THE GENERALLY COVARIANT DIRAC EQUATION AND THE SOLUTION OF WEYL'S EQUATION IN A FLUID COSMOLOGY.

TIMOTHY CAMPBELL CHAPMAN

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LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L'AVONS REÇUE
THE GENERALLY COVARIANT DIRAC EQUATION AND
THE SOLUTION OF WEYL'S EQUATION IN A FLUID
COSMOLOGY

by

Timothy Campbell Chapman

A Dissertation
submitted to the Faculty of Graduate Studies
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1977
ABSTRACT

THE GENERALLY COVARIANT DIRAC EQUATION AND
THE SOLUTION OF WEYL'S EQUATION IN A FLUID
COSMOLOGY

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Timothy Campbell Chapman

The special relativistic 4-component Dirac equation is reviewed in its manifestly covariant form. The equation is then generalized to noninertial space-times where the Dirac $\gamma$-matrices become position dependent.

A comprehensive introduction and study of the generally covariant 4-component (bispinor) Dirac equation is given. It is developed for general Riemannian curved space-times. The spin-covariant derivative of 4-component spinors is introduced by means of the Fock-Ivanenko coefficients. The iterated form of the Dirac equation is given in the curved space and the Gordon decomposition of the current is performed. The weak field limit of both the Dirac equation and its iterated form are given. The weak field-WKB limit of the Klein-Gordon equation is shown to be geodesic.

The general tetrad structure on space-time is given and explicit tetrads in a 'canonical gauge' are calculated for any given metric.

A formalism is developed to determine the Dirac wave functions in flat noninertial space-times using the canonical gauge tetrads and the representation independent generally covariant Dirac equation. Examples are given for the following accelerated
frames: rotating, Galilean, generalized hyperbolic, and static, homogeneous gravitational fields.

A model for incorporating nonminimal couplings is presented for the Einstein-Dirac self-consistent field system. This is a direct coupling Lagrangian theory.

Dirac particles are studied in a new and unpublished background metric of Glass and Wilkinson. The nondiagonal metric is axially and cylindrically symmetric and describes a Bianchi type III fluid with negative pressures. The Weyl equation is constructed and shown to be completely separable if a simple normal mode assumption is made for the two cyclic variables. The Weyl solutions are found, with no approximations, in terms of Whittaker and generalized spheroidal functions. It is also shown that the full Dirac equation separates; part of the solutions are equivalent to the Weyl case. The Dirac equation for this metric is also given in the Newman-Penrose formalism.

The equivalence of the spinor and bispinor formulations of the general Dirac theory is shown in an appendix.
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Financial assistance from the governments of Canada and Ontario in the form of scholarships, teaching assistanceships, and research assistanceships was most generous.

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NOTATION

For relativity we usually adopt the notation and conventions of Adler, Bazin, and Schiffer (1965); also see below. In Appendix B we follow the notation of Hawking and Ellis (1973).

For relativistic quantum mechanics we follow Bjorken and Drell (1964) except as noted below.

Latin indices $i, j, k \ldots$ run over three spatial coordinate labels, usually, 1, 2, 3.

Greek indices $\alpha, \beta, \gamma \ldots$ run over the four space-time inertial coordinate labels 0, 1, 2, 3 or $cT, X, Y, Z$. Greek indices $\mu, \nu, \rho \ldots$ run over the four coordinate labels in a general coordinate system.

There is summation over repeated indices (Einstein summation convention).

Partial derivatives:

$$\partial_{(x)} = \frac{\partial}{\partial x^{(x)}} \quad \partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = \gamma_{\mu}$$

Lorentz metric:

$$\eta^{(\alpha)(\beta)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Pauli spin matrices:

$$\sigma^{(i)} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \sigma^{(2)} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^{(3)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma^{(\varphi)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

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Dirac \( \gamma \)-matrices:

\[
\gamma^{(0)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad \gamma^{(1)} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\gamma^{(2)} = \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & -i & 0 & 0 \\
-i & 0 & 0 & 0
\end{pmatrix}, \quad \gamma^{(3)} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\gamma^{(5)} = \gamma^{(5)} = i \gamma^{(0)} \gamma^{(1)} \gamma^{(2)} \gamma^{(3)} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

Anticommutator:

\[
\{ \gamma^{(x)}, \gamma^{(y)} \} = \gamma^{(x)} \gamma^{(y)} + \gamma^{(y)} \gamma^{(x)}
\]

Commutator:

\[
[ \gamma^{(x)}, \gamma^{(y)} ] = \gamma^{(x)} \gamma^{(y)} - \gamma^{(y)} \gamma^{(x)}
\]

Ricci's Identity:

\[
\delta \gamma^{(x)}_{\beta \gamma} - \delta \gamma^{(x)}_{\gamma \beta} = R^x_{\eta \beta \gamma} \delta \gamma^{(x)}_{\eta}
\]

Contracted Riemann tensor:

\[
R_{\mu \nu} = R^x_{\mu \nu \eta} \delta \gamma^{(x)}_{\eta}
\]
Electromagnetic Vector Potential:

$A_\mu := (\delta, A_i)$. 

Electromagnetic Field Tensor:

$F_{\mu\nu} := A_{\mu\nu} - A_{\nu\mu}$. 

Electronic charge:

$q = \pm |e| = \pm 4.8032 \times 10^{-10} \text{ esu}$. 

Mass of the electron:

$m = 9.11 \times 10^{-31} \text{ gm}$. 

Speed of light:

$c = 2.998 \times 10^{10} \text{ cm/sec}$. 

Planck's constant:

$h = 2\pi\hbar = 6.625 \times 10^{-17} \text{ erg-sec}$. 

Newton's gravitational constant:

$G = 6.67 \times 10^{-8} \text{ dyn-cm/gm}^2$. 
CHAPTER I
INTRODUCTION

When Einstein (1915) published his general theory of relativity he was interested in the classical aspects of gravitation and its concomitant effects on planetary systems and electrodynamics. The theory was expressible in the tensor language of a pseudo-Riemannian space-time. Such a state of affairs would have been sufficient if all physically interesting quantities had only tensor transformation properties. However, more than a decade after the development of Einstein's theory, Dirac (1928) proposed his special relativistic theory of a quantum mechanical electron. This introduced a new object into the space-time arena, a four-component spinor whose transformation properties were more intricate than that of a tensor.

By using the principle of general covariance, expanded to include quantities which also had nontrivial spin transformations, it was possible to make the Dirac equation conform to general relativity. In particular, Fock and Ivanenko (1929) seem to have been among the first to determine the form of the spin-covariant derivative. At that time, there was a fairly active effort to understand the general relativistic form of quantum theory by many authors (e.g. Weyl 1929, Schrödinger 1932, Pauli 1933, Dirac 1935, 1958). The final version of the four-component theory was provided by Bargmann (1932). In addition, the generalization to Riemannian space-time of the two component spinor Dirac equation was studied by Infeld and van der Waerden (1933) and later updated by Bade and
Jehle (1953) and Bergmann (1957).

However, interest in the subject waned. No less an expert than Dirac (1967) stated the conventional wisdom: 'There is no need to make the (relativistic theory of the electron) conform to general relativity, since general relativity is required only when one is dealing with gravitation, and gravitational forces are quite unimportant in atomic phenomena.' It was generally conceded that, although gravitational influences were quite weak, the gravitational field would have to be reckoned with eventually in a wider interaction picture.

A major impetus to the subject occurred with Kerr's (1963) 'spinning Schwarzschild' solution. Carter (1968) found that the Hamilton–Jacobi equation separated in the Kerr metric; this was wholly unexpected because of the complexity of the field. But in retrospect, it occurred to Carter (1973) that one could 'derive' the Kerr metric as being the simplest non-spherically symmetric generalization of the Schwarzschild geometry in which the Klein–Gordon equation would separate. The whole question of separable coordinate systems in curved spaces for the Hamilton–Jacobi equation etc. commanded some interest (e.g. Woochouse 1975, Deitz 1976).

Physically, the Kerr solution turned out to be very interesting. Working with the perturbed Newman–Penrose (1962) formalism Teukolsky (1973) was able to give a single equation describing the perturbations of spin 0, ½, 1, and 2 massless fields. He discovered that this equation separated in the Kerr metric (see also Unruh 1973 for the neutrino case). Chandrasekhar (1976) found that the Dirac equation also separated; Page (1976) generalized this
result to the Kerr-Newman case. These techniques enabled researchers to see if various waves would super-radiate against Kerr black holes; it appears, roughly, that integer spin fields give a reflection coefficient larger than one while spin-$\frac{1}{2}$ waves (Dirac, neutrino) do not (e.g. for the neutrino case see Unruh 1973, Chandrasekhar and Detweiler 1977).

The separability of these perturbation equations in the Kerr background is very striking and somewhat perplexing. In fact, at the Eighth International Conference on General Relativity and Gravitation, 1977, J. Plebański stated that the separability properties in Kerr space are explainable (somehow) by sheaf cohomology theory on complexified manifolds for type D metrics (otherwise understood as complexifying perplexity).

Many astrophysical situations are believed to exist where gravitation takes on a dominant rôle for the description of matter. For example, Hawking (1975) has initiated quantum-mechanical studies concerning particle creation by black hole formation (Unruh 1976, Page 1976, Gibbons 1977). The calculations here for the Kerr black hole are in the realm of second quantization (Unruh 1974b) just as they are in the cosmological cases (Parker 1977). The whole subject including the quantization of gravity (e.g. 'supergravity', see Van Nieuwenhuizen 1977) is at the forefront of modern research.

The work considered here is concerned with the first quantized generally covariant Dirac equation. Except in Chapter VII and Appendix A we confine attention to the case where the Dirac particle is a test particle in a given background. Part of this
work gives a comprehensive introduction, with new extensions, to the Dirac theory in Riemannian space-time. Other parts have already been published (Leiter and Chapman 1975, Chapman and Leiter 1976).

One of the main results here is the solution of Weyl's equation in the fluid cosmology of Glass and Wilkinson (1976). The solution is exact unlike many of the solutions of wave equations in general relativity. (For various solutions of typical problems see Unruh 1974a, Unruh 1976, Rowan and Stephenson 1976, Einstein and Finkelstein 1977.) Moreover, the full Dirac equation is shown to separate in this metric.

In Chapter II we review the standard covariance argument regarding the four-component Dirac equation in Minkowski space (following Bjorken and Drell 1964) to show the failure of such arguments in general relativity. Our motivation for establishing a suitable framework for study is supplied by the spirit of general relativity; that is, we want only to employ manifestly covariant equations. We demand that the wave function transform under the Lorentz group as a scalar and that the $\gamma$-matrices and derivatives of the wave function transform as suitable Lorentz four-vectors. The Dirac equation is then generalized to noninertial frames where the $\gamma$-matrices become position dependent; the generalization of the anticommutation relation arises in a natural way for the non-Lorenzian space-times. The Dirac equation we construct is not representation independent with respect to coordinate dependent spin transformations.

In order to have a representation independent Dirac
equation, Chapter III is devoted to deriving the standard properties of the generally covariant Dirac equation. Our motivation for this construction is provided by the analogy with the noninertial case. When a gravitational field is involved one no longer has zero curvature, so there cannot exist global transformations between the local inertial frames. In this case the global transformation coefficients between frames in special relativity are replaced by the global notion of tetrads. Thus the generalization of the anti-commutation rules for the point dependent $\gamma$-matrices is analogous to the method described in Chapter II. The principle of general covariance is extended to include quantities with a spin-transformation nature. The major construct in this regard is the spin-covariant derivative which involves the use of the Fock-Ivanenko coefficients. The form of the spin-covariant derivative of the four-component wave function and its Dirac adjoint are given. From the requirement that the spin-covariant derivative of the $\gamma$-matrix vanish, a number of equivalent forms of the Fock-Ivanenko coefficients are derived.

Chapter IV completes the introduction to the generally covariant Dirac equation. In analogy with the Riemann curvature tensor, a so-called spin-curvature tensor is constructed. A number of relations is given connecting the spin-curvature tensor and $\gamma$-matrices to the usual constructs of Riemannian space-time. These are used to express the iterated Dirac equation in curved space-time. The Gordon decomposition of the conserved current is performed for this equation to show the analogy between the Fock-Ivanenko coefficients and the electromagnetic vector potential. Both the Dirac
equation and its iterated form are shown in the linearized gravitational theory. The spinless Klein-Gordon is shown to give a particle limit that is geodesic if a small quantum potential term is neglected.

The general tetrad structure on space-time is given in Chapter V. We calculate explicit tetrads for any given metric; the 'canonical gauge' of de Oliveira and Tiomno (1962) is used.

We return in Chapter VI to the problem of determining the Dirac wave equation in flat noninertial space-times. A formalism is developed to do this. An algorithm is given to calculate the equivalence transformation relating the wave functions in noninertial and inertial frames. As examples we calculate the exact equivalence transformations for the following accelerated frames: rotating, Galilean acceleration, generalized hyperbolic motion, and the static homogeneous gravitational fields.

In Chapter VII a model for nonminimal coupling is presented for the Einstein-Dirac self-consistent system. It is argued that Dirac particles do not apply to the strong principle of equivalence essentially because their spin couples directly to the Riemann tensor. Since the weak principle of equivalence has the only experimental verification it is postulated that there may be a scalar field that couples nonminimally to the Riemann scalar in the Lagrangian for the Einstein-Dirac field system. This produces a direct coupling theory in contrast to an indirect coupling theory such as the Brans-Dicke (1961) version. The Euler-Lagrange equations are given for our system. The Dirac equation is modified by the addition of a term depending on the Riemann scalar and functional derivatives of the
postulated scalar field, $S$. The Einstein equation is modified by additional terms on the right hand side; the interesting notion of a variable gravitational 'constant' is seen. A general model for $S$ is also given in terms of the bilinear covariants of the theory.

Appendix A gives the orthodox variational principles for the Einstein-Dirac fields. The variation for the gravitational field is done with respect to the tetrad fields, or, more precisely, the curved space $\gamma$-matrices. The equivalence between this formulation and the usual variation with respect to the metric is given. The Dirac equation and its adjoint form are found; also the energy-momentum density tensor is given for the Dirac fields.

Finally, Chapter VIII deals with the Dirac equation in the background of the new fluid cosmological solution found by Glass and Wilkinson (1976). This metric is discussed in Appendix B. The metric describes a Bianchi type III cylindrically symmetric fluid cosmology with zero shear and nonvanishing expansion. It is shown that the weak, dominant, and strong energy conditions are fulfilled in both the anisotropic and isotropic case.

From the tetrads for this metric the relevant quantities in the Dirac equation are calculated in Chapter VIII. Two important vectors found in this connection are shown to be curl free. This enables the massless Dirac equation to be written in a simple way. The condition describing physical left-handed neutrinos gives the Weyl equation. Since the Glass-Wilkinson metric is independent of the $z$ and $\phi$ coordinates an assumption is made to the dependence of these modes in the wave function for the Weyl equation. The technique of separation of variables is used for the remaining $t$ and $\gamma$
dependent parts of the wave function. The method is successful and two sets of two equations coupled, respectively, in the $t$ and $y$ functions are found with suitable separation constants. The wave functions in $y$ are solved for and are found to be of the Whittaker function type. The wave functions in $t$ are also solved for and shown to be related to solutions of the generalized spheroidal equation. No normalization or boundary conditions are applied to the solutions.

In Chapter VIII it is also found that the full set of Dirac equations separate. The wave functions in $y$ are found to be mass independent so that they are the same as in the neutrino case. The wave function components in $t$ obey a complicated set of differential equations for which it seems difficult to extract individual second order differential equations for each component. The great simplification of the system is shown when the mass is put to zero. The resulting equations, under the physical neutrino condition, are shown to be equivalent to the Weyl equations.

Appendix C gives the Dirac equation written in the Newman-Penrose formalism for the Glass-Wilkinson metric.

Appendix D shows the equivalence of the four-component (bispinor) formalism of the Dirac theory used here with that of the spinor formulation.
CHAPTER II

DIRAC EQUATION IN FLAT SPACE-TIME

The flat space-time Dirac equation in Cartesian coordinates is

\[ -i \gamma^{(\alpha)} \Psi^{(\alpha)} (X) + \hbar \nabla \Psi (X) = 0, \]

where \( X := mc^2/\hbar \). The wave function \( \Psi \) is dependent on \( X^{(\alpha)} = (cT, X, Y, Z) \); we use capital letters for the flat inertial coordinates and put parentheses on the indices to distinguish them clearly from the curvilinear coordinates, \( x^\mu \). In the case of curved space-time the \( X^{(\alpha)} \) coordinates refer to a locally Lorentz frame in the tangent space. The \( \gamma^{(\alpha)} \) are the Dirac \( \gamma \)-matrices in the standard flat representation (Bjorken and Drell 1964) and obey the anticommutation relation

\[ \left\{ \gamma^{(\alpha)}, \gamma^{(\beta)} \right\} = 2 \eta^{(\alpha)(\beta)} 1, \]

where \( \eta^{(\alpha)(\beta)} \) is the Lorentz metric of signature \((-2)\) and \( 1 \) is the \( 4 \times 4 \) unit matrix.

In special relativity one is concerned with Lorentz transformations, \( L^{(\alpha)}_{(\beta)} \), between two inertial frames of reference, \( X^{(\alpha)} \) and \( X^{(\beta)} \). They are related by

\[ X^{(\alpha)} = L^{(\alpha)}_{(\beta)} X^{(\beta)}. \]
The transformation depends only on the constant relative velocity of the frames so that the coordinate differentials are related by

\[ dX^{(\alpha)} = \land^{(\alpha)} {\beta} dX^{(\beta)}. \]  

(2.4)

In effect we have

\[ \frac{\partial X^{(\alpha)}}{\partial X^{(\beta)}} = \land^{(\alpha)} {\beta}. \]  

(2.5)

In special relativity the invariant interval is given by

\[ ds^2 = \eta_{(\mu) (\nu)} dX^{(\mu)} dX^{(\nu)}. \]  

(2.6a)

\[ = \eta_{(\mu) (\nu)} \land^{(\mu)} {\alpha} \land^{(\nu)} {\beta} dX^{(\alpha)} dX^{(\beta)}. \]  

(2.6b)

\[ = \land^{(\nu) (\alpha)} \land^{(\nu) (\beta)} dX^{(\alpha)} dX^{(\beta)}. \]  

(2.6c)

\[ = \eta_{(\alpha) (\beta)} dX^{(\alpha)} dX^{(\beta)}. \]  

(2.6d)

so that

\[ \land^{(\nu) (\alpha)} \land^{(\nu) (\beta)} = \eta_{(\alpha) (\beta)}. \]  

(2.7)

\[ \land^{(\nu) (\alpha)} \land^{(\nu) (\beta)} = \delta^{(\beta)}_{(\alpha)}. \]  

(2.8)

Applying \( \land^{(\alpha)} \) on both sides of equation (2.3) and noting equation (2.8) we find that
\[
X^{(\epsilon)} = \bigwedge_{(\alpha)} (^{(\epsilon)} X)^{(\alpha)}
\]  \hspace{1cm} (2.9)

is the inverse transformation. Similarly it can be shown that

\[
\bigwedge_{(\alpha)} (^{(\epsilon)} X) \bigwedge_{(\beta)} (^{(\epsilon)} X) = \eta^{(\alpha)} |^{(\beta)}
\]  \hspace{1cm} (2.10)

and, in the sense of equation (2.5), that

\[
\frac{\partial X^{(\beta)}}{\partial X^{(\alpha)}} := \bigwedge_{(\alpha)} (^{(\beta)} X)
\]  \hspace{1cm} (2.11)

The Lorentz covariance of the Dirac equation is usually argued (e.g. Bjorken and Drell 1964) in the following way. Under any Lorentz transformation, \( \bigwedge_{(\alpha)} \), the Dirac \( \gamma \)-matrices are to remain unchanged with the prescribed rôle of merely a book-keeping device. We denote the Lorentz transformed quantities by a prime so that the new form invariant equation is

\[
-i \gamma^{(\alpha)} \gamma^{(\alpha)} \Psi' + \kappa \Psi' = 0.
\]  \hspace{1cm} (2.12)

If we demand that

\[
\gamma^{(\alpha)} = \bigwedge_{(\alpha)} (^{(\beta)} \gamma^{(\beta)}
\]  \hspace{1cm} (2.13)

and assume

\[
\Psi' (X') = S \Psi (X)
\]  \hspace{1cm} (2.14)
where $S$ is a nonsingular matrix (we suppress all matrix indices), then equation (2.12) is equivalent to equation (2.1) if

$$\gamma^{(\alpha)} = \gamma^{(\alpha)} = \Lambda^{(\alpha)}_{(\beta)} SS^{(\beta)} S^{-1}$$  \hspace{1cm} (2.15)

(Here $S$ appears as a similarity transformation dependent only on the Lorentz transformation, i.e. $\gamma_{(\alpha)} S = 0$.) Given a Lorentz transformation, $\Lambda^{(\alpha)}_{(\beta)}$, then one solves equation (2.15) for $S$ in order to relate the wave functions in the different frames through (2.14).

It is well known (Sakurai 1967) that equation (2.1) is representation independent so that under an arbitrary nonsingular similarity transformation $T$ along with a Lorentz transformation we would have

$$dX^{(\alpha)} = dX^{(\alpha)}$$  \hspace{1cm} (2.16a)

$$\gamma^{(\alpha)} = \gamma^{(\alpha)}$$  \hspace{1cm} (2.16b)

$$\Psi^{(\alpha)} = T \Psi^{(\alpha)} = (TS) \Psi^{(\alpha)}$$  \hspace{1cm} (2.16c)

$$\gamma^{(\alpha)} = \Lambda^{(\alpha)}_{(\beta)} (TS) \gamma^{(\beta)} (TS)^{-1}$$  \hspace{1cm} (2.16d)

Since $T$ is arbitrary a certain simplification occurs if $T = S^{-1}$ so that equation (2.16) becomes

$$dX^{(\alpha)} = \Lambda^{(\alpha)}_{(\beta)} dX^{(\beta)}$$  \hspace{1cm} (2.17a)
\[ \xi^{(\mu)} = \bigwedge_{(\lambda)}^{(\beta)} \xi^{(\rho)} \quad \text{(2.17b)} \]

\[ \Psi'' = \Psi \quad \text{(2.17c)} \]

\[ \chi^{(\mu)} = \bigwedge_{(\lambda)}^{(\rho)} \chi^{(\beta)} \quad \text{(2.17d)} \]

Note that the new \( \gamma \)-matrices, \( \gamma^{(\mu)} \), are not equal to the original \( \gamma^{(\mu)} \) since \( \gamma^{(\mu)} = T \gamma^{(\mu)} T^{-1} = S \gamma^{(\mu)} S \); the \( \gamma \)-matrices now transform as Lorentz 4-vectors. Because of the orthogonality properties of the \( \bigwedge_{(\lambda)}^{(\rho)} \) the new \( \gamma \)-matrices obey the anticommutation relation

\[ \left[ \gamma^{(\mu)}_{(\nu)} \gamma^{(\nu)}_{(\beta)} \right] = 2 \eta^{(\mu)(\nu)} I \quad \text{(2.18)} \]

The covariance argument becomes trivial if we consider \( \gamma^{(\mu)} \) and \( \xi^{(\nu)} \) to transform as Lorentz 4-vectors and \( \Psi(X) \) as a scalar in flat space-time. The anticommutation relation (2.2) still holds as it must (Feynman 1961). Clearly, the transformation properties of bilinear covariants (e.g. the current) remain unchanged.

This latter view is better suited in making equation (2.1) generally covariant. As a precursor and motivation to the curved space case, a generalization to flat noninertial frames will now be considered.

Given a transformation from an inertial frame, \( X^{(\alpha)} = (cT, X, Y, Z) \), to a noninertial one with coordinates \( x^{\mu} = (ct, x, y, z) \) then \( x^{\mu} = x^{\mu}(X) \). We will restrict consideration to transformations between Cartesian type frames. Using primes now to denote quantities
in the noninertial frame and our consideration of manifest covariance we write the Dirac equation as

\[ \gamma'^{\mu} \partial_{\mu} \Psi'(x) + \chi \Psi'(x) = 0 \]  

(2.19)

where

\[ \Psi'(x) = \Psi(x), \]  

(2.20)

\[ \gamma'^{\mu}(x) = \frac{\partial x'^{\mu}}{\partial x^{(\kappa)}} \gamma^{(\kappa)}. \]  

(2.21)

The metric, $g_{\mu\nu}(x)$, in the noninertial frame is not Minkowskian because of the invariance of the interval:

\[ ds^2 = \eta_{(\alpha)(\beta)} \, dX^{(\alpha)} \, dX^{(\beta)} \]  

(2.22a)

\[ = \eta_{(\alpha)(\beta)} \frac{\partial X^{(\alpha)}}{\partial x^\mu} \frac{\partial X^{(\beta)}}{\partial x^\nu} \, dx^\mu \, dx^\nu \]  

(2.22b)

\[ = g_{\mu\nu} \, dx^\mu \, dx^\nu. \]  

(2.22c)

As a consequence, the natural generalization of the anticommutation relation (2.2) for the new primed $\gamma$-matrices becomes

\[ \{ \gamma'^{\mu}, \gamma'^{\nu} \} = 2 g'^{\mu\nu}(x) I. \]  

(2.23)

Here $g'^{\mu\nu}$ is the contravariant form of the metric tensor which is related to covariant one, $g_{\mu\nu}$, by $g'^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}$.

In the Dirac equation, one is not only concerned with coordinate transformations but also with spin transformations.
The Dirac particles possess an internal structure for which a space-time description is not appropriate so that one associates a complex four-dimensional spin space at each point in space-time in which $4 \times 4$ matrices operate on the four-component spinors. The spin transformations may generally be coordinate dependent but coordinate transformations in space-time and spin transformations in the spin space are generally independent of each other (Penrose and Rindler).

Since the $\gamma$'s are matrices we may multiply through equation (2.23) with a nonsingular matrix $N$ to give

$$N \gamma^\nu N^{-1} N \gamma^\nu N^{-1} + N \gamma^\nu N^{-1} N \gamma^\nu N^{-1} = 2 \gamma^\nu \mathbf{I},$$

or

$$\{ \gamma^\nu, \gamma^\nu \} = 2 \gamma^\nu \mathbf{I}, \tag{2.25}$$

where

$$\gamma^\nu := N \gamma^\nu N^{-1}. \tag{2.26}$$

On comparing equations (2.23) and (2.25) it is evident that the metric structure of the anticommutation relation for the $\gamma$-matrices remains unchanged with the introduction of the spin transformation of equation (2.26). For now, $N$ can be considered as any nonsingular $4 \times 4$ matrix and as a scalar under coordinate transformations. The value for $N$ will depend on what is chosen for $\gamma^\mu$; in general, $N$ will be coordinate dependent.

Using equation (2.26) we can write (2.19) in terms of $\gamma^\mu$ as

$$-i \gamma^\mu (\partial_\mu - N_{\lambda\mu} N^{-1}) \nabla \Psi + \alpha N \nabla \Psi = 0. \tag{2.27}$$
The set of equations (2.27) and (2.25) is not of the same form as
equations (2.19) and (2.23) so that the Dirac equation (2.27) is not
representation independent with respect to coordinate dependent spin
transformations on the \( \gamma \)-matrices. For describing quantum processes
in flat noninertial frames it would be preferrable to use a Dirac
equation which is covariant under both spin and coordinate trans-
formations and which reflects the general relativistic flavour of
the physics involved.

We shall return to these points in Chapter VI where a
satisfactory formulation which relates wave functions in noninertial
and inertial frames will be given. (For an alternative approach see
Schmutzer and Plebański 1977.)
CHAPTER III
GENERALLY COVARIANT DIRAC EQUATION

In this Chapter we present the generally covariant Dirac equation. The motivation we have in this development is that the treatment of the Dirac equation in Chapter II for noninertial frames lends itself to straightforward generalization to curved space.

Equations which are manifestly Lorentz covariant can be made valid in regions with gravitational fields by invoking the principle of general covariance (Weinberg 1972); i.e. we replace \( \eta_{\mu\nu} \) with \( g_{\mu\nu} \), all derivatives with covariant derivatives, and Lorentz tensors (or tensor densities) with objects that transform like tensors (or tensor densities) under general coordinate transformations. However, in order to generalize equation (2.1) to include gravitational effects, this method must be expanded in order to deal with structures having spin transformation properties.

The problem of devising a covariant procedure for differentiation in curved space is handled by assuming a vector law of parallel transport involving the Christoffel symbols, \( \{_{\lambda}^{\mu\nu} \} \). One arrives at the form of the covariant derivatives (\( ; \)) with respect to the metric \( g_{\mu\nu} \) for contravariant and covariant vectors, respectively:

\[
\nabla_{\mu}^{;\nu} = \nabla_{\mu}^{\nu} + \{^{\nu}_{\lambda} \} \nabla^{\lambda}
\quad (3.1a)
\]

\[
\nabla_{\nu}^{\mu} = \nabla_{\nu} + \{^{\nu}_{\lambda} \} \nabla^{\lambda}
\quad (3.1b)
\]

The covariant derivative of higher rank tensors is easily generalized.
The Christoffel symbol is

\[
\{ ^\lambda_\mu_\nu \} = \frac{1}{2} q^{\lambda \xi} \left( q_{\xi \nu, \mu} + q_{\nu \mu, \xi} - q_{\xi \mu, \nu} \right),
\]

(3.2)

and has the following inhomogeneous coordinate transformation

\[
\{ ^\alpha_\beta_\gamma \} = \frac{\partial x^\lambda}{\partial x^\alpha} \frac{\partial x^\mu}{\partial x^\beta} \frac{\partial x^\nu}{\partial x^\gamma} \{ ^\lambda_\mu_\nu \} - \frac{\partial x^\lambda}{\partial x^\alpha} \frac{\partial x^\mu}{\partial x^\beta} \frac{\partial x^\nu}{\partial x^\gamma} \frac{\partial x^\sigma}{\partial x^\lambda} \frac{\partial x^\rho}{\partial x^\mu} \frac{\partial x^\sigma}{\partial x^\nu}.
\]

(3.3)

In devising a suitable derivative operator (Adler 1970) in curved space for a four-component spinor, \( \psi(x) \), one takes the analogy of the vector transplantation law over to bispinors and has the definition for the spin-covariant derivative, \( \nabla_\mu \), of the \( \psi \) as

\[
\nabla_\mu \psi(x) = \psi(x)_{,\mu} + \Gamma_\mu^\lambda(x) \psi(x).
\]

(3.4)

We treat \( \psi \) as a coordinate scalar under the covariant derivative so that \( \psi_\mu = \psi_{,\mu} \). The \( \Gamma_\mu^\lambda \) are called the Fock-Ivanenko (1929) coefficients; they are zero for a Lorentz frame.

We assume that the generalization of the anticommutation relation (2.2) should go over to Riemannian space-time with metric \( g_{\mu \nu} \) in the fashion of the noninertial case of equation (2.23). We put

\[
\{ Y^\mu_\lambda(x), Y^\nu_\xi(x) \} = 2 g^{\lambda \xi}(x) \Gamma.
\]

(3.5)

The connection between the \( \gamma \)-matrices, \( Y^\mu_\lambda(x) \), of the curved space-time to those in the tangent space should be linearly related as suggested by equation (2.21). We put

\[
Y^\mu_\lambda(x) = \lambda^\mu_\lambda(x) Y^\lambda_\xi(x).
\]

(3.6)
The $\chi_{\mu}^{(\alpha)}(x)$ are known as tetrads; they transform as a Lorentz 4-vector on the $(\alpha)$ index under Lorentz transformations. The tetrads will be discussed in Chapter IV; it is sufficient to note now that equation (3.5) is true given (3.6) if $\chi_{\mu}^{(\alpha)}\gamma^{(\alpha)} = g_{\mu}^{\nu}$ (in analogy with (2.22)). The indices on the tetrads can be raised and lowered in the usual way:

\begin{align}
\chi_{\mu}^{(\alpha)} = g_{\mu}^{\nu}\chi_{\nu}^{(\alpha)} = \eta^{(\alpha)(\beta)}\chi_{\beta}^{(\alpha)} \\
\chi_{\mu(\alpha)} = g_{\mu\nu}\chi_{\nu(\alpha)} = \eta^{(\alpha)(\beta)}\chi_{\mu(\beta)}
\end{align}

(3.7a)

(3.7b)

The generally covariant Dirac equation can thus be presented as

\[-i\gamma^{\mu}\nabla_{\mu}\psi + \kappa \psi = 0\]

(3.3)

where equation (3.5) is satisfied. The gravitational effects enter through $\gamma^{\mu}$ and $\Gamma_{\mu}$. We have constructed equation (3.3) to be manifestly general covariant so that $\gamma^{\mu}$ and $\Gamma_{\mu}$ transform as vectors under general coordinate transformations. We noted in Chapter II that it would be convenient to have this Dirac equation also representation independent. Using carets to denote spin-transformed quantities and demanding representation independence, equation (3.8) should become

\[-i\hat{\gamma}^{\mu}\hat{\nabla}_{\mu}\hat{\psi} + \kappa \hat{\psi} = 0\]

(3.9)
where \[ \{ \hat{\gamma}^\mu, \hat{\gamma}^\nu \} = 2 \eta^{\mu \nu} \mathbb{I}. \quad (3.10) \]

Under a spin transformation \( B(x) \) in spin space we have \( \hat{\gamma}^\mu \rightarrow \hat{\gamma}^\mu' = B \hat{\gamma}^\mu B^{-1} \). In order for equations (3.9) and (3.8) to be equivalent we must have

\[ \hat{\psi} = B \psi, \quad (3.11) \]

and

\[ B \nabla_\mu \psi = \nabla_\mu \hat{\psi}. \quad (3.12) \]

These conditions imply that the \( \Gamma_\mu \) has the inhomogeneous spin transformation,

\[ \hat{\Gamma}_\mu = B \Gamma_\mu B^{-1} - B \mu B^{-1}. \quad (3.13) \]

Consider the quantity, \( \bar{\psi} = \psi^\dagger \gamma^{(0)} \), where \( \dagger \) means the Hermitian conjugate. We shall call \( \bar{\psi} \) the Dirac adjoint of \( \psi \). From relativistic electron theory we know that \( \bar{\psi} \psi \) should be both spin and coordinate scalars so that under the spin transformation \( B \) we should have \( \bar{\psi} \rightarrow \bar{\psi} B^{-1} \). Using the standard Liebnitz properties of the spin-covariant derivative (Brill and Wheeler 1957) on \( \bar{\psi} \psi \) we find

\[ \nabla_\mu \bar{\psi} = \bar{\psi} \nabla_\mu - \bar{\psi} \Gamma_\mu . \quad (3.14) \]

Similarly, for \( \gamma^\mu \), which transforms as \( B \gamma^\mu B^{-1} \), we must have
\[ \nabla_\mu \gamma^\nu = \gamma^\nu_{\;\mu} + [\Gamma^\nu_{\mu\rho}, \gamma^\rho] \]  
\[ = \gamma^\nu_{\;\mu} + \left( \chi^\nu_{\mu} \right) \gamma^\lambda + \Gamma^\nu_{\mu \lambda} - \gamma^\nu \Gamma^\lambda_{\mu}. \]  

(3.15a)  

(3.15b)

It is important to emphasize the operational nature of the spin-covariant derivative. Clearly, the effect of the spin-covariant derivative on any tensor quantity is the same as that of the covariant derivative. The spin-covariant operator can act on the various spin associated quantities depending on their spin and tensor transformation nature, viz. equations (3.4), (3.14), and (3.15). In this sense, then, we have a generalized quotient theorem. For example, the Dirac equation (3.8) indicates that \( \nabla_\mu \psi \) transforms as a covariant vector under coordinate transformations and a 4-component spinor (i.e. a bispinor) under spin transformations (see (3.12)).

From the tensor analysis of standard general relativity the covariant derivative of the \( g_{\mu\nu} \) tensor is zero, i.e. \( \nabla_{\mu} g_{\nu\rho} = 0 \). We certainly want to retain this feature here. Upon consideration of equation (3.5), a sufficient condition to ensure this result is

\[ \nabla_{\xi} \gamma^\nu = 0. \]  

(3.16)

(It should be noted that this condition is not the most general that could be envisaged. Novello (1973) has shown that, if

\[ \nabla_{\xi} \gamma^\nu = \left[ \nabla_{\rho}, \gamma^\nu \right] \]  

(3.17)
where \( V_{\ell} \) is any arbitrary element of the Clifford algebra, then \( g^{\mu\nu} = 0 \). Our condition (3.16) has either \( V_{\ell} = 0 \) or \( V_{\ell} \) being some purely vector field.)

From equations (3.16), (3.15), and (3.6) we infer that

\[
\left[ \Gamma_{\ell}^\alpha, \gamma^{(\beta)} \right] = - (\chi^{(\beta)}_{\mu} \gamma_{\mu\alpha}) \gamma^{(\alpha)}
\]  

(3.18)

We want to solve equation (3.18) for \( \Gamma_{\ell}^\alpha \); it will be determined up to an arbitrary vector field times the 4 x 4 identity matrix since, by Schur's lemma, this would commute with the \( \gamma^{(\beta)} \) in the commutator of equation (3.18). (Because of this fact one may minimally couple the electromagnetic field to Dirac's equation (3.8) by the replacement \( \Gamma_{\mu}^\alpha \rightarrow \Gamma_{\mu}^\alpha + (ie/\hbar c) A^\mu I \) where \( q \) is the charge on the particle, i.e. \( q = -|e| \) for the electron and \( A^\mu \) is the vector potential.) The solution of equation (3.18) follows immediately from the Clifford algebra properties (Schweber 1961) contained in the anticommutation relation (2.2) for the \( \gamma^{(d)} \). The only combination of \( \gamma^{(d)} \) which will commute with the \( \gamma^{(\beta)} \) is

\[
\sigma^{(d)(\beta)} := \frac{1}{\gamma} \left[ \gamma^{(\alpha)}, \gamma^{(\beta)} \right] \]  

(3.19a)

where

\[
\sigma^{\mu\nu} := \lambda^{(\alpha)}_{\mu} \lambda_{(\nu)}^{(\beta)} \sigma^{(d)(\beta)} = \gamma^{\mu\nu} - q^{\mu\nu} I
\]  

(3.19b)

So we are led to make the ansatz that \( \Gamma_{\ell}^\alpha = C(\lambda^{(\alpha)}_{\mu} \lambda_{(\mu)}^{(\beta)}; \rho) \) where \( C \) is a factor to be determined by consistency. Substitution of this \( \Gamma_{\ell}^\alpha \) into equation (3.18) and use of the fact that
\[
\left[ \gamma^{(\mu)}, \sigma^{(\nu)(\rho)} \right] = 2 \left( \gamma^{(\beta)}, \eta^{(\mu)(\kappa)} \right) \delta^{(\kappa)(\rho)} \eta^{(\mu)(\beta)}
\]

(3.20)

shows that \( C = \frac{1}{2} \); we have finally that

\[
\gamma^{\rho} = \frac{1}{4} \left( \lambda^\mu_{(\alpha)} \lambda^\nu_{(\beta)} \gamma^{\alpha \beta}, \rho - \lambda^\mu_{(\alpha)} \lambda^\nu_{(\beta)} \left\{ \gamma^{\alpha \beta}, \rho \right\} \right) \sigma^{(\alpha \beta)}
\]

(3.21)

Other equivalent forms of the Fock–Ivanenko coefficient are possible:

\[
\gamma^{\rho} = \frac{1}{4} \gamma^\mu \gamma_{\mu \rho}
\]

(3.22a)

\[
= -\frac{1}{4} \gamma^\mu \gamma_{\mu \rho}
\]

(3.22b)

\[
= \frac{1}{8} \left[ \gamma^\mu, \gamma_{\mu \rho} \right]
\]

(3.22c)

\[
= \frac{1}{8} \left( \gamma^\mu \gamma_{\mu \rho}, \gamma_{\mu \rho} \gamma^\mu - \left\{ \gamma^\nu, \gamma_{\mu \rho} \right\} \left( \lambda^\gamma_{(\beta)}, \gamma^\kappa_{(\rho)} - \lambda^\kappa_{(\rho)}, \gamma^\gamma_{(\beta)} \right) \right)
\]

(3.22d)

\[
= \frac{1}{8} \left( \left[ \gamma^\rho, \gamma_{\mu \rho} \right] + 2 q \lambda^\rho \left\{ \gamma^\nu, \gamma_{\mu \rho} \right\} \gamma^\lambda \gamma^\nu \right)
\]

(3.22e)

\[
= -\frac{1}{4} q \lambda^\rho \left( \lambda^\nu_{(\beta)}, \gamma^\mu_{(\rho)} - \left\{ \rho^\nu, \gamma^\mu_{(\beta)} \right\} \right) \gamma^\lambda \gamma^\nu
\]

(3.22f)

Note the fact that

\[
\gamma^\rho \gamma_{\rho \nu} = 0.
\]

(3.23)

Since the tetrad are real then the Hermitian conjugate of the coordinate dependent \( \gamma \)-matrices is just as in special relativity,
\[ \gamma^{+\mu} = \gamma^{(0)} \gamma^\mu \gamma^{(0)} \]  

(3.24)

Thus, the Fock-Ivanenko coefficient has the Hermitian conjugate

\[ \gamma^+_{\mu} = -\gamma^{(0)} \gamma^\mu \gamma^{(0)} \]  

(3.25)

For \( \bar{\psi} = \psi^+ \gamma^{(0)} \) one can write down the generally covariant Dirac adjoint equation as

\[ i (\nabla_{\mu} \bar{\psi}) \gamma^\mu + \gamma \bar{\psi} = 0, \]  

(3.26)

where \( \nabla_{\mu} \bar{\psi} \) is given by equation (3.14).

In the usual manner we find the conserved current of the Dirac equation as

\[ j^\mu := \sqrt{-g} \bar{\psi} \gamma^\mu \psi. \]  

(3.27)

The \( j^\mu \) is conserved in the sense that

\[ \nabla_{\mu} (\bar{\psi} \gamma^\mu \psi) = 0, \]  

(3.28)

for

\[ \partial_{\mu} (\sqrt{-g} \bar{\psi} \gamma^\mu \psi) = 0. \]  

(3.29)
CHAPTER IV
ITERATION AND LINEARIZATION

In the context of studying quantum systems in general relativity it proves useful to consider the usual quantities of Riemannian space-time in terms of spin-tensor constructs. This gives the mathematical apparatus (Schrödinger 1932) for studying physical situations where the spin of an elemental system couples to the external gravitational field.

The Riemann curvature tensor, $R^a_{\eta\beta\gamma}$, is defined in terms of a vector field $\xi^\alpha$ through Ricci's identity by

$$\xi^\alpha_{,\beta,\gamma} - \xi^\alpha_{,\gamma,\beta} = R^\alpha_{\eta\rho\sigma} \xi^\eta_{,\rho}$$ \hspace{1cm} (4.1)

where $R^\alpha_{\eta\rho\sigma} = \{\partial_{\alpha}, \{\eta,_{\rho}\}_\sigma + \{\eta,_{\sigma}\}_\rho - \{\eta,_{\rho}\}_\sigma\}$ \hspace{1cm} (4.2)

In analogy to the definition of the Riemann curvature tensor one defines an antisymmetric spin-curvature tensor $K_{\mu\nu}$ as

$$K_{\mu\nu} := \nabla_\nu (\nabla_\mu \Psi) - \nabla_\mu (\nabla_\nu \Psi)$$ \hspace{1cm} (4.3)

Using the operational properties of $\nabla_\mu$ we find

$$K_{\mu\nu} = \Gamma_{\mu,\nu} - \Gamma_{\nu,\mu} + [\Gamma_{\nu}, \Gamma_{\mu}]$$ \hspace{1cm} (4.4)

Since $\nabla_\mu \nabla^\mu = 0$, a relation between the Riemann curvature tensor and spin-curvature is found from
\[ 0 = \nabla_\rho (\nabla_\alpha \gamma_\mu) - \nabla_\beta (\nabla_\alpha \gamma_\mu) \]  
(4.5)

\[ = R_{\mu\lambda\alpha\beta} \gamma^\lambda + [K_{\alpha\beta}, \gamma_\mu] \]  
(4.6)

A solution to equation (4.6) for \( K_{\alpha\beta} \) is

\[ K_{\mu\lambda\alpha\beta} = \frac{1}{4} R_{\mu\lambda\alpha\beta} \sigma^{\alpha\beta} \]  
(4.7)

Due to the commutator in equation (4.6) this solution can admit an antisymmetric second rank tensor multiplied by the 4 x 4 identity matrix as an additive part. It was seen in Chapter III, along similar arguments, that minimal coupling in Dirac's equation allowed the natural inclusion of the vector potentials \( A_\mu \) in the Fock-Ivanenko coefficients. In this case, the spin-curvature in equation (4.4) or (4.7) would have the additional piece \((ie/4c)F_{\mu\nu}\) where \( F_{\mu\nu} \) is the electromagnetic field tensor,

\[ F_{\mu\nu} := A_{\mu,\nu} - A_{\nu,\mu} \]  
(4.8)

Pagels (1965) has made fruitful analogies with the spin-curvature and electromagnetic field.

It can be shown that, because \( \nabla_\epsilon \sigma^{\mu\nu} = 0 \), then

\[ \nabla_\lambda K_{\alpha\beta} + \nabla_\alpha K_{\rho\lambda} + \nabla_\rho K_{\lambda\alpha} = 0 \]  
(4.9)

With equation (4.7) one is able to show that equation (4.9) is
satisfied by the Bianchi identities.

Multiplying on the right of equation (4.6) with $\gamma_\mu$, using the fact that

$$
\sigma^{(\alpha)(\beta)} = \gamma^{(\alpha)} \gamma^{(\beta)} - \eta^{(\alpha)(\beta)}
$$

(4.10)

and taking the trace we have

$$
R_{\lambda \mu \alpha \beta} = \frac{1}{2} \text{Tr} (K_{\alpha \beta} \sigma_{\mu \lambda})
$$

(4.11)

By the Abelian property of the trace and antisymmetry of $K_{\alpha \beta}$ and $\sigma_{\mu \lambda}$ we have that the usual symmetries of the Riemann tensor can be visualized in this one single expression.

It is possible to express the usual tensor constructs of Riemannian space-time in terms of spin-tensor quantities such as $K_{\mu \nu}$ etc. For example, we provide the following facts without proof:

$$
\gamma_{\alpha} K_{\alpha \beta} \gamma^{\beta} = 0,
$$

(4.12)

$$
[K_{\mu \nu}, \sigma^{\mu \nu}] = 0,
$$

(4.13)

$$
R_{\mu \rho \lambda \beta} = \frac{1}{2} \{ \gamma_\rho, [K_{\lambda \beta}, \gamma_\beta] \},
$$

(4.14)

$$
R_{\mu \rho \lambda \beta} = \frac{1}{2} \text{Tr} (K_{\lambda \beta} \sigma_{\mu \rho}),
$$

(4.15)

$$
R_{\mu \beta} = \frac{1}{2} \{ \gamma_\lambda, [K_{\mu \rho}, \gamma_\rho] \},
$$

(4.16)
\[ R_{\mu\nu} = \frac{1}{i} \text{Tr} (K_{\alpha\beta} \sigma^{\alpha\beta}) \]  

(4.17)

\[ R \Sigma = \gamma^\alpha K_{\alpha\beta} \gamma^\beta - K_{\alpha\beta} \sigma^{\alpha\beta} \]  

(4.18)

\[ R = -\frac{1}{i} \text{Tr} (K_{\alpha\beta} \sigma^{\alpha\beta}) \]  

(4.19)

where we have the conventions,

\[ R_{\mu\rho} := q^{\rho\phi} R_{\mu\phi\beta} = R_{\phi\mu\rho} \]  

(4.20)

\[ R := q^{\mu\rho} R_{\mu\rho} = R_{\rho\mu} \]  

(4.21)

We may write the Einstein tensor, \( G_{\mu\nu} \), in this formalism as

\[ G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} q_{\mu\nu} R \]  

(4.22)

\[ = \text{Tr} \left[ \frac{1}{2} (q_{\mu\alpha} K_{\rho\nu} - \frac{1}{2} q_{\mu\nu} K_{\rho\alpha}) \sigma^{\alpha\beta} \right] \]  

(4.23)

The full Einstein field equations are

\[ G_{\mu\nu} = (-8\pi G/c^2) T_{\mu\nu}, \]  

(4.24)

where \( T_{\mu\nu} \) is the energy-momentum tensor.

There is another important and nontrivial relation which can be algebraically worked out from equation (4.6) and Riemann tensor symmetries:
\[ R I = -\frac{1}{4} R_{\mu\nu\rho\sigma} \sigma^{\mu\nu} \sigma^{\rho\sigma} = -2 K_{\mu\nu} \sigma^{\mu\nu}. \] (4.25)

The generally covariant Dirac equation (3.8) may be iterated in the familiar special relativistic fashion to give

\[ \gamma^\nu \nabla_\mu (\nabla_\nu \psi) - \frac{i}{2} K_{\mu\nu} \sigma^{\mu\nu} \psi - \frac{1}{2} \left( i q / \hbar c \right) F_{\mu\nu} \sigma^{\mu\nu} \psi + \gamma^\nu \chi \psi = 0, \] (4.26)

where we have included the electromagnetic field tensor, \( F_{\mu\nu} \), for comparison with the spin-curvature, \( K_{\mu\nu} \). Equation (4.26) can be further reduced using equation (4.25) and the Einstein field equations to show that

\[ \frac{\gamma^1}{c} K_{\mu\nu} \sigma^{\mu\nu} \psi = \left( \frac{2}{\pi} \frac{G}{\sigma^2} \right) T \psi. \] (4.27)

where \( T \) is the trace of the energy momentum tensor. The coupling of the electron spin with the curvature gives a small contribution since \( \chi^1 + 2 \pi G c^2 T = \chi^1 \left( 1 + 2 \pi G c^2 m^{-2} c^{-4} T \right) = \chi^1 \left( 1 + 7 \times 10^{-48} \text{gm}^{-1} \text{cm}^3 \text{T} \right); \) high densities of the order \( 10^{13} \text{gm} \text{cm}^{-3} \) would be significant.

We also note the following expansion,

\[ \gamma^\nu \nabla_\mu (\nabla_\nu \psi) = \gamma^\nu \left( \psi; \nabla_\mu + \Gamma^\nu_{\rho\mu} \psi + \Gamma^\nu_{\mu,\sigma} \psi + \Gamma^\nu_{\rho,\sigma} \psi \right), \] (4.28)

where

\[ \Gamma^\nu_{\rho\mu} := \nabla_\rho \psi + \left( i q / \hbar c \right) A^\rho \psi. \] (4.29)

In Chapter III we found that there is a conserved current, \( j^\mu \), for the Dirac field. This current may be written:
Here we have performed the usual Gordon decomposition of the current \( j_\mu \); the first term of equation (4.30) in the square brackets corresponds to the conserved current of the iterated equation (4.26). Note again the close relationship of \( \tilde{\Gamma}_\mu \) with \( A_\mu \) in equation (4.30), just as is evident through \( K_\mu \nu \) and \( F_\mu \nu \) in equation (4.26).

Both the Dirac equation and its iterated counterpart, coupled to a linearized gravitational field, have been studied by Lawrence (1969, 1970) and others. Lawrence indicated that the iterated equation would lead to a Papapetrou-type-equation (Papapetrou 1951) in a WKB limit.

We will show that the Klein-Gordon particle follows geodesics in the linearized gravitational field limit. The argument is a generalization of the nonrelativistic quantum case.

First we establish the linearized limit for our Dirac equation (3.8) and its iterated form (4.26). We will neglect the minimal coupling to the electromagnetic field.

In the domain of weak gravitational fields we write the metric as
\[ q^{\mu \nu} = \eta^{\mu \nu} + h^{\mu \nu} \]  

(4.31)

where \( h^{\mu \nu} \ll 1 \) and \( \eta^{\mu \nu} \) is the Lorentz metric. Written out in full:

\[
q^{\mu \nu} = \begin{bmatrix}
1 + h^{00} & h^{01} & h^{02} & h^{03} \\
-1 - h^{10} & h^{11} & -h^{12} & h^{13} \\
-1 - h^{20} & -h^{21} & 1 + h^{22} & h^{23} \\
-1 - h^{30} & -h^{31} & -h^{32} & 1 + h^{33}
\end{bmatrix}
\]  

(4.32)

As long as we restrict ourselves to first order in \( h \) we must raise and lower all indices using \( \eta^{\mu \nu} \), viz.

\[
\eta_{\lambda \rho} h^{\rho \nu} = h^{\nu \lambda}, \quad \eta_{\lambda \rho} \frac{\partial}{\partial \chi_{\rho}} = \frac{\partial}{\partial \chi_{\lambda}}
\]  

(4.33)

The contravariant form of the metric must be

\[
q_{\mu \nu} = \begin{bmatrix}
1 - h^{00} & h^{01} & h^{02} & h^{03} \\
-1 + h^{10} & h^{11} & -h^{12} & -h^{13} \\
-1 + h^{20} & -h^{21} & 1 - h^{22} & -h^{23} \\
-1 + h^{30} & -h^{31} & -h^{32} & 1 - h^{33}
\end{bmatrix}
\]  

(4.34)

in order that \( q^{\mu \rho} q_{\rho \beta} = \delta_{\beta}^{\mu} \). Thus, if we define

\[
q_{\mu \nu} := \eta_{\mu \nu} + h_{\mu \nu}
\]  

(4.36)

then

\[
h_{\mu \nu} := -\eta_{\mu \alpha} \eta_{\nu \beta} h^{\alpha \beta}
\]  

(4.36)

We find from the above to \( O(h^2) \) that

\[
ge^{\mu} := \text{det} q_{\mu \nu} = -1 + h^{00} - h^{11} - h^{22} - h^{33}
\]  

(4.37)

Define

\[
h_{\alpha} := -\eta_{\alpha \beta} h^{\beta \rho} := -h^{\alpha}_{\beta}
\]  

(4.38)
then
\[ q = - (1 + h), \]  \hspace{1cm} (4.39)

\[ q^{-1} = - (1 - h). \]  \hspace{1cm} (4.40)

The Christoffel symbols of equation (3.2) become
\[ \{ ^\lambda \nu \mu \} = - \frac{1}{2} \left( \eta_{\nu \mu} h^\lambda, \nu + \eta_{\nu \sigma} h^\lambda, \nu \right) - \eta_{\nu \mu \sigma} \eta_{\nu \beta} \eta_{\nu \gamma} \sigma_{\gamma} \sigma_{\nu} \]  \hspace{1cm} (4.41)

\[ \{ ^\gamma \nu \mu \} = - \frac{1}{2} \eta_{\nu \mu} h^\gamma, \mu. \]  \hspace{1cm} (4.42)

Consider the tetrads for the linearized case. We can put them in the form $e^{\mu(\alpha)} = \eta^{\mu \nu} + k^{\mu \nu}$ where there is no difference between global and local indices. For $g^{\mu \nu} = \eta^{\mu \nu} + h^{\mu \nu}$ we can always rotate the tetrads so that $h^{\mu \nu} = k^{\mu \nu} + k^{\nu \mu}$, i.e. the $k^{\mu \nu}$ are symmetric. This constitutes six conditions on the tetrads. In order to obey the completeness relation (5.8) for the metric we have:

\[ e^{\mu(\alpha)} = \eta^{\mu \alpha} + \frac{1}{2} h^{\mu \alpha}, \]  \hspace{1cm} (4.43)

\[ e^{\alpha(\nu)} = \delta_{\alpha}^{\mu} + \frac{1}{2} h^{\mu \alpha}, \]  \hspace{1cm} (4.44)

\[ e^{\mu(\alpha)} = \eta^{\mu \alpha} + \frac{1}{2} h^{\mu \alpha} \]  \hspace{1cm} (4.45)

\[ e^{\alpha(\nu)} = \eta^{\mu \alpha} - \frac{1}{2} \eta_{\mu \sigma} \eta_{\nu \sigma} \]  \hspace{1cm} (4.46)

\[ e^{\nu(\mu)} = \delta^{\nu}_{\mu} + \frac{1}{2} \eta^{\mu \rho} r_{\nu \rho}. \]  \hspace{1cm} (4.47)
\[ = \delta^\alpha_{\mu} - \frac{1}{2} \eta^\alpha_{\rho} h^{\rho}_{\sigma} \hphantom{\epsilon} (4.48) \]

For the linear limit we put
\[ \gamma^{(\alpha)} := \gamma^\alpha \hphantom{\epsilon} (4.49) \]
\[ \sigma^{(\alpha)}(\beta) := \sigma^\alpha \beta \hphantom{\epsilon} (4.50) \]

The Fock-Ivanenko coefficient (3.21) has the limit
\[ \sigma^\alpha_{\mu} = \frac{1}{4} \gamma^{\alpha}_{\mu, \beta} \sigma^\beta \hphantom{\epsilon} (4.51) \]

In the Dirac equation we have terms like \( \gamma^\alpha_{\mu} \) which have the limit
\[ \gamma^\alpha_{\mu} \gamma^\mu = \frac{1}{4} \left( \gamma^\alpha_{\mu, \nu} - \gamma^\nu_{\mu, \alpha} \right) \gamma^\alpha \hphantom{\epsilon} (4.52) \]

But for the linear theory it is always possible to choose the standard Weyl gauge,
\[ \gamma^\nu_{\mu, \nu} - \frac{1}{2} \gamma^\nu_{\nu, \mu} = 0. \hphantom{\epsilon} (4.53) \]

Thus equation (4.52) becomes
\[ \gamma^\alpha_{\mu} \gamma^\mu = -\frac{1}{8} \gamma^\nu_{\nu, \mu} \gamma^\alpha \hphantom{\epsilon} (4.54) \]
Finally, the generally covariant Dirac equation

\[ \gamma^\alpha \partial_\alpha \psi + \gamma^\alpha \Gamma_\alpha \psi + i \kappa \psi = 0 \]  

may be written in the linear theory as

\[ \tilde{\gamma}^\alpha (\partial_\alpha + \frac{i}{2} h^\alpha_{\mu\nu} \partial_\nu - \frac{i}{8} h_{\alpha,\mu}) \psi + i \kappa \psi = 0 . \]  

Letting \[ \bar{\psi} := e^{\alpha \psi} \left( \frac{1}{\sqrt{\gamma}} \right) \psi \]  

then equation (4.55) becomes

\[ \tilde{\gamma}^\alpha (\partial_\alpha + \frac{i}{2} h^\alpha_{\mu\nu} \partial_\nu) \bar{\psi} + i \kappa \bar{\psi} = 0 . \]  

The iterated equation (4.26) with the expansion of \[ \tilde{\gamma}^\alpha \nabla_\alpha (\nabla \psi) \]

in equation (4.28) may be found in the linear limit using equations (4.51) and (4.53) as:

\[ \tilde{\gamma}^\alpha (\psi_{\mu\nu} + \frac{i}{2} h_{\mu\nu} \sigma^{\alpha\beta} \psi_\beta) + \frac{i}{4} \alpha^2 \psi + \kappa^2 \psi = 0 . \]  

In the linear limit the Ricci scalar is

\[ R = - \frac{i}{2} h_{\alpha,\alpha} \]  

so that equation (4.57) becomes

\[ \psi_{\mu\nu} + \tilde{\gamma}^\alpha \psi_{\mu,\nu} + \frac{i}{2} \tilde{\gamma}^\alpha h_{\mu\nu,\beta} \sigma^{\alpha\beta} \psi_\beta = \frac{i}{8} h_{\alpha,\alpha} \psi + \kappa^2 \psi = 0 . \]  

(4.59)
If we put \( \psi := \exp(\hbar c/8) \psi \) and again use the Weyl gauge then equation (4.59) finally is
\[
\frac{i}{\hbar c} \left( \partial \gamma \partial \sigma + \frac{i}{2} \hbar c \partial \sigma \partial \gamma + \frac{i}{2} \hbar \gamma \partial \sigma \partial \gamma \right) \psi + \chi^2 \psi = 0 \quad (4.60)
\]

The methods of Lawrence (1969, 1970) yield the same equations of motion in the WKB limit using equation (4.60) or (4.57) with the \( \hbar c R \) term neglected.

Consider the generally covariant Klein-Gordon equation which describes particles with no spin:
\[
\partial^\mu \partial_\mu \phi + m^2 \phi = 0 \quad (4.61)
\]

Consider the situation where the wave function of a quasi-classical physical system has the form
\[
\phi = A \exp(i S/\hbar) \quad (4.62)
\]

where \( A \) and \( S \) are real. Substituting this into the Klein-Gordon equation and equating imaginary and real parts to zero, we have
\[
\begin{align*}
2A,_{\nu} S^\nu + AS,_{\nu} S^\nu - \hbar A S,_{\nu} S^\nu + \hbar c A,_{\nu} S^\nu & = 0, \quad (4.63) \\
A,_{\nu} S^\nu - \hbar A S,_{\nu} S^\nu + \hbar c A,_{\nu} S^\nu + m^2 c^2 A & = 0. \quad (4.64)
\end{align*}
\]

These may be written as
\((\sqrt{-g} A^2 S^\nu), \nu = 0\),

\[(4.65)\]

\[\delta_\nu S^\nu - m^2 c^2 = \hbar^2 \left(\frac{\sqrt{-g} A}{\sqrt{-g} A}\right)_\nu = \hbar^2 \frac{\Box A}{A}\]

\[(4.66)\]

Equation (4.65) has the usual interpretation of representing current conservation since the probability density \(\rho\) is \(\rho \phi = A\).

Equation (4.66) has the usual interpretation of the Hamilton-Jacobi equation. As usual we put

\[p_\mu := S_{,\mu}\]

so that, equation (4.66) becomes

\[p_\mu p^\mu = m^2 c^2 + \hbar^2 \frac{\Box A}{A}\]

\[(4.68)\]

A vector \(q_\mu\) is geodesic if

\[q_\mu ; \nu q^\nu + \frac{1}{2} q^\nu q_\nu = 0\]

\[(4.69)\]

Here we are able to take advantage of (4.67) to write

\[p_\mu ; \nu p^\nu = p_\nu ; \mu p^\nu = \frac{1}{2} (p_\nu p^\nu) ; \mu\]

\[(4.70)\]

Using equation (4.68) in this expression we find

\[p_\mu ; \nu p^\nu = \frac{1}{2} \hbar^2 \left(\frac{\Box A}{A}\right) ; \mu\]

\[(4.71)\]
If\[ p^v - \alpha^2 \dot{\xi} \, dS \quad (4.72) \]
then equation (4.71) implies geodesic motion, and if
\[ \Box A = 0 \quad (4.73) \]
then the motion is affinely parameterized so we recover the geodesic equation of motion
\[ \frac{d^2 x^\alpha}{ds^2} + \left\{ \frac{\alpha}{\beta} \right\} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \quad (4.74) \]
This classical behaviour is analogous to the Schrödinger case (see, for example, Landau and Lifshitz 1965).

\[ \]

\[ I \text{ originally showed equations (4.71) and (4.74) by the methods of Lawrence (1969); Dr. E. N. Glass gave the elegant method used in the above.} \]
CHAPTER V

TETRADS IN CURVED SPACE-TIME

In Chapter III we introduced the notion of a tetrad in regard to the curved space-time $\mathcal{C}$-matrices, $\mathcal{C}_\alpha(x)$, of equation (3.6) and the Fock-Ivanenko coefficients, $\Gamma^\alpha_\mu$, of equation (3.21). Here we deal with tetrads in a general way as a method to relate tensorial quantities in different frames of reference. The main result of this Chapter is to give an algorithm for the explicit calculation of tetrads for any given metric; they will be used for calculational purposes in Chapter VI.

At each point on a smooth manifold of four dimensions with indefinite metric $g_{\mu\nu}$ (Riemannian space-time) one is able to construct a locally flat reference frame as a set of Lorentzian differentials $dx^{(\alpha)}$. At the same point there are the general coordinates, $x^\mu$, associated with the metric $g_{\mu\nu}(x)$ whose 'coordinate differentials, $dx^\mu$, are assumed to be related to $dx^{(\alpha)}$ by

$$d x^\mu = \frac{\partial x^\mu}{\partial x^{(\alpha)}} \, dx^{(\alpha)} \quad (5.1)$$

The $\frac{\partial x^\mu}{\partial x^{(\alpha)}}(x)$ are the four ($\alpha = 0, 1, 2, 3$) tetrad (vierbein or four-leg) vectors in the Riemannian space-time. They relate tensor quantities in the Riemann space-time to the observer's locally Lorentz frame much in the same way scale factors can be used to project down 'physical components' of tensors in a curvilinear coordinate system to a Cartesian system (Synge and Schild 1949). In a true gravitational field the Riemann tensor is nonzero; the one-forms $dx^\mu = \frac{\partial x^\mu}{\partial x^{(\alpha)}} \, dx^{(\alpha)}$
are not exact in an arbitrary curved space. In flat space the Riemann tensor is zero so that \( \gamma^\alpha/\beta \) become tetrads; for the case of special relativity the tetrads are just the constant velocity-dependent Lorentz transformation coefficients.

The invariant interval may be written

\[
 ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{(\alpha)(\beta)} d\lambda^{(\alpha)} d\lambda^{(\beta)}. \tag{5.2}
\]

Using equations (3.7), (5.1), and (5.2) we see that

\[
 \lambda_{(\alpha)}^\mu \lambda^{\alpha}_{(\beta)} = \eta_{(\alpha)(\beta)}. \tag{5.3}
\]

For equation (5.1) to be invertible we put

\[
 d\lambda^{(\alpha)} = \lambda^{(\alpha)} \quad d\lambda^{(\beta)}. \tag{5.4}
\]

With equation (5.1) in (5.4) we find

\[
 \lambda_{\nu}^{\alpha} \quad \lambda_{\nu}^{(\alpha)} = \delta_{\nu}^{\alpha}, \tag{5.5}
\]

and with equation (5.4) in (5.2) that

\[
 \lambda_{\nu}^{(\beta)} = \lambda_{\nu}^{(\beta)}. \tag{5.7}
\]

Thus, the fundamental relationships giving the tetrad structure on space-time are:
\[ \lambda{_{(\alpha)}}{_{\epsilon}} \lambda{_{(\nu)}}{_{(\rho)}} = \eta{_{(\nu)}}{_{(\rho)}} \] (5.3)

\[ \lambda{_{(\alpha)}}{_{\epsilon}} = g{^{\epsilon}}{_{\mu}} \] (5.8)

From these relations we can deduce:

\[ \{ \epsilon_{\alpha} \} = \lambda{_{(\alpha)}}{_{\epsilon}} \lambda{_{(\mu)}}{_{\epsilon}}{_{\rho}} \] (5.9)

\[ \lambda{_{(\alpha)}}{_{\epsilon}} \lambda{_{(\beta)}}{_{\rho}} = - \lambda{_{(\alpha)}}{_{\epsilon}} j_{\beta}{_{\epsilon}}{_{\rho}} \lambda{_{(\rho)}}{_{\epsilon}} \] (5.10)

Equation (5.3) holds if the tetrads are Lorentz rotated so that six components of \( \lambda{_{(\alpha)}}{_{\epsilon}} \) can be freely chosen in order to determine the ten independent components of \( g{^{\mu}}{_{\nu}} \). We find it advantageous to follow the 'canonical gauge' of de Oliveira and Tiomno (1962) and put \( \lambda{_{(1)}}{_{\epsilon}} = \lambda{_{(2)}}{_{\epsilon}} = \lambda{_{(3)}}{_{\epsilon}} = 0 \). This defines the tetrad \( b{_{(\alpha)}}{_{\epsilon}} \).

So now equations (5.3) and (5.8) become

\[ b{_{(\alpha)}}{_{\epsilon}} b{_{(\mu)}}{_{\rho}} = \eta{_{(\nu)}}{_{(\rho)}} \] (5.11)

\[ b{_{(\alpha)}}{_{\epsilon}} b{_{(\nu)}}{_{(\rho)}} = g{^{\epsilon}}{_{\mu}} \] (5.12)

We have solved (5.12) explicitly. On choosing square root signs so \( \det b{_{(\alpha)}}{_{\epsilon}} = (-g^{1/2}) \), where \( g = \det g{^{\mu}}{_{\nu}} \), the result is:

\[ b{_{(\alpha)}}{_{\epsilon}} = \begin{bmatrix}
A & 0 & 0 & 0 \\
q^{10}/A & B & 0 & 0 \\
q^{12}/A & -\omega^{12}/B & C & 0 \\
q^{13}/A & -\omega^{13}/B & D & E
\end{bmatrix} \] (5.13)
where \( \omega^{ij} = g^{ij} - q^i_0 q^j_0 / q^0_0 \), \( (5.14a) \)

\[ A = (q^{00})^{1/2} \]  
 \( (5.14b) \)

\[ B = (-\omega^{ii})^{1/2} \]  
 \( (5.14c) \)

\[ C = \left[ \omega^{ii} \omega^{jj} - (\omega^{ij})^2 \right]^{1/2} B^{-1} \]  
 \( (5.14d) \)

\[ D = (C^{\mu})^{-1} (\omega^{ii} \omega^{33} + \omega^{12} \omega^{13}) \]  
 \( (5.14e) \)

\[ E = (-\omega^{33} + (\omega^{13})^2 / w^2 - \gamma^2)^{1/2} \]  
 \( (5.14f) \)

For diagonal metrics equation \( (5.13) \) collapses to

\[ \sigma^{\mu}_{(\alpha)} = \left[ \eta_{(\alpha)(\beta)} q^{\mu\alpha} \delta_{(\alpha)}^{\beta} \right]^{1/2} \text{ (no sum) } \]  
 \( (5.15) \)

(From the appearance of this expression one sometimes refers to tetrads as being the 'square root' of the metric.)

In some applications (e.g. Chapter VI) of the Dirac equation one may calculate, given the metric, the tetrads from equation \( (5.13) \) and then the Fock-Ivanenko coefficients directly from equation \( (3.21) \). However, this procedure can sometimes become overwhelming and it is useful (e.g. Chapter VIII) to take advantage of certain algebraic properties of the Dirac \( \gamma \)-matrices. The term of interest is the \( \gamma^{\mu}_{\mu} \) term in the Dirac equation \( (3.3) \).

A useful quantity is the gamma-five matrix:
\[ \gamma^{(s)} = \gamma_{(s)} = \gamma^{(r)} \gamma^{(1)} \gamma^{(2)} \gamma^{(3)} \]  

\[ = i \frac{1}{4!} \epsilon_{\mu \nu \rho}(\sigma) \gamma^{(r)} \gamma^{(\nu)} \gamma^{(\rho)} \gamma^{(\sigma)} \]  

\[ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \]

We note the usual properties:

\[ \{ \gamma^{(r)} \gamma^{(s)} \} = 0 \]  

\[ \gamma^{(s)} \gamma^{(k)} = \frac{1}{3!} \epsilon_{\lambda(\mu)(\nu)(\rho)} \gamma^{(k)} \gamma^{(\nu)} \gamma^{(\rho)} \]  

where \( \epsilon_{\lambda(\mu)(\nu)(\rho)} \) is the completely antisymmetric unit symbol such that \( \epsilon_{(\omega)(\lambda)(\mu)(\nu)(\rho)} = 1 \). Using well known properties of the alternating symbol, equation (3.20) and the fact that

\[ \{ \gamma^{(r)} \} = \frac{1}{2} \epsilon_{\lambda(\mu)\gamma(\delta)} \gamma^{(\delta)} \gamma^{(s)} \]  

then we find that (de Oliveira and Tjomno 1962, Lawrence, Leiter, and Szamosi 1973):

\[ \gamma^{\mu} \gamma^{(r)} = \gamma^{(r)} [ A_{(r)}(x) + i \gamma^{(s)} B_{(r)}(x) ] \]  

where

\[ A_{(r)}(x) = \frac{i}{2} (-q) \gamma^{\mu} \gamma^{(r)}(x) \]  

\[ B_{(r)}(x) = \gamma^{\mu} \gamma^{(r)}(x) \]
\[ B_{(s)}(\lambda) = \frac{1}{4} \epsilon_{\lambda_1(\lambda_2(\lambda_3(\lambda_4)))} \lambda_1^{(\lambda_2)} \lambda_2^{(\lambda_3)} \lambda_3^{(\lambda_4)} \lambda_4^{(\lambda)} . \] (5.22)

In fact, for \( A_\mu = \lambda_\mu (A) \), we find

\[ \chi^\mu A_\mu = \frac{1}{2} \{ \chi^\mu, \Gamma_\mu \} , \] (5.23)

\[ \chi^\mu \cdot \chi^{(s)} B_\mu = \frac{1}{2} \{ \chi^\mu, \Gamma_\mu \} . \] (5.24)

For a diagonal metric

\[ B_\mu = 0 , \] (5.25)

\[ \chi^\nu \Gamma_\nu = \chi^\nu \left[ \ln (-g)^{\nu\mu} \right] \chi^\mu - \frac{1}{4} \chi^\nu g^{\alpha\beta} g_{\alpha\beta} . \] (5.26)
CHAPTER VI
APPLICATIONS TO FLAT NONINERTIAL FRAMES

We now return to the question posed in Chapter II concerning how one can formulate the description of 'quantum processes' in accelerative frames which is (i) covariant and (ii) representation independent. We should first recapitulate the state of affairs up to now. Wave functions describing a physical system transform as scalars for all Lorentz observers. The generalization of the Dirac equation to flat noninertial frames gave a covariant equation (2.27) but which was not representation independent. The relevant equations are:

\[-i\gamma^\mu (\partial_\mu + N_{\mu} N^{-1}) N \Psi' + N \gamma^\mu \Psi' = 0,\]  

(2.27)

\[\{\gamma^\mu, \gamma^\nu\} = 2\epsilon^{\mu\nu\alpha}\Gamma_\alpha\]  

(2.25)

\[\gamma^\mu = N \gamma^{\prime\mu} N^{-1},\]  

(2.26)

\[\gamma^{\prime\mu} = \frac{\partial \chi^{\mu}}{\partial X^{(\alpha)}} \gamma^{(\alpha)}.\]  

(2.21)

But in Chapter III we precisely constructed a Dirac equation which satisfied both of the above conditions. Therefore, we shall consider this equation to be better suited than equation (2.27) to describe quantum mechanics in flat noninertial frames. We have

\[-i\gamma^\mu (\partial_\mu + \Gamma_\mu) \Psi + N \Psi = 0\]  

(3.3)
where we now prescribe that the $\Phi$-matrices appearing in equations (3.3) and (2.27) be the ones constructed from the canonical tetrads according to

$$\gamma^{\mu} = \beta^{\mu}_{(\alpha)} \gamma^{(\alpha)}$$

which, of course, satisfies equation (2.25). For the choice of tetrad (which can be determined for any metric from equation (5.13)) the Fock-Ivanenko coefficient expression (3.21) becomes

$$\Gamma_{\mu} = \frac{1}{4} \left[ \frac{\gamma^{\nu}}{\beta^{\nu}_{(\alpha)}} b^{\beta}_{(\mu)} - b^{\nu}_{(\mu)} \right] \frac{\gamma^{(\alpha)}}{\beta^{(\alpha)}} g^{(\nu)(\beta)}$$

(6.2)

The choice of equation (3.3) over (2.27) is merely a formal constraint since the former will describe the same physical system as the latter if

$$\psi(x) = N(x) \overline{\psi}(x) = N(x) \overline{\psi}(x)$$

(6.3)

$$\Gamma_{\mu}(x) = -N(x)_{\mu} N^{-1}(x)$$

(6.4)

where we have used equation (2.20) in (6.3). Note that all derivatives in equations (6.4), (6.2), (3.3), and (2.27) are taken with respect to the same metric $g_{\mu\nu}$ of the particular noninertial frame.

In other words, the wave function $\psi(x)$ for the observer in the noninertial frame is related to the wave function $\overline{\psi}(x)$ of the inertial frame observer (who also looks at the same physical situation) through an equivalence relation, $N(x)$, given by equation...
(6.3). The problem is to know what \( N(x) \) is. However, equation (6.4) gives a simple differential equation for \( N(x) \) since \( \Gamma_\mu(x) \) is known from the given metric. Formally the solution of equation (6.4) is

\[
N(x) = \exp \left( - \int x^\mu \Gamma_\mu(x) \, dx^\nu \right). 
\]  

(6.5)

In general \( \gamma^\mu \neq \delta^\mu_\nu \); but, since the \( dx^\mu/dx^{(x)} \) in equation (2.22) relate to the metric in the same fashion as the \( b^\mu_{(x)} \) do in equation (5.12), then they must differ from each other by a coordinate-dependent Lorentz transformation, \( M^{(\beta)}_{(x)}(x) \), as

\[
\frac{dx^\nu}{dx^{(x)}} := M^{(\beta)}_{(x)}(x) \frac{b^\nu_{(x)}(x)}{b^{\beta}_{(x)}(x)}. 
\]  

(6.6)

Recalling equation (2.26) and using a minor extension of Pauli's fundamental theorem (Good 1955) to coordinate-dependent transformations we know that there exists a nonsingular equivalence matrix \( N(x) \) such that

\[
\gamma^{(\beta)}_{(x)}(x) = b^{\nu}_{(\beta)}(x) M_{(x)}^{(\beta)}(x) \delta^{\nu}_{(\beta)}(x). 
\]  

(6.7)

\[
= N_{(x)}^{-1} \gamma^{\nu}_{(x)}(x) N(x). 
\]  

(6.3)

We shall now calculate some equivalence transformations for different accelerated frames. The inertial Lorentz frame will generally be denoted by the Cartesian coordinates \( x^{(\nu)} = (ct, x, y, z) \) and the flat noninertial frame by \( x^{(\beta)} = (ct, x, y, z) \).

The first example, which we shall consider in detail, is
a rotating frame which has a positive uniform angular velocity $\omega$ about the $Z$ axis. The coordinate transformation from the inertial Lorentz frame, $X^{(x)}$, to the rotating one, $x^A$, is well known (e.g. Bow 1972); it is

$$ t = T, \quad (6.9a) $$

$$ x = X \cos \theta + Y \sin \theta, \quad (6.9b) $$

$$ y = -X \sin \theta + Y \cos \theta, \quad (6.9c) $$

$$ z = Z, \quad (6.9d) $$

where $\theta = \omega t = \omega T. \quad (6.10) $$

The transformation coefficients are found directly from the coordinate transformation:

$$ \frac{\partial x^A}{\partial X^{(x)}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{\omega}{c} Y & \cos \theta & \sin \theta & 0 \\ -\frac{\omega}{c} X & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.11) $$

The contravariant metric can be calculated from equation (6.11) and (2.22) as

$$ g^{\mu\nu} = \begin{bmatrix} 1 & \frac{\omega}{c} Y & -\frac{\omega}{c} X & 0 \\ \frac{\omega}{c} Y & \frac{\omega}{c}^2 Y^2 - 1 & -\frac{\omega}{c} X & 0 \\ -\frac{\omega}{c} X & -\frac{\omega}{c} X & \frac{\omega}{c}^2 X^2 - 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (6.12) $$
Thus the line element may be written

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

\[ = \left[ 1 - \frac{\omega^2}{c^2} (x^2 + y^2) \right] c^2 dt^2 + 2 \omega y dt dx - 2 \omega x dt dy - dx^2 - dy^2 - dz^2. \]  

(6.13)

The canonical gauge tetrads may be calculated from (5.13) as

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\omega}{c^2} & 0 & 0 \\
0 & 0 & \frac{\omega}{c^2} & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

(6.14)

Incidentally, the coordinate dependent Lorentz transformation relating \( e^\mu / s^\nu \) and \( b^\mu (s) \) is

\[
M(s) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \Theta & -\sin \Theta & 0 \\
0 & \sin \Theta & \cos \Theta & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

(6.15)

The Christoffel symbols for the metric of equation (6.13) may be most easily calculated from equation (3.3) since the Christoffel symbols in the Lorentz frame are all zero. The nonzero ones in the accelerated frame are:

\[
\begin{align*}
\{^0_0\} &= -\frac{\omega}{c^2} x, & \{^0_1\} &= -\frac{\omega}{c^2} y, & \{^0_2\} &= -\frac{\omega}{c}, & \{^2_0\} &= \frac{3}{c} \\
\end{align*}
\]

(6.16)

The Fock-Ivanenko coefficients may be calculated from equations (6.16), (6.14), (6.12), and (6.2) as:

\[
\sum_i \Gamma_i = 0 
\]

(6.17a)
\[ \nabla \sigma^{(1)(2)} = -\frac{\alpha}{2c} \begin{pmatrix} \sigma^{(3)} & 0 \\ 0 & \sigma^{(3)} \end{pmatrix} \]  \hspace{1cm} (6.17b)

We now use equation (6.5) to calculate the equivalence relation:

\[ N = \exp \left[ \frac{i \theta}{\hbar} \begin{pmatrix} \sigma^{(1)} & 0 \\ 0 & \sigma^{(1)} \end{pmatrix} \right] \]  \hspace{1cm} (6.18)

From equations (6.18) and (6.3) one can see the connection between the wave function \( \Psi(x) \) in the rotating frame and the wave function \( \Psi'(x) \) in the inertial frame as

\[ \Psi = \exp \left[ \frac{i \theta}{\hbar} \begin{pmatrix} \sigma^{(3)} & 0 \\ 0 & \sigma^{(3)} \end{pmatrix} \right] \Psi' \]  \hspace{1cm} (6.19)

The resemblance between this equivalence relation and that found in a case of Bjorken and Drell (1964, p. 23), for example, is of no significance since the cases we treat are basically different and are handled by a different covariance procedure. If different tetrads, rather than the canonical gauge ones, were chosen then the equivalence transformations would have been altered.

As a second example consider a Galilean accelerating frame defined by the transformations:

\[ t = T, \]  \hspace{1cm} (6.20a)

\[ x = \chi, \]  \hspace{1cm} (6.20b)
\[ y = \gamma \quad (6.20c) \]

\[ z = \frac{z}{\lambda^2} \quad (6.20d) \]

The interval is given by

\[ ds^2 = [1 - (at/c)^2] c^2 dt^2 - 2at dt dt' - dx^2 - dy^2 - dz^2. \quad (6.21) \]

The contravariant form of the metric is

\[ q^{\mu \nu}_0 = \begin{bmatrix} 1 & 0 & 0 & -at/c \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -at/c & 0 & 0 & -1 + (at/c)^2 \end{bmatrix} \quad (6.22) \]

so that the tetrads are

\[ b^\mu_{(a)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -at/c & 0 & 0 & 1 \end{bmatrix} \quad (6.23) \]

In this case

\[ \frac{\partial x^\mu}{\partial x^{(a)}} = b^\mu_{(a)} \quad (6.24) \]

The only nonzero Christoffel symbol is

\[ \frac{\Gamma^{\gamma}_{\alpha \beta}}{\Gamma^{\gamma}_{\alpha \beta}} = \frac{a}{c^2}. \quad (6.25) \]

The Fock-Ivanenko coefficients all vanish. Thus the equivalence transformation is a constant and, for normalized wave functions,
it can be taken to be the identity.

As a third example consider the case of uniform acceleration \( g \) along the \( Z \) axis. By this we mean that the acceleration in the accelerating frame is constant at its spatial origin. This motion is called hyperbolic motion and is determined by the condition

\[
\tau = d\left( \nu \left( 1 - v^2/c^2 \right)^{-1/2} \right)/dT = \text{const}.
\]

(6.26)

where \( \nu \) is the velocity of the spatial origin of the \( x^\prime \) system with respect to the inertial \( x^{(\omega)} \) system.

The coordinates of a particle undergoing hyperbolic motion are easily found from the integration of equation (6.26). Möller's (1952) transformations in this case can be generalized to the situation where the accelerating frame has velocity \( v_0 \) in the \( Z \) direction at \( T = 0 \):

\[
T = (c/\tau) \left( \sinh H' - h \right) = \left( c/\tau \right) \left( \nu v/c - h \right),
\]

(6.27a)

\[
X = \kappa,
\]

(6.27b)

\[
Y = \gamma,
\]

(6.27c)

\[
Z = (c^2/\tau) \left( \cosh H' - \gamma_0 \right) = \left( c^2/\tau \right) \left( \gamma - \gamma_0 \right),
\]

(6.27d)

where

\[
h := v_0/c \left( 1 - \frac{v_0^2}{c^2} \right)^{1/2},
\]

(6.28)

\[
H' := \text{sh}^{-1} h + \tau \gamma/c,
\]

(6.29)
\[ \gamma = \left( \frac{c}{q} \right) \left( \sinh^{-1} \left( \lambda + q T/c \right) - \sinh^{-1} \lambda \right), \quad (6.30) \]

and the proper time of the particle is given by

\[ \gamma d\gamma = d\tau, \quad (6.31) \]

where

\[ \gamma = \left( 1 - \frac{v^2}{c^2} \right)^{-1/2}, \quad (6.32) \]

\[ \tau = \frac{dZ}{d\tau}. \quad (6.33) \]

Note also

\[ \gamma = \cosh \mathcal{H}', \quad (6.34) \]

\[ \frac{v}{c} = \tanh \mathcal{H}', \quad (6.35) \]

\[ \gamma \frac{v}{c} = \sinh \mathcal{H}'. \quad (6.36) \]

Møller (1952) desired to introduce a system of coordinates \( x^\alpha \) which is the relativistic analogue of a classical rigid frame of Cartesian axes following the particle in its motion, so that the particle is constantly situated at the origin of this frame of reference, and the space axes have constant directions. A rigid frame of reference is a frame in which the distance between two reference points, as measured by standard measuring rods at rest in the system, is constant in time. These conditions enabled Møller to construct such frames in a general manner; he claims that these, along with the rotating frame, are essentially the only possible
rigid systems of reference in flat space-time. He specialized the frame of reference to the hyperbolic case which we shall generalize to include an initial velocity \( v_0 \).

In lieu of Møller's treatment we present a heuristic derivation of the frame-to-frame transformation from the inertial to the generalized hyperbolic frame. This is done in direct analogy with the ordinary Lorentz transformations

\[ T = \gamma_0 t + \gamma_0 v_0 z/c, \]  
\[ X = x, \]  
\[ Y = \gamma y, \]  
\[ Z = \gamma_0 z + \gamma_0 v_0 t. \]  

(The \( \gamma_0 \) and \( v_0 \) in equation (6.37) denote the usual constant Lorentz quantities and are of no relation to previously used quantities in this Chapter.)

Equations (6.27) refer to a point (origin) which is undergoing hyperbolic motion with respect to some inertial frame \( x^{(i)} \); the particle had initial velocity \( v_0 \) at \( T = 0 \). On this particle we wish to construct a coordinated reference frame \( x^{(r)} \). To do this, we consider the usual Lorentz transformation (6.37) and generalize the quantities \( \gamma_0 \) and \( v_0 \) to become 'dynamic' variables \( \gamma \) and \( v \) in the form of equations (6.34) and (6.35). The particle's proper time \( \tau \) must now be extended to mean a coordinate time \( t \) for the accelerating
frame; thus we change \( t \) to \( t' \) in equation (6.29) so that \( H' \rightarrow H = \sinh^{-1} h + \frac{gt}{c} \). Thus, for example, the generalization of the term \( \gamma, v, z/c \) in equation (6.37) is \( \gamma v z/c^2 = (z/c) \sinh H \). In this way we substitute all these generalizations into equation (6.37) to give the desired full set of transformations describing the generalized hyperbolic frame \( x^\mu \) as:

\[
T = \left( \frac{c}{q^2} \right) \left[ (1 + q^2 z/c^2) \sinh H - \frac{1}{H} \right],
\]
(6.38a)

\[
X = \chi,
\]
(6.38b)

\[
Y = \gamma, \quad \gamma
\]
(6.38c)

\[
Z = \left( \frac{c^2}{q^2} \right) \left[ (1 + q^2 z/c^2) \cosh H - \gamma \right],
\]
(6.38d)

where \( h \) is still given by equation (6.28). From equations (6.29) and (6.36) we have

\[
H = \omega + \frac{q t}{c} = \sinh^{-1} (\gamma v/c),
\]
(6.39)

where \( \omega = \sinh^{-1} h \).

(6.40)

For no initial velocity \( v_0 \), then \( h = 0 \) so that the frame-to-frame transformations of Möller (1952, p. 256) are recovered.

The invariant interval is given by

\[
ds^2 = \left( 1 + q^2 z/c^2 \right) c^2 dt^2 - dx^2 - dy_0^2 - dz^2,
\]
(6.41)
which, of course, is independent of any initial velocity for the
accelerating frame and gives the special relativistic limit for \( g = 0 \).

The only nonvanishing Christoffel symbols are

\[
\{^0_3\} = (1 + q \frac{\dot{z}}{c^2})^{-1} \frac{q}{c^2} \tag{6.42a}
\]

\[
\{^3_{00}\} = (1 + q \frac{\dot{z}}{c^2}) \frac{q}{c^2} \tag{6.42b}
\]

The canonical tetrads are found from equation (5.15) as

\[
B^M_{\ (u)} = \begin{bmatrix}
(1 + q \frac{\dot{z}}{c^2})^{-1} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \tag{6.43}
\]

The transformation coefficients are

\[
\frac{\partial X^M}{\partial X^\mu_{(u)}} = \begin{bmatrix}
(1 + q \frac{\dot{z}}{c^2})^{-1} \cosh H & 0 & 0 & -(1 + q \frac{\dot{z}}{c^2})^{-1} \sinh H \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\sinh H & 0 & 0 & \cosh H
\end{bmatrix} \tag{6.44}
\]

The Fock–Ivanenko coefficients are

\[
\Gamma^1_i = 0 \tag{6.45}
\]

\[
\Gamma^0_o = (q / 2 c^2) \sigma^{(0)(3)} \tag{6.46}
\]

The equivalence transformation we find is

\[
N = \exp \left[ (-q \frac{t}{2c}) \sigma^{(0)(3)} \right] \tag{6.47a}
\]

\[
= \exp \left[ \frac{1}{2} (\omega - H) \sigma^{(0)(3)} \right] \tag{6.47b}
\]
where we have used equations (6.39) and (6.40). This equivalence transformation again looks similar to a case treated by Bjorken and Drell (1964), but, as was mentioned above, it is of no significance. Note that this N goes to the identity when only a Lorentz transformation is used (ω = H).

As a fourth and final example consider Rohrlich's (1963) so-called static homogeneous gravitational fields (SHGF). A SHGF is defined by a diagonal metric where the coefficients have only z-dependence. He deals with a flat space so that the Riemann curvature tensor is zero and demands that the nonrelativistic limit for the g_{00} term is \((1 + 2\varphi_{\text{NR}})\) where the gravitational potential \(\varphi\) for a force in the negative z-direction is

\[
\varphi_{\text{NR}} = \varphi_z \quad ; \quad F_{\text{NR}} = -m\nabla \varphi_{\text{NR}} = -m\varphi_{zz}.
\] (6.48)

The line element \((c = 1)\) turns out to be

\[
ds^2 = u^2 dt^2 - \left(\frac{u}{\varphi}\right)^2 dz^2 - dy^2 - dx^2,
\] (6.49)

where the prime denotes differentiation with respect to \(z\). The function \(u(z)\) is an arbitrary real function restricted only by requirements of continuity and the nonrelativistic limit \(u_{\text{NR}}(z) = 1 + 2gz\). For example, Møller's metric is given by \(u = 1 + gz/c^2\). Clearly there are an infinity of line elements for the SHGF that obey the \(u_{\text{NR}}(z)\) limit.

The only nonvanishing Christoffel symbols are
\[
\{^{3}_{00}\} = u q_x^2 / u', \quad \{^{3}_{33}\} = u'' / u', \quad \{^{0}_{30}\} = u / u.
\] (6.50)

The canonical tetrads are

\[
\mathbf{b}^{\mu}_{(a)} = \begin{bmatrix}
u^{-1} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & q / u'
\end{bmatrix}
\] (6.51)

The only nonvanishing Fock-Ivanenko coefficient is

\[
\Gamma_0 = \left( \frac{q}{2 c^2} \right) \sigma^{(o)(3)}
\] (6.52)

which is the same, since it is independent of \( u \), as was found for Möller's metric.
CHAPTER VII

NONMINIMAL EINSTEIN–DIRAC SYSTEM

The Einstein field equations in their standard form (4.24) may be derived from the following action: \( I = I_g + I_m \), where \( I_g \) and \( I_m \) are the actions of the gravitational field and matter respectively. The equations are found from the principle of least action; under a variation of the metric field we have \( \delta I = 0 \). Misner, Thorne, and Wheeler (1973, p. 491) point out that of the fourteen independent curvature invariants only the Riemann scalar, \( R \), is linear in the second derivatives of the metric coefficients; the choice for the Einstein action must be

\[
I_g = \int_{\mathcal{D}} R \sqrt{-g} d^4x
\]  

(7.1)

In Appendix A we have given the outline of the variation giving the standard Einstein–Dirac system. The variation was done with respect to the fields, \( \gamma^a(x) \), instead of the metric, \( g^{\mu\nu} \), because this is a natural choice when dealing with the Dirac theory. In any event, the variation done either way is equivalent.

Over the years many different physical motivations have prompted researchers to modify the Einstein equations. One of the earliest attempts was done by Einstein (1917) himself. In order that his static universe not have negative density or pressure he modified his equations to read

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}
\]  

(7.2)
where \( \Lambda \) was called the cosmological constant. The corresponding Lagrangian was changed to

\[
\int_{D^+} \left( R - 2 \Lambda \right) \sqrt{-g} \, d^4x \tag{7.3}
\]

Observations constrain \( |\Lambda| < 10^{-54} \) cm\(^{-1} \); in the following we take \( \Lambda \neq 0 \).

If, in addition to gravity being described by the metric \( g_{\mu \nu} \), one considers other long range fields associated with gravity then they would fall into two groups: direct or indirect coupling. Direct coupling to matter would entail slight deviations from the standard theory in terms of geodesic deviation or point dependent masses. Misner, Thorne, and Wheeler (1973, p.1064) have described experiments which restrict direct coupling theories; one can find no extra long range second-rank tensor or vector fields. An indirect coupling field would not couple to matter thus leaving the geodesic equation unaltered but would play a role in the field equations describing the geometry of space-time.

The most famous indirect coupling theory of gravity is that of Brans and Dicke (1961). The motivation there was to introduce a scalar field \( \phi \) to account for long range inertial effects in the Machian sense. The action for the \( \phi \) field and gravitation is

\[
\int_{D^+} \left( \phi \, R + \omega \phi \phi_\alpha \phi^{\alpha} \phi^{-1} \right) \sqrt{-g} \, d^4x \tag{7.4}
\]

where \( \omega \) is the dimensionless 'Dicke coupling constant'. In the variation, the \( \phi \) field is considered as an independent function to be varied. The field equations for \( \phi \) are
\[ \Box \phi = \frac{C}{3+2\omega} T^{\text{matter}} \]  

(7.5)

Einstein's equations now read

\[ G_{\mu\nu} = (C \phi^4) \Box \phi - (\omega \phi^2)(\phi,\mu \phi,\nu - \frac{1}{2} g_{\mu\nu} \phi^2 \Box \phi) \]  

(7.6)

Note that \( \phi \) plays the role of a variable gravitational 'constant'.

The Brans-Dicke theory becomes the Einstein theory in the limit \( \omega \to \infty \).

With improvements both to experimental tests of gravitation theories and the theoretical aspects (e.g. Will and Nordvedt 1972) it seemed that only the Einstein and Brans-Dicke theories held up as viable gravitational theories. However, as \( \omega \) gets pushed higher, the Brans-Dicke theory falls into disfavour; some researchers (e.g. Ohanian 1976) have given up on it.

Nevertheless, we shall wish to argue through the principle of equivalence to indicate that features of the generally covariant Dirac theory open an avenue for modification of the Einstein equations. Since there are several versions of the principle of equivalence, we shall take the statements of Ohanian (1973) as a basis for discussion. The weak principle of equivalence is: 'The gravitational mass of any system equals its inertial mass.' The well known Eötvös type experiments have established this equality to about one part in \( 10^6 \) so that this principle has a firm basis in fact. The strong principle of equivalence is: 'At each point of space-time it is possible to find a coordinate transformation such that the gravitational field variables disappear from the field equations of matter.'
This means that a local geodesic frame of reference in a gravitational field may be set up in which first derivatives of the metric are zero and the local metric takes on its Lorentzian values; thus, the laws of physics will take on their special-relativistic form and that no gravitational effects would enter at all. This gives a prescription for the construction of Lagrangian densities and is called the minimal coupling principle: 'The Lagrangian density in the presence of gravitation is obtained from the corresponding Lagrangian density of special relativity (assumed known) by replacing $\eta(\omega^\alpha)$ by $\delta^{\alpha\beta}$ and ordinary derivatives by (suitable) covariant derivatives.'

It is easy to see that the strong principle of equivalence as stated above does not apply to the iterated Dirac equation (4.26). Equation (4.28) indicates that there are derivative terms of the Fock-Ivanenko coefficients which is equivalent to second order terms in the metric. Lawrence (1970) has shown that these terms imply nongeodesic motion in the weak field limit. The coupling of Dirac particles with the gravitational field has been also emphasised by Unruh (1971). In ordinary gravitational fields this spin-curvature coupling is extremely small as can be seen by dimensional arguments (Weinberg 1972, p.133) but in strong fields associated with the late phase of gravitational collapse a type of 'vacuum polarization' takes place because of tidal forces (Misner, Thorne, Wheeler 1973, p.1121).

Dirac particles possess an intrinsic spin so that any statement of the strong principle of equivalence which involves test particles will not apply. Whether or not these particles couple minimally or nonminimally is empirically open. For these
reasons it is useful to study the theoretical nature of nonminimal couplings in the Einstein-Dirac system.

The nonminimal Einstein-Dirac theory we will be interested in is the one due to Leiter and Chapman (1975). This modification is obtained from the standard action of Appendix A by adding an interaction term which couples an arbitrary scalar function of the Dirac matter bilinear covariants directly to the gravitational scalar curvature. This will couple matter directly to the metric and its first and second derivatives in the modified Einstein equations which are generated; hence, this is a direct coupling field.

We will not include the contribution of any other matter fields as this just generates an additional energy-momentum tensor through the variation of the matter field variables in the usual way. In this connection note that the Einstein and Dirac fields here have to be determined in a self-consistent field fashion.

Following the above reasoning, our nonminimal Einstein-Dirac action is

$$I = \int d^4x F^{\alpha \beta} \left[ R^\alpha_\beta (ch/c) h - i \overline{\Psi} \nabla_\alpha \Psi + i \nabla_\mu \overline{\nabla_\alpha \Psi} + 2 \nabla_\alpha \overline{\nabla_\beta \Psi} \right] + \eta_{CRS} \right]$$

(7.7)

where the nonminimal coupling term is $\eta_{CRS}$. Recall $C$ is the gravitational coupling constant of equation (A.23); $\eta$ is the nonminimal-coupling constant. For simplicity, and in order to keep our modified equations from being higher than second order, we choose the scalar functions $S(\overline{\Psi}, \Psi, \Psi^\alpha, ...,)$ to be independent of derivatives of $\Psi$.

A variation of the Dirac adjoint wave function $\overline{\Psi}$ yields
the nonminimally coupled Dirac equation as

\[ -i \gamma^\mu \partial_\mu \psi + mc \psi + \frac{i}{2} \eta c R \partial S / \partial \bar{\psi} = 0. \]  

(7.8)

Similarly, the Dirac adjoint equation is

\[ i \hbar \partial_\mu \bar{\psi} \gamma^\mu + mc \bar{\psi} + \frac{i}{2} \eta c R \partial S / \partial \bar{\psi} = 0. \]  

(7.9)

Note that current conservation

\[ \nabla_\mu (\bar{\psi} \gamma^\mu \psi) = 0 \]  

(7.10)

requires that

\[ \frac{\partial S}{\partial \bar{\psi}} = \frac{\partial S}{\partial \psi}. \]  

(7.11)

This is valid since \( S \) is a real scalar function of the various bilinear covariants of \( \psi \) as demanded by the action principle for real energy-momentum tensors. If we substitute the new Dirac equations into the \( \mathcal{L}_{\text{dirac}} \) of equation (A.22) then equation (7.11) tells us that we no longer have \( \mathcal{L}_{\text{dirac}} = 0 \) but,

\[ \mathcal{L}_{\text{dirac}} = -\eta c R \bar{\psi} \partial S / \partial \bar{\psi}. \]  

(7.12)

A term like this will now show up in variations of equation (7.7) with respect to \( g_{\mu \nu} \) whereas it did not contribute before (Appendix A).

A variation of the metric tensor, \( \delta g_{\mu \nu} \), generates the
modified Einstein equations. Many of the terms one encounters are familiar from the usual approach. When a variation meets the \( \gamma(x) \)-matrix then it is very helpful to recall,

\[
\delta \gamma^\mu = \frac{1}{2} \gamma_\alpha \delta g_\mu^\alpha
\]  

(A.17)

As an example of a variation which differs from the standard case, consider the variation of the nonminimal term. Define

\[
I_{NM} = \int_{\mathcal{M}} d^4x \sqrt{-g} \eta \: C \: R \: S
\]  

(7.13)

We have

\[
\delta I_{NM} = \int_{\mathcal{M}} d^4x \eta \left[ C (\partial_\mu S \partial_\nu S + 2 R \delta_{\mu\nu} S + 2 R \partial_{\mu} S \partial_{\nu} S) + 2 R \delta_{\mu\nu} \delta_{\rho\sigma} \partial_\rho S \partial_\sigma S \right].
\]  

(7.14)

After using the divergence theorem twice and a rearrangement of terms, the first term of equation (7.14) becomes

\[
\int_{\mathcal{M}} d^4x \eta \left( C \sqrt{-g} \: \delta_{\mu\nu} S \right) = -\int_{\mathcal{M}} d^4x \eta \left( C \sqrt{-g} \: H_{\mu\nu} \delta_{\mu\nu} S \right),
\]  

(7.15)

where

\[
H_{\mu\nu} = \frac{1}{\sqrt{-g}} \left[ \left( R \sqrt{-g} \frac{\partial S}{\partial g_{\mu\nu}} \right)_x - R \sqrt{-g} \frac{\partial S}{\partial g_{\mu\nu}} \right].
\]  

(7.16)

The second and third terms give the Einstein tensor in the usual way:

\[
\int_{\mathcal{M}} d^4x \eta \left( C (SR \sqrt{-g} + 2 R \partial_{\mu} S \partial_{\nu} S) \right) = \int_{\mathcal{M}} d^4x \eta \left( C \sqrt{-g} \: G_{\mu\nu} \delta_{\mu\nu} S \right).
\]  

(7.17)

The last term of equation (7.14) necessitates a longer calculation.
than normal but it is straightforward. Doing a variation in which \( \delta \eta^{\mu} = 0 \) on the boundary of \( D^4 \), using a local geodesic coordinate system, and using the divergence theorem we find

\[
\int d^4 x \sqrt{-g} \eta \left( \bar{S} q^{\mu \nu} \delta q_{\mu \nu} - S_{,\nu} q^{\mu \nu} \delta q_{\mu}^{\nu} \right) \delta \left( \eta^{\mu} \right) \left[ \left( S q^{\alpha \beta} \delta q^{\alpha \beta}_{,\mu} \right)_{,\nu} - S_{,\nu} q^{\alpha \beta} \delta q^{\alpha \beta}_{,\mu} \right]
\]

\( \eta \)\n
\[
= \int d^4 x \sqrt{-g} \eta \left( \bar{S} q^{\mu \nu} \delta q^{\mu \nu} _{,\nu} - S_{,\nu} q^{\mu \nu} \delta q^{\mu \nu}_{,\nu} \right) \delta \left( \eta^{\mu} \right)
\]

\( \eta \)\n
\[
= \int d^4 x \sqrt{-g} \eta \left( S_{,\nu} q^{\mu \nu} - g^{\mu \nu} S_{,\nu} q^{\mu \nu} \right) \delta q^{\mu \nu}
\]

(7.20)

The results of equations (7.20), (7.17) and (7.15) with all the other variations with respect to \( \eta^{\mu} \) in the action (7.7) yield the modified Einstein equations as:

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \frac{c}{1 + \eta \bar{S}} \left( R_{\mu \nu}^{\text{Dirac}} - \frac{1}{2} \eta g_{\mu \nu} \frac{\delta S}{\delta \gamma} + \eta H_{\mu \nu} + \eta \left( g_{\mu \nu} S_{,\mu}^{\gamma} - S^{\gamma}_{,\mu} \right) \frac{\delta q^{\mu \nu}}{\delta \gamma} \right),
\]

(7.21)

where \( H_{\mu \nu} \) is given by equation (7.16) and \( T_{\mu \nu}^{\text{Dirac}} \) is given by equation (A.27). For \( \eta \to 0 \) we recover the standard Einstein theory.

One may compare equations (7.21) and (7.6) of the Brans-Dicke theory. The resemblance of the last term in each case is due to the similar construction of their Lagrangians. There is the important difference, however, that here \( S \) is not considered to be an independent field to be varied. In this theory we also have an
effective gravitational constant of the form $C(1 + \eta \psi \bar{\psi})$.

Taking the trace of both sides of equation (7.21) and using (A.28) we have the Riemann curvature scalar:

$$R = \frac{-C(m \bar{\psi} \psi + \eta (H_{\mu}^\mu + 3 S_{\mu}^{\nu},_{\mu}))}{1 + \eta C(S - \frac{3}{2} \bar{\psi} \partial S \partial \bar{\psi})} \quad (7.22)$$

In this model theory we will choose the scalar function $S$ to be real, positive definite, and invariant under charge conjugation so that particles and their antiparticles feel the same gravitation. A general model for $S$ would then take the form

$$S = f(\bar{\psi} \psi) + h(\bar{\psi} (\bar{\psi} \gamma^5) \psi) + k(\bar{\psi} \gamma^\mu \psi) + l(\bar{\psi} \gamma^5 \gamma^\mu \gamma^\nu \psi) + m(\bar{\psi} \gamma^5 \gamma^\mu \gamma^\nu \gamma^\sigma \psi) + n(\bar{\psi} \gamma^5 \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\tau \psi) \quad (7.23)$$

where $f$, $h$, $k$, $l$, $m$, and $n$ are scalar functions of their indicated bilinear covariant arguments. Thus the Dirac equation takes the final form,

$$\left[ -i \hbar \gamma^\mu \nabla_\mu + mc + \frac{1}{2} \eta c R \left( \psi' + h' \gamma^5 \gamma^\mu \psi \right) + \left( k' \right)_{\mu} \bar{\gamma}^\mu \psi \right] \psi = 0 \quad (7.24)$$

where the primes imply the derivative of a given function with respect to its indicated bilinear covariant argument (e.g. $(k')_{\mu}^{\gamma} := \partial k / \partial (\bar{\psi} \gamma^\mu \psi)$).

The nonminimal coupling generates, respectively, mass and pseudoscalar mass corrections, vector and pseudovector forces, and tensor and pseudotensor forces. The specific forms of the functions
f, h, k, l, p, and r probably will have to depend on experimental or theoretical fits for specific problems. For example, Glass and Leiter (1977) have argued in an electromagnetic version of this theory that the low flux of solar neutrinos at the Earth could give a bound on some of the nonminimal couplings.

The strengths of the above effects are quite unknown in the presently understood arena of interaction physics. However, it appears here that, with these additional interactions for quantum particles and a variable gravitational 'constant', the nature of space-time singularities may be altered. No account of this will be given here.
CHAPTER VIII
WEYL EQUATION SOLUTIONS IN A FLUID COSMOLOGY

In this Chapter we shall show that the full Dirac equation with mass is amenable to the technique of separation of variables for a new cosmological metric. In particular we shall show that the Weyl equations (physical neutrino equations of the massless Dirac equation) can be explicitly solved in this metric.

The background metric that we shall be using is that of Glass and Wilkinson (1976). Physically, the metric is a solution to Einstein's equations for a fluid cosmology. Mathematically, the nondiagonal metric has axial and cylindrical symmetry with respect to two Killing vectors $\partial_{\xi}$ and $\partial_{\eta}$; it depends on the coordinate time $t$ and a 'radial' coordinate $y$. The metric is a class of shear-free Bianchi type III solutions with nonzero expansion (Collins 1977).

For details of the Glass-Wilkinson metric with some further developments see Appendix B. (See also Glass 1977.)

The metric to be used as a background in the generally covariant Dirac equation is described by the following line element

$$ds^2 = f^4 dt^2 - \xi_0^2 \xi^{-10} \xi^2 d\xi_0^2 - \xi_0^2 \xi^8 f^{-10} \xi^2 d\phi^2$$

$$- \xi_0^6 \xi^8 f^{-8} (dz - \xi_0^2 d\phi)^2$$

(8.1)

where $c = 1$ and $(0,1,2,3) \sim (t,y,z,\phi)$. We will not use the canonical gauge tetrads but will construct tetrads, which obey the completeness relations (5.3) and (5.8), from the basis given in (5.12); they are:
\[ \chi^{\mu(\sigma)} = \begin{bmatrix} f^{-1} & 0 & 0 & 0 \\ 0 & \Sigma \gamma_5 & 0 & 0 \\ 0 & 0 & c f^5 \Lambda_0 \gamma_2 & 0 \\ 0 & 0 & 0 & c f^5 \Sigma^{-1} \end{bmatrix} \]  
(8.2)

\[ \lambda_{\mu(\omega)} = \begin{bmatrix} f^2 & 0 & 0 & 0 \\ 0 & \Sigma^{-1} \gamma_5 & 0 & 0 \\ 0 & 0 & d^{-1} f^{-4} \gamma_5 \Sigma^{-1} \\ 0 & 0 & \Sigma^{-1} c f^5 \gamma_5 - d^{-1} f^{-4} \Lambda_0 \gamma_2 \end{bmatrix} \]  
(8.3)

Recall that the Dirac equation may be written

\[ (-i Y^\mu \partial_\mu - i Y^\mu \Gamma^\mu + \mathfrak{R}) \Psi = 0 \]  
(3.8)

where

\[ Y^\mu \Gamma^\mu = Y^{(e)}(A(e)(x) + i Y^{(s)} B(e)(x)), \]  
(5.20)

\[ A(e)(x) = \frac{1}{2} (-g^{-1})^{1/2} (\sqrt{-g} \chi^{(a)} \Sigma_m^{(a)} \gamma_m)_{\mu} \]  
(5.21)

\[ B(e)(x) = \frac{1}{4} \epsilon^{(e)(s)(a)(b)} Y^{(a)} \chi^{(s)} \gamma_m \chi^{(b)}_{\mu} \]  
(5.22)

By direct calculation from the tetrad we find

\[ A(e) = -7 f f^{-3} \delta_\rho - b f^5 \Lambda_0 \delta_\rho \]  
(8.4a)

\[ B(e) = \frac{1}{2} a f^6 \Lambda_0 \delta_\rho \]  
(8.4b)

\[ A_\mu = -7 f f^{-1} \delta_\mu + \gamma \Sigma^{-1} \Lambda_0 \delta_\mu \]  
(8.5a)
\[ \beta_\mu = \frac{1}{2} \, a \int \delta (x - a) \, \delta_\mu. \] 

(8.5b)

It is of interest to know if \( \lambda_\mu \) and \( \beta_\mu \) are curl-free and a simple calculation shows that they are. Thus we may write

\[ \lambda_\mu := A_\mu, \] 

(8.6)

\[ \beta_\mu := B_\mu. \] 

(8.7)

Integration, using the generic formula (B.29), gives, to within a constant,

\[ \exp \lambda = \int \tilde{\Sigma}^{1/2}. \] 

(8.8)

\[ \exp B = \left[ \int \left( \int \left( \frac{2}{a} \right)^{1/2} \right) \right] \frac{a L}{2} \tilde{\lambda}^2. \] 

(8.9)

Since the \( \lambda_\mu \) term may be written as the gradient of a scalar we let

\[ \psi := \exp (-\lambda) \chi = \int \tilde{\Sigma}^{-1/2} \chi. \] 

(8.10)

so that the Dirac equation becomes

\[ \gamma^\mu \partial_\mu \psi + i \gamma^\mu \gamma^{(5)} \beta_\mu \psi + i \chi \psi = 0. \] 

(8.11)

We consider the Weyl neutrino case now, and leave the massive Dirac case until later. The Weyl two-component theory of 

(\( a \))
a physical 'left-handed' neutrino is equivalent to a massless Dirac (four-component) description in which some constraints have been imposed on the amplitudes (Schweber 1961). Let the solution of the massless \((\kappa = 0)\) Dirac equation be \(\psi_\nu'\) where the subscript \(\nu\) only indicates 'neutrino'. This will be equivalent to the Weyl equation if we demand

\[
(\mathbf{I} + \gamma^{(5)}) \psi_\nu' = 0
\]  

Equation (8.12) reads,

\[
\gamma^\nu \partial_\nu \psi_\nu' + i \gamma^\nu \gamma^{(5)} \beta_\nu \psi_\nu' = 0,
\]  

for \(\kappa = 0\). But from equation (8.12) we have

\[
\gamma^\nu \partial_\nu \psi_\nu' - i \gamma^\nu \beta_\nu \psi_\nu' = 0.
\]  

Equation (8.7) allows us to write

\[
\gamma^\nu \partial_\nu \psi_\nu' - i \gamma^\nu B_\nu \psi_\nu' = 0
\]  

Given some \(f\) then define

\[
\psi_\nu' = \exp(i B) \psi_\nu
\]  

The Weyl neutrino equation to study is
\[ \gamma^\nu \partial_\nu \Psi = 0 \quad (8.17) \]

The requirement of equation (8.12) allows us to write

\[ \Psi = \begin{pmatrix} \eta \\ -\eta \end{pmatrix} \quad (8.18) \]

where \( \eta \) is a spinor.

Using the tetrads of equation (8.2) and (8.3) we evaluate the curved \( \gamma \)-matrices of equation (3.6). By putting equation (8.18) into (8.17) we find, after multiplying through by \( f^5 \), the only relevant equation

\[
\begin{align*}
f^{-1} \eta_{,0} + b_1 \gamma_{i} \sigma^{(i)} \eta_{,0} + c \lambda \gamma_{i} \eta^{(i)} \eta_{,2} \\
+d_1 \gamma^{i} \sigma^{(i)} \eta_{,2} + c \gamma^{-1} \sigma^{(2)} \eta_{,3} = 0. \quad (8.19)
\end{align*}
\]

We shall assume (cf. Unruh 1974b) that, since the metric is independent of \( z \) and \( \phi \), the \( z \) and \( \phi \) modes obey, respectively,

\[
\begin{align*}
\partial_2 \eta &= -im \eta \quad (8.20a) \\
\partial_3 \eta &= -in \eta \quad (8.20b)
\end{align*}
\]

where \( m \) and \( n \) are constants.

We try the technique of separation of variables on equation (8.19) by putting
\[ \eta = e^{-i(m + n \phi)} \begin{pmatrix} \frac{\partial T_1}{\partial t} \, \phi_1 \left( \frac{\partial}{\partial \eta} \right) \\ \frac{\partial T_2}{\partial t} \, \phi_2 \left( \frac{\partial}{\partial \eta} \right) \end{pmatrix} \]  

(8.21)

Substitution of this into equation (8.19) results in the following set of two equations:

\[ \begin{align*}
\phi_1 \left[ f^{-1} \frac{d}{dt} T_1^* - i m d f^{-1} T_1 \right] &= T_2 \left[ -b \frac{d}{dy} \phi_2 - c \rho_2 \phi_2 - n \rho_2 \phi_2 + n \rho_2 \phi_2 \right], \\
\phi_2 \left[ f^{-1} \frac{d}{dt} T_2^* + i m d f^{-1} T_2 \right] &= T_1 \left[ -b \frac{d}{dy} \phi_2 - c \rho_2 \phi_2 - n \rho_2 \phi_2 + n \rho_2 \phi_2 \right].
\end{align*} \]  

(8.22a, 8.22b)

Clearly these equations separate. Let \( k_1 \) and \( k_2 \) be the separation constants. The Weyl neutrino equations reduce to the two sets of two equations coupled in \( T_1, T_2 \) and \( \phi_1, \phi_2 \) respectively:

\[ \begin{align*}
f^{-1} \frac{d}{dt} T_1^* - i m d f^{-1} T_1 &= k_1 \, T_1, \\
f^{-1} \frac{d}{dt} T_2^* + i m d f^{-1} T_2 &= k_2 \, T_2.
\end{align*} \]  

(8.23a, 8.23b)

\[ \begin{align*}
b \frac{d}{dy} \phi_2 - c \rho_2 \phi_2 &= -k_1 \, \phi_1, \\
b \frac{d}{dy} \phi_2 + c \rho_2 \phi_2 &= -k_2 \, \phi_2,
\end{align*} \]  

(8.24a, 8.24b)

where \( \sigma = n + m \rho_2 \).  

(8.25)

The above four equations, (8.23) and (8.24), are our fundamental neutrino equations. A more complicated set of \( T \) equations would have resulted if equation (8.14) was used because an extra term
of the form $-12a \mathcal{L}_0 \mathbf{u}_1$ would have been added to the lefthand sides of equation (8.23).

We consider now the fundamental $\psi$ equations (8.24). Because of the function of $\psi$ which multiplies the derivative operator it is convenient to define a new variable $w$ by:

$$dw = -2 \mathcal{L}_0 \psi T^{-1} d\psi,$$

$$w = -\ln \left( \mathcal{L}_0^{-1} + \psi^2 \right).$$

(8.26)

(8.27)

Thus we may write

$$\psi = e^{-w} - \mathcal{L}_0^{-1} \psi,$$  \hspace{1cm} (8.28a)

$$\Sigma_1^2 \psi = \alpha e^w + \nu,$$  \hspace{1cm} (8.28b)

where

$$\alpha := (n - mk^2) \mathcal{L}_0^{-1}.$$  \hspace{1cm} (8.29)

In terms of the $w$ variable the first order $\psi$ equations are

$$-2b \mathcal{L}_0 \frac{d\psi_1}{dw} + c \psi e^w \psi_1 + \nu \psi_1 = -k_1 \psi_2,$$  \hspace{1cm} (8.30a)

$$-2b \mathcal{L}_0 \frac{d\psi_2}{dw} - c \psi e^w \psi_2 - \nu \psi_2 = -k_1 \psi_1.$$  \hspace{1cm} (8.30b)

We can decouple the equations (8.30) to give the master $\psi$ equation describing the second order $\psi_1$ or $\psi_2$ equation:
\[
\frac{d^2 y}{dw^2} = \left( A e^{2w} + B e^w + C^2 \right) y, \quad (8.31)
\]

where
\[
A^2 = \frac{c^2 \alpha^2}{4b^2 \lambda_0^2}, \quad (8.32a)
\]
\[
B = \frac{\text{sinc} \cdot 2b \lambda_0 \alpha + c^2 m + c^2 \beta}{4b^2 \lambda_0^2}, \quad (8.32b)
\]
\[
C^2 = \frac{c^2 m^2 + \lambda}{4b^2 \lambda_0^2}, \quad (8.32c)
\]
\[
\lambda = \hbar, \hbar_c, \quad (8.32d)
\]
\[
s_{\pm n} = \begin{cases} +1 & \text{for } y = y_1, \\ -1 & \text{for } y = y_2 \end{cases}, \quad (8.32e)
\]

Equation (8.31) is now in a standard form as in equation 2.273(14) of Kamke (1943). The general solution to equation (8.31) is
\[
y = e^{-\omega n} \mathcal{W}(B/2A, C, L_A e^{-\omega}) \quad (8.32f)
\]

where \(\mathcal{W}(k, m, x)\) is the solution to Whittaker's differential equation,
\[
4x^2 \mathcal{W}'' = \left( x^2 - 4kx + 4m^2 - 1 \right) \mathcal{W} \quad (8.33)
\]

Recall (Mathews and Walker 1965) that Whittaker's equation has the solution
\[
\mathcal{W}(k, m, x) = C_1 M_{\nu, m}(x) + C_2 M_{\nu, -m}(x) \quad (8.34)
\]
where $2m$ is not an integer and $C_i$ and $C_\kappa$ are arbitrary constants and

$$M_{k,m}(\kappa) := C \kappa^{-\frac{\kappa}{2}} e^{-\kappa/2} F_1(m+\frac{1}{2}-h,\frac{1}{2}m+1;\kappa) \quad (8.36)$$

where $F_1(\ldots)$ is the confluent hypergeometric function. For all $k$ and $m$, Whittaker’s equation is solved by

$$W_{k,m} := \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}m-h)} M_{k,m} + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}m-h)} M_{k,-m}, \quad (8.37)$$

where $\Gamma(\ldots)$ is the gamma function and $W_{k,m}$ are Whittaker functions.

In principle the solution to the fundamental $\Psi$ equations (8.24) has been given. However, $\lambda$ is a characteristic value of the master equation (8.31) so that the $\Psi$ solutions will depend on $\lambda$ as determined by the boundary values of the problem. Since the Glass-Wilkinson metric has not as yet been considered in full physical detail this problem will be deferred.

We consider now the fundamental $\Upsilon$ equations (8.23) for the neutrino. It is more convenient to replace the time derivative with one with respect to the function $f$. Thus the $\Upsilon$ equations become:

$$\int f^* \frac{d\Upsilon_1}{df} - i m f \frac{d}{df} \Upsilon_1 = k_1 \Upsilon_1, \quad (8.38a)$$

$$\int f^* \frac{d\Upsilon_2}{df} + i m f \frac{d}{df} \Upsilon_2 = k_2 \Upsilon_1. \quad (8.38b)$$

The $f$ equations that are to be considered here have the generic form (8.29). They describe the isotropic negative pressure case of equations (8.31) or a wide class of anisotropic pressure.
cases of equation (B.30) for $\delta^2 \approx 3.1263$.

By a change of variable we can absorb the coefficient multiplying the $f$-derivative operator; we define,

$$du = \int f^{-1} df.$$  \hfill (8.39)

Substituting $f$ from equation (B.29) we find

$$u = -\frac{1}{\theta^2 q^2} \ln \left[ \frac{q^2 + (q^2 + f^2)^{1/2}}{q} \right].$$  \hfill (8.40)

Simple algebra reveals the relations,

$$f q^{-1} = -\operatorname{csch} (q^2 q u)$$ \hfill (8.41a)

$$(q^2 + f^2)^{1/2} = -q \operatorname{coth} (q^2 q u)$$ \hfill (8.41b)

If we define a new variable,

$$v = q^2 q u$$ \hfill (8.42)

then the $T_i$ equations become

$$\frac{d\sigma T_i}{dv} + \frac{imd}{p^2 q^2} \sinh v \ T_i = \frac{h_1}{\theta^2} \ T_{i+1}$$ \hfill (8.43a)

$$\frac{d\sigma T_i}{dv} - \frac{imd}{p^2 q^2} \sinh v \ T_i = \frac{h_2}{\theta^2} \ T_i$$ \hfill (8.43b)

These equations may be decoupled into two second order equations.
only of $\tau_1$ or $\tau_2$. The master equation describing them is

$$\frac{d^2 \tau}{dv^2} + \left( sgn \gamma \cosh v - \gamma^2 \sinh^2 v - \lambda \right) \tau = 0 \quad (8.44)$$

where

$$\gamma : = \frac{m d}{\rho^2 q^2} \quad , \quad (8.45a)$$

$$\lambda : = \frac{\lambda}{\rho^4 q^2} \quad , \quad (8.45b)$$

$$sgn : = \begin{cases} +1 & \text{for } \tau = \tau_1, \\ -1 & \text{for } \tau = \tau_2 \end{cases} \quad (8.45c)$$

Finally we change the variable to

$$w : = \cosh v \quad . \quad (8.45d)$$

This will transform equation (8.44) to the new master equation,

$$\left( w^2 - 1 \right) \frac{d^2 \tau}{dw^2} + w \frac{d\tau}{dw} + \left( -\gamma^2 w^2 + sgn \gamma w + \gamma^2 - \lambda \right) \tau = 0 \quad . \quad (8.47)$$

Equation (8.47) has regular singular points at $w = \pm 1$ and
an irregular singular point at $w = \infty$. A series solution around the
ordinary point $x = 0$ can be found.

In fact, equation (8.47) is part of a class called the
generalized spheroidal equation. A special case of it (Wilson 1928a)
involves a restricted three body problem of an electron in the
field of two protons (Wilson 1928b). The generalized spheroidal
equation has been solved by Fisher (1937).

Using the parameters of equation (8.47) in the solution of the generalized spheroidal equation, we can write the solutions of (8.47) as

\[
\Phi = \Phi \left[ \frac{1}{2}, -\Lambda, 2i \gamma; \frac{1}{2}, -\frac{1}{8} \text{sgn} \gamma, w \right]. \tag{8.48a}
\]

Other solutions are also

\[
\Phi \left[ \frac{1}{2}, -\Lambda, 2i \gamma; \gamma, -\frac{1}{8} \text{sgn} \gamma, w \right], \tag{8.48b}
\]

\[
\Phi \left[ \frac{1}{2}, -\Lambda, -2i \gamma; 0, \frac{1}{8} \text{sgn} \gamma, w \right], \tag{8.48c}
\]

\[
\Phi \left[ \frac{1}{2}, -\Lambda, 2i \gamma; 0, \frac{1}{8} \text{sgn} \gamma, -w \right]. \tag{8.48d}
\]

Explicitly, one type of solution is

\[
\Phi = e^w \sum_{n=0}^{\infty} d_n \left( \frac{1-w}{2} \right)^n \binom{\frac{1}{2}}{n} \binom{-\frac{1}{2}}{n} \text{F} \left( \frac{1}{2}, -\frac{1}{8} \text{sgn} \gamma; \frac{1}{8} \text{sgn} \gamma, n+\frac{1}{8}, \frac{1-w}{2} \right). \tag{8.49}
\]

where \( d_{-1} = 0 \) and the three term recursion formula is

\[
(n+1)(n+\frac{1}{2}) d_{n+1} - 4\gamma(n-\frac{1}{8} \text{sgn} \gamma) \frac{n-\frac{1}{8} \text{sgn} \gamma+1}{n-\frac{1}{2}} d_n
\]

\[
= \left[ n(n-(\text{sgn} \gamma)\gamma - \Lambda + \frac{1}{4} (\text{sgn} \gamma)^2 + \gamma (\text{sgn} \gamma - 1) \right] d_n. \tag{8.50}
\]

The function \( F \) is the hypergeometric series which is defined by
\[ F(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha \beta}{\Gamma(\gamma)} x + \frac{\alpha(\alpha + 1) \beta(\beta + 1)}{2! \Gamma(\gamma + 1)} x^2 + \frac{\alpha(\alpha + 1)(\alpha + 2) \beta(\beta + 1)(\beta + 2)}{3! \Gamma(\gamma + 1)(\gamma + 2)} x^3 \ldots \quad (8.51) \]

For \( T = T_1 \), we have \( \text{sgn} = +1 \). The first slot in the hypergeometric function of equation (8.49) is zero so \( F = 1 \). Thus the solution is

\[ T_1 = e^{yw} \sum_{n=0}^{\infty} d_n \left( \frac{1-w}{z} \right)^n \quad (8.52) \]

where \( d_{-1} = 0 \) and

\[ (\eta + 1)(\eta + \frac{1}{2}) d_{\eta + 1} - 4\eta \frac{\eta(\eta + 1)}{\eta + \frac{1}{2}} d_{\eta - 1} = (\eta (\eta - 4\eta - \Lambda) - \Lambda) d_{\eta}. \quad (8.53) \]

For \( T = T_2 \), we have \( \text{sgn} = -1 \) so that

\[ T_2 = e^{yw} \sum_{n=0}^{\infty} d_n \left( \frac{1-w}{z} \right)^n F(-1; \eta + \frac{1}{2}, \eta + \frac{1}{2}; 1; \frac{1-w}{z}). \quad (8.54) \]

Because of the \(-1\) in the first slot of the hypergeometric series, the series stops after the second term; hence,

\[ T_2 = e^{yw} \left( \frac{1-w}{z} \right)^{\frac{\eta}{2}} \sum_{n=0}^{\infty} d_n \left( \frac{1-w}{z} \right)^n, \quad (8.55) \]

where \( d_{-1} = 0 \) and

\[ (\eta + 1)(\eta + \frac{1}{2}) d_{\eta + 1} - 4\eta \frac{\eta(\eta + 1)}{\eta + \frac{1}{2}} d_{\eta - 1} \]

\[ = \left[ \eta(\eta + 2 - 4\eta - \Lambda - 2\eta + 1) \right] d_{\eta}. \quad (8.56) \]
Another type of solution is

\[ \mathcal{T} = \sum_{n=0}^{\infty} \alpha_n (ix)^{n+1/2} M \left( \frac{n+1}{2}, \frac{n+1/2}{2}, \frac{n}{2} \right) \]  \hspace{1cm} (8.57)

where \( \chi = 2i \chi \), \( w - 1 \), \hspace{1cm} (8.58a)

\[-4\chi(n+1)(n+1/2) \alpha_{n+1} \left[ \gamma(n+1/2) - \chi \gamma(sq-1) \right] \alpha_{n+1} + [n-1/2 (sq+1)] \alpha_n \] \hspace{1cm} (8.58b)

and the \( M \) function is defined by equation (8.36).

For \( \mathcal{T} = \mathcal{T}_1 \) we have \( sgn = +1 \) so that

\[ \mathcal{T}_1 = \sum_{n=0}^{\infty} \alpha_n (ix)^{n+1/2} M \left( \frac{n+1}{2}, \frac{n+1/2}{2}, \frac{n}{2} \right) \] \hspace{1cm} (8.59)

where \( \chi = 2i \chi \), \( w - 1 \), \hspace{1cm} (8.60a)

\[-4\chi(n+1)(n+1/2) \alpha_{n+1} \left[ \gamma(n+1/2) - \chi \gamma(sq-1) \right] \alpha_{n+1} + (n+1) \alpha_n \] \hspace{1cm} (8.60b)

Since the indices on \( M \) are the same we get a more compact form of the solution

\[ \mathcal{T}_1 = \sum_{n=0}^{\infty} \alpha_n (ix)^{n+1/2} \Gamma\left( \frac{1}{2}, n+1/2, i \chi \right) \] \hspace{1cm} (8.61)

where \( \Gamma (...) \) is the confluent hypergeometric function.

For \( \mathcal{T} = \mathcal{T}_2 \) we have \( sgn = -1 \) so that

\[ \mathcal{T}_2 = \sum_{n=0}^{\infty} \alpha_n (ix)^{n+1/2} M \left( \frac{n+1}{2}, \frac{n+1/2}{2}, \frac{n}{2} \right) \] \hspace{1cm} (8.62)
where \( \chi = 2i \chi (w - 1) \) \hspace{1cm} (8.63a)

\[-4\gamma(n+1)(n+1)\alpha_{m+1} = [n(n+4\chi)-\Lambda+2\gamma]\alpha_m + n\alpha_{m-1} \] \hspace{1cm} (8.63b)

Again the indices on \( M \) allow us to write

\[ \varpi_2 = \sum_{n=0}^{\infty} \alpha_n (ix)^n \bar{\varphi}_{\frac{1}{2}} \right)_F (-\frac{1}{2}, \frac{1}{2}; i \chi) . \] \hspace{1cm} (8.64)

This completes the \( \varpi_2 \) and \( \varpi_3 \) solutions of the Weyl equation.

Again we have not imposed boundary conditions on our solutions nor normalized the wave functions.

We reconsider now the full Dirac equation

\[ \gamma^\mu \gamma_\mu \psi + i \gamma^5 \gamma \gamma_\mu \gamma \psi + i \nu \psi = 0 . \] \hspace{1cm} (8.11)

In this case we let the four component spinor \( \psi \) be

\[ \psi := \begin{pmatrix} f_1 \\ f_2 \\ q_1 \\ q_2 \end{pmatrix} \exp -i (\gamma x + \gamma \phi) , \] \hspace{1cm} (8.65)

Again using the tetrads of equations (8.2) and (8.3) and \( B_m \) from equation (8.5b) we find that, after multiplying through by \( \gamma^5 \), equation (8.11) becomes:

\[ \mathcal{L} f_1 - \tilde{\mathcal{L}} q_1 + i F_{q_1} = 0 \] \hspace{1cm} (8.66a)

\[ \mathcal{L} f_2 - \tilde{\mathcal{L}} q_2 - i F_{q_2} = 0 \] \hspace{1cm} (8.66b)
\[
\mathcal{L}_t q_1 - \mathcal{L}_y f_2 + i F_f_2 = 0 \quad , \quad (8.66c)
\]
\[
\mathcal{L}_t^* q_2 - \mathcal{L}_y^* f_1 - i \tilde{F} f_2 = 0 \quad , \quad (8.66d)
\]

where \( \mathcal{L}_t \) and \( \mathcal{L}_y \) are purely \( t \) and \( y \) operators respectively, viz.

\[
\mathcal{L}_t := f^{-1} \partial \sigma + i f^{1/2} \Sigma \quad , \quad (8.67a)
\]
\[
\mathcal{L}_y := b_y^{-1} \Sigma \partial \sigma - c \Sigma^{-1} \sigma \quad , \quad (8.67b)
\]
\[
\mathcal{L}_y^* := b_y^{-1} \Sigma \partial \sigma + c \Sigma^{-1} \sigma \quad , \quad (8.67c)
\]
\[
F := m d f^{-1} + \frac{1}{2} a f \mathcal{R} \quad , \quad (8.67d)
\]
\[
\tilde{F} := m d f^{-1} - \frac{1}{2} a f \mathcal{R} \quad , \quad (8.67e)
\]

We try the technique of separation of variables by putting the following into equations (8.66):

\[
f_1 := T_1(t) Y_1(y) \quad , \quad f_2 := T_2(t) Y_2(y) \quad , \quad q_1 := T_3(t) Y_3(y) \quad , \quad q_2 := T_4(t) Y_4(y) \quad . \quad (8.68)
\]

The result is

\[
Y_1 \mathcal{L}_t T_1 - T_4 \mathcal{L}_y Y_4 + i F T_3 Y_3 = 0 \quad , \quad (8.69a)
\]
\[
Y_2 \mathcal{L}_t T_2 - T_3 \mathcal{L}_y Y_3 - i \tilde{F} T_4 Y_4 = 0 \quad , \quad (8.69b)
\]
\[ Y_3 \mathcal{L}_t T_3 - T_2 \mathcal{L}_y Y_1 + iF_1 T_1 Y_1 = 0, \quad (8.69c) \]
\[ Y_4 \mathcal{L}_t^* T_4 - T_1 \mathcal{L}_y^* Y_1 - i\tilde{F}_2 T_2 Y_2 = 0. \quad (8.69d) \]

Put
\[ Y_1 = Y_3, \quad Y_2 = Y_4. \quad (8.70) \]

Therefore
\[ Y_1 \left( \mathcal{L}_t T_1 + iF_1 T_1 \right) = T_4 \mathcal{L}_y Y_2, \quad (8.71a) \]
\[ Y_2 \left( \mathcal{L}_t T_2 - i\tilde{F}_2 T_2 \right) = T_3 \mathcal{L}_y^* Y_1, \quad (8.71b) \]
\[ Y_1 \left( \mathcal{L}_t^* T_3 + iF_1 T_1 \right) = T_2 \mathcal{L}_y Y_2, \quad (8.71c) \]
\[ Y_2 \left( \mathcal{L}_t^* T_4 - i\tilde{F}_2 T_2 \right) = T_1 \mathcal{L}_y^* Y_1. \quad (8.71d) \]

These equations clearly separate and imply
\[ \mathcal{L}_t T_1 + iF_1 T_1 = -K_1 T_1, \quad (8.72a) \]
\[ \mathcal{L}_t T_2 - i\tilde{F}_2 T_2 = -K_2 T_2, \quad (8.72b) \]
\[ \mathcal{L}_t^* T_3 + iF_1 T_1 = -K_3 T_2, \quad (8.72c) \]
\[ \mathcal{L}_t^* T_4 - i\tilde{F}_2 T_2 = -K_4 T_1, \quad (8.72d) \]
\[ \mathcal{L}_y Y_2 = -K_1 Y_1, \quad \mathcal{L}_y^* Y_2 = -K_3 Y_1, \quad (8.73a) \]
\[ \mathcal{L}_y Y_1 = -K_2 Y_2, \quad \mathcal{L}_y^* Y_1 = -K_4 Y_2, \quad (8.73b) \]
where $K_1, \ldots, K_4$ are four constants of separation. From equations (8.73) we infer that

$$K_1 = K_3 \quad \text{and} \quad K_2 = K_4 \quad (8.74)$$

The full set of Dirac equations for the Glass-Wilkinson metric are:

\[
\begin{align*}
\frac{df}{dt} T_1 + \frac{df}{dt} x T_1 + i (mdf' - \frac{i}{2} af \gamma_0) T_3 &= -K_1 T_4, \quad (8.75a) \\
\frac{df}{dt} T_2 + i \frac{df}{dt} x T_2 - i (mdf' - \frac{i}{2} af \gamma_0) T_4 &= -K_2 T_3, \quad (8.75b) \\
\frac{df}{dt} T_3 - i \frac{df}{dt} x T_3 + i (mdf' + \frac{i}{2} af \gamma_0) T_1 &= -K_1 T_2, \quad (8.75c) \\
\frac{df}{dt} T_4 - i \frac{df}{dt} x T_4 - i (mdf' + \frac{i}{2} af \gamma_0) T_2 &= -K_2 T_1, \quad (8.75d) \\
\end{align*}
\]

\[
\begin{align*}
by_1 \gamma^1 \frac{dy_1}{dy} - c \Sigma^1 \sigma Y_1 &= -K_1 Y_1, \quad (8.76a) \\
by \gamma^1 \frac{dy_1}{dy} + c \Sigma^1 \sigma Y_1 &= -K_2 Y_2. \quad (8.76b)
\end{align*}
\]

Equations (8.76) do not depend on the mass and are, in fact, the same as in the neutrino case of equations (8.24) if we identify $Y \leftrightarrow y$ and $K \leftrightarrow k$. Thus, the solutions found for the neutrino case have the same form here.

Equations (8.75) for the $T$'s do not uncouple conveniently to give second order equations only in $T_1, T_2, T_3,$ or $T_4$.

The probable reason for this is that the separation conditions were
extremely weak; all we required was \( Y_1 = Y_2 \) and \( Y_3 = Y_4 \). No relations
between the \( T \)'s ever showed up to give more consistency relations so
that there remains two separation constants. The system of equations
can be reduced to the following two sets of two equations in \( T_1, T_2 \)
and \( T_3, T_4 \):

\[
\mathbf{L}_1 T_2 = i L_2 T_4, \quad \mathbf{L}_4^* T_4 = i L_2 T_2, \quad (8.77a)
\]

\[
\mathbf{L}_3 T_3 = i L_3 T_3, \quad \mathbf{L}_3^* T_3 = i L_3 T_3, \quad (8.77b)
\]

where

\[
\mathbf{L}_1 = a \mathbf{L}_1 \mathbf{L}_0 \frac{d}{dt} + i f^{-2} \mathbf{L}_0 \mathbf{L}_1 \mathbf{L}_0 + \frac{1}{f} \mathbf{L}_1 \frac{d}{dt} \mathbf{F}, \quad (8.78a)
\]

\[
\mathbf{L}_3 = a \mathbf{L}_1 \mathbf{L}_0 \frac{d}{dt} + i f^{-2} \mathbf{L}_0 \mathbf{L}_1 \mathbf{L}_0 \mathbf{L}_1 \mathbf{L}_0 - \frac{1}{f} \mathbf{L}_1 \frac{d}{dt} \mathbf{F}, \quad (8.78b)
\]

\[
\mathbf{L}_2 = \frac{1}{f} \frac{d^2}{dt^2} - 7 f^{-1} \frac{d}{dt} + \frac{1}{f} \mathbf{L}_1 \frac{d}{dt} \mathbf{L}_1 \mathbf{L}_0 + \frac{1}{f} \mathbf{L}_1 \frac{d}{dt} \mathbf{L}_1 \mathbf{L}_0 + i S f^{-1} \mathbf{L}_1 \mathbf{L}_0 + K_1 K_2. \quad (8.78c)
\]

In the case of zero mass \( (\gamma = 0) \) then a number of
simplifications occur. For example, the operators \( \mathbf{L}_1 \) and \( \mathbf{L}_1^* \) then
commute since

\[
\mathbf{L}_1 \mathbf{L}_1^* - \mathbf{L}_1^* \mathbf{L}_1 = 10 i f^{-1} f \mathbf{L}_1 \mathbf{L}_0 \quad (8.79)
\]

Furthermore, the operators \( L_1, \mathbf{L}_1, \) and \( L_2 \) become real. Equations
\((8.77)\) then imply that \( T_1 \) is proportional to \( T_4 \) (and similarly for
\( T_2 \) and \( T_3 \)) and that the proportionality constant is \( \pm 1 \). The left-
handed neutrino condition would set the proportionality constant
as \(-1\). Thus, in the massless case, the equations uncouple; for the
physical neutrino we get the equations

$$\frac{dT_i}{dt} - i(\hbar \frac{d}{dt} + \frac{1}{2} \alpha \frac{Q_0}{\lambda}) T_i = K_i T_i \quad (8.50a)$$

$$\frac{dT_2}{dt} + i(\hbar \frac{d}{dt} + \frac{1}{2} \alpha \frac{Q_0}{\lambda}) T_2 = K_2 T_1 \quad (8.50b)$$

These are just the form of the neutrino equations alluded to in the discussion after equation (8.25).

A few concluding remarks are in order. We have found the solutions to the Weyl equation in the Glass-Wilkinson metric (and half of the Dirac equations). Much of the physics of the problem enters in at the level of prescribing boundary conditions on the solutions. For the moment such considerations are not well understood in general relativity and future study with an exact solution like the one presented here could be most enlightening.

There may be some other physical interest in these neutrino solutions for such a cosmology. In the standard Big Bang model of the universe there are an estimated ~$10^3$ neutrinos per baryon, and other considerations (Bludman 1976) suggest there may be more neutrinos that would be cosmologically significant. The Glass-Wilkinson metric describes a class of physically acceptable fluid cosmologies with anisotropic or isotropic pressures (albeit negative generally). A fluid model like this may be better suited than the empty space models (Kasner 1921, Misner 1969) for a description of the early universe with respect to anisotropy damping (Natzner and Misner 1973), angular-momentum transfer, and particle production (Parker 1977).
APPENDIX A

VARIATIONAL PRINCIPLES

In this Appendix we give the standard variational principle background to Chapter VII for both the Einstein and Dirac fields. Such variations with respect to the tetrad fields, incidentally, have modern interest in and applications to supergravity (van Nieuwenhuizen 1977) which we shall ignore.

The usual derivation of the Einstein field equations (e.g. Adler, Bazin, and Schiffer 1965, pp. 305–309) from a variational principle uses a Palatini technique in which variations are taken independently with the metric tensor, \( \mathbf{g}_{\mu\nu} \), and the Christoffel symbols, \( \Gamma^{\alpha}_{\beta\delta} \). For \( D^+ \) an arbitrary region in four-space, the integral to be varied is

\[
\mathcal{J} = \int_{D^+} \mathcal{R} \, d^4x \quad (A.1)
\]

where

\[
\mathcal{R} := \sqrt{-g} \, R \quad (A.2)
\]

It can be shown that the tensor density \( g_{\mu\nu} \sqrt{-g} \) can be obtained as the variational derivative of \( \mathcal{J} \) under a variation of the metric in \( D^+ \) which vanishes on the boundary of \( D^+ \).

In Chapter IV we noted the basic analogy of the Fock-Ivanenko coefficients, \( \Gamma_\mu \), and the Christoffel symbols, \( \Gamma^\alpha_{\beta\delta} \). For the Palatini variation to be used here we adopt \( \Gamma_\mu \) as a field to be varied. Also, in Chapter V, we noticed that the tetrads are as basic to the description of space-time as is the metric; we shall
take the tetrads $X^\mu_\alpha$ (or, more precisely, the $Y^\mu = X^\mu_\alpha Y^{(\alpha)}$) as a field to be varied. The Riemann scalar quantity that we use here is

$$R = \frac{1}{2} \text{Tr} (K_{\nu \mu} \sigma^{\mu \nu})$$  \hspace{1cm} (4.19)

We write equation (A.1) as

$$J = \int_{\mathcal{D}} \frac{1}{2} \text{Tr} (K_{\nu \mu} \sigma^{\mu \nu}) \sqrt{q} \, d^4 x. \hspace{1cm} (A.3)$$

In order that the integral $J$ should be an extremum one requires that the variations in $J$ caused by $Y^\nu$ and $\Gamma_{\nu}$ should separately vanish; thus,

$$0 = \delta J = \int_{\mathcal{D}} \sqrt{q} \, d^4 x \left( \frac{1}{2} \text{Tr} [\delta K_{\nu \mu} \sigma^{\mu \nu}] + \frac{1}{2} \text{Tr} [K_{\nu \mu} \delta \sigma^{\mu \nu}] + \frac{R}{\sqrt{q}} \delta \sqrt{q} \right) \hspace{1cm} (A.4)$$

Using equation (A.4), which expresses the spin-curvature in terms of the Fock-Ivanenko coefficients, and a geodesic coordinate system in which $\Gamma_{\nu} = 0$, we find that the first term in equation (A.4) becomes

$$\frac{1}{2} \text{Tr} (\delta K_{\nu \mu} \sigma^{\mu \nu}) \sqrt{q} = \sqrt{q} \text{Tr} (\delta \Gamma_{\nu} \sigma^{\mu \nu}) \hspace{1cm} (A.5)$$

In the geodesic frame all derivative operations are equivalent so that, using the fact that $\delta \Gamma_{\nu}$ vanishes on the surface of $\mathcal{D}$, we find

$$\int_{\mathcal{D}} \frac{1}{2} \text{Tr} (\delta K_{\nu \mu} \sigma^{\mu \nu}) \sqrt{q} \, d^4 x = - \text{Tr} \int_{\mathcal{D}} \nabla_{\mu} \sigma^{\mu \nu} \delta \Gamma_{\nu} \sqrt{q} \, d^4 x \hspace{1cm} (A.6)$$
Consider the second term of equation (A.4). Since $K_{\mu\nu}\delta\gamma^\mu = K_{\mu\nu}\delta q^\mu$ then

$$
\text{Tr}(K_{\gamma\mu}\delta q^\mu) = \text{Tr}([\gamma^\mu K_{\mu\rho}]\delta q^\rho).
$$

(A.7)

From equation (A.6) it is easy to see that

$$
R_{\mu\nu}^\gamma = [\gamma^\mu, K_{\mu\nu}],
$$

(A.8)

and hence

$$
\frac{1}{2}g_8 \text{Tr}[[K_{\gamma\mu}\delta q^\mu]] = \frac{1}{2}g_8 \text{Tr}(R_{\gamma\mu\nu}\delta q^\mu\delta q^\nu).
$$

(A.9)

Now consider the last term of equation (A.4),

$$
R_{\gamma\mu\nu}^\gamma = \frac{R}{4}\text{Tr}\left(\frac{\delta q^\mu}{\delta q^\nu} \frac{\delta q^\nu}{\delta q^\mu} \delta q^\mu \delta q^\nu\right).
$$

(A.10)

The first term in the round brackets is well known and the second is easily found from equation (3.5); hence,

$$
R_{\gamma\mu\nu}^\gamma = \text{Tr}(\frac{1}{2}g_8 q_{\gamma\mu\nu}\delta q^\mu \delta q^\nu).
$$

(A.11)

Putting equations (A.11), (A.9), and (A.7) into (A.6) we have

$$
0 = \delta\mathcal{J} = \int_{\gamma^4} d^4x \left[ (R_{\gamma\mu\nu} - \frac{i}{2} q_{\gamma\mu\nu} R) \text{Tr}(\frac{1}{2} \gamma^\mu \delta q^\nu) - \text{Tr}(\nabla_{\gamma\mu}\delta q^\rho \delta \Gamma^\rho) \right].
$$

(A.12)

The last term is the only one that contains variations of the $\Gamma_i$; for the integral to be an extremum we must have
\[ \text{Tr}(\nabla_\mu \sigma^{\mu\nu}) = 0 \]  \hspace{1cm} (A.13)

But it is sufficient that
\[ \nabla_\mu \gamma^\mu = 0 \]  \hspace{1cm} (A.14)

which gives, as one expects in the Palatini technique, the usual relation (3.16) between \( \gamma^\mu \) and \( \Gamma^\nu_\mu \).

By equation (4.22) we have that the Einstein tensor must vanish in the absence of matter in order that \( \delta J = 0 \). It is easy to show that this formulation is equivalent to variation of the metric since
\[ G_{\mu\nu} \text{Tr}(\frac{1}{2} \gamma^\mu \delta \gamma^\nu) = G_{\mu\nu} \delta \tilde{q}^\nu, \]  \hspace{1cm} (A.15)

where we have noted that
\[ \delta \{\gamma^\mu, \gamma^\nu\} = 2 \delta \tilde{q}^\nu, \]  \hspace{1cm} (A.16)

The simplest solution of equation (A.16) is (Pauli 1933, Laurent 1959)
\[ \delta \gamma^\mu = \frac{1}{2} \gamma_\alpha \delta q^{\mu\alpha}, \]  \hspace{1cm} (A.17)

We note the following without proof:
\[ \gamma^\mu \delta \Gamma_\mu \gamma^\nu = 0, \quad (A.18) \]
\[ \delta \Gamma_\mu = \frac{1}{4} \sigma^{\alpha \beta} \delta \Gamma_{\alpha \beta} \quad (A.19) \]
\[ = \frac{1}{4} g_{\alpha \beta} \delta \{ \frac{\delta}{\delta \gamma} \} \sigma^{\alpha \beta}, \quad (A.20) \]
\[ \{ \gamma^\nu, \delta \Gamma_{\mu} \} = 0. \quad (A.21) \]

Now we consider the Einstein field equations for which we include the contribution due to the Dirac field. The Lagrangian density and energy-momentum tensor can be generalized from special to general relativity by replacement of all partial derivatives with spin-covariant derivatives and the \( \gamma \)-matrices become position dependent. We use the symmetric energy-momentum tensor and adjust constants so units will match in Einstein's equations.

The Lagrangian density is

\[ \mathcal{L} = (\mathcal{C} c^{-3}) (-\bar{\psi} \gamma^\mu \nabla_\mu \psi + i \nabla_\mu \bar{\psi} \gamma^\mu \psi + 2 \chi \bar{\psi} \psi) \quad (A.22) \]

where \[ \mathcal{C} := -8\pi G/c^2. \quad (A.23) \]

The Lagrangian is

\[ \mathcal{L} = \int \mathcal{L}, \quad (A.24) \]

To arrive at the generally covariant Dirac equation (3.8)
or its Dirac adjoint form of equation (3.26) we vary the Lagrangian with respect to \( \bar{\Psi} \) and \( \Psi \) respectively. In the Einstein-Dirac system the fields \( g_{\mu\nu} \) (or equivalently \( \gamma^\nu \)), \( \bar{\Psi} \), and \( \Psi \) are considered independent. Consider the variation of \( L \) with respect to \( \Psi \). In order that we have an extremum we must have

\[
o = \delta L = \int_{D^4} \sqrt{-g} \ d^4x \ \delta \mathcal{L}_{\text{Dirac}}
\]

\[
= \int_{D^4} \sqrt{-g} \left( -i \bar{\Psi} \gamma^\mu \partial_\mu \delta \Psi - i \bar{\Psi} \gamma^\mu \nabla_\mu \delta \Psi + 2 \bar{\Psi} \delta \Psi \right). \tag{A.25}
\]

In the first term of this expression we may write \(- i \int_{D^4} \sqrt{-g} \bar{\Psi} \gamma^\mu \partial_\mu \delta \Psi \), \(- \lambda (\sqrt{-g} \bar{\Psi} \gamma^\mu \delta \Psi) \). By the divergence theorem, the first term of this expression can be made to vanish since we assume the variation of \( \Psi \) to vanish on the surface of \( D^4 \). We use the divergence formula on the second term of the above expression and note that

\[
\gamma^\mu_{\ ij} + \nabla_\mu \gamma^\nu - \gamma^\nu \nabla_\mu = 0. \tag{3.15}
\]

This collapses equation (A.25) to

\[
o = \int_{D^4} \sqrt{-g} \ 2 \left[ i \nabla_\mu \bar{\Psi} \gamma^\mu + \chi \bar{\Psi} \right] \delta \Psi. \tag{A.26}
\]

The only way this is possible is to have

\[
i \nabla_\mu \bar{\Psi} \gamma^\mu + \chi \bar{\Psi} = 0. \tag{3.26}
\]

In a similar fashion we find
\[-i \gamma^\nu \nabla_\mu \Psi + m \Psi = 0. \tag{3.8}\]

The energy-momentum tensor is

\[ T_{\mu\nu}^{\text{Dirac}} = \frac{i}{4c} \left( \overline{\Psi} V_\mu \nabla_\nu \Psi - \nabla_\nu \overline{\Psi} V_\mu \Psi + \overline{\Psi} \nabla_\mu \nabla_\nu \Psi - \nabla_\mu \overline{\Psi} \nabla_\nu \Psi \right). \tag{4.27}\]

As a consequence of the Dirac equation, the trace of \( T_{\mu\nu}^{\text{Dirac}} \) is,

\[ T^{\text{Dirac}} = T^{\text{Dirac}}_\mu = m \overline{\Psi} \Psi. \tag{4.28}\]

The Einstein equation with the Dirac contribution reads

\[
\mathcal{R}_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} \mathcal{R} \Rightarrow
\]

\[ = -2\pi i L^* \left( \overline{\Psi} V_\mu \nabla_\nu \Psi - \nabla_\nu \overline{\Psi} V_\mu \Psi + \overline{\Psi} \nabla_\mu \nabla_\nu \Psi - \nabla_\mu \overline{\Psi} \nabla_\nu \Psi \right) \tag{4.29}\]

where \( L^* \) is the Planck distance,

\[ L^* := \left( \frac{\hbar c}{G} \right)^{1/2} = 1.6 \times 10^{-33} \text{ cm}. \tag{4.30}\]

The physical significance of the Planck length from 'quantum gravity' is that violent fluctuations in the geometry are predicted at distances of that order.
APPENDIX B

THE GLASS-WILKINSON METRIC

If we use the conventions of Hawking and Ellis (1973) with \( c = G = 1 \) but signature \(-2\) then the line element for the Glass-Wilkinson (1976) metric is

\[
dS^2 = f^4 dt^2 - \delta^6 f^{-10} L^2 \psi^2 dy^2 - \delta^5 f^{-10} L^2 d\phi^2
\]

\[
- \delta^6 f^{-8} \left( \frac{dz}{\Lambda_0} - \frac{1}{2} \psi \frac{d\phi}{f} \right)^2.
\]

(B.1)

The covariant and contravariant metric tensors are:

\[
q_{\mu\nu} = \begin{bmatrix}
    f^4 & 0 & 0 & 0 \\
    0 & -\delta^6 f^{-10} L^2 & 0 & 0 \\
    0 & 0 & -\delta^5 f^{-8} \Lambda_0 \psi^2 & 0 \\
    0 & 0 & 0 & -\delta^5 f^{-10} L^2 - \delta^5 f^{-8} \Lambda_0 \psi^2
\end{bmatrix}
\]

(B.2)

\[
q^{\mu\nu} = \begin{bmatrix}
    f^{-4} & 0 & 0 & 0 \\
    0 & -\delta^6 f^{-10} L^2 & 0 & 0 \\
    0 & 0 & -\delta^5 f^{-8} \Lambda_0 \psi^2 & 0 \\
    0 & 0 & 0 & -\delta^5 f^{-10} L^2 - \delta^5 f^{-8} \Lambda_0 \psi^2
\end{bmatrix}
\]

(B.3)

where \( \delta^6 f^{-10} \psi^2 \),

\[
\delta^5 f^{-8} \Lambda_0 \psi^2,
\]

\[
\delta^5 f^{-10} L^2 - \delta^5 f^{-8} \Lambda_0 \psi^2.
\]

(B.4)
\( \epsilon \) is a positive constant (= \(49/116\) in the original theory),
\( \delta \) is an undetermined constant but subject to energy
conditions (see below),
\[ f = f(t), \]
\[ \Sigma = k^2 - \beta \, y^2, \quad k^2 \text{ constant, } \beta = \mp \ln = \text{constant}. \]
We shall take \( \beta = -\ln \) so that there are no singularities in the
metric; thus,
\[ \Sigma = k^2 + \ln \, y^2. \quad (B.5) \]

The sixteen nonvanishing Christoffel symbols in the coord-
inates where \((0,1,2,3) \sim (t,y,z,\phi)\) are (Wainwright 1976):

\[ \{_{00}\} = 2 \, \mp f^{-1}, \quad (B.6a) \]

\[ \{_{10}\} = -5 \, f^{-1}, \quad (B.6b) \]

\[ \{_{20}\} = -5 \, \epsilon \, g \, f^{-1} \, y^2 \Sigma^2, \quad (B.6c) \]

\[ \{_{11}\} = -2 \, \ln \, y^2 \Sigma^{-1} + y^{-1}, \quad (B.6d) \]

\[ \{_{21}\} = -4 \, f^{-1}, \quad (B.6e) \]

\[ \{_{20}\} = -\epsilon \, f^2 \, \ln \, y^3 \Sigma^{-2}, \quad (B.6f) \]

\[ \{_{11}\} = -\epsilon \, f \, \ln \, y^4 \Sigma^{-2}, \quad (B.6g) \]
\[ \{^0_{11}\} = -4 \hat{f} \delta^5 f^{-13} \]  
\hspace{5cm} (B.6h) 

\[ \{^3_{30}\} = \hat{f} f \mathcal{L}_0 \mathcal{Y} \]  
\hspace{5cm} (B.6i) 

\[ \{^3_{30}\} = -5 \hat{f} \mathcal{Y} \]  
\hspace{5cm} (B.6j) 

\[ \{^5_{31}\} = \varepsilon^1 \delta^5 \delta^1 \mathcal{L}_0 \mathcal{Y}^2 - \lambda \mathcal{L}_0 \mathcal{Y}^2 \]  
\hspace{5cm} (B.6k) 

\[ \{^3_{31}\} = \varepsilon^1 \delta^3 \mathcal{L}_0 \mathcal{Y}^2 - 2 \mathcal{L}_0 \mathcal{Y} \]  
\hspace{5cm} (B.6l) 

\[ \{^5_{31}\} = 4 \hat{f} \delta^5 f^{-13} \mathcal{L}_0 \mathcal{Y} \]  
\hspace{5cm} (B.6m) 

\[ \{^1_{32}\} = \varepsilon^1 \delta^1 \mathcal{L}_0 \mathcal{Y} \mathcal{J} \]  
\hspace{5cm} (B.6n) 

\[ \{^0_{32}\} = -5 \varepsilon^1 \delta^5 f^{-15} \mathcal{Y}^2 - 4 \hat{f} \varepsilon^5 f^{-13} \mathcal{L}_0 \mathcal{Y} \]  
\hspace{5cm} (B.6o) 

\[ \{^3_{33}\} = -2 \varepsilon^1 \delta^1 \mathcal{L}_0 \mathcal{Y}^2 - 2 \mathcal{L}_0 \mathcal{Y} \mathcal{J} \]  
\hspace{5cm} (B.6p) 

where \[ \hat{f} := \frac{df}{dt} \]  
\hspace{5cm} (B.7) 

The components of the Ricci tensor are:

\[ R_{\rho\sigma} = 108 \hat{f} \mathcal{Y} f^{-2} + 14 \hat{f} \mathcal{Y} \]  
\hspace{5cm} (B.8a) 

\[ R_{\nu} = \mathcal{L}_0 \mathcal{Y} [-2 \hat{f} \varepsilon \mathcal{L}_0 \mathcal{J} + 2 \mathcal{L}_0 \mathcal{Y} - 2 \delta^5 f^{-13} \mathcal{L}_0 \mathcal{Y}] \]  
\hspace{5cm} (B.8b) 

\[ \mathcal{L}_0 = \mathcal{L}_0 \mathcal{Y} \mathcal{J} \]
\[ R_{22} = 2 \varepsilon^2 \delta^{-1} \frac{\partial^2}{\partial \alpha^2} + 4 \varepsilon^4 \delta^{14} (- \frac{\partial^2}{\partial \alpha^2} + 17 \frac{\partial^4}{\partial \alpha^4}) \]  
(B.8c)

\[ R_{32} = - \partial_\alpha y^2 R_{22} \]  
(B.8d)

\[ R_{33} = 8 \varepsilon^4 \delta^4 R_{\mu \nu} + 2 \partial_\alpha y^4 R_{22} \]  
(B.8e)

The mixed components of the Ricci tensor are:

\[ R_{\alpha} = -108 \varepsilon^2 \frac{\partial^2}{\partial \alpha^2} + 14 \frac{\partial^4}{\partial \alpha^4} \]  
(B.9a)

\[ R^i = 5 \varepsilon^2 \frac{\partial^2}{\partial \alpha^2} - 85 \varepsilon^4 \frac{\partial^4}{\partial \alpha^4} + 2 \varepsilon^8 \delta^{12} \frac{\partial^2}{\partial \alpha^2} + 4 \varepsilon^8 \delta^{10} \frac{\partial^4}{\partial \alpha^4} \]  
(B.9b)

\[ R^2 = 4 \varepsilon^2 \frac{\partial^2}{\partial \alpha^2} - 68 \varepsilon^4 \frac{\partial^4}{\partial \alpha^4} - 2 \varepsilon^8 \delta^{12} \frac{\partial^2}{\partial \alpha^2} \]  
(B.9c)

\[ R^3 = R^1 \]  
(B.9d)

\[ R^3 = (8 \varepsilon^2 - 17 \varepsilon^4) \frac{\partial^2}{\partial \alpha^2} + 4 \varepsilon^8 \delta^{10} \frac{\partial^4}{\partial \alpha^4} (\epsilon + \varepsilon^2) \frac{\partial^2}{\partial \alpha^2} \]  
(B.9e)

The energy-momentum tensor may be written

\[ T_{\mu \nu} = \varepsilon \varepsilon_{\mu \nu} + p_1 A_{\mu} A_{\nu} + p_2 B_{\mu} B_{\nu} + p_3 C_{\mu} C_{\nu} \]  
(B.10)

where

\[ -1 = - \varepsilon A_{\mu} = B_{\mu} = C_{\mu} \]  
(B.11)

A suitable basis for the metric (B.1) is:
\[ u^\nu = f^{-1} \delta^\nu \]  \hspace{1cm} (B.12a)

\[ u_\mu dx^\mu = f dt \]  \hspace{1cm} (B.12b)

\[ A^\mu = -b f^5 \delta^\mu_1 \]  \hspace{1cm} (B.12c)

\[ A_\mu dx^\mu = b^{-1} f^{-5} \delta^\mu_1 \Sigma^1 \]  \hspace{1cm} (B.12d)

\[ B^\mu = -c f^5 \Sigma^1 \left( \delta^\mu_3 + \mathcal{R}_0 \delta^\mu_2 \right) \]  \hspace{1cm} (B.12e)

\[ B_\mu dx^\mu = c^{-1} f^{-5} \Sigma d\phi \]  \hspace{1cm} (B.12f)

\[ C^\mu = -d f^4 \delta^\mu_2 \]  \hspace{1cm} (B.12g)

\[ C_\mu dx^\mu = d^{-1} f^{-4} (dz - \mathcal{R}_0 \delta^1 d\phi) \]  \hspace{1cm} (B.12h)

where we define:

\[ a := e^{-4} \delta^{-3} \]  \hspace{1cm} (B.13a)

\[ b := e^{-7/2} \delta^{-3} \]  \hspace{1cm} (B.13b)

\[ c := e^{-7/2} \delta^{-5/2} \]  \hspace{1cm} (B.13c)

\[ d := e^{-3} \delta^{-5/2} \]  \hspace{1cm} (B.13d)
In the above coordinate basis then

\[ T^0_0 = w, \quad T^1_1 = -p_1, \quad T^2_2 = -p_2, \quad T^3_3 = -p_3, \quad (B.14a) \]

\[ T = w - p_0 - p_1 - p_3, \quad (B.14b) \]

where \( w \) is the density of the fluid and the \( p_i \) relate to the pressures.

In the notation of Hawking and Ellis (1973) the Einstein equations can be written

\[ R^\mu_\nu = 8\pi \left( T^\mu_\nu - \frac{1}{2} g^\mu_\nu T \right). \quad (B.15) \]

The mixed components of the Ricci tensor can be written

\[ R^0_0 = 4\pi (w + 2p_1 + p_2), \quad (B.16a) \]

\[ R^1_1 = R^2_2 = 4\pi (-w + p_2), \quad (B.16b) \]

\[ R^2_2 = 4\pi (-w + 2p_1 - p_2), \quad (B.16c) \]

where equation (B.9d) has been used to show that

\[ p_1 = p_3. \quad (B.17) \]

We can solve equations (B.16) to give

\[ 16\pi w = R^0_0 - R^2_2 - 2R^1_1, \quad (B.18a) \]
For the type of energy-momentum tensor (B.10) that we are considering here, certain energy conditions may be considered. The conditions are:

Weak energy condition: \( w \geq 0, \quad w + p_i \geq 0 \), \( \quad (B.19) \)

Dominant energy condition: \( w \geq 0, \quad -w \leq p_i \leq w \), \( \quad (B.20) \)

Strong energy condition: \( w + p_i \geq 0, \quad w + \xi p_i \geq 0 \). \( \quad (B.21) \)

From the original Glass and Wilkinson (1976) theory, the function \( f \) obeyed the equation

\[
\frac{df}{dt} = \frac{1}{\pi \sqrt{2}} \sqrt{\frac{s}{S}} \frac{G}{\sqrt{2}} \left[ \sqrt{1 + f^2} \right]^{1/2} \quad (B.22)
\]

For \( \xi = 49/116 \) we have any of the energy conditions satisfied for \( \delta^2 \geq 8.1263 \). Note that \( p_i = p_3 < 0 \) always but we may have \( p_i \geq 0 \) for \( \delta^2 \leq 32.5051 \).

Collins (1977) pointed out that the function \( f \) need not obey (B.22) which we have just seen is a physical solution for restricted values of \( \delta \). Different equations for \( f \) which give acceptable energy conditions will define a class of solutions of Einstein's equations. For example, we may prescribe \( f \) by the requirement that the pressure tensor be isotropic, i.e., \( R_{1} = R_{2} = R_{3} \) so that \( p_i = p_1 = p_3 \). We already have that \( R_{i} = R_{3} \) by equation (B.9d) so we put
\[ R_1' = R_2' \]  

\[ \frac{5}{2} f' + \gamma \xi^2 + 4 \epsilon B \delta \frac{\partial}{\partial f} \gamma \left( \xi + f^2 \right) = 0 \]  

(B.23)

This gives the second order nonlinear equation for \( f \),

\[ \frac{5}{2} f' + \gamma \xi^2 + 4 \epsilon B \delta \frac{\partial}{\partial f} \gamma \left( \xi + f^2 \right) = 0 \]  

(B.24)

A first integral of this equation is

\[ f = \sqrt{\alpha f^8 + \beta f^4} \]  

(B.25)

where \( \alpha \) and \( \beta \) are constants to be determined. Substitution of equation (B.25) into (B.24) implies

\[ \alpha = \frac{1}{2} \frac{4 \xi^2}{\gamma} \]  

(B.26a)

\[ \beta = \frac{4 \xi^2}{\gamma} \]  

(B.26b)

Thus, for the isotropic case

\[ f = \sqrt{\frac{4 \xi^2}{\gamma} f^8 \left( \frac{8 \xi^2}{\gamma} \epsilon + f^2 \right)^{4/3}} \]  

(B.27)

For this choice of \( f \) we find the elegant result that all the energy conditions are satisfied for all values of \( \delta \). Furthermore, now we have

\[ p_i < 0 \]  

(B.28)
For use in Chapter VII we will consider both cases for \( f \) of equations (B.22) and (B.27) by the generic form

\[
\hat{f} = \hat{f}^{2-\delta} \left( q_{\theta}^2 + f^2 \right)^{1/2}
\]  

(B.29)

where, for the anisotropic case of equation (B.22) we have

\[
\hat{f}^2 = \frac{2}{\gamma} \mathbb{J}_0 e^{-5/2} \delta^{-2}
\]  

(B.30a)

\[
q_{\theta}^2 = 1
\]  

(B.30b)

and, for the isotropic case of equation (B.27) we have

\[
\hat{f}^2 = 2^{-1/2} \mathbb{J}_0
\]  

(B.31a)

\[
q_{\theta}^2 = \frac{2}{\gamma} \delta
\]  

(B.31b)
APPENDIX C

DIRAC EQUATION IN NEWMAN-PENROSE FORMALISM

Dirac's equations (Penrose 1968) in spinor form \( (\kappa = c = 1) \) are:

\[
\nabla_{\hat{\alpha}} P^\hat{\alpha} + i \mu_e Q_{\hat{\alpha}} = 0 \tag{C.1}
\]

\[
\nabla_{\hat{\alpha}} Q^\hat{\alpha} + i \mu_e P_{\hat{\alpha}} = 0 \tag{C.2}
\]

where \( \mu_e = 2^{\frac{-\kappa}{2}} m, \ Q_{\hat{\alpha}} = (Q^\alpha)^*, \ * \) denotes complex conjugation, \( \nabla_{\hat{\alpha}} \) is the symbol for covariant differentiation, and \( P^\hat{\alpha} \) or \( Q^\hat{\alpha} \) is a two component spinor. (See Appendix D for a discussion of spinors.)

The sourceless neutrino equation (Teukolsky 1973) is

\[
\nabla^{\hat{\alpha}} \phi_{\hat{\alpha}} = 0 \tag{C.3}
\]

(For an alternate scheme see Gibbons (1977) which makes use of the Geroch-Held-Penrose (1973) methods and spin-weighted spherical harmonics (Goldberg et al. 1967).)

We wish to exhibit the spinor Dirac and neutrino equations in the Newman-Penrose (1962) formalism for general relativity. We will then just exhibit the Dirac equation in this framework for the Glass-Wilkinson metric of Appendix B.

The formalism of Newman and Penrose is well known (Piran 1964) so we will not go into details. At each point in space-time we construct four null vectors \( n^\hat{\mu}, m^\mu, m^{\mu*} \) which satisfy the
orthogonality relations:

\[ -l^\mu \eta_{\mu} = m^\mu m_{\mu} = -1 \]  \hspace{1cm} (C.4)

\[ l^\mu m_{\mu} = l^\mu m_{\mu} = \eta^\mu m_{\mu} = \eta^\mu m_{\mu} = 0 \]  \hspace{1cm} (C.5)

Vectors \( l^\mu \) and \( n^\mu \) are both real and \( m^\mu \) is complex. They give the completeness relation

\[ q^\mu_\nu = l^\mu n_\nu + n^\mu l_\nu - m^\mu m_\nu - m^\mu m_\nu. \]  \hspace{1cm} (C.6)

The intrinsic or covariant directional derivatives are:

\[ D_\mu = l^\mu d_\mu \]  \hspace{1cm} (C.7a)

\[ \Delta_\mu = n^\mu d_\mu \]  \hspace{1cm} (C.7b)

\[ \delta_\mu = m^\mu d_\mu \]  \hspace{1cm} (C.7c)

where \( d_\mu \) is the tensor covariant derivative.

Newman and Penrose have defined twelve complex spin coefficients:

\[ \chi := l_{\mu ; \nu} m^\mu l^\nu \]  \hspace{1cm} (C.8a)

\[ \Pi := -n_{\mu ; \nu} m^\mu m^\nu l^\nu \]  \hspace{1cm} (C.8b)

\[ \varepsilon := \frac{1}{2} \left( l_{\mu ; \nu} n^\mu l^\nu - m_{\mu ; \nu} m^\mu m^\nu l^\nu \right). \]  \hspace{1cm} (C.8c)
\( \rho := \lambda_{\mu \nu} \eta^{\lambda \nu} \), \hspace{1cm} (C.3d)

\( \lambda := -\gamma_{\mu \nu} \eta^{\mu \nu} \gamma_{\mu \nu} \), \hspace{1cm} (C.3e)

\( \alpha := \frac{1}{2} \left( \lambda_{\mu \nu} \eta^{\mu \nu} - \lambda_{\mu \nu} \eta^{\mu \nu} \right) \), \hspace{1cm} (C.3f)

\( \sigma := \lambda_{\mu \nu} \gamma^{\mu \nu} \gamma^{\mu \nu} \), \hspace{1cm} (C.3g)

\( \mu := -\gamma_{\mu \nu} \gamma^{\mu \nu} \gamma^{\mu \nu} \), \hspace{1cm} (C.3h)

\( \beta := \frac{1}{2} \left( \lambda_{\mu \nu} \eta^{\mu \nu} - \lambda_{\mu \nu} \eta^{\mu \nu} \right) \), \hspace{1cm} (C.3i)

\( \nu := -\gamma_{\mu \nu} \gamma^{\mu \nu} \gamma^{\mu \nu} \), \hspace{1cm} (C.3j)

\( \psi := \frac{1}{2} \left( \lambda_{\mu \nu} \eta^{\mu \nu} - \lambda_{\mu \nu} \eta^{\mu \nu} \right) \), \hspace{1cm} (C.3k)

\( \gamma := \lambda_{\mu \nu} \gamma^{\mu \nu} \gamma^{\mu \nu} \), \hspace{1cm} (C.3l)

Letting

\[
F_{1} = P^{o}, \quad F_{2} = P^{1}, \quad G_{1} = Q^{1}, \quad G_{2} = -Q^{o}, \hspace{1cm} (C.9)
\]

Chandrasekhar (1976) has shown that Dirac's equation in Newman-Penrose formalism may be written as the following set of four coupled equations:

\[
F_{1} = P^{o}, \quad F_{2} = P^{1}, \quad G_{1} = Q^{1}, \quad G_{2} = -Q^{o}, \hspace{1cm} (C.9)
\]
\begin{align}
(D + \epsilon - \rho) F_i & \quad + (\delta + \pi - \alpha) F_L = i \mu_e G_i, \quad \text{(C.10a)} \\
(D + \mu - \delta) F_i & \quad + (\delta + \pi - \alpha) F_L = i \mu_e G_L, \quad \text{(C.10b)} \\
(D + \epsilon - \delta) G_i & \quad - (\delta + \pi - \alpha) G_L = i \mu_e F_i, \quad \text{(C.10c)} \\
(D + \mu - \alpha) G_i & \quad - (\delta + \pi - \epsilon) G_L = i \mu_e F_L. \quad \text{(C.10d)}
\end{align}

The neutrino equations are found from equations (C.10a) and (C.10b) by the setting of \( \mu_L = 0 \) and the replacements: \( F_i \rightarrow \chi_i \) and \( F \rightarrow -\chi_L \).

For the Glass-Wilkinson metric (A.1) we found suitable null tetrads \((l^\mu, n^\mu, m^\mu, m^*\mu)\) from the basis of equations (B.12) as:

\begin{align}
\sqrt{2} l^\mu &= (f^1, b f^5 \gamma^i \Sigma^i, 0, 0), \quad \text{(C.11a)} \\
\sqrt{2} n^\mu &= (f^1, -b f^5 \gamma^i \Sigma^i, 0, 0), \quad \text{(C.11b)} \\
\sqrt{2} m^\mu &= (0, 0, -c f^5 \Sigma^i, -i d f^4, -c f^5 \Sigma^i), \quad \text{(C.11c)} \\
\sqrt{2} m^*\mu &= (f^1, -b^{-1} f^5 \gamma^i \Sigma^i, 0, 0), \quad \text{(C.12a)} \\
\sqrt{2} n^\mu &= (f^1, b^{-1} f^5 \gamma^i \Sigma^i, 0, 0), \quad \text{(C.12b)} \\
\sqrt{2} m^\mu &= (0, 0, i d f^4, c f^5 \Sigma^i, -i d f^4 \Sigma^i), \quad \text{(C.12c)}
\end{align}

Recall that \( a, b, c, \) and \( d \) are given in equations (B.13).
When we calculate the spin coefficients we find:

\[ \chi = \gamma = \Pi = \alpha = \beta = \epsilon = \eta = 0 \]  
\[ 2 \sqrt{2} \lambda = - \dot{f} f^3 - 2 b f^5 \lambda_0 - i 2 \alpha f^6 \lambda_0 \] \[ 2 \sqrt{2} \sigma = \dot{f} f^3 - 2 b f^5 \lambda_0 + i 2 \alpha f^6 \lambda_0 \]  
\[ 2 \sqrt{2} \gamma = 5 \dot{f} f^3 + i \alpha f^6 \lambda_0 \]  
\[ \epsilon = - \gamma \]  
\[ 2 \sqrt{2} \nu = - 9 \dot{f} f^3 - 2 b f^5 \lambda_0 \]  
\[ 2 \sqrt{2} \rho = 9 \dot{f} f^3 - 2 b f^5 \lambda_0 \]  

Incidently, we notice from the spin coefficients that

\[ \chi = \rho - \rho^* = \gamma = \mu - \mu^* = \eta = \zeta = 0 \]  

which implies that \( l_{\mu} \) and \( n_{\mu} \) obey the Frobenius condition (Flaherty 1976, p.146) for hypersurface orthogonality:

\[ l_{[\mu \nu}, l_{\delta]} = n_{[\mu \nu}, n_{\delta]} = 0 \]  

(antisymmetrize over all three indices). Although we shall not use them, one may find null coordinates \( u \) and \( v \) since it is possible to
write \[ l^\mu = L u^\mu \quad \text{and} \quad \eta^\mu = N v^\mu \quad (C.16) \]

where \( L \) and \( N \) are functions of \( t \) and \( y \). For the \( \tau \) equation (B.29) we find that \( u \) and \( v \) may be expressed as

\[ \sqrt{\rho} \rho \frac{q}{\eta} u = \Sigma^{-\frac{\rho}{\eta^2} \frac{q}{\eta^2} \Sigma b} f \left[ q^2 + (q^2 + f^2)^{\frac{3}{2}} \right]^{-1} \quad (C.17a) \]

\[ \sqrt{\rho} \rho \frac{q}{\eta} v = \Sigma^{-\frac{\rho}{\eta^2} \frac{q}{\eta^2} \Sigma a} f \left[ q^2 + (q^2 + f^2)^{\frac{3}{2}} \right]^{-1} \quad (C.17b) \]

The intrinsic derivatives are

\[ f^{-5} \sqrt{\rho} \frac{\partial}{\partial t} = f^{-5} \frac{\partial}{\partial t} + b \psi^{-1} \Sigma \frac{\partial}{\partial y} \quad (C.18a) \]

\[ f^{-5} \sqrt{\rho} \Delta = f^{-5} \frac{\partial}{\partial t} - b \psi^{-1} \Sigma \frac{\partial}{\partial y} \quad (C.18b) \]

\[ f^{-5} \sqrt{\rho} \delta = (-c \Sigma \sqrt{\rho} \psi^{-1} - i df^{-1}) \frac{\partial}{\partial z} - c \Sigma^{-1} \frac{\partial}{\partial \phi} \quad (C.18c) \]

For this metric, Dirac's equations (C.10) reduce to

\[ (\gamma - \rho) F_1 + \delta^* F_1 = i \mu e G_1 \quad (C.19a) \]

\[ (\Delta - \gamma^* \rho) F_2 + \delta F_1 = i \mu e G_2 \quad (C.19b) \]

\[ (\gamma - \rho) G_2 - \delta G_1 = i \mu e F_2 \quad (C.19c) \]

\[ (\Delta - \gamma^* \rho) G_1 - \delta^* G_2 = i \mu e F_1 \quad (C.19d) \]
If we put
\[ F_i = f^{-1} \Sigma^{-1/2} f_i \]  
(C.20),

and similarly for \( F_i, G_i, \) and \( G_i \) then the Dirac system is

\[ \dot{d} \Sigma f_i + \Sigma \gamma^5 \Sigma f_i - i2 \Sigma \eta_i e f^{-5} \Sigma q_i + \hbar \partial_d f_i - 2c \Sigma^{-1} \partial_0 f_i = 0, \]  
(C.21a)

\[ \dot{d} \Sigma f_i - \Sigma \gamma^5 \Sigma f_i - i2 \Sigma \eta_i e f^{-5} \Sigma q_i + \hbar \partial_d f_i - 2c \Sigma^{-1} \partial_0 f_i = 0, \]  
(C.21b)

\[ \dot{d} \Sigma q_i + \Sigma \gamma^5 \Sigma q_i - i2 \Sigma \eta_i e f^{-5} \Sigma f_i - \hbar \partial_d q_i + 2c \Sigma^{-1} \partial_0 q_i = 0, \]  
(C.21c)

\[ \dot{d} \Sigma q_i - \Sigma \gamma^5 \Sigma q_i - i2 \Sigma \eta_i e f^{-5} \Sigma f_i - \hbar \partial_d q_i + 2c \Sigma^{-1} \partial_0 q_i = 0, \]  
(C.21d)

where
\[ \dot{d} := 2f^{-1} \partial_d - i\alpha f \Sigma, \]  
(C.22)

\[ \Sigma \gamma^5 := 2b \eta_i \Sigma \partial_q, \]  
(C.23)

\[ \hbar := -2c \Sigma \eta_i \Sigma^{-1} + i2 \partial_d f^{-1} \]  
(C.24)

Under the assumption that the four components of the wave function have a \( z \) and \( \phi \) dependence of the form \( \exp\{i(mz + n\phi)\} \) there is the usual slight simplification in equations (C.21); however, separation in the \( t \) and \( y \) variables is not evident except under stronger restrictions (e.g. \( m = 0 \)).
APPENDIX D

EQUIVALENCE OF SPINOR AND BISPINOR FORMALISMS

In spinor form Dirac's equations with the minimally coupled electromagnetic vector potential (Page 1976a, Gibbons 1977) are

\[
(\nabla_{A\tilde{B}} - i e A_{A\tilde{B}}) \rho^A = -i m z^{-1/2} \lambda_{\tilde{B}}, \tag{D.1}
\]

\[
(\nabla_{A\tilde{B}} + i e A_{A\tilde{B}}) \lambda^A = -i m z^{-1/2} \rho_{\tilde{B}}. \tag{D.2}
\]

The isomorphism between the real space-time vector \( A_{\mu} \) and the Hermitian spinor \( A_{A\tilde{B}} \) in spin space is provided by the van der Waerden symbol \( \sigma_{A\tilde{B}}^{\mu} \) according to

\[
A_{A\tilde{B}} = \sigma_{A\tilde{B}}^{\mu} A_{\mu}. \tag{D.3}
\]

The spinor notation used here is that of Bade and Jehle (1953). Since matrix representations of the quantities in the spinor Dirac theory is advantageous we will regard the relative positions of dotted and undotted indices as important (contrast Pirani 1964 and Penrose and Rindler). For example, the transpose of the quantity \( B_{A\tilde{B}} \) is \( B_{\tilde{A}C} \) where the position of the dot is fixed; the complex conjugate of \( B_{A\tilde{B}} \) is \( B_{\tilde{B}C} \). The van der Waerden symbol is Hermitian so that

\[
\sigma_{A\tilde{B}}^{\mu} = \sigma_{\tilde{B}A}^{\mu}. \tag{D.4}
\]
As usual, quantities with Greek indices transform appropriately in the space-time and are raised or lowered with the metric tensor $g_{\mu\nu}$.

In terms of the tetrads $X^\mu_{(A)}$, we have

$$
\sigma^\mu_{A\dot{B}} = X^\mu_{(A)} \sigma_{A\dot{B}}^{(A)}.
$$

(D.5)

A suitable representation from the spinor theory for the $\sigma^{(A)}_{A\dot{B}}$ is

$$
\begin{align*}
\sigma^{(0)}_{A\dot{B}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma^{(1)}_{A\dot{B}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma^{(2)}_{A\dot{B}} &= \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma^{(3)}_{A\dot{B}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
$$

(D.6)

The $(\alpha)$ index is raised or lowered with the tetrad metric $\eta_{(\alpha)(\beta)}$ chosen to be diag.(1,-1,-1,-1) while the spinor indices are raised or lowered according to the rules:

$$
\begin{align*}
\rho^A &= \varepsilon_{AB} \rho^B, \\
\rho_A &= \varepsilon_{BA} \rho^B, \\
\text{etc.}
\end{align*}
$$

(D.7)

where the spin space metric $\xi^{AB}$ is

$$
\xi^{AB} = \xi_{A\dot{B}} = \xi_{AB} = \xi_{A\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

(D.8)

Thus, from equation (D.6), we have

$$
\begin{align*}
\sigma^{(0)A\dot{B}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma^{(1)A\dot{B}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma^{(2)A\dot{B}} &= \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma^{(3)A\dot{B}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
$$

(D.9)
\[ \tilde{\sigma}^{(2)} A^B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \tilde{\sigma}^{(1)} A^B = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix} \] (D.9)

From the known relations of the tetrads (5.3) and (5.3) and the properties of the spinor algebra for the \( \sigma^{(\mu) A^B} \) we have

\[ \sigma^{\mu} = \sigma^A \gamma^A \gamma^\nu \] (D.10)

From Penrose and Rindler we quote the essentials of spinor analysis. The covariant differentiation of spinors \( \psi^A \) and \( \rho^B \) are

\[ \nabla^A \nabla^\mu = \frac{\partial}{\partial \mu} \psi^A + \gamma^A \gamma^\nu \rho^B \] (D.13)

where

\[ \gamma^A \gamma^A = 0 \] (D.14)

The complex conjugate case just has all the spinor indices dotted.

The spinor connection is \( \gamma^A_{\mu B} \). The covariant constancy of the \( \tilde{\sigma}^{\mu A^B} \) implies

\[ \tilde{\sigma} = \tilde{\sigma}^{A^B} = \sigma^A \gamma^A + \sigma^A \nabla^A \gamma^B \gamma^B + \sigma^A \nabla^B \gamma^A \gamma^B \] (D.15)

This determines the spinor connection as

\[ \gamma^A_{\mu B} = \frac{1}{2} \sigma^A \nabla^A \gamma^B \gamma^B (\sigma^A \gamma^A) - \sigma^A \gamma^A \gamma^B \] (D.16a)
\[ = \frac{1}{2} \mathcal{L}_{AB} \left( \sigma^B \delta^{cB} \cdot \mu + \sigma^B \left( \mu^{cB} \right) \right). \]  
(D.16b)

There is also the notation

\[ \nabla_{BA} \lambda^A = \sigma^{\mu \nu} \lambda_1^A. \]  
(D.17)

Now, in bispinor form, Dirac's equation is

\[ \gamma^\mu \left( \partial_\mu + i c A_\mu \right) \psi + i m \psi = 0. \]  
(D.18)

where again we have \( \hbar = c = 1 \) and \( A_\mu \) is the electromagnetic vector potential. Recall also

\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} I, \]  
(D.19)

\[ \Gamma_{\mu} = \frac{1}{4} \gamma^\nu \gamma_{\nu} \gamma_\mu \]  
(D.20a)

\[ = - \frac{1}{4} \gamma_{\nu} \gamma_\mu \gamma^\nu \]  
(D.20b)

\[ \gamma^\mu = \chi^{(\omega)} \gamma^{(\omega)} \]  
(D.21a)

\[ \nabla_\nu \gamma^\mu = 0. \]  
(D.21b)

We now jump to 4-dimensional representations of the Clifford algebra and make a choice for the \( \chi \)-matrix representation (Schweber, 1961, p.112):
\[ \gamma^{(o)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^{(o)} \\ \sigma^{(o)} & 0 \end{pmatrix}, \quad (D.22a) \]

\[ \gamma^{(k)} = \begin{pmatrix} 0 & \sigma^{(k)} \\ -\sigma^{(k)} & 0 \end{pmatrix}, \quad (D.22b) \]

\[ \gamma^{(5)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (D.22c) \]

where 1 is the 2 x 2 unit matrix and the \( \sigma^{(k)} \) are the Pauli matrices.

Define

\[ \gamma^{(a)} = \begin{pmatrix} 0 & \sigma^{(a)} \\ -\sigma^{(a)} & 0 \end{pmatrix}, \quad (D.23) \]

\[ \sigma^{(a)} = \begin{pmatrix} 0 & \sigma^{(o)} \\ \sigma^{(o)} & 0 \end{pmatrix}, \quad (D.24a) \]

\[ \tilde{\sigma}^{(a)} = \begin{pmatrix} 0 & \sigma^{(h)} \\ -\sigma^{(h)} & 0 \end{pmatrix}, \quad (D.24b) \]

\[ \sigma^M := \gamma^{(a)} \sigma^{(a)}, \quad (D.25a) \]

\[ \tilde{\sigma}^M := \gamma^{(a)} \tilde{\sigma}^{(a)}, \quad (D.25b) \]

Therefore, \[ \gamma^M = \begin{pmatrix} 0 & \sigma^M \\ -\tilde{\sigma}^M & 0 \end{pmatrix}, \quad (D.26) \]

It is easy to see that \( \sigma^M \) and \( \tilde{\sigma}^M \) are Hermitean; furthermore (Sachs 1967),

\[ \tilde{\sigma}^M = \epsilon \sigma^{\dagger M} \epsilon \quad (D.27) \]
where  \( \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)  \hspace{1cm} (D.28)

and (Bergmann 1957):

\[
\sigma^\lambda \bar{\sigma}^\lambda = -4  \hspace{1cm} (D.29a)
\]
\[
\sigma_\mu \bar{\sigma}^\mu = -2  \hspace{1cm} (D.29b)
\]
\[
\sigma_\mu A \bar{\sigma}^\mu = -2 \text{ Tr } A  \hspace{1cm} (D.29c)
\]

From the definition of \( \sigma \) and \( \bar{\sigma} \) in (D.24) one notices that (D.29) gives the spinor indexed equivalents (this relation we denote by \( \leftrightarrow \)) viz.

\[
\sigma^\mu \leftrightarrow \sqrt{2} \sigma^{-\mu} \gamma^\nu  \hspace{1cm} (D.30)
\]
\[
\bar{\sigma}^\mu \leftrightarrow -\sqrt{2} \sigma^{-\mu} \bar{\gamma}^\nu  \hspace{1cm} (D.31)
\]

The situation is as follows: given quantities in the 4-component bispinor representation, for example the anticommutation relations (D.19), then partitioned matrix multiplication will give quantities in terms of the matrices \( \sigma^\mu \) and \( \bar{\sigma}^\mu \). So, for this example, substitution of the \( \gamma \)-representation (D.26) into (D.19) gives

\[
\gamma^\nu \bar{1} = -\frac{1}{2} \left( \sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu \right)  \hspace{1cm} (D.32a)
\]
\[
\tilde{g}_{\mu} = -\frac{1}{2} \left( \tilde{\sigma}^\nu \tilde{\sigma}^\mu + \tilde{\sigma}^\mu \tilde{\sigma}^\nu \right) \quad \text{(D.32b)}
\]

One equation is derivable from the other by using (D.27); this redundancy is common in this method. Of course matrix multiplication implied in (D.32) but the indices are suppressed. From the key relations (D.30) and (D.31) one may make the index structure explicit for the matrix multiplication; thereafter, the index manipulation obeys the spinor index rules. Thus, if we take equation (D.32a), and make the obvious substitutions in the \(\leftrightarrow\) operation we have

\[
\tilde{q}_{\mu} \delta^\nu_A = \frac{-1}{\sqrt{2}} \left( \sqrt{2} \left( \sigma^{\mu}_{\lambda\chi} \sigma^\lambda_Y \sigma^\chi_Y + \sigma^{\nu}_{\lambda\chi} \sigma^\lambda_Y \sigma^\chi_Y \right) \right) \quad \text{(D.33)}
\]

If we put \(\chi = \lambda\), know that \(\delta^\lambda_A = 2\), and use the Hermiticity relation (D.4) then we recover exactly equation (D.10).

With the chosen \(\chi\)-representation, the Fock-Ivanenko coefficients have the form

\[
\Gamma^\mu = \begin{pmatrix}
\tilde{\Gamma}^\mu & 0 \\
0 & \tilde{\Gamma}^\mu
\end{pmatrix}
\]

where

\[
\tilde{\Gamma}^\mu = -\frac{1}{4} \tilde{\sigma}^\nu \sigma_{\nu;\mu} \quad \text{(D.35a)}
\]

\[
= \frac{1}{4} \tilde{\sigma}^\nu \sigma_{\nu;\mu} \quad \text{(D.35b)}
\]

and

\[
\tilde{\Gamma}^\mu = -\frac{1}{4} \sigma^\nu \tilde{\sigma}_{\nu;\mu} \quad \text{(D.36a)}
\]

\[
= \frac{1}{4} \sigma^\nu \sigma_{\nu;\mu} \quad \text{(D.36b)}
\]
The quantity \( \Gamma^\mu \) is redundant since

\[
\Gamma^\mu = - \varepsilon \Gamma^\mu \varepsilon \tag{D.37}
\]

Using the key substitutions (D.30) and (D.31) respectively in (D.35):

\[
\Gamma^\mu = \int_{\mu} \Gamma^\mu = \frac{1}{2} \sigma^\nu \sigma^{\nu \lambda} \tag{D.38a}
\]

\[
= - \frac{1}{2} \sigma^\nu \sigma^{\nu \lambda} \tag{D.38b}
\]

which are just the relations (D.16) for \( \gamma^\mu \), so

\[
\Gamma^\mu = \gamma^\mu \tag{D.39}
\]

Also

\[
\Gamma^\mu = \int_{\mu} \Gamma^\mu = \frac{1}{2} \sigma^\nu \sigma^{\nu \lambda} \tag{D.40a}
\]

\[
= - \frac{1}{2} \sigma^\nu \sigma^{\nu \lambda} \tag{D.40b}
\]

Because of (D.37) we find

\[
\Gamma^\mu = - \Gamma^\mu \tag{D.41}
\]

But from (D.40b) and (D.39a) we have, respectively

\[
\Gamma^\mu_{\lambda \beta} = - \frac{1}{2} \sigma^\nu \sigma^{\lambda \nu} \sigma^\beta \tag{D.42a}
\]

\[
\Gamma^\mu_{\lambda \beta} = \frac{1}{2} \sigma^\nu \sigma^{\beta \nu} \tag{D.42b}
\]
And so from (D.41) we have that

\[ \tilde{\Gamma}^A_{\mu} = \tilde{\Gamma}^B_{\mu} \]  (D.43)

which is equivalent to (D.13). The relative order of the spinor indices is irrelevant for this expression. Recall

\[ \gamma^\mu \tilde{\Gamma}^\rho \gamma_{\mu} = 0 \]  (3.23)

From equations (D.26) and (D.34) the above gives

\[ \sigma^\mu \tilde{\Gamma}^\rho \sigma_{\mu} = \tilde{\sigma}^\mu \tilde{\Gamma}^\rho \sigma_{\mu} = 0 \]  (D.44)

Using relation (D.29c) we find

\[ T^\rho \tilde{\Gamma}^\rho = 0 \]  (D.45)

In spinor language this is just the trace free condition on the spinor connection (D.14).

We represent the bispinor \( \Psi \) as

\[ \Psi = \begin{pmatrix} U \\ L \end{pmatrix} \]  (D.46)

where \( U \) and \( L \) are the 2 \( \times \) 2 upper and lower components of \( \Psi \). Putting (D.46), (D.34), and (D.26) into the Dirac equation (D.13) we have

\[ \sigma^\mu \left( \partial_\mu + \tilde{\Gamma}_\mu - i e A_\mu \right) U = -i m U \]  (D.47a)
\[ \bar{\sigma}^\mu (\partial_\mu + i \Gamma_\mu - ie A_\mu) U = \iota m L. \] (D.47b)

From our equivalence method (\( \equiv \)) to change from matrix multiplication to spinor algebra we infer

\[ \bar{\Gamma} \bar{\sigma}^\mu \bar{\nu} \left( \partial_\mu L^c + \Gamma^c_{\mu \nu} \bar{L}^a - ie A_\mu L^a \right) = -\iota m \bar{U}_\bar{\nu} \] (D.48a)

\[ -\bar{\Gamma} \bar{\sigma}^\mu \bar{\nu} \left( \partial_\mu \bar{U}^c - \Gamma^c_{\mu \nu} \bar{U}^a - ie A_\mu \bar{U}^a \right) = \iota m L^a \] (D.48b)

or, taking the complex conjugate of (D.48b) and rearranging indices,

\[ \bar{\sigma}^\mu \bar{\nu} \left( \partial_\mu \bar{U}^c + \Gamma^c_{\mu \nu} \bar{U}^a + ie A_\mu \bar{U}^a \right) = -\iota \bar{L}^a \bar{m} L_\bar{\nu}. \] (D.48c)

Equation (D.48) is equivalent to the Dirac equations (D.1) and (D.2) if

\[ \Psi = \begin{pmatrix} U \\ L \end{pmatrix} \leftrightarrow \begin{pmatrix} U^c \\ L^a \end{pmatrix}. \] (D.49)

The appropriateness of the covariant derivatives definitions (D.11), (D.12), and (D.17) is seen from the \( \Gamma_\mu \) construction. For example, the covariant constancy of the \( \sigma^{\mu \nu} \) is equivalent to the vanishing of the spin-covariant derivative of \( \gamma^\mu \) by (D.21b).
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