\rho)} \right\}, \quad (2.36)
\]
and
\[
\text{Var}(\hat{\rho}_A) \equiv \frac{2(1-\rho)^2}{n_0^2} \left\{ \frac{[1+(n_0-1)\rho]^2}{N-K} + \frac{(1-\rho)(1+(2n_0-1)\rho)}{K-1} \right\}
\]
\[
+ \frac{\frac{1}{n_0} \rho^2}{(K-1)^2} \right\}, \quad (2.37)
\]
where \( n_0 \) and \( \frac{1}{n_0} \) are as defined by (2.13) and (2.15) respectively. The derivation of \( \text{Var}(\hat{\rho}_A) \) was first given by Smith (1957) and thus will be referred to as Smith's expression for the variance of ANOVA estimator.

2.3.3 Other Moment Estimators

Another moment estimator of \( \rho \) may be obtained if the variation among \( \bar{y}_i \)'s is not too large, where in such a case \( \frac{s^2_y}{\bar{y}} \) will not differ much from MSA. If this is the case, then taking the ratio of \( \frac{s^2_y}{\bar{y}} \) to MSW, replacing them with their expectations and solving for \( \rho \), we have
\[
\hat{\rho}_s = \frac{s^2_y - \lambda_1 \text{MSW}}{s^2_y + (1-\lambda_1)\text{MSW}}, \quad (2.38)
\]
where $\lambda_1$ is as defined by (2.11).

This estimator was first proposed by Smith (1957) by considering the uniform weighting for each group and later Srivastava (1984) obtained it in a different context when he proposed several new estimators for the interclass correlation. Since $\text{Cov}(S_{y}^{2}/y, \text{MSW}) = 0$, the large sample bias and variance of $\hat{\rho}_{S}$ can be obtained, using the delta method, by letting $x = [S_{y}^{2}/y, \text{MSW}]^T$ and replacing $\hat{\rho}_{A}$ and MSA by $\hat{\rho}_{S}$ and $S_{y}^{2}/y$, respectively, in (2.32) and (2.33). The mean and variance of $S_{y}^{2}/y$ are given by (2.20) and (2.21) respectively. Furthermore, the first and second derivatives of $\hat{\rho}_{S}$ with respect to $S_{y}^{2}/y$ and MSA, evaluated at the mean values, are

$$D_1(\hat{\rho}_{S}) = \frac{1-\rho}{\sigma^2}, \quad D_2(\hat{\rho}_{S}) = \frac{-[\rho+(1-\rho)\lambda_1]}{\sigma^2},$$

$$D_{11}(\hat{\rho}_{S}) = \frac{-2(1-\rho)}{\sigma^4} \quad \text{and} \quad D_{22}(\hat{\rho}_{S}) = \frac{2(1-\lambda_1)[\rho+(1-\rho)\lambda_1]}{\sigma^4}.$$

Thus the large sample bias and variance of $\hat{\rho}_{S}$ are respectively given by

$$\text{Bias}(\hat{\rho}_{S}) \equiv -2(1-\rho) \left\{ \frac{(K-2)(1-\rho)^2}{K(K-1)} \right\} U$$

$$+ \left[ \frac{[(\rho+(1-\rho)\lambda_1]}{K-1} - \frac{(1-\rho)(1-\lambda_1)}{N-K} \right]$$

(2.38)

and

$$\text{Var}(\hat{\rho}_{S}) \equiv \frac{2(1-\rho)^2}{K-1} \left\{ \frac{(K-2)(1-\rho)^2}{K} U + \frac{(N-1)(\rho+(1-\rho)\lambda_1)^2}{N-K} \right\}.$$  

(2.40)

The expression for the variance of $\hat{\rho}_{S}$ was derived before, independently, by Smith (1957) and Keen (1987). The
expression (2.40) is identical to Keen's expression (2.22) but there seems to a misprint in Smith's expression.

In an attempt to extract an estimator of \( \rho \) which is not tied to the structure of the underlying ANOVA model (2.1), Karlin et al. (1981), considering \( n_i \)'s as random variables, suggested the following moment estimator of \( \rho \): 

\[
\hat{\rho}_K = \frac{B^* (1-\text{E}(1/n)) - W^* (1-K^{-1}) \text{E}(1/n)}{[B^* + (1-K^{-1}) W^* ][1-\text{E}(1/n)]},
\] 

(2.41)

where 

\[
B^* = \frac{(K-1)}{K} \frac{s_y^2}{n}, \quad W^* = \frac{1}{K} \sum_{i=1}^K \frac{1}{n_i} \sum_{j=1}^{n_i} (y_{ij} - y_i^-)^2,
\]

\( \text{E}(1/n) \) is the reciprocal of the harmonic mean of the \( n_i \)'s and the expectation is taken with respect to the chance mechanism generating the \( n_i \)'s.

In order to find the bias and variance of \( \hat{\rho}_K \) by delta method, we need to know the means and variances of \( B^* \) and \( W^* \). Since \( B^* \) and \( W^* \) are functions of observations \( y_{ij} \)'s and group sizes \( n_i \)'s, their means and variances must be obtained by conditioning on the distribution of the group sizes \( n_i \)'s. Before we proceed further, it should be noted that the conditional means, variances and covariance of \( B^* \) and \( W^* \) for given \( n_i \)'s are

\[
\text{E}(B^*|n_i) = \frac{K-1}{K} \text{E}(\frac{s_y^2}{n_i} | n_i) = \frac{K-1}{K} [\rho + (1-\rho)\lambda_1] \sigma_y^2,
\]

\[
\text{E}(W^*|n_i) = \frac{1}{K} \sum_{i=1}^K \frac{1}{n_i} \text{E}(SSW_{i1}|n_i) = (1-\lambda_1)(1-\rho) \sigma_y^2.
\]

\[
\text{Var}(B^*|n_i) = \frac{(K-1)^2}{K^2} \text{Var}(\frac{s_y^2}{n_i} | n_i) = \frac{2(K-1)^2}{K^2} \left[ \frac{(\rho + (1-\rho)\lambda_1)^2}{\lambda_2} + \frac{(K-2)(1-\rho)^2}{(K-1)} (\lambda_2 - \lambda_1) \right],
\]

44
\[
\text{Var}(\hat{W}^*|n_1^*) = \frac{1}{K} \sum_{i=1}^{K} \frac{1}{n_1^*} \text{Var}((SSW)_i|n_1^*) = \frac{2(1-\rho)^2 \sigma^4}{K} (\lambda_1 - \lambda_2^*).
\]

Thus
\[
E(B^*) = E[E(B^*|n_1^*]) = \frac{K-1}{K} [\rho + (1-\rho)E(1/n)] \sigma^2,
\]

\[
\text{Var}(B^*) = E[\text{Var}(B^*|n_1^*)] + \text{Var}[E(B^*|n_1^*)] = \frac{2(K-1)\sigma^4}{K^2} \left[ (\rho + (1-\rho)E(1/n))^2 + \frac{3(K-1)(1-\rho)^2}{2K} \text{Var}(1/n) \right],
\]

\[
E(CW^*) = E[E(CW^*|n_1^*)] = [1-E(1/n)](1-\rho) \sigma^2.
\]

\[
\text{Var}(W^*) = E[\text{Var}(W^*|n_1^*)] + \text{Var}[E(W^*|n_1^*)] = \frac{(1-\rho)^2 \sigma^4}{K} \left[ 2E(1/n)[1-E(1/n)] - \text{Var}(1/n) \right],
\]

and
\[
\text{Cov}(B^*, W^*) = 0.
\]

(2.43)

where \( \text{Var}(1/n) \) is the variance of the \( n_1^{-1} \)'s. Further, by letting \( X = [B^*, W^*]^T \), the first and second derivatives of \( \hat{\rho}_K \) with respect to \( B^* \) and \( W^* \), evaluated at the mean values, are given by
\[
D_1(\hat{\rho}_K) = \frac{KC(1-\rho)}{(K-1)\sigma^2}, \quad D_2(\hat{\rho}_K) = \frac{[-\rho + (1-\rho)E(1/n)]}{[1-E(1/n)]\sigma^2},
\]

\[
D_{11}(\hat{\rho}_K) = \frac{-2K^2(1-\rho)}{(K-1)^2\sigma^4}, \quad D_{22}(\hat{\rho}_K) = \frac{2[\rho + (1-\rho)E(1/n)]}{[1-E(1/n)]\sigma^4}.
\]

(2.44)
Thus by using the delta method and the independence of $B^*$ and $W^*$, the large sample bias and variance of $\hat{\rho}_K$ are given respectively by

$$\text{Bias}(\hat{\rho}_K) \approx \frac{-2(1-\rho)}{k} \left\{ \frac{[\rho+(1-\rho)E(1/n)][kp+(1-\rho)E(1/n)]}{k-1} ight. \right. \left. + \frac{(1-\rho)(1.5 - [\rho+(1-\rho)E(1/n)])}{[1-E(1/n)]} \text{Var}(1/n) \right\}, \quad (2.45)$$

and

$$\text{Var}(\hat{\rho}_K) \approx \frac{2(1-\rho)^2}{k[1-E(1/n)]} \left\{ \frac{[\rho+(1-\rho)E(1/n)]^2[E(1/n)]}{k-1} \right. \right. \left. \right. \left. + \frac{\left[3(1-\rho)^2[1-E(1/n)] - [\rho+(1-\rho)E(1/n)]^2 \right]}{2[1-E(1/n)]} \text{Var}(1/n) \right\}. \quad (2.46)$$

Unless the exact distribution of sibship sizes is known, it is not feasible to use $\hat{\rho}_K$. However, $\hat{\rho}_K$ may be used on replacing $E(1/n)$ by $\lambda_1 = \frac{1}{k} \sum_{i=1}^{K} \frac{1}{n_i}$, so the $\hat{\rho}_K$ becomes

$$\hat{\rho}_E = \frac{B^*(1-\lambda_1) - W^*(1-K^{-1})\lambda_1}{[B^*+(1-K^{-1})W^*](1-\lambda_1)} \quad (2.47)$$

which can be considered as an empirical version of Karlin's estimator $\hat{\rho}_K$. Since the $n_i$'s in (2.47) are fixed, the expressions (2.42) can be used for the means and variances of $B^*$ and $W^*$ in finding the large sample bias and variance of $\hat{\rho}_E$. Furthermore, the first and second order partial derivatives of $\hat{\rho}_E$ with respect to $B^*$ and $W^*$, evaluated at the mean values, can be obtained from (2.44) by replacing $E(1/n)$ with $\lambda_1$. Thus the large sample bias and variance of $\hat{\rho}_E$, derived by delta method, are given by
\[ \text{Bias}(\hat{\rho}_E) \equiv -2(1-\rho) \left[ \frac{(K-2)(1-\rho)^2}{K(K-1)} U \right. \\
+ \left[ \rho + (1-\rho)\lambda_1 \right] \left[ \frac{\rho + (1-\rho)\lambda_1}{K-1} - \frac{(1-\rho)(\lambda_1 - \lambda_2)}{K(1-\lambda_1)} \right] \] 

and

\[ \text{Var}(\hat{\rho}_E) \equiv \frac{2(1-\rho)^2}{K} \left[ \frac{(K-2)(1-\rho)^2}{(K-1)} U \right. \\
+ \left[ \rho + (1-\rho)\lambda_1 \right]^2 \left[ \frac{K}{K-1} + \frac{(\lambda_1 - \lambda_2)^2}{(1-\lambda_1)^2} \right] \right], \] (2.48)

where \( U \) is the sample variance of \( n_i^{-1} \)'s as given by (2.22).

### 2.3.4 Pairwise Estimators

Considering the set of all pairs of observations \((y_{ij}, y_{im}), j \neq m = 1, 2, \ldots, n_i; i = 1, 2, \ldots, K\), Pearson and Filon (1938) proposed the pairwise estimator:

\[ r_p = \frac{K \sum_{i=1}^{n_i} \sum_{j=m=1}^{K} (y_{ij} - \bar{y})(y_{im} - \bar{y})}{\sum_{i=1}^{K} \sum_{j=1}^{n_i} (y_{ij} - \bar{y})^2 / (N-1)}. \] (2.50)

The \( r_p \) is perhaps the oldest estimator of intraclass correlation and it is the Pearson Product-Moment correlation computed over all possible pairs of observations. As pointed out by Fieller and Smith (1951), \( r_p \) tends to give too much weight to groups of large sizes. The simulation study of Donner and Koval (1980a) also show that \( r_p \) is not suited for the estimation of \( \rho \geq 0.1 \) because it decline rapidly in effectiveness as \( \rho \) increases which can be attributed to the increasing negative bias of this estimator with \( \rho \).
To overcome this disadvantage, Karlin et al. (1981) have proposed a general weighted pairwise estimator given by

$$\hat{\rho}_W = \frac{\sum_{i=1}^{K} W_i \sum_{j \neq m=1}^{n_1} (y_{ij} - \hat{\mu})(y_{im} - \hat{\mu})}{\sum_{i=1}^{K} W_i (n_1 - 1) \sum_{j=1}^{n_1} (y_{ij} - \hat{\mu})^2}$$

(2.51)

where $\hat{\mu} = \sum_{i=1}^{K} n_i (n_1 - 1) W_i y_{i1} / K$ and the weights $W_i$'s are constrained to satisfy $\sum_{i=1}^{K} n_i (n_1 - 1) W_i = 1$. Clearly, this weighted estimator is unsuitable when groups of size 1 are present in the analysis because in such a case the weights are impossible to obtain.

Karlin et al. (1981) proposed three different methods of weighting. The first method, called the sib-pair method, assigns equal weights to each pair of observations. Since there are $\sum_{i=1}^{K} n_i (n_1 - 1)$ such pairs, the weight for each pair is $W_i = 1 / \sum_{i=1}^{K} n_i (n_1 - 1)$. The second method of weighting, called the group method, assigns equal weight to each group, independent of the number of observations in that group. This can be done by setting $W_i = 1 / Kn_i (n_1 - 1)$. The third method of weighting, called the individual method, assigns weights to each pair according to the number of pairs in which an individual appears. Since every individual from the $i$th group will appear in $N(n_1 - 1)$ pairs, the weight for each pair in the $i$th group is $W_i = 1 / N(n_1 - 1)$. As mentioned by Karlin et al. (1981), the sib-pair method emphasizes contributions from large groups most, whereas the group method emphasizes contributions from large groups least, in
fact, treating all groups equivalently. The individual method lies between the other two in its emphasis on large groups. However, when applied by Namboordiri et al. (1984) to a data set on the familial aggregation of lipids, there seemed to be little variation in the results. Here we choose the individual method of weighting which is an intermediate method of weighting and gives an individual estimator of intraclass correlation given by

$$\hat{\rho}_I = \frac{\sum_{i=1}^{K} \frac{1}{n_i-1} \sum_{j=m=1}^{n_i} (y_{ij} - \bar{y}_i)(y_{jm} - \bar{y}_j)}{\sum_{i=1}^{K} \frac{n_i}{n_i^2} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2}.$$  \hspace{1cm} (2.52)

In order to find the bias and variance of $\hat{\rho}_I$ by the delta method, we need to know the means, variances and covariance of $A$ and $SST$. Since

$$A = SSA - \sum_{i=1}^{K} \frac{SSW_i}{n_i(n_i-1)}$$
and

$$SST = SSA + SSW,$$

$$EC(A) = EC(SSA) - \sum_{i=1}^{K} \frac{EC(SSW_i)}{n_i(n_i-1)}$$

$$= [n_0(K-1)\rho - (1-\rho)]\sigma^2,$$

$$Var(A) = Var(SSA) + \sum_{i=1}^{K} \frac{Var(SSW_i)}{n_i(n_i-1)^2}$$

$$= 2\left[\rho^2 + (K-1)(1-\rho)[1+(2n_0-1)\rho]\right] + \left[\sum_{i=1}^{K} \frac{1}{n_i(n_i-1)}(1-\rho)^2\right] \sigma^4,$$

$$EC(SST) = EC(SSA) + EC(SSW)$$

$$= [(N-1)(1-\rho) + n_0(K-1)\rho] \sigma^2.$$
\[
\text{Var}(\text{SST}) = \text{Var}(\text{SSA}) + \text{Var}(\text{SSW})
\]
\[
= 2 \left[ \frac{N - 1}{2} \rho^2 + 2n_0 (K-1) \rho (1-\rho) + (N-1)(1-\rho)^2 \right] \sigma^4
\]

and
\[
\text{Cov}(A, \text{SST}) = \text{Var}(\text{SSA}) - \frac{1}{K} \sum_{i=1}^{K} \frac{1}{n_i-1} \text{Var}(\text{SSW}_i)
\]
\[
= 2 \left[ \frac{N - 1}{2} \rho^2 + 2n_0 (K-1) \rho (1-\rho) - (1-\rho)^2 \right] \sigma^4.
\]

Thus, once again on using (1.6) and (1.7), the large sample bias and variance of \( \hat{\rho}_1 \) are, respectively, given by
\[
\text{Bias}(\hat{\rho}_1) \cong \frac{1}{[\text{EC}(\text{SST})]^2} \left[ \frac{\text{EC}(A)}{\text{EC}(\text{SST})} \text{Var}(\text{SST}) - \text{Cov}(A, \text{SST}) \right] \tag{2.53}
\]

and
\[
\text{Var}(\hat{\rho}_1) \cong \frac{1}{[\text{EC}(\text{SST})]^2} \left[ \text{Var}(A) + \frac{[\text{EC}(A)]^2}{[\text{EC}(\text{SST})]^2} \text{Var}(\text{SST})
\]
\[
- \frac{2 \text{EC}(A)}{\text{EC}(\text{SST})} \text{Cov}(A, \text{SST}) \right] \tag{2.54}
\]

2.4 Comparison of the Estimators

The expressions for the large sample biases and variances of the estimators of intraclass correlation have quite complicated forms and their direct comparisons are not possible. Therefore, the sampling properties of the estimators were investigated by examining the biases and variances of the estimators for twelve particular designs representing a wide spectrum of unbalancedness. These designs are listed in Table 2.1. Among these twelve designs, design numbers 2, 4, 6, 8, 10 and 12 have a severe degree of unbalancedness. For each design, the bias and
TABLE 2.1

Unbalanced Designs

<table>
<thead>
<tr>
<th>Design NO.</th>
<th>No. of Groups</th>
<th>Group Sizes ($n_i$'s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>4, 6</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2, 100</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>5, 10, 15</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>5, 5, 500</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>2, 4, 6, 8, 10</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>5, 10, 50, 100, 500</td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>3, 5, 9, 12, 15, 18, 21, 24, 27, 30</td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td>5, 5, 25, 25, 50, 50, 100, 100, 500</td>
</tr>
<tr>
<td>9</td>
<td>25</td>
<td>5 e 10, 5 e 20, 5 e 30, 5 e 40, 5 e 50</td>
</tr>
<tr>
<td>10</td>
<td>25</td>
<td>22 e 4, 200, 400, 800</td>
</tr>
<tr>
<td>11</td>
<td>50</td>
<td>10 e 5, 10 e 10, 10 e 15, 10 e 20, 10 e 25</td>
</tr>
<tr>
<td>12</td>
<td>50</td>
<td>45 e 5, 200, 300, 400, 500, 800</td>
</tr>
</tbody>
</table>

m e n means that n appears m times (e.g., 2 e 3 means 3, 3).

The variance of each estimator was calculated for $\rho = 0$ to 0.9. Since no closed form solutions for the maximum likelihood estimator $\hat{\rho}_M$ exist, the expression for the large sample bias of $\hat{\rho}_M$ is very complicated to derive and is still not available. In calculating the bias and variance of $\hat{\rho}_K$, the $E(1/n)$ and $\text{Var}(1/n)$ in expressions (2.45) and (2.46) were replaced by their unbiased estimators $\lambda_1$ (2.11) and $U$ (2.22), respectively.

The results of the study for these designs are given in Table 2.2 through Table 2.7. Since the bias of the maximum likelihood estimator is not known, this estimator of intraclass correlation is compared with other estimators in
<table>
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<tr>
<th>Design</th>
<th>( \rho )</th>
<th>( \hat{\rho}_M )</th>
<th>( \hat{\rho}_A )</th>
<th>( \hat{\rho}_S )</th>
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<th>( \hat{\rho}_E )</th>
<th>( \hat{\rho}_I )</th>
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<td>0.0977</td>
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52
### TABLE 2.3

The large sample bias (B) and variance (V) for various point estimators of intraclass correlation for \( K = 3 \).

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TABLE 2.4

The large sample bias (B) and variance (V) for various point estimators of intraclass correlation for \( K = 5 \).

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54
TABLE 2.5

The large sample bias (B) and variance (V) for various point estimators of intraclass correlation for K = 10.

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## TABLE 2.6

The large sample bias (B) and variance (V) for various point estimators of intraclass correlation for $K = 25$.

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--- Not available \* $\epsilon (-0.00005,0)$ \* * $\leq 0.00005$
TABLE 2.7

The large sample bias (B) and variance (V) for various point estimators of intraclass correlation for \( K = 50 \).

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--- Not available \( \ast \in (-.00005,0) \) \( ** \leq .00005 \)
terms of its large sample variance only. As can be seen from these tables, the estimator $\hat{\rho}_M$ has smaller variance for most of the presented cases except when $K$ is small ($2, 3, \text{ and } 5$) and design is severely unbalanced. In such a case, Karlin’s individual estimator $\hat{\rho}_I$ has smaller variance provided $0.2 \leq \rho \leq 0.7$.

All the estimators, except $\hat{\rho}_M$, can also be compared in terms of their biases and mean square errors. The mean square error (MSE) of an estimator $\hat{\theta}$ of the parameter $\theta$ is defined as

$$\text{MSEC}(\hat{\theta}) = \text{E}(\hat{\theta} - \theta)^2 = \text{Var}(\hat{\theta}) + [\text{Bias}(\hat{\theta})]^2.$$ 

Clearly if $\hat{\theta}$ is an unbiased estimator for $\theta$ then $\text{MSEC}(\hat{\theta}) = \text{Var}(\hat{\theta})$.

As can be seen from Tables 2.2 - 2.7, all estimators, on average, underestimate $\rho$. Karlin’s empirical estimator $\hat{\rho}_E$ is relatively less biased for moderately large numbers of groups ($K = 10, 20, 50$) provided $\rho \geq 0.2$. Otherwise Karlin’s individual estimator $\hat{\rho}_I$ is less biased. For small values of $K$ ($2, 3, 5$), the estimator $\hat{\rho}_I$ has a smaller mean square error provided $\rho \leq 0.5$, Otherwise Karlin’s empirical estimator $\hat{\rho}_E$ has a smaller mean square error. For moderately large values of $K$ ($10, 25, 50$), Smith’s estimator $\hat{\rho}_S$ has a smaller mean square error provided $\rho \geq 0.2$. The estimator $\hat{\rho}_I$ performs very well for small values of $\rho$ ($\leq 0.2$) but declines rapidly in its effectiveness as $\rho$ increases. A more detailed examination of the results shows that this can be attributed to a rapidly increasing negative bias as $\rho$
increases from 0 to 0.6.

2.5 Discussion

The major purpose of this chapter was to investigate the sampling properties of several point estimators of the intraclass correlation for a wide variety of unbalanced designs. The simulation studies of Donner and Koval (1980a) and Keen (1987) compare several estimators of the intraclass correlation in terms of their biases and variances. The results of their studies are valid only for North American families. The former compare the biases and variances of the ANOVA estimator $\hat{\rho}_A$, the maximum likelihood estimator $\hat{\rho}_M$ and pairwise estimator $\hat{\rho}_P$. They recommend the use of $\hat{\rho}_M$ when no prior information concerning the intraclass correlation is available or the intraclass correlation is thought to be large. If the value of $\rho$ is likely to be small ($<0.5$), they recommend the use of the ANOVA estimator with groups having only one member deleted from the analysis. The latter compares $\hat{\rho}_A$, $\hat{\rho}_S$, $\hat{\rho}_M$ and an estimator based on the approaches of Wald (1940) and Bhargava (1946). He recommends the use of $\hat{\rho}_A$ for small values of the intraclass correlation and $\hat{\rho}_S$ for large values of the intraclass correlation. In this investigation, the comparison of several other estimators of the intraclass correlation is under consideration and it is based on the expressions for large sample biases and variances of the estimators. The major findings of this quantitative investigation are summarized as follows:
(1) As was expected, the biases and variances of all the estimators decreases as the number of groups $K$ increases.

(2) For moderately large values of $K$ (10, 25, 50), all the estimators perform much better in terms of biases, variances and mean square errors for the designs having less variation in group sizes relative to the designs having severe degree of unbalancedness.

(3) The maximum likelihood estimator $\hat{\rho}_M$ has the smallest variance among all the estimators for moderately large numbers of groups ($K = 10, 25, 50$). For small number of groups ($K = 2, 3, 5$) and relatively less unbalanced designs, the $\hat{\rho}_M$ has smaller variance provided $\rho \geq 0.2$. When comparing the results of severely unbalanced designs with relatively less unbalanced designs, the variance of $\hat{\rho}_M$ is smaller for small values of $K$ and severely unbalanced designs (2, 4, 6). The reverse is true for moderately large values of $K$ and less unbalanced designs (7, 9, 11).

(4) Karlin's individual estimator $\hat{\rho}_I$ is less biased and has a smaller mean square error for small numbers of groups ($K = 2, 3, 5$), especially for severely unbalanced designs (2, 4, 6). For moderately large numbers of groups ($K = 10, 25, 50$), it performs better only for small values of the intraclass correlation ($\rho \leq 0.1$).

(5) For large numbers of groups, Smith's estimator $\hat{\rho}_S$ performs much better than other estimators in terms of mean square error provided $\rho \geq 0.5$. For these groups,
its performance is even better for severely unbalanced designs where it has smaller mean square error for $\rho \geq 0.2$.

Karlin's empirical estimator $\hat{\rho}_E$ performs better than its counterpart $\hat{\rho}_K$ for most combinations of $K$ and $\rho$. However for moderately large values of $K$ (10, 25, 50), both estimators have almost identical values of the biases and variances.

Based on the discussion above, it is recommended that for small numbers of groups, Karlin's individual estimator should be used for the estimation of intraclass correlation.

However for large numbers of groups, Karlin's empirical estimator is recommended for $\rho \leq 0.5$, otherwise Smith's estimator is recommended. If no prior knowledge concerning the value of intraclass correlation exists, the use of Karlin's individual estimator is recommended for small numbers of groups ($K \leq 10$) and the use of maximum likelihood estimator is recommended otherwise.

In this chapter, the sampling properties of several point estimators for the intraclass correlation are investigated. In the next chapter, the problem of testing hypotheses regarding intraclass correlation is considered. Several procedures for testing hypotheses on the intraclass correlation parameter are discussed and compared in terms of their empirical levels and powers.
CHAPTER 3
SIGNIFICANCE TESTING PROCEDURES FOR INTRACLASS CORRELATION

3.1 Introduction

The estimation of the intraclass correlation coefficient, $\rho$, was discussed in chapter 2. The sampling properties of several estimators of $\rho$ were investigated for several values of $\rho$, $k$, and a wide variety of unbalanced designs. An equally important problem is to test the hypotheses regarding the intraclass correlation coefficient.

It may be of interest to test whether individuals belonging to the same group are more alike with respect to the attribute in question than are individuals from different groups. This can be investigated by testing $H_0: \rho = 0$ against $H_1: \rho > 0$. In family studies, if the expected contribution of genetic correlation to the familial aggregation is no more than $\rho_0$, it may be of interest to determine whether or not there is an additional contribution to the magnitude of the intraclass correlation that can be attributed to a common family environment. This suggests testing the hypothesis $H_0: \rho = \rho_0$ against $H_1: \rho > \rho_0$ and the rejection of this null hypothesis is an indication of very strong familial aggregation with respect to the characteristic under study. The purpose of this chapter is to provide and compare several testing procedures for the null hypothesis $H_0: \rho = \rho_0$ against the alternative $H_1: \rho > \rho_0$ (or $H_1: \rho \neq \rho_0$), where $\rho_0$ ($>0$) is a specified constant. In a review on the inference procedures for intraclass
correlation. Donner (1988) had mentioned a few procedures to
test the said hypothesis but the subject is not complete.
Although one can find several procedures cited in the
literature of variance components which can be used to test
the said hypotheses, nothing is known about the performance
of these procedures. Several procedures that are tractable
in practice will be discussed and compared in terms of their
empirical significance levels and powers. The obtained
results on testing different hypotheses regarding \( \rho \) will be
compared with the conclusions of Donner and Wells (1988) and
Keen (1987), concerning the methods of constructing
confidence intervals for \( \rho \). Such comparisons are warranted
because of the connection between hypothesis testing and
interval estimation.

3.2 Procedures for Testing \( H_0: \rho = \rho_0 \)

In practice it is of interest to test \( H_0: \rho = \rho_0 \) against
\( H_1: \rho > \rho_0 \) (or \( H_1: \rho \neq \rho_0 \)), where \( \rho_0 \geq 0 \) is a specified value
of \( \rho \). Among fifteen test procedures presented in this
section, four are based on F-distribution of the test
statistics, seven are obtained by comparing the estimators
to their large sample standard errors, two are based on the
CC0D procedure of Neyman (1959), one is based on the
normalizing transformation due to Fisher (1925), and the
last is the likelihood ratio test. These test procedures
are discussed below.

3.2.1 Tests Based on the Large Sample Theory of Maximum
Likelihood

If \( Y_1, Y_2, Y_3, \ldots, Y_K \) denote a simple random sample (SRS)
from the multivariate normal model (2.4), the log-likelihood function, \( l \), of the sample is given by (2.24) which is a function of three unknown parameters \( \mu \), \( \sigma^2 \) and \( \rho \). The maximum likelihood (ML) estimators \( \hat{\rho}(\rho) \) of \( \mu \) and \( \hat{\sigma}^2(\rho) \) of \( \sigma^2 \) are given by (2.25) and (2.26), respectively. It is shown that the maximum likelihood estimate \( \hat{\rho}_M \) of \( \rho \) can be obtained numerically by maximizing a single variable function \( \ell^* \) given by (2.27) and the asymptotic variance of \( \hat{\rho}_M \), called \( \text{Var}(\hat{\rho}_M) \), is given by (2.28).

Let \( \theta = [\theta_1, \theta_2, \theta_3]^T = [\rho, \mu, \sigma^2]^T \). The five asymptotically optimal tests for testing \( H_0: \rho = \rho_0 \) can be constructed by implementing the results of section 1.6.4.

Replacing \( \rho \) in \( \text{Var}(\hat{\rho}_M) \) by the specified value \( \rho_0 \) under \( H_0 \), the test statistic (1.19) gives

\[
Z_M = \frac{\hat{\rho}_M - \rho_0}{\left[ \text{Var}(\hat{\rho}_M) \right]^{1/2}}
\]

which is asymptotically normally distributed with zero mean and unit variance under \( H_0 \). Hence an asymptotic test of size \( \alpha \) is to reject \( H_0: \rho = \rho_0 \) in favor of the one-sided alternative \( H_1: \rho > \rho_0 \) if \( Z_M > Z_{1-\alpha} \), where \( Z_{1-\alpha} \) is the \( 100(1-\alpha) \) percentile point of the standard normal distribution.

Wald (1943) suggested that the \( \rho \) in \( \text{Var}(\hat{\rho}_M) \) should be replaced by its ML estimator \( \hat{\rho}_M \) such that

\[
\text{Var}(\hat{\rho}_M) = \text{Var}(\hat{\rho}_M) \big|_{\rho=\hat{\rho}_M}
\]

is a consistent estimator of \( \text{Var}(\hat{\rho}_M) \). This suggests using the statistic
\[ Z^*_W = \frac{\hat{\rho}_M - \rho_0}{[\text{Var}(\hat{\rho}_M)]^{1/2}}. \]  

(3.2)

which is also asymptotically normally distributed with zero mean and unit variance under \( H_0 \). Thus, another asymptotic test of size \( \alpha \) is to reject \( H_0: \rho = \rho_0 \) in favour of \( H_1: \rho > \rho_0 \) if \( Z^*_W > Z_{1-\alpha} \).

A likelihood ratio (LR) test of the hypothesis \( H_0: \rho = \rho_0 \) can be developed by using (1.20), which gives the statistic

\[ \lambda = N \ln \left[ \frac{\hat{\sigma}^2(\rho_0^*)}{\hat{\sigma}^2(\hat{\rho}_M^*)} \right] + (N-K) \ln \left[ \frac{1-\rho_0^*}{1-\hat{\rho}_M^*} \right] \]

\[ + \sum_{i=1}^{K} \ln \left[ \frac{1+(n_i-1)\rho_0}{1+(n_i-1)\hat{\rho}_M} \right], \]

(3.3)

where \( \hat{\sigma}^2(\cdot) \) is as defined by (2.26). Under \( H_0 \), the likelihood ratio test statistic \( \lambda \) is asymptotically distributed as chi-square with one degree of freedom. Thus for the two sided alternative \( H_1: \rho \neq \rho_0 \), the null hypothesis \( H_0: \rho = \rho_0 \) should be rejected if \( \lambda > \chi^2_{(1-\alpha)}(1) \), where \( \chi^2_{(1-\alpha)}(1) \) is the \( 100(1-\alpha) \) percentile point of the chi-square distribution with one degree of freedom. The likelihood ratio statistic \( \lambda \) should not be used for one sided alternatives because the problem of approximating its distribution for the one sided alternative still remains to be solved.

The locally optimal \( C(\alpha) \) (or partial score) statistics, due to Neyman (1959), are also of interest for the said hypothesis. Moran (1970) proved their asymptotic equivalence to the \( Z^*_M, Z^*_W \) and \( \lambda \) tests. Denote
\[ \psi = \psi(\rho, \mu, \sigma^2) = \frac{\partial l}{\partial \rho}, \quad r_1 = r_1(\rho, \mu, \sigma^2) = \frac{\partial l}{\partial \mu}, \]
and
\[ r_2 = r_2(\rho, \mu, \sigma^2) = \frac{\partial l}{\partial \sigma^2}. \]

Before we proceed further, we need to know the elements of Fisher's information matrix \( \mathbf{E}(\theta) \) as defined by (1.15). If we denote the \((i,j)\)th element of \( \mathbf{E}(\theta) \) by \( e_{ij} \), the elements of Fisher's information matrix are given by

\[ e_{11} = -E \left[ \frac{\partial^2 l}{\partial \rho^2} \right] = \frac{1}{2(1-\rho^2)} \sum_{i=1}^{K} \frac{n_i(n_i-1)(1+(n_i-1)\rho^2)}{[1+(n_i-1)\rho]^2}, \]

\[ e_{12} = e_{21} = -E \left[ \frac{\partial^2 l}{\partial \rho \partial \mu} \right] = 0, \]

\[ e_{13} = e_{31} = -E \left[ \frac{\partial^2 l}{\partial \rho \partial \sigma^2} \right] = -\frac{\rho}{2(1-\rho^2)} \sum_{i=1}^{K} \frac{1}{n_i} \frac{(n_i-1)}{[1+(n_i-1)\rho]} \]

\[ e_{22} = -E \left[ \frac{\partial^2 l}{\partial \mu^2} \right] = \frac{1}{\sigma^2} \sum_{i=1}^{K} \frac{n_i}{[1+(n_i-1)\rho]} \]

\[ e_{23} = e_{32} = -E \left[ \frac{\partial^2 l}{\partial \mu \partial \sigma^2} \right] = 0 \]

and

\[ e_{33} = -E \left[ \frac{\partial^2 l}{\partial \sigma^4} \right] = \frac{N}{2\sigma^4}. \] (3.4)

The CCW test statistics, from (1.27), are based on the function

\[ T = T(\rho, \mu, \sigma^2) = \psi - [r_1 \quad r_2] \begin{bmatrix} e_{22} & e_{23} \\ e_{32} & e_{33} \end{bmatrix}^{-1} \begin{bmatrix} e_{21} \\ e_{31} \end{bmatrix} \]

\[ = \psi + \frac{\rho \sigma^2}{N(1-\rho)} \left[ \sum_{i=1}^{K} \frac{n_i(n_i-1)}{[1+(n_i-1)\rho]} \right] r_2 \] (3.5)
where

\[ \psi = \frac{1}{2} \left\{ \frac{\rho}{1-\rho} \sum_{i=1}^{K} \frac{n_i(n_i-1)}{[1+(n_i-1)\rho]} \right\} \]

\[ - \frac{1}{\sigma^2} \left[ \frac{SSW}{(1-\rho)^2} - \sum_{i=1}^{K} \frac{n_i(n_i-1)(\bar{y}_i - \mu)^2}{[1+(n_i-1)\rho]^2} \right] \]

and

\[ \gamma^2 = \frac{-1}{2} \left[ \frac{N}{\sigma^2} - \frac{1}{\sigma^4} \left[ \frac{SSW}{(1-\rho)} + \sum_{i=1}^{K} \frac{n_i(y_i - \mu)^2}{[1+(n_i-1)\rho]} \right] \right]. \]

It is known that E(\Gamma) = 0 and, from (1.28), Var(\Gamma) = [Var(\hat{\rho}_M^2)]^{-1}. Notice that \( \hat{T} \) depends on the unknown nuisance parameters \( \mu \) and \( \sigma^2 \) which are not specified by the null hypothesis and it is still not appropriate to use. Moran (1970) suggested that the nuisance parameters in \( \hat{T} \) should be replaced by their \( \sqrt{n} \)-consistent estimators. It is well known, from (1.13), that the maximum likelihood estimators are \( \sqrt{n} \)-consistent. The maximum likelihood estimators of \( \mu \) and \( \sigma^2 \), under \( H_0 \), are given by

\[ \hat{\mu}(\rho_0) = \frac{1}{\sum_{i=1}^{K} \frac{n_i}{[1+(n_i-1)\rho_0]}} \left( \sum_{i=1}^{K} \frac{n_i \bar{y}_i}{[1+(n_i-1)\rho_0]} \right), \]

\[ \hat{\sigma}^2(\rho_0) = \frac{1}{N} \left[ \frac{SSW}{(1-\rho_0)^2} + \sum_{i=1}^{K} \frac{n_i(y_i - \hat{\mu}(\rho_0))^2}{[1+(n_i-1)\rho_0]} \right]. \]

(3.6)

If the maximum likelihood estimators of \( \mu \) and \( \sigma^2 \) are used in \( \hat{T} \) (3.5), then \( \hat{T} \), under \( H_0 \), is reduced to

\[ T_M = \psi(\rho_0, \hat{\mu}(\rho_0), \hat{\sigma}^2(\rho_0)) = \frac{1}{2} \left\{ \frac{\rho_0}{1-\rho_0} \sum_{i=1}^{K} \frac{n_i(n_i-1)}{[1+(n_i-1)\rho_0]} \right\} \]

\[ - \frac{1}{\hat{\sigma}^2(\rho_0)} \left[ \frac{SSW}{(1-\rho_0)^2} - \sum_{i=1}^{K} \frac{n_i(n_i-1)(\bar{y}_i - \hat{\mu}(\rho_0))^2}{[1+(n_i-1)\rho_0]^2} \right]. \]
Thus the $C(\alpha)$ test statistic, from (1.32), for testing $H_0: \rho = \rho_0$ is given by

$$Z_{CM} = T_M \left[ \text{Var}(\hat{\rho}_M) \big| \rho = \rho_0 \right]^{1/2}. \tag{3.7}$$

which is asymptotically normally distributed with zero mean and unit variance under $H_0$. Thus an asymptotic test of size $\alpha$ for the one sided alternative $H_1: \rho > \rho_0$ is to reject $H_0$ if $Z_{CM} > Z_{1-\alpha}$, where $Z_{1-\alpha}$ is as defined before.

The other appropriate choice for $\sqrt{n}$-consistent estimators of $\mu$ and $\sigma^2$ is the ANOVA estimators. The ANOVA estimators of $\mu$ and $\sigma^2$ have closed forms and are known to be $\sqrt{n}$-consistent estimators (see, e.g., Cramér, 1946). The ANOVA estimators of $\mu$ and $\sigma^2$ are given by

$$\hat{\mu} = \bar{y} = \frac{1}{N} \sum_{i=1}^{K} n_i \bar{y}_i$$

and

$$\hat{\sigma}^2 = \frac{1}{n_0} \left[ \text{MSA} + (n_0 - 1) \text{MSW} \right], \tag{3.8}$$

respectively, where $n_0$ is as defined by (2.13). If the ANOVA estimators (3.8) are used in $T$, then $T$, under $H_0$, becomes

$$T_A = \frac{-n_0}{2(\text{MSA} + (n_0 - 1) \text{MSW})} \left[ \frac{\text{SSW}}{(1-\rho_0)^2} - \frac{K n_1 (n_1 - 1) \bar{y}_1 - \bar{y}^2}{\sum_{i=1}^{K} \frac{1}{1+(n_i - 1)\rho_0^2}} \right]$$

$$- \frac{\rho_0}{K(1-\rho_0)^2} \left[ \frac{K n_1 (n_1 - 1)}{\sum_{i=1}^{K} \frac{1}{1+(n_i - 1)\rho_0^2}} \left[ \frac{\text{SSW}}{(1-\rho_0)^2} + \frac{K n_1 (\bar{y}_1 - \bar{y})^2}{\sum_{i=1}^{K} \frac{1}{1+(n_i - 1)\rho_0^2}} \right] \right] \right].$$

Thus, another $C(\alpha)$ test statistic, from (1.30), for testing $H_0: \rho = \rho_0$ is given by

$$Z_{CA} = T_A \left[ \text{Var}(\hat{\rho}_M) \big| \rho = \rho_0 \right]^{1/2}. \tag{3.9}$$
which is also asymptotically normally distributed with zero mean and unit variance under $H_0$. Thus, an asymptotic test of size $\alpha$ is to reject $H_0: \rho = \rho_0$ in favour of $H_1: \rho > \rho_0$ if $Z_{CA} > Z_{1-\alpha}$.

3.2.2 Exact and Approximate F-Tests

Several tests can be constructed by using the summary statistics provided by the ANOVA table (2.2), where the size of the test is obtained from the table of F-distribution with (K-1) and (N-K) degrees of freedom. The first of these tests is due to Wald (1940) who showed that under $H_0$, the quantity

$$F_E = \frac{(1-\rho_0) \sum_{i=1}^{K} \frac{n_i}{1+(n_i-1)\rho_0} (\bar{y}_i - \hat{\mu}(\rho_0))^2}{(K-1) MSW}$$

(3.10)

has an exact F-distribution with (K-1) and (N-K) degrees of freedom, where $\hat{\mu}(\rho_0)$ is as defined by (3.8). Thus an exact test of size $\alpha$ is to reject $H_0: \rho = \rho_0$ in favour of the one-sided alternative $H_1: \rho > \rho_0$ if $F_E > F_{1-\alpha}(K-1, N-K)$, where $F_{1-\alpha}(K-1, N-K)$ is the 100(1-\alpha) percentile point of the F-distribution with (K-1) and (N-K) degrees of freedom.

Another exact test was derived independently by Bhargava (1946) and Spjøtvoll (1967) and is given by

$$F_E = \frac{(N-K) \sum_{i=1}^{K} \frac{n_i}{1+n_i A_0} (\bar{y}_i - \bar{y}_0)^2}{(K-1) \sum_{i=1}^{K} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2}$$

where

$$\bar{y}_0 = \frac{K \sum_{i=1}^{K} \frac{n_i}{1+n_i A_0} \bar{y}_i}{\sum_{i=1}^{K} \frac{n_i}{1+n_i A_0}}$$
and

\[ \Delta_0 = \frac{\rho_0}{1-\rho_0} \]

is the specified value of the variance ratio

\[ \Delta = \frac{\sigma_a^2}{\sigma_e^2} = \frac{\rho}{1-\rho} . \]

Under \( H_0 \), the statistic \( F_E^* \) also has an exact F-distribution with \((K-1)\) and \((N-K)\) degrees of freedom. It can be easily shown that \( F_E^* \) is equivalent to Wald's \( F_E \).

The following three tests will have approximate F-distributions and are shown to compare favourably with \( F_E \):

1. The usual analysis of variance ratio statistic

\[ F = \frac{\text{MSE}}{\text{MSW}} = \frac{1+(n_0-1)\hat{\rho}_A}{1-\hat{\rho}_A} \quad \text{(3.11)} \]

can be used to test the null hypothesis \( H_0: \rho = 0 \) against the alternative \( H_1: \rho > 0 \). The statistic \( F \) (3.11) can be modified to test a general hypothesis \( H_0: \rho = \rho_0 \) by considering the quantity

\[ Q_1 = \frac{\text{SSA}}{1+(n_0-1)\rho_0 \sigma_e^2} . \]

The quantity \( Q_1 \) does not have an exact chi-square distribution unless all \( n_i \)'s are equal. However, it may be approximated by a chi-squared distribution with \((K-1)\) degree of freedom. Furthermore the quantity

\[ Q_2 = \frac{\text{SSW}}{(1-\rho)\sigma_e^2} \]

has exact chi-square distribution with \((N-K)\) degree of freedom. Since \( Q_1 \) and \( Q_2 \) are independent, the quantity
\[
\frac{Q_1}{(K-1)} / \frac{Q_2}{(N-K)} = \frac{(1-\rho) F}{[1+(n_0-1)\rho]}
\]

may be have an approximate F-distribution with \((K-1)\) and \((N-K)\) degrees of freedom. Thus, the modified F-ANOVA statistic to test \(H_0: \rho = \rho_0\) is

\[
F_A = \frac{(1-\rho_0) F}{[1+(n_0-1)\rho_0]}
\]

This modification to \(F\) was suggested by Donner (1979) for the unbalanced case. Thus an approximate test of size \(\alpha\) is to reject \(H_0: \rho = \rho_0\) in favour of \(H_1: \rho > \rho_0\) if \(F_A > F_{1-\alpha}(K-1,N-K)\).

The statistic \(F_A\) (3.12) does not have an exact F-distribution in the unbalanced case, unless \(\rho_0 = 0\). Thus the accuracy of the approximation presumably declines as the specified value of \(\rho\) increases.

(2) Thomas and Hultquist (1978) have shown that the quantity

\[
F_T = \frac{(1-\rho_0) s_y^2}{[\rho_0 \cdot (1-\rho_0) \lambda_1] MSW}
\]

has approximate F-distribution with \((K-1)\) and \((N-K)\) degrees of freedom under \(H_0\), where \(\lambda_1\) and \(s_y^2\) are as defined in section 2.3. Thus, an approximate test of size \(\alpha\) for the one-sided alternative \(H_1: \rho > \rho_0\) is to reject \(H_0\) if \(F_T > F_{1-\alpha}(K-1,N-K)\).

(3) In cases when the variation among group sizes is negligible, we may replace \(\hat{\mu}(\rho_0)\) in (3.10) by \(\bar{y}\) given by (2.9). Therefore, we may propose the statistic
\[
F_p = \frac{(1 - \rho_0) \sum_{i=1}^{K} \alpha_i (\rho_0) (\bar{y}_i - \bar{y})^2}{(K-1) MSW} \tag{3.14}
\]

to test \(H_0: \rho = \rho_0\). Thus \(F_p\) may also have an approximate F-distribution with \((K-1)\) and \((N-K)\) degrees of freedom. Therefore another approximate test of size \(\alpha\) is to reject \(H_0\) in favour of \(H_1: \rho > \rho_0\) if \(F_p > F_{1-\alpha}(K-1,N-K)\).

It is interesting to note that in the case of equal group sizes (i.e., \(n_i = n, \quad i = 1,2,\ldots,K\)), the statistics \(F_1, F_2, F_T\) and \(F_p\) reduce to a common quantity

\[
F^* = \frac{n(1 - \rho_0) \sum_{i=1}^{K} (\bar{y}_i - \bar{y})^2}{[1+(n-1)\rho_0] \sum_{i=1}^{K} \sum_{j=1}^{n} (y_{ij} - \bar{y}_i)^2},
\]

which was given by Fisher (1925) for balanced designs.

### 3.2.3 Other Tests

The five other test statistics can be derived by using the estimators \(\hat{\rho}_A\) (2.30), \(\hat{\rho}_S\) (2.38), \(\hat{\rho}_K\) (2.41), \(\hat{\rho}_E\) (2.47) and \(\hat{\rho}_I\) (2.42) of \(\rho\) along with their asymptotic variances (2.37), (2.40), (2.45), (2.49) and (2.44), respectively. Thus, by comparing the estimator to the square root of its asymptotic variance under \(H_0\), the statistics

\[
Z_i = \frac{\hat{\rho}_i - \rho_0}{\left[\text{Var}(\hat{\rho}_i)\right]^{1/2}}, \quad i = A, S, K, E \text{ and } I \tag{3.15}
\]

are assumed to be asymptotically normally distributed with zero means and unit variances. Hence five asymptotic tests each of size \(\alpha\) can be constructed by comparing the values of
the test statistics with $Z_{1-\alpha}$, where $Z_{1-\alpha}$ is the 100(1-\alpha)
percentile point of the standard normal distribution. A
large value of the test statistic is an indication that
$H_0: \rho = \rho_0$ is rejected in favour of the one-sided alternative
$H_1: \rho > \rho_0$, at a preassigned level $\alpha$.

Another test statistic for testing $H_0$ may be formulated
by using a transformation on $\hat{\rho}_A$ (2.30) by Fisher (1925).
Fisher showed that the quantity
\[
\phi(\hat{\rho}_A) = \frac{1}{2} \ln \left[ \frac{1+(n_0-1)\hat{\rho}_A}{1-\hat{\rho}_A} \right] = \frac{1}{2} \ln(F)
\]
has an approximate normal distribution with mean
\[
\phi(\rho) = \frac{1}{2} \ln \left[ \frac{1+(n_0-1)\rho}{1-\rho} \right]
\]
and variance
\[
\nu = \frac{1}{2} \left[ \frac{1}{k-1} + \frac{1}{N-K} \right].
\]
Therefore, under $H_0$, the test statistic
\[
Z_F = \frac{\phi(\hat{\rho}_A) - \phi(\rho_0)}{\sqrt{\nu}} \tag{3.16}
\]
is approximately normally distributed with zero mean and
unit variance. Thus another asymptotic test of size $\alpha$ is to
reject $H_0: \rho = \rho_0$ in favour of $H_1: \rho > \rho_0$ if $Z_F > Z_{1-\alpha}$, where
$Z_{1-\alpha}$ is as defined before.

All the fifteen test statistics described in this
section can also be used to test $H_0: \rho = \rho_0$ against the two-
sided alternative $H_1: \rho \neq \rho_0$. The rejection criteria of $Z
statistics for the said hypothesis, at preassigned level $\alpha$, is to reject $H_0$ if $|Z_1| > Z_{1-\alpha/2}$ (i = M,W,CM,CA,A,S,K,E,F),
where $Z_{1-\alpha/2}$ is the $100(1-\alpha/2)$ percentile point of the standard normal distribution. The rejection criteria for $F$ statistics for two sided alternatives are to reject $H_0: \rho = \rho_0$ if $F < F_{\alpha/2}(K-1,N-K)$ or $F > F_{1-\alpha/2}(K-1,N-K)$ ($i = E,A,T,P$), where $F_{\alpha/2}(K-1,N-K)$ and $F_{1-\alpha/2}(K-1,N-K)$ are, respectively, the $100(\alpha/2)$ and $100(1-\alpha/2)$ percentile points of the $F$-distribution with $(K-1)$ and $(N-K)$ degrees of freedom.

3.3 A Monte Carlo Investigation

The investigation of the theoretical properties of the test procedures discussed in section 3.2 is not feasible in the case of finite samples. Therefore, a Monte Carlo study was designed to determine the empirical levels and powers of these tests. The difficulty of infinitely many group sizes was overcome by focusing the aim of our investigation on the analysis of familial data and generating the sibship sizes from the distribution reflective of those that occur in practice. Brass (1959) has shown that the zero-truncated negative binomial distribution fits the observed distribution of sibship sizes very well in a wide variety of human populations. The probability mass function of this distribution with parameters $m$ and $P$ is given by

$$P(n) = \frac{(m+n-1)!}{n!(m-1)!} \left[ \frac{P}{Q} \right]^{-n} \left[ Q^m - 1 \right]^{-1}, \quad n = 1, 2, \ldots; Q=1+P.$$  

(3.17)

This sibship size distribution has been used before by Rosner et al. (1977), Donner and Koval (1980a), Keen (1987) and others to generate sibship sizes in simulation studies.
In this present simulation study, the parameters \( m = 2.84 \) and \( P = 0.93 \) (values reported by Brass (1958) for the population of United States in 1950) were chosen to generate sibship sizes by subroutine GEEDA from IMSL (1987) for \( K = 25, 50 \) and 100. In order to ensure the correlation structure of the model (2.4) for each family, the method of conditioning was used to generate the sample observations \( y_{ij} \), by implementing the algorithm of Donner and Koval (1980a). This method is described in the algorithm for each of \( K \) families as follows:

For \( i = 1, 2, \ldots, K \)

(i) Generate a set of \( n_i \) independent standard normal deviates \( z_{ij} \), \( j = 1, 2, \ldots, n_i \).

(ii) Set \( y_{i1} = z_{i1} \).

(iii) Calculate the remaining scores iteratively as follows:

\[
y_{ij} = M_{ij} + z_{ij} \sqrt{V_j}, \quad j = 2, 3, \ldots, n_i,
\]

where

\[
M_{ij} = \frac{\rho}{[1+(j-2)\rho]} \sum_{g=1}^{j-1} y_{ig}
\]

and

\[
V_j = \frac{(1-\rho)(1+(j-1)\rho)}{(1+(j-2)\rho)}.
\]

The \( M_{ij} \) and \( V_j \) given above are the conditional mean and variance, respectively, of \( y_{ij} \) given \( y_{i1}, y_{i2}, \ldots, y_{i(j-1)} \) as derived from the standard multivariate normal theory (see, e.g., Anderson, 1984). The collection of standard normal deviates was generated by subroutine RNNOA and the maximum likelihood estimate of \( \rho \) was obtained by maximizing \( r^* \), given by (2.27), using subroutine DUVMGS from IMSL (1987).
The empirical levels and powers of the tests were all based on 1000 repeated samples for different combinations of K and \( p \). It appears that for some simulated samples, the subroutine DUVMGSS was unable to locate a satisfactory local maximum of \( l^* \). This confirms a comment made by Smith (1980a,b) that the convergence of any iterative procedure for determining the maximum likelihood estimate is not guaranteed. If for some simulated samples, the optimization criteria were not met then this sample was discarded and replaced by another generated sample. This process continued until 1000 samples were available, each of which gave a satisfactory maximum likelihood estimate of \( p \).

The test procedure \( Z_1 \) (see 3.15), based on the individual estimator \( \hat{\rho}_1 \) of Karlin et al. (1981), was excluded from this comparison because its variance requires all group sizes \( n_i \) to be greater than one. But when using zero truncated negative binomial distribution for this purpose, the expected percentage of sibships of size one is almost 25 percent.

3.4 Results

The proportion of the tests for which \( H_0 \) was rejected was recorded and called the empirical levels. Tables 3.1 - 3.3 give the empirical levels of the test procedures, except LR test \( \lambda \), for testing \( H_0: \rho = \rho_0 \) against one sided alternative \( H_1: \rho > \rho_0 \), where \( \rho_0 = 0, 0.3 \) and 0.5, respectively. These tables show that the \( Z_M \) (3.1), \( Z_S \) (see 3.15) and \( Z_K \) (See 3.15) tests give empirical levels that are
TABLE 3.1

Empirical levels of the tests based on 1000 runs at $\alpha = 0.01, 0.05, 0.10$ and $K = 25, 50, 100$ for testing $H_0: \rho = 0$ against $H_1: \rho > 0$.

<table>
<thead>
<tr>
<th>K</th>
<th>Test</th>
<th>25</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\alpha$</td>
<td>0.01</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.01</td>
<td>0.05</td>
</tr>
<tr>
<td>25</td>
<td>$Z_M$</td>
<td>0.028</td>
<td>0.034</td>
<td>0.038</td>
</tr>
<tr>
<td></td>
<td>$Z_W$</td>
<td>0.008</td>
<td>0.018</td>
<td>0.050</td>
</tr>
<tr>
<td></td>
<td>$Z_{CM}$</td>
<td>0.016</td>
<td>0.038</td>
<td>0.072</td>
</tr>
<tr>
<td></td>
<td>$Z_{CA}$</td>
<td>0.015</td>
<td>0.037</td>
<td>0.067</td>
</tr>
<tr>
<td></td>
<td>$F_E$</td>
<td>0.010</td>
<td>0.054</td>
<td>0.107</td>
</tr>
<tr>
<td></td>
<td>$F_A$</td>
<td>0.010</td>
<td>0.054</td>
<td>0.107</td>
</tr>
<tr>
<td></td>
<td>$F_T$</td>
<td>0.021</td>
<td>0.077</td>
<td>0.134</td>
</tr>
<tr>
<td></td>
<td>$F_P$</td>
<td>0.012</td>
<td>0.080</td>
<td>0.121</td>
</tr>
<tr>
<td></td>
<td>$Z_A$</td>
<td>0.012</td>
<td>0.056</td>
<td>0.109</td>
</tr>
<tr>
<td></td>
<td>$Z_S$</td>
<td>0.002</td>
<td>0.037</td>
<td>0.082</td>
</tr>
<tr>
<td></td>
<td>$Z_K$</td>
<td>0.003</td>
<td>0.050</td>
<td>0.113</td>
</tr>
<tr>
<td></td>
<td>$Z_E$</td>
<td>0.002</td>
<td>0.042</td>
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<td></td>
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<td>0.008</td>
<td>0.045</td>
<td>0.097</td>
</tr>
</tbody>
</table>

* The observed empirical level is more than 2 standard deviations from the true level $\alpha$. 
TABLE 3.2:

Empirical levels of the tests based on 1000 runs at $\alpha = 0.01, 0.05, 0.10$

and $K = 25, 50, 100$ for testing $H_0: \rho = 0.3$ against $H_1: \rho > 0.3$.

<table>
<thead>
<tr>
<th>Test</th>
<th>25</th>
<th></th>
<th>50</th>
<th></th>
<th>100</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha$</td>
<td></td>
<td>$\alpha$</td>
<td></td>
<td>$\alpha$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>0.05</td>
<td>0.10</td>
<td>0.01</td>
<td>0.05</td>
<td>0.10</td>
</tr>
<tr>
<td>$Z_M$</td>
<td>0.004*</td>
<td>0.030*</td>
<td>0.068*</td>
<td>0.005*</td>
<td>0.030*</td>
<td>0.070*</td>
</tr>
<tr>
<td>$Z_N$</td>
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<td>0.039*</td>
<td>0.072*</td>
<td>0.012*</td>
<td>0.035*</td>
<td>0.078*</td>
</tr>
<tr>
<td>$Z_{CM}$</td>
<td>0.013*</td>
<td>0.039*</td>
<td>0.075*</td>
<td>0.013*</td>
<td>0.042*</td>
<td>0.083</td>
</tr>
<tr>
<td>$Z_{CA}$</td>
<td>0.002*</td>
<td>0.020*</td>
<td>0.051*</td>
<td>0.005*</td>
<td>0.026*</td>
<td>0.062*</td>
</tr>
<tr>
<td>$F_E$</td>
<td>0.007*</td>
<td>0.050*</td>
<td>0.098</td>
<td>0.012*</td>
<td>0.040*</td>
<td>0.093</td>
</tr>
<tr>
<td>$F_A$</td>
<td>0.011*</td>
<td>0.050*</td>
<td>0.100</td>
<td>0.017*</td>
<td>0.050*</td>
<td>0.103</td>
</tr>
<tr>
<td>$F_T$</td>
<td>0.011*</td>
<td>0.052*</td>
<td>0.120</td>
<td>0.010*</td>
<td>0.047*</td>
<td>0.094</td>
</tr>
<tr>
<td>$F_P$</td>
<td>0.008*</td>
<td>0.051*</td>
<td>0.099</td>
<td>0.012*</td>
<td>0.042*</td>
<td>0.094</td>
</tr>
<tr>
<td>$Z_A$</td>
<td>0.007*</td>
<td>0.027*</td>
<td>0.076*</td>
<td>0.005*</td>
<td>0.035*</td>
<td>0.081</td>
</tr>
<tr>
<td>$Z_S$</td>
<td>0.001*</td>
<td>0.020*</td>
<td>0.072*</td>
<td>0.001*</td>
<td>0.022*</td>
<td>0.073*</td>
</tr>
<tr>
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<td>0.025*</td>
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</tr>
<tr>
<td>$Z_E$</td>
<td>0.000*</td>
<td>0.023*</td>
<td>0.076*</td>
<td>0.002*</td>
<td>0.031*</td>
<td>0.074*</td>
</tr>
<tr>
<td>$Z_T$</td>
<td>0.010*</td>
<td>0.045*</td>
<td>0.093</td>
<td>0.013*</td>
<td>0.046*</td>
<td>0.097</td>
</tr>
</tbody>
</table>

* The observed empirical level is more than 2 standard deviations from the true level $\alpha$. 
TABLE 3.3

Empirical levels of the tests based on 1000 runs at $\alpha = 0.01, 0.05, 0.10$
and $K = 25, 50, 100$ for testing $H_0: \rho = 0.5$ against $H_1: \rho > 0.5$.

<table>
<thead>
<tr>
<th>K</th>
<th>25</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha$</td>
<td>$\alpha$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>Test</td>
<td>0.01</td>
<td>0.05</td>
<td>0.10</td>
</tr>
<tr>
<td>$Z_M$</td>
<td>.001*</td>
<td>.015*</td>
<td>.051*</td>
</tr>
<tr>
<td>$Z_W$</td>
<td>.015</td>
<td>.048</td>
<td>.085</td>
</tr>
<tr>
<td>$Z_{CM}$</td>
<td>.008</td>
<td>.040</td>
<td>.078*</td>
</tr>
<tr>
<td>$Z_{CA}$</td>
<td>.000*</td>
<td>.011*</td>
<td>.042*</td>
</tr>
<tr>
<td>$F_E$</td>
<td>.007</td>
<td>.043</td>
<td>.101</td>
</tr>
<tr>
<td>$F_A$</td>
<td>.013</td>
<td>.060</td>
<td>.101</td>
</tr>
<tr>
<td>$F_T$</td>
<td>.012</td>
<td>.049</td>
<td>.111</td>
</tr>
<tr>
<td>$F_E$</td>
<td>.007</td>
<td>.044</td>
<td>.101</td>
</tr>
<tr>
<td>$Z_A$</td>
<td>.000*</td>
<td>.016</td>
<td>.060</td>
</tr>
<tr>
<td>$Z_S$</td>
<td>.000*</td>
<td>.015</td>
<td>.054*</td>
</tr>
<tr>
<td>$Z_K$</td>
<td>.000*</td>
<td>.018</td>
<td>.068*</td>
</tr>
<tr>
<td>$Z_E$</td>
<td>.000*</td>
<td>.011*</td>
<td>.049*</td>
</tr>
<tr>
<td>$Z_F$</td>
<td>.009</td>
<td>.047</td>
<td>.091</td>
</tr>
</tbody>
</table>

* The observed empirical level is more than 2 standard deviations from the true level $\alpha$. 
considerably different from the nominal levels $\alpha = 0.01$, 0.05 and 0.10. The $F_T$ (3.13) test is too liberal for $\rho_0 = 0$. Furthermore, the test procedures $Z_{CA}$ (3.9) and $Z_{-T}$ (see 3.15) for $\rho_0 \geq 0.3$ and $Z_W$ (3.2) for $\rho_0 \leq 0.3$ are very conservative as they give empirical levels which are smaller than the nominal levels. The empirical levels of all three other procedures show satisfactory agreement with the stated nominal levels. It is quite important now to determine which test is to be used for one sided alternatives, using powers as the criterion.

The empirical power curves of the tests are presented in Figures 3.1 (a) through 3.3 (f) where each curve is identified by a number as follows:

1. $Z_M$
2. $Z_W$
3. $Z_{CA}$
4. $F_E$
5. $F_T$
6. $F_P$
7. $Z_A$
8. $Z_S$
9. $Z_K$
10. $Z_E$
11. $Z_T$

For testing $H_0: \rho = 0$ against $H_1: \rho > 0$, the graph of empirical powers for the tests are shown in Figures 3.1.
(a) - (h). These figures show that the procedure $Z_M$, based on the standardized maximum likelihood estimator of $\rho$, retains higher power, which might be due to its anti-conservative behaviour for testing $H_0: \rho = 0$. The test procedures $F_E$ (3.10) and $F_A$ (3.12) are identical for the said hypothesis, and are asymptotically as powerful as the procedures $F_P$ (3.13) and $Z_A$ (see 3.15). The $C(a)$ test procedures $Z_{CM}$ (3.7) and $Z_{CA}$ (3.9) seem to be equally powerful and are very competitive with the $Z_M$ test for this hypothesis. The test procedures $F_T$, $Z_S$, $Z_K$ and $Z_E$ are least powerful for testing the said hypothesis.

Figures 3.2 (a) - (f) show the graphs of empirical powers for testing $H_0: \rho = 0.3$ against $H_1: \rho > 0.3$, and Figures 3.3 (a) - (f) for testing $H_0: \rho = 0.5$ against $H_1: \rho > 0.5$. Clearly, all the test procedures seem to be asymptotically equally powerful, except the $Z_S$, $Z_K$ and $Z_P$ tests which are least powerful. However, the $F_A$ for $H_1: \rho > 0.5$ seems to be more powerful than all the other procedures.

For testing against two sided alternatives, Tables 3.4 (a) - (d) compare the empirical levels and powers of the test procedures (including the LR test $\lambda$) for testing $H_0: \rho = 0.3$ against $H_1: \rho \neq 0.3$ and Tables 3.5 (a) - (d) for testing $H_0: \rho = 0.5$ against $H_1: \rho \neq 0.5$. These tables show that the empirical levels of the test procedures $Z_V$ and $Z_F$ are considerably different from the stated nominal levels. The same is noticed for the empirical levels of $Z_M$ and $F_A$ procedures, when $H_0: \rho = 0.5$ is being tested. All the other
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0$ VS $H_1: \rho > 0$

$K = 25$ AND $\alpha = 0.01$

Figure 3.1 (a)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0$ VS $H_1: \rho > 0$

$K=25$ AND $\alpha=0.05$

Figure 3.1 (b)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0 \text{ vs } H_1: \rho > 0$

$K = 25 \text{ and } \alpha = 0.10$

Figure 3.1 (c)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0$ VS $H_1: \rho > 0$

$K = 50$ AND $\alpha = 0.01$

Figure 3.1 (d)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0$ VS $H_1: \rho > 0$

$K=50$ AND $\alpha=0.05$

Figure 3.1 (e)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0$ VS $H_1: \rho > 0$

$K=50$ AND $\alpha=0.10$

Figure 3.1 (f)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho=0$ VS $H_1: \rho>0$

$K=100$ AND $\alpha=0.01$

Figure 3.1 (g)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0$ VS $H_1: \rho > 0$

$K = 100$ AND $\alpha = 0.05$

Figure 3.1 (h)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0.3$ VS $H_1: \rho > 0.3$

$K=25$ AND $\alpha=0.01$

Figure 3.2 (a)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0.3$ VS $H_1: \rho > 0.3$

$K = 25$ AND $\alpha = 0.10$

Figure 3.2 (b)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0.3$ VS $H_1: \rho > 0.3$

$K = 50$ AND $\alpha = 0.01$

Figure 3.2 (c)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0.3$ VS $H_1: \rho > 0.3$

$K = 50$ AND $\alpha = 0.05$

Figure 3.2 (d)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0.3$ VS $H_1: \rho > 0.3$

$K = 100$ AND $\alpha = 0.01$

Figure 3.2 (e)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho=0.3$ VS $H_1: \rho>0.3$

$K=100$ AND $\alpha=0.05$

Figure 3.2 (f)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0.5$ VS $H_1: \rho > 0.5$

$K = 25$ AND $\alpha = 0.01$

Figure 3.3 (a)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0.5$ VS $H_1: \rho > 0.5$

$K = 25$ AND $\alpha = 0.05$

Figure 3.3 (b)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho=0.5$ vs $H_1: \rho>0.5$

$K=25$ AND $\alpha=0.10$

Figure 3.3 (c)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0.5$ VS $H_1: \rho > 0.5$

$K = 50$ AND $\alpha = 0.05$

**Figure 3.3 (d)**
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0.5$ VS $H_1: \rho > 0.5$

$K = 50$ AND $\alpha = 0.10$

Figure 3.3 (e)
POWERS OF THE TESTS FOR $\rho$

$H_0: \rho = 0.5$ VS $H_1: \rho > 0.5$

$K = 100$ AND $\alpha = 0.01$

Figure 3.3 (f)
TABLE 3.4 (a)

Empirical levels and powers of the tests based on 1000 runs at $\alpha = 0.01$ and $K = 25$ for testing $H_0: \rho = 0.3$ against $H_1: \rho = 0.3$.

<table>
<thead>
<tr>
<th>Test</th>
<th>Actual values of $\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.493</td>
</tr>
<tr>
<td>$Z_M$</td>
<td>0.422</td>
</tr>
<tr>
<td>$Z_W$</td>
<td>0.790</td>
</tr>
<tr>
<td>$Z_{CM}$</td>
<td>0.114</td>
</tr>
<tr>
<td>$Z_{CA}$</td>
<td>0.400</td>
</tr>
<tr>
<td>$F_E$</td>
<td>0.243</td>
</tr>
<tr>
<td>$F_A$</td>
<td>0.350</td>
</tr>
<tr>
<td>$F_T$</td>
<td>0.199</td>
</tr>
<tr>
<td>$F_P$</td>
<td>0.239</td>
</tr>
<tr>
<td>$Z_A$</td>
<td>0.344</td>
</tr>
<tr>
<td>$Z_S$</td>
<td>0.393</td>
</tr>
<tr>
<td>$Z_K$</td>
<td>0.174</td>
</tr>
<tr>
<td>$Z_E$</td>
<td>0.278</td>
</tr>
<tr>
<td>$Z_F$</td>
<td>0.446</td>
</tr>
</tbody>
</table>

* The observed empirical level is more than 2 standard deviations from the true level $\alpha$. 
TABLE 3.4 (b)

Empirical levels and powers of the tests based on 1000 runs at $\alpha = 0.01$ and $k = 50$ for testing $H_0: \rho = 0.3$ against $H_1: \rho \neq 0.3$.

<table>
<thead>
<tr>
<th>Test</th>
<th>$\lambda$</th>
<th>$Z_M$</th>
<th>$Z_W$</th>
<th>$Z_{CM}$</th>
<th>$Z_{CA}$</th>
<th>$F_E$</th>
<th>$F_A$</th>
<th>$F_T$</th>
<th>$F_P$</th>
<th>$Z_A$</th>
<th>$Z_S$</th>
<th>$Z_K$</th>
<th>$Z_E$</th>
<th>$Z_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.847</td>
<td>0.361</td>
<td>0.077</td>
<td>0.017</td>
<td>0.056</td>
<td>0.348</td>
<td>0.788</td>
<td>0.979</td>
<td>1.000</td>
<td>0.844</td>
<td>0.359</td>
<td>0.077</td>
<td>0.013</td>
<td>0.035</td>
</tr>
<tr>
<td></td>
<td>0.936</td>
<td>0.547</td>
<td>0.156</td>
<td>0.038</td>
<td>0.072</td>
<td>0.378</td>
<td>0.813</td>
<td>0.984</td>
<td>1.000</td>
<td>0.914</td>
<td>0.185</td>
<td>0.033</td>
<td>0.011</td>
<td>0.080</td>
</tr>
<tr>
<td></td>
<td>0.910</td>
<td>0.361</td>
<td>0.076</td>
<td>0.014</td>
<td>0.036</td>
<td>0.258</td>
<td>0.700</td>
<td>0.951</td>
<td>0.999</td>
<td>0.585</td>
<td>0.235</td>
<td>0.055</td>
<td>0.011</td>
<td>0.060</td>
</tr>
<tr>
<td></td>
<td>0.767</td>
<td>0.317</td>
<td>0.069</td>
<td>0.017</td>
<td>0.084</td>
<td>0.413</td>
<td>0.830</td>
<td>0.984</td>
<td>1.000</td>
<td>0.443</td>
<td>0.187</td>
<td>0.053</td>
<td>0.012</td>
<td>0.047</td>
</tr>
<tr>
<td></td>
<td>0.582</td>
<td>0.233</td>
<td>0.055</td>
<td>0.012</td>
<td>0.060</td>
<td>0.354</td>
<td>0.801</td>
<td>0.982</td>
<td>1.000</td>
<td>0.760</td>
<td>0.310</td>
<td>0.067</td>
<td>0.014</td>
<td>0.040</td>
</tr>
<tr>
<td></td>
<td>0.525</td>
<td>0.259</td>
<td>0.074</td>
<td>0.012</td>
<td>0.012</td>
<td>0.113</td>
<td>0.483</td>
<td>0.808</td>
<td>0.998</td>
<td>0.417</td>
<td>0.171</td>
<td>0.041</td>
<td>0.005</td>
<td>0.008</td>
</tr>
<tr>
<td></td>
<td>0.509</td>
<td>0.233</td>
<td>0.058</td>
<td>0.011</td>
<td>0.012</td>
<td>0.098</td>
<td>0.440</td>
<td>0.877</td>
<td>0.996</td>
<td>0.509</td>
<td>0.233</td>
<td>0.058</td>
<td>0.011</td>
<td>0.012</td>
</tr>
<tr>
<td></td>
<td>0.823</td>
<td>0.383</td>
<td>0.085</td>
<td>0.019</td>
<td>0.073</td>
<td>0.374</td>
<td>0.818</td>
<td>0.980</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* The observed empirical level is more than 2 standard deviations from the true level $\alpha$. 
<table>
<thead>
<tr>
<th>Test</th>
<th>$Z_N$</th>
<th>$Z_W$</th>
<th>$Z_{CA}$</th>
<th>$F_E$</th>
<th>$F_A$</th>
<th>$F_T$</th>
<th>$Z_A$</th>
<th>$Z_S$</th>
<th>$Z_K$</th>
<th>$Z_E$</th>
<th>$Z_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.785</td>
<td>0.776</td>
<td>0.870</td>
<td>0.490</td>
<td>0.408</td>
<td>0.483</td>
<td>0.340</td>
<td>0.369</td>
<td>0.492</td>
<td>0.185</td>
<td>0.303</td>
</tr>
<tr>
<td>0.1</td>
<td>0.402</td>
<td>0.398</td>
<td>0.299</td>
<td>0.246</td>
<td>0.227</td>
<td>0.238</td>
<td>0.157</td>
<td>0.153</td>
<td>0.167</td>
<td>0.270</td>
<td>0.383</td>
</tr>
<tr>
<td>0.2</td>
<td>0.153</td>
<td>0.143</td>
<td>0.089</td>
<td>0.114</td>
<td>0.115</td>
<td>0.113</td>
<td>0.094</td>
<td>0.094</td>
<td>0.103</td>
<td>0.120</td>
<td>0.101</td>
</tr>
<tr>
<td>0.3</td>
<td>0.093</td>
<td>0.087</td>
<td>0.059</td>
<td>0.114</td>
<td>0.109</td>
<td>0.113</td>
<td>0.059</td>
<td>0.059</td>
<td>0.063</td>
<td>0.075</td>
<td>0.065</td>
</tr>
<tr>
<td>0.3+</td>
<td>0.094</td>
<td>0.098</td>
<td>0.089</td>
<td>0.069</td>
<td>0.058</td>
<td>0.036</td>
<td>0.087</td>
<td>0.087</td>
<td>0.094</td>
<td>0.087</td>
<td>0.087</td>
</tr>
</tbody>
</table>

The observed empirical level is more than 2 standard deviations from the true level $\alpha$. The table shows the empirical levels and powers of tests for $H_0: \rho = 0.3$ against $H_1: \rho = 0.3$ with $K = 25$ for testing $H_0: \rho = 0.3$ against $H_1: \rho = 0.3$.

For $\alpha = 0.05$, the table shows the actual values of $\rho$ for different tests.

The table is based on 1000 runs.
<table>
<thead>
<tr>
<th>Test</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.053</td>
<td>0.530</td>
<td>0.224</td>
<td>0.056</td>
<td>0.169</td>
<td>0.502</td>
<td>0.613</td>
<td>0.993</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_M$</td>
<td>0.936</td>
<td>0.647</td>
<td>0.245</td>
<td>0.061</td>
<td>0.134</td>
<td>0.527</td>
<td>0.895</td>
<td>0.991</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_w$</td>
<td>0.970</td>
<td>0.719</td>
<td>0.298</td>
<td>0.079*</td>
<td>0.180</td>
<td>0.586</td>
<td>0.913</td>
<td>0.993</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_{CM}$</td>
<td>0.906</td>
<td>0.517</td>
<td>0.150</td>
<td>0.049</td>
<td>0.201</td>
<td>0.605</td>
<td>0.917</td>
<td>0.995</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_{CA}$</td>
<td>0.938</td>
<td>0.516</td>
<td>0.215</td>
<td>0.053</td>
<td>0.129</td>
<td>0.489</td>
<td>0.885</td>
<td>0.990</td>
<td>1.000</td>
</tr>
<tr>
<td>$F_e$</td>
<td>0.823</td>
<td>0.460</td>
<td>0.152</td>
<td>0.049</td>
<td>0.171</td>
<td>0.563</td>
<td>0.909</td>
<td>0.999</td>
<td>1.000</td>
</tr>
<tr>
<td>$F_A$</td>
<td>0.925</td>
<td>0.572</td>
<td>0.195</td>
<td>0.053</td>
<td>0.211</td>
<td>0.631</td>
<td>0.927</td>
<td>0.999</td>
<td>1.000</td>
</tr>
<tr>
<td>$F_T$</td>
<td>0.889</td>
<td>0.398</td>
<td>0.143</td>
<td>0.054</td>
<td>0.160</td>
<td>0.495</td>
<td>0.873</td>
<td>0.990</td>
<td>1.000</td>
</tr>
<tr>
<td>$F_P$</td>
<td>0.822</td>
<td>0.456</td>
<td>0.161</td>
<td>0.050</td>
<td>0.173</td>
<td>0.585</td>
<td>0.909</td>
<td>0.999</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_A$</td>
<td>0.925</td>
<td>0.572</td>
<td>0.192</td>
<td>0.053</td>
<td>0.155</td>
<td>0.535</td>
<td>0.996</td>
<td>0.992</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_S$</td>
<td>0.729</td>
<td>0.440</td>
<td>0.173</td>
<td>0.050</td>
<td>0.089</td>
<td>0.380</td>
<td>0.802</td>
<td>0.979</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_K$</td>
<td>0.827</td>
<td>0.359</td>
<td>0.138</td>
<td>0.031*</td>
<td>0.067</td>
<td>0.308</td>
<td>0.716</td>
<td>0.998</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_E$</td>
<td>0.689</td>
<td>0.430</td>
<td>0.174</td>
<td>0.047</td>
<td>0.087</td>
<td>0.340</td>
<td>0.756</td>
<td>0.976</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_F$</td>
<td>0.833</td>
<td>0.611</td>
<td>0.224</td>
<td>0.059*</td>
<td>0.191</td>
<td>0.600</td>
<td>0.922</td>
<td>0.995</td>
<td>1.000</td>
</tr>
</tbody>
</table>

* The observed empirical level is more than 2 standard deviations from the true level $\alpha$.  

**TABLE 3.4 (d)**  
Empirical levels and powers of the tests based on 1000 runs at $\alpha = 0.05$ and $K = 50$ for testing $H_0: \rho = 0.3$ against $H_1: \rho \neq 0.3$.  

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TABLE 3.5 (a)

Empirical levels and powers of the tests based on 1000 runs at $\alpha = 0.01$ and $K = 25$ for testing $H_0: \rho = 0.5$ against $H_1: \rho \neq 0.5$.

<table>
<thead>
<tr>
<th>Test</th>
<th>$0.0$</th>
<th>$0.1$</th>
<th>$0.2$</th>
<th>$0.3$</th>
<th>$0.4$</th>
<th>$0.5^*$</th>
<th>$0.6$</th>
<th>$0.7$</th>
<th>$0.8$</th>
<th>$0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.919</td>
<td>0.998</td>
<td>0.387</td>
<td>0.161</td>
<td>0.051</td>
<td>0.017</td>
<td>0.031</td>
<td>0.245</td>
<td>0.739</td>
<td>0.995</td>
</tr>
<tr>
<td>$Z_M$</td>
<td>0.970</td>
<td>0.841</td>
<td>0.555</td>
<td>0.274</td>
<td>0.095</td>
<td>0.030</td>
<td>0.005</td>
<td>0.037</td>
<td>0.355</td>
<td>0.950</td>
</tr>
<tr>
<td>$Z_W$</td>
<td>0.980</td>
<td>0.817</td>
<td>0.527</td>
<td>0.253</td>
<td>0.085</td>
<td>0.036</td>
<td>0.083</td>
<td>0.384</td>
<td>0.831</td>
<td>0.999</td>
</tr>
<tr>
<td>$Z_{CM}$</td>
<td>0.920</td>
<td>0.352</td>
<td>0.161</td>
<td>0.052</td>
<td>0.014</td>
<td>0.009</td>
<td>0.049</td>
<td>0.292</td>
<td>0.772</td>
<td>0.995</td>
</tr>
<tr>
<td>$Z_{CA}$</td>
<td>0.941</td>
<td>0.781</td>
<td>0.499</td>
<td>0.258</td>
<td>0.095</td>
<td>0.023</td>
<td>0.001</td>
<td>0.027</td>
<td>0.305</td>
<td>0.910</td>
</tr>
<tr>
<td>$F_E$</td>
<td>0.702</td>
<td>0.479</td>
<td>0.280</td>
<td>0.115</td>
<td>0.035</td>
<td>0.013</td>
<td>0.040</td>
<td>0.288</td>
<td>0.757</td>
<td>0.997</td>
</tr>
<tr>
<td>$F_A$</td>
<td>0.885</td>
<td>0.645</td>
<td>0.383</td>
<td>0.152</td>
<td>0.041</td>
<td>0.019</td>
<td>0.067</td>
<td>0.313</td>
<td>0.766</td>
<td>0.994</td>
</tr>
<tr>
<td>$F_T$</td>
<td>0.823</td>
<td>0.430</td>
<td>0.239</td>
<td>0.112</td>
<td>0.039</td>
<td>0.014</td>
<td>0.042</td>
<td>0.268</td>
<td>0.723</td>
<td>0.995</td>
</tr>
<tr>
<td>$F_P$</td>
<td>0.702</td>
<td>0.479</td>
<td>0.280</td>
<td>0.115</td>
<td>0.035</td>
<td>0.013</td>
<td>0.040</td>
<td>0.289</td>
<td>0.759</td>
<td>0.997</td>
</tr>
<tr>
<td>$Z_A$</td>
<td>0.930</td>
<td>0.745</td>
<td>0.442</td>
<td>0.195</td>
<td>0.074</td>
<td>0.015</td>
<td>0.005</td>
<td>0.036</td>
<td>0.307</td>
<td>0.822</td>
</tr>
<tr>
<td>$Z_S$</td>
<td>0.773</td>
<td>0.640</td>
<td>0.444</td>
<td>0.234</td>
<td>0.101</td>
<td>0.032</td>
<td>0.008</td>
<td>0.013</td>
<td>0.215</td>
<td>0.890</td>
</tr>
<tr>
<td>$Z_K$</td>
<td>0.885</td>
<td>0.502</td>
<td>0.310</td>
<td>0.154</td>
<td>0.055</td>
<td>0.018</td>
<td>0.003</td>
<td>0.003</td>
<td>0.095</td>
<td>0.795</td>
</tr>
<tr>
<td>$Z_E$</td>
<td>0.766</td>
<td>0.611</td>
<td>0.413</td>
<td>0.222</td>
<td>0.085</td>
<td>0.026</td>
<td>0.008</td>
<td>0.005</td>
<td>0.125</td>
<td>0.826</td>
</tr>
<tr>
<td>$Z_F$</td>
<td>0.933</td>
<td>0.755</td>
<td>0.451</td>
<td>0.201</td>
<td>0.075</td>
<td>0.023</td>
<td>0.054</td>
<td>0.270</td>
<td>0.738</td>
<td>0.993</td>
</tr>
</tbody>
</table>

* The observed empirical level is more than $2$ standard deviations from the true level $\alpha$. 
### TABLE 3.5 (b)

Empirical levels and powers of the tests based on 1000 runs at $\alpha = 0.01$ and $k = 50$ for testing $H_0: \rho = 0.5$ against $H_1: \rho = 0.5$.

<table>
<thead>
<tr>
<th>Test</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5*</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.999</td>
<td>0.974</td>
<td>0.768</td>
<td>0.360</td>
<td>0.080</td>
<td>0.016</td>
<td>0.082</td>
<td>0.533</td>
<td>0.969</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_M$</td>
<td>1.000</td>
<td>0.984</td>
<td>0.863</td>
<td>0.488</td>
<td>0.141</td>
<td>0.021*</td>
<td>0.028</td>
<td>0.318</td>
<td>0.917</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_W$</td>
<td>1.000</td>
<td>0.982</td>
<td>0.805</td>
<td>0.406</td>
<td>0.102</td>
<td>0.025*</td>
<td>0.116</td>
<td>0.630</td>
<td>0.982</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_{CM}$</td>
<td>0.999</td>
<td>0.908</td>
<td>0.582</td>
<td>0.225</td>
<td>0.042</td>
<td>0.013</td>
<td>0.088</td>
<td>0.575</td>
<td>0.975</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_{CA}$</td>
<td>1.000</td>
<td>0.984</td>
<td>0.847</td>
<td>0.470</td>
<td>0.135</td>
<td>0.017*</td>
<td>0.023</td>
<td>0.299</td>
<td>0.910</td>
<td>1.000</td>
</tr>
<tr>
<td>$F_E$</td>
<td>0.985</td>
<td>0.888</td>
<td>0.823</td>
<td>0.283</td>
<td>0.080</td>
<td>0.010</td>
<td>0.086</td>
<td>0.561</td>
<td>0.972</td>
<td>1.000</td>
</tr>
<tr>
<td>$F_A$</td>
<td>1.000</td>
<td>0.973</td>
<td>0.777</td>
<td>0.367</td>
<td>0.090</td>
<td>0.020*</td>
<td>0.108</td>
<td>0.601</td>
<td>0.969</td>
<td>1.000</td>
</tr>
<tr>
<td>$F_T$</td>
<td>0.962</td>
<td>0.813</td>
<td>0.558</td>
<td>0.252</td>
<td>0.064</td>
<td>0.011</td>
<td>0.082</td>
<td>0.517</td>
<td>0.966</td>
<td>1.000</td>
</tr>
<tr>
<td>$F_P$</td>
<td>0.985</td>
<td>0.885</td>
<td>0.818</td>
<td>0.282</td>
<td>0.059</td>
<td>0.010</td>
<td>0.086</td>
<td>0.563</td>
<td>0.972</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_{A}$</td>
<td>1.000</td>
<td>0.982</td>
<td>0.818</td>
<td>0.418</td>
<td>0.114</td>
<td>0.018</td>
<td>0.029</td>
<td>0.302</td>
<td>0.869</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_{S}$</td>
<td>0.983</td>
<td>0.917</td>
<td>0.719</td>
<td>0.404</td>
<td>0.118</td>
<td>0.019*</td>
<td>0.016</td>
<td>0.219</td>
<td>0.872</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_{K}$</td>
<td>0.962</td>
<td>0.841</td>
<td>0.600</td>
<td>0.309</td>
<td>0.092</td>
<td>0.009</td>
<td>0.008</td>
<td>0.133</td>
<td>0.777</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_{E}$</td>
<td>0.978</td>
<td>0.889</td>
<td>0.683</td>
<td>0.379</td>
<td>0.109</td>
<td>0.018</td>
<td>0.013</td>
<td>0.195</td>
<td>0.830</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_{F}$</td>
<td>1.000</td>
<td>0.982</td>
<td>0.827</td>
<td>0.428</td>
<td>0.115</td>
<td>0.022*</td>
<td>0.001</td>
<td>0.574</td>
<td>0.966</td>
<td>1.000</td>
</tr>
</tbody>
</table>

* The observed empirical level is more than 2 standard deviations from the true level $\alpha$. 
TABLE 3.5 (c)

Empirical levels and powers of the tests based on 1000 runs at $\alpha = 0.05$ and $K = 25$ for testing $H_0: \rho = 0.5$ against $H_1: \rho \neq 0.5$.

<table>
<thead>
<tr>
<th>Test</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5*</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.980</td>
<td>0.987</td>
<td>0.986</td>
<td>0.372</td>
<td>0.157</td>
<td>0.063</td>
<td>0.120</td>
<td>0.453</td>
<td>0.853</td>
<td>0.999</td>
</tr>
<tr>
<td>$Z_M$</td>
<td>0.991</td>
<td>0.928</td>
<td>0.756</td>
<td>0.407</td>
<td>0.214</td>
<td>0.076*</td>
<td>0.054</td>
<td>0.285</td>
<td>0.763</td>
<td>0.997</td>
</tr>
<tr>
<td>$Z_W$</td>
<td>0.990</td>
<td>0.920</td>
<td>0.708</td>
<td>0.418</td>
<td>0.186</td>
<td>0.088*</td>
<td>0.167</td>
<td>0.546</td>
<td>0.805</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_{CM}$</td>
<td>0.923</td>
<td>0.757</td>
<td>0.484</td>
<td>0.258</td>
<td>0.098</td>
<td>0.047</td>
<td>0.133</td>
<td>0.498</td>
<td>0.880</td>
<td>0.998</td>
</tr>
<tr>
<td>$Z_{CA}$</td>
<td>0.982</td>
<td>0.913</td>
<td>0.723</td>
<td>0.435</td>
<td>0.208</td>
<td>0.070*</td>
<td>0.046</td>
<td>0.245</td>
<td>0.729</td>
<td>0.992</td>
</tr>
<tr>
<td>$F_E$</td>
<td>0.892</td>
<td>0.726</td>
<td>0.501</td>
<td>0.279</td>
<td>0.112</td>
<td>0.053</td>
<td>0.146</td>
<td>0.405</td>
<td>0.890</td>
<td>1.000</td>
</tr>
<tr>
<td>$F_A$</td>
<td>0.973</td>
<td>0.863</td>
<td>0.619</td>
<td>0.348</td>
<td>0.143</td>
<td>0.067*</td>
<td>0.165</td>
<td>0.522</td>
<td>0.889</td>
<td>0.997</td>
</tr>
<tr>
<td>$F_T$</td>
<td>0.815</td>
<td>0.674</td>
<td>0.478</td>
<td>0.270</td>
<td>0.117</td>
<td>0.059</td>
<td>0.152</td>
<td>0.471</td>
<td>0.874</td>
<td>1.000</td>
</tr>
<tr>
<td>$F_P$</td>
<td>0.880</td>
<td>0.725</td>
<td>0.501</td>
<td>0.278</td>
<td>0.111</td>
<td>0.052</td>
<td>0.147</td>
<td>0.498</td>
<td>0.891</td>
<td>1.000</td>
</tr>
<tr>
<td>$Z_A$</td>
<td>0.982</td>
<td>0.897</td>
<td>0.671</td>
<td>0.393</td>
<td>0.160</td>
<td>0.058</td>
<td>0.257</td>
<td>0.724</td>
<td>0.993</td>
<td></td>
</tr>
<tr>
<td>$Z_S$</td>
<td>0.966</td>
<td>0.775</td>
<td>0.608</td>
<td>0.388</td>
<td>0.198</td>
<td>0.062</td>
<td>0.045</td>
<td>0.222</td>
<td>0.693</td>
<td>0.995</td>
</tr>
<tr>
<td>$Z_K$</td>
<td>0.823</td>
<td>0.681</td>
<td>0.489</td>
<td>0.273</td>
<td>0.129</td>
<td>0.043</td>
<td>0.037</td>
<td>0.185</td>
<td>0.640</td>
<td>0.987</td>
</tr>
<tr>
<td>$Z_E$</td>
<td>0.872</td>
<td>0.757</td>
<td>0.583</td>
<td>0.305</td>
<td>0.184</td>
<td>0.050</td>
<td>0.042</td>
<td>0.182</td>
<td>0.640</td>
<td>0.987</td>
</tr>
<tr>
<td>$Z_F$</td>
<td>0.984</td>
<td>0.905</td>
<td>0.680</td>
<td>0.405</td>
<td>0.173</td>
<td>0.080*</td>
<td>0.150</td>
<td>0.481</td>
<td>0.866</td>
<td>0.997</td>
</tr>
</tbody>
</table>

* The observed empirical level is more than 2 standard deviations from the true level $\alpha$. 
TABLE 3.5 (d)

Empirical levels and powers of the tests based on 1000 runs at \( \alpha = 0.05 \) and \( K = 50 \) for testing \( H_0: \rho = 0.5 \) against \( H_1: \rho \neq 0.5 \).

<table>
<thead>
<tr>
<th>Test</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5$^*$</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>1.000</td>
<td>0.997</td>
<td>0.923</td>
<td>0.616</td>
<td>0.230</td>
<td>0.058</td>
<td>0.205</td>
<td>0.753</td>
<td>0.889</td>
<td>1.000</td>
</tr>
<tr>
<td>( Z_M )</td>
<td>1.000</td>
<td>0.999</td>
<td>0.949</td>
<td>0.695</td>
<td>0.294</td>
<td>0.066$^*$</td>
<td>0.124</td>
<td>0.540</td>
<td>0.882</td>
<td>1.000</td>
</tr>
<tr>
<td>( Z_W )</td>
<td>1.000</td>
<td>0.998</td>
<td>0.930</td>
<td>0.628</td>
<td>0.247</td>
<td>0.070$^*$</td>
<td>0.254</td>
<td>0.793</td>
<td>0.991</td>
<td>1.000</td>
</tr>
<tr>
<td>( Z_{CM} )</td>
<td>1.000</td>
<td>0.997</td>
<td>0.880</td>
<td>0.533</td>
<td>0.177</td>
<td>0.052</td>
<td>0.236</td>
<td>0.774</td>
<td>0.990</td>
<td>1.000</td>
</tr>
<tr>
<td>( Z_{CA} )</td>
<td>1.000</td>
<td>0.998</td>
<td>0.939</td>
<td>0.675</td>
<td>0.288</td>
<td>0.082</td>
<td>0.117</td>
<td>0.625</td>
<td>0.980</td>
<td>1.000</td>
</tr>
<tr>
<td>( F_E )</td>
<td>0.999</td>
<td>0.977</td>
<td>0.837</td>
<td>0.522</td>
<td>0.178</td>
<td>0.052</td>
<td>0.228</td>
<td>0.766</td>
<td>0.992</td>
<td>1.000</td>
</tr>
<tr>
<td>( F_A )</td>
<td>1.000</td>
<td>0.996</td>
<td>0.919</td>
<td>0.606</td>
<td>0.229</td>
<td>0.076$^*$</td>
<td>0.270</td>
<td>0.792</td>
<td>0.991</td>
<td>1.000</td>
</tr>
<tr>
<td>( F_T )</td>
<td>0.992</td>
<td>0.949</td>
<td>0.781</td>
<td>0.485</td>
<td>0.177</td>
<td>0.052</td>
<td>0.210</td>
<td>0.744</td>
<td>0.990</td>
<td>1.000</td>
</tr>
<tr>
<td>( F_P )</td>
<td>0.999</td>
<td>0.977</td>
<td>0.836</td>
<td>0.521</td>
<td>0.178</td>
<td>0.052</td>
<td>0.228</td>
<td>0.787</td>
<td>0.992</td>
<td>1.000</td>
</tr>
<tr>
<td>( Z_A )</td>
<td>1.000</td>
<td>0.997</td>
<td>0.928</td>
<td>0.634</td>
<td>0.248</td>
<td>0.056</td>
<td>0.131</td>
<td>0.622</td>
<td>0.974</td>
<td>1.000</td>
</tr>
<tr>
<td>( Z_S )</td>
<td>0.997</td>
<td>0.976</td>
<td>0.859</td>
<td>0.576</td>
<td>0.249</td>
<td>0.057</td>
<td>0.113</td>
<td>0.579</td>
<td>0.973</td>
<td>1.000</td>
</tr>
<tr>
<td>( Z_K )</td>
<td>0.989</td>
<td>0.942</td>
<td>0.777</td>
<td>0.504</td>
<td>0.202</td>
<td>0.044</td>
<td>0.087</td>
<td>0.487</td>
<td>0.958</td>
<td>1.000</td>
</tr>
<tr>
<td>( Z_F )</td>
<td>0.991</td>
<td>0.952</td>
<td>0.818</td>
<td>0.551</td>
<td>0.243</td>
<td>0.063</td>
<td>0.102</td>
<td>0.540</td>
<td>0.960</td>
<td>1.000</td>
</tr>
<tr>
<td>( Z_E )</td>
<td>1.000</td>
<td>0.998</td>
<td>0.935</td>
<td>0.656</td>
<td>0.263</td>
<td>0.080$^*$</td>
<td>0.248</td>
<td>0.770</td>
<td>0.980</td>
<td>1.000</td>
</tr>
</tbody>
</table>

$^*$ The observed empirical level is more than 2 standard deviations from the true level \( \alpha \).
procedures seem to give empirical levels that are close to the selected nominal level $\alpha$. In terms of powers, the test procedure $F_A$ is more powerful for $\rho > \rho_0$. However $Z_W$ and $Z_{CA}$ seem to be more powerful when $\rho < \rho_0$ for testing $H_0: \rho = 0.3$ and $H_0: \rho = 0.5$, respectively.

3.5 Discussion

The main purpose of this investigation was to construct test procedures for testing the hypotheses regarding intraclass correlation and to assess their performance in terms of their empirical powers. The major findings of this investigation are summarized as follows:

1. The test procedures $Z_S$, $Z_K$ and $Z_E$ show very poor performance in all the presented cases for both one and two sided alternatives.

2. The procedure $Z_F$, based on Fisher's transformation, performs reasonably well for one sided alternatives but it tends to give empirical levels which are substantially greater than the nominal levels for two sided alternatives. When used to construct confidence intervals for $\rho$, Donner and Wells (1986) and Keen (1987) reported that $Z_F$ is inferior to $F_T$ and $Z_A$ test procedures.

3. The procedure $F_T$, based on Thomas and Hultquist's (1978) approximation, seems to hold selected nominal levels but it is substantially less powerful than $F_E$, $F_A$ and $F_P$ tests.

4. Wald's exact procedure $F_E$, the proposed procedure $F_P$
and the procedure $Z_A$ give good estimates of the levels of significance and very comparable powers. They seem to be asymptotically equally powerful in all the presented cases.

(5) The performance of the LR procedure $\lambda$ is satisfactory for two sided alternatives, though its empirical levels are greater than the nominal levels in all the presented cases.

(6) The procedure $Z_M$, based on the ML estimator, did not hold significance levels well in most of the presented cases, though it is more powerful when the null hypothesis $H_0: \rho = 0$ was tested.

(7) The modified ANOVA procedure $F_A$ gives empirical levels which are considerably different from the true nominal levels in all the the presented cases, except when $H_0: \rho = 0$ was tested in which case it is identical to $F_E$ procedure.

(8) The procedure $Z_{CM'}$ based on the large sample properties of the efficient scores (which are asymptotically equivalent to $Z_M$, $Z_W$, $Z_{CA}$ and $\lambda$ procedures), tends to perform very well in moderately large samples ($K \geq 50$). It performs substantially better than $Z_M$, $Z_W$, $Z_{CA}$ and $\lambda$ procedures. Use of this procedure is strongly recommended for one sided alternatives because it holds nominal levels and gives comparable powers.

It may be argued that values of $\rho > 0.5$ are uncommon in most applications to family studies, and hence the situations in which $Z_{CM}$ may be applicable are limited.
However, Hennekens et al. (1980) reported an estimate of the intraclass correlation on cholesterol level for children of men with myocardial infarction as high as 0.71. Thus, one may recommend using the procedure $Z_{CM}$ in situations where the suspected value of $p$ is large.

It should be noted that in the simulation studies on the testing of hypotheses about interclass correlation, Konishi (1985) demonstrated that the simulated powers are markedly affected by the choice of $m$ and $P$ in generating sibship sizes from the zero-truncated negative binomial distribution. Therefore, it is emphasized here that the conclusions of this investigation are conditional on sibship size distributions typical of North American families. Investigations that utilize different combinations of $m$ and $P$, and hence different degrees of unbalance in cluster sizes, may provide different conclusions.

In this chapter, several procedures for testing hypotheses regarding intraclass correlation in a single population were presented and compared in terms of their powers. When data on more than one population are available, it is of interest to test the equality of intraclass correlations in several populations. In the next chapter, the inference procedures for intraclass correlations are considered when data from several populations are available.
CHAPTER 4
INFERENCE PROCEDURES FOR INTRACLASS CORRELATIONS IN MULTIPLE SAMPLES

4.1 Introduction

The estimation and testing procedures for intraclass correlation using a single random sample from multivariate normal populations were discussed in chapters 2 and 3, respectively. In chapter 2, the sampling properties of several estimators of the intraclass correlation were investigated. In chapter 3, several procedures for testing hypotheses regarding intraclass correlation were presented and the power comparisons were achieved by Monte Carlo simulations. This chapter deals with the inference procedures for intraclass correlations when sample data are available from several multivariate normal populations. Such problems arise in practice when information is collected under different experimental or observational conditions. For example, Maxwell and Pilliner (1960) examined the similarity of intelligence quotient (I.Q.) among 11-year old Scottish children in families of different size with known structure. Martarell et al. (1978) examined the sibling correlations with respect to nutrition level among Guatemalan and American children, while Hennekens et al. (1980) investigated the familial aggregation of cholesterol levels among children with and without a history of myocardial infarction. An application of multiple sample inference procedures is in the statistical analysis of

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twinship data where families may be divided into several classes according to the type of their twinships (i.e., monozygotic and dizygotic). Using several covariates (possibly parent scores), Muñoz et al. (1988) have presented the methods of multiple regression in the presence of heterogeneous intraclass correlations among siblings and illustrated the method by an application to the analysis of twinship data.

Donner (1985, 1988) has presented the methodology for estimation of intraclass correlations in multiple samples but under a restrictive model. More specifically, he made the assumption of a common environmental effect (error variance) for all the populations that are being compared. His assumption allows a flexible but not very general theory for making inferences for the intraclass correlations.

The estimation of intraclass correlations, under a very general model, is considered when simple random samples are drawn from several multivariate normal populations. The expressions for the asymptotic variances of the ANOVA and the maximum likelihood estimators of the intraclass correlations are derived. It is usually of critical interest in family studies to test that the intraclass correlations are common across conditions. But, unfortunately, this cannot be tested in a routine manner for unbalanced data. The comparisons of this kind are usually made descriptively or on the basis of a sub-sampling procedure designed to produce equal-sized samples. The latter approach, while valid, is clearly inefficient because
there is some loss of information when observations are
discarded from the available data. Several test statistics
for testing the homogeneity of intraclass correlations are
derived and compared in terms of their empirical powers.
Although some such tests for the homogeneity of intraclass
correlations for two populations have been discussed by
Donner and Bull (1983), and Donner (1985), the performance
of the tests for unbalanced data is still unknown. In
particular, the two versions of C(00) test of Neyman (1959)
are derived and compared with the likelihood ratio (LR) and
several other test statistics on the homogeneity of
intraclass correlations.

4.2 The Models

Suppose that the data under M different conditions
(i.e., form M populations) are available. Let \( y_{hij} \) be the
score of \( j \)th individual in the \( i \)th group under \( h \)th condition
The "mixed effect" model for investigating the intraclass
correlations states that

\[
y_{hij} = \mu + c_h + a_{hi} + \varepsilon_{hij}, \quad h = 1,2,\ldots,M, i = 1,2,\ldots,K_h,
\]

\[
\quad j = 1,2,\ldots,n_{hi},
\]

where \( K_h \) is the number of groups under condition \( h \), \( n_{hi} \) is
the number of individuals in the \( i \)th group under condition
\( h \), \( \mu \) is overall mean, \( c_h \) is a fixed effect of condition \( h \),
the group effects \( \{a_{hi}\} \) are \( N(0, \sigma^2_{ah}) \) and the error effects
\( \{\varepsilon_{hij}\} \) are \( N(0, \sigma^2_{eh}) \). Furthermore it is assumed that the
\( \{a_{hi}\} \) and \( \{\varepsilon_{hij}\} \) are mutually independent. Under model
(4.1), the mean and variance of any observation under
condition \( h \) \( (h = 1, 2, \ldots, M) \), respectively, are

\[
\mu_h = \text{E}(y_{hij}) = \mu + c_h, \quad i = 1, 2, \ldots, K_h, \quad j = 1, 2, \ldots, n_{hi}
\]

and

\[
\sigma_h^2 = \text{Var}(y_{hij}) = \sigma_{ah}^2 + \sigma_{eh}^2, \quad i = 1, 2, \ldots, K_h, \quad j = 1, 2, \ldots, n_{hi}.
\]  

(4.2)

Furthermore, the correlation between any two randomly chosen observations in the same group under condition \( h \) \( (h = 1, 2, \ldots, M) \) is

\[
\rho_h = \text{Cov}(y_{hij}, y_{him}) = \frac{\sigma_{ah}^2}{\sigma_{eh}^2} = \frac{\sigma_{ah}^2}{\sigma_{ah}^2 + \sigma_{eh}^2}, \quad j \neq m,
\]

\[
i = 1, 2, \ldots, K_h, \quad j, m = 1, 2, \ldots, n_{hi}.
\]  

(4.3)

and all the other covariances are zero.

Donner (1985, 1986) assumes that the errors \( \varepsilon_{hij} \) have a common variance \( \sigma_e^2 \) (i.e., \( \sigma_{eh}^2 = \sigma_e^2 \), \( h = 1, 2, \ldots, M \), independent of condition \( h \). This means that the environmental effect on the individuals is same across all the populations. If this is the case, then homogeneity of the variance components \( \sigma_{ah}^2 \) across the conditions also implies homogeneity of the intraclass correlations, that is

\[
\rho_1^a = \rho_2^a = \ldots = \rho_M^a.
\]

Similar to the one sample case, we may consider the following multivariate normal model: Assume that the observations in the \( ith \) group under condition \( h \) follow a multivariate normal distribution with mean vector \( \mu_{hi} \) and covariance matrix \( \Sigma_{hi} \). More specifically, if a simple random sample of \( n_{hi} \) observations from the \( ith \) group under condition \( h \) is denoted by

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\[ y_{hi} = [y_{h1i}, y_{h2i}, \ldots, y_{hin_{hi}}]^T, \quad h = 1, 2, \ldots, M, \quad i = 1, 2, \ldots, K, \]

then

\[ y_{hi} \sim \mathcal{MN}_{n_{hi}} \left( \mu_{hi}, \Sigma_{hi} \right), \quad h = 1, 2, \ldots, M, \quad i = 1, 2, \ldots, K, \tag{4.4} \]

where

\[ \mu_{hi} = \mu_h \sim_{n_{hi}} x_i, \]

\[ \Sigma_{hi} = \text{Cov}(y_{hi}) = \left[ (1 - \rho_h) I_{n_{hi}} + \rho_h J_{n_{hi} \times n_{hi}} \right] \sigma_h^2, \]

\( I_p \) is an identity matrix of order \( p \) and \( J_{pxq} \) is an \( (p \times q) \) matrix each element of which is 1. Thus

\[ y_h = [y_{h1}, y_{h2}, y_{h3}, \ldots, y_{hk_h}]^T, \quad h = 1, 2, \ldots, M, \]

represents a simple random sample from the \( h \)th multivariate normal population (under condition \( h \)) and

\[ Y = [y_1, y_2, y_3, \ldots, y_M]^T \]

is the matrix of the combined sample from all the populations.

Under model (4.4), for \( h = 1, 2, \ldots, M \) and \( i = 1, 2, \ldots, K_h \), we have

\[ \text{E}(y_{hij}) = \mu_h, \quad j = 1, 2, \ldots, n_{hi}, \]

\[ \text{Var}(y_{hij}) = \sigma_h^2, \quad j = 1, 2, \ldots, n_{hi}, \]

\[ \text{Cov}(y_{hij}, y_{him}) = \rho_h \sigma_h^2, \quad j \neq m, j, m = 1, 2, \ldots, n_{hi} \]

and all the other covariances are zero. Thus the parameters \( \mu_h, \sigma_h^2 \) and \( \rho_h \) are constant over all groups under the same condition. It is reasonable to assume that the values of \( \rho_h \) (\( h = 1, 2, \ldots, M \)) are non-negative particularly in the analysis of familial data. Therefore the models (4.1) and (4.4) are essentially equivalent. In the remainder of this
chapter, the distinction between the two models will be made only if the need arises.

4.3 Estimation of Intraclass Correlations from Multiple Samples

Let us assume that a simple random sample is taken under the above models. In the remainder of this chapter, we shall use the following notations adapted from Dunn and Clark (1974):

\[ K = \sum_{h=1}^{M} K_h \] be the number of groups in total combined sample,

\[ N_h = \sum_{i=1}^{K_h} n_{hi} \] be the number of individuals in hth sample,

\[ N = \sum_{h=1}^{M} N_h \] be the number of individuals in total combined sample,

\[ \bar{y}_{hi.} = \frac{1}{n_{hi}} \sum_{j=1}^{n_{hi}} y_{hij} \] be the sample mean of observations in ith group within the hth sample,

\[ \bar{y}_{h..} = \frac{1}{N_h} \sum_{i=1}^{K_h} \bar{y}_{hi.} \] be the sample mean of observations in the hth sample,

\[ \bar{y}_{..} = \frac{1}{N} \sum_{h=1}^{M} N_h \bar{y}_{h..} \] be the sample mean of all observations in the total combined sample.

\[ SSG_h = \sum_{i=1}^{K_h} n_{hi} (\bar{y}_{hi.} - \bar{y}_{h..})^2 \] be the sum of squares among groups within the hth sample,

\[ SSE_h = \sum_{i=1}^{K_h} \sum_{j=1}^{n_{hi}} (y_{hij} - \bar{y}_{hi.})^2 \] be the error sum of squares within the hth sample,

\[ SSC = \sum_{h=1}^{M} N_h (\bar{y}_{hi.} - \bar{y}_{..})^2 \] be the sum of squares among conditions (samples).
SSG = \sum_{h=1}^{M} SSG_h \text{ be the sum of squares among groups within conditions (samples),}

SSE = \sum_{h=1}^{M} SSE_h \text{ be the error sum of squares,}

\text{SST} = \sum_{h=1}^{M} \sum_{i=1}^{K_h} \sum_{j=1}^{n_{hi}} (y_{hij} - \bar{y})^2 \text{ be the total sum of squares,}

n_{0h} = \frac{1}{K_h - 1} \left[ N_h - \sum_{i=1}^{K_h} n_{hi} / N_h \right], \quad h = 1, 2, \ldots, M,

and

n_A = \frac{1}{K - M} \sum_{h=1}^{M} (K_h - 1)n_{0h} = \frac{1}{K - M} \left[ N - \sum_{h=1}^{M} \sum_{i=1}^{K_h} n_{hi} / N_h \right].

4.3.1 ANOVA Estimators

It should be noted that model (4.1) under condition h (h = 1, 2, \ldots, M) is equivalent to the ANOVA model (2.1) for the single sample case. Thus under each different condition, the results of section 2.3 are directly applicable here. Treating MSE_h and MSG_h as MSW and MSA, respectively, it is known, from (2.29), that for h = 1, 2, \ldots, M:

\text{EC}(\bar{y}_{h})' = \mu_h,'

\text{EC}(MSG_h) = \frac{1}{K_h - 1} \text{EC}(SSG_h) = \sigma_{eh}^2 + n_{0h} \sigma_{ah}^2 = [1 + (n_{0h} - 1) \rho_h] \sigma_h^2

and

\text{EC}(MSE_h) = \frac{1}{N_h - K_h} \text{EC}(SSE_h) = \sigma_{eh}^2 = (1 - \rho_h) \sigma_h^2.

Thus by analogy of equations (2.30), the ANOVA estimators of \mu_h, \sigma_h^2\text{ and }\rho_h \text{ (h = 1, 2, \ldots, M), based on the method of variance components, are given by}
\[ \hat{\mu}_h = \bar{y}_h \text{, } h = 1, 2, \ldots, M. \]

\[ \hat{\sigma}_h^2 = \frac{1}{n_{0h}} \left[ \text{MSG}_h + (n_{0h} - 1) \text{MSE}_h \right], \text{ } h = 1, 2, \ldots, M \]

and

\[ \hat{\rho}_h = \frac{\text{MSG}_h - \text{MSE}_h}{\text{MSG}_h + (n_{0h} - 1) \text{MSE}_h}, \text{ } h = 1, 2, \ldots, M. \] \hspace{1cm} (4.5)

respectively. The large sample variance of \( \hat{\rho}_h \), from (2.37), is thus given by

\[ \text{Var}(\hat{\rho}_h) \approx \frac{2(1-\rho_h)^2}{n_{0h}} \left\{ \frac{[1+(n_{0h} - 1)\rho_h]^2}{n_h - K_h} + \frac{\hat{\rho}_h^2}{K_h - 1} + \frac{(1-\rho_h)[1+(n_{0h} - 1)\rho_h]}{K_h - 1} \right\}, \] \hspace{1cm} (4.6)

where

\[ \hat{\rho}_h = K_h \left( \sum_{i=1}^{n_{hi}} - 2n_{hi}^{-1} \sum_{i=1}^{K_h} n_{hi}^{-1} + n_{hi}^{-2} \left( \sum_{i=1}^{K_h} n_{hi} \right)^2 \right). \] \hspace{1cm} (4.7)

The ANOVA table corresponding to model (4.1), as given by Dunn and Clark (1974), is:

<table>
<thead>
<tr>
<th>S. O. V.</th>
<th>D. F.</th>
<th>SUM OF SQUARES</th>
<th>MEAN SQUARES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Among Conditions</td>
<td>M-1</td>
<td>SSC</td>
<td>MSC = SSC/(M-1)</td>
</tr>
<tr>
<td>Among Groups</td>
<td>K-M</td>
<td>SSG</td>
<td>MSG = SSG/(K-M)</td>
</tr>
<tr>
<td>Within Samples</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>N-K</td>
<td>SSE</td>
<td>MSE = SSE/(N-K)</td>
</tr>
<tr>
<td>Total</td>
<td>N-1</td>
<td>SST</td>
<td>---</td>
</tr>
</tbody>
</table>

(4.8)

The expected mean squares \( \text{E}(\text{MSC}) \), \( \text{E}(\text{MSG}) \) and \( \text{E}(\text{MSE}) \)
corresponding to model (4.1) are

\[
E(MSC) = \frac{1}{K-1} \left\{ \sum_{h=1}^{M} \sigma_{eh}^2 \left[ 1 - \frac{N_h}{N} \right] + \sum_{h=1}^{M} \sigma_{ah}^2 \left[ \frac{1}{N_h} - \frac{1}{N} \right] \right\} \sum_{i=1}^{K_h} n_{hi}^2 \\
+ \sum_{h=1}^{M} N_h c_h^2 - \frac{1}{N} \left( \sum_{h=1}^{M} N_h c_h \right)^2
\]

\[
E(MSG) = \frac{1}{K-M} \sum_{h=1}^{M} E(SSG_h) = \frac{1}{K-M} \sum_{h=1}^{M} \left( K_h - 1 \right) \left( \sigma_{eh}^2 + n_{Oh} \sigma_{ah}^2 \right) \\
= \frac{1}{K-M} \sum_{h=1}^{M} \left( K_h - 1 \right) \left( 1 + (n_{Oh} - 1) \rho_h \right) \sigma_{ah}^2
\]

and

\[
E(MSE) = \frac{1}{N-K} \sum_{h=1}^{M} E(SSE_h) = \frac{1}{N-K} \sum_{h=1}^{M} \left( N_h - K_h \right) \sigma_{eh}^2 \\
= \frac{1}{N-K} \sum_{h=1}^{M} \left( N_h - K_h \right) \left( 1 - \rho_h \right) \sigma_{ah}^2
\]

(4.9)

Donner (1985, 1986) assumed that \( \sigma_{eh}^2 = \sigma_e^2 \), \( h = 1, 2, \ldots, M \). Thus, under this assumption, the homogeneity of variance components \( \sigma_{ah}^2 \) across all the populations (i.e., \( \sigma_{ah}^2 = \sigma_a^2 \), \( h = 1, 2, \ldots, M \)) also implies the homogeneity of intraclass correlations (i.e., \( \rho_h = \rho_c \), \( h = 1, 2, \ldots, M \)). For this particular situation, the common value of the intraclass correlations \( \rho_c \) can be expressed as

\[
\rho_c = \frac{\sigma_a^2}{\sigma_e^2} = \frac{\sigma_a^2}{\sigma_a^2 + \sigma_e^2}
\]

(4.10)

where \( \sigma^2 = \sigma_a^2 + \sigma_e^2 \) is common variance of \( y_{hij} \)'s across all the populations. Furthermore, the expected mean squares (4.9) are reduced to
$$\text{EC(MSC)} = \sigma_e^2 + \frac{\sigma_a^2}{M-1} \sum_{h=1}^{M} \left[ \frac{1}{N_h} - \frac{1}{N} \right] \left[ \frac{K_h}{\sum_{i=1}^{N_h} h_i} \right] + \frac{1}{M-1} \sum_{h=1}^{M} N_h c_h^2$$

$$- \frac{1}{NCM-15} \left[ \sum_{h=1}^{M} N_h c_h \right]^2,$$

$$\text{EC(MSG)} = \sigma_e^2 + n_A \sigma_a^2 = [1+(n_A-1)\rho_c] \sigma^2$$

and

$$\text{EC(MSE)} = \sigma_e^2 = (1-\rho_c) \sigma^2.$$

Thus the ANOVA estimators of $\sigma_a^2$, $\sigma_e^2$, $\sigma^2$ and $\rho_c$ are given by

(Donner, 1985, 1986)

$$\hat{\sigma}_a^2 = \frac{1}{n_A} [\text{MSG} - \text{MSE}],$$

$$\hat{\sigma}_e^2 = \text{MSE},$$

$$\hat{\sigma}^2 = \hat{\sigma}_a^2 + \hat{\sigma}_e^2 = \frac{1}{n_A} [\text{MSG} + (n_A-1)\text{MSE}]$$

and

$$\hat{\rho}_c = \frac{\hat{\sigma}_a^2}{\hat{\sigma}_a^2 + \hat{\sigma}_e^2} = \frac{\text{MSG} - \text{MSE}}{\text{MSG} + (n_A-1)\text{MSE}}$$

respectively. By using the results of SSA and SSW from section 2.3 for $SSG_h$ and $SSE_h$, respectively, we have

$$\text{Var(MSG)} = \frac{1}{(K-MD)^2} \sum_{h=1}^{M} \text{Var} (SSG_h)$$

$$= \frac{2}{(K-MD)^2} \sum_{h=1}^{M} \left[ \hat{\phi}_h \rho_c^2 + (K_h-1)(1-\rho_c)[1+(2n_{0h}-1)\rho_c] \right]$$

$$= \frac{2}{(K-MD)^2} \left[ \hat{\phi}_h \rho_c^2 + (K-MD)(1-\rho_c)[1+(2n_{0h}-1)\rho_c] \right],$$

$$\text{Var(MSE)} = \frac{1}{(N-K)^2} \sum_{h=1}^{M} \text{Var} (SSE_h) = \frac{2(1-\rho_c)^2 \sigma^4}{N-K},$$
where
\[
\tilde{\rho} = \sum_{h=1}^{M} \tilde{\rho}_h
\]
and \(\tilde{\rho}_h\) is defined by (4.7). Furthermore \(\text{Cov}(\text{MSG}, \text{MSE}) = 0\) and the first order partial derivatives of \(\tilde{\rho}_c\) (4.12) with respect to MSG and MSE, evaluated at their mean values, are
\[
\frac{\partial \tilde{\rho}_c}{\partial \text{MSG}} = \frac{1-\rho_c}{\sigma_A^2}, \quad \frac{\partial \tilde{\rho}_c}{\partial \text{MSE}} = \frac{[1+(n_A-1)\rho_c]}{\sigma_A^2}.
\]
where
\[
\tilde{\mathbf{M}} = [\text{E}(\text{MSG}), \text{E}(\text{MSE})]^T.
\]
Thus from (1.7), the large sample variance of \(\tilde{\rho}_c\), derived by the delta method, is given by
\[
\text{Var}(\tilde{\rho}_c) \approx \frac{2(1-\rho)^2}{n_A^2} \left\{ \frac{[1+(n_A-1)\rho]^2}{N-K} + \frac{(1-\rho)[1+(2n_A-1)\rho]}{K-M} + \frac{\tilde{\rho}^2}{(K-M)d^2} \right\}.
\]
(4.13)

If we assume that the intraclass correlations are homogeneous across all the populations (e.g., \(\rho_h = \rho, h = 1, 2, \ldots, M\)) without making any assumption regarding the homogeneity of environmental variance components \(\sigma^2_{eh}\) \((h = 1, 2, \ldots, M)\), the ANOVA estimator for the common value \(\rho\) cannot be obtained from (4.9) in closed form. Thus, a weighted estimator of \(\rho\) is suggested as
\[
\tilde{\rho} = \frac{1}{M} \sum_{h=1}^{M} W_h \tilde{\rho}_h,
\]
(4.14)
where
\[ W_h = \frac{1}{\text{var}(\tilde{\rho}_h)}, \quad h = 1, 2, \ldots, M. \] (4.15)

The true values of \( W_h, \quad h = 1, 2, \ldots, M \), are usually not known and may be estimated by replacing \( \rho_h \) by \( \tilde{\rho}_h \) (4.5) in \( \text{var}(\tilde{\rho}_h) \), given by (4.6), for real applications.

Clearly, from (4.14) and (4.15), the large sample variance of \( \tilde{\rho} \) is

\[ \text{Var}(\tilde{\rho}) \approx \sum_{h=1}^{M} W_h^2 \text{Var}(\tilde{\rho}_h) / \left[ \sum_{h=1}^{M} W_h \right]^2 \]

\[ = \left[ \sum_{h=1}^{M} [\text{Var}(\tilde{\rho}_h)]^{-1} \right]^{-1}. \quad (4.18) \]

### 4.3.2 Maximum Likelihood Estimators

Under model (4.4), the log-likelihood function of the \( h \)-th sample is given by

\[ \ell_h = -\frac{N_h}{2} \ln(2\pi \sigma_h^2) - \frac{1}{2} \left( \frac{\sum_{i=1}^{K_h} \ln[1+(n_{hi}-1)\rho_h]}{\sum_{i=1}^{K_h} [1+(n_{hi}-1)\rho_h]} \right) \]

\[ - \frac{1}{2} \left( \frac{\sum_{i=1}^{K_h} (\tilde{y}_{hi} - \mu_h)^2}{\sum_{i=1}^{K_h} [1+(n_{hi}-1)\rho_h]} \right). \quad (4.17) \]

Thus the log-likelihood function of the total combined sample is

\[ \ell = -\frac{N}{2} \ln(2\pi) - \frac{1}{2} \sum_{h=1}^{M} N_h \ln(\sigma_h^2) - \frac{1}{2} \sum_{h=1}^{M} (N_h - K_h) \ln(1-\rho_h) \]

\[ - \frac{1}{2} \sum_{h=1}^{M} \sum_{i=1}^{K_h} \ln[1+(n_{hi}-1)\rho_h] \]

\[ - \frac{1}{2} \sum_{h=1}^{M} \sum_{i=1}^{K_h} \frac{\sum_{i=1}^{K_h} (\tilde{y}_{hi} - \mu_h)^2}{\sum_{i=1}^{K_h} [1+(n_{hi}-1)\rho_h]}. \quad (4.18) \]

Taking the derivatives of \( \ell \) with respect to \( \mu_h \) and \( \sigma_h^2 \),
equating them to zero and solving for these parameters, the maximum likelihood estimators of $\mu_h$ and $\sigma_h^2$, as functions of $\rho_h$, are given by

$$\hat{\mu}_h(\rho_h) = \frac{1}{\sum_{i=1}^{n_{hi}} \frac{n_{hi} \bar{y}_{hi}}{\sum_{i=1}^{n_{hi}} [1+(n_{hi}-1)\rho_h]}}$$

\[ \text{where } h = 1, 2, \ldots, M, \]

and

$$\hat{\sigma}_h^2(\rho_h) = \frac{1}{n_h} \left[ \frac{\text{SSE}_h}{1-\rho_h} + \sum_{h=1}^{K_h} \frac{n_{hi}(\bar{y}_{hi} - \hat{\mu}_h(\rho_h))^2}{[1+(n_{hi}-1)\rho_h]} \right]$$

\[ \text{where } h = 1, 2, \ldots, M, \]

(4.19)

respectively. Now, following the method of Richards (1961), discussed in section 1.6.1, for maximizing a function of several variables and substituting (4.19) in (4.18), we have

$$l^* = -\frac{N}{2} \left[ 1 + \ln(2\pi) \right] - \frac{1}{2} \sum_{h=1}^{M} n_h \ln(\hat{\sigma}_h^2(\rho_h))$$

$$- \frac{1}{2} \sum_{h=1}^{M} (N_h - K_h) \ln(1-\rho_h) - \frac{1}{2} \sum_{h=1}^{M} K_h \ln[1+(n_{hi}-1)\rho_h]$$

(4.20)

which is now a function of $\rho_h$ ($h = 1, 2, \ldots, M$ only).

For any given $h$, the results of Section 2.3 are directly applicable here. Thus, the maximum likelihood estimate $\hat{\rho}_h$ of $\rho_h$ ($h = 1, 2, \ldots, M$) can be obtained by maximizing a single variable function

$$l^*_h = -\frac{N_h}{2} \left[ 1 + \ln(2\pi) + \ln(\hat{\sigma}_h^2(\rho_h)) \right] - \frac{N_h - K_h}{2} \ln(1-\rho_h)$$

$$- \frac{1}{2} \sum_{i=1}^{K_h} \ln[1+(n_{hi}-1)\rho_h]$$

(4.21)
and the large sample variance of \( \hat{\rho}_h \), obtained by inverting the Fisher's information matrix, based on \( I_h \), is given by

\[
\text{Var}(\hat{\rho}_h) = 2(1-\rho_h)^2 \left\{ \frac{K_h \sum h_{hi}(n_{hi} - 1)(1+n_{hi} - 1)\rho_h^2}{\sum_{i=1}^N \frac{n_{hi}(n_{hi} - 1)}{1+(n_{hi} - 1)\rho_h^2}} \right\}.
\]

Under the tentative assumption of the homogeneity of intraclass correlations in all the \( M \) populations, the log-likelihood function \( l \) (4.18) becomes

\[
l_c = -\frac{N}{2} \ln(2\pi) - \frac{1}{2} \sum_{h=1}^M N_h \ln(c_h^2) - \frac{1}{2} (N - K) \ln(1-\rho)
\]

\[
- \frac{1}{2} \sum_{h=1}^M \sum_{i=1}^{K_h} \ln[1+(n_{hi} - 1)\rho]
\]

\[
- \frac{1}{2} \sum_{h=1}^M \frac{1}{\sigma_h^2} \left[ \text{SSE}_h - \frac{K_h \sum_{i=1}^{n_{hi}} (\hat{y}_{hi} - \mu_h)^2}{1-\rho} \right].
\]

(4.23)

Once again using the method of Richards (1961) and substituting (4.19) in (4.23), we have a single variable function

\[
\ell_c^* = -\frac{N}{2} \left[ 1 + \ln(2\pi) \right] - \frac{1}{2} \sum_{h=1}^M N_h \ln(c_h^2(\rho))
\]

\[
- \frac{N - K}{2} \ln(1-\rho) - \frac{1}{2} \sum_{h=1}^M \sum_{i=1}^{n_{hi}} \ln[1+(n_{hi} - 1)\rho],
\]

(4.24)

which is a function of \( \rho \) only. Thus the maximum likelihood estimate \( \hat{\rho} \) of the common value \( \rho \) of intraclass correlations can be obtained by maximizing \( \ell_c^* \) (4.24). The large sample variance of \( \hat{\rho} \) can be obtained by inverting the Fisher's information matrix corresponding to log-likelihood function \( l_c \) (4.23) which is a function of \( (2M+1) \) parameters. Let \( \text{E}(\theta) \) be the Fisher's (or expected) information matrix. Then
\[
E(\Theta) = \left\| -E\left[ \frac{\partial^2 l_c}{\partial \theta_r \partial \theta_s} \right] \right\|, \quad r, s = 1, 2, \ldots, 2M+1.
\]

where \( \theta_r = \mu_r \), \( \theta_{M+r} = \sigma_r^2 \), \( r = 1, 2, \ldots, M \), and \( \theta_{2M+1} = \rho \).

Denoting the \((r,s)\)th element of \( E(\Theta) \) by \( e_{r,s} \), the elements of Fisher's information matrix are given by

\[
e_{r,r} = \frac{1}{\sigma_r^2} \sum_{i=1}^{K_r} \frac{n_{ri}}{1 + (n_{ri} - 1)\rho}, \quad r = 1, 2, \ldots, M,
\]

\[
e_{r,s} = 0, \quad r \neq s; \quad r = 1, 2, \ldots, M; \quad s = 1, 2, \ldots, 2M+1,
\]

\[
e_{r,r} = \frac{N_r}{2\sigma_r^2}, \quad r = M+1, M+2, \ldots, 2M,
\]

\[
e_{r,s} = 0, \quad r \neq s; \quad r, s = M+1, M+2, \ldots, 2M,
\]

\[
e_{M+r, 2M+1} = \frac{-\rho}{2(1-\rho)^2} \sum_{i=1}^{K_r} \frac{n_{ri}(n_{ri} - 1)}{1 + (n_{ri} - 1)\rho}, \quad r = 1, 2, \ldots, M
\]

and

\[
e_{2M+1, 2M+1} = \frac{1}{2(1-\rho)^2} \sum_{h=1}^{K_h} \frac{n_{hi}(n_{hi} - 1)(1 + (n_{hi} - 1)\rho^2)}{[1 + (n_{hi} - 1)\rho]^2}.
\]

Hence

\[
E(\Theta) = \begin{bmatrix}
E_{11} & 0 & 0 \\
0 & E_{22} & E_{23} \\
0 & E_{32} & E_{33}
\end{bmatrix}
\]

and by standard algebra of partitioned matrices

\[
E^{-1}(\Theta) = \begin{bmatrix}
E_{11}^{-1} & 0 & 0 \\
0 & E_{22.3}^{-1} & -E_{22}^{-1}E_{23}^{-1}E_{33.2} \\
0 & -E_{33}^{-1}E_{32}^{-1}E_{22.3} & E_{33.2}^{-1}
\end{bmatrix},
\]

where
\[ E_{11}(\Theta) = \text{Diag}(e_{1,1}, e_{2,2}, \ldots, e_{M,N}), \]

\[ E_{22}(\Theta) = \text{Diag}(e_{M+1,M+1}, e_{M+2,M+2}, \ldots, e_{2M,2M}), \]

\[ E_{23}(\Theta) = E_{32}(\Theta)^T = [e_{M+1,2M+1}, e_{M+2,2M+1}, \ldots, e_{2M,2M+1}]^T \]

\[ E_{33}(\Theta) = e_{2M+1,2M+1}. \]

\[ E_{22,3}(\Theta) = E_{22}(\Theta) - E_{23}(\Theta) E_{33}(\Theta)^{-1} E_{32}(\Theta), \]

and

\[ E_{33,2}(\Theta) = E_{33}(\Theta) - E_{32}(\Theta) E_{22}(\Theta)^{-1} E_{23}(\Theta). \]

(4.28)

Thus the large sample variance of the maximum likelihood estimator \( \hat{\rho} \) of \( \rho \) is given by

\[
\text{Var}(\hat{\rho}) \approx E_{33,2}(\Theta)
\]

\[
= [2(1-\rho)^2] \left\{ \sum_{h=1}^{M} \sum_{i=1}^{K} \frac{n_{hi}(n_{hi}-1)(1+(n_{hi}-1)\rho)}{[1+(n_{hi}-1)\rho]^2} \right. \\
\left. - \rho^2 \sum_{h=1}^{M} \frac{1}{N_h} \left[ \sum_{i=1}^{K} \frac{n_{hi}(n_{hi}-1)}{[1+(n_{hi}-1)\rho]} \right]^{-1} \right\}.
\]

(4.27)

For \( M = 2 \) and \( n_{hi} = n_h, h = 1,2, \) the expression (4.27) reduces to the variance expression for the maximum likelihood estimator of the common intraclass correlation given by Donner and Bull (1983) and, for \( M=1, \) it reduces to the expression (2.18) for the variance of the maximum likelihood estimator of \( \rho \) in single sample. (Donner and Koval, 1980b).

4.4 Homogeneity Tests for Intraclass Correlations

Here data, denoted by \( \gamma_h, (h = 1,2,\ldots,M), \) regarding intraclass correlations are available from \( M \) independent
multivariate normal populations or under $M$ different conditions, and it is of interest to test the homogeneity of intraclass correlations. In particular, we construct test procedures for the composite null hypothesis

$$H_0: \rho_1 = \rho_2 = \ldots = \rho_M$$

against

$$H_1: \rho_h \neq \rho_m \text{ for some } h \neq m.$$  

Let $\rho$ be the common value of the intraclass correlations across the $M$ populations. Six procedures for testing the said hypothesis will be presented. Two of them are based on the $C(\alpha)$ procedures of Neyman (1959), one is likelihood ratio (LR) test, one is based on the approximation of Fisher (1925) and the other two are based on a weighted method.

### 4.4.1 Likelihood Ratio Test

The log-likelihood function of the total combined sample is $\ell_c$ (4.23) under the null hypothesis and is $\ell$ (4.18) under the alternate hypothesis. Denote the maximum value of $\ell_c$ by $\ell_0$ and of $\ell$ by $\ell_1$. Then it follows from the large sample theory of the method of maximum likelihood that the likelihood ratio (LR) statistic

$$\chi^2_L = -2 (\ell_0 - \ell_1)$$

$$= \sum_{h=1}^{M} N_h \ln \left[ \frac{\hat{\sigma}^2_h(\hat{\rho})}{\hat{\sigma}^2_h(\rho_h)} \right] + \sum_{h=1}^{M} (N_h - K_h) \ln \left[ \frac{1 - \hat{\rho}}{1 - \rho_h} \right]$$

$$+ \sum_{h=1}^{M} \sum_{i=1}^{K_h} \ln \left[ \frac{1 + C_{hi} \hat{\rho}}{1 + C_{hi} \rho_h} \right]$$  

is approximately distributed as chi-squared with $(M-1)$ degrees of freedom, where $\hat{\sigma}^2_h(\cdot), h = 1, 2, \ldots, M$ are as defined by (4.18). Thus, an approximate test of size $\alpha$ is
to reject \( H_0 \) if \( \chi^2_L > \chi^2_{(1-\alpha)(M-1)} \), where \( \chi^2_{(1-\alpha)(M-1)} \) is the 100(1-\alpha) percentile point of the chi-squared distribution with (M-1) degrees of freedom.

### 4.4.2 C(\alpha) tests

Let us assume that under the alternate hypothesis, \( \rho_h = \rho + \delta_h \), \( h = 1,2,\ldots,M \), with \( \delta_M = 0 \). Then testing the null hypothesis of homogeneity of intraclass correlations reduces to testing \( H_0: \delta_h = 0 \) for all \( h \). Tarone (1985) uses this technique to derive C(\alpha) test for the homogeneity of odds ratio in 2 x 2 contingency tables.

Under the above reparameterization, the log-likelihood function of the total combined sample \( l \) (4.18) becomes

\[
l'_c = -\frac{N}{2} \ln(2\pi) - \frac{1}{2} \sum_{h=1}^{M} N_h \ln(\sigma^2_h) - \frac{1}{2} \sum_{h=1}^{M} (N_h - K_h) \ln(1 - \rho - \delta_h) \\
- \frac{1}{2} \sum_{h=1}^{M} \frac{K_h}{\sigma^2_h} \ln(1 + (n_{hi} - 1)(\delta_h + \rho)) \\
- \frac{1}{2} \sum_{h=1}^{M} \frac{1}{\sigma^2_h} \left[ \frac{\text{SSE}_h}{1-\delta_h-\rho} + \sum_{i=1}^{K_h} \frac{n_{hi} (\bar{y}_{hi} - \mu_h)^2}{1 + (n_{hi} - 1)(\delta_h + \rho)} \right]. \quad (4.29)
\]

Let

\[
\delta = [\delta_1, \delta_2, \ldots, \delta_{M-1}]^T, \\
\gamma = [\gamma_1, \gamma_2, \ldots, \gamma_{2M+1}]^T \\
= [\mu_1, \mu_2, \ldots, \mu_M, \sigma^2_1, \ldots, \sigma^2_M, \rho]^T
\]

and denote the efficient scores by

\[
D_{h}(\gamma) = \left[ \frac{\delta}{\delta \gamma_h} l'_c \right]_{\delta=0}, \quad h = 1,2,\ldots,M-1,
\]

and

\[
R_{j}(\gamma) = \left[ \frac{\delta}{\delta \gamma_j} l'_c \right]_{\delta=0}, \quad j = 1,2,\ldots,2M+1.
\]

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The C(\omega) test statistics are based on the functions

\[ T_h = T_h(\gamma) = D_h(\gamma) - \sum_{j=1}^{2M+1} \beta_{h,j} R_j(\gamma), \quad h = 1, 2, \ldots, M-1, \]

where \( \beta_{h,j}, \quad j = 1, 2, \ldots, 2M+1 \), are the regression coefficients of \( D_h(\gamma) \) on \( R_j(\gamma), \quad j = 1, 2, \ldots, 2M+1 \), respectively. Following Moran (1970), these regression coefficients can be obtained by solving, under \( H_0 \), a homogeneous system of \( 2M+1 \) equations given by

\[-2 \left\{ E[D_h(\gamma)R_i(\gamma)] - \sum_{j=1}^{2M+1} \beta_{h,j} E[R_i(\gamma)R_j(\gamma)] \right\} = 0, \quad i = 1, 2, \ldots, 2M+1\]

or equivalently

\[
\begin{align*}
\left\{ E\left[ -\frac{\delta^2 l'_c}{\delta \theta_h \delta \mu_1} \right] - \sum_{j=1}^{M} \beta_{h,j} E\left[ -\frac{\delta^2 l'_c}{\delta \mu_1 \delta \mu_j} \right] - \sum_{j=1}^{M} \beta_{h,M+j} E\left[ -\frac{\delta^2 l'_c}{\delta \mu_1 \delta \sigma_j} \right] \right\} & = 0, \quad i = 1, 2, \ldots, M, \\
\left\{ E\left[ -\frac{\delta^2 l'_c}{\delta \theta_h \delta \mu_2} \right] - \sum_{j=1}^{M} \beta_{h,j} E\left[ -\frac{\delta^2 l'_c}{\delta \mu_2 \delta \mu_j} \right] - \sum_{j=1}^{M} \beta_{h,M+j} E\left[ -\frac{\delta^2 l'_c}{\delta \mu_2 \delta \sigma_j} \right] \right\} & = 0, \quad i = 1, 2, \ldots, M, \\
\left\{ E\left[ -\frac{\delta^2 l'_c}{\delta \theta_h \delta \sigma_1} \right] - \sum_{j=1}^{M} \beta_{h,j} E\left[ -\frac{\delta^2 l'_c}{\delta \sigma_1 \delta \mu_j} \right] - \sum_{j=1}^{M} \beta_{h,M+j} E\left[ -\frac{\delta^2 l'_c}{\delta \sigma_1 \delta \sigma_j} \right] \right\} & = 0, \\
\left\{ E\left[ -\frac{\delta^2 l'_c}{\delta \theta_h \delta \sigma_2} \right] - \sum_{j=1}^{M} \beta_{h,j} E\left[ -\frac{\delta^2 l'_c}{\delta \sigma_2 \delta \mu_j} \right] - \sum_{j=1}^{M} \beta_{h,M+j} E\left[ -\frac{\delta^2 l'_c}{\delta \sigma_2 \delta \sigma_j} \right] \right\} & = 0.
\end{align*}
\]

The first order derivatives of \( l'_c \), under \( H_0 \), are given by

\[ (4.31) \]
\[
D_h(\gamma) = \frac{-\rho}{2(1-\rho)\sigma_h^2} \sum_{i=1}^{K_h} \frac{n_{hi}(n_{hi}-1)}{[1+(n_{hi}-1)\rho]}
- \frac{1}{2\sigma_h^2} \left[ \frac{\text{SSE}_h}{(1-\rho)^2} - \frac{K_h}{\Sigma} \frac{n_{hi}(n_{hi}-1)(\bar{y}_{hi} - \mu_h)^2}{[1+(n_{hi}-1)\rho]^2} \right],
\]
\[h = 1, 2, \ldots, M-1,\]

\[
R_h(\gamma) = \frac{1}{\sigma_h^2} \sum_{i=1}^{K_h} \frac{n_{hi}(\bar{y}_{hi} - \mu_h)^2}{[1+(n_{hi}-1)\rho]},
\]
\[h = 1, 2, \ldots, M,\]

\[
R_{M+h}(\gamma) = -\frac{N_h}{2\sigma_h^2} + \frac{1}{2\sigma_h^4} \left[ \frac{\text{SSE}_h}{1-\rho} + \frac{K_h}{\Sigma} \frac{n_{hi}(\bar{y}_{hi} - \mu_h)^2}{[1+(n_{hi}-1)\rho]} \right],
\]
\[h = 1, 2, \ldots, M,\]

and

\[
R_{2M+1}(\gamma) = \frac{-1}{\sigma_h^2} \sum_{h=1}^{M} \left\{ \frac{\rho}{(1-\rho)\sigma_h^2} \frac{K_h}{\Sigma} \frac{n_{hi}(n_{hi}-1)}{[1+(n_{hi}-1)\rho]}
+ \frac{1}{\sigma_h^2} \left[ \frac{\text{SSE}_h}{(1-\rho)^2} - \frac{K_h}{\Sigma} \frac{n_{hi}(n_{hi}-1)(\bar{y}_{hi} - \mu_h)^2}{[1+(n_{hi}-1)\rho]^2} \right] \right\}.
\]

(4.32)

Further let

\[
A = -E \left\| \frac{\delta^2 I_c'}{\delta r \delta s} \right\|, \quad r, s = 1, 2, \ldots, M-1,
\]

\[
B = -E \left\| \frac{\delta^2 I_c'}{\delta s \delta r} \right\|, \quad r = 1, 2, \ldots, M-1,
\]
\[s = 1, 2, \ldots, 2M+1,\]

and

\[
C = -E \left\| \frac{\delta^2 I_c'}{\delta r' \delta s'} \right\|, \quad r, s = 1, 2, \ldots, 2M+1
\]

(4.33)

then, under \( H_0 \), the \((r, s)\)th elements of \( A \), \( B \) and \( C \) denoted by \( A_{r,s} , B_{r,s} \) and \( C_{r,s} \), respectively, are given by

\[
A_{r,r} = \frac{1}{2(1-\rho)^2} \sum_{i=1}^{K_r} \frac{n_{ri}(n_{ri}-1)[1+(n_{ri}-1)\rho^2]}{[1+(n_{ri}-1)\rho]^2}, \quad r = 1, 2, \ldots, M-1.
\]
\( A_{r,m} = 0, \quad r \neq m, r = 1, 2, \ldots, M-1, \)

\( B_{r,s} = 0, \quad s \neq M+r, \quad r = 1, 2, \ldots, M-1, \quad s = 1, 2, \ldots, 2M, \)

\[ B_{r,M+r} = \frac{-\rho}{\sum_{i=1}^{K_r} \frac{n_{r_i}(n_{r_i}-1)}{2(1-\rho)(1+cn_{r_i}-1)\rho}}, \quad r = 1, 2, \ldots, M-1, \]

\[ B_{r,2M+1} = A_{r,r}, \quad r = 1, 2, \ldots, M-1, \]

\[ C_{r,s} = e_{r,s}, \quad r, s = 1, 2, \ldots, 2M+1 \quad (4.34) \]

and \( e_{r,s} \) as given by (4.25). Now on substituting (4.34) in (4.31), the solution of (2M+1) equations is

\[ \beta_{h,j} = 0, \quad j = 1, 2, \ldots, M, \]

\[ \beta_{h,M+j} = -\xi_h \left[ \frac{\sum_{i=1}^{M} [\text{Var}(\hat{\rho}_{i})]^{-1}}{\text{Var}(\hat{\rho}_j)} \right]^{-1}, \quad j \neq h, \quad j = 1, 2, \ldots, M, \]

\[ \beta_{h,M+h} = \xi_h - \xi_h \left[ \frac{\sum_{i=1}^{M} [\text{Var}(\hat{\rho}_{i})]^{-1}}{\text{Var}(\hat{\rho}_h)} \right]^{-1} \]

and

\[ \beta_{h,2M+1} = \left[ \frac{\sum_{i=1}^{M} [\text{Var}(\hat{\rho}_{i})]^{-1}}{\text{Var}(\hat{\rho}_h)} \right]^{-1}, \quad (4.35) \]

where

\[ \xi_h = \frac{c_{M+h,2M+1}}{c_{M+h,M+h}} = \frac{\rho}{\sum_{i=1}^{K_h} \frac{n_{h_i}(n_{h_i}-1)}{N_h(1-\rho)(1+cn_{h_i}-1)\rho}}. \quad (4.36) \]

Furthermore, it should be noted that

\[ \frac{\delta\epsilon'_c}{\delta h} \bigg|_{\delta=0} = \frac{\delta\epsilon'_h}{\delta h} \bigg|_{\rho_h = \rho} \quad \text{and} \quad \frac{\delta\epsilon'_c}{\delta \gamma} \bigg|_{\delta=0} = \frac{\delta\epsilon'_h}{\delta \gamma} \bigg|_{\rho_h = \rho}, \quad j = 1, 2, \ldots, 2M. \quad (4.37) \]

Thus substituting (4.35), (4.36) and (4.37) in (4.30), we get

\[ \tau_h(\gamma) = \psi_h(\gamma) - \psi(\gamma) \left[ \frac{\sum_{h=1}^{M} [\text{Var}(\hat{\rho}_{h})]^{-1}}{\text{Var}(\hat{\rho}_h)} \right]^{-1}, \quad (4.38) \]
where

\[ \psi_h(\gamma) = \frac{\partial \ell_h}{\partial \rho_h} \bigg|_{\rho_h = \rho} - \xi_h \frac{\partial \ell_h}{\partial \sigma^2_h} \bigg|_{\rho_h = \rho} \quad h = 1, 2, \ldots, M. \]

\[ = \frac{1}{2\sigma^2_h} \left[ \frac{\rho}{N_h(1-\rho)} \left[ \sum_{i=1}^{n_{hi}} \frac{n_{hi}(n_{hi}-1)}{1+(n_{hi}-1)\rho} \right] \right. \]

\[ \times \left[ \frac{\text{SSE}_h}{1-\rho} + \sum_{i=1}^{K_h} \frac{n_{hi}(\gamma_{hi} - \mu_h)^2}{1+(n_{hi}-1)\rho} \right] \]

\[ - \left[ \frac{\text{SSE}_h}{(1-\rho)^2} - \sum_{i=1}^{K_h} \frac{n_{hi}(n_{hi}-1)(\gamma_{hi} - \mu_h)^2}{1+(n_{hi}-1)\rho} \right]. \quad (4.39) \]

and

\[ \psi(\gamma) = \sum_{h=1}^{M} \psi_h(\gamma). \quad (4.40) \]

Let

\[ \tilde{\gamma} = T(\gamma) = [T_1, T_2, T_3, \ldots, T_{M-1}]^T, \]

then \( \tilde{\gamma} \) is distributed as multivariate normal with mean vector \( \bar{\gamma} \) and covariance matrix \( \Sigma = [A - B^T C^{-1} B] \). Thus, from Theorem 1.7, the quantity

\[ \chi^2_C(\gamma) = \tilde{\gamma}^T \Sigma^{-1} \tilde{\gamma} \quad (4.41) \]

has an approximate chi-squared distribution with \((M-1)\) degrees of freedom under \( H_0 \). If \( V_{ij} \) and \( \psi_{ij} \) denote the \((i,j)^{\text{th}}\) elements of \( \psi \) and \( \psi^{-1} \), respectively, then from (4.34),

\[ V_{ii} = \left[ \sum_{h=1}^{M} \left( \text{Var}(\hat{\rho}_h) \right)^{-1} \right] \left( \text{Var}(\hat{\rho}_1) \right)^{-1} \left[ I - \left( \text{Var}(\hat{\rho}_1) \right)^{-1} \right], \]

\[ i = 1, 2, \ldots, M-1, \]

\[ V_{ij} = - \left[ \sum_{h=1}^{M} \left( \text{Var}(\hat{\rho}_h) \right)^{-1} \right] \left( \text{Var}(\hat{\rho}_i) \right)^{-1} \left( \text{Var}(\hat{\rho}_j) \right)^{-1}, \]

\[ i \neq j, i, j = 1, 2, \ldots, M-1. \]
\[ v^{i1} = \left[ \left( \text{Var} \left( \hat{\rho}_1 \right) \right)^{-1} + \left( \text{Var} \left( \hat{\rho}_M \right) \right)^{-1} \right]^{-1}, \quad i = 1, 2, \ldots, M-1 \]

and

\[ v^{ij} = \text{Var} \left( \hat{\rho}_j \right), \quad i \neq j, \quad i, j = 1, 2, \ldots, M-1. \]

Thus the quantity (4.41) becomes

\[ \chi^2_C(\gamma) = \sum_{h=1}^{M-1} T_h^2(\gamma) \text{Var} \left( \hat{\rho}_h \right) + \left[ \sum_{h=1}^{M-1} T_h(\gamma) \right]^2 \text{Var} \left( \hat{\rho}_M \right). \tag{4.42} \]

Since \[ \sum_{h=1}^{M} T_h(\gamma) = 0 \] implies \[ \sum_{h=1}^{M-1} T_h(\gamma) = -T_M(\gamma), \]

the expression (4.42) becomes

\[ \chi^2_C(\gamma) = \sum_{h=1}^{M} T_h^2(\gamma) \text{Var} \left( \hat{\rho}_h \right). \tag{4.43} \]

Substituting (4.38) in (4.43), we get

\[ \chi^2_C(\gamma) = \sum_{h=1}^{M} \psi_h^2(\gamma) \text{Var} \left( \hat{\rho}_h \right) - \left[ \sum_{h=1}^{M} \left( \text{Var} \left( \hat{\rho}_h \right) \right)^{-1} \right]^{-1} \left[ \sum_{h=1}^{M} \psi_h(\gamma) \right]^2, \tag{4.44} \]

where \[ \psi_h(\gamma), \quad h = 1, 2, \ldots, 2M+1, \] are defined by (4.39).

Notice that the \( \chi^2_C(\gamma) \) (4.44) depends on the nuisance parameters \( \gamma \) and is still inappropriate to use. Moran (1970) suggested that these nuisance parameters should be replaced by any \( \gamma_n \)-consistent estimators. If \( \gamma^*_n \) is some \( \gamma_n \)-consistent estimator of \( \gamma \), then the test statistic \( \chi^2_C(\gamma^*_n) \) has an approximate chi-squared distribution with \( (M-1) \) degrees of freedom. Thus an approximate test of size \( \alpha \) is to reject \( H_0 \) if \( \chi^2_C(\gamma^*_n) > \chi^2_{(1-\alpha)}(M-1) \), where \( \chi^2_{(1-\alpha)}(M-1) \) is the 100(1-\( \alpha \)) percentile point of the chi-squared distribution with \( (M-1) \) degrees of freedom.
If the maximum likelihood estimators \( \hat{\mu}_h(\hat{\rho}) \), \( \hat{\sigma}_h^2(\hat{\rho}) \), \( h = 1, 2, \ldots, M \), given by (4.19), and \( \hat{\rho} \), obtained by maximizing \( \ell_c^2 \) (4.24), are used in (4.44), then the second term on the right hand side of (4.44) vanishes because \( \sum_{h=1}^{M} \psi_h(\hat{\gamma}) = 0 \) and it is reduced to

\[
\chi_{CM}^2 = \sum_{h=1}^{M} \psi_h(\hat{\gamma}) \text{ Var}(\hat{\rho}_h), \tag{4.45}
\]

where, from (4.39),

\[
\psi_h(\hat{\gamma}) \approx \frac{\partial \ell_h}{\partial \rho_h} \bigg|_{\gamma = \hat{\gamma}} \approx \frac{1}{2} \left[ \frac{\hat{\rho}}{1 - \hat{\rho}} \sum_{i=1}^{K_h} \frac{n_{hi}(n_{hi} - 1)}{[1 + (n_{hi} - 1)\hat{\rho}]} \right] \]

\[
- \frac{1}{\hat{\sigma}_h^2(\hat{\rho})} \left[ \frac{\text{SSE}_h}{(1 - \hat{\rho})^2} - \frac{\sum_{i=1}^{K_h} n_{hi}(n_{hi} - 1)(\bar{y}_{hi} - \hat{\mu}_h(\hat{\rho}))^2}{1 + (n_{hi} - 1)\hat{\rho}^2} \right].
\]

The ANOVA estimators of \( \mu_h \), \( \sigma_h^2 \), \( h = 1, 2, \ldots, M \), and \( \rho \) have closed forms and it is well known that they are also \( \sqrt{n} \)-consistent estimators. The ANOVA estimators \( \hat{\mu}_h \) and \( \hat{\sigma}_h^2 \), \( (h = 1, 2, \ldots, M) \) are given by (4.5) and the ANOVA estimator \( \hat{\rho} \) of \( \rho \) is given by (4.14). If the ANOVA estimators are substituted in (4.44), then \( \psi_h(\hat{\gamma}) \) (4.39) becomes

\[
\psi_h(\hat{\gamma}) \approx \frac{n_A}{2(\text{MSG}_h + (n_{oh} - 1)\text{MSE}_h)} \left[ \frac{\hat{\rho}}{n_h(1 - \hat{\rho})} \left[ \sum_{i=1}^{K_h} n_{hi}(n_{hi} - 1) \right] \right] \]

\[
\times \left[ \frac{\text{SSE}_h}{(1 - \hat{\rho})^2} - \frac{\sum_{i=1}^{K_h} n_{hi}(n_{hi} - 1)(\bar{y}_{hi} - \hat{\mu}_h(\hat{\rho}))^2}{1 + (n_{hi} - 1)\hat{\rho}^2} \right] \]

\[
- \left[ \frac{\text{SSE}_h}{(1 - \hat{\rho})^2} - \frac{\sum_{i=1}^{K_h} n_{hi}(n_{hi} - 1)(\bar{y}_{hi} - \hat{\mu}_h(\hat{\rho}))^2}{1 + (n_{hi} - 1)\hat{\rho}^2} \right]. \tag{4.46}
\]

When the ANOVA estimators of the parameters are used in
(4.44), the resulting test statistic \( \chi_{NC}^2 \) will be denoted by \( \chi_{CA}^2 \).

4.4.3 Other Homogeneity Tests

1) Using the Fisher's (1925) normalizing transformation on ANOVA estimator \( \tilde{\rho}_h \) (4.5) of \( \rho_h \), it is known that the quantity

\[
\varphi_h(\tilde{\rho}_h) = \frac{1}{2} \ln \left[ \frac{1 + (n_{Oh}^{-1})\tilde{\rho}_h}{1 - \tilde{\rho}_h} \right]
\]

is asymptotically normally distributed with mean \( \varphi_h(\rho_h) \) and variance

\[
\frac{1}{2} \left[ \frac{1}{k_h^{-1}} + \frac{1}{N_h^{-1}k_h} \right] = \frac{N_h^{-1}}{2(k_h^{-1} - 1)(N_h^{-1} - k_h)}.
\]

Therefore the test statistic

\[
\chi_F^2 = \sum_{h=1}^{M} W_h \left[ \varphi_h(\tilde{\rho}_h) - \varphi_h(\tilde{\rho}) \right]^2
- \frac{1}{W} \left[ \sum_{h=1}^{M} W_h \left[ \varphi_h(\tilde{\rho}_h) - \varphi_h(\tilde{\rho}) \right] \right]^2,
\]

\[
W_h = \frac{2(2 - 1)(N_h^{-1} - k_h)}{N_h^{-1}}, \quad h = 1, 2, \ldots, M.
\]

has an approximate chi-squared distribution with \((M-1)\) degrees of freedom under \( H_0 \). Thus an appropriate test of size \( \alpha \) is to reject \( H_0 \) in favour of \( H_1 \) if \( \chi_F^2 > \chi_{(1-\alpha)(M-1)}^2 \), where \( \chi_{(1-\alpha)(M-1)}^2 \) is the 100\((1-\alpha)\) percentile point of the chi-squared distribution with \((M-1)\) degrees of freedom.

2) The ANOVA estimator \( \tilde{\rho}_h \) (4.5) is asymptotically normally distributed with mean \( \rho_h \) and variance as given by (4.6).
Furthermore, \( \tilde{\rho} \) (4.14) is the ANOVA estimator of common value \( \rho \) of the intraclass correlations. I suggest the test the statistic

\[
\chi^2_A = \sum_{h=1}^{M} \hat{w}_h [\tilde{\rho}_h - \tilde{\rho}]^2.
\]

(4.48)

where

\[
\tilde{w}_h = \frac{1}{[\text{Var}(\tilde{\rho}_h | \rho_h = \rho)]}
\]

which has asymptotically chi-squared distribution with \((M-1)\) degrees of freedom under \(H_0\). Thus, another test of size \(\alpha\) is to reject \(H_0\) in favour of \(H_1\) if

\[
\chi^2_F > \chi^2_{(1-\alpha)(M-1)}.
\]

3) The maximum likelihood estimator of \(\hat{\rho}_h\) is asymptotically normally distributed with mean \(\rho_h\) and variance \(\text{Var}(\hat{\rho}_h)\), given by (4.22). Thus, under \(H_0\), the statistic

\[
\chi^2_M = \sum_{h=1}^{M} \hat{w}_h [\hat{\rho}_h - \hat{\rho}_W]^2
\]

(4.49)

has approximate chi-squared distribution with \((M-1)\) degrees of freedom, where

\[
\hat{w}_h = \frac{1}{[\text{Var}(\hat{\rho}_h | \rho_h = \rho)]}, \quad \hat{\rho}_W = \frac{1}{\hat{w}} \sum_{h=1}^{M} \hat{w}_h \hat{\rho}_h
\]

and \(\hat{w} = \sum_{h=1}^{M} \hat{w}_h\). Thus another test of size \(\alpha\) is to reject \(H_0\) in favour of \(H_1\) if

\[
\chi^2_F > \chi^2_{(1-\alpha)(M-1)}, \quad \text{where} \quad \chi^2_{(1-\alpha)(M-1)} \text{ is as defined before.}
\]

4.5 Comparison of Homogeneity Tests

The large sample properties of the homogeneity tests for intraclass correlations were investigated with respect to
their empirical significance levels and powers by conducting a Monte Carlo study. Once again, by focusing the aim of this investigation on the analysis of familial data, the sibship sizes were generated from zero-truncated negative binomial distribution using the probability mass function (3.17) for \( m = 2.84 \) and \( P = 0.93 \). In order to ensure the correlation structure of model (4.1) for each family within each sample, the method of conditioning was used to generate the sample observations \( y_{hij} \) for each iteration. This method is described in the algorithm for each of \( K_h \) families within \( h \)-th sample \( (h = 1, 2, \ldots, M) \) as follows:

For \( i = 1, 2, \ldots, K_h \), and \( h = 1, 2, \ldots, M \),

(i) Generate a set of \( n_{hi} \) independent standard normal deviates \( Z_{hij}, j = 1, 2, \ldots, n_{hi} \),

(ii) Set \( y_{hii} = Z_{hii} \),

(iii) Calculate the remaining scores iteratively as follows:

\[
y_{hij} = \mu_{hij} + Z_{hij} \sigma^2_{hij},
\]

where

\[
\mu_{hij} = \frac{\rho_h}{[1+(j-2)\rho_h]} \sum_{g=1}^{j-1} y_{hig}
\]

and

\[
\sigma^2_{hij} = \frac{(1-\rho_h)(1+(j-1)\rho_h)}{[1+(j-2)\rho_h]}.
\]

The \( \mu_{hij} \) and \( \sigma^2_{hij} \) given above, are the conditional mean and variance, respectively, of \( y_{hij} \) given \( y_{hii}, y_{hij}, \ldots, y_{hij(j-1)} \) as derived from the standard multivariate normal theory (see, e.g., Anderson, 1988). The maximum likelihood estimates of \( \rho_h, h = 1, 2, \ldots, M, \) and \( \rho \)
were obtained by maximizing $\chi^2_h$ (4.21), $h = 1, 2, \ldots, M$, and $\chi^2_c$ (4.24), respectively, by using IMSL (1987) subroutine DUVMGS. The estimates of the significance levels and powers are all based on 1000 repeated samples for different combinations of $M$, $K_1 = K_2 = \ldots = K_M$ and $\rho_h$, $h = 1, 2, \ldots, M$. It has been noticed that, as in section 3.3, the subroutine DUVMGS was unable to find satisfactory ML estimates of $\rho_h$, $h = 1, 2, \ldots, M$, or $\rho$ for some generated samples. If for some simulated sample, the optimization criteria are not met, then this sample was discarded and was replaced by another new sample. This process was continued until 1000 samples were available, each of which give satisfactory ML estimates of $\rho_h$, $h = 1, 2, \ldots, M$, and $\rho$.

Table 4.1 gives the empirical levels of $\chi^2_L$ (4.28), $\chi^2_{CM}$ (4.45), $\chi^2_{CA}$ (4.44 along with 4.46), $\chi^2_F$ (4.47), $\chi^2_A$ (4.48) and $\chi^2_M$ (4.49) procedures corrsponds to nominal levels $\alpha = 0.01$, 0.05 and 0.10 for testing $H_0: \rho_1 = \rho_2$ against $H_1: \rho_1 \neq \rho_2$. This table shows that the empirical levels of $\chi^2_L$ and $\chi^2_M$ seem to be greater than the nominal levels for $\rho_1 = \rho_2 = 0, 0.1$. Furthermore the C(ω) test $\chi^2_{CA}$ for $\rho_1 = \rho_2 = 0$ and $\chi^2_F$ for $\rho_1 = \rho_2 = 0.5$ give empirical levels that are significantly different from the stated nominal levels. The empirical levels of all the other procedures show satisfactory agreement with the true nominal levels. It is quite important now to compare these tests using powers as the criterion.

For testing $H_0: \rho_1 = \rho_2$ against $H_1: \rho_1 \neq \rho_2$, the empirical power curves of the tests are presented in Figures 4.1.
TABLE 4.1

Empirical levels based on 1000 runs of $\chi^2_L$, $\chi^2_{CM}$, $\chi^2_{CA}$, $\chi^2_{F}$, $\chi^2_A$ and $\chi^2_M$ tests for testing $H_0: \rho_1 = \rho_2$ versus $H_1: \rho_1 \neq \rho_2$, where $\rho_1 = \rho_2 = 0$, 0.1, 0.3 and 0.5.

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<th>$\rho_1=\rho_2$</th>
<th>$\alpha$</th>
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<th>$\chi^2_{CA}$</th>
<th>$\chi^2_{F}$</th>
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<td>0.094*</td>
<td>0.036</td>
<td>0.095*</td>
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<td>0.119</td>
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</tbody>
</table>

*The empirical level is more than 2 standard deviations away from the true level $\alpha$. 

141
through 4.8. Each power curve in these figures is identified as follows:

\[ \chi^2 = \text{---} \quad \circ \quad \text{---} \quad \bullet \quad \text{---} \quad \rightarrow \]
\[ \chi^2_{CM} = \text{---} \quad \Delta \quad \text{---} \quad \rightarrow \]
\[ \chi^2_{CA} = \text{---} \quad \ast \quad \text{---} \quad \ast \quad \text{---} \quad \rightarrow \]
\[ \chi^2_F = \text{---} \quad \ast \quad \text{---} \quad \ast \quad \text{---} \quad \rightarrow \]
\[ \chi^2_A = \text{---} \quad \ast \quad \text{---} \quad \ast \quad \text{---} \quad \rightarrow \]
\[ \chi^2_M = \text{---} \quad \ast \quad \text{---} \quad \ast \quad \text{---} \quad \rightarrow \]

These figures show that the likelihood ratio procedure \( \chi^2_L \) retains higher powers for testing \( H_0: \rho_1 = \rho_2 = 0 \) and 0.1. This might be due to its anti-conservative behaviour as it tends to give empirical levels that are consistently greater than the selected nominal levels for the said hypotheses. For testing the null hypotheses \( H_0: \rho_1 = \rho_2 = 0.3 \) and 0.5, the procedure based on Fisher's transformation \( \chi^2_F \) seem to be more powerful than all the other procedures.

The empirical levels and powers of the tests for testing \( H_0: \rho_1 = \rho_2 = \rho_3 \) against \( H_1: \rho_1 \neq \rho_j \), for some \( i \neq j \), are presented in Tables 4.2 through 4.7. These tables show that the empirical levels of \( \chi^2_L \) procedure are greater than the nominal levels for \( H_0: \rho_1 = \rho_2 = \rho_3 = 0.0, 0.1, 0.2 \) and \( \chi^2_F \) for \( H_0: \rho_1 = \rho_2 = \rho_3 = 0.3, 0.5 \). All the other procedures give satisfactory empirical levels, except \( \chi^2_M \) which gives statistically significant empirical levels when \( H_0: \rho_1 = \rho_2 = \rho_3 = 0.0 \) is tested. Furthermore, the procedure \( \chi^2_L \) retains higher powers provided \( \rho_1 \leq 0.2 \), \( i = 1, 2, 3 \).
POWERS OF THE HOMOGENEITY TESTS

\( K_1 = K_2 = 25 \) AND \( \alpha = 0.01, 0.05, 0.10 \)

\( H_0: \rho_1 = \rho_2 \) VS \( H_1: \rho_1 \neq \rho_2 \)

\( M = 2 \) SAMPLES

\( \rho_1 = 0 \)

Figure 4.1
POWERS OF THE HOMOGENEITY TESTS

\( K_1 = K_2 = 50 \) AND \( \alpha = 0.01, 0.05, 0.10 \)

\( H_0: \rho_1 = \rho_2 \) VS \( H_1: \rho_1 \neq \rho_2 \)

\( M = 2 \) SAMPLES

\( \rho_1 = 0 \)

\( \rho_2 \)

Figure 4.2
POWERS OF THE HOMOGENEITY TESTS

$K_1 = K_2 = 25$ AND $\alpha = 0.01, 0.05, 0.10$

$H_0: \rho_1 = \rho_2$ VS $H_1: \rho_1 \neq \rho_2$

$M = 2$ SAMPLES

$\rho_1 = 0.1$

Figure 4.3
POWERS OF THE HOMOGENEITY TESTS

\(K_1 = K_2 = 50\) AND \(\alpha = 0.01, 0.05, 0.10\)

\(H_0: \rho_1 = \rho_2\) VS \(H_1: \rho_1 \neq \rho_2\)

\(M = 2\) SAMPLES

\(\rho_1 = 0.1\)

\(\rho_2\)

Figure 4.4
POWERS OF THE HOMOGENEITY TESTS

\[ K_1 = K_2 = 25 \text{ AND } \alpha = 0.01, 0.05, 0.10 \]

\[ H_0: \rho_1 = \rho_2 \text{ VS } H_1: \rho_1 \neq \rho_2 \]

\[ \text{M=2 SAMPLES} \]

\[ \rho_1 = 0.3 \]

Figure 4.5
POWERS OF THE HOMOGENEITY TESTS

\[ K_1 = K_2 = 50 \text{ AND } \alpha = 0.01, 0.05, 0.10 \]

\[ H_0: \rho_1 = \rho_2 \text{ VS } H_1: \rho_1 \neq \rho_2 \]

\[ M = 2 \text{ SAMPLES} \]

\[ \rho_1 = 0.3 \]

Figure 4.6
POWERS OF THE HOMOGENEITY TESTS

$K_1=K_2=25$ AND $\alpha=0.01, 0.05, 0.10$

$H_0: \rho_1=\rho_2$ VS $H_1: \rho_1 \neq \rho_2$

$M=2$ SAMPLES

$\rho_1=0.5$

$\rho_2$

Figure 4.7
POWERS OF THE HOMOGENEITY TESTS

$K_1 = K_2 = 50$ AND $\alpha = 0.01, 0.05, 0.10$

$H_0: \rho_1 = \rho_2$ VS $H_1: \rho_1 \neq \rho_2$

$M = 2$ SAMPLES

$\rho_1 = 0.5$

Figure 4.8
### Table 4.2

Empirical levels and powers of $\chi^2_L$, $\chi^2_{CM}$, $\chi^2_{CA}$, $\chi^2_F$, $\chi^2_A$ and $\chi^2_M$ test procedures for testing $H_0: \rho_1 = \rho_2 = \rho_3$ against $H_1: \rho_i \neq \rho_j$, for some $i \neq j$, based on 1000 runs for $k_1 = k_2 = k_3 = 25$ and $\alpha = 0.01$.

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*The empirical level is more than 2 standard deviations away from the true level $\alpha$.  

151
### Table 4.3

Empirical levels and powers of $\chi^2_L$, $\chi^2_{CM}$, $\chi^2_{CA}$, $\chi^2_F$, $\chi^2_A$ and $\chi^2_M$ test procedures for testing $H_0: \rho_1 = \rho_2 = \rho_3$ against $H_1: \rho_i \neq \rho_j$, for some $i \neq j$, based on 1000 runs for $K_1 = K_2 = K_3 = 25$ and $\alpha = 0.05$.

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*The empirical level is more than 2 standard deviations away from the true level $\alpha$. 

152
TABLE 4.4

Empirical levels and powers of $\chi^2_L$, $\chi^2_{CM}$, $\chi^2_{CA}$, $\chi^2_F$, $\chi^2_A$ and $\chi^2_M$ test procedures for testing $H_0: \rho_1 = \rho_2 = \rho_3$ against $H_1: \rho_i \neq \rho_j$, for some $i \neq j$, based on 1000 runs for $K_1 = K_2 = K_3 = 25$ and $\alpha = 0.10$.

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*The empirical level is more than 2 standard deviations away from the true level $\alpha$. 

153
TABLE 4.5

Empirical levels and powers of $\chi^2$, $\chi^2_{CM}$, $\chi^2_{CA}$, $\chi^2_F$, $\chi^2_A$ and $\chi^2_M$ test procedures for testing $H_0: \rho_1 = \rho_2 = \rho_3$ against $H_1: \rho_i \neq \rho_j$, for some $i \neq j$, based on 1000 runs for $k_1 = k_2 = k_3 = 60$ and $\alpha = 0.01$.

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*The empirical level is more than 2 standard deviations away from the true level $\alpha$.  

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TABLE 4.6

Empirical levels and powers of $\chi^2_L$, $\chi^2_{CM}$, $\chi^2_{CA}$, $\chi^2_F$, $\chi^2_A$ and $\chi^2_M$ test procedures for testing $H_0: \rho_1 = \rho_2 = \rho_3$ against $H_1: \rho_i \neq \rho_j$, for some $i \neq j$, based on 1000 runs for $K_1 = K_2 = K_3 = 50$ and $\alpha = 0.05$.

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*The empirical level is more than 2 standard deviations away from the true level $\alpha$. 

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*The empirical level is more than 2 standard deviations away from the true level \( \alpha \).
otherwise $\chi^2_F$ is most powerful.

4.6 Discussion

The inference procedures for intraclass correlations in multiple samples are presented in this chapter. The ANOVA estimators of intraclass correlations and the expressions for their asymptotic variances are provided under a very general model which differs from the previously presented model of Donner (1985, 1986). It is shown that the maximum likelihood estimates of intraclass correlations can be obtained by maximizing single variable functions. The expressions for asymptotic variances of the maximum likelihood estimators of intraclass correlations are also derived. Several procedures for testing the homogeneity of intraclass correlations in multiple samples are derived and their performance is assessed in term of their empirical powers by a Monte Carlo study. The major findings of this quantitative investigation are summarized as follows:

1. The likelihood ratio procedure $\chi^2_L$ is more powerful than other procedures provided $\rho_1 = 0.1, \rho = 1, 2, 3$, though it gives empirical levels that are considerably greater than the selected nominal levels in most of the presented cases. It seems that the distribution of $\chi^2_L$ depends on the common value of intraclass correlations under the null hypothesis of homogeneity, especially when $K \leq 25$. The same has been noticed by Khatri et al. (1989) when the null hypothesis of homogeneity of the intraclass correlations in two populations (i.e.,
(2) The procedure $\chi^2_F$, based on Fisher's transformation, performs very well for most of the presented cases, especially for $M = 2$ populations. However, it gives empirical levels which are substantially greater than the nominal levels when $H_0: \rho_1 = \rho_2 = \rho_3 = 0.3$ and 0.5 were tested.

(3) The procedures $\chi^2_{CM}$ and $\chi^2_A$ give good estimates of nominal levels and very comparable powers. They seem to be asymptotically equally powerful in most of the presented cases.

(4) The procedures $\chi^2_{CA}$ and $\chi^2_M$ did not hold significance levels well in most of the presented cases, though $\chi^2_{CA}$ gives comparable powers when $H_0: \rho_1 = \rho_2 = \rho_3 = 0.3$ and 0.5 were tested.

It should be noted that the null distributions of many test statistics are not same for all the presented cases. Thus, the recommendations here should be based on the power comparisons by keeping the null distributions of test statistics in mind. On the basis of this Monte Carlo study, the use of $\chi^2_F$ is strongly recommended when the values of intraclass correlation in each population are thought to be small (i.e., $\rho_i \leq 0.2$, $i = 1, 2, \ldots, M$), because it holds nominal levels and gives comparable powers. If the values of intraclass correlations are thought to be moderate in all the populations (i.e., $0.3 \leq \rho_i \leq 0.5$, $i = 1, 2, 3$), the use of $\chi^2_{CM}$ and $\chi^2_A$ is recommended. It is also recommended that the procedure $\chi^2_{CM}$ should be used if no prior knowledge
concerning the values of intraclass correlations exists.

Please note that these recommendations are specific to the sibship-size distribution used in the Monte Carlo study. Investigations that utilize different sibship-size distribution or different values of the parameters of zero-truncated negative binomial distribution may provide different conclusions.

In the next chapter, the estimation of three sibling correlations (brother-brother, sister-sister, and brother-sister correlations) from multivariate normal population will be considered.
CHAPTER 5

MAXIMUM LIKELIHOOD ESTIMATION OF SIBLING CORRELATIONS

(Parent score's are excluded)

5.1 Introduction

In chapters 2, 3 and 4, the inference procedures for sib-sib (intra-class) correlation were discussed and compared, where the siblings (individuals) within each family (group) were not distinguished by their sex. However, when data on siblings are classified according to their sex, it is very important to make use of this additional information in the development of inferences regarding sibling correlations. This problem is of fundamental importance to epidemiologists who want to know the sex effects on the reported values of sibling correlations. This can be done by making tests of significance comparing the values of sibling correlations estimated from the same sample. The tests of this kind are usually performed descriptively. For example, Higgins and Keller (1975) investigated the effect of sex on sibling correlations with respect to forced expiratory volume. Roberts et al. (1978) compared the coefficients of resemblance for stature among siblings of each sex, while Martarell et al. (1978) compared sibling correlations with respect to the prevalence of ossification centres. Donner et al. (1984) discussed several procedures for testing the effect of sex differences on sibling correlations. The maximum likelihood estimation is possibly the most widely adopted method for parameter estimation but no simple
algorithm to find the maximum likelihood estimators of sibling correlations for unbalanced data is available. In this chapter, the maximum likelihood estimation of the three sibling correlations (brother-brother, sister-sister, and brother-sister correlations) for unbalanced familial data from multivariate normal populations will be considered. When the data are balanced, Elston (1975) derived the expressions for maximum likelihood estimators of $\rho_b$ (brother-brother), $\rho_s$ (sister-sister), and $\rho_{bs}$ (brother-sister) correlations. He also provided the expressions for the asymptotic variances and covariances of the estimators. Estimation of such correlations via maximum likelihood estimators of variance components is the most commonly used technique for unbalanced data. Smith (1980 a,b) and Bener and Huda (1987) propose closely related iterative procedures to compute the maximum likelihood estimates of the variance components of the model and hence sibling correlations. Their proposed procedures are quite complicated to adopt. Furthermore, the variances and covariances of the maximum likelihood estimators of parameters are not known to the best of our knowledge.

In this chapter, a technique due to Richards (1961) is utilized by which the estimates of the three sibling correlations and the variance parameters are obtained by maximizing a function of fewer number of parameters. Two cases will be considered. In the first case it is assumed that the within brothers and within sisters variances are not equal (non-homogeneous), and in the other it is assumed
that the within sib variances are equal (homogeneous). The expressions for the asymptotic variances and covariances of the maximum likelihood estimators are obtained by inverting the Fisher's information matrices. The published arterial blood pressure family data of Miall and Oldham (1955) will be used to illustrate the methodology.

5.2 The Model

Let

\[ \mathbf{y}_i = [\mathbf{y}_{ib}^T; \mathbf{y}_{is}^T]^T \]

\[ = [y_{i1}, y_{i2}, \ldots, y_{ib_1}; y_{ib_1+1}, \ldots, y_{in_1}]^T, \]

\[ i = 1, 2, \ldots, K \]

be a vector of measurements from the \( i \)th family, where \( K \) is the number of sampled families, and for \( i = 1, 2, \ldots, K \):

\( b_i \) = number of brothers in the \( i \)th family,

\( s_i \) = number of sisters in the \( i \)th family,

\( n_i = b_i + s_i \) = number of offspring in the \( i \)th family,

\( N_b = \sum_{i=1}^{K} b_i \) = total number of brothers in the sample,

\( N_s = \sum_{i=1}^{K} s_i \) = total number of sisters in the sample,

\( N = N_b + N_s \) = total number of offspring in the sample.

\[ y_{ij} = \begin{cases} 
\text{score of } j \text{th brother in the } i \text{th family}, & j = 1, 2, \ldots, b_i, \\
\text{score of } j \text{th sister in the } i \text{th family}, & j = b_i + 1, \ldots, n_i. 
\end{cases} \]

It is assumed that the \( \mathbf{y}_i \)'s are independently distributed multivariate normal vectors with means \( \mu_i \) and dispersion matrices \( \Omega_i \). That is,

\[ \mathbf{y}_i \sim \mathcal{MN}_{n_i} (\mu_i, \Omega_i), i = 1, 2, \ldots, K. \]  \hspace{1cm} (5.1)

where

\[ \mu_i = [\mu_{ib}^T; \mu_{is}^T]^T = [\mu_b, \mu_b, \ldots, \mu_b; \mu_s, \ldots, \mu_s]^T \]
is a column vector of length $n_i$ and

$$
\Omega_i = \begin{bmatrix}
(1-\rho_b)I_{b_1} + \rho_b I_{b_1}x_{b_1} & \sigma_b^2 & \sigma_b \sigma_s \\
I_{b_1}x_{b_1} & \rho_{bs} \sigma_b \sigma_s & \sigma_s^2 \\
I_{s_1}x_{s_1} & \rho_{bs} \sigma_b \sigma_s & (1-\rho_s)I_{s_1} + \rho_s I_{s_1}x_{s_1} & \sigma_s^2
\end{bmatrix}.
$$

(5.2)

Here, $I_p$ is an identity matrix of order $p$ and $I_{pxq}$ is a $(p \times q)$ matrix each element of which is 1. Furthermore, it is assumed that the sibling correlations $\rho_b, \rho_s$ and $\rho_{bs}$ are constant over all families and the observations $y_{i_1}'s$ are independent of the sibling sizes $n_i, i = 1, 2, \ldots, K$.

Under model (5.1), we have for $i = 1, 2, \ldots, K$:

$$
E(y_{ij}) = \begin{bmatrix}
\mu_b, & j = 1, 2, \ldots, b_1, \\
\mu_s, & j = b_1 + 1, \ldots, n_i,
\end{bmatrix}
$$

$$
\text{Var}(y_{ij}) = \begin{bmatrix}
\sigma_b^2, & j = 1, 2, \ldots, b_1, \\
\sigma_s^2, & j = b_1 + 1, \ldots, n_i
\end{bmatrix},
$$

$$
\text{Cov}(y_{ij}, y_{im}) = \begin{bmatrix}
\rho_b \sigma_b \sigma_s, & j = 1, 2, \ldots, b_1, \\
\rho_s \sigma_b \sigma_s, & j = b_1 + 1, \ldots, n_i, \\
\rho_{bs} \sigma_b \sigma_s, & j = 1, 2, \ldots, b_1, \\
\rho_{bs} \sigma_b \sigma_s, & m = b_1 + 1, \ldots, n_i 
\end{bmatrix}.
$$

(5.3)

and all the other covariances are zero.

The dispersion matrices $\Omega_i$ ($i = 1, 2, \ldots, K$) are positive definite by their definition. A necessary and sufficient condition for the positive definiteness of $\Omega_i$ is given by the following lemma.

**Lemma:** If

$$
\rho_{bs} < \left[ \rho_b + b_0^{-1} (1-\rho_b) \right] \left[ \rho_s + s_0^{-1} (1-\rho_s) \right],
$$

where

$$
b_0 = \max_i b_i \quad \text{and} \quad s_0 = \max_i s_i.
$$
then Ω₁ is positive definite.

Proof: Let

\[ \bar{y}_{ib} = \frac{1}{b_1} \sum_{j=1}^{b_1} y_{ij}, \quad i = 1, 2, \ldots, K \]

and

\[ \bar{y}_{is} = \frac{1}{s_1} \sum_{j=b_1+1}^{n_1} y_{ij}, \quad i = 1, 2, \ldots, K. \]

From (5.1), the mean and dispersion matrix of

\[ \bar{x}_i = [\bar{y}_{ib}, \bar{y}_{is}]^T \]

are given respectively as

\[ \mu_1^x = [\mu_b, \mu_s]^T \]

\[ \Omega_1^x = \begin{bmatrix} [1+(b_1-1)\rho_b] \sigma_b^2/b_1 & \rho_{bs} \sigma_b \sigma_s \\ \rho_{bs} \sigma_b \sigma_s & [1+(s_1-1)\rho_s] \sigma_s^2/s_1 \end{bmatrix}. \]

Following Srivastava and Katapa (1988), it can be shown that

\[ |\Omega_1| = b_1 s_1 \left( \sigma_b^2 b_1^{-1} \left( \sigma_s^2 s_1^{-1} (1-\rho_b) b_1^{-1} (1-\rho_s) s_1^{-1} \right) \right) |\Omega_1^x|. \]

\[ b_1, s_1, \sigma_b^2, \sigma_s^2, (1-\rho_b), (1-\rho_s) \geq 0. \]

Hence Ω₁ is positive definite if Ω₁^x is positive definite.

But

\[ |\Omega_1^x| = \sigma_b^2 \sigma_s^2 \left[ (\rho_b + b_1^{-1} (1-\rho_b)) [\rho_s + s_1^{-1} (1-\rho_s)] - \rho_{bs}^2 \right]. \]

Thus \(|\Omega_1^x| > 0\) implies that

\[ \rho_{bs}^2 < [\rho_b + b_1^{-1} (1-\rho_b)][\rho_s + s_1^{-1} (1-\rho_s)]. \]

The above condition is simultaneously satisfied whenever

\[ \rho_{bs}^2 < [\rho_b + b_0^{-1} (1-\rho_b)][\rho_s + s_0^{-1} (1-\rho_s)], \]

where

\[ b_0 = \max b_i \quad \text{and} \quad s_0 = \max s_i. \]
5.3 Maximum likelihood Estimation

Case (1): $\sigma_b^2 \neq \sigma_s^2$

Under the multivariate normal model (5.1), the likelihood function of the sample of $K$ families can be written as

$$L_1 = (2\pi)^{-N/2} \prod_{i=1}^{K} |\Omega_1|^{-1/2} \exp(-\frac{1}{2} Q_1),$$  \hspace{1cm} (5.9)

where

$$Q_1 = \sum_{i=1}^{K} [\bar{Y}_i - \mu_i]^{\top} \Omega_1^{-1} [\bar{Y}_i - \mu_i].$$

Clearly, from (5.6) and (5.7),

$$|\Omega_1| = (\sigma_b^2 b_1 (\sigma_s^2) s_1 (1-\rho_b) b_1^{-1} (1-\rho_s) s_1^{-1} w_i),$$  \hspace{1cm} (5.10)

where

$$w_i \equiv w_i(\rho_b, \rho_s, \rho_{bs}) = [1+(b_1-1)\rho_b][1+(s_1-1)\rho_s] - b_1 s_1 \rho_{bs}^2,$$

$$i = 1, 2, \ldots, K.$$  \hspace{1cm} (5.11)

In terms of sample statistics, the quadratic exponent in (5.9) will be conveniently written in the following form:

$$Q_1 = \frac{SSW_b}{(1-\rho_b)\sigma_b^2} + \frac{SSW_s}{(1-\rho_s)\sigma_s^2} + \sum_{i=1}^{K} \frac{[\bar{Y}_i - \mu_i]}{\Omega_1^{-1}} [\bar{Y}_i - \mu_i].$$

$$= \frac{1}{\sigma_b^2} \left[ \frac{SSW_b}{1-\rho_b} + \sum_{i=1}^{K} \frac{b_1 w_i (\bar{Y}_{i1} - \mu_b)^2}{w_i} \right]$$

$$+ \frac{1}{\sigma_s^2} \left[ \frac{SSW_s}{1-\rho_s} + \sum_{i=1}^{K} \frac{w_i (\bar{Y}_{is} - \mu_s)^2}{w_i} \right]$$

$$- \frac{2\rho_{bs}}{\sigma_b \sigma_s} \left[ \sum_{i=1}^{K} \frac{b_1 s_1 (\bar{Y}_{i1} - \mu_b)(\bar{Y}_{is} - \mu_s)}{w_i} \right].$$  \hspace{1cm} (5.12)

where
\[ SSW_b = \sum_{i=1}^{K} \sum_{j=1}^{b_i} (y_{ij} - \bar{y}_{ib})^2, \quad SSW_s = \sum_{i=1}^{K} \sum_{j=b_i+1}^{n_i} (y_{ij} - \bar{y}_{is})^2, \]

\[ u_i = u_i(\rho_b) = 1 + (b_i - 1)\rho_b, \quad i = 1, 2, \ldots, K \]

and

\[ v_i = v_i(\rho_s) = 1 + (s_i - 1)\rho_s, \quad i = 1, 2, \ldots, K. \]

(5.13)

Substituting (5.10) and (5.12) in (5.9), the log-likelihood function of the sample becomes

\[ l_i = -\frac{N}{2} \ln(2\pi) - \frac{N_b}{2} \ln(\sigma_b^2) - \frac{N_s}{2} \ln(\sigma_s^2) - \frac{N_b - K}{2} \ln(1 - \rho_b) \]

\[ - \frac{N_s - K}{2} \ln(1 - \rho_s) - \frac{1}{2} \sum_{i=1}^{K} \ln(w_i) - \frac{1}{2} Q_i, \]

(5.14)

where \( w_i, \quad i = 1, 2, \ldots, K, \) are given by (5.11). Differentiating (5.14) with respect to \( \mu_b \) and \( \mu_s \) and equating them to zero, we have

\[ \frac{1}{\sigma_b} \sum_{i=1}^{K} \frac{K b_i v_i (\bar{y}_{ib} - \mu_b)}{w_i} - \frac{\rho_{bs}}{\sigma_s} \sum_{i=1}^{K} \frac{K b_i s_i (\bar{y}_{is} - \mu_s)}{w_i} = 0, \]

(5.15)

\[ \frac{1}{\sigma_s} \sum_{i=1}^{K} \frac{K s_i u_i (\bar{y}_{is} - \mu_s)}{w_i} - \frac{\rho_{bs}}{\sigma_b} \sum_{i=1}^{K} \frac{K b_i s_i (\bar{y}_{ib} - \mu_b)}{w_i} = 0. \]

The maximum likelihood estimators of \( \mu_b \) and \( \mu_s \), as functions of \( \sigma_b, \sigma_s, \rho_b, \rho_s \) and \( \rho_{bs} \), can be obtained by solving (5.15) for \( \mu_b \) and \( \mu_s \), and are given by

\[ \hat{\mu}_b \equiv \hat{\mu}_b(\sigma_b, \sigma_s, \rho_b, \rho_s, \rho_{bs}) \]

\[ = \frac{1}{A_1} \left[ \left( \sum \frac{s_i u_i}{w_i} \right) \left( \sum \frac{b_i v_i \bar{y}_{ib}}{w_i} \right) - \rho_{bs} \left( \sum \frac{b_i s_i}{w_i} \right) \left( \sum \frac{b_i s_i \bar{y}_{ib}}{w_i} \right) \right] \]

\[ + \frac{\rho_{bs} \sigma_b}{\sigma_s} \left[ \left( \sum \frac{b_i s_i}{w_i} \right) \left( \sum \frac{s_i u_i \bar{y}_{is}}{w_i} \right) - \left( \sum \frac{s_i u_i}{w_i} \right) \left( \sum \frac{b_i s_i \bar{y}_{is}}{w_i} \right) \right] \]

(5.16)

and
\[ \hat{\mu}_s = \mu_s (\sigma_b, \sigma_s, \rho_b, \rho_s, \rho_{bs}) \]
\[ = \frac{1}{\hat{\nu}_s} \left\{ \sum b_i \sqrt{\frac{v_i}{w_i}} \left[ \sum s_i u_i \bar{y}_{is} \right] - \rho_{bs} \left[ \sum b_i s_i \bar{y}_{is} \right] \right\} \]
\[ + \frac{\rho_{bs} \sigma_s}{\sigma_b} \left[ \left( \sum b_i s_i \bar{y}_{ib} \right) \left( \sum b_i v_i \bar{y}_{ib} \right) - \left( \sum b_i v_i \bar{y}_{ib} \right) \left( \sum b_i s_i \bar{y}_{ib} \right) \right], \]
(5.17)

where
\[ \hat{\nu}_s = \left[ \sum b_i v_i \right] \left[ \sum s_i u_i \right] - \rho_{bs}^2 \left[ \sum b_i s_i \right]^2. \]
(5.18)

(All summations are over \( i = 1, 2, \ldots, K \)).

Now following the procedure by Richards (1981) (see Section 1.6.1) and substituting (5.18) and (5.17) in \( \ell_1 \) given by (5.14), we get
\[ \ell^*_1 = -\frac{N}{2} \ln(2\pi) - \frac{N_b}{2} \ln(\sigma_b^2) - \frac{N_s}{2} \ln(\sigma_s^2) - \frac{N_{b-K}}{2} \ln(1-\rho_b) \]
\[ - \frac{N_{b-K}}{2} \ln(1-\rho_s) - \frac{1}{2} \sum_{i=1}^{K} \ln(w_i) - \frac{1}{2} Q^*_{1}, \]
(5.19)

where
\[ Q^*_{1} = \frac{1}{\sigma_b^2} \left[ \frac{SSW_{b}}{1-\rho_b} \sum_{i=1}^{K} b_i v_i (\bar{y}_{ib} - \hat{\mu}_b)^2 \right] \]
\[ + \frac{1}{\sigma_s^2} \left[ \frac{SSW_{s}}{1-\rho_s} \sum_{i=1}^{K} s_i u_i (\bar{y}_{is} - \hat{\mu}_s)^2 \right] \]
\[ - 2 \rho_{bs} \frac{1}{\sigma_b \sigma_s} \left[ \sum_{i=1}^{K} b_i s_i (\bar{y}_{ib} - \hat{\mu}_b)(\bar{y}_{is} - \hat{\mu}_s) \right], \]
which is a function of five parameters. The estimates \( \hat{\sigma}_b^2 \),
\( \hat{\sigma}_b^2, \hat{\rho}_b, \hat{\rho}_s \) and \( \hat{\rho}_{bs} \) of \( \sigma_b^2, \sigma_s^2, \rho_b, \rho_s \) and \( \rho_{bs} \), respectively, can be obtained by numerically maximizing \( \hat{\ell}_1^* \). Once the local maximum is found, the estimates \( \hat{\mu}_b \) and \( \hat{\mu}_s \) of \( \mu_b \) and \( \mu_s \), respectively, can be obtained by substituting \( \hat{\sigma}_b, \hat{\sigma}_s, \hat{\rho}_b, \hat{\rho}_s \) and \( \hat{\rho}_{bs} \) in (5.16) and (5.17) for their corresponding parameters.

Case (ii): \( \sigma_b^2 = \sigma_s^2 = \sigma_0^2 \) (say)

In case (i) a general situation was presented when the two types of individuals constituting the sampling unit are heterogeneous. However, the situation can be significantly simplified under the assumption of homogeneity of the within brothers and within sisters variances (i.e., \( \sigma_b^2 = \sigma_s^2 \)). This may be acceptable in family studies, especially when it is recognised that the variation due to shared household environment is approximately homogeneous across the population. In such a case the log-likelihood function (5.14) becomes

\[
\hat{\ell}_2 = -\frac{N}{2} \left[ \ln(2\pi \sigma_0^2) \right] - \frac{N_b-K}{2} \ln(1-\rho_b) - \frac{N_s-K}{2} \ln(1-\rho_s) \\
- \frac{1}{2} \sum_{i=1}^{K} \ln(w_i) - \frac{1}{2} Q_2,
\]

(5.20)

where

\[
Q_2 = \frac{1}{\sigma_0^2} \left\{ \frac{SSW_b}{1-\rho_b} + \frac{SSW_s}{1-\rho_s} + \sum_{i=1}^{K} \frac{b_i v_i (\bar{y}_{ib} - \mu_b)^2}{w_i} \right. \\
+ \sum_{i=1}^{K} \frac{s_i u_i (\bar{y}_{is} - \mu_s)^2}{w_i} \\
- 2\rho_{bs} \sum_{i=1}^{K} \frac{b_i s_i (\bar{y}_{ib} - \mu_b) (\bar{y}_{is} - \mu_s)}{w_i} \right\}.
\]

Differentiating (5.20) with respect to \( \mu_b, \mu_s \) and \( \sigma_0^2 \), and
equating them to zero, we get a system of three equations

\[
\begin{align*}
K \sum_{i=1}^{s} \frac{b_{i} v_{i} (\bar{y}_{ib} - \mu_{b})}{w_{i}} - \rho_{bs} \sum_{i=1}^{s} \frac{b_{i} s_{i} (\bar{y}_{is} - \mu_{s})}{w_{i}} &= 0, \\
K \sum_{i=1}^{s} \frac{s_{i} u_{i} (\bar{y}_{is} - \mu_{s})}{w_{i}} - \rho_{bs} \sum_{i=1}^{s} \frac{b_{i} s_{i} (\bar{y}_{ib} - \mu_{b})}{w_{i}} &= 0, \\
N - Q_{2} &= 0.
\end{align*}
\] (5.21)

The maximum likelihood estimators \( \tilde{\mu}_{b}, \tilde{\mu}_{s} \) and \( \sigma_{0}^{2} \), as functions of \( \rho_{b}, \rho_{s} \) and \( \rho_{bs} \) of \( \mu_{b}, \mu_{s} \) and \( \sigma_{0}^{2} \), respectively, can be obtained by solving the system of equations (5.21) and are given by

\[
\tilde{\mu}_{b} = \tilde{\mu}_{b}(\rho_{b}, \rho_{s}, \rho_{bs}) = \frac{1}{A_{b}} \left\{ \left[ \sum \frac{s_{i} u_{i}}{w_{i}} \right] \left[ \sum \frac{b_{i} v_{i} (\bar{y}_{ib})}{w_{i}} \right] - \rho_{bs} \left[ \sum \frac{b_{i} s_{i} (\bar{y}_{ib})}{w_{i}} \right] \left[ \sum \frac{b_{i} s_{i} (\bar{\bar{y}}_{ib})}{w_{i}} \right] \right\},
\] (5.22)

\[
\tilde{\mu}_{s} = \tilde{\mu}_{s}(\rho_{b}, \rho_{s}, \rho_{bs}) = \frac{1}{A_{s}} \left\{ \left[ \sum \frac{b_{i} v_{i}}{w_{i}} \right] \left[ \sum \frac{s_{i} u_{i} (\bar{y}_{is})}{w_{i}} \right] - \rho_{bs} \left[ \sum \frac{b_{i} s_{i} (\bar{y}_{is})}{w_{i}} \right] \left[ \sum \frac{b_{i} s_{i} (\bar{\bar{y}}_{is})}{w_{i}} \right] \right\} + \rho_{bs} \left[ \left[ \sum \frac{b_{i} s_{i} (\bar{y}_{ib})}{w_{i}} \right] \left[ \sum \frac{b_{i} v_{i} (\bar{y}_{ib})}{w_{i}} \right] - \left[ \sum \frac{b_{i} v_{i} (\bar{\bar{y}}_{ib})}{w_{i}} \right] \left[ \sum \frac{b_{i} s_{i} (\bar{\bar{y}}_{ib})}{w_{i}} \right] \right] \right\},
\] (5.23)

and

\[
\tilde{\sigma}_{0}^{2} = \tilde{\sigma}_{0}^{2}(\rho_{b}, \rho_{s}, \rho_{bs}) = \frac{1}{N} \left\{ \frac{SSW_{b}}{1-\rho_{b}} + \frac{SSW_{s}}{1-\rho_{s}} + \sum_{i=1}^{s} \frac{K b_{i} v_{i} (\bar{y}_{ib} - \tilde{\mu}_{b})^{2}}{w_{i}} + \sum_{i=1}^{s} \frac{K s_{i} u_{i} (\bar{y}_{is} - \tilde{\mu}_{s})^{2}}{w_{i}} + \frac{K b_{i} s_{i} (\bar{y}_{ib} - \tilde{\mu}_{b}) (\bar{y}_{is} - \tilde{\mu}_{s})}{w_{i}} - 2 \rho_{bs} \sum \frac{K b_{i} s_{i} (\bar{y}_{ib} - \tilde{\mu}_{b}) (\bar{y}_{is} - \tilde{\mu}_{s})}{w_{i}} \right\}.
\] (5.24)
where $\Delta_1$ is given by (5.18). Once again, following Richards (1961), by substituting $\tilde{\mu}_b$, $\tilde{\mu}_s$ and $\tilde{\sigma}_0^2$ in (5.20), we get:

$$
\xi_2^* = -\frac{N}{2} \left[ 1 + \ln(2\pi \tilde{\sigma}_0^2) \right] - \frac{N_{b}-K}{2} \ln(1-\tilde{\rho}_b) - \frac{N_{s}-K}{2} \ln(1-\tilde{\rho}_s) - \frac{1}{2} \sum_{i=1}^{K} \ln(w_i),
$$

(5.25)

which is now a function of three unknown parameters only. The maximum likelihood estimates $\tilde{\rho}_b$, $\tilde{\rho}_s$ and $\tilde{\rho}_{bs}$ of the parameters $\rho_b$, $\rho_s$ and $\rho_{bs}$, respectively, can be obtained numerically by maximizing $\xi_2^*$, and once they are found, the maximum likelihood estimates of $\mu_b$, $\mu_s$ and $\sigma_0^2$ can be obtained from (5.22), (5.23) and (5.24), respectively, by substituting $\tilde{\rho}_b$, $\tilde{\rho}_s$ and $\tilde{\rho}_{bs}$ for their corresponding parameters.

5.4 Asymptotic Results

In this section, the asymptotic distributions for the maximum likelihood estimates of the parameters for case (i) and case (ii) are established separately by using the results of section 1.6.2.

Case (i): $\sigma_b^2 \neq \sigma_s^2$

Let us denote the Fisher's information (or expected) matrix of $(7 \times 1)$ random vector $\hat{\theta} = [\hat{\mu}_b, \hat{\mu}_s, \hat{\sigma}_b, \hat{\sigma}_s, \hat{\rho}_b, \hat{\rho}_s, \hat{\rho}_{bs}]^\top$ by

$$
\Sigma(\hat{\theta}) = -E \left| \left| \frac{\partial^2 \xi_1}{\partial \theta_p \partial \theta_q} \right| \right|, \quad p, q = 1, 2, \ldots, 7, 
$$

(5.28)

$\theta_1 \equiv \mu_b$, $\theta_2 \equiv \mu_s$, $\theta_3 \equiv \sigma_b^2$, $\theta_4 \equiv \sigma_s^2$, $\theta_5 \equiv \rho_b$, $\theta_6 \equiv \rho_s$, $\theta_7 \equiv \rho_{bs}$.

In order to find the elements of $\Sigma(\hat{\theta})$, we need to know the
first and second order partial derivatives of $t_1$ (5.14) with respect to all the seven parameters. Let us denote

$$D_p = \frac{\partial t_1}{\partial \theta_p}, \quad p = 1, 2, \ldots, 7, \quad (5.27)$$

where $\theta_1 = \mu_b$, $\theta_2 = \mu_s$, $\theta_3 = \sigma_b^2$, $\theta_4 = \sigma_s^2$, $\theta_5 = \rho_b$, $\theta_6 = \rho_s$, $\theta_7 = \rho_{bs}$. The first order partial derivatives of $t_1$ are given by

$$D_1 = \frac{1}{\sigma_b^2} \sum \frac{b_i v_i z_i b}{w_i} - \frac{\rho_{bs}}{\sigma_b \sigma_s} \sum \frac{b_i s_i z_i s}{w_i},$$

$$D_2 = \frac{1}{\sigma_s^2} \sum \frac{s_i u_i z_i s}{w_i} - \frac{\rho_{bs}}{\sigma_b \sigma_s} \sum \frac{b_i s_i z_i b}{w_i},$$

$$D_3 = -\frac{N_b}{2\sigma_b^2} + \frac{1}{2\sigma_b^4} \left[ \frac{SSW_B}{1-\rho_b} + \sum \frac{b_i v_i z_i b}{w_i} \right]$$

$$- \frac{\rho_{bs}}{2\sigma_b \sigma_s^3} \sum \frac{b_i s_i z_i b z_i s}{w_i},$$

$$D_4 = -\frac{N_s}{2\sigma_s^2} + \frac{1}{2\sigma_s^4} \left[ \frac{SSW_S}{1-\rho_s} + \sum \frac{s_i u_i z_i s}{w_i} \right]$$

$$- \frac{\rho_{bs}}{2\sigma_b \sigma_s^3} \sum \frac{b_i s_i z_i b z_i s}{w_i},$$

$$D_5 = \frac{1}{2} \left[ \frac{N_b-K}{(1-\rho_b)^2} - \sum \frac{(b_i-1)v_i}{w_i} \right]$$

$$- \frac{1}{2\sigma_b^2} \left[ \frac{SSW_B}{(1-\rho_b)^2} - \sum \frac{b_i (b_i-1)v_i z_i^2}{w_i^2} \right]$$

$$\frac{\rho_{bs}}{2\sigma_s^2} \sum \frac{b_i s_i^2 (b_i-1)z_i z_i s}{w_i^2} - \frac{\rho_{bs}}{\sigma_b \sigma_s} \sum \frac{b_i s_i^2 (b_i-1)v_i z_i b z_i s}{w_i^2},$$

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\[ D_6 = \frac{1}{2} \left[ \frac{N_s - K}{(1 - \rho_s)} - \sum \frac{(s_i - 1)u_{i1}}{w_i} \right] \]

\[ - \frac{1}{2 \sigma_s^2} \left[ \frac{SSW_s}{(1 - \rho_s)^2} - \sum \frac{s_i(s_i - 1)u_{i1}^2 z_{i1}^2}{w_i} \right] \]

\[ + \frac{\rho_{bs}^2}{2 \sigma_b^2} \sum \frac{b_{1s} (s_i - 1) z_{i1}^2}{w_i} - \frac{\rho_{bs}}{\sigma_b \sigma_s} \sum \frac{b_{1s} (s_i - 1) u_{i1} z_{i1} z_{i1}}{w_i} \]

and

\[ D_7 = \rho_{bs} \sum \frac{b_{1s} s_i}{w_i} - \frac{\rho_{bs}}{\sigma_b^2} \sum \frac{b_{1s} v_{i1} z_{i1}^2}{w_i} - \frac{\rho_{bs}}{\sigma_s^2} \sum \frac{b_{1s} u_{i1} z_{i1}^2}{w_i} \]

\[ + \frac{1}{\sigma_b \sigma_s} \left[ 2 \rho_{bs} \sum \frac{b_{1s} y_{i1} z_{i1} z_{i1}}{w_i} \right] \]

(5.28)

where

\[ z_{i1} = (\bar{y}_{i1} - \mu_b), \quad i = 1, 2, \ldots, K \]

and

\[ z_{is} = (\bar{y}_{is} - \mu_s), \quad i = 1, 2, \ldots, K. \]

(5.29)

(All summations are over \( i = 1, 2, \ldots, K \).)

Furthermore, let us denote

\[ D_{pq} = \frac{\partial^2 \ell_1}{\partial \theta_p \partial \theta_q}, \quad p, q = 1, 2, \ldots, 7, \] (5.30)

where \( \theta_1 = \mu_b, \quad \theta_2 = \mu_s, \quad \theta_3 = \sigma_b^2, \quad \theta_4 = \sigma_s^2, \quad \theta_5 = \rho_b, \quad \theta_6 = \rho_s, \]

\( \theta_7 = \rho_{bs}. \) The second order partial derivatives of \( \ell_1 \) (5.14) are given by

\[ D_{11} = -\frac{1}{\sigma_b^2} \sum \frac{b_{1i} y_{i1}}{w_i}, \quad D_{12} = \frac{\rho_{bs}}{\sigma_b \sigma_s} \sum \frac{b_{1s} s_i}{w_i}, \]

\[ D_{13} = -\frac{1}{\sigma_b^4} \sum \frac{b_{1i} y_{i1} z_{i1}}{w_i} + \frac{\rho_{bs}}{2 \sigma_b \sigma_s} \sum \frac{b_{1s} y_{i1} z_{i1}}{w_i}, \]

\[ D_{13} = -\frac{1}{\sigma_b^2} \sum \frac{b_{1i} y_{i1} z_{i1}}{w_i} + \frac{\rho_{bs}}{2 \sigma_b \sigma_s} \sum \frac{b_{1s} y_{i1} z_{i1}}{w_i}, \]

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\[ D_{14} = \frac{\rho_{bs}}{2\sigma_b \sigma_s} \sum \frac{b_i s_i z_is}{w_i} \]

\[ D_{15} = -\frac{1}{\sigma_b^2} \sum \frac{b_i (b_i - 1) v_{i ib}}{w_i} + \frac{\rho_{bs}}{\sigma_b \sigma_s} \sum \frac{b_i s_i (b_i - 1) v_{i iz_is}}{w_i} \]

\[ D_{16} = -\frac{\rho_{bs}}{\sigma_b^2} \sum \frac{b_i s_i (s_i - 1) z_{i ib}}{w_i} + \frac{\rho_{bs}}{\sigma_b \sigma_s} \sum \frac{b_i s_i (s_i - 1) u_{i iz_is}}{w_i} \]

\[ D_{17} = \frac{2\rho_{bs}}{\sigma_b^2} \sum \frac{b_i^2 s_i v_{i ib}}{w_i} - \frac{1}{\sigma_b \sigma_s} \left[ 2\rho_{bs} \sum \frac{b_i s_i z_is}{w_i} + \sum \frac{b_i s_i z_is}{w_i} \right] \]

\[ D_{22} = -\frac{1}{\sigma_s^2} \sum \frac{s_i u_{i ib}}{w_i} \]

\[ D_{23} = \frac{\rho_{bs}}{2\sigma_b \sigma_s} \sum \frac{b_i s_i z_{ib}}{w_i} \]

\[ D_{24} = -\frac{1}{\sigma_s^4} \sum \frac{s_i u_{i iz_is}}{w_i} + \frac{\rho_{bs}}{2\sigma_b \sigma_s} \sum \frac{b_i s_i z_{ib}}{w_i} \]

\[ D_{25} = -\frac{\rho_{bs}}{\sigma_s^2} \sum \frac{b_i s_i^2 (b_i - 1) z_{i iz_is}}{w_i} + \frac{\rho_{bs}}{\sigma_b \sigma_s} \sum \frac{b_i s_i (b_i - 1) v_{i iz_is}}{w_i} \]

\[ D_{26} = -\frac{1}{\sigma_s^2} \sum \frac{s_i (s_i - 1) u_{i iz_is}}{w_i} + \frac{\rho_{bs}}{\sigma_b \sigma_s} \sum \frac{b_i s_i (s_i - 1) u_{i iz_is}}{w_i} \]

\[ D_{27} = \frac{2\rho_{bs}}{\sigma_s^2} \sum \frac{b_i^2 s_i v_{i iz_is}}{w_i} - \frac{1}{\sigma_b \sigma_s} \left[ 2\rho_{bs} \sum \frac{b_i s_i z_{ib}}{w_i} + \sum \frac{b_i s_i z_{ib}}{w_i} \right] \]

\[ D_{33} = \frac{N_b}{\sigma_b^4} - \frac{1}{\sigma_b^6} \left[ SSW_b \left[ 1 - \rho_b \right] + \sum \frac{b_i v_{i iz_ib}}{w_i} \right] + \frac{3\rho_{bs}}{4\sigma_b \sigma_s} \sum \frac{b_i s_i z_{ib} z_{i is}}{w_i} \]
\[ D_{34} = \frac{\rho_{bs}}{4 \sigma^3_b \sigma_s} \sum b_{1} s_{1} z_{1} b_{1} z_{1} s_{1} \frac{w_i}{w_1} \]

\[ D_{35} = \frac{1}{2 \sigma_b^4} \left[ \frac{SSW_b}{(1-\rho_b)} - \sum b_{1} (b_{1} - 1) v_{1}^2 \frac{w_i}{w_1} \right] \]

\[ + \frac{\rho_{bs}}{2 \sigma_b^3 \sigma_s} \sum b_{1} s_{1} (b_{1} - 1) v_{1} z_{1} b_{1} z_{1} s_{1} \frac{w_i}{w_1} \]

\[ D_{36} = -\frac{\rho_{bs}}{2 \sigma_b^4} \sum b_{1} s_{1} (s_{1} - 1) z_{1}^2 \frac{w_i}{w_1} \]

\[ + \frac{\rho_{bs}}{2 \sigma_b^3 \sigma_s} \sum b_{1} s_{1} (s_{1} - 1) u_{1} z_{1} b_{1} z_{1} s_{1} \frac{w_i}{w_1} \]

\[ D_{37} = \frac{\rho_{bs}}{\sigma_b^4} \sum b_{1} s_{1} z_{1} z_{1}^2 \frac{w_i}{w_1} - \frac{\rho_{bs}}{\sigma_b^3 \sigma_s} \sum b_{1} s_{1} z_{1} b_{1} z_{1} s_{1} \frac{w_i}{w_1} \]

\[ - \frac{1}{2 \sigma_b^3 \sigma_s} \sum b_{1} s_{1} z_{1} b_{1} z_{1} s_{1} \frac{w_i}{w_1} \]

\[ D_{44} = \frac{N_s}{2 \sigma_s^4} - \frac{1}{\sigma_s^4} \left[ \frac{SSW_s}{1-\rho_s} + \sum s_{1} u_{1} z_{1}^2 \frac{w_i}{w_1} \right] + \frac{3 \rho_{bs}}{4 \sigma_b \sigma_s^5} \sum b_{1} s_{1} z_{1} b_{1} z_{1} s_{1} \frac{w_i}{w_1} \]

\[ D_{45} = -\frac{\rho_{bs}}{2 \sigma_s^4} \sum b_{1} s_{1} (b_{1} - 1) z_{1}^2 \frac{w_i}{w_1} \]

\[ + \frac{\rho_{bs}}{2 \sigma_b^3 \sigma_s} \sum b_{1} s_{1} (b_{1} - 1) v_{1} z_{1} b_{1} z_{1} s_{1} \frac{w_i}{w_1} \]

\[ D_{48} = \frac{1}{2 \sigma_s^4} \left[ \frac{SSW_s}{(1-\rho_s)^2} - \sum s_{1} (s_{1} - 1) u_{1} z_{1}^2 \frac{w_i}{w_1} \right] \]

\[ + \frac{\rho_{bs}}{2 \sigma_b^3 \sigma_s} \sum b_{1} s_{1} (s_{1} - 1) u_{1} z_{1} b_{1} z_{1} s_{1} \frac{w_i}{w_1} \]
\[ D_{47} = \frac{\rho_{bs}}{\sigma_s^4} \sum \frac{b_i s_i u_i z_i^2}{w_i} - \frac{\rho_{bs}}{\sigma_b \sigma_s^3} \sum \frac{b_i s_i^2 z_{ib} z_{is}}{w_i} \\
- \frac{1}{2 \sigma_b \sigma_s^3} \sum \frac{b_i s_i z_{ib} z_{is}}{w_i} , \]

\[ D_{55} = \frac{1}{2} \left[ \frac{N_b - K}{(1 - \rho_b)^2} + \sum \frac{(b_i - 1)^2 v_{i1}^2}{w_i} \right] \\
- \frac{1}{\sigma_b^2} \left[ \frac{SSW_b}{(1 - \rho_b)^3} + \sum \frac{b_i (b_i - 1)^2 v_{i1}^2}{w_i} \right] \]

\[ + \frac{\rho_{bs}}{\sigma_b^2} \sum \frac{b_i s_i (b_i - 1)^2 v_{i1} z_{is}}{w_i} \]

\[ + \frac{2 \rho_{bs}}{\sigma_b \sigma_s} \sum \frac{b_i s_i (b_i - 1)^2 v_{i1}^2}{w_i} \]

\[ D_{56} = \frac{\rho_{bs}^2}{2} \sum \frac{b_i s_i (b_i - 1) (s_i - 1)}{w_i} \]

\[ - \frac{\rho_{bs}^2}{\sigma_b^2} \sum \frac{b_i s_i (b_i - 1) (s_i - 1) v_{i1} z_{ib}^2}{w_i} \]

\[ - \frac{\rho_{bs}^2}{\sigma_s^2} \sum \frac{b_i s_i^2 (b_i - 1) (s_i - 1) u_i z_{is}^2}{w_i} \]

\[ + \frac{\rho_{bs}}{\sigma_b \sigma_s} \sum \frac{b_i s_i (b_i - 1) (s_i - 1) [u_i v_i + b_i s_i \rho_{bs}^2] z_{ib} z_{is}}{w_i} , \]

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\[ D_{57} = -\rho_{bs} \sum \frac{b_{i}s_{1}(s_{1}-1)u_{i}}{w_{i}^{2}} + \frac{2\rho_{bs}}{\sigma_{b}^{2}} \sum \frac{b_{i}s_{1}(s_{1}-1)u_{i}^{2}z_{ib}z_{is}}{w_{i}^{3}} \]

\[ + \frac{\rho_{bs}}{\sigma_{s}^{2}} \sum \frac{b_{i}s_{1}(s_{1}-1)(u_{i}v_{i} + b_{i}s_{1}\rho_{bs})z_{is}}{w_{i}^{3}} \]

\[ - \frac{4\rho_{bs}}{\sigma_{b}\sigma_{s}} \sum \frac{b_{i}s_{1}(s_{1}-1)v_{i}z_{ib}z_{is}}{w_{i}^{3}} \]

\[ - \frac{1}{\sigma_{b}\sigma_{s}} \sum \frac{b_{i}s_{1}(s_{1}-1)v_{i}z_{ib}z_{is}}{w_{i}^{2}} , \]

\[ D_{68} = \frac{1}{2} \left[ \frac{N_{s} - K}{(1-\rho_{b})^{2}} + \sum \frac{(s_{1}-1)^{2}u_{i}^{2}}{w_{i}^{2}} \right] \]

\[ - \rho_{bs} \frac{2}{\sigma_{b}^{2}} \sum \frac{b_{i}s_{1}(s_{1}-1)^{2}u_{i}^{2}z_{ib}}{w_{i}^{3}} \]

\[ - \frac{1}{\sigma_{s}^{2}} \left[ \frac{SSW_{s}}{(1-\rho_{s})^{3}} + \sum \frac{s_{i}(s_{1}-1)^{2}u_{i}^{2}z_{ib}z_{is}}{w_{i}^{3}} \right] \]

\[ + \frac{2\rho_{bs}}{\sigma_{b}\sigma_{s}} \sum \frac{b_{i}s_{1}(s_{1}-1)^{2}u_{i}^{2}z_{ib}z_{is}}{w_{i}^{3}} , \]

\[ D_{67} = -\rho_{bs} \sum \frac{b_{i}s_{1}(s_{1}-1)u_{i}}{w_{i}^{2}} + \frac{2\rho_{bs}}{\sigma_{s}^{2}} \sum \frac{b_{i}s_{1}(s_{1}-1)u_{i}^{2}z_{is}}{w_{i}^{3}} \]

\[ + \frac{\rho_{bs}}{\sigma_{b}^{2}} \sum \frac{b_{i}s_{1}(s_{1}-1)(u_{i}v_{i} + b_{i}s_{1}\rho_{bs})z_{ib}z_{is}}{w_{i}^{3}} \]

\[ - \frac{4\rho_{bs}}{\sigma_{b}\sigma_{s}} \sum \frac{b_{i}s_{1}(s_{1}-1)u_{i}^{2}z_{ib}z_{is}}{w_{i}^{3}} \]

\[ - \frac{1}{\sigma_{b}\sigma_{s}} \sum \frac{b_{i}s_{1}(s_{1}-1)u_{i}^{2}z_{ib}z_{is}}{w_{i}^{2}} , \]

and

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\[ D_{77} = 2 \rho_{bs}^2 \sum \frac{b_{i1}^2}{w_1^2} + \sum \frac{b_{i1}}{w_1} - \frac{4 \rho_{bs}^2}{\sigma_b^2} \sum \frac{b_{i1}^2}{w_1^3} + \frac{1}{\sigma_b^2} \sum \frac{b_{i1}^2}{w_1^2} - \frac{4 \rho_{bs}^2}{\sigma_s^2} \sum \frac{b_{i1}^2}{w_1^3} - \frac{1}{\sigma_s^2} \sum \frac{b_{i1}^2}{w_1^2} + \frac{6 \rho_{bs}}{\sigma_b \sigma_s} \sum \frac{b_{i1}^2}{w_1^2} \]

(5.31)

where \( z_{ib} \) and \( z_{is} \) \((i = 1, 2, \ldots, K)\) are given by (5.29).

(All summations are over \( i = 1, 2, \ldots, K \)).

Before we proceed further, notice that

\[ \text{EC}(z_{ib}) = 0, \quad \text{EC}(z_{is}) = 0, \quad i = 1, 2, \ldots, K, \]

\[ \text{EC}(z_{ib}^2) = [1 + (b_i - 1) \rho_b] \sigma_b^2 / b_i, \quad i = 1, 2, \ldots, K, \]

\[ \text{EC}(z_{is}^2) = [1 + (s_i - 1) \rho_s] \sigma_s^2 / s_i, \quad i = 1, 2, \ldots, K, \]

\[ \text{EC}(z_{ib}z_{is}) = \rho_{bs} \sigma_b \sigma_s, \quad i = 1, 2, \ldots, K, \quad (5.32) \]

\[ \text{EC}(SSW_b) = (N_b - K)(1 - \rho_b) \sigma_b^2 \]

and

\[ \text{EC}(SSW_s) = (N_s - K)(1 - \rho_s) \sigma_s^2. \]

Denoting the \( p, q \)th element of matrix \( \mathbf{D} \) (5.28) by \( a_{pq} \), the elements of Fisher's information matrix are

\[ a_{pq} = a_{qp} = -\text{EC}(D_{pq}), \quad p, q = 1, 2, \ldots, 7. \]

Thus by using (5.31) and (5.32), we get
\[ a_{11} = \frac{1}{\sigma_b^2} \sum \frac{b_i w_i}{v_i}, \quad a_{12} = \frac{-\rho_{b s}}{\sigma_b \sigma_s} \sum \frac{b_i s_i}{w_i}, \]

\[ a_{22} = \frac{1}{\sigma_s^2} \sum \frac{s_i u_i}{w_i}, \quad a_{pq} = 0, \quad p = 1, 2; \quad q = 3, 4, 5, 6, 7, \]

\[ a_{33} = \frac{1}{4\sigma_b^2} \left[ 2N_b + \rho_{b s}^2 \sum \frac{b_i s_i}{w_i} \right], \quad a_{34} = \frac{-\rho_{b s}^2}{4\sigma_b^2 \sigma_s^2} \sum \frac{b_i s_i}{w_i}, \]

\[ a_{35} = \frac{-1}{2\sigma_b^2} \left[ \frac{N_b - K}{1 - \rho_b^2} - \sum \frac{(b_i - 1)v_i}{w_i} \right], \quad a_{36} = 0, \]

\[ a_{37} = \frac{-\rho_{b s}}{2\sigma_s^2} \sum \frac{b_i s_i}{w_i}, \quad a_{44} = \frac{1}{4\sigma_s^4} \left[ 2N_s + \rho_{b s}^2 \sum \frac{b_i s_i}{w_i} \right], \]

\[ a_{45} = 0, \quad a_{46} = \frac{-1}{2\sigma_s^2} \left[ \frac{N_s - K}{1 - \rho_s^2} - \sum \frac{(s_i - 1)u_i}{w_i} \right], \]

\[ a_{47} = \frac{-\rho_{b s}}{2\sigma_s^2} \sum \frac{b_i s_i}{w_i}, \quad a_{55} = \frac{1}{2} \left[ \frac{N_b - K}{(1 - \rho_b)^2} + \sum \frac{(b_i - 1)^2 v_i^2}{w_i^2} \right], \]

\[ a_{56} = \frac{\rho_{b s}^2}{2} \sum \frac{b_i s_i (b_i - 1)(s_i - 1)}{w_i^2}, \]

\[ a_{57} = -\rho_{b s} \sum \frac{b_i s_i (b_i - 1)v_i}{w_i}, \]

\[ a_{66} = \frac{1}{2} \left[ \frac{N_s - K}{(1 - \rho_s)^2} + \sum \frac{(s_i - 1)^2 u_i^2}{w_i^2} \right], \]

\[ a_{67} = -\rho_{b s} \sum \frac{b_i s_i (s_i - 1)u_i}{w_i}, \]

and

\[ a_{77} = \sum \frac{b_i s_i (u_i v_i + b_i s_i \rho_{b s}^2)}{w_i^2}. \]

(All summations are over \( i = 1, 2, \ldots, K \)).
Therefore,

\[ \Sigma_{\Theta} = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}, \]

where

\[ \Sigma_{11}(\Theta) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \]

and

\[ \Sigma_{22}(\Theta) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} a_{33} & a_{34} & a_{35} & a_{37} \\ a_{43} & a_{44} & a_{45} & a_{47} \\ a_{53} & a_{54} & a_{55} & a_{57} \\ a_{63} & a_{64} & a_{65} & a_{67} \\ a_{73} & a_{74} & a_{75} & a_{77} \end{bmatrix}. \]

Now from the standard matrix algebra of partitioned matrices, we have

\[ \Sigma^{-1}(\Theta) = \begin{bmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix}, \quad (5.34) \]

where

\[ \Sigma_{11}^{-1}(\Theta) = \frac{1}{\Lambda_1 \sigma_b^2 \sigma_s^2} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}, \quad (5.35) \]

\[ \Sigma_{22}^{-1}(\Theta) = \begin{bmatrix} A_{11.2}^{-1} & -A_{11.2}^{-1} A_{12} A_{22}^{-1} \\ -A_{11.2}^{-1} A_{22} A_{12}^{-1} & A_{11.2}^{-1} \end{bmatrix}, \quad (5.36) \]

\[ A_{11.2}(\Theta) = A_{11} - A_{12} A_{22}^{-1} A_{21}, \]

\[ A_{22.1}(\Theta) = A_{22} - A_{21} A_{11}^{-1} A_{12} \]

and \( \Lambda_1 \) is given by (5.18).

We have the following theorems:

**Theorem 5.1.** Conditional on \( n_1 \), as \( K \to \infty \), the \((7 \times 1)\)
random vector \( \hat{\Theta} = [\hat{\mu}_b, \hat{\mu}_s, \hat{\sigma}_b^2, \hat{\sigma}_s^2, \hat{\rho}_b, \hat{\rho}_s, \hat{\rho}_{bs}]^\top \) is
asymptotically distributed as multivariate normal with mean
vector \( \Theta = [\mu_b, \mu_s, \sigma_b^2, \sigma_s^2, \rho_b, \rho_s, \rho_{bs}]^T \) and dispersion matrix \( \Sigma^{-1}(\Theta) \), where \( \Sigma^{-1}(\Theta) \) is given by (5.34).

Theorem 5.2. Conditional on \( n_1 \), as \( K \rightarrow \infty \), the \((2 \times d)\) random vector \( \hat{\Theta}^{(1)} = [\hat{\mu}_b, \hat{\mu}_s]^T \) is asymptotically distributed as multivariate normal with mean vector \( \Theta^{(1)} = [\mu_b, \mu_s]^T \) and dispersion matrix \( \Sigma_{11}^{-1}(\Theta) \), where \( \Sigma_{11}^{-1}(\Theta) \) is given by (5.35).

Theorem 5.3. Conditional on \( n_1 \), as \( K \rightarrow \infty \), the \((5 \times d)\) random vector \( \hat{\Theta}^{(2)} = [\hat{\sigma}_b^2, \hat{\sigma}_s^2, \hat{\rho}_b, \hat{\rho}_s, \hat{\rho}_{bs}]^T \) is asymptotically distributed as multivariate normal with mean vector \( \Theta^{(2)} = [\sigma_b^2, \sigma_s^2, \rho_b, \rho_s, \rho_{bs}]^T \) and dispersion matrix \( \Sigma_{22}^{-1}(\Theta) \), where \( \Sigma_{22}^{-1}(\Theta) \) is given by (5.36).

Furthermore, from (5.36), the asymptotic dispersion matrix of the random vector \( [\hat{\rho}_b, \hat{\rho}_s, \hat{\rho}_{bs}]^T \) is \( \Lambda_{22,1}(\Theta) \), which on simplification gives:

\[
\text{Var}(\hat{\rho}_b) \approx \frac{df - e^2}{\Delta_2}, \quad \text{Var}(\hat{\rho}_s) \approx \frac{af - c^2}{\Delta_2},
\]

\[
\text{Var}(\hat{\rho}_{bs}) \approx \frac{ad - b^2}{\Delta_2}, \quad \text{Cov}(\hat{\rho}_b, \hat{\rho}_s) \approx \frac{ce - bf}{\Delta_2},
\]

\[
\text{Cov}(\hat{\rho}_b, \hat{\rho}_{bs}) \approx \frac{be - cd}{\Delta_2}, \quad \text{Cov}(\hat{\rho}_s, \hat{\rho}_{bs}) \approx \frac{cd - be}{\Delta_2},
\]

where

\[
a = \frac{1}{2} \left[ \left( \frac{N_b - K}{\sigma_b^2} - \sum \frac{b_i - 1}{w_1} \right)^2 \right] - \frac{1}{H} \left[ 2N_s + \rho_{bs} \sum \frac{b_i s_1}{w_1} \right]
\]

\[
\times \left[ \frac{N_b - K}{1 - \rho_b} - \sum \frac{(b_i - 1) v_1}{w_1} \right] \left[ \frac{N_b - K}{1 - \rho_b} - \sum \frac{(b_i - 1) v_1}{w_1} \right]^{-1}.
\]
\[b = \frac{\rho_{bs}}{2} \left\{ \sum \frac{b_i s_i (b_i - 1) (s_i - 1)}{w_1^2} - \frac{1}{H} \left[ \sum \frac{b_i s_i}{w_1} \right] \right\} \times \left[ \frac{N_b - K}{1 - \rho_b} - \sum \frac{(b_i - 1)v_1}{w_1} \right] \left[ \frac{N_s - K}{1 - \rho_s} - \sum \frac{(s_i - 1)u_1}{w_1} \right].\]

\[c = -\rho_{bs} \left\{ \frac{\sum b_i s_i (b_i - 1) v_1}{w_1^2} + \frac{1}{H} \left[ \sum \frac{b_i s_i}{w_1} \right] \right\} \times \left[ \frac{N_b - K}{1 - \rho_b} - \sum \frac{(b_i - 1)v_1}{w_1} \right] \left[ N_s + \rho_{bs}^2 \sum \frac{b_i s_i}{w_1} \right].\]

\[d = \frac{1}{2} \left\{ \left[ \frac{N_s - K}{(1 - \rho_s)^2} + \sum \frac{(s_i - 1)^2 u_1^2}{w_1^2} \right] - \frac{1}{H} \left[ 2N_b + \rho_{bs}^2 \sum \frac{b_i s_i}{w_1} \right] \right\} \times \left[ \frac{N_b - K}{1 - \rho_b} - \sum \frac{(b_i - 1)v_1}{w_1} \right]^2.\]

\[e = -\rho_{bs} \left\{ \sum \frac{b_i s_i (s_i - 1) u_1}{w_1^2} + \frac{1}{H} \left[ \sum \frac{b_i s_i}{w_1} \right] \right\} \times \left[ N_b + \rho_{bs}^2 \sum \frac{b_i s_i}{w_1} \right] \left[ \frac{N_s - K}{1 - \rho_s} - \sum \frac{(s_i - 1)u_1}{w_1} \right].\]

\[f = \left\{ \sum \frac{b_i s_i (u_1 v_1 + b_i s_i \rho_{bs})}{w_1^2} - \rho_{bs}^2 \frac{\sum b_i s_i}{w_1} \right\}^2 \times \left[ N + 2\rho_{bs}^2 \sum \frac{b_i s_i}{w_1} \right].\]

\[H = 2N_b N_s + N \rho_{bs}^2 \sum \frac{b_i s_i}{w_1}\]

and

\[\Delta_2 = a(d^2 - e^2) + b(c^2 - f) + c(2be - cd).\]

(All summations are over \(i = 1, 2, \ldots, K\).

For the case when there is no distinction between siblings as brothers and sisters, the asymptotic variance of
\( \hat{\rho}_b \) (\( \hat{\rho}_s \)) is reduced to the formula for the large sample variance of the maximum likelihood estimator of sib-sib (intraclass) correlation (2.28).

Case (ii): \( \sigma^2_b = \sigma^2_s (\equiv \sigma^2_0) \)

Let us denote the Fisher's information (or expected) matrix of \((6 \times 1)\) random vector \( \tilde{\psi} = [\tilde{\mu}_b, \tilde{\mu}_s, \tilde{\sigma}^2_o, \tilde{\rho}_b, \tilde{\rho}_s, \tilde{\rho}_{bs}]^T \) by

\[
\mathcal{W}_Y = -E \left| \left| \frac{\partial^2 \ell}{\partial \psi_p \partial \psi_q} \right| \right|, \quad p, q = 1, 2, \ldots, 8,
\]

\( \psi_1 \equiv \mu_b, \psi_2 \equiv \mu_s, \psi_3 \equiv \sigma^2_o, \psi_4 \equiv \rho_b, \psi_5 \equiv \rho_s, \psi_6 \equiv \rho_{bs} \).

In order to find the elements of \( \mathcal{W}_Y \), we need to know the first and second order partial derivatives of \( \ell_2 \) (5.20) with respect to all the six parameters. Let us denote

\[
\nabla_p = \frac{\partial \ell_2}{\partial \psi_p}, \quad p = 1, 2, \ldots, 8,
\]

where \( \psi_1 \equiv \mu_b, \psi_2 \equiv \mu_s, \psi_3 \equiv \sigma^2_o, \psi_4 \equiv \rho_b, \psi_5 \equiv \rho_s, \psi_6 \equiv \rho_{bs} \). The first order partial derivatives of \( \ell_2 \) are given by

\[
\nabla_1 = \frac{1}{\sigma^2_o} \left[ \sum \frac{b_i^{v_i z} z_{ib}}{w_i} - \rho_{bs} \sum \frac{b_i^{v_i z} z_{is}}{w_i} \right],
\]

\[
\nabla_2 = \frac{1}{\sigma^2_o} \left[ \sum \frac{s_i u_i z_{is}}{w_i} - \rho_{bs} \sum \frac{b_i^{s_i z} z_{ib}}{w_i} \right],
\]

\[
\nabla_3 = -\frac{N}{2\sigma^2_o} + \frac{1}{2\sigma^2_o} \left[ \frac{SSV_b}{1-\rho_b} + \frac{SSV_s}{1-\rho_s} + \sum \frac{b_i^{v_i z} z_{ib}^2}{w_i} \right.
\]
\[
\left. + \sum \frac{s_i u_i z_{is}^2}{w_i} - 2\rho_{bs} \sum \frac{b_i^{s_i z} z_{ib} z_{is}}{w_i} \right].
\]
\[
\nu_4 = \frac{1}{2} \left[ \frac{N_{b-K}}{(1-\rho_b)} - \sum \frac{(b_{i-1})v_i}{w_i} \right] - \frac{1}{2\sigma_0^2} \left[ \frac{SSW_b}{(1-\rho_b)^2} - \sum \frac{b_i(b_{i-1})v_{i1}^2}{w_i^2} \right] - \rho_{bs} \sum \frac{b_{i1}s_{i1}(b_{i-1})z_{1is}}{w_i^2} + 2\rho_{bs} \sum \frac{b_{i1}s_{i1}(b_{i-1})v_{i1}z_{1ib}z_{is}}{w_i^2} 
\]

\[
\nu_5 = \frac{1}{2} \left[ \frac{N_{s-K}}{(1-\rho_s)} - \sum \frac{(s_{i-1})u_i}{w_i} \right] - \frac{1}{2\sigma_0^2} \left[ \frac{SSW_s}{(1-\rho_s)^2} - \sum \frac{s_{i1}(s_{i-1})u_{1is}^2}{w_i^2} \right] - \rho_{bs} \sum \frac{b_{i1}s_{i1}(s_{i-1})z_{ib}^2}{w_i^2} + 2\rho_{bs} \sum \frac{b_{i1}s_{i1}(s_{i-1})u_{1ib}z_{ib}z_{is}}{w_i^2} 
\]

and

\[
\nu_6 = \rho_{bs} \sum \frac{b_{i1}s_{i1}}{w_i} - \frac{\rho_{bs}}{\sigma_0^2} \left[ \sum \frac{b_{i1}^2s_{i1}v_{i1}^2}{w_i^2} + \sum \frac{b_{i1}s_{i1}u_{1is}^2}{w_i^2} \right] - 2\rho_{bs} \sum \frac{b_{i1}s_{i1}z_{ib}^2}{w_i^2} + \frac{1}{\sigma_0^2} \sum \frac{b_{i1}s_{i1}z_{ib}z_{is}}{w_i^2} ,
\]

(5.39)

where \(z_{ib}\) and \(z_{is}\) \((i = 1, 2, \ldots, K)\) are as defined by (5.29).

(All summations are over \(i = 1, 2, \ldots, K\).

Furthermore, let us denote

\[
\nabla_{pq} = \frac{\partial^2 l_2}{\partial \psi_p \partial \psi_q} \quad p, q = 1, 2, \ldots, 6, \quad (5.40)
\]

where \(\psi_1 \equiv \mu_b\), \(\psi_2 \equiv \mu_s\), \(\psi_3 \equiv \sigma_0^2\), \(\psi_4 \equiv \rho_b\), \(\psi_5 \equiv \rho_s\), \(\psi_6 \equiv \rho_{bs}\). The second order partial derivatives of \(l_2\) (5.20) are given by

\[
\nabla_{11} = -\frac{1}{\sigma_o^2} \sum \frac{b_{i1}v_{i1}}{w_i} , \quad \nabla_{12} = \frac{\rho_{bs}}{\sigma_o^2} \sum \frac{b_{i1}s_{i1}}{w_i} ,
\]

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\[ v_{13} = -\frac{1}{\sigma_o^4} \left[ \sum \frac{b_i v_{1zib}}{w_1} - \rho_{bs} \sum \frac{b_i s_{1zis}}{w_1} \right], \]

\[ v_{14} = -\frac{1}{\sigma_o^2} \left[ \sum \frac{b_i (b_i - 1) v_{1zib}^2}{w_1} - \rho_{bs} \sum \frac{b_i s_{1zis} (b_i - 1) v_{1zis}}{w_1^2} \right], \]

\[ v_{15} = -\frac{\rho_{bs}}{\sigma_o^2} \left[ \rho_{bs} \sum \frac{b_i s_{1zib}^2}{w_1^2} - \sum \frac{b_i s_{1zis} (b_i - 1) u_{1zib}}{w_1^2} \right], \]

\[ v_{16} = \frac{1}{\sigma_o^2} \left[ 2\rho_{bs} \left[ \sum \frac{b_i s_{1zis}^2}{w_1^2} - \rho_{bs} \sum \frac{b_i s_{1zib}^2}{w_1^2} \right] \right. \]
\[ \left. - \sum \frac{b_i s_{1zis}}{w_1} \right]. \]

\[ v_{22} = -\frac{1}{\sigma_o^2} \sum \frac{s_{1u_1}}{w_1}, \]

\[ v_{23} = -\frac{1}{\sigma_o^4} \left[ \sum \frac{s_{1u_1 zis}}{w_1} - \rho_{bs} \sum \frac{b_i s_{1zib}}{w_1} \right], \]

\[ v_{24} = -\frac{\rho_{bs}}{\sigma_o^2} \left[ \rho_{bs} \sum \frac{b_i s_{1zis}^2 (b_i - 1) z_{is}}{w_1^2} - \sum \frac{b_i s_{1zib}^2 v_{1zib}}{w_1^2} \right], \]

\[ v_{25} = -\frac{1}{\sigma_o^2} \left[ \sum \frac{s_{1zib}}{w_1^2} - \rho_{bs} \sum \frac{b_i s_{1zis} (b_i - 1) u_{1zib}}{w_1^2} \right], \]

\[ v_{26} = \frac{1}{\sigma_o^2} \left[ 2\rho_{bs} \left[ \sum \frac{b_i s_{1zis} u_{1zis}}{w_1^2} - \rho_{bs} \sum \frac{b_i s_{1zib}^2}{w_1^2} \right] \right. \]
\[ \left. - \sum \frac{b_i s_{1zib}}{w_1} \right]. \]
\[ v_{33} = \frac{N}{2 \sigma_o} - \frac{1}{\sigma_o} \left[ \frac{\text{SSW}_b}{1 - \rho_b} + \frac{\text{SSW}_s}{1 - \rho_s} + \sum \frac{b_i v_i^2}{w_1} + \sum \frac{s_i u_i z_{is}^2}{w_1} \right. \]
\[ \left. - 2 \rho_{bs} \sum \frac{b_i s_i z_{ib} z_{is}}{w_1} \right]. \]

\[ v_{34} = \frac{1}{2 \sigma_o} \left[ \frac{\text{SSW}_b}{(1 - \rho_b)^2} - \sum \frac{b_i (b_i - 1) v_i^2}{w_1} \right. \]
\[ \left. - \rho_{bs} \sum \frac{b_i s_i (b_i - 1) z_{is}^2}{w_1} + 2 \rho_{bs} \sum \frac{b_i s_i (b_i - 1) v_i z_{ib} z_{is}}{w_1} \right]. \]

\[ v_{35} = \frac{1}{2 \sigma_o} \left[ \frac{\text{SSW}_s}{(1 - \rho_s)^2} - \sum \frac{s_i (s_i - 1) u_i^2}{w_1} \right. \]
\[ \left. - \rho_{bs} \sum \frac{b_i s_i (s_i - 1) z_{ib}^2}{w_1} + 2 \rho_{bs} \sum \frac{b_i s_i (s_i - 1) u_i z_{ib} z_{is}}{w_1} \right]. \]

\[ v_{36} = \frac{1}{2 \sigma_o} \left[ \rho_{bs} \left( \sum \frac{b_i s_i v_i z_{ib}^2}{w_1} + \sum \frac{b_i s_i u_i z_{is}^2}{w_1} \right) \right. \]
\[ \left. - 2 \rho_{bs} \sum \frac{b_i s_i z_{ib} z_{is}^2}{w_1} - \sum \frac{b_i s_i z_{ib} z_{is}}{w_1} \right]. \]

\[ v_{44} = \frac{1}{2} \left[ \frac{N_b - K}{(1 - \rho_b)^3} + \sum \frac{(b_i - 1)^2 v_i^2}{w_1} \right. \]
\[ \left. - \frac{1}{\sigma_o} \left[ \frac{\text{SSW}_b}{(1 - \rho_b)^3} + \sum \frac{b_i (b_i - 1)^3 v_i^2 z_{ib}^2}{w_1} \right. \right. \]
\[ \left. + \rho_{bs} \sum \frac{b_i s_i (b_i - 1)^2 v_i z_{is}^2}{w_1} \right. \]
\[ \left. - 2 \rho_{bs} \sum \frac{b_i s_i (b_i - 1)^2 v_i z_{ib} z_{is}}{w_1} \right]. \]
\[ v_{45} = \frac{\rho_{bs}}{2} \sum_i \frac{b_{i1} s_i (b_i - 1)(s_i - 1)}{w_i^2} \]

\[ - \frac{1}{\sigma^2_o} \left\{ \rho_{bs} \frac{b_{i1} s_i (b_i - 1)(s_i - 1)v_i z_i b}{w_i^3} \right\} \]

\[ + \frac{b_{i1} s_i^2 (b_i - 1)(s_i - 1)u_i z_i s}{w_i^3} \]

\[ - \rho_{bs} \sum_i \frac{b_{i1} s_i (b_i - 1)(s_i - 1)[u_i v_i + b_{i1} s_i \rho_{bs} z_i b z_i s]}{w_i^3} \right\}. \]

\[ v_{46} = - \rho_{bs} \sum_i \frac{b_{i1} s_i (b_i - 1)v_i}{w_i^2} + \frac{2\rho_{bs}}{\sigma^2_o} \sum_i \frac{b_{i1} s_i^2 (b_i - 1)v_i^2 z_i b}{w_i^3} \]

\[ + \frac{\rho_{bs}}{\sigma^2_o} \sum_i \frac{b_{i1} s_i^2 (b_i - 1)[u_i v_i + b_{i1} s_i \rho_{bs} z_i b z_i s]}{w_i^3} \]

\[ - \frac{4\rho_{bs}^2}{\sigma^2_o} \sum_i \frac{b_{i1} s_i^2 (b_i - 1)v_i z_i b z_i s}{w_i^3} \]

\[ - \frac{1}{\sigma^2_o} \sum_i \frac{b_{i1} s_i^2 (b_i - 1)v_i z_i b z_i s}{w_i^3} \right\}. \]

\[ v_{55} = \frac{1}{2} \left\{ \frac{N_{s-K}}{(1-\rho_s)^2} + \sum \frac{(s_i - 1)^2 u_i^2}{w_i^2} \right\} \]

\[ - \frac{1}{\sigma^2_o} \left\{ \frac{SSW_s}{(1-\rho_s)^3} + \sum \frac{s_i (s_i - 1)^2 u_i z_i s}{w_i^3} \right\} \]

\[ + \rho_{bs} \sum_i \frac{b_{i1} s_i^2 (s_i - 1)^2 u_i z_i b}{w_i^3} \]

\[ - 2\rho_{bs} \sum_i \frac{b_{i1} s_i (s_i - 1)^2 u_i z_i b z_i s}{w_i^3} \right\}. \]
\[ v_{56} = - \rho_{bs} \sum \frac{b_{1s}^2(s_i-1)u_{1i}}{w_i} + \frac{2 \rho_{bs}}{\sigma_o^2} \sum \frac{b_{1s}^2(s_i-1) u_{1i}^2 z_{1is}}{w_i} \]

\[ + \rho_{bs} \sum \frac{b_{1s}^2(s_i-1)u_{1i}v_{1i} + b_{1s}^2 \rho_{bs}}{w_i^2} z_{1ib} \]

\[ - \frac{4 \rho_{bs}}{\sigma_o^2} \sum \frac{b_{1s}^2(s_i-1)u_{1i} z_{1ib} z_{is}}{w_i^3} \]

\[ - \frac{1}{\sigma_o^2} \sum \frac{b_{1s}^2(s_i-1)u_{1i} z_{1ib} z_{is}}{w_i^2} \]

and

\[ v_{66} = 2 \rho_{bs} \sum \frac{b_{1s}^2}{w_i} + \sum \frac{b_{1s}^2}{w_i} \]

\[ - \frac{4 \rho_{bs}}{\sigma_o^2} \left[ \sum \frac{b_{1s}^2 v_{1i} z_{1ib}^2}{w_i} + \sum \frac{b_{1s}^2 u_{1i} z_{1is}^2}{w_i} \right] \]

\[ - \frac{1}{\sigma_o^2} \left[ \sum \frac{b_{1s}^2 v_{1i} z_{1ib}^2}{w_i} + \sum \frac{b_{1s}^2 u_{1i} z_{1is}^2}{w_i} \right] \]

\[ + \frac{8 \rho_{bs}}{\sigma_o^2} \sum \frac{b_{1s}^2 z_{1ib} z_{is}^2}{w_i^3} + \frac{6 \rho_{bs}}{\sigma_o^2} \sum \frac{b_{1s}^2 z_{1ib} z_{is}^2}{w_i^2} \]

\[ (5.41) \]

where \( z_{ib} \) and \( z_{is} \) \((i = 1, 2, \ldots, K)\) are given by \((5.29)\).

(All summations are over \( i = 1, 2, \ldots, K \).)

Furthermore, replacing \( \sigma_b \) and \( \sigma_s \) by \( \sigma_o \) in \((5.32)\), we have:

\[ EC(z_{1b}) = 0, \quad EC(z_{1s}) = 0, \quad i = 1, 2, \ldots, K, \]

\[ EC(z_{1b}^2) = \left[ 1 + (s_i - 1) \rho_b \right] \sigma_o^2 b_i, \quad i = 1, 2, \ldots, K, \]

\[ EC(z_{1s}^2) = \left[ 1 + (s_i - 1) \rho_s \right] \sigma_o^2 / s_i, \quad i = 1, 2, \ldots, K, \]
\[ E(z_{ib}z_{is}) = \rho_{bs}\sigma_o^2, \quad i = 1, 2, \ldots, K, \quad (5.42) \]

\[ E(SSW_b) = (N_b - K)(1 - \rho_b)\sigma_o^2 \]

and

\[ E(SSW_s) = (N_s - K)(1 - \rho_s)\sigma_o^2. \]

Denoting the \((p, q)\)th element of matrix \(\Lambda(\Psi)\) (5.37) by \(\epsilon_{pq}\), the elements of Fisher's information matrix are

\[ \epsilon_{pq} = \epsilon_{qp} = -E(V_{pq}), \quad p, q = 1, 2, \ldots, 6. \quad (5.43) \]

Thus by using (5.41) and (5.42), we get

\[ \epsilon_{11} = \frac{1}{\sigma_o^2} \sum \frac{b_i v_i}{w_i}, \quad \epsilon_{12} = \frac{-\rho_{bs}}{\sigma_o^2} \sum \frac{b_i s_i}{w_i}, \]

\[ \epsilon_{22} = \frac{1}{\sigma_o^2} \sum \frac{s_i u_i}{w_i}, \quad \epsilon_{pq} = 0, \quad p = 1, 2; \quad q = 3, 4, 5, 6, \]

\[ \epsilon_{33} = \frac{N}{2\sigma_o^4}, \quad \epsilon_{34} = \frac{-1}{2\sigma_o^2} \left[ \frac{N_b - K}{1 - \rho_b} - \sum \frac{(b_i - 1)v_i}{w_i} \right], \]

\[ \epsilon_{35} = \frac{-1}{2\sigma_o^2} \left[ \frac{N - K}{1 - \rho_s} - \sum \frac{(s_i - 1)u_i}{w_i} \right], \quad \epsilon_{36} = \frac{-\rho_{bs}}{\sigma_o^2} \sum \frac{b_i s_i}{w_i}, \]

\[ \epsilon_{pq} = a_{pq}', \quad p' = p + 1, \quad q' = q + 1, \quad p, q = 4, 5, 6, \]

where \(a_{pq}'s\) are given by (5.33). Therefore,

\[ \Lambda(\Psi) = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix}, \]

where

\[ \Lambda_{11}(\Psi) = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix} \]

and
Once again from the standard algebra of partitioned matrices, we have

$$
\Lambda^{-1}(\Psi) = \begin{bmatrix}
\Lambda_{11}^{-1} & 0 \\
0 & \Lambda_{22}^{-1}
\end{bmatrix},
$$

(5.44)

where

$$
\Lambda_{11}^{-1}(\Psi) = \frac{1}{\Lambda_1 \sigma_0^2} \begin{bmatrix}
e_{22} & -e_{12} \\
-e_{21} & e_{11}
\end{bmatrix},
$$

(5.45)

$$
\Lambda_{22}^{-1}(\Psi) = \begin{bmatrix}
-\tilde{B}_{11.2}^{-1} & -\tilde{B}_{12}^{-1} & -\tilde{B}_{22}^{-1} \\
-\tilde{B}_{21}^{-1} & B_{11.2}^{-1} & B_{12}^{-1} \\
-\tilde{B}_{22}^{-1} & B_{21}^{-1} & B_{22.1}
\end{bmatrix},
$$

(5.46)

$$
\tilde{B}_{11.2}(\Psi) = B_{11} - B_{12} B_{22}^{-1} B_{21},
$$

$$
\tilde{B}_{22.1}(\Psi) = B_{22} - B_{21} B_{11}^{-1} B_{12}
$$

and $\Lambda_1$ is given by (5.18).

We have the following results.

**Theorem 5.4:** Conditional on $n_1$, as $K \to \infty$, the (6x1) random vector $\tilde{\Psi} = [\tilde{\mu}_b, \tilde{\mu}_s, \tilde{\sigma}_o^2, \tilde{\rho}_b, \tilde{\rho}_s, \tilde{\rho}_{bs}]^T$ is asymptotically distributed as multivariate normal with mean vector $\bar{\Psi} = [\mu_b, \mu_s, \sigma_o^2, \rho_b, \rho_s, \rho_{bs}]^T$ and dispersion matrix $\Lambda^{-1}(\Psi)$, where $\Lambda^{-1}(\Psi)$ is given by (5.44).

**Theorem 5.5:** Conditional on $n_1$, as $K \to \infty$, the (2x1) random vector $\tilde{\Psi}^{(1)} = [\tilde{\mu}_b, \tilde{\mu}_s]^T$ is asymptotically distributed as multivariate normal with mean vector $\bar{\Psi}^{(1)} = [\mu_b, \mu_s]^T$ and
dispersion matrix $A_{11}^{-1}(\Psi)$, where $A_{11}^{-1}(\Psi)$ is given by (5.45).

Theorem 5.6: Conditional on $n_1$, as $K \to \infty$, the $(4 \times 1)$ random vector $\hat{\mathbf{\psi}}(2) = [\hat{\sigma}_0^2, \hat{\rho}_b, \hat{\rho}_s, \hat{\rho}_{bs}]^T$ is asymptotically distributed as multivariate normal with mean vector $\mathbf{\bar{\psi}}(2) = [\sigma_0^2, \rho_b, \rho_s, \rho_{bs}]^T$ and dispersion matrix $A_{22}^{-1}(\Psi)$, where $A_{22}^{-1}(\Psi)$ is given by (5.48).

Furthermore, from (5.48), the asymptotic dispersion matrix of the random vector $[\hat{\rho}_b, \hat{\rho}_s, \hat{\rho}_{bs}]^T$ is $B_{22}^{-1}(\Psi)$, which on simplification gives:

$$\text{Var}(\hat{\rho}_b) \approx \frac{gt - r^2}{A_3},$$

$$\text{Var}(\hat{\rho}_s) \approx \frac{gt - r^2}{A_3},$$

$$\text{Var}(\hat{\rho}_{bs}) \approx \frac{A_3}{A_3},$$

$$\text{Cov}(\hat{\rho}_b, \hat{\rho}_s) \approx \frac{rg - hr}{A_3},$$

$$\text{Cov}(\hat{\rho}_b, \hat{\rho}_{bs}) \approx \frac{A_3}{A_3},$$

$$\text{Cov}(\hat{\rho}_s, \hat{\rho}_{bs}) \approx \frac{A_3}{A_3},$$

where

$$g = \frac{1}{2} \left\{ \left[ \frac{N_b - K}{K - \rho_b} \right]^2 + \sum \frac{(b_i - 1)c_i w_i^2}{v_i^4} \right\},$$

$$h = \frac{1}{2} \left\{ \left[ \frac{N_b - K}{K - \rho_b} - \frac{1}{N} \sum \frac{(b_i - 1)c_i v_i}{w_i} \right]^2 \right\},$$

$$r = -\rho_{bs} \left\{ \sum \frac{b_i c_i (b_i - 1)c_i v_i}{w_i^2} + \frac{1}{N} \left[ \sum \frac{b_i c_i}{w_i} \right] \right\},$$

$$x = \left[ \frac{N_b - K}{K - \rho_b} - \frac{1}{N} \sum \frac{(b_i - 1)c_i v_i}{w_i} \right].$$
\[ p = \frac{1}{2} \left\{ \left[ \frac{N_s - K}{(1 - \rho_s)^2} + \sum \frac{(s_i - 1)^2 u_i^2}{w_i^2} \right] \right. \\
\left. - \frac{1}{N} \left[ \frac{N_s - K}{1 - \rho_s} - \sum \frac{(s_i - 1)u_i}{w_i} \right]^2 \right\}. \]

\[ q = -\rho_{bs} \left\{ \sum \frac{b_i s_i (s_i - 1)u_i}{w_i^2} + \frac{1}{N} \left[ \sum \frac{b_i s_i}{w_i} \right] \right. \\
\left. \times \left[ \frac{N_s - K}{1 - \rho_s} - \sum \frac{(s_i - 1)u_i}{w_i} \right] \right\}. \]

\[ t = \left\{ \sum \frac{b_i s_i (u_i v_i + b_i s_i^2 \rho_{bs})}{w_i^2} - \frac{2\rho_{bs}}{N} \left[ \sum \frac{b_i s_i}{w_i} \right]^2 \right\}. \]

and

\[ \Delta_3 = g(pt - q^2) + h(rq - ht) + r(hq - rp). \]

(All summations are over \( i = 1, 2, \ldots, K \).)

5.5 Example

Here, an example is presented using published arterial blood pressure data which results from a survey done by Miall and Oldham (1955). The purpose of the survey was to assess the correlations in systolic and diastolic blood pressures among family members living within 25 miles of Rhondda Fach valley in South Wales. The purpose of the following analysis is to obtain the maximum likelihood estimates of sibling correlations and to test their significance.

Observations were made on each subject and his/her first degree relatives (parents, siblings, and children). Each observation consists of systolic and diastolic blood pressures to the nearest 5mm Hg or 10mm Hg below the level.
indicated. After identifying each family member as either a father, mother, son (brother) and daughter (sister), the data on brothers (sons) and sisters (daughters) are used for this analysis. In families consisting of three generations, the data on youngest generation were omitted and data on single generation families were omitted entirely. Among 250 sampled families, only 215 contain information on brothers and sisters. Because of an impossible low systolic blood pressure (15mm Hg) for a daughter (sister), another family was omitted so there remained 215 families for this analysis. Before proceeding further it is necessary to remove the effects of differences in age and sex of sibling scores. The well known z-score transformation \( Z = (Y-M)/S \) was used to adjust the arterial blood pressures, where \( M \) and \( S \) refers to age and sex specific mean and standard deviations in the age groups \( \leq 10, 11-20, 21-30, 31-45 \) and over 45.

The maximum likelihood estimates of the sibling correlations and other parameters were obtained using IMSL (1987) subroutine BCONF from (MATH/LIBRARY). This subroutine uses a quasi-Newton method to minimize a function of several variables subject to user supplied bounds on the variables. The criterion chosen to stop the cycle of iteration was that the Euclidean distance between the estimated vectors in the last two steps be less than \( 10^{-5} \). The estimates are obtained by maximizing \( \tilde{\zeta}_1 \) (5.19) for the non-homogeneous case and \( \tilde{\zeta}_2 \) (5.25) for the homogeneous case under the restriction (5.8), and are presented in Table 5.1.
TABLE 5.1

The maximum likelihood estimates (MLE) of the familial parameters for non-homogeneous and homogeneous case.

<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>MLE SBP</th>
<th>MLE DBP</th>
<th>PARAMETER</th>
<th>MLE SBP</th>
<th>MLE DBP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_b$</td>
<td>-0.0020</td>
<td>0.0079</td>
<td>$\mu_b$</td>
<td>-0.0021</td>
<td>0.0078</td>
</tr>
<tr>
<td>$\mu_s$</td>
<td>0.0010</td>
<td>0.0016</td>
<td>$\mu_s$</td>
<td>0.0008</td>
<td>0.0015</td>
</tr>
<tr>
<td>$\sigma_b^2$</td>
<td>0.9853</td>
<td>0.9797</td>
<td>$\sigma_b^2$</td>
<td>0.9893</td>
<td>0.9824</td>
</tr>
<tr>
<td>$\sigma_s^2$</td>
<td>0.9947</td>
<td>0.9850</td>
<td>$\sigma_s^2$</td>
<td>-</td>
<td>--</td>
</tr>
<tr>
<td>$\rho_b$</td>
<td>0.1450</td>
<td>0.1632</td>
<td>$\rho_b$</td>
<td>0.1453</td>
<td>0.1634</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>0.3200</td>
<td>0.2844</td>
<td>$\rho_s$</td>
<td>0.3184</td>
<td>0.2844</td>
</tr>
<tr>
<td>$\rho_{bs}$</td>
<td>0.1776</td>
<td>0.2149</td>
<td>$\rho_{bs}$</td>
<td>0.1767</td>
<td>0.2149</td>
</tr>
</tbody>
</table>

SBP - Systolic Blood Pressure,
DBP - Diastolic Blood Pressure.

Because of the homogeneous effects of the shared household environment on both types of siblings, one should expect that the sibling correlations will not differ markedly for both homogeneous and non-homogeneous cases. On comparing the entries in Table 5.1, this seems to be true for the given data set because the estimated correlations in both cases are almost identical. Tables 5.2 and 5.3 give estimated variances and covariances of the maximum likelihood estimators of sibling correlations for non-homogeneous and homogeneous case, respectively. These estimates of the variances and covariances are obtained by replacing unknown parameters by their maximum likelihood estimation.
TABLE 5.2
The estimated variances and covariances of the maximum likelihood estimates (MLE) of sibling correlations for non-homogeneous case (i.e., $\sigma_b^2 \neq \sigma_s^2$).

<table>
<thead>
<tr>
<th>MLE</th>
<th>SYSTOLIC BLOOD PRESSURE</th>
<th></th>
<th></th>
<th>DIASTOLIC BLOOD PRESSURE</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{\rho}_b$</td>
<td>$\hat{\rho}_s$</td>
<td>$\hat{\rho}_{bs}$</td>
<td></td>
<td>$\hat{\rho}_b$</td>
<td>$\hat{\rho}_s$</td>
</tr>
<tr>
<td>$\hat{\rho}_b$</td>
<td>0.0054</td>
<td>0.0002</td>
<td>0.0010</td>
<td>0.0054</td>
<td>0.0003</td>
<td>0.0012</td>
</tr>
<tr>
<td>$\hat{\rho}_s$</td>
<td>0.0002</td>
<td>0.0048</td>
<td>0.0009</td>
<td>0.0003</td>
<td>0.0047</td>
<td>0.0011</td>
</tr>
<tr>
<td>$\hat{\rho}_{bs}$</td>
<td>0.0010</td>
<td>0.0008</td>
<td>0.0029</td>
<td>0.0012</td>
<td>0.0011</td>
<td>0.0027</td>
</tr>
</tbody>
</table>

TABLE 5.3
The estimated variances and covariances of the maximum likelihood estimates (MLE) of sibling correlations for homogeneous case (i.e., $\sigma_b^2 = \sigma_s^2 = \sigma_o^2$).

<table>
<thead>
<tr>
<th>MLE</th>
<th>SYSTOLIC BLOOD PRESSURE</th>
<th></th>
<th></th>
<th>DIASTOLIC BLOOD PRESSURE</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{\rho}_b$</td>
<td>$\hat{\rho}_s$</td>
<td>$\hat{\rho}_{bs}$</td>
<td></td>
<td>$\hat{\rho}_b$</td>
<td>$\hat{\rho}_s$</td>
</tr>
<tr>
<td>$\hat{\rho}_b$</td>
<td>0.0054</td>
<td>0.0003</td>
<td>0.0010</td>
<td>0.0054</td>
<td>0.0003</td>
<td>0.0012</td>
</tr>
<tr>
<td>$\hat{\rho}_s$</td>
<td>0.0003</td>
<td>0.0048</td>
<td>0.0008</td>
<td>0.0003</td>
<td>0.0047</td>
<td>0.0011</td>
</tr>
<tr>
<td>$\hat{\rho}_{bs}$</td>
<td>0.0010</td>
<td>0.0008</td>
<td>0.0029</td>
<td>0.0012</td>
<td>0.0011</td>
<td>0.0027</td>
</tr>
</tbody>
</table>

estimates in asymptotic expressions for variances and covariances given in section 5.4. The entries of these tables are needed for testing the hypotheses regarding significance of sibling correlations which will be discussed
in the following section.

5.6 Tests of significance

Here some procedures for testing the significance of sibling correlations are presented. These tests are based on the asymptotic theory of maximum likelihood estimators (see e.g., Section 1.6). Notice that the variances and covariances of the maximum likelihood estimators of sibling correlations depends upon three unknown parameters $\rho_b$, $\rho_s$ and $\rho_{bs}$. Wald (1943) suggested replacing these parameters in variances and covariances expressions by their maximum likelihood estimates to get the consistent estimates of the variances and covariances. From (1.19), using Wald's criteria, an appropriate test statistic to test $H_0: \rho_i = 0$ is

$$Z_i = \frac{\hat{\rho}_i}{[\text{Var}(\hat{\rho}_i)]^{1/2}}, \quad (i = b, s, bs),$$

which is asymptotically normally distributed with zero means and unit variances under $H_0$. Thus an asymptotic test of size $\alpha$ is to reject $H_0: \rho_i = 0$ in favour of $H_1: \rho_i > 0$ if $Z > Z_{1-\alpha}$, where $Z_{1-\alpha}$ is the $100(1-\alpha)$ percentile point of the standard normal distribution. Another hypothesis of interest regarding sibling correlations is to test that there is no significant difference between brother-brother and sister-sister correlations (i.e., $H_0: \rho_b - \rho_s = 0$ against $H_1: \rho_b - \rho_s \neq 0$). This hypothesis may be tested by using the statistic

$$Z_d = \frac{\hat{\rho}_b - \hat{\rho}_s}{[\text{Var}(\hat{\rho}_b - \hat{\rho}_s)]^{1/2}} = \frac{\hat{\rho}_b - \hat{\rho}_s}{[\text{Var}(\hat{\rho}_b) + \text{Var}(\hat{\rho}_s) - 2\text{Cov}(\hat{\rho}_b, \hat{\rho}_s)]^{1/2}}$$
which, under $H_0$, is asymptotically normally distributed with zero mean and unit variance. Thus an approximate test of size $\alpha$ is to reject $H_0: \rho_b - \rho_s = 0$ in favour of $H_1: \rho_b - \rho_s \neq 0$ if $|Z_d| > Z_{1-\alpha/2}$.

Before proceeding further, it is necessary to find out which of the results (non-homogeneous or homogeneous case) should be used for testing hypotheses regarding sibling correlations. That is, it is desired to test $H_0: \sigma_b^2 = \sigma_s^2$ against $H_1: \sigma_b^2 \neq \sigma_s^2$. The tests of this kind are referred to as testing the equality of variances of two correlated variables. A test statistic for such a hypothesis has been given by Pitman (1939) for bivariate normal populations. In this particular case, this hypothesis may be tested by using the likelihood ratio test statistic. Let $\ell_1$ and $\ell_0$ denote the maximum values of log-likelihood functions $\ell_1$ (5.14) and $\ell_2$ (5.20), respectively. Then under $H_0$, the likelihood ratio test statistic

$$\lambda = -2(\ell_0 - \ell_1)$$

(5.47)

is asymptotically distributed as chi-squared with one degree of freedom. Thus an approximate test of size $\alpha$ is to reject $H_0: \sigma_b^2 = \sigma_s^2$ in favour of $H_1: \sigma_b^2 \neq \sigma_s^2$ if $\lambda > \chi^2_1(\alpha)$, where $\chi^2_1(\alpha)$ is the $100(1-\alpha)$ percentile point of the chi-squared distribution with one degree of freedom. For this particular data set, the values of the $\lambda$ statistic for systolic and diastolic blood pressures are 0.0048 and 0.0024, respectively. The p-values for testing the said hypothesis are greater than 0.90 for either type of blood pressures.
Thus, the hypothesis of equality of variances is not rejected for either type of blood pressures and based on this evidence, the results of homogeneous case must be used for testing hypotheses regarding sibling correlations. Table 5.4 shows the results for testing the significance of sibling correlations for homogeneous case using systolic and diastolic blood pressures. All correlations seem to be

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Null Hypothesis} & \text{Alternate Hypothesis} & \text{Value of } Z \text{ for SBP} & \text{Value of } Z \text{ for DBP} \\
\hline
H_0: \rho_b = 0 & H_1: \rho_b > 0 & Z_b = 1.9773^* & Z_b = 2.2236^* \\
H_0: \rho_s = 0 & H_1: \rho_s > 0 & Z_s = 4.6946^{**} & Z_s = 4.1484^{**} \\
H_0: \rho_b = \rho_s & H_1: \rho_b \neq \rho_s & Z_{bs} = 3.2812^{**} & Z_{bs} = 4.1208^{**} \\
H_0: \rho_b = \rho_s & H_1: \rho_b \neq \rho_s & Z_d = -1.7854 & Z_d = -1.2414 \\
\hline
\end{array}
\]

Note: \( \text{Var}(\hat{\rho}_b - \hat{\rho}_s) \approx [\text{Var}(\hat{\rho}_b) + \text{Var}(\hat{\rho}_s) - 2\text{Cov}(\hat{\rho}_b, \hat{\rho}_s)] \).

*P-Value < 0.05, **P-Value < 0.01.

significantly greater than zero for both types of blood pressures which indicates the possibility of a strong familial aggregation for arterial blood pressures. Further there is not enough evidence to reject the hypothesis \( H_0: \rho_b = \rho_s \) for either type of blood pressures, thus there seems to be no significant sex effect on brother-brother
and sister-sister correlations.

5.7 Concluding Remarks

The maximum likelihood estimation of sibling correlations using a model that allows for the sex differences among siblings was presented in this chapter. Smith (1980 a,b), and Bener and Huda (1987) estimated the variance components of the model and hence sibling correlations. Their iterative procedures to find the maximum likelihood estimates of variance components of the model are quite difficult to implement. The maximum likelihood estimation procedure presented in this chapter is easy to implement, and is different from their iterative procedures because the direct maximum likelihood estimation of sibling correlations and other parameters is considered.

Further when estimating the variance components for brothers (sisters), the brother-sister correlation was ignored by Smith (1980 a,b) and hence there seems to be some loss of information which is very difficult to assess for general applications. The expressions for the large sample variances and covariances of the maximum likelihood estimators are also presented here which, to the best of our knowledge, are not known in the literature. In the next chapter, the maximum likelihood estimation of sibling correlations and other parameters will be considered in the presence of their mother score.
CHAPTER 6

REGRESSION MODELS APPROACH TO THE ANALYSIS OF FAMILIAL CORRELATIONS (Mother's Score is included)

6.1 Introduction

In chapter 5, the maximum likelihood estimation of three sibling correlations (brother-brother, sister-sister, and brother-sister correlations) was considered. In family studies, often information on a single parent (usually mother) or both parents are also collected with the siblings. This chapter deals with the situation when data on mother and her children are available. The maximum likelihood estimation of familial correlations $\rho_b$ (brother-brother), $\rho_s$ (sister-sister), $\rho_{bs}$ (brother-sister), $\rho_{mb}$ (mother-brother) and $\rho_{ms}$ (mother-sister), and other parameters, using a linear model approach, for unbalanced data from multivariate normal populations is considered. A linear model approach to the analysis of familial correlations was first introduced by Kempthorne and Tandon (1953), whereby the technique of estimating such correlations was almost distribution free. Mak and Ng (1981) used a multivariate regression model with error terms possessing an intra-class correlation structure to find the maximum likelihood estimates of these correlations when the mother's score is used as a regressor variable. Utilizing the scores of both parents and children, Shoukri and Ward (1989) generalized Mak and Ng's model and used the concept of genetic correlation between parents to construct ensemble
estimators for the interclass and intraclass correlations. One limitation of the above models is that the offspring (children) within each family are not distinguished by sex. This real situation is of fundamental importance to epidemiologists and animal breeders who may be interested in measuring the effect of the sex difference on the reported values of the sibling correlations in the presence of their mother’s score.

When offspring are classified by sex, Elston (1975) gave expressions for the maximum likelihood estimators of the set of correlations in the case of balanced data. The case of unbalanced data was dealt with by Smith (1980 a,b) within the framework of the models of variance components.

In this chapter, the Kempthorne-Tandon model is generalized to the case when offspring scores are classified by sex. Using the technique of Richards (1981) to maximize a function of several variables, the maximum likelihood estimates of five familial correlations, namely: $\rho_b$ (brother-brother), $\rho_s$ (sister-sister), $\rho_{bs}$ (brother-sister), $\rho_{mb}$ (mother-brother) and $\rho_{ms}$ (mother-sister), and other parameters are evaluated. The asymptotic variances and covariances of the estimators are derived by first inverting the Fisher's (1925) information matrix and then applying the delta method. The well known arterial blood pressure data collected by Miall and Oldham (1955) is used to illustrate the methodology.

6.2 The Models

For the purpose of estimation by maximum likelihood, it
is assumed that a simple random sample of $K$ independent families is available, and within each family we have a varying number of individuals constituting the source of measurements. Let

$$\xi_i = [x_i; y_i^T]^T = [x_i; y_{ib}^T; y_{is}^T]^T,$$

$$= [x_i; y_{i1}; y_{i2}; \ldots y_{ib_i}; y_{ib_i+1}; \ldots y_{in_i}]^T,$$

$$i = 1, 2, \ldots, K$$

be a vector of measurements from the $i$th family, where $K$ is the number of sampled families and for $i = 1, 2, \ldots, K$:

$b_i$ = number of brothers in the $i$th family,
$s_i$ = number of sisters in the $i$th family,
$n_i = b_i + s_i$ = number of offspring in the $i$th family,
$K$
$N_b = \sum_{i=1}^K b_i$ = total number of brothers in the sample,
$K$
$N_s = \sum_{i=1}^K s_i$ = total number of sisters in the sample,

$N = N_b + N_s$ = total number of offspring in the sample,

$x_i$ = mother score in the $i$th family,

$$y_{ij} = \begin{cases} 
\text{score of } j\text{th brother in the } i\text{th family}, 
\text{if } j = 1, 2, \ldots, b_i, \\
\text{score of } j\text{th sister in the } i\text{th family}, 
\text{if } j = b_i+1, \ldots, n_i.
\end{cases}$$

It is assumed that the $\xi_i$'s are independently distributed multivariate normal vectors with mean $\gamma_i$ and dispersion matrix $\Psi_i$. That is,

$$\xi_i \sim MN_{n_i+1}(\gamma_i, \Psi_i), \quad i = 1, 2, \ldots, K,$$

where

$$\gamma_i = [\mu_m; \mu_{ib}^T]^T = [\mu_m; \mu_{ib}^T; \mu_{is}^T]^T,$$

$$= [\mu_m; \mu_b, \mu_{b'}, \ldots, \mu_b; \mu_s, \mu_s', \ldots, \mu_s]^T.$$
\[ \Psi_1 = \begin{bmatrix} \sigma_m^2 & \psi_{mo} \\ \psi_{mo}^T & \Omega_1 \end{bmatrix} \]  

(6.2)

\[ \psi_{mo} = [I_{1 \times b_1} \rho_{mb} \sigma_m \sigma_b, I_{1 \times s_1} \rho_{ms} \sigma_m \sigma_s] \]

and \( \Omega_1 \) is dispersion matrix of random vector \( \Psi_1 \) as given by (5.2). Here \( \gamma_i \) is a column vector of length \((n_i + 1)\), \( \tilde{\Psi}_1 \) is \((n_i + 1 \times n_i + 1)\) matrix and \( J_{p \times q} \) is a \((p \times q)\) matrix, each element of which is unity. Furthermore it is assumed that the familial correlations \( \rho_{mb}, \rho_{ms}, \rho_b, \rho_s \) and \( \rho_{bs} \) are constant over all families and the observations \( \tilde{\Psi}_1 \)'s are independent of the family size \( (n_i + 1) \), \( i = 1, 2, \ldots, K \).

Clearly, by theorem 1.7, the marginal distribution of random vector \( \Psi_i \) is multivariate normal as given by (5.1). Under the multivariate normal model (6.1), we have for \( i = 1, 2, \ldots, K \):

\[ \text{EC}(x_i) = \mu_m, \quad \text{Var}(x_i) = \sigma_m^2, \]

\[ \text{EC}(y_{ij}) = \begin{bmatrix} \mu_b, & j = 1, 2, \ldots, b_i, \\ \mu_s, & j = b_i + 1, \ldots, n_i, \end{bmatrix} \]

\[ \text{Var}(y_{ij}) = \begin{bmatrix} \sigma_b^2, & j = 1, 2, \ldots, b_i, \\ \sigma_s^2, & j = b_i + 1, \ldots, n_i, \end{bmatrix} \]

\[ \text{Cov}(x_i, y_{ij}) = \begin{bmatrix} \rho_{mb} \sigma_m \sigma_b, & j = 1, 2, \ldots, b_i, \\ \rho_{ms} \sigma_m \sigma_s, & j = b_i + 1, \ldots, n_i, \end{bmatrix} \]

\[ \text{Cov}(y_{ij}, y_{im}) = \begin{bmatrix} \rho_b \sigma_b^2, & j = m = 1, 2, \ldots, b_i, \\ \rho_s \sigma_s^2, & j = m = b_i + 1, \ldots, n_i, \\ \rho_{bs} \sigma_b \sigma_s, & j = 1, 2, \ldots, b_i, \\ m = b_i + 1, \ldots, n_i \end{bmatrix} \]  

(6.3)
and all the other covariances are zero.

Generalizing the linear regression model of Kempthorne and Tandon (1963) to represent sibling scores in terms of their mother score, it is assumed that for \( i = 1, 2, \ldots, K \):

\[
y_{ij} = \begin{cases} 
\beta_{0b} + \beta_{1b}x_{i1} + \epsilon_{ij}, & j = 1, 2, \ldots, b_i, \\
\beta_{0s} + \beta_{1s}x_{i1} + \epsilon_{ij}, & j = b_i+1, \ldots, n_i.
\end{cases}
\] (6.4)

Here it is assumed that the errors \( \epsilon_{ij} \)'s are normally distributed such that for \( i = 1, 2, \ldots, K \):

\[
\text{E}(\epsilon_{ij}) = 0, \quad j = 1, 2, \ldots, b_i, b_i+1, \ldots, n_i,
\]

\[
\text{Var}(\epsilon_{ij}) = \begin{cases} 
\sigma_1^2, & j = 1, 2, \ldots, b_i, \\
\sigma_2^2, & j = b_i+1, \ldots, n_i.
\end{cases}
\]

\[
\text{Cov}(\epsilon_{ij}, \epsilon_{im}) = \begin{cases} 
\rho_1 \sigma_1^2, & j \neq m = 1, 2, \ldots, b_i, \\
\rho_2 \sigma_2^2, & j \neq m = b_i+1, \ldots, n_i, \\
\rho_{12} \sigma_1 \sigma_2, & j = 1, 2, \ldots, b_i, \\
m = b_i+1, \ldots, n_i.
\end{cases}
\]

Furthermore, the \( x_i \)'s are independently normally distributed with mean \( \mu_m \) and the variance \( \sigma_m^2 \), and are independent of \( \epsilon_{ij} \)'s. In terms of the parameters of linear model (6.4), we have for \( i = 1, 2, \ldots, K \):

\[
\text{E}(y_{ij}) = \begin{cases} 
\beta_{0b} + \beta_{1b} \mu_m, & j = 1, 2, \ldots, b_i, \\
\beta_{0s} + \beta_{1s} \mu_m, & j = b_i+1, \ldots, n_i.
\end{cases}
\]

\[
\text{Var}(y_{ij}) = \begin{cases} 
\beta^2_{1b} \sigma_m^2 + \sigma_1^2, & j = 1, 2, \ldots, b_i, \\
\beta^2_{1s} \sigma_m^2 + \sigma_2^2, & j = b_i+1, \ldots, n_i.
\end{cases}
\] (6.5)

\[
\text{Cov}(y_{ij}, y_{ik}) = \begin{cases} 
\beta^2_{1b} \sigma_m^2 + \rho_1 \sigma_1^2, & j \neq m = 1, 2, \ldots, b_i, \\
\beta^2_{1s} \sigma_m^2 + \rho_2 \sigma_2^2, & j \neq m = b_i+1, \ldots, n_i, \\
\beta_{1b} \beta_{1s} \sigma_m^2 + \rho_{12} \sigma_1 \sigma_2, & j = 1, 2, \ldots, b_i, \\
k = b_i+1, \ldots, n_i.
\end{cases}
\]
By comparing (6.3) and (6.5), the parameters of the model described in (6.1) can be written in terms of the parameters of linear model (6.4) as follows:

\[
\begin{align*}
\mu_b &= \beta_{0b} + \beta_{1b} \mu_m, \\
\sigma^2_b &= \beta_{1b}^2 \sigma^2_m + \sigma^2_1, \\
\sigma^2_s &= \beta_{is}^2 \sigma^2_m + \sigma^2_2, \\
\rho_b &= \frac{\beta_{1b}^2 \sigma^2_m + \rho_{1b} \sigma_1^2 \sigma_2^2}{\beta_{1b}^2 \sigma^2_m + \sigma^2_1}, \\
\rho_s &= \frac{\beta_{is}^2 \sigma^2_m + \rho_{is} \sigma^2_2}{\beta_{is}^2 \sigma^2_m + \sigma^2_2}, \\
\rho_{bs} &= \frac{\beta_{1b} \beta_{is} \sigma^2_m + \rho_{1b} \sigma_1^2 \sigma^2_2}{(\beta_{1b}^2 \sigma^2_m + \sigma^2_1)(\beta_{is}^2 \sigma^2_m + \sigma^2_2)^{1/2}}, \\
\rho_{mb} &= \frac{\beta_{1b} \sigma^2_m}{(\beta_{1b}^2 \sigma^2_m + \sigma^2_1)^{1/2}}, \\
\rho_{ms} &= \frac{\beta_{is} \sigma^2_m}{(\beta_{is}^2 \sigma^2_m + \sigma^2_2)^{1/2}}.
\end{align*}
\tag{6.6}
\]

Conversely,

\[
\begin{align*}
\beta_{0b} &= \mu_b - \frac{\rho_{mb} \sigma^2_b}{\sigma_m} \mu_m, \\
\beta_{0s} &= \mu_s - \frac{\rho_{ms} \sigma^2_s}{\sigma_m} \mu_m, \\
\beta_{1b} &= \frac{\rho_{mb} \sigma_b}{\sigma_m}, \\
\beta_{is} &= \frac{\rho_{ms} \sigma_s}{\sigma_m}, \\
\sigma^2_1 &= (1 - \rho_{mb}^2) \sigma^2_b, \\
\sigma^2_2 &= (1 - \rho_{ms}^2) \sigma^2_s, \\
\rho_1 &= \frac{(\rho_b - \rho_{mb}^2)}{(1 - \rho_{mb}^2)}, \\
\rho_2 &= \frac{(\rho_s - \rho_{ms}^2)}{(1 - \rho_{ms}^2)}, \\
\rho_{12} &= \frac{\rho_{bs} - \rho_{mb} \rho_{ms}}{(1 - \rho_{mb}^2)(1 - \rho_{ms}^2)^{1/2}}.
\end{align*}
\tag{6.7}
\]
The correlations ρ₁, ρ₂ and ρ₁₂ are in fact the partial correlation coefficients, respectively, between brother-brother, sister-sister and brother-sister. In the next section we derive the maximum likelihood estimates of the parameters of model (6.4), and hence the maximum likelihood estimates of the parameters of model (6.1).

A necessary condition for the positive definiteness of \( \mathbf{Y}_1 \) is given by the following lemma.

Lemma: If
\[
\rho_{12}^2 < \left[ \rho_1 + b_0^{-1}(1-\rho_1) \right] \left[ \rho_2 + s_0^{-1}(1-\rho_2) \right]
\]
or equivalently
\[
[\rho_{bs} - \rho_{mb}^2 \rho_{ms}]^2 < \left[ \rho_b + b_0^{-1}(1-\rho_b) - \rho_{mb}^2 \right] \left[ \rho_s + s_0^{-1}(1-\rho_s) - \rho_{ms}^2 \right].
\]
where
\[
b_0 = \max b_i \quad \text{and} \quad s_0 = \max s_i,
\]
then \( \mathbf{Y}_1 \) is positive definite.

Proof: Let \( \bar{y}_{ib} \) and \( \bar{y}_{is} \) \((i = 1, 2, \ldots, K)\) be the sample means of brother and sister scores, respectively, in the \( i \)th family as defined by (5.4). From the multivariate normal model (6.1), the conditional distribution of the random vector \( \mathbf{Y}_i = [\bar{y}_{ib}, \bar{y}_{is}]^\top \) given \( x_i \) is bivariate normal with mean vector
\[
\mu_{i}^{XY} = [\beta_{0b} + \beta_{1b} x_i, \beta_{0s} + \beta_{1s} x_i]^\top
\]
and dispersion matrix
\[ \Omega_i^{**} = \begin{bmatrix} [1+(b_1-1)\rho_1] \sigma_1^2/b_1 & \rho_{12} \sigma_1 \sigma_2 \\ \rho_{12} \sigma_1 \sigma_2 & [1+(s_1-1)\rho_2] \sigma_2^2/s_1 \end{bmatrix}. \] (6.8)

Following Srivastava and Katapa (1986), it can be shown that
\[ |\Omega_i^{**}| = b_1 s_1 \sigma_m^2 (\sigma_1^2 b_1^{-1} (\sigma_2^2 s_1^{-1} (1-\rho_1) b_1^{-1} (1-\rho_2) s_1^{-1} |\Omega_i^*|, \]
\[ b_1, s_1, \sigma_m, \sigma_1^2, \sigma_2^2, (1-\rho_1), (1-\rho_2) \geq 0. \] (6.9)

Hence \( \Omega_i^{**} \) is positive definite if \( \Omega_i^* \) is positive definite.

But
\[ |\Omega_i^{**}| = \sigma_1^2 \sigma_2^2 \{[\rho_1 + b_1^{-1}(1-\rho_1)][\rho_2 + s_1^{-1}(1-\rho_2)] - \rho_{12}^2\}. \] (6.10)

Thus \( |\Omega_i^{**}| > 0 \) implies that
\[ \rho_{12}^2 < [\rho_1 + b_1^{-1}(1-\rho_1)][\rho_2 + s_1^{-1}(1-\rho_2)]. \]

The above condition is simultaneously satisfied whenever
\[ \rho_{12}^2 < [\rho_1 + b_0^{-1}(1-\rho_1)][\rho_2 + s_0^{-1}(1-\rho_2)], \] (6.11)

or equivalently
\[ [\rho_{bs} - \rho_{mb}\rho_{ms}] < [\rho_b + b_0^{-1}(1-\rho_b) - \rho_{mb}][\rho_s + s_0^{-1}(1-\rho_s) - \rho_{ms}], \] (6.12)

where
\[ b_0 = \max_i b_i \quad \text{and} \quad s_0 = \max_i s_i. \]

6.3 Maximum Likelihood Estimation

Under the linear model (6.4), with the normality assumption on the mother score, the overall likelihood function of the sample of \( K \) families will be written as:

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\[ \mathcal{L}(\mu, \sigma^2; \beta_{0b}, \beta_{os}, \beta_{ib}, \beta_{is}, \sigma_1^2, \sigma_2^2, \rho_1, \rho_2, \rho_{12}; \mathbf{x}, \mathbf{y}) = \mathcal{L}(\mu, \sigma^2; \mathbf{x}) \mathcal{L}(\beta_{0b}, \beta_{os}, \beta_{ib}, \beta_{is}, \sigma_1^2, \sigma_2^2, \rho_1, \rho_2, \rho_{12}; \mathbf{y} | \mathbf{x}) \]  

(8.13)

where \( \mathcal{L}_1 \) is the marginal likelihood function of the \( x_i \)'s and \( \mathcal{L}_2 \) is the conditional likelihood function of the \( y_{ij} \)'s given \( x_i \)'s. The likelihood function \( \mathcal{L} \) will be maximized for those estimates of \( \mu, \sigma^2, \beta_{0b}, \beta_{os}, \beta_{ib}, \beta_{is}, \sigma_1^2, \sigma_2^2, \rho_1, \rho_2 \) and \( \rho_{12} \) which will maximize both \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) simultaneously. Note that \( \mathcal{L}_1 \) does not involve \( \beta_{0b}, \beta_{os}, \beta_{ib}, \beta_{is}, \sigma_1^2, \sigma_2^2, \rho_1, \rho_2 \) and \( \rho_{12} \), and \( \mathcal{L}_2 \) does not involve \( \mu \) and \( \sigma^2 \). Thus the maximum likelihood estimates can be obtained by maximizing \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) separately. Since \( \mathcal{L}_1 \) is the likelihood function based on a simple random sample \( (x_1, x_2, \ldots, x_k) \) from \( \mathcal{N}(\mu, \sigma^2) \), \( \mathcal{L}_1 \) is maximum when

\[ \hat{\mu} = \frac{1}{K} \sum_{i=1}^{K} x_i \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{K} \sum_{i=1}^{K} (x_i - \hat{\mu})^2. \]  

(8.14)

Moreover, \( \mathcal{L}_2 \) is the likelihood function based on the conditional density function of \( n_1 \)-component random vector \( \mathbf{X}_1 \) given \( X_i = x_i \), which is multivariate normal, by Theorem 1.8, with mean vector \( \gamma_i^* \) and dispersion matrix \( \Psi_i^* \), where

\[ \gamma_i^* = \begin{bmatrix} (\beta_{0b} + \beta_{ib} x_i) x_i \, I_{b_1 x_i} \\ (\beta_{os} + \beta_{is} x_i) x_i \, I_{s_1 x_i} \end{bmatrix} \]  

(8.15)

and

\[ \Psi_i^* = \begin{bmatrix} I_{b_1 x_i} \sigma_1^2 + \rho_{1b} I_{b_1 x_i} & I_{b_1 x_i} \rho_{12} \sigma_1 \sigma_2 \\ I_{b_1 x_i} \rho_{12} \sigma_1 \sigma_2 & I_{s_1 x_i} \sigma_2^2 + (1 - \rho_{1s}) I_{s_1 x_i} \rho_{12} \sigma_1 \sigma_2 \end{bmatrix}. \]  

(8.16)

Thus \( \mathcal{L}_2 \) can be written as
\[ L_2 = (2\pi)^{-N/2} \prod_{i=1}^{K} |\Psi_i^*|^{-1/2} \exp\left(-\frac{1}{2} Q_1\right), \]  
where

\[ Q_1 = \sum_{i=1}^{K} \left[ \Psi_i^* - \Psi_i \right]^T \Psi_i^{-1} \left[ \Psi_i^* - \Psi_i \right]. \]

From (8.10) and (8.18), it can be shown that

\[ |\Psi_i^*| = b_1 s_1 (\sigma_1^2)^{b_1-1} (\sigma_2^2)^{s_1-1} (1-\rho_1)^{b_1-1} (1-\rho_2)^{s_1-1} |\Omega_i^{\Psi*}| \]

\[ = b_1 s_1 (\sigma_1^2)^{b_1-1} (\sigma_2^2)^{s_1-1} (1-\rho_1)^{b_1-1} (1-\rho_2)^{s_1-1} w_i, \]  

where

\[ w_i = w_i (\rho_1, \rho_2, \rho_{12}) \]

\[ = [1+(b_1-1)\rho_1][1+(s_1-1)\rho_2] - b_1 s_1 \rho_{12}, i = 1, 2, \ldots, K. \]

(8.19)

In terms of the sample statistics, the quadratic exponent in (8.17) will be conveniently written as

\[ Q_1 = \frac{SSW_b}{(1-\rho_1)\sigma_1^2} + \frac{SSW_s}{(1-\rho_2)\sigma_2^2} + \sum_{i=1}^{K} \left[ \bar{y}_i^* - \bar{\mu}_1 \right]^T \Omega_i^{\Psi*} \left[ \bar{y}_i^* - \bar{\mu}_1 \right]. \]

\[ = \frac{1}{\sigma_1^2} \left[ \frac{SSW_b}{1-\rho_1} + \sum_{i=1}^{K} \frac{b_1 u_i (\bar{y}_{ib} - \beta_{0b} - \beta_{1b} x_i)^2}{w_i} \right] \]

\[ + \frac{1}{\sigma_2^2} \left[ \frac{SSW_s}{1-\rho_2} + \sum_{i=1}^{K} \frac{s_1 u_i (\bar{y}_{is} - \beta_{0s} - \beta_{1s} x_i)^2}{w_i} \right] \]

\[ - \frac{2\rho_{12}}{\sigma_1^2 \sigma_2} \left[ \sum_{i=1}^{K} \frac{b_1 s_1 (\bar{y}_{ib} - \beta_{0b} - \beta_{1b} x_i)(\bar{y}_{is} - \beta_{0s} - \beta_{1s} x_i)}{w_i} \right]. \]

(8.20)

where

\[ u_i = u_i (\rho_1) = 1+(b_1-1)\rho_1, i = 1, 2, \ldots, K, \]

\[ v_i = v_i (\rho_2) = 1+(s_1-1)\rho_2, i = 1, 2, \ldots, K, \]

and \(SSW_b\) and \(SSW_s\) are as defined by (5.13). Substituting
(6.18) and (6.20) in (6.17), the natural logarithm of $L_2$ is given as

$$
L_2 = -\frac{N}{2} \ln(2\pi) - \frac{N_B}{2} \ln\sigma_1^2 - \frac{N_S}{2} \ln\sigma_2^2 - \frac{N_{b-K}}{2} \ln(1-\rho_1)
- \frac{N_s-K}{2} \ln(1-\rho_2) - \frac{1}{2} \sum_{i=1}^{K} \ln(w_i) - \frac{1}{2} Q_1. 
$$

(6.22)

Differentiating (6.22) with respect to $\beta_{0b}$, $\beta_{os}$, $\beta_{ib}$, $\beta_{is}$, equating to zero and solving for these parameters we get the following system of linear equations:

$$
\begin{bmatrix}
M_2 & -C_{1M_1} & M_6 & -C_{1M_4} \\
-C_{2M_1} & M_3 & -C_{2M_4} & M_7 \\
M_6 & -C_{1M_4} & M_9 & -C_{1M_5} \\
-C_{2M_4} & M_7 & -C_{2M_5} & M_9
\end{bmatrix}
\begin{bmatrix}
\beta_{0b} \\
\beta_{os} \\
\beta_{ib} \\
\beta_{is}
\end{bmatrix}
=
\begin{bmatrix}
M_{14} - C_{1M_{11}} \\
M_{15} - C_{2M_{10}} \\
M_{16} - C_{1M_{13}} \\
M_{17} - C_{2M_{12}}
\end{bmatrix}.
$$

(6.23)

where

$$
c_1 = \frac{\rho_{12}\sigma_1}{\sigma_2},
c_2 = \frac{\rho_{12}\sigma_2}{\sigma_1},
M_1 = \sum \frac{b_1s_1}{w_1},
$$

$$
N_2 = \sum \frac{b_1v_1}{w_1},
M_3 = \sum \frac{s_1u_1}{w_1},
M_4 = \sum \frac{b_1s_1x_1}{w_1},
$$

$$
M_5 = \sum \frac{b_1s_1x_1^2}{w_1},
M_6 = \sum \frac{b_1v_1x_1}{w_1},
M_7 = \sum \frac{s_1u_1x_1}{w_1},
$$

$$
M_8 = \sum \frac{b_1v_1x_1^2}{w_1},
M_9 = \sum \frac{s_1u_1x_1^2}{w_1},
M_{10} = \sum \frac{b_1s_1y_{1b}}{w_1},
$$

$$
M_{11} = \sum \frac{b_1s_1y_{1s}}{w_1},
M_{12} = \sum \frac{b_1s_1x_1y_{1b}}{w_1},
M_{13} = \sum \frac{b_1s_1x_1y_{1s}}{w_1},
$$

$$
M_{14} = \sum \frac{b_1v_1y_{1b}}{w_1},
M_{15} = \sum \frac{b_1s_1y_{1s}}{w_1},
M_{16} = \sum \frac{b_1v_1x_1y_{1b}}{w_1},
$$

and
\[ \hat{m}_{17} = \sum \frac{s_i u_{ib} x_{ib} \bar{y}_{is}}{w_i} \]  

(6.24)

(All summations are over \( i = 1, 2, \ldots, K \)).

This system of equations can be solved for \( \hat{\beta} \)'s easily by using Cramer's rule or some Fortran subroutine written for this purpose (e.g., LSAR9 from IMSL, 1987). Note that \( \hat{\beta} \)'s are functions of the five unknown parameters \( \sigma_1^2, \sigma_2^2, \rho_1, \rho_2 \) and \( \rho_{12} \). Now following the procedure of Richards (1981) for maximizing a function of several variables and substituting \( \hat{\beta} \)'s for \( \beta \)'s in \( l_2 \) (6.22) gives:

\[ l_2^* = -\frac{N_2}{2} \ln(2\pi) - \frac{N_b}{2} \ln(\sigma_1^2) - \frac{N_s}{2} \ln(\sigma_2^2) - \frac{N_b-K}{2} \ln(1-\rho_1) \]

\[ -\frac{N_s-K}{2} \ln(1-\rho_2) - \frac{1}{2} \sum_{i=1}^{K} \ln(w_i) - \frac{1}{2} Q_1^* \]  

(6.25)

where

\[ Q_1^* = \frac{1}{\sigma_1^2} \left[ \frac{SSW_d}{1-\rho_1} + \sum_{i=1}^{K} \frac{b_{1i} v_i (\bar{y}_{ib} - \hat{\beta}_0 - \hat{\beta}_{1b} x_1)^2}{w_i} \right] \]

\[ + \frac{1}{\sigma_2^2} \left[ \frac{SSW_s}{1-\rho_2} + \sum_{i=1}^{K} \frac{b_{1i} v_i (\bar{y}_{is} - \hat{\beta}_0 - \hat{\beta}_{1s} x_1)^2}{w_i} \right] \]

\[ - \frac{2\rho_{12}}{\sigma_1^2 \sigma_2^2} \left[ \sum_{i=1}^{K} \frac{b_{1i} s_i (\bar{y}_{ib} - \hat{\beta}_0 - \hat{\beta}_{1b} x_1) (\bar{y}_{is} - \hat{\beta}_0 - \hat{\beta}_{1s} x_1)}{w_i} \right] \]

which is now a function of five parameters. The maximum likelihood estimators \( \hat{\sigma}_1^2, \hat{\sigma}_2^2, \hat{\rho}_1, \hat{\rho}_2 \) and \( \hat{\rho}_{12} \) of \( \sigma_1^2, \sigma_2^2, \rho_1, \rho_2 \) and \( \rho_{12} \), respectively, can be obtained by numerically maximizing \( l_2^* \) given by (6.25). Once the local maximum is found, the maximum likelihood estimates \( \hat{\beta} \)'s of \( \beta \)'s can be
obtained by substituting $\hat{\sigma}_1^2$, $\hat{\sigma}_2^2$, $\hat{\rho}_1$, $\hat{\rho}_2$, and $\hat{\rho}_{12}$ in the solution of (6.23). Since the parameters of the linear model (6.4) are in one-to-one correspondence with the parameters of the multivariate normal model (6.1), the maximum likelihood estimates of $\mu_b$, $\mu_s$, $\sigma_b^2$, $\sigma_s^2$, $\rho_b$, $\rho_s$, $\rho_{bs}$, $\rho_{mb}$ and $\rho_{ms}$ can be obtained from (6.8) by using the invariance property of maximum likelihood estimators. Thus, the maximum likelihood estimators of model parameters are:

$$\hat{\mu}_b = \hat{\beta}_{0b} + \hat{\beta}_{1b} \hat{\mu}_m,$$
$$\hat{\mu}_s = \hat{\beta}_{0s} + \hat{\beta}_{1s} \hat{\mu}_m,$$

$$\hat{\sigma}_b^2 = \hat{\beta}_{1b}^2 \hat{\sigma}_m^2 + \hat{\sigma}_1^2,$$
$$\hat{\sigma}_s^2 = \hat{\beta}_{1s}^2 \hat{\sigma}_m^2 + \hat{\sigma}_2^2,$$

$$\hat{\rho}_b = \frac{(\hat{\beta}_{1b}^2 \hat{\sigma}_m^2 + \hat{\rho}_{12} \hat{\sigma}_1 \hat{\sigma}_2)}{(\hat{\beta}_{1b}^2 \hat{\sigma}_m^2 + \hat{\sigma}_1^2)},$$
$$\hat{\rho}_s = \frac{(\hat{\beta}_{1s}^2 \hat{\sigma}_m^2 + \hat{\rho}_{12} \hat{\sigma}_1 \hat{\sigma}_2)}{(\hat{\beta}_{1s}^2 \hat{\sigma}_m^2 + \hat{\sigma}_2^2)}.$$

$$\hat{\rho}_{bs} = \frac{(\hat{\beta}_{1s} \hat{\beta}_{1b} \hat{\sigma}_m^2 + \hat{\rho}_{12} \hat{\sigma}_1 \hat{\sigma}_2)}{([\hat{\beta}_{1b}^2 \hat{\sigma}_m^2 + \hat{\sigma}_1^2]([\hat{\beta}_{1s}^2 \hat{\sigma}_m^2 + \hat{\sigma}_2^2])^{1/2}},$$

$$\hat{\rho}_{mb} = \frac{\hat{\beta}_{1b} \hat{\sigma}_m}{([\hat{\beta}_{1b}^2 \hat{\sigma}_m^2 + \hat{\sigma}_1^2])^{1/2}},$$

$$\hat{\rho}_{ms} = \frac{\hat{\beta}_{1s} \hat{\sigma}_m}{([\hat{\beta}_{1s}^2 \hat{\sigma}_m^2 + \hat{\sigma}_2^2])^{1/2}}.$$

### 6.4 Asymptotic Results

In this section, the asymptotic distributions of the maximum likelihood estimators of familial correlations and other parameters are established. We start first by establishing the asymptotic distribution of the parameters.
of the linear model (6.4). Let us denote the Fisher’s information matrix of (9x1) random vector

\[ \hat{\sigma} = [ \hat{\beta}_{0b}, \hat{\beta}_{0s}, \hat{\beta}_{1b}, \hat{\beta}_{1s}, \hat{\gamma}^2, \hat{\gamma}^2, \hat{\gamma}^2, \rho_1, \rho_2, \rho_{12} ]^T \]

by

\[ \mathbb{E}(\hat{\sigma}) = - E \left[ \begin{array}{c} \sigma_p^2 \sigma_q^2 \end{array} \right], \quad p, q = 1, 2, \ldots, 9, \quad (6.27) \]

where \( \theta_1 = \beta_{0b}, \theta_2 = \beta_{0s}, \theta_3 = \beta_{1b}, \theta_4 = \beta_{1s}, \theta_5 = \gamma^2, \theta_6 = \gamma^2, \)
\( \theta_7 = \rho_1, \theta_8 = \rho_2, \theta_9 = \rho_{12}. \) In order to find the elements of \( \mathbb{E}(\hat{\sigma}) \), we need to know the first and second order partial derivatives of \( L_2 \) (6.22) with respect to all the nine parameters. Let us denote

\[ D_p = \frac{\partial L_2}{\partial \theta_p}, \quad p = 1, 2, \ldots, 9, \quad (6.28) \]

where \( \theta_1 = \beta_{0b}, \theta_2 = \beta_{0s}, \theta_3 = \beta_{1b}, \theta_4 = \beta_{1s}, \theta_5 = \gamma^2, \theta_6 = \gamma^2, \)
\( \theta_7 = \rho_1, \theta_8 = \rho_2, \theta_9 = \rho_{12}. \) The first order partial derivatives of \( L_2 \) are given by

\[ D_1 = \frac{1}{\sigma_1^2} \sum \frac{b_1 y_i z_{ib}}{w_i} - \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_1 s_i z_{is}}{w_i}, \]
\[ D_2 = \frac{1}{\sigma_2^2} \sum \frac{s_i u_i z_{is}}{w_i} - \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_1 s_i z_{ib}}{w_i}, \]
\[ D_3 = \frac{1}{\sigma_1^2} \sum \frac{b_1 y_i x_i u_i z_{ib}}{w_i} - \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_1 s_i x_i z_{is}}{w_i}, \]
\[ D_4 = \frac{1}{\sigma_2^2} \sum \frac{s_i u_i x_i z_{is}}{w_i} - \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_1 s_i x_i z_{ib}}{w_i}, \]
\[ D_5 = - \frac{N_b}{2\sigma_1^2} + \frac{1}{2\sigma_1^4} \left[ \frac{SSW_b}{1 - \rho_1} + \sum \frac{b_1 y_i z_{ib}^2}{w_i} \right] - \frac{\rho_{12}}{2\sigma_1 \sigma_2} \sum \frac{b_1 s_i z_{ib} z_{is}}{w_i}, \]

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\[ D_6 = - \frac{N_s}{2 \sigma_2^2} + \frac{1}{2 \sigma_2^4} \left[ \frac{SSW_s}{1 - \rho_2^2} + \sum \frac{s_{11} y_{is}^2}{w_1} \right] \]

\[ - \frac{\rho_{12}}{2 \sigma_1 \sigma_2} \sum \frac{b_{11} s_{11} z_{ib}^2 z_{is}^2}{w_1} \]

\[ D_7 = \frac{1}{2} \left[ \frac{N_s - K}{1 - \rho_2^2} - \sum \frac{(b_{11} - 1) y_{i1}^2}{w_1} \right] \]

\[ - \frac{1}{2 \sigma_2^2} \left[ \frac{SSW_s}{1 - \rho_2^2} - \sum \frac{b_{11} (b_{11} - 1) y_{i1}^2}{w_1} \right] \]

\[ + \frac{\rho_{12}}{2 \sigma_2^2} \sum \frac{b_{11} y_{i1} z_{i1}^2 z_{ib}^2}{w_1} - \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_{11} y_{i1} z_{ib} z_{is}^2}{w_1} \]

\[ D_8 = \frac{1}{2} \left[ \frac{N_s - K}{1 - \rho_2^2} - \sum \frac{(s_{11} - 1) y_{i1}^2}{w_1} \right] \]

\[ - \frac{1}{2 \sigma_2^2} \left[ \frac{SSW_s}{1 - \rho_2^2} - \sum \frac{s_{11} (s_{11} - 1) y_{i1}^2}{w_1} \right] \]

\[ + \frac{\rho_{12}}{2 \sigma_1 \sigma_2} \sum \frac{b_{11} s_{11} y_{i1} z_{i1}^2}{w_1} - \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_{11} s_{11} y_{i1} z_{ib} z_{is}^2}{w_1} \]

and

\[ D_9 = \rho_{12} \sum \frac{b_{11} y_{i1}}{w_1} - \frac{\rho_{12}}{\sigma_2^2} \sum \frac{b_{11} y_{i1} z_{i1}^2}{w_1} - \frac{\rho_{12}}{\sigma_2^2} \sum \frac{b_{11} y_{i1} z_{ib} z_{is}^2}{w_1} \]

\[ + \frac{1}{\sigma_1 \sigma_2} \left[ 2 \rho_{12} \sum \frac{b_{11} s_{11} y_{i1} z_{i1}^2}{w_1} + \sum \frac{b_{11} y_{i1} z_{ib} z_{is}^2}{w_1} \right] \]  

(6.29)

(All summations are over \( i = 1, 2, \ldots, K \))

where

\[ z_{ib} = \langle \tilde{y}_{ib} - \beta_{0b} - \beta_{1b} x_i \rangle, \quad i = 1, 2, \ldots, K, \]

\[ z_{is} = \langle \tilde{y}_{is} - \beta_{0s} - \beta_{1s} x_i \rangle, \quad i = 1, 2, \ldots, K, \]

\( w_1 \)'s are defined by (6.19), and \( u_1 \)'s and \( v_1 \)'s are defined by
Furthermore, let us denote

\[ D_{pq} = D_{qp} = \frac{\partial^2 l_2}{\partial \theta_p \partial \theta_q}, \quad p, q = 1, 2, \ldots, 8, \]  

(6.31)

where \( \theta_1 = \beta_0 \), \( \theta_2 = \beta_0 \), \( \theta_3 = \beta_1 \), \( \theta_4 = \beta_1 \), \( \theta_5 = \sigma_1 \), \( \theta_6 = \sigma_2 \), \( \theta_7 = \rho_1 \), \( \theta_8 = \rho_2 \), \( \theta_9 = \rho_{12} \). The second order partial derivatives of \( l_2 \) (6.22) are given by

\[ D_{11} = -\frac{1}{\sigma_1^2} \sum b_1 v_1 \frac{w_1}{w_1}, \quad D_{12} = \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum b_{1s_1} \frac{v_1}{w_1}, \]

\[ D_{13} = -\frac{1}{\sigma_1^2} \sum b_1 v_1 x_1 \frac{w_1}{w_1}, \quad D_{14} = \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum b_{1s_1} x_1 \frac{w_1}{w_1}, \]

\[ D_{15} = -\frac{1}{\sigma_1^4} \sum b_1 v_1 z_{1b} \frac{w_1}{w_1} + \frac{\rho_{12}}{2 \sigma_1^3 \sigma_2} \sum b_{1s_1} z_{1s} \frac{w_1}{w_1}, \]

\[ D_{16} = \frac{\rho_{12}}{2 \sigma_1^3 \sigma_2} \sum b_{1s_1} z_{1s} \frac{w_1}{w_1}, \]

\[ D_{17} = -\frac{1}{\sigma_1^2} \sum \frac{b_1 (c_1 - 1) v_1^2 z_{1b}}{w_1^2} + \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_{1s_1} (c_1 - 1) v_1 z_{1s}}{w_1^2}, \]

\[ D_{18} = -\frac{\rho_{12}^2}{\sigma_1^2} \sum \frac{b_{1s_1} (c_1 - 1) z_{1b}}{w_1^2} + \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_{1s_1} (c_1 - 1) u_1 z_{1s}}{w_1^2}, \]

\[ D_{19} = \frac{2 \rho_{12}}{\sigma_1^2} \sum \frac{b_{1s_1} v_1 z_{1b}}{w_1^2} - \frac{1}{\sigma_1 \sigma_2} \left[ \frac{2 \rho_{12}^2}{w_1} \sum \frac{b_{1s_1} z_{1s}}{w_1^2} + \sum \frac{b_{1s_1} z_{1s}}{w_1} \right], \]

\[ D_{22} = -\frac{1}{\sigma_2^2} \sum \frac{s_1 u_1}{w_1}, \quad D_{23} = \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_{1s_1} x_1}{w_1}, \]

\[ D_{24} = -\frac{1}{\sigma_2^2} \sum \frac{s_1 u_1 x_1}{w_1}, \quad D_{25} = \frac{\rho_{12}}{2 \sigma_1 \sigma_2} \sum \frac{b_{1s_1} z_{1b}}{w_1}. \]
\[ D_{28} = -\frac{1}{\sigma_2^4} \sum \frac{s_i y_i z_{1s}}{w_1} + \frac{\rho_{12}}{2 \sigma_1 \sigma_2^3} \sum \frac{b_{1s}^{2}z_{1b}}{w_1} , \]

\[ D_{27} = -\frac{\rho_{12}^2}{\sigma_2^2} \sum \frac{b_{1s}^{2}(b_{1-1})y_{1s}}{w_1^2} + \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_{1s}(b_{1-1}y_{1b})}{w_1} , \]

\[ D_{28} = -\frac{1}{\sigma_2^2} \sum \frac{s_i (s_i - 1)y_i z_{1s}}{w_1} + \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_{1s}(s_i - 1)y_{1b}}{w_1} , \]

\[ D_{29} = \frac{2\rho_{12}}{\sigma_2^2} \sum \frac{b_{1s}^{2}y_i z_{1s}}{w_1^2} - \frac{1}{\sigma_1 \sigma_2} \left[ 2\rho_{12}^2 \sum \frac{b_{1s}^{2}z_{1b}}{w_1^2} \right. \]

\[ \left. + \sum \frac{b_{1s}^{2}z_{1b}}{w_1} \right] , \]

\[ D_{33} = -\frac{1}{\sigma_1^3} \sum \frac{b_{1s} x_i z_{1s}}{w_1} , \quad D_{34} = \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_{1s} x_i z_{1s}}{w_1} , \]

\[ D_{35} = -\frac{1}{\sigma_1^4} \sum \frac{b_{1s} x_i z_{1b}}{w_1} + \frac{\rho_{12}}{2 \sigma_1 \sigma_2^3} \sum \frac{b_{1s} x_i z_{1s}}{w_1} , \]

\[ D_{36} = \frac{\rho_{12}}{2 \sigma_1 \sigma_2^3} \sum \frac{b_{1s} x_i z_{1s}}{w_1} , \]

\[ D_{37} = -\frac{1}{\sigma_1^2} \sum \frac{b_{1s}(b_{1-1})y_{1s} x_{1b}}{w_1^2} + \frac{\rho_{12}}{\sigma_1 \sigma_2^3} \sum \frac{b_{1s}(b_{1-1}y_{1s} x_{1s})}{w_1} , \]

\[ D_{38} = -\frac{\rho_{12}^2}{\sigma_1^2} \sum \frac{b_{1s}^{2}(s_i - 1)x_{1b}}{w_1^2} + \frac{\rho_{12}}{\sigma_1 \sigma_2^3} \sum \frac{b_{1s}^{2}(s_i - 1)y_{1s} x_{1s}}{w_1} , \]

\[ D_{39} = \frac{2\rho_{12}}{\sigma_1^2} \sum \frac{b_{1s}^{2}x_{1s} z_{1b}}{w_1^2} - \frac{1}{\sigma_1 \sigma_2} \left[ 2\rho_{12}^2 \sum \frac{b_{1s}^{2}x_{1s} z_{1s}}{w_1^2} \right. \]

\[ \left. + \sum \frac{b_{1s}^{2}x_{1s} z_{1s}}{w_1} \right] , \]

\[ D_{44} = -\frac{1}{\sigma_2^2} \sum \frac{s_i y_i x_i}{w_1} , \quad D_{45} = \frac{\rho_{12}}{2 \sigma_1 \sigma_2^3} \sum \frac{b_{1s} x_i z_{1b}}{w_1} . \]
\[ D_{46} = -\frac{1}{\sigma_2^4} \sum \frac{s_i u_i x_i z_{is}}{w_i} + \frac{\rho_{12}}{2 \sigma_1 \sigma_2^3} \sum \frac{b_1 s_i x_i z_{ib}}{w_i} , \]

\[ D_{47} = -\frac{\rho_{12}}{\sigma_2^2} \sum \frac{b_1 s_i (b_i - 1) x_i z_{is}}{w_i} + \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_1 s_i (b_i - 1) y_i x_i z_{ib}}{w_i} , \]

\[ D_{48} = -\frac{1}{\sigma_2^2} \sum \frac{s_i (s_i - 1) u_i^2 x_i z_{is}}{w_i^2} + \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_1 s_i (s_i - 1) y_i x_i z_{ib}}{w_i^2} , \]

\[ D_{49} = \frac{2 \rho_{12}}{\sigma_2^2} \sum \frac{b_1 s_i^2 u_i x_i z_{is}}{w_i^2} \cdot \frac{1}{\sigma_1 \sigma_2} \left[ 2 \rho_{12} \sum \frac{b_1^2 s_i^2 y_i z_{ib}}{w_i^2} \right. \]

\[ \left. + \sum \frac{b_1 s_i x_i z_{ib}}{w_i} \right] , \]

\[ D_{55} = \frac{N_b}{2 \sigma_1^4} - \frac{1}{\sigma_1^6} \left[ \frac{SSW_b}{1 - \rho_1} + \sum \frac{b_1 y_i z_{ib}^2}{w_i} \right] + \frac{3 \rho_{12}}{4 \sigma_1^3 \sigma_2} \sum \frac{b_1 s_i z_{ib} z_{is}}{w_i} , \]

\[ D_{56} = \frac{\rho_{12}}{4 \sigma_1^3 \sigma_2} \sum \frac{b_1 s_i z_{ib} z_{is}}{w_i} , \]

\[ D_{57} = \frac{1}{2 \sigma_1^4} \left[ \frac{SSW_b}{(1 - \rho_1)^2} - \sum \frac{b_1 (b_i - 1) y_i^2 z_{ib}^2}{w_i^2} \right] \]

\[ + \frac{\rho_{12}}{2 \sigma_1 \sigma_2} \sum \frac{b_1 s_i (b_i - 1) y_i z_{ib} z_{is}}{w_i^2} , \]

\[ D_{58} = -\frac{\rho_{12}}{2 \sigma_1^4} \sum \frac{b_1^2 s_i (s_i - 1) z_{ib}^2}{w_i^2} \]

\[ + \frac{\rho_{12}}{2 \sigma_1^3 \sigma_2} \sum \frac{b_1 s_i (s_i - 1) y_i z_{ib} z_{is}}{w_i} , \]

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\[ D_{68} = \frac{\rho_{12}}{\sigma_1^4} \sum \frac{b_1s_1y_1^{2}z_{1b}}{w_1^2} - \frac{\rho_2}{\sigma_1^3 \sigma_2} \sum \frac{b_1s_1^2z_{1b}^{2}z_{1s}}{w_1^2} - \frac{1}{2\sigma_1^3 \sigma_2} \sum \frac{b_1s_1z_{1b}z_{1s}}{w_1} \]

\[ D_{66} = \frac{N_s}{2\sigma_2^4} \left[ \frac{SSW_s}{1-\rho_2} + \sum \frac{s_1y_1z_{1s}^2}{w_1} \right] + \frac{3\rho_{12}}{4\sigma_1 \sigma_2} \sum \frac{b_1s_1z_{1b}z_{1s}}{w_1} \]

\[ D_{67} = -\frac{\rho_{12}}{\sigma_2^4} \sum \frac{b_1s_1(b_1-1)y_1z_{1s}^{2}}{w_1^2} + \frac{\rho_{12}}{2\sigma_1 \sigma_2} \sum \frac{b_1s_1(b_1-1)y_1z_{1b}z_{1s}}{w_1^2} \]

\[ D_{68} = \frac{1}{2\sigma_2^4} \left[ \frac{SSW_s}{(1-\rho_2)^2} - \sum \frac{s_1(s_1-1)y_1z_{1s}^{2}}{w_1^2} \right] + \frac{\rho_{12}}{2\sigma_1 \sigma_2} \sum \frac{b_1s_1(s_1-1)y_1z_{1b}z_{1s}}{w_1^2} \]

\[ D_{69} = \frac{\rho_{12}}{\sigma_2^4} \sum \frac{b_1s_1^2y_1^{2}z_{1s}}{w_1^2} - \frac{\rho_2}{\sigma_1^3 \sigma_2} \sum \frac{b_1s_1^2z_{1b}z_{1s}}{w_1^2} - \frac{1}{2\sigma_1 \sigma_2} \sum \frac{b_1s_1z_{1b}z_{1s}}{w_1} \]

\[ D_{77} = \frac{1}{2} \left[ \frac{N_b-K}{(1-\rho_1)^3} + \sum \frac{(b_1-1)y_1^2}{w_1^2} \right] - \frac{1}{\sigma_1^3} \left[ \frac{SSW_b}{(1-\rho_1)^3} + \sum \frac{b_1(b_1-1)y_1^2}{w_1^3} \right] - \frac{\rho_{12}}{\sigma_2^2} \sum \frac{b_1s_1^2(b_1-1)y_1z_{1s}^2}{w_1^3} \]

\[ + \frac{2\rho_{12}}{\sigma_1 \sigma_2} \sum \frac{b_1s_1(b_1-1)y_1z_{1b}z_{1s}}{w_1^3} \]
\[
\begin{align*}
D_{78} &= \frac{\rho_{12}^2}{2} \sum_{i} \frac{b_{1}s_1(b_1-1)(s_1-1)}{y_i^2} \\
&\quad - \frac{\rho_{12}^2}{\sigma_1^2} \sum_{i} \frac{b_{1}s_1(b_1-1)(s_1-1)y_i z_{1b}^2}{y_i^3} \\
&\quad - \frac{\rho_{12}^2}{\sigma_2^2} \sum_{i} \frac{b_{1}s_1(b_1-1)(s_1-1)y_i z_{1s}^2}{y_i^3} \\
&\quad + \frac{\rho_{12}}{\sigma_1 \sigma_2} \sum_{i} \frac{b_{1}s_1(b_1-1)(s_1-1)y_i z_{1b} z_{1s}}{y_i^3}.
\end{align*}
\]

\[
\begin{align*}
D_{78} &= -\rho_{12} \sum_{i} \frac{b_{1}s_1(b_1-1)y_i}{y_i^2} + \frac{2\rho_{12}}{\sigma_1^2} \sum_{i} \frac{b_{1}s_1(b_1-1)y_i z_{1b}^2}{y_i^3} \\
&\quad + \frac{\rho_{12}}{\sigma_2^2} \sum_{i} \frac{b_{1}s_1^2(b_1-1)y_i z_{1s} z_{1b}^2}{y_i^3} \\
&\quad - \frac{4\rho_{12}^2}{\sigma_1 \sigma_2} \sum_{i} \frac{b_{1}s_1^2(b_1-1)y_i z_{1b} z_{1s}}{y_i^3} \\
&\quad - \frac{1}{\sigma_1 \sigma_2} \sum_{i} \frac{b_{1}s_1(b_1-1)y_i z_{1b} z_{1s}}{y_i^2}.
\end{align*}
\]

\[
\begin{align*}
D_{88} &= \frac{1}{2} \left[ \frac{N_s - K}{(1-\rho_2^2)^2} + \sum \frac{(s_1-1)^2 y_i^2}{y_i^2} \right] \\
&\quad - \frac{\rho_{12}^2}{\sigma_1^2} \sum_{i} \frac{b_{1}s_1(s_1-1)^2 y_i z_{1b}^2}{y_i^3} \\
&\quad - \frac{1}{\sigma_2^2} \left[ \frac{SSW_s}{(1-\rho_2^3)} + \sum \frac{s_1(s_1-1)^2 y_i z_{1s}^2}{y_i^3} \right] \\
&\quad + \frac{2\rho_{12}}{\sigma_1 \sigma_2} \sum_{i} \frac{b_{1}s_1(s_1-1)^2 y_i z_{1b} z_{1s}}{y_i^3}.
\end{align*}
\]
\[ D_{gg} = - \rho_{12} \sum \frac{b_{i} s_{i}(s_{i}-1) u_{i}}{w_{i}^{2}} + \frac{2 \rho_{12}}{\sigma_{2}} \sum \frac{b_{i} s_{i}^{2}(s_{i}-1) u_{z_{i}^{2}}}{w_{i}^{3}} \]

\[ + \frac{\rho_{12}}{\sigma_{1}^{2}} \sum \frac{b_{i}^{2} s_{i}(s_{i}-1) u_{i} z_{i b} z_{i s}}{w_{i}^{3}} \]

\[ - \frac{4 \rho_{12}^{2}}{\sigma_{1} \sigma_{2}} \sum \frac{b_{i}^{2} s_{i}(s_{i}-1) u_{i} z_{i b} z_{i s}}{w_{i}^{3}} \]

\[- \frac{1}{\sigma_{1} \sigma_{2}} \sum \frac{b_{i} s_{i}(s_{i}-1) u_{i} z_{i b} z_{i s}}{w_{i}^{2}} \]

and

\[ D_{gg} = 2 \rho_{12}^{2} \sum \frac{b_{i}^{2} s_{i}^{2}}{w_{i}^{2}} + \sum \frac{b_{i} s_{i}}{w_{i}^{2}} - \frac{4 \rho_{12}^{2}}{\sigma_{1}^{2}} \sum \frac{b_{i}^{3} s_{i} v_{i} z_{i b}^{2}}{w_{i}^{3}} \]

\[- \frac{1}{\sigma_{1}^{2}} \sum \frac{b_{i}^{2} s_{i} v_{i} z_{i b}^{2}}{w_{i}^{2}} - \frac{4 \rho_{12}^{2}}{\sigma_{2}^{2}} \sum \frac{b_{i}^{3} s_{i} u_{i} z_{i s}^{2}}{w_{i}^{3}} \]

\[- \frac{1}{\sigma_{2}^{2}} \sum \frac{b_{i}^{2} s_{i} u_{i} z_{i s}^{2}}{w_{i}^{2}} + \frac{8 \rho_{12}^{3}}{\sigma_{1} \sigma_{2}^{2}} \sum \frac{b_{i}^{3} s_{i} z_{i b} z_{i s}^{3}}{w_{i}^{3}} \]

\[ + \frac{8 \rho_{12}^{2}}{\sigma_{1} \sigma_{2}^{2}} \sum \frac{b_{i}^{2} s_{i} z_{i b} z_{i s}^{2}}{w_{i}^{2}} \]

(6.32)

where \( z_{i b} \) and \( z_{i s} \) (\( i = 1, 2, \ldots, K \)) are given by (6.30).

(All summations are over \( i = 1, 2, \ldots, K \)).

Before we proceed further, notice that for given \( x_i \)

\[ E(z_{i b}^{2}) = 0, \quad E(z_{i s}^{2}) = 0, \quad i = 1, 2, \ldots, K, \]

\[ E(z_{i b}^{2}) = (1+c_{i} b_{i}-1) \rho_{1} \sigma_{1}^{2} b_{i}, \quad i = 1, 2, \ldots, K, \]

\[ E(z_{i s}^{2}) = (1+c_{i} s_{i}-1) \rho_{2} \sigma_{2}^{2} s_{i}, \quad i = 1, 2, \ldots, K, \]
\[ E(Z_{ib}Z_{is}) = \rho_{12} \sigma_1 \sigma_2, \quad i = 1, 2, \ldots, K, \]

\[ E(SSW_b) = (N_b - K)(1 - \rho_1^2) \sigma_1^2 \]

and

\[ E(SSW_s) = (N_s - K)(1 - \rho_2^2) \sigma_2^2. \]

Denoting the \((p, q)\)th element of matrix \( E(q) \) (6.27) by \( e_{pq} \),

the elements of Fisher's information matrix are

\[ e_{pq} = e_{qp} = -E(D_{pq}), \quad p, q = 1, 2, \ldots, 9. \]

Thus by using (6.32) and (6.33), we get

\[ e_{ij} = 0, \quad i = 1, 2, 3, 4; \quad j = 5, 6, \ldots, 9. \]

\[ e_{11} = \frac{1}{\sigma_1^2} M_2, \quad e_{12} = -\frac{\rho_{12}}{\sigma_1 \sigma_2} M_1, \quad e_{13} = \frac{1}{\sigma_1^2} M_6, \]

\[ e_{14} = -\frac{\rho_{12}}{\sigma_1 \sigma_2} M_4, \quad e_{22} = \frac{1}{\sigma_2^2} M_3, \quad e_{23} = -\frac{\rho_{12}}{\sigma_1 \sigma_2} M_4, \]

\[ e_{24} = \frac{1}{\sigma_2^2} M_7, \quad e_{33} = \frac{1}{\sigma_1^2} M_8, \quad e_{34} = -\frac{\rho_{12}}{\sigma_1 \sigma_2} M_5, \]

\[ e_{44} = \frac{1}{\sigma_2^2} M_9, \quad e_{55} = \frac{1}{4\sigma_1^4} \left[ 2N_b + M_1\rho_{12}^2 \right], \]

\[ e_{56} = -\frac{\rho_{12}^2}{4\sigma_1^2 \sigma_2^2} M_1, \quad e_{57} = -\frac{1}{2\sigma_1^2} \left[ \frac{N_b - K}{1 - \rho_1} - \sum \frac{(b_i - 1)v_i}{w_i} \right], \]

\[ e_{58} = 0, \quad e_{59} = -\frac{\rho_{12}}{2\sigma_1^2} M_1, \]

\[ e_{66} = \frac{1}{4\sigma_2^4} \left[ 2N_s + M_1\rho_{12}^2 \right], \quad e_{67} = 0. \]
\[ e_{68} = -\frac{1}{2\sigma_2^2} \left[ \frac{N_s - K}{1 - \rho_2} - \sum \frac{(s_i - 1) y_i}{y_i^2} \right], \quad e_{69} = -\frac{\rho_{12}}{2\sigma_2^2} M_1, \]

\[ e_{77} = \frac{1}{2} \left[ \frac{N_s - K}{(1 - \rho_1)^2} + \sum \frac{(b_i - 1)^2 y_i^2}{y_i^2} \right], \]

\[ e_{78} = \frac{\rho_{12}^2}{2} \sum \frac{b_i s_i (b_i - 1) (s_i - 1)}{y_i^2}, \]

\[ e_{79} = -\rho_{12} \sum \frac{b_i s_i (b_i - 1) y_i}{y_i^2}, \]

\[ e_{88} = \frac{1}{2} \left[ \frac{N_s - K}{(1 - \rho_2)^2} + \sum \frac{(s_i - 1)^2 y_i^2}{y_i^2} \right], \]

\[ e_{89} = -\rho_{12} \sum \frac{b_i s_i (s_i - 1) y_i}{y_i^2} \]

and

\[ e_{99} = \sum \frac{b_i s_i [y_i y_i + b_i s_i \rho_{12}^2]}{y_i^2}, \]

where \( M_i, i = 1, 2, \ldots, 17 \) are given by (6.24).

(All summations are over \( i = 1, 2, \ldots, K \)).

Therefore,

\[ E(\omega) = \begin{bmatrix} E_{11} & 0 \\ 0 & E_{22} \end{bmatrix} \quad \text{and} \quad E^{-1}(\omega) = \begin{bmatrix} E^{-1}_{11} & 0 \\ 0 & E^{-1}_{22} \end{bmatrix}, \quad (6.34) \]

where

\[ E_{11}(\omega) = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix} \quad (6.35) \]

and
We have the following theorems:

**Theorem 6.1.** Conditional on $n_1$, as $K \rightarrow \infty$, the $(9 \times d)$ random vector $\hat{\mathbf{g}} = [\hat{\beta}_0, \hat{\beta}_0 s, \hat{\beta}_1 b, \hat{\beta}_1 s, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_{12}]^T$ is asymptotically distributed as multivariate normal with mean vector $\mathbf{g} = [\beta_0, \beta_0 s, \beta_1 b, \beta_1 s, \sigma_1, \sigma_2, \rho_1, \rho_2, \rho_{12}]^T$ and dispersion matrix $\Sigma^{-1}(\mathbf{g})$, where $\Sigma(\mathbf{g})$ is given by (6.34).

**Theorem 6.2.** Conditional on $n_1$, as $K \rightarrow \infty$, the $(4 \times d)$ random vector $\hat{\mathbf{g}}^{(1)} = [\hat{\beta}_0, \hat{\beta}_0 s, \hat{\beta}_1 b, \hat{\beta}_1 s]^T$ is asymptotically distributed as multivariate normal with mean vector $\mathbf{g}^{(1)} = [\beta_0, \beta_0 s, \beta_1 b, \beta_1 s]^T$ and dispersion matrix $\Sigma_{11}^{-1}(\mathbf{g})$, where $\Sigma_{11}(\mathbf{g})$ is given by (6.35).

**Theorem 6.3.** Conditional on $n_1$, as $K \rightarrow \infty$, the $(5 \times d)$ random vector $\hat{\mathbf{g}}^{(2)} = [\hat{\sigma}_1, \hat{\sigma}_2, \hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_{12}]^T$ is asymptotically distributed as multivariate normal with mean vector $\mathbf{g}^{(2)} = [\sigma_1, \sigma_2, \rho_1, \rho_2, \rho_{12}]^T$ and dispersion matrix $\Sigma_{22}^{-1}(\mathbf{g})$, where $\Sigma_{22}(\mathbf{g})$ is given by (6.36).

Furthermore, since $\Sigma_{12}(\mathbf{g}) = \Sigma_{21}(\mathbf{g}) = 0$, the random vectors $\hat{\mathbf{g}}^{(1)} = [\hat{\beta}_0, \hat{\beta}_0 s, \hat{\beta}_1 b, \hat{\beta}_1 s]^T$ and $\hat{\mathbf{g}}^{(2)} = [\hat{\sigma}_1, \hat{\sigma}_2, \hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_{12}]^T$ are distributed independently by Theorem 1.9. It is also well known that the $\hat{\gamma} = [\hat{\mu}_m, \hat{\sigma}_m]^T$ is asymptotically
distributed as bivariate normal such that

\[ E(\hat{\mu}_m) = \mu_m, \quad E(\hat{\sigma}_m^2) = \sigma_m^2, \quad \text{Var}(\hat{\mu}_m) = \frac{\sigma_m^2}{K}, \]

\[ \text{Var}(\hat{\sigma}_m^2) \approx \frac{2\sigma_m^4}{K} \quad \text{and} \quad \text{Cov}(\hat{\mu}_m, \hat{\sigma}_m^2) = 0. \]

Using the delta method (see Section 1.4), the asymptotic variances of \( \hat{\mu}_b, \hat{\mu}_s, \hat{\sigma}_b, \hat{\sigma}_s, \rho_b, \hat{\rho}_b, \hat{\rho}_bs, \hat{\rho}_{mb}, \) and \( \hat{\rho}_{ms} \), to the first order of approximation, are respectively given by:

\[ \text{Var}(\hat{\mu}_b) \approx \text{Var}(\hat{\beta}_{0b}) + \mu_m^2 \text{Var}(\hat{\beta}_{1b}) + \beta_{1b}^2 \text{Var}(\hat{\mu}_m) \]

\[ + 2\mu_m \text{Cov}(\hat{\beta}_{0b}, \hat{\beta}_{1b}). \]

\[ \text{Var}(\hat{\mu}_s) \approx \text{Var}(\hat{\beta}_{0s}) + \mu_m^2 \text{Var}(\hat{\beta}_{1s}) + \beta_{1s}^2 \text{Var}(\hat{\mu}_m) \]

\[ + 2\mu_m \text{Cov}(\hat{\beta}_{0s}, \hat{\beta}_{1s}). \]

\[ \text{Var}(\hat{\sigma}_b^2) \approx 4\beta_{1b}^2 \sigma_m^4 \text{Var}(\hat{\beta}_{1b}) + \beta_{1b}^4 \text{Var}(\hat{\sigma}_m^2) + \text{Var}(\sigma_1^2). \]

\[ \text{Var}(\hat{\sigma}_s^2) \approx 4\beta_{1s}^2 \sigma_m^4 \text{Var}(\hat{\beta}_{1s}) + \beta_{1s}^4 \text{Var}(\hat{\sigma}_m^2) + \text{Var}(\sigma_2^2). \]

\[ \text{Var}(\hat{\rho}_b) \approx \frac{\beta_{1b}(1-\rho_1)^2}{\sigma_1^4 \sigma_2^4} \cdot \left\{ 4\sigma_m^4 \text{Var}(\hat{\beta}_{1b}) + \beta_{1b}^2 \sigma_m^4 \text{Var}(\hat{\sigma}_1^2) \right\} \]

\[ + \beta_{1b}^2 \sigma_1^4 \text{Var}(\hat{\sigma}_m^2) \] + \frac{\sigma_1^2}{\sigma_2^4} \left\{ \sigma_1^2 \text{Var}(\hat{\rho}_1) \right\} \]

\[ - \frac{2\beta_{1b}^2 \sigma_2^2 (1-\rho_1)}{\sigma_1} \text{Cov}(\hat{\sigma}_1^2, \hat{\rho}_1) \} , \]
\[
\text{Var}(\hat{\rho}_s) \equiv \frac{\beta_{1s}^2 (1-\rho_2)^2}{\sigma_m^4 \sqrt{\sigma_1^2 \text{Var}(\hat{\rho}_{1s}) + \sigma_m^4 \text{Var}(\hat{\rho}_2)}} \\
+ \beta_{1s}^2 \sigma_m^2 \text{Var}(\hat{\rho}_m) \} + \frac{\sigma_2^2}{\sigma_2^2} \left\{ \frac{\sigma_2^2}{2} \text{Var}(\rho_2) \right. \\
- \frac{2\beta_{1s}^2 \sigma_m^2 (1-\rho_2)}{\sigma_2^2} \text{Cov}(\hat{\rho}_2, \hat{\rho}_2) \right\},
\]

\[
\text{Var}(\hat{\rho}_{bs}) \equiv \frac{H_{1s}^2}{\sigma_m^2} \left\{ 2 \sigma_1^2 \text{Var}(\hat{\rho}_{1b}) + \frac{\beta_{1b}^2}{2\sigma_1^2} \text{Var}(\hat{\rho}_1^2) \right\} \\
+ \frac{H_{1s}^2}{\sigma_{g_2}^2} \left\{ \sigma_2^2 \text{Var}(\hat{\rho}_{1s}) + \frac{\beta_{1s}^2}{2\sigma_2^2} \text{Var}(\hat{\rho}_2) \right\} \\
+ \frac{1}{2\sigma_1^2 \sigma_2^2} \left\{ \frac{\beta_{1b}^2 H_{1s}^2}{\sigma_1^2} + \frac{\beta_{1s}^2 H_{2s}^2}{\sigma_2^2} \right\} \text{Var}(\hat{\rho}_m) \\
+ \frac{H_{1s}^2 \sigma_m^4}{\sigma_{g_2}^2} \left\{ 2\sigma_1^2 \sigma_2 \text{Cov}(\hat{\rho}_{1b}, \hat{\rho}_{1s}) \right. \\
\left. + \frac{\beta_{1b} \beta_{1s}}{2\sigma_1^2} \text{Cov}(\hat{\rho}_1^2, \hat{\rho}_2) \right\},
\]

\[
\text{Var}(\hat{\rho}_{mb}) \equiv \frac{1}{\sigma_1^2} \left\{ \sigma_m^4 \text{Var}(\hat{\rho}_{1b}) + \frac{\beta_{1b}^2 \sigma_m^2}{4} \text{Var}(\hat{\rho}_1^2) \\
+ \frac{\beta_{1b}^2 \sigma_1^2}{4\sigma_m^2} \text{Cov}(\hat{\rho}_1^2, \hat{\rho}_2) \right\},
\]

and
\[ \text{Var}(\hat{\rho}_{ms}) \approx \frac{1}{6} \left\{ \frac{\beta^2}{4 \sigma_m^2} \text{Var}(\hat{\rho}_{1s}) + \frac{\beta^2 \sigma_m^2}{4} \text{Var}(\rho^2) + \frac{\beta^2 \sigma_m^4}{4 \sigma_m^2} \text{Cov}(\rho^2, \rho_m^2) \right\}. \]

where

\[ G_1 = \beta_1 \sigma_m^2 + \sigma_1^2 \quad G_2 = \beta_1 \sigma_m^2 + \sigma_2^2 \]

\[ H_1 = \beta_1 \sigma_1^2 - \beta_1 \rho_{12} \sigma_2 \quad \text{and} \quad H_2 = \beta_1 \rho_{12} - \beta_1 \rho_{12} \sigma_1. \]

Therefore, the asymptotic distributions of \( \hat{\mu}_b, \hat{\mu}_s, \hat{\sigma}_b, \hat{\sigma}_s, \hat{\rho}_b, \hat{\rho}_s \) and \( \hat{\rho}_{bs} \) are normal with respective means \( \mu_b, \mu_s, \sigma_b, \sigma_s, \rho_b, \rho_s \) and \( \rho_{bs} \) and variances given by (6.37). Furthermore, the asymptotic covariances between \( \hat{\rho}_b \) and \( \hat{\rho}_s \), and between \( \hat{\rho}_{mb} \) and \( \hat{\rho}_{ms} \), by delta method, are given by

\[ \text{Cov}(\hat{\rho}_b, \hat{\rho}_s) \approx \frac{\beta_1 \rho_{12} (1-\rho_1) (1-\rho_2)}{G_1 G_2} \left\{ \frac{\beta_1 \rho_{12}}{G_1} \text{Var}(\rho^2) + 4 \sigma_m^2 \text{Cov}(\hat{\rho}_b, \hat{\rho}_s) \right\} + \sigma_1^2 \sigma_2^2 \left[ \frac{\beta_1 \rho_{12}}{G_1} \text{Var}(\rho^2) + 4 \sigma_m^2 \text{Cov}(\hat{\rho}_b, \hat{\rho}_s) \right] \]

\[ - \frac{1}{G_1 G_2} \left\{ \frac{\beta_1 \rho_{12} \sigma_m^2 (1-\rho_1)}{G_1} \text{Cov}(\rho^2, \rho_1) - \sigma_1^2 \sigma_2^2 \text{Cov}(\rho_1, \rho_2) \right\} \]

and

\[ \text{Cov}(\hat{\rho}_{mb}, \hat{\rho}_{ms}) \approx \frac{1}{G_1 G_2 G_3} \left\{ \frac{\beta_1 \rho_{12}}{G_1} \text{Var}(\rho^2) + 4 \sigma_m^2 \text{Cov}(\hat{\rho}_{mb}, \hat{\rho}_{ms}) \right\} + \sigma_m^2 \text{Cov}(\hat{\rho}_{mb}, \hat{\rho}_{ms}) \]

\[ + \frac{\beta_1 \rho_{12} \sigma_m^2}{4} \text{Cov}(\rho^2, \rho_2) \].

where \( G_1 \) and \( G_2 \) are given by (6.38).
6.5 EXAMPLE

The methodology is now illustrated by using the arterial blood pressure data of Miall and Oldham (1955). The data set is described in section 5.5 and have been analyzed before by Miall and Oldham (1955), Donner and Koval (1981) and Keen (1987). Among 215 families, selected in section 5.5, which contain complete information on brothers and sisters, only 165 have information on their mother's blood pressures. We used these 165 families for this analysis to estimate familial parameters and test the significance of familial correlations. In order to allow for differences in age and sex, once again the arterial blood pressures are adjusted by Z-score transformation \( Z = (y - \mu_y) / \sigma_y \), where \( \mu_y \) and \( \sigma_y \) refers to the age and sex specific mean and standard deviation, respectively, in the age groups \( \leq 10 \), 11-20, 21-35, 36-50, 51-65 and over 65. Miall and Oldham (1955), and Keen (1987) used the method of polynomial regression to adjust the blood pressure scores by age and sex. Donner and Koval (1981) used the Z-score method but they categorized individuals into age groups irrespective of their sex and their analysis was only for systolic blood pressure.

The maximum likelihood estimates of the parameters of model (6.4) were obtained by using IMSL (1987) subroutine BCONF from (MATH/LIBRARY). This subroutine uses a quasi-Newton method to minimize a function of several variables, subject to user supplied bounds on the variables.

The criterion chosen to stop the cycle of iteration was
that the Euclidean distance between the estimated vectors in
the last two steps be less than $10^{-5}$. The given estimates
are obtained by maximizing $L$ (6.25) under the restriction
(6.11). The estimates of familial parameters are obtained
by using the relationships given by (6.26). The estimates
of the model and the familial parameters are presented in
Table 6.1. The given standard errors of the estimates are
obtained by taking the square roots of the estimated
variances of the maximum likelihood estimators which were
obtained by replacing parameters in the results of section
6.4 by their corresponding maximum likelihood estimates.
These estimates along with their standard errors are needed
for testing hypotheses regarding familial correlations which
will be discussed in the following section. These estimates
in Table 6.1 show a slight disagreement with the estimates
obtained by Keen (1987). This is because of using a
different method for adjusting blood pressure scores. Even
though Donner and Koval (1981) did not take into account the
sex differences in the adjustment of data, their obtained
estimates are fairly close to our estimates.

6.6 Tests of Significance

Based on the asymptotic theory of maximum likelihood
estimators as discussed in Section 1.6, the procedures for
testing the significance of familial correlations are
presented here. From (1.19) using Wald's (1943) criteria
(see Section 1.6) an appropriate test statistic to test
$H_0: \rho_1 = 0$ is given by
<table>
<thead>
<tr>
<th>Parameter</th>
<th>SYSTOLIC</th>
<th></th>
<th>DIASTOLIC</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{ob}$</td>
<td>-0.0301</td>
<td>0.0635</td>
<td>-0.0025</td>
<td>0.0653</td>
</tr>
<tr>
<td>$\beta_{os}$</td>
<td>-0.0732</td>
<td>0.0712</td>
<td>-0.1051</td>
<td>0.0688</td>
</tr>
<tr>
<td>$\beta_{1b}$</td>
<td>0.2277</td>
<td>0.0576</td>
<td>0.1513</td>
<td>0.0618</td>
</tr>
<tr>
<td>$\beta_{1s}$</td>
<td>0.1565</td>
<td>0.0710</td>
<td>0.2326</td>
<td>0.0745</td>
</tr>
<tr>
<td>$\sigma_1^2$</td>
<td>0.9255</td>
<td>0.0822</td>
<td>0.9547</td>
<td>0.0852</td>
</tr>
<tr>
<td>$\sigma_2^2$</td>
<td>0.8977</td>
<td>0.0862</td>
<td>0.8838</td>
<td>0.0827</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>0.0962</td>
<td>0.0798</td>
<td>0.1423</td>
<td>0.0805</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>0.3578</td>
<td>0.0760</td>
<td>0.3068</td>
<td>0.0777</td>
</tr>
<tr>
<td>$\rho_{12}$</td>
<td>0.1559</td>
<td>0.0612</td>
<td>0.2106</td>
<td>0.0583</td>
</tr>
<tr>
<td>Familial</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_m$</td>
<td>0.0880</td>
<td>0.0811</td>
<td>0.1000</td>
<td>0.0805</td>
</tr>
<tr>
<td>$\sigma_m^2$</td>
<td>1.0659</td>
<td>0.1196</td>
<td>1.0581</td>
<td>0.1176</td>
</tr>
<tr>
<td>$\mu_b$</td>
<td>-0.0106</td>
<td>0.0858</td>
<td>0.0126</td>
<td>0.0683</td>
</tr>
<tr>
<td>$\mu_s$</td>
<td>-0.0597</td>
<td>0.0721</td>
<td>-0.0819</td>
<td>0.0708</td>
</tr>
<tr>
<td>$\sigma_b^2$</td>
<td>0.9827</td>
<td>0.0872</td>
<td>0.9791</td>
<td>0.0878</td>
</tr>
<tr>
<td>$\sigma_s^2$</td>
<td>0.9243</td>
<td>0.0866</td>
<td>0.9416</td>
<td>0.0809</td>
</tr>
<tr>
<td>$\rho_b$</td>
<td>0.1480</td>
<td>0.0790</td>
<td>0.1377</td>
<td>0.0801</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>0.3860</td>
<td>0.0749</td>
<td>0.3493</td>
<td>0.0763</td>
</tr>
<tr>
<td>$\rho_{bs}$</td>
<td>0.1898</td>
<td>0.0612</td>
<td>0.2406</td>
<td>0.0580</td>
</tr>
<tr>
<td>$\rho_{mb}$</td>
<td>0.2393</td>
<td>0.0593</td>
<td>0.1580</td>
<td>0.0639</td>
</tr>
<tr>
<td>$\rho_{ms}$</td>
<td>0.1896</td>
<td>0.0757</td>
<td>0.2477</td>
<td>0.0763</td>
</tr>
<tr>
<td>$\rho_b - \rho_s$</td>
<td>-0.2390</td>
<td>0.1068</td>
<td>-0.1858</td>
<td>0.1070</td>
</tr>
<tr>
<td>$\rho_{mb} - \rho_{ms}$</td>
<td>0.0697</td>
<td>0.0885</td>
<td>-0.0897</td>
<td>0.0885</td>
</tr>
</tbody>
</table>
\[
Z_i = \frac{\hat{\rho}_i}{\text{S.E.}(\hat{\rho}_i)}, \quad (i = b, s, bs, mb, ms),
\]

where \(\text{S.E.}(\hat{\rho}_i) = [\text{Var}(\hat{\rho}_i)]^{1/2}\). The \(Z_i\)'s are asymptotically normally distributed random variables with zero means and unit variances under \(H_0\). Thus an asymptotic test of size \(\alpha\) is to reject \(H_0: \rho_i = 0\) in favour of \(H_1: \rho_i > 0\) if \(|Z_i| > Z_{1-\alpha}\), where \(Z_{1-\alpha}\) is the \(100(1-\alpha)\) percentile point of the standard normal distribution. Other hypotheses of interest are to test \(H_0: \rho_b = \rho_s\) against \(H_1: \rho_b \neq \rho_s\), and \(H_0: \rho_{mb} = \rho_{ms}\) against \(H_1: \rho_{mb} \neq \rho_{ms}\). These hypotheses can be tested by using the test statistics

\[
Z = \frac{\hat{\theta}}{\text{S.E.}(\hat{\theta})},
\]

where \(\hat{\theta} = \rho_b - \rho_s\) or \(\rho_{mb} - \rho_{ms}\) which, under \(H_0\), also has an asymptotic normal distribution with zero mean and unit variance. Thus an appropriate test of size \(\alpha\) is to reject \(H_0\) if \(|Z| > Z_{1-\alpha/2}\). Table 6.2 shows the results for testing several hypotheses regarding familial correlations. All the correlations are significantly greater than zero which indicates the possibility of a strong familial aggregation for both systolic and diastolic blood pressures. The hypothesis \(H_0: \rho_{mb} = \rho_{ms}\) is not rejected which may indicate a balanced maternal effect on the inherited characters (systolic and diastolic blood pressures) for both types of offspring. Furthermore, the hypothesis \(H_0: \rho_b = \rho_s\) is rejected for systolic blood pressure, but there is not enough evidence to reject it for diastolic blood pressure.
TABLE 6.2

Tests regarding familial correlations using systolic and diastolic blood pressures.

<table>
<thead>
<tr>
<th>NULL HYPOTHESIS</th>
<th>ALTERNATE HYPOTHESIS</th>
<th>Z-VALUES</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0: \rho_b = 0$</td>
<td>$H_1: \rho_b &gt; 0$</td>
<td>1.8734*</td>
<td>2.0437*</td>
<td></td>
</tr>
<tr>
<td>$H_0: \rho_s = 0$</td>
<td>$H_1: \rho_s &gt; 0$</td>
<td>5.1539**</td>
<td>4.5780**</td>
<td></td>
</tr>
<tr>
<td>$H_0: \rho_{bs} = 0$</td>
<td>$H_1: \rho_{bs} &gt; 0$</td>
<td>3.1013**</td>
<td>4.1453**</td>
<td></td>
</tr>
<tr>
<td>$H_0: \rho_{mb} = 0$</td>
<td>$H_1: \rho_{mb} &gt; 0$</td>
<td>4.0354**</td>
<td>2.4726**</td>
<td></td>
</tr>
<tr>
<td>$H_0: \rho_{ms} = 0$</td>
<td>$H_1: \rho_{ms} &gt; 0$</td>
<td>2.2404*</td>
<td>3.2464**</td>
<td></td>
</tr>
<tr>
<td>$H_0: \rho_b = \rho_s$</td>
<td>$H_1: \rho_b \neq \rho_s$</td>
<td>-2.2285*</td>
<td>-1.7345</td>
<td></td>
</tr>
<tr>
<td>$H_0: \rho_{mb} = \rho_{ms}$</td>
<td>$H_1: \rho_{mb} \neq \rho_{ms}$</td>
<td>0.8065</td>
<td>-1.0370</td>
<td></td>
</tr>
</tbody>
</table>

*P-value < 0.05, **P-value < 0.01

6.7 DISCUSSION

The linear regression model of Kempthorne and Tandon (1953) was generalized and used to analyze the arterial blood pressure data to gauge the pattern of familial aggregation when offspring are classified by sex. The suggested procedures for testing the significance of familial correlations are believed to be valid when large number of families are available. However, to be able to assess the performance of such procedures, a Monte Carlo study design is needed. Such a complete validation procedure is not the aim of this chapter, and a thorough investigation may be needed to assess the performances of these tests.
It should be noted that the reliability of the estimated parameters will depend largely on how well models (6.4) are fitted to the data. In the context of regression modelling there are several known techniques which provide diagnostic tools for assessing the adequacy of the fit. Traditionally, this is done with single-number summaries such as the multiple correlation or the standard deviation of residuals. Neither is a satisfactory general purpose tool. Alternatively, well-chosen graphical displays are often useful for diagnostic purposes. The most commonly used graphs are the residual plots. In the regression models discussed in this chapter, the residuals should exhibit tendencies that tend to confirm the assumptions we have made. Non-normality of the errors is usually detected by plotting the ordered residuals against expected order statistics from standard normal distribution. Fortunately, such plots are not sensitive to the degree of dependency among the residuals. Figures 6.1 through 6.4 show the mentioned plots for the systolic and diastolic blood pressures of brothers and sisters. As can be seen, the plots do not exhibit a denial of the normality assumptions. Non-constancy of the variance of the error structure may be detected graphically, in standard regression situations, by plotting the residuals against the fitted observations, or against the mother score. The plots of residuals against the fitted values of the systolic and diastolic blood pressures of brothers and sisters are not shown but no detectable trend in either graph is noticed.
LINEAR MODEL FOR SISTERS
QUANTILE PLOT FOR SBP

Figure 6.1
LINEAR MODEL FOR SISTERS
QUANTILE PLOT FOR DBP

Figure 6.2
LINEAR MODEL FOR BROTHERS
QUANTILE PLOT FOR SBP

Figure 6.3
LINEAR MODEL FOR BROTHERS
QUANTILE PLOT FOR DBP

Figure 6.4
CHAPTER 7

SUMMARY AND RECOMMENDATIONS FOR FUTURE RESEARCH

7.1 Summary

The theory of point estimation and hypotheses testing for the intraclass (sib-sib) correlation coefficient, \( \rho \), when siblings are not distinguished by sex were discussed in Chapters 2, 3, and 4. The more realistic models which distinguish the sex of progeny were dealt in Chapters 5 and 6. The terminology used, however, is not limited to the analysis of familial data and, hopefully, this will encourage the use of these correlations in several other fields of research.

In Chapter 2, a study was performed to investigate the sampling properties of several point estimators of the intraclass correlation for a wide variety of unbalanced designs. The expressions for the large sample bias and variance of several point estimators of the intraclass correlation were derived and compared. Based on this investigation, the use of Karlin's pairwise (individual) estimator was recommended for small number of groups with severe degree of unbalancedness. However for moderately large number of groups \( (K > 10) \), Karlin's empirical estimator was recommended when the true value of \( \rho \) is thought to be less than or equal to 0.5, otherwise Smith's estimator was recommended. If no prior knowledge concerning
the value of intraclass correlation exist, the use of Karlin's individual estimator is recommended for small number of groups \( K \leq 10 \) and the use of maximum likelihood estimator is recommended otherwise.

The procedures for testing the hypothesis that the intraclass correlation is equal to a specified value were dealt in Chapter 3. Several procedures for testing the said hypotheses were proposed and compared by Monte Carlo studies for a family size distribution which is likely to occur in practice. For one sided alternatives, the use of the Neyman's \( C(o) \) (or partial score) test procedure, using maximum likelihood estimates for nuisance parameters, was recommended because it hold nominal levels and gives compatible powers, especially, when the number of sampled families are large \( K \geq 50 \). For two sided alternatives, the modified F-ANOVA test procedure was recommended as it is consistently more powerful for \( \rho > \rho_0 \), the specified value of \( \rho \), otherwise Wald's test procedure, based on the maximum likelihood estimator, was recommended.

The estimation of intraclass correlations in multiple samples by the methods of ANOVA and the maximum likelihood was discussed in Chapter 4. The expressions for the asymptotic variances of the estimators were derived. The results on point estimation of the intraclass correlations in multiple samples yeilds the single sample case, discussed in Chapter 2. Several procedures for testing the homogeneity of intraclass
correlations in multiple samples were proposed and compared by extensive Monte Carlo studies. The use of a test based on the variance stabilizing transformation of Fisher (1925) was strongly recommended provided that the values of intraclass correlations are less than or equal to 0.5. The use of Neyman's C(0) test, using maximum likelihood estimates for nuisance parameters, was recommended if the values of intraclass correlations are thought to be large or no prior knowledge concerning these values exists.

The maximum likelihood estimation of sibling (brother-brother, sister-sister and brother-sister) correlations was considered in Chapter 5. Two cases were discussed. The first assumes that the within brothers and within sisters variances are not equal (non-homogeneous), and the other assumes that the within variances are equal (homogeneous). It was shown that the maximum likelihood estimates of the parameters can be obtained by maximizing functions of fewer parameters. Procedures to test the significance of sibling correlations were also discussed and the methodology was presented on published arterial blood pressure data collected by Miall and Oldham (1955).

In Chapter 6, the maximum likelihood estimation of familial correlations, using a linear model approach, was discussed. The maximum likelihood estimates of two mother-sib (mother-brother and mother-sister) and three sib-sib (brother-brother, sister-sister and brother-sister) correlations were obtained by first finding the maximum
likelihood estimates of parameters of the linear model and then of the familial correlations. The expressions for the asymptotic variances and covariances of the estimators were provided and the procedures to test the significance of these correlations were discussed. The previously mentioned arterial blood pressure data set was used to illustrate the methodology.

The Monte Carlo studies were not done to investigate the properties of test procedures regarding familial correlations discussed in Chapters 5 and 6. This is in contrast to the extensive Monte Carlo studies for intraclass correlations in Chapters 3 and 4. However, with regard to the simulation procedures in Chapters 3 and 4, one can do similar kind of studies when the offspring are classified by sex into groups of sons (brothers) and daughters (sisters).

7.2 Recommendations for Future Research

The Monte Carlo simulations on testing hypotheses regarding intraclass correlation in Chapter 3, and on testing the homogeneity of intraclass correlations in Chapter 4, were conditional on sibship size distributions typical for North American families. Investigations that utilize different sibship size distributions may provide different conclusions. More investigation needs to be done concerning the choice of sibship size distribution which may be different from zero-truncated negative binomial distribution.
As a special case in Chapter 5, a assumption was made regarding the homogeneity of within brothers variance ($\sigma_b^2$) and within sisters variance ($\sigma_s^2$). This assumption is of its own interest and must be tested before testing the significance of sibling correlations. The likelihood ratio test statistic in order to test $H_0: \sigma_b^2 = \sigma_s^2$ is given by (5.47). Another test statistic which may be used to test the said hypothesis is

$$Z = \frac{\hat{\sigma}_b^2 - \hat{\sigma}_s^2}{\left[ \text{Var}(\hat{\sigma}_b^2) + \text{Var}(\hat{\sigma}_s^2) - 2 \text{Cov}(\hat{\sigma}_b^2, \hat{\sigma}_s^2) \right]^{1/2}},$$

which, under $H_0$, is asymptotically normally distributed with zero mean and unit variance. Pitman (1939) proposed a test statistic to test the homogeneity of correlated variances in bivariate normal populations which may be extended to the present situation of multivariate normal populations. The other test statistics of interest are based on the $C(\alpha)$ theory of Neyman (1959) which may be derived on the lines of Section 4.4.2. More work is needed to develop these tests and to investigate their performance.

Within the framework of linear models, Shoukri and Ward (1989) have proposed ensemble estimates for the familial correlations which utilize the scores of both parents but their model did not distinguish the sex of progeny. The results of Chapter 6 may be generalized to the models consisting of the scores of both parents and children of both sexes. Using the scores of both parents as regressor
variables, the linear model

$$y_{ij} = \begin{cases} 
\beta_{0b} + \beta_{1b}x_{if} + \beta_{2b}x_{im} + \epsilon_{ij}, & j = 1, 2, \ldots, b_i, \\
\beta_{0s} + \beta_{1s}x_{if} + \beta_{2s}x_{im} + \epsilon_{ij}, & j = b_i + 1, \ldots, n_i, \\
i = 1, 2, \ldots, K
\end{cases}$$

may be used to find the ensemble estimates of familial correlations. Here $x_{if}$ and $x_{im}$ are the scores of father and mother, respectively, in the $i$th family. The set of familial correlations that are to estimated are $\rho_{fb}$ (father-brother), $\rho_{fs}$ (father-sister), $\rho_{mb}$ (mother-brother), $\rho_{ms}$ (mother-sister), $\rho_b$ (brother-brother), $\rho_s$ (sister-sister) and $\rho_{bs}$ (brother-sister). The expressions for the asymptotic variances and covariances of the maximum likelihood estimators of the familial correlations also need to be developed.

The bootstrap techniques, introduced by Efron (1979), are now used extensively to estimate the probability distributions of the point estimators and null distributions of the test statistics. The standard applications of bootstrapping involve simple resampling from the observed data. Suppose that our data consists of a random sample of size $n$ from an unknown probability distribution function $F$,

$$X_1, X_2, \ldots, X_n \sim F.$$

The goals are accomplished by drawing bootstrap samples from the observed data $S = [x_1, x_2, \ldots, x_n]$ (or some transformation of the data). The bootstrap method for the one-sample problem is as follows:
1) Take a random sample of size n from the probability distribution function F and call it the observed data.

2) Construct the empirical distribution function \( \hat{F} \) by putting mass \( 1/n \) at each observed point \( x_1, x_2, \ldots, x_n \).

3) With fixed \( \hat{F} \), draw a random sample with replacement of size n from \( \hat{F} \), say \( \hat{S}^* = [\hat{x}_1^*, \hat{x}_2^*, \ldots, \hat{x}_n^*] \), and call it the bootstrap sample.

4) Approximate the sampling distribution of the estimator or the null distribution of test statistic by drawing B bootstrap samples from \( \hat{F} \), keeping \( \hat{F} \) fixed at its observed value.

For more details of bootstrap method see, e.g., Efron and Tibshirani (1988). In order to make further refinements to the solution offered by the delta method, the bootstrap techniques may be used to approximate the probability distributions of the point estimators of familial correlations. These techniques may also be used to estimate the null and alternate distributions of several statistics for testing hypotheses regarding familial correlations.
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