Algorithmic Aspects of Some Problems in Computational Biology

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Algorithmic Aspects of Some Problems in Computational Biology

by

Md. Shafiul Alam

A Dissertation

Submitted to the Faculty of Graduate Studies through the School of Computer Science in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy at the University of Windsor

Windsor, Ontario, Canada

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Algorithmic Aspects of Some Problems in Computational Biology

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Declaration of Co-Authorship / Previous Publication

I. Co-Authorship Declaration

I hereby declare that this thesis incorporates material that is result of joint research, as follows: This thesis incorporates the outcome of a joint research undertaken in collaboration with Dr. A. Sarker under the supervision of my thesis supervisor Professor A. Mukhopadhyay. The collaboration is covered in Chapter 3 of the thesis. In that investigation the key ideas, primary contributions, etc. were performed by myself working under my thesis supervisor Professor A. Mukhopadhyay. The contribution of co-author Dr. A. Sarker was primarily a theorem for determining the total number of layer graphs for a cycle. But the theorem is not included in this dissertation. It is used to confirm the total number of layer graphs for different cycles.

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Abstract

Given a sequence of pairs of numbers \( (a_i, l_i), i = 1, 2, \ldots, n \), with \( l_i > 0 \), and another pair of numbers \( L \) and \( U \), the length-constrained maximum density segment problem is to find a subsequence \([a_i, a_j]\) whose density \( \sum_{s=i}^{j} a_s / \sum_{s=i}^{j} l_s \) is the maximum under the constraint \( L \leq \sum_{s=i}^{j} l_s \leq U \). It has application to DNA sequence analysis in Computational Biology, particularly in the determination of the percentage of CG contents in a DNA sequence. A linear time geometric algorithm is presented that is more powerful than the existing linear time algorithms.

The method is extended to solve the \( k \) length-constrained maximum density segments problem in \( O(nk), O((n + k) \lg^2(U - L)) \) and \( O(n(U - L)) \) time when \( k \in O(\lg^2(U - L)) \), \( k \in \omega(\lg^2(U - L)) \cap o(n(U - L) / \lg^2(U - L)) \) and \( k \in \Omega(n(U - L) / \lg^2(U - L)) \cap O(n(U - L)) \) respectively. Previously, there was no known algorithm with non-trivial time complexity for this problem. We present a linear time algorithm to solve the length-constrained maximum sum segment problem. It is extended to solve the \( k \) length-constrained maximum sum segments problem in \( O(n + k) \) time. The algorithms are extended to solve the problem of finding all the length-constrained segments satisfying user specified sum or density lower bound in \( O(n + h) \) time, where \( h \) is the size of the output.

The point placement problem is to determine the positions of a linear set of points uniquely up to translation and reflection from the fewest possible distance queries between
pairs of points. The motivation comes from a problem known as the restriction site mapping.

If the points are necessarily distinct the lower bound and the upper bound for 2 rounds are $17n/16$ and $4n/3 + O(\sqrt{n})$ respectively, where $n$ is the number of points. We present 2-round algorithms with queries $10n/7 + O(1)$, $4n/3 + O(1)$ and $9n/7 + O(1)$ respectively.

The lower bound for 2 rounds is improved from $17n/16$ to $9n/8$.

We also present a modification of a geometric method called MSPocket for detection of ligand binding sites on protein surfaces. Experimentation using 48 benchmark dataset of bound protein structures shows that the success rate of our method is slightly better than that of MSPocket.
Dedicated to my parents Md. Khorsheed Alam and Rokeya Begum, my son Md. Ashraful Alam, and my daughters Shaima Alam and Tasnim Alam.
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I would like to thank my wife Amena Akter, my son Md. Ashraful Alam, and my daughters Shaima Alam and Tasnim Alam for their love, understanding and cooperation.
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Chapter 1

Introduction

1.1 The Problems of This Study

Deoxyribonucleic acid (DNA), Ribonucleic acid (RNA) and protein are the three essential macromolecules of all living cells. DNA is made of nucleotides (Figure 1.1). It does not usually exist as a single molecule in living organisms. Instead, it exists in pair, which are held together tightly in the form of a double helix.

A nucleotide consists of a sugar, a phosphate group and one of a set of 4 nucleobases. The nucleobase is attached to the sugar. The nucleobases are adenine, cytosine, guanine and thymine (or uracil). They are represented by the four letters A, C, G and T (or U). The nucleotides are linked in a chain to form the DNA. Consequently, DNA can be specified as a sequence of nucleobases. In fact, it is written as a sequence of letters representing the nucleobases. It is called the primary structure of DNA. DNA has secondary and tertiary structures as well. DNA contains information which is used by a living cell to manufacture proteins. A triplet of consecutive nucleobases corresponds to a specific amino acid. The triplet is called a codon. More than one codon can correspond to an amino acid; but one
codon can correspond to exactly one amino acid. The sequence of nucleobases of a DNA is translated into a sequence of amino acids.

Figure 1.1: DNA molecular structure [Figure from Wikipedia: http://upload.wikimedia.org/wikipedia/commons/e/e4/DNA_chemical_structure.svg].

Different composition of DNA sequence is associated with different properties of DNA. For example, genes are found in most of the cases in GC-richest regions of a DNA sequence. DNA sequences are analyzed to find biologically significant regions. In this dissertation we propose some algorithms for sequence analysis.
Molecules are three-dimensional entities. Their functions often depend on their three-dimensional structures. The energy associated with each set of possible atomic positions create intramolecular motions in large and complicated molecules such as proteins. The set of all conformations generated by the intramolecular motion is called the conformational space of the molecule. The difference between the energies of two conformations is due to various nonbonded interactions. The relation between those interactions and the conformational state is very important in molecular biology.

It is essential to determine the conformational space of a molecule from experimental data about its conformational state. The data are obtained by x-ray crystallography, nuclear magnetic resonance (NMR) spectroscopy, etc. While x-ray crystallography can produce the conformational space effectively, there are problems with this technique. For example, many macromolecules does not form good crystals. Much of the experimental data, about the conformational state of molecules, obtained by other methods, and the majority of energy functions can be expressed in terms of intermolecular distances. The conformational space can be described in terms of interatomic distances. The approach that determines the conformational space from the intermolecular distance data and chirality constraints is called distance geometry approach [25, 42]. Distance geometry can also be used to evaluate the effectiveness of other methods for determining molecular conformation space.

When the distances lie in between prescribed bounds the problem is called a bound embedding problem, otherwise it is called a distance embedding problem which is a spe-
cial case of the bound embedding problem. The one-dimensional version of the distance embedding problem without chirality constraint is the point placement problem on a line.

A polypeptide is a linked chain of amino acids. A protein consists of one or more polypeptides. Usually a protein folds into globular or fibrous form that facilitates a biological function. Proteins participate in almost all functions of a cell. Many proteins are enzymes and work as a catalytic agent in metabolism. Proteins also perform mechanical functions, cell signaling, immune responses, etc. Proteins can also work together to perform a task. Functions of proteins are the results of their interactions with other molecules. The interactions usually occur in the concave regions on the surface of a protein.

Like other molecular surfaces, protein surface is the solvent excluded surface, called Connolly surface. It is a purely geometric feature. Connolly [24] proposed a purely geometric method to compute it. Sanner et al. [60] proposed an improved method to calculate it. His method is also based on geometry. Consequently, geometry plays a major role in making a pocket suitable for ligand binding site. Among the best performing methods are those methods that uses purely geometric features of the pocket to predict the ligand binding site. We study one such method called MSPocket [75].

Geometry is involved in all the problems, directly or indirectly (by transformation into a geometric problem), that we study in this dissertation. Techniques of computational geometry can play a big role in solving these problems and other problems in structural biology involving geometry. The problems are described in the following subsections.
1.1.1 Sequence Analysis

Let $A$ be a sequence $(a_i, l_i) \ (i = 1, ..., n)$ of $n$ ordered pairs of real numbers $a_i \ (i = 1, ..., n)$ called values and positive real numbers $l_i > 0 \ (i = 1, ..., n)$ called lengths, and $L$ and $U$ be two positive real parameters $0 < L \leq U$. A segment, denoted by $A[i, j], 1 \leq i \leq j \leq n$, of $A$ is a consecutive subsequence of $A$ between the indices $i$ and $j$. The length $l[i, j]$, sum $s[i, j]$ and density $d[i, j]$ of a segment $A[i, j]$ are $l[i, j] = \sum_{t=i}^{j} l_t$, $s[i, j] = \sum_{t=i}^{j} a_t$ and
\[ d[i, j] = \frac{\sum_{l=i}^{j} a_l}{\sum_{l=i}^{j} l} \] respectively. \( A[i, j] \) is feasible if \( L \leq l[i, j] \leq U \). The length-constrained maximum sum segment problem is to find a feasible segment of maximum sum. The \( k \) length-constrained maximum sum segments problem is to find \( k \) feasible segments such that their sums are the \( k \) largest. The length-constrained maximum density segment problem is to find a feasible segment of maximum density. The \( k \) length-constrained maximum density segments problem is to find \( k \) feasible segments such that their densities are the \( k \) largest.

We present algorithms for length-constrained maximum sum segment and maximum density segment problems, in particular, and the problems of finding length-constrained heaviest segments, in general, for a sequence of real numbers. The length-constrained maximum density segment problem and the \( k \) length-constrained maximum density segments problem have been transformed into geometric slope selection problems.

The algorithms have potential applications in different areas of biomolecular sequence analysis, including finding CG-rich regions, TA and CG-deficient regions, CpG islands and regions rich in periodical three-base patterns, post processing sequence alignment, annotating multiple sequence alignments, and computing length constrained ungapped local alignment. They have applications in other areas also, such as pattern recognition, digital image processing and data mining [1, 34, 35].

### 1.1.2 Point Placement Problem

The point placement problem is to determine the positions of a set of \( n \) distinct points, \( P = \{p_1, p_2, p_3, \ldots, p_n\} \), on a line uniquely, up to translation and reflection, from the fewest
possible distance queries between pairs of points. This problem is closely related to the restriction site mapping problem in DNA sequence and turnpike problem in computational geometry. In the latter problems, the points are deduced from a set of interpoint distances between unlabeled points. While the distances in point placement problem are between labeled points. The point placement problem has application in the graph embedding problem.

The higher dimensional version of the point placement problem has application in the area of molecular conformation. In both the cases the interpoint distances of labeled points are specified. The molecular configuration of a rigid molecule is unique upto translation. It is not unique upto reflection. The same interatomic distances of a molecule can also occur in its mirror images. So, in order to uniquely determine the configuration of a rigid molecule, one needs to specify the interatomic distances and the chirality of a single asymmetric centre. This problem is called a distance embedding problem. On the other hand, solution to the three-dimensional version of the point placement problem is unique upto translation and reflection. So, the distances can uniquely determine the positions. If the distances are the interatomic distances of a rigid molecule, then the solution to the point placement problem will be the configuration of either the molecule or its mirror image.

If the molecule is mobile then the distances lie in between prescribed bounds. For this case the inputs are the lower and upper bounds on the interatomic distances and the chirality of quadruples of atoms. The problem is to determine the conformation space of
the molecule. This problem is called a bound embedding problem.

1.1.3 Ligand Binding

Molecular surface is a 3D Euclidean surface. Ligand binding sites are situated at dents on this surface. Consequently, geometry plays a major role in making a pocket suitable for ligand binding site. Among the best performing methods for predicting ligand binding site are those methods that uses purely geometric features of the pocket.

1.2 Contributions

The main contributions of this dissertation are described below:

- Sequence analysis problems: For the maximum sum segment problem with non-uniform length there is an algorithm with time and space complexities in $O(n)$. An algorithm with time complexity in $O(n)$ and space complexity in $O(\frac{U-L}{l_{\min}})$ is presented in this dissertation. For the maximum density segment problem with non-uniform length there is a combinatorial solution with time complexity in $O(n)$ and space complexity in $O(\frac{U}{l_{\min}})$. We present a simple geometric algorithm with the same time complexity and $O(\min\{\frac{U-L}{l_{\min}}, \frac{L}{l_{\min}}\})$ space complexity.

The algorithms are extended to respectively solve the $k$ length-constrained maximum sum segments problem in $O(n + k)$ time, and the $k$ length-constrained maximum density segments problem in $O(nk)$, $O((n + k)(\lg(U - L))^2)$ and $O(n(U - L))$ time when $k \in O(\lg^2(U - L))$, $k \in \omega(\lg^2(U - L)) \cap o(n(U - L)/\lg^2(U - L))$ and $k \in$
\(\Omega(n(U - L)/t g^2(U - L)) \cap O(n(U - L))\) respectively. They are extended to find all the length-constrained segments satisfying user specified sum or density lower bound in \(O(n + h)\) time, where \(h\) is the size of the output. Previously, there was no known linear time algorithm for these problems. We indicate the extensions of our algorithms to higher dimensions.

- Point placement problem: A 2-round algorithm is presented which solves the point placement problem with \(9n/7 + O(1)\) queries, where \(n\) is the number of points. The lower bound on 2-round algorithms is improved from \(17n/16\) to \(9n/8\). This improves the current best results for 2-round algorithm reported in [20].

- Ligand binding: We study a geometric method called MSPocket [75] for the detection of ligand binding site. It is one of the few best performing method for predicting ligand binding site. In this dissertation MSPocket is modified by replacing one constraint. Experiment on a set of 48 benchmark dataset of bound proteins shows that our method has a slightly better success rate than that of MSPocket.

1.3 Structure of the Thesis

Chapter 2 deals with the problems of maximum sum/density segment problems, \(k\)-maximum sum/density segment problems and segments satisfying sum/density requirement problems. In Chapters 3 and 4 we present some improved algorithms and an improved lower bound for the point placement problem. Chapter 5 presents the study of ligand binding. Results
are summarized in Chapter 6.
Chapter 2

Maximum Density Segment

2.1 Introduction

Let $A$ be a sequence $(a_i, l_i) (i = 1, ..., n)$ of $n$ ordered pairs of real numbers $a_i$ ($i = 1, ..., n$) called values and positive real numbers $l_i > 0$ ($i = 1, ..., n$) called lengths, and $L$ and $U$ be two positive real parameters $0 < L \leq U$. A segment, denoted by $A[i, j], 1 \leq i \leq j \leq n$, of $A$ is a consecutive subsequence of $A$ between the indices $i$ and $j$. The length $l[i, j]$, sum $s[i, j]$ and density $d[i, j]$ of a segment $A[i, j]$ are $l[i, j] = \sum_{t=i}^{j} l_t$, $s[i, j] = \sum_{t=i}^{j} a_t$ and $d[i, j] = \frac{\sum_{t=i}^{j} a_t}{\sum_{t=i}^{j} l_t}$ respectively. A feasible segment of $A$ is a segment $A[i, j]$ such that $L \leq l[i, j] \leq U$. The prefix sums of the sequence are defined as $s_0 = 0$ and $s_i = a_1 + a_2 + \ldots + a_i$ for $i = 1, ..., n$. $s_i$ ($i = 1, ..., n$) can be computed in linear time by noting that $s_i = s_{i-1} + a_i$. Once $s_i$’s are known, $s[i, j]$ ($1 \leq i \leq j \leq n$) can be computed in constant time since $s[i, j] = s_j - s_{i-1}$.

In this chapter we study some problems concerning the determination of length-constrained heaviest segments in a sequence of real numbers. The problems are formally described below:
Definition 2.1.1 Let $A$ be a sequence of pairs of real numbers $(a_i, l_i)$, $i = 1, 2, \ldots, n$, with $l_i > 0$, and $L$ and $U$ be a pair of real numbers with $L \leq U$. (a) The length-constrained maximum sum segment problem is to find a segment $A[i, j]$ whose sum $s[i, j]$ is the maximum under the constraint $L \leq l[i, j] \leq U$. (b) The length-constrained maximum density segment problem is to find a segment $A[i, j]$ whose density $d[i, j]$ is the maximum under the constraint $L \leq l[i, j] \leq U$.

The maximum sum segment problem, with uniform length ($l_i = 1$ for all $i$) and no restriction on segment length, was first studied by Grenander [39]. This problem arose while researching in the area of pattern recognition in digitized images. The original problem, as proposed by Grenander, was in 2-dimensions. In that setting the maximum sum subarray was an estimator for the maximum likelihood of a pattern in a digital image. He also simplified the problem to 1-dimension. The problem also has applications in other areas such as graphics, data mining [1, 34, 35] and bioinformatics [7]. An optimal linear time algorithm for the problem proposed by Kadane is described by Bentley [9] and Gries [40]. Its space complexity is $O(1)$. The two-dimensional version of the problem is to find a connected rectangular submatrix of maximum sum from a two-dimensional rectangular input matrix of real numbers [9]. Here the lengths are uniform, i.e., $l_i = 1$ for all $i$, and there is no restriction on the size of the submatrix. The problem has been extended to higher dimensions [70]. In higher dimensions the problem is called the maximum sum subarray problem. The higher dimensional problem has applications in the area of data mining (for dimensions less than
4) and Monte-Carlo simulation (dimensions being high) [70]. It can be solved by reducing it to 1-dimensional problems [10, 70]. For a 2-dimensional $m \times n$ matrix there are $O(m^2)$ column intervals. Each of them is solved using Kadane’s linear time algorithm for maximum sum segment problem. Hence, its time complexity is $O(m^2 n)$ [10, 70]. For this case, i.e., 2-dimensions with uniform length and no restriction on length, there is a better algorithm based on a distance matrix multiplication technique [70, 69]. Its running time is subcubic.

Huang [44] introduced the restriction of length cutoff $L$ in the setting of biomolecular sequence analysis to avoid reporting extremely short segments. He gave a linear time algorithm for computing the maximum sum segment of length at least $L$, but no restriction on the upper bound of its length, i.e., $U = n$. He had observed that the segments reported by the algorithm are usually much larger than $L$. From this observation Lin et al. [50] argued that the segments reported by the method may contain some poor and irrelevant segments. To avoid this they introduced the restriction of upper bound $U$ on the length of the segment. They proposed a linear time algorithm for the problem when there is only the upper bound $U$ on the length of the segment, but no lower bound, i.e., $L = 0$. They combined that algorithm with Huang’s [44] technique to develop a linear time algorithm for arbitrary $L$ and $U$. Its space requirement is also linear. Fan et al. [31] gave an $O(n)$ time and $O(U)$ space algorithm for the problem for the case of uniform length. In this chapter, we present an algorithm for this general problem with time complexity in $O(n)$ and space complexity in $O(U - L)$. It can be modified in a straightforward way to solve the problem
with non-uniform length in $O(n)$ time and $O\left(\frac{U-L}{l_{\text{min}}}\right)$ space. We indicate the extension of this algorithm to solve the problem in higher dimensions by using the technique of reducing the problem to 1-dimension [10, 70].

The $k$ maximum sum segments problem was introduced by Bae and Takaoka [8]. There was no restriction on the segment length. A natural extension of this problem is the $k$ length-constrained maximal sum segments problem. The problem is defined as follows:

**Definition 2.1.2** Given a sequence $A$ of real numbers $a_i$, $i = 1, 2, \ldots, n$, a pair of real numbers $L \leq U$ and an integer $k$ such that $1 \leq k \leq (n-U+1)(U-L+1)+\frac{1}{2}(U-L)(U-L+1)$, the $k$ length-constrained maximum sum segments problem is to find $k$ segments of $A$ of length at least $L$ and at most $U$ such that their sums are the $k$ largest among all the possible segments of $A$ of length at least $L$ and at most $U$.

When there is no restriction on segment length, i.e., $L = 0$ and $U = n$, Cheng et al. [18, 19] gave an $O(n + k \log \min\{n, k\})$ time algorithm, and Brodal and Jorgensen [13] gave an optimal $O(n+k)$ time algorithm for this. The latter algorithm constructs a partially persistent [29] binary maximum heap that implicitly contains all the $\binom{n}{2} + n$ number of sums for all possible segments in $O(n)$ time. The heap is a modified version of the self-adjusting heap of Sleator and Tarjan [64]. The $k$ maximum sums are selected from the heap using the linear time heap selection algorithm of Frederickson [33]. Brodal and Jorgensen [13] extended their algorithm to higher dimension by using the technique of reducing the problem to 1-dimension [10, 70]. Liu and Chao [51] gave an $O(n + k)$ time and $O(n)$ space
algorithm for the \( k \) length-constrained maximum sum segments problem. Combining with the technique of Brodal and Jorgensen [13] we extend our algorithm for the maximum sum segment problem to solve the \( k \) length-constrained (i.e., arbitrary \( L \) and \( U \)) maximum sum segments problem. Its time and space complexities are in \( O(n + k) \) and \( O(U - L + k) \) respectively.

For the maximum density segment problem, when the lengths are uniform and there is no restriction on the segment length, the maximum element in the sequence will be the solution and it can be found in a straightforward way in \( n - 1 \) comparisons and \( O(1) \) space. When \( U = L \) the problem is trivially solvable in \( O(n) \) time since there are \( n - U + 1 \) feasible segments. When the lengths are uniform, \( U \neq L \) and no upper bound \( (U \geq n - L) \), Huang [44] showed that the length of the maximum density segment is at most \( 2L - 1 \). So, this case is equivalent to the case when \( U = 2L - 1 \) and can be solved in \( O(nL) \) using brute force method since the number of feasible segments is \( O(nL) \). For this case, Lin et al. [50] gave an \( O(n \log L) \) time algorithm by using a method of right skew decomposition of the sequence. When the lengths are uniform, and \( U \) and \( L \) are arbitrary, Goldwasser et al. [37] gave an \( O(n) \) time algorithm. For the general case, where the lengths are not uniform and \( U \) and \( L \) are arbitrary, Goldwasser et al. [38] extended the right skew decomposition method of Lin et al. [50] to develop an \( O(n) \)-time and space algorithm. A combinatorial solution with time-complexity in \( O(n) \) and space complexity in \( O(U) \) was proposed by [21, 22]. The algorithm works in an online manner. In the same paper it was pointed out that the
linearity claim of a geometric approach by Kim [46] is flawed. Lee et al. [48] fixed the flaw of Kim’s algorithm by exploiting the property of decomposability of tangent query. Its time and space complexities are in $O(n)$. In this chapter, we present a simple modification to Kim’s algorithm to address the flaw, while retaining the simplicity, elegance and linearity of his geometric approach. Our algorithm’s time and space complexities are in $O(n)$ and $O(\min\{U - L, L\})$ respectively, and it works in an online manner. 

The $k$ maximum sum segments problem was introduced by Bae and Takaoka [8]. A natural extension of this problem is the $k$ length-constrained maximal density segments problem. The problem is defined as follows:

**Definition 2.1.3** Given a sequence $A$ of real numbers $a_i$, $i = 1, 2, \ldots, n$, a pair of real numbers $L \leq U$ and an integer $k$ such that $1 \leq k \leq (n - U + 1)(U - L + 1) + \frac{1}{2}(U - L)(U - L + 1)$, the $k$ length-constrained maximum density segments problem is to find $k$ segments of $A$ of length at least $L$ and at most $U$ such that their densities are the $k$ largest among all the possible segments of $A$ of length at least $L$ and at most $U$.

We extend our algorithm to solve the $k$ length-constrained maximum density segments problem in $O(nk)$, $O((n + k)(\log(U - L))^2)$ and $O(n(U - L))$ time when $k \in O(\log^2(U - L))$, $k \in \omega(\log^2(U - L)) \cap o(n(U - L)/\log^2(U - L))$ and $k \in \Omega(n(U - L)/\log^2(U - L)) \cap O(n(U - L))$ respectively.

Huang [44] introduced the problem of finding segments of a sequence satisfying a sum

\footnote{The algorithm was presented at the 20th Annual Fall Workshop on Computational Geometry 2010 [4].}
requirement. The content requirement is expressed as the count of equal length oligomers in biomolecular sequence. We shall call this problem as the required sum segments problem. A natural extension of this is the required density segments problem. The problems are defined as follows:

**Definition 2.1.4** Let $A$ be a sequence of real numbers $a_i$, $i = 1, 2, \ldots, n$, $\sigma$ and $\delta$ be a pair of real numbers, and $L$ and $U$ be another pair of real numbers with $L \leq U$. (a) The length constrained segments satisfying a sum lower bound problem is to find all the segments $A[i, j]$ such that $s[i, j] \geq \sigma$. (b) The length constrained segments satisfying a density lower bound problem is to find all the segments $A[i, j]$ such that $d[i, j] \geq \delta$.

For the former problem when there is only a lower bound on the length of the sequence and no upper bound on its length, Huang [44] gave a linear time algorithm for a related problem using dynamic programming technique. His algorithm finds all the optimal segments of length at least $L$ satisfying a sum lower bound. Modifying the technique of Liu and Chao [51], we solve both the length-constrained segments satisfying a sum lower bound problem and the length-constrained segments satisfying a density lower bound problem in $O(n + h)$ time and $O(U - L + h)$ space, where $h$ is the size of the output. Previously, there was no known algorithm with non-trivial result for these problems.

All of our algorithms can be used to solve the corresponding higher dimensional problems by reducing them to 1-dimensional problems in the way described in [10, 70]. They can also be extended to solve the problems with non-uniform length. We note that for $k$ maximum
sum segments problem there is another version of the problem where there is no restriction
on the segment length (i.e., \( L = 0 \) and \( U = n \)) but the segments are not allowed to
overlap. For this case there are linear time algorithms for 1-dimension [16, 17, 59]. In this
dissertation, we shall not pursue this line. In all of our algorithms in this dissertation, the
segments are allowed to overlap.

According to [55, 68], the compositional heterogeneity of a genomic sequence is strongly
correlated to its CG content regardless of the size of the genome. It is also found that
gene length [30], gene density [76], patterns of codon usage [61], distribution of different
types of repetitive elements [30, 66], number of isochores [11], length of isochores [55] and
recombination rate within chromosomes [36] are related to CG content. The algorithms can
be used directly to find length-constrained CG-rich regions with the maximum sum and
average or with some user specified content requirement in a DNA sequence.

The nucleotide composition of a newly determined DNA sequence is analyzed to locate
its biologically meaningful segments including finding CG-rich regions [32, 41], TA and CG-
deficient regions [56], CpG islands [41], regions rich in periodical three-base pattern [62, 71],
post processing sequence alignment [74], annotating multiple sequence alignments [68] and
computing length-constrained ungapped local alignment [6]. Our algorithms have potential
applications in those areas.

In Section 2.2 we briefly describe Kim’s [46] algorithm for the maximum density seg-
ment problem. Our algorithms for the maximum density, \( k \) maximum density segments,
maximum sum segments and $k$ maximum sum segments of a sequence are presented in Sections 2.3, 2.4, 2.5 and 2.6 respectively. Section 2.7 describes our algorithms for finding all the segments of a sequence having sum or density bounded below by some user specified value. Concluding remarks are given in Section 2.8.

### 2.2 Kim’s Algorithm for Maximum Density Segment

We describe Kim’s [46] algorithm for the maximum density segment problem using uniform length. He reduced the problem to a geometric one thus. The element indices and corresponding prefix sums give $n + 1$ points in the plane $p_0 = (0, s_0), p_1 = (1, s_1), p_2 = (2, s_2), \ldots, p_n = (n, s_n)$, sorted by their $x$-coordinates. The density of a segment $A[i, j]$ can then be interpreted as the slope of the line segment through the points $(i - 1, s_{i-1})$ and $(j, s_j)$. The problem then is to find $p_i$ and $p_j$ such that $\frac{s_{i-1}}{i-1} - \frac{s_j}{j}$ has the largest slope.

Without any restriction on the segment length, the maximum density segment problem is solved by computing the largest slope defined by a pair of the above points. We can use any of a number of $O(n \log n)$ slope selection algorithms for this problem ([23] or [45] for example). The constraints on the segment length add a new dimension to the problem.

For a given right endpoint $p_j$, the set of candidate left endpoints $p_i$ has $i$ in the index-window $I_j = [0, j - L]$ when $L \leq j < U$ and in $I_j = [j - U, j - L]$ when $j \geq U$. If we maintain the lower convex hull of the points in this index-window, then the largest slope is found by drawing a tangent from $p_j$ to a point $p_i$ on this lower hull. The maximum density
segment for a fixed $j$ is then $a_{t+1}, a_{t+2}, \ldots, a_j$. As $j$ goes from $L$ to $n$ the maximum of all slopes found gives the desired maximum density segment.

Based on the above formulation, Kim proposed an algorithm that claimed to be able to perform all the dynamic updates to the lower convex hull as the index-window moves from the left to the right in $O(n)$ amortized time. This claim is inaccurate. Figure 2.1 shows the lower convex hull ($lch$, for short) of the points inside the index-window $I(j)$, where $p_x, p_z$ and $p_y$ are the leftmost, bottommost and rightmost points on the $lch$. Kim maintains the portion of $lch$ from $p_y$ to $p_z$ in one array and the portion of the $lch$ from $p_x$ to $p_z$ in another array.

![Figure 2.1: The lower convex hull of the points in the index-window $I_j$](image)

Now, it is crucial to the correctness of Kim’s algorithm that, as the window $I_j$ slides to the right the algorithm remains updated about the new value of $p_z$. Kim’s algorithm correctly updates $p_z$, except in the case shown in Figure 2.2.

In this case, as the window slides to the next position the hull update cannot be done in $O(1)$ time as Figure 2.3 shows.
2.3 Algorithm for Maximum Density Segment

First, we describe our algorithm for the case of uniform length, i.e., \( l_i = 1 \) for \( i = 1, \ldots, n \).

The main idea underlying the new algorithm is to consider the right end point \( p_j \) (for \( j = L, L + 1, \ldots, n \)) of all the feasible segments \( \overline{p_i p_j} \) in batches of a fixed size. For each \( p_j \), instead of computing a single lower convex hull of the feasible set of left end points \( p_i \), we compute two lower convex hulls - a left one and a right one that start at 2 adjacent points \( p_{m-1} \) and \( p_m \), \( j - U < m \leq j - L \) (Figure 2.4). The right lower hulls are computed incrementally in a left-to-right (LR) pass for a batched set \( p_j \), and the left hulls in a right-to-left (RL) pass for the same batched set. Thus, the problem that arises in Kim’s algorithm
from the dynamic convex hull update as a result of deletion on the left is avoided. The correctness of this scheme follows from the following observation for the property of a set:

\[ \max_{p_i \in G^j} \text{slope}(p_ip_j) = \max \{ \max_{p_i \in G_1^j} \text{slope}(p_ip_j), \max_{p_i \in G_2^j} \text{slope}(p_ip_j) \}. \]

We consider the right end points \( p_j, j > U \), in batches of size \( U - L + 1 \). The details of the LR and RL passes for a batch of points \( p_j, j \in [k, k + U - L], k > U \), are described below. For each pass, we first describe the algorithm informally, and then follow it up with a formal description.

2.3.1 The LR pass

In this pass, we consider the right end points \( p_j, j \in [k, k + U - L] \), in left-to-right fashion. For each new right end point \( p_j, j \in [k, k + U - L] \), we incrementally compute the lower convex hull \( H_r = LCH(\{p_{k-L}, p_{k-L+1}, \ldots, p_j-L\}) \). In other words, for each \( p_j \), \( H_r \) is updated by insertion of a new point \( p_{j-L} \) on the right end of it. Following Kim [46], we maintain 2
parameters to aid the incremental computation: a tangent line $l$ to the current hull $H_r$ with the maximum slope found so far, and the point of contact $\alpha$ of $l$ with the current hull $H_r$. We always represent $l$ by a pair of points. The slope of $l$ is the current maximum density for this batch of $p_j$.

Initially, $H_r = \{p_{k-L}\}$, $l = \overline{p_{k-L}p_k}$ and $\alpha = p_{k-L}$. For the current right end point $p_j$, $j \in [k, k + U - L - 1]$, let $H_r$, $l$ and $\alpha$ be as shown in Figure 2.5. For the next right end point $p_{j+1}$, we update $H_r$, $l$ and $\alpha$. $H_r$ is updated by inserting the point $p_{j+1-L}$ on the right, i.e., $H_r = LCH(H_r \cup \{p_{j+1-L}\})$. The updated $H_r$ is traversed counterclockwise from $\alpha$ (or from the newly inserted hull point $p_{j+1-L}$ if $\alpha$ is deleted from $H_r$) to find the new tangent line $l$ having the maximum slope found so far, and the new point of contact $\alpha$ on $H_r$ with the updated $l$. There are 4 cases as follows:

![Figure 2.5: LR pass: The lower hull, $l$ and $\alpha$ for the right end point $p_j$, $j \in [k, k + U - L - 1]$](image)

**Case 1:** Both $p_{j+1-L}$ and $p_{j+1}$ are above $l$ (Figure 2.6).

$H_r$ is updated. $H_r$ is traversed counterclockwise from $\alpha$ to the point of contact of the
tangent from \( p_{j+1} \) to this new \( H_r \), while these tangent and point of contact are set to be the new \( l \) and \( \alpha \) respectively.

**Figure 2.6:** LR pass: Both \( p_{j+1-L} \) and \( p_{j+1} \) are above \( l \)

**Case 2:** \( p_{j+1-L} \) is above, and \( p_{j+1} \) is on or below \( l \) (Figure 2.7).

\( H_r \) is updated. However, \( \alpha \) and \( l \) remain unchanged.

**Figure 2.7:** LR pass: \( p_{j+1-L} \) is above, while \( p_{j+1} \) is on or below \( l \)

**Case 3:** \( p_{j+1-L} \) is on or below \( l \) (Figure 2.8).

\( H_r \) is updated. Let \( l' \) be a line through \( p_{j+1-L} \) and parallel to \( l \). Let \( p_{j+1} \) be above
$l'$; reset $l = p_{j+1-L}p_{j+1}$ and $\alpha = p_{j+1-L}$.

Figure 2.8: LR pass: $p_{j+1-L}$ is on or below $l$, and $p_{j+1}$ is above $l'$

**Case 4:** $p_{j+1-L}$ is on or below $l$, and $p_{j+1}$ is on or below $l'$ (Figure 2.9).

$H_r$ is updated. Set $l$ to $l'$ and $\alpha = p_{j+1-L}$.

Figure 2.9: LR pass: $p_{j+1-L}$ is on or below $l$, and $p_{j+1}$ is on or below $l'$

Each point in the left window $\{p_{k-L}, p_{k+1-L}, \ldots, p_{k+U-2L}\}$ is added to an $H_r$ once, and deleted at most once from a subsequent $H_r$. For a new point $p_j$, if $\alpha$ does not move right, the cost of computation is constant and is charged to the point $p_{j-L}$ that is added to the
hull. We note that $\alpha$ never moves clockwise. Now we consider the case in which $\alpha$ move counterclockwise. Each point on $H_r$ is accessed at most once during the recomputation of $\alpha$, since it never moves clockwise. The cost of recomputing $\alpha$ is charged to the points passed over as we move counterclockwise on the updated $H_r$ from the current $\alpha$, and the cost of deleting the points on $H_r$ on the left of $\alpha$ are charged to them. Thus, each point $p_i$ in the left window is charged at most 3 times: 2 times for insertion into and deletion from $H_r$ and once for being passed over by $\alpha$.

Since $\alpha$ never moves backward in this pass, we do not need to maintain the part of $H_r$ that lies on the left of $\alpha$. The algorithm for the LR pass, called MDS-LRPASS, is given in Algorithm 1.

We note that at the end of traversal of $H_r$ in step 2.1.2.1, if an edge of $H_r$ coincides with the new tangent line, we select the right end point of that edge as the point of contact $p_i$.

### 2.3.2 The RL pass

This pass needs more careful handling. In this pass, we consider the right end points $p_j$, $j \in [k, k+U-L-1]$, in right-to-left fashion. For each new right end point $p_j$, $j \in [k, k+U-L-1]$, we incrementally compute the lower convex hull $H_l = LCH(\{p_{j-U}, p_{j-U+1}, \ldots, p_{k-L-1}\})$. In other words, for each $p_j$, $H_l$ is updated by insertion of a new point $p_{j-U}$ on the left end of it. As in LR pass, we maintain 2 parameters to aid the incremental computation: a tangent line $l$ to the current hull $H_l$ with the maximum slope found so far, and the point
Algorithm 1 Algorithm for LR Pass

1: procedure MDS-LRPASS(s, L, U, k)

Input: \(s\) is the array of prefix sum for the input sequence. \(L\) and \(U\) are respectively lower and upper bounds. \(k\) is the index of the first element of the current batch of right end elements.

Output: Maximum density segment \(l\) in LR pass for the current batch of elements.

2: \(H_r \leftarrow LCH(\{p_{k-L}\})\) \(\triangleright\) \(p_i\) is the point \((i, s_i)\). \(H_r\) is the \(lch\).
3: \(\alpha \leftarrow p_{k-L}\) \(\triangleright\) \(\alpha\) is the left end point of the current maximum slope line segment.
4: \(l \leftarrow \alpha \overrightarrow{p_k}\) \(\triangleright\) \(l\) is the current maximum slope line segment. It is stored as a pair of points.
5: for \(j \leftarrow k + 1\) to \(k + U - L\) do
6:     if \(p_{j-L}\) is above \(l\) \(\text{then}\)
7:         \(H_r \leftarrow LCH(H_r \cup \{p_{j-L}\}\)
8:         if \(p_j\) is above \(H_r\) \(\triangleright\) \(l\) and \(\alpha\) are not updated if \(p_j\) is on or below \(l\).
9:         Starting from \(\alpha\), traverse \(H_r\) counterclockwise to find the new point of contact \(p_i\) on it with the tangent line passing through \(p_j\), and delete from \(H_r\) those points that are passed over by \(\alpha\).
10:         \(\alpha \leftarrow p_i\)
11:         \(l \leftarrow \alpha \overrightarrow{p_j}\)
12:     end if
13: else
14:     \(H_r \leftarrow \{p_{j-L}\}\)
15:     \(\alpha \leftarrow p_{j-L}\)
16:     Set \(l\) to the line parallel to \(l\) and passing through \(\alpha\)
17:     if \(p_j\) is above \(l\) \(\text{then}\) \(\triangleright\) \(l\) is not updated further if \(p_j\) is on or below \(l\)
18:         \(l \leftarrow \alpha \overrightarrow{p_j}\)
19:     end if
20: end if
21: end for
22: return \(l\)
23: end procedure
of contact $\alpha$ of $l$ with the current hull $H_l$. We always represent $l$ by a pair of points. The slope of $l$ is the current maximum density for this batch of $p_j$.

Initially, $H_l = \{p_{k-L-1}\}$, $l = \overline{p_{k-L-1}p_{k+U-L-1}}$ and $\alpha = p_{k-L-1}$. For the current right end point $p_j$, $j \in [k + 1, k + U - L - 1]$, let $H_l$, $l$ and $\alpha$ be as shown in Figure 2.10. For the next right end point $p_{j-1}$, we update $H_l$, $l$ and $\alpha$. $H_l$ is updated by inserting the point $p_{j-1-U}$ on the left, i.e., $H_l = LCH(\{p_{j-1-U}\} \cup H_l)$. The updated $H_l$ is traversed counterclockwise from $\alpha$ (or from the newly inserted hull point $p_{j-1-U}$ - if $\alpha$ is deleted from $H_l$) to find the new tangent line $l$ having the maximum slope found so far, and the new point of contact $\alpha$ on $H_l$ with the updated $l$. Again, there are 4 cases as follows:

$$\begin{align*}
\text{Case 1: } p_{j-1-U} &\text{ is on or above } l, \text{ and } p_{j-1} \text{ is above } l \text{ (Figure 2.11).}
\end{align*}$$

$H_l$ is updated. We traverse $H_l$ counterclockwise from $\alpha$ to find a tangent to it from $p_{j-1}$. We reset $l$ to this tangent line and $\alpha$ to the point of contact between updated $l$ and $H_l$. 

Figure 2.10: *RL pass: The lower hull, $l$ and $\alpha$ for the right end point $p_j$, $j \in [k + 1, k + U - L - 1]*

\begin{align*}
\text{Case 1: } p_{j-1-U} &\text{ is on or above } l, \text{ and } p_{j-1} \text{ is above } l \text{ (Figure 2.11).}
\end{align*}
Figure 2.11: RL pass: $p_{j-1-U}$ on or above $l$, and $p_{j-1}$ is above $l$

Case 2: $p_{j-1-U}$ is on or above $l$, and $p_{j-1}$ is on or below $l$ (Figure 2.12).

$H_l$ is updated. However, $\alpha$ and $l$ remain unchanged.

Figure 2.12: RL pass: $p_{j-1-U}$ is on or above $l$ and $p_{j-1}$ is on or below $l$

Case 3: $p_{j-1-U}$ is below $l$ (Figure 2.13).

$H_l$ is updated. Let $l'$ be a line through $p_{j-1-U}$ and parallel to $l$. Let $p_{j-1}$ be above $l'$.

There will be only one point, viz., $p_{j-1-U}$, on the updated $H_l$ that is on the left side of $\alpha$. We traverse the updated $H_l$ from $p_{j-1-U}$ counterclockwise from $\alpha$ to the point of contact of the tangent from $p_{j-1}$ to the new $H_l$, while $\alpha$ and $l$ are updated to the new
tangent and the point of contact respectively. In this case, on the left of \( \alpha \) at most one point, viz., the newly added point \( p_{j-1-U} \), is checked to find \( \alpha \). Consequently, \( \alpha \) can move left by at most one point.

**Figure 2.13**: RL pass: \( p_{j-1-U} \) is below \( l \) and \( p_j \) is above \( l' \)

**Case 4**: \( p_{j-1-U} \) is below \( l \), and \( p_{j-1} \) is on or below \( l' \) (Figure 2.14).

\( H_l \) is updated as in Case 3. We reset \( l \) to \( l' \) and \( \alpha \) to \( p_{j-1-U} \).

**Figure 2.14**: RL pass: \( p_{j-1-U} \) is below \( l \) and \( p_j-1 \) is on or below \( l' \)

Time complexity analysis for this pass is exactly the same as that for the LR pass, except that for a new point \( p_j \), \( \alpha \) may move clockwise on \( H_l \) exactly by one position. If it
does move clockwise, then it moves to \( p_{j-U} \). This cost is charged to the new point \( p_{j-U} \) in the left window. Thus, each point \( p_i \) in the left window is charged at most 4 times: 2 times for insertion into and deletion from \( H_l \), once when \( \alpha \) moves clockwise to it and once when \( \alpha \) passes over it.

We note that once \( \alpha \) moves clockwise and passes over a point \( p_i \) on \( H_l \), it never moves back to that point again, or to any point lying on its left in the current \( H_l \). Consequently, those points cannot be in contention for \( \alpha \) anymore. We delete them from current \( H_l \) and do not consider them for future \( H_l \). The algorithm for the RL pass, called MDS-RLPASS, is given in Algorithm 2.

We note that at the end of traversal of \( H_r \) in steps 2.1.1 and 2.2.4.1, if an edge of \( H_r \) coincides with the new tangent line, we select the right end point of that edge as the point of contact \( \alpha \).

### 2.3.3 The Algorithm

For the first batch of \( U - L + 1 \) points \( p_j, j \in [L,U] \), we make an LR pass only. For the remaining points at the end, right end points are \( p_j, j \in [k,n] \). First, we make an LR pass with \( p_i, i \in [k-L+1,n-L+1] \) and \( \text{LCH}(\{p_{k-L}, p_{k-L+1}, \ldots, p_{n-L}\}) \). Next, we make an RL pass as follows: Left end points are \( p_i, i \in [k-U+1,k-L] \). Construct \( \text{LCH} \) \( H_l \) for the left end points \( p_{k-L-1}, p_{k-L-2}, \ldots, p_{n-U} \). Draw tangent from \( p_n \) to this hull. The tangent line is \( l \) and the point of contact is \( \alpha \). Delete from \( H_l \) the points that are on the right side of \( \alpha \). Now we make RL pass starting from right end point \( p_{n-1} \) and left end point \( p_{n-U-1} \).
Algorithm 2 Algorithm for RL pass

1: procedure MDS-RLPASS(s, L, U, k)
2:   \textbf{Input:} s is the array of prefix sum for the input sequence. L and U are respectively lower and upper bounds. k is the index of the first element of the current batch of right end elements.
3:   \textbf{Output:} Maximum density segment l in RL pass for the current batch of elements.
4: \begin{align*}
5:   &H_l \leftarrow LCH(\{p_{k-L-1}\}) \quad \triangleright p_i \text{ is the point } (i, s_i). \quad H_l \text{ is the lch.} \\
6:   &\alpha \leftarrow p_{k-L-1} \quad \triangleright \alpha \text{ is the left end point of the current maximum slope line segment.} \\
7:   &l \leftarrow \alpha p_{k+U-L-1} \quad \triangleright l \text{ is the current maximum slope line segment. It is stored as a pair of points.} \\
8:   &\textbf{for} j \leftarrow k + U - L - 2 \textbf{ to } k \textbf{ do} \\
9:   &\textbf{if} p_{j-U} \text{ is on or above } l, \text{ and } p_j \text{ is above } l \textbf{ then} \quad \triangleright \text{If } p_{j-U} \text{ is on or above } l, \text{ and } p_j \text{ is on or below } l, \text{ then none of } H_l, l \text{ and } \alpha \text{ is updated} \\
10:   &\quad \text{Starting from } \alpha, \text{ traverse } H_l \text{ counterclockwise to find the new point of contact } p_i \text{ on it with the tangent line passing through } p_j, \text{ and delete from } H_l \text{ those points that are passed over by } \alpha \\
11:   &\quad \alpha \leftarrow p_i \\
12:   &\quad l \leftarrow \alpha p_j \\
13:   &\textbf{else} \quad \triangleright p_{j-U} \text{ is below } l \\
14:   &\quad H_l \leftarrow LCH(\{p_{j-U}\} \cup H_l) \\
15:   &\quad \alpha \leftarrow p_{j-U} \\
16:   &\quad \text{Set } l \text{ to the line parallel to } l \text{ and passing through } \alpha \\
17:   &\quad \textbf{if} p_j \text{ is above } l \textbf{ then} \quad \triangleright \text{If } p_j \text{ is on or below updated } l, \text{ then none of } \alpha \text{ and } l \text{ is updated again} \\
18:   &\quad \quad \text{Starting from } \alpha, \text{ traverse } H_l \text{ counterclockwise to find the new point of contact } p_i \text{ on it with the tangent line passing through } p_j, \text{ and delete from } H_l \text{ those points that are passed over by } \alpha \\
19:   &\quad \quad \alpha \leftarrow p_i \\
20:   &\quad \quad l \leftarrow \alpha p_j \\
21:   &\textbf{end if} \\
22: \end{align*}
23: \textbf{end if} 
24: \textbf{end for} 
25: return l 
26: end procedure
It stops when $j = k$ and $i = k - U$. We call this algorithm as MDS-RIGHTRESIDUAL.

The algorithms for maximum density segment, called MDS, is given in Algorithm 3.

**Algorithm 3** Algorithm for maximum density segment problem

1: procedure MDS($A, L, U$)  
2: \hspace{1em} \textbf{input}: $A$ is the input sequence. $L$ and $U$ are respectively lower and upper bounds.  
3: \hspace{1em} \textbf{output}: Maximum density segment $l$ of $A$.  
4: \hspace{2em} $n \leftarrow |A|$ \hspace{1em} \triangleright n is the number of elements in $A$  
5: \hspace{2em} $s_0 \leftarrow 0$ \hspace{1em} \triangleright s is the array of prefix sum for the input sequence $A$.  
6: \hspace{2em} for $i = 1, i \leftarrow i + 1$ till $k \leq n$ do  
7: \hspace{3em} $s_0 \leftarrow s_i + A[i]$  
8: \hspace{2em} end for  
9: \hspace{2em} $l \leftarrow$ MDS-LRPASS($s, L, U, L$) \hspace{1em} \triangleright l is the current maximum density line segment. It is stored as a pair of points.  
10: \hspace{2em} $b \leftarrow U - L + 1$  
11: \hspace{2em} for $k = U + 1, k \leftarrow k + b$ till $k \leq n$ do \hspace{1em} \triangleright One iteration for each batch of $U - L + 1$ elements and final iteration for the batch of residual elements (if any). Exit when $k > n$  
12: \hspace{3em} $l' \leftarrow$ MDS-RLPASS($s, L, U, k$) \hspace{1em} \triangleright l' is the maximum slope line segment returned by MDS-RLPASS  
13: \hspace{3em} if $\text{slope}(l) < \text{slope}(l')$ then  
14: \hspace{4em} $l \leftarrow l'$  
15: \hspace{3em} end if  
16: \hspace{3em} $l' \leftarrow$ MDS-LRPASS($s, L, U, k$) \hspace{1em} \triangleright l' is the maximum slope line segment returned by MDS-LRPASS  
17: \hspace{3em} if $\text{slope}(l) < \text{slope}(l')$ then  
18: \hspace{4em} $l \leftarrow l'$  
19: \hspace{3em} end if  
20: \hspace{3em} if $k - s < n$ then  
21: \hspace{4em} $l' \leftarrow$ MDS-RIGHTRESIDUAL($s, L, U, k - b, n$) \hspace{1em} \triangleright l' is the maximum slope line segment returned by MDS-RIGHTRESIDUAL  
22: \hspace{4em} if $\text{slope}(l) < \text{slope}(l')$ then  
23: \hspace{5em} $l \leftarrow l'$  
24: \hspace{4em} end if  
25: \hspace{3em} end if  
26: \hspace{2em} end for  
27: \hspace{2em} return $l$  
28: end procedure
2.3.4 Analysis

Each batch of $U - L + 1$ points in the left index window is considered at most twice by MDS algorithm: once for an LR pass of a batch of $U - L + 1$ right end points and once for an RL pass of a batch of $U - L$ right end points. As mentioned above the cost charged to each of these left end points are constant for each pass. Each of the right end points is accessed at most twice and that cost is charged to the respective point. Consequently, the time complexity is in $O(n)$. Thus, we have the following theorem:

**Theorem 1** Given a sequence $A$ of $n$ real numbers and two real numbers $L$ and $U$ with $1 \leq L \leq U \leq n$, MDS algorithm finds the maximum density segment of $A$ from among all the segments of $A$ of length at least $L$ and at most $U$ in $O(n)$ time and $O(U - L)$ space in an online manner.

2.3.5 Improved Algorithm

The MDS algorithm works for all $L$ and $U$ with $0 \leq L \leq U \leq n$. Huang [44] proved the following result:

**Observation 2** If $R \subset A$ is a maximum density segment of length at least $2L$, then $R$ can be obtained by merging 2 adjacent segments of length at least $L$ with the highest densities.

We prove a similar result below:

**Lemma 2** Let $A$ be a sequence of length $n$, and $L$ be a positive number such that $L \leq n$. 

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One of the maximum density segments of \( A \) of length at least \( L \) must be of length at most \( 2L - 1 \).

**Proof.** Let \( R = A[i, j] \) be a maximum density segment of length \( l \geq 2L \). Let \( \rho \) denotes the density function. Let \( \rho(R) = d \). We shall show that there is a subsegment \( R' \subset R \) of length at least \( L \) such that \( \rho(R') \geq d \).

Let us consider the subsegment \( R_1 = A[i, i + L - 1] \) of \( R \) of length \( L \). Let \( \rho(R_1) = d_1 \). If \( d_1 \geq d \), then \( R_1 \) is the required subsegment \( R' \) and we are done.

Otherwise, we consider \( R_2 = R - R_1 \). Let \( \rho(R_2) = d_2 \). Since \(|R| \geq 2L\), we have

\[
|R_2| = l - L \geq 2L - L = L
\]

and

\[
d_2 = \frac{ld - Ld_1}{l - L} = d + \frac{L(d - d_1)}{l - L} > d, \text{ since } d > d_1 \text{ and } l > L
\]

Therefore, \( R_2 \) is the required subsegment \( R' \). □

When \( U \geq 2L \), for a right end point \( p_j \) we do not need to consider the left end points \( p_i \) such that \( i \leq j - 2L \) by the above lemma. For this case, we improve our above algorithm by restricting the size of the batch of right end points \( p_j \) to \( L \). If the batch consists of \( p_j \), \( j \in [k, k + L] \), the set of left end points are \( p_i \), where \( i \in [k - L, k] \) for the LR pass and \( i \in [k - 2L, L] \) for the RL pass. The improved algorithm, called MDS-IMPROVED, is given in Algorithm 4.

We have the following theorem:

**Theorem 3** Given a sequence \( A \) of \( n \) real numbers and two real numbers \( L \) and \( U \) with
Algorithm 4 Improved algorithm for maximum density segment problem

1: procedure MDS-IMPROVED(A, L, U)
   Input: A is the input sequence. L and U are respectively lower and upper bounds.
   Output: Maximum density segment l of A.
2: if $U > 2L - 1$ then
3:     $U \leftarrow 2L - 1$ \Comment{If U is larger than 2L − 1, reset it to 2L − 1.}
4: end if
5: $l \leftarrow$ MDS(A, L, U) \Comment{Solve the problem by MDS.}
6: return $l$
7: end procedure

1 ≤ L ≤ U ≤ n, MDS-IMPROVED algorithm finds the maximum density segment of $A$
from among all the segments of $A$ of length at least L and at most U in $O(n)$ time and
$O(\min\{U - L, L\})$ space in an online manner.

2.3.6 Implementation

In the implementation of our algorithms described above, no division is needed except once
such as for reporting the final result of maximum density for the whole problem. We always
represent the tangent line $l$ by the pair of points $p_i$ and $p_j$ through which it passes. To
represent the line $l'$ passing through a point $p'_i$ and parallel to $l$, we determine the translation
that translates $p_i$ to $p'_i$, and make the same translation to $p_j$ to find the point $p'_j$. Then the
line $l'$ is represented by the pair of points $p'_i$ and $p'_j$. We do not need to maintain the slope
$\mu$. Instead we compare the slopes of a pair of line segments. To compare the slopes of 2
line segments, all we need to do is to determine if it is a left turn or a right turn. This can
be done without division.

Our algorithm and Chung and Lu’s [22] algorithm’s run time have been compared using
random number sequence data and real DNA sequence data (see Tables 2.1 and 2.2
respectively). It is found that our algorithm performs better when the difference between $U$ and $L$ is very large in comparison to the number of inputs $N$, except for one case of DNA sequence data. In other cases Chung and Lu’s algorithm performs better.

### 2.3.7 Non-uniform Length

The above algorithm for uniform length can be extended to solve the general problem where the lengths $l_i, i = 1, ..., n$, are arbitrary. For this we define the cumulative lengths $L_i, i = 0, ..., n,$ as $L_0 = 0$ and $L_i = l_1 + ... + l_i,$ for $i = 1, ..., n$. Then the density $\mu_{i,j}$ of a segment $A[i,j]$ can be written as

$$\mu_{i,j} = \frac{s_j - s_{i-1}}{L_j - L_{i-1}}.$$ 

For each element $a_i, i = 1, ..., n,$ in the sequence $A$ we get the point $(L_i, s_i), i = 1, ..., n,$
in the plane. We have the initial point \((L_0, s_0) = (0, 0)\). Then the problem to find the feasible segments with the maximum density is reduced to finding the feasible pairs of points with the maximum slope. And this can be solved by a simple modification to our above algorithm for uniform lengths. The only difference is that the abscissas of the consecutive points \((L_i, s_i)\) and \((L_{i+1}, s_{i+1})\) are \(l_{i+1}\) distance away instead of unit distance away. Its time and space complexity will be \(O(n)\) and \(O(U-L)\) respectively. Thus, we have the following theorem:

**Theorem 4** Given a sequence \(A\) of \(n\) pairs of real numbers \((a_i, l_i), i = 1, \ldots, n\), and two real numbers \(L\) and \(U\) with \(1 \leq L \leq U \leq n\), our geometric algorithm as described above finds the maximum density segment of \(A\) from among all the segments of \(A\) of length at least \(L\) and at most \(U\) in \(O(n)\) time and \(O(U-L)\) space in an online manner.

### 2.3.8 Extension to Higher Dimensions

Using the method of [10, 70] the 2-dimensional problem is reduced to \(\binom{m}{2} + m\) 1-dimensional problems for an \(m \times n\) input matrix. We solve each of them using the above algorithm. The time complexity will be \(O(m^2n)\).

**Theorem 5** Given a 2-dimensional \(m \times n\) matrix \(A\) of pairs of real numbers \((a_{ij}, l_{ij}), i = 1, \ldots, m; j = 1, \ldots, n\), and two real numbers \(L\) and \(U\) with \(1 \leq L \leq U \leq n\), there exists an algorithm to find the maximum density subarray of \(A\) from among all the subarrays of \(A\) of length at least \(L\) and at most \(U\) in \(O(m^2n)\) time and \(O(mU)\) space.
Proof. Similar to the proof of Theorem 3 of Brodal and Jorgensen [13] and is omitted. □

The above algorithm can be extended to any dimension $d$ in a straight forward way.

**Theorem 6** Given a $d$-dimensional $n_1 \times n_2 \times \ldots \times n_d$ matrix $A$ of pairs of real numbers and two real numbers $L$ and $U$ with $1 \leq L \leq U \leq n$, there exists an algorithm to find the maximum density subarray of $A$ from among all the subarrays of $A$ of length at least $L$ and at most $U$ in $O(n_1 \Pi_{i=2}^{d} n_i^2)$ time and $O(U \Pi_{i=2}^{d} n_i)$ space.

**Proof.** Similar to the proof of Theorem 4 of Brodal and Jorgensen [13] and is omitted. □

To avoid being repetitive, we note that the algorithms described in the following sections can all be extended to higher dimensions using the same reduction technique.

2.4 $k$ Maximum Density Segments

2.4.1 Algorithm for $k \in O(\lg^2(U - L))$

Let us assume that $k \in O(\lg^2(U - L))$. The above MDS algorithm is repeated $k$ times for each batch of $U - L + 1$ points to find at least $k$ maximum density segments with right end points in the batch. In each iteration the maximum density segment for the iteration is found. Keeping the left end point of the maximum density segment found in the current iteration fixed, all the feasible segments with right end point within the current batch of $U - L + 1$ points are selected and the left end point is deleted at the end of the iteration. The algorithm for a batch of $U - L + 1$ points is described in Algorithm 5. We call this algorithm as KMDS-SMALLK. $X$ and $Y$ are the sets of respectively $2U - 2L + 1$ and
$U - L + 1$ number of left and right end points of MDS algorithm. $D$ is the set of $k$ number of candidate maximum density segments found so far.

**Algorithm 5** Algorithm for k-maximum density segment problem when $k \in O(\lg^2(U - L))$

1: **procedure** KMDS-SMALLK($X, Y, L, U, D, k$)
   
   **Input:** $X$ and $Y$ are the sets of left and right end points respectively. $L$ and $U$ are lower and upper bounds respectively. $k$ is the required number of maximum density segments.
   
   **Output:** The set $D$ of $k$ number of candidate maximum density segments and corresponding densities found so far.

2: **for** $i = 1$ to $k$ **do**

3: Find the maximum density segment from among the feasible segments with left end points in $X$ and right end points in $Y$ using the MDS algorithm (Algorithm 3).

4: Let $x$ be the left end point of this segment. For all feasible segments with left end point $x$, i.e., $(x, y'), y' \in Y$, insert $(x, y', d(x, y'))$ in $D$.

5: From $D$ select the $k$ maximum density segments using a linear time selection algorithm [12] and update $D$ with these $k$ elements.

6: Delete $x$ from $X$.

7: **end for**

8: **return** $D$

9: **end procedure**

Clearly, each iteration costs $O(U - L)$ time. $k$ iterations in a pass cost $O(k(U - L))$ time. The total cost per right end point is in $O(k)$. Thus, we have the following theorem:

**Theorem 7** Given a sequence $A$ of $n$ real numbers, two integers $L$ and $U$ with $1 \leq L \leq U \leq n$, and an integer $k \in O(\lg^2(U - L))$, MDS-SMALLK algorithm finds the $k$ maximum density segments of $A$ from among all the segments of $A$ of length at least $L$ and at most $U$ in $O(kn)$ time and $O(U - L)$ space in an online manner.

**2.4.2 Algorithm for** $k \in \omega(\lg^2(U - L)) \cap o(n(U - L)/\lg^2(U - L))$

For $k \in \omega(\lg^2(U - L)) \cap o(n(U - L)/\lg^2(U - L))$ we present an improved algorithm. For simplicity, we assume that the sequence elements are of unit length, i.e., $l_i = 1, i = 1, ..., n$. 

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As before, for each batch of \(U - L + 1\) points we make LR and RL passes to consider all the feasible segments with right points in the batch. But the segments are processed in a different way. Let the batch of right end points be in the index window \([b, b + U - L]\).

In the LR pass the left end points of all the feasible segments are in the index window \([b - U + 1, b - L + 1]\), and in RL pass they are in \([b - L + 2, b + U - 2L + 1]\). Then the LR and RL passes will consider all the feasible segments with right end points in the index window \([b, b + U - L]\). The LR and RL passes are similar. We describe LR pass for the batch.

**Grouping the feasible segments**

Here we describe the grouping of feasible segments for the LR pass. A group of feasible segments is represented by the pair \(I_l \times I_r\) where \(I_l\) and \(I_r\) are the index windows for respectively the consecutive left end points and the consecutive right end points according to the \(x\)-coordinate such that all the combinations of segments with left end points in \(I_l\) and right end points in \(I_r\) are feasible. The main advantage of this grouping is that for each right end point \(p_j\) with \(j \in I_r\), all the segments \(\overline{p_i p_j}\) with \(i \in I_l\) is a feasible segment. All the segments represented by a group \(I_l \times I_r\) will be processed in one batch. Processing of a group of feasible segments will be described next. We shall call \(I_l\) the left index window and \(I_r\) the right index window.

For example, in Figure 2.15 the large square identifies a group of 64 segments which will be represented by \([b - U + 8, b - U + 15] \times [b, b + 7]\). We do not construct the segments explicitly, just identify the pair of index windows.
We scan $I_l$ in left-to-right fashion. First, for the single right end point $p_b$, we make a group of all the feasible segments with the single left end point $p_{b-U+1}$. The group of feasible segments is $[b - U + 1, b - 1] \times [b, b]$ (Figure 2.15).

Next, we make the following 2 groups of feasible segments: $[b - U + 2, b - U + 3] \times [b, b+1]$ and $[b - U + 3, b - U + 3] \times [b + 2, b + 2]$. This completes scanning 2 more left end points $p_i \in [p_{b-U+2}, p_{b-U+3}]$. After this scan all the feasible segments with consecutive 3 left end points starting from $p_{b-U+1}$ and consecutive 3 right end points starting from $p_b$ have been completely grouped.

**Figure 2.15: Grouping of feasible segments**
Next, we make the following 4 groups of feasible segments: 

\[ [b-U+4, b-U+7] \times [b, b+3], \]
\[ [b-U+5, b-U+5] \times [b+4, b+4], \]
\[ [b-U+6, b-U+7] \times [b+4, b+5] \]
\[ [b-U+7, b-U+7] \times [b+6, b+6]. \]

This completes scanning 4 more left end points \( p_i \in [p_{b-U+2}, p_{b-U+3}] \).

After this scan all the feasible segments with consecutive 7 left end points starting from \( p_{b-U+1} \) and consecutive 7 right end points starting from \( p_b \) have been completely grouped.

At the end of the \( i \)-th step we have grouped all the combinations of segments generated by \( 2^i - 1 \) consecutive right end points and the same number of consecutive left end points such that they are feasible. We note that for each of the groups of feasible segments generated by the above algorithm, the left and right index windows have the same length, and that the length of the index windows are in powers of 2. For simplicity, let us assume that \( U - L + 1 = 2^s - 1 \) for some positive integer \( s \). After \( s \) steps all the feasible segments with consecutive \( 2^s - 1 \) left end points starting from \( p_{b-U+1} \) and ending at \( p_{b-L+1} \), and the same number of consecutive right end points starting from \( p_b \) and ending at \( p_{b+U-L} \) have been completely grouped. Thus, all the feasible segments corresponding to the LR pass have been completely grouped. We note that all the \( G_i \)'s are mutually disjoint in the sense that no pair of \( G_i \)'s have any common segment.

**Lemma 8** The above algorithm constructs groups of feasible segments \( G_i, i = 1, ..., U-L+1 \) such that \( \bigcup_{i=1}^{U-L+1} G_i \) is the set of all feasible segments in the LR pass and all the \( G_i \)'s are mutually disjoint.

The two characteristics of \( G_i \)'s mentioned in Lemma 8 ensures that the segments in each
group can be processed independently of the other groups and that we need to process the $G_i$s only.

In the above grouping procedure we do not consider the segments, but their indices. It will take constant time to construct a group. For $2^s - 1$ right end points, $2^s - 1$ groups of feasible segments will be created.

From Figure 2.15 we see that the above grouping can be done by the recursive algorithm KMDS-GROUPING in Algorithm 6. The inputs $l_s$, $r_s$ and $m$ are respectively the starting indices of left and right windows, and the length of them.

**Algorithm 6** Algorithm for grouping points

```
1: procedure KMDS-GROUPING($l_s, r_s, m$)
   Input: $l_s$ and $r_s$ and $m$ are the starting indices of left and right windows respectively. $m$ is the length of them.
   Output: The set of pairs of left and right intervals for the indices of the input sequences.
   For each pair length of left and right intervals are the same and are of the form $2^i$ where $i$ is an integer.
2:    if $m > 1$ then
3:       $m' \leftarrow \frac{m+1}{2}$
4:       return KMDS-GROUPING($l_s, r_s, m' - 1$) \cup \{([$l_s + m' - 1, l_s + m - 1]$), [$r_s, r_s + m' - 1]$)$\}$ \cup KMDS-GROUPING($l_s + m', r_s + m', m' - 1$)
5:    else
6:       return \{([$l_s, l_s$], [$r_s, r_s$])$\}$
7:    end if
8: end procedure
```

Grouping in the RL pass will be similar. Thus, we have the following lemma:

**Lemma 9** All the groups $G_i$ for each of LR and RL passes can be created in $O(U - L)$ time and space.
Organizing the points

Now we describe the processing of a group of feasible segments. Let $G = I_l \times I_r$ be a group of feasible segments where $|I_l| = |I_r| = m = 2^t$ for some positive integer $t$. Then $|G| = |I_l| \times |I_r| = 2^{2t}$. Let $Q$ and $R$ be the sets of points having index windows $I_l$ and $I_r$ respectively. Then $|Q| = |R| = m = 2^t$.

First, we organize the points in $Q$. We use Overmars and van Leeuwen’s [57] algorithm, with a simple modification, to construct the $lch$ (lower convex hull) of $Q$ by composition. In our computation $Q$ will not change. We do not need insertion or deletion operation for the convex hull.

By construction of the geometric problem all the points are already sorted by $x$-coordinates, and are vertically separated (i.e., no pair of points lie on the same vertical line). In fact, all the $n$ input points are separated by unit distance in $x$-coordinate, and consequently all the points of $Q$ are separated by unit distance in $x$-coordinates.

The convex hull is constructed iteratively. In the first iteration, we construct $2^{t-1}$ $lch$s of 2 consecutive points each. In the 2nd iteration, we construct $2^{t-2}$ $lch$s of $2^2$ consecutive points each by composing pairs of adjacent constituent $lch$s of 2 consecutive points each. In the 3rd iteration, we construct $2^{t-3}$ $lch$s of $2^3$ consecutive points each by composing pairs of adjacent constituent $lch$s of $2^2$ consecutive points each. We continue this for $t$ iterations.

The information of all of these constituent $lch$s as well as the $lch$ of $Q$ is stored in a balanced binary search tree, say $C$. This tree will be called LCH Tree. Its leaf vertices represent
the points of $Q$. Direct parents of the leaves represent the next higher level of $lchs$. Direct parents of these parents represent the next higher level of $lchs$ and so on. The root represent the $lch$ of $Q$. We denote the $lch$ of $Q$ by $H^1$ and a $lch$ at $i$-th level and $j$-th position from left by $H^1_{ij}$.

First, we describe a naive algorithm for finding the $k$-maximum density segments. In Overmars and Leeuwen’s [57] algorithm each vertex $u$ of $C$ is associated with a concatenable queue [2] to store the information about a portion of the $lch$ of all the points stored in the leaves of the subtree rooted at $u$. This was necessary to update $C$ after each insertion and deletion. But we do not insert or delete any points in $Q$. Once constructed we do not need to modify $C$. With each vertex $u \in C$, we associate an array $Q_u$ instead of a concatenable queue. $Q_u$ stores the same information as the concatenable queue, viz., the left or right part of the contour of the $lch$, of the points at the leaves of the subtree rooted at $u$, that is not a part of the contour of the $lch$ associated with the father of $(u)$. Contents of a vertex $c$ of $C$ are as follows:

1. $f(c)$ - a pointer to the father of $c$ (if any).

2. $\text{lchild}(c)$ - a pointer to the left child of $c$.

3. $\text{rchild}(c)$ - a pointer to the right child of $c$.

4. $Q_c$ - an array containing the $lch$ of the set of points in the subtree of $c$.

5. $b_l$ - left end point of the bridge.
6. $b_r$ - right end point of the bridge.

The time for the construction of the arrays at any level of $C$ from its immediate lower level is bounded by $O(m)$ and there are $\log m$ levels in the tree. Thus, we have the following Lemma which is similar to Proposition 4.1 of Overmars and Leeuwen [57]:

**Lemma 10** The tree $C$ for a set of $m$ points can be constructed in $O(m \log m)$ time and $O(m)$ space.

**Proof.** Similar to the proof of Corollary 3.3 and Proposition 4.1 of Overmars and Leeuwen [57] and is omitted. □

The time needed for the construction of the convex hull is blown up by a factor of $\log m$. But it will help searching the $k$ maximum density segments efficiently.

**Searching**

Now we describe searching for $k$ maximum density segments for the group of segments in $G$.

Let us assume that the LCH Tree $C$ of all levels of $lch$s of $Q$ have already been constructed.

For a right end point $p_j \in R$, the maximum density segment is found by drawing tangent to the top most level $lch$ $H^1$ (Figure 2.16). For simplicity, we assume that there is only one point of contact always. But this assumption is not essential for the method being described in the following. Because, if there are multiple points of contact, say $s$ number of points of contact $p_i, p_{i'}, p_{i''}, \ldots$ etc., then all of them will correspond to the same density. If necessary, all of them will be selected first at no extra cost. Only then, the search needs
to find another segment of lower density by following all of \( p_i, p_i', p_i'', \ldots \). If this total cost is averaged over the \( p_i \)s, then it will be the same as that of following each of some \( s \) points with different tangents separately.

Let the single point of contact be \( p_{j1}^j \) (\( p_{j-L-12} \) in Figure 2.16). We want to find the next maximum density segment with the same right end point \( p_j \). Let the left end point of this segment be \( p_{j2}^j \) (\( p_{j-L-11} \) in Figure 2.16). We need to find it. Clearly, it lies either on the contour of, or interior to \( H^1 \). It will be:

- either, one of the 2-adjacent points \( p_{j1}' \) and \( p_{j1}'' \) of \( p_{j1} \) (one on the left and the other on the right of \( p_{j1} \)) on \( H^1 \);

- or, interior to \( H^1 \) and its \( x \)-coordinate lies between the \( x \)-coordinates of \( p_{j1}' \) and \( p_{j2}'' \), i.e., \( x_{j1}' < x_{j2} < x_{j2}'' \).

We search neighbourhood of \( p_{j1} \) by successively reducing the size of the neighbourhood until we reach 2 adjacent points with \( x \)-coordinates \( x_{j1} - 1 \) and \( x_{j1} + 1 \).

By construction, the contour of lower hull \( H^1 \) consists of a portion of the contour of each of \( H^2_1 \) and \( H^2_2 \). They are joined by an edge, called bridge [57], between the 2 nearest end points of those portions. So, \( p_{j1}^j \) must lie either on \( H^2_1 \) or on \( H^2_2 \). Let it lie on \( H^2_2 \).

By construction of \( H \), any pair of \( lch \)s at the same level are mutually disjoint, \( H^i_{j1} \cap H^i_{j2} = \phi \) for all \( j_1 \) and \( j_2 \) with \( j_1 \neq j_2 \). Since \( p_{j1}^j \) lies on the contour of \( H^2_2 \), \( p_{j2}^j \) can either be the point of contact of tangent from \( p_j \) to \( H^1_1 \), or on the contour or interior of \( H^2_2 \). To find
the point of contact with \( H_1^2 \), the contour of \( H_1^2 \) can be searched in \( O(\lg m) \) time using binary search on the array associated with the corresponding vertex \( c_1^2 \) in \( C \). The second case is the same as the initial problem except for the \( lch \) changed from \( H^1 \) to \( H_2^2 \). Thus, the problem is solved recursively. There are \( \lg m \) recursions. In each recursion the tangent point to the contour of one \( lch \) is found by using binary search on the array of points of the contour. Total time for searching \( p_j^2 \) is \( O(\lg^2 m) \).

For each point \( p_j \in R \), we find the length constrained maximum density segment with \( p_j \) as the right end point. This is done in \( O(\lg m) \) time by drawing tangent from \( p_j \) to the top level \( lch \ H^1 \) (stored at the root of \( C \)). The tangent point is found by using a binary search of the array associated with the root of \( C \). For each \( p_j \) a vertex \( v_{j_1} \) is constructed for the maximum density segment w.r.t. \( p_j \). Since the tangents to \( H^1 \) from multiple points in \( R \) may have the same point of contact, the same point in \( H^1 \) may be left end points for multiple vertices \( v_{j_1}, v_{k_1}, \ldots, etc. \), having distinct right end points \( p_j, p_k, \ldots, etc. \) respectively.
A maximum heap $T$ is constructed using $v_{j_1}$, $j \in I_r$, as its vertices and the density of a segment as the order of the heap (Figure 2.17). The heap is initially constructed as a balanced binary search tree with the exception that each vertex has a null middle children. The middle children will point to an implicit heap that will be initialized and expanded as needed.

From the heap the $k$ maximum density elements are selected by using Frederickson’s [33] heap selection algorithm. A middle child will be explicitly constructed only when Frederickson’s [33] algorithm reaches there. Each of these vertices will have a maximum of $\lg m$ number of children. A child vertex will be created for each vertex of $C$ that is visited during search. After the initial construction of $T$, we will never create a left child or a right child of any of its initial vertices.

Let $t$ be any vertex of $T$. Let $p_i$ and $p_j$ be the left and right end points corresponding to $t$. Let $p_i$ be the tangent point on the $lch \ H^q_r$. Then $t$ will contain a pointer to $c^q_r$, where $c^q_r$ represents the $lch \ H^q_r$. For each vertex $u_{j_1}$ of $T$, $j \in I_r$, all the vertices of its middle
subtrees as well as itself represent the segments for which right end points are the same point \( p_j \). Contents of a vertex \( t \) of \( T \) are as follows:

1. \( f(t) \) - a pointer to the father of \( t \) (if any).

2. \( lchild(t) \) - a pointer to the left child of \( t \).

3. \( rchild(t) \) - a pointer to the right child of \( t \).

4. \( c(t) \) - a pointer to the vertex \( c \) of \( C \), where by searching the \( lch \) at \( c \) the search has selected \( p_i \) as the left end point of a segment

5. \( p_j \) - right end point of a segment.

6. \( p_i \) - left end point of a segment with right end point \( p_j \). As mentioned before, its value is selected by searching the vertex pointed by \( c(t) \).

7. \( \rho \) - slope of \( p_i p_j \).

Let Fredrickson's [33] algorithm wants to access the children of a vertex \( v \) of \( T \). Accessing the left and right children is straightforward. To access the middle children, it creates them first. Let \( v \) corresponds to the tangent point on \( H^q_{p_j} \) w.r.t. \( p_j \). First, we search for the next maximum density segment w.r.t. \( p_j \). We search for the tangent point on \( lchs \) at lower levels than \( H^q_{p_j} \). The search is conducted by the recursive algorithm discussed above. A child vertex of \( v \) is created for the tangent point on each of the lower level \( lchs \) from \( p_j \).
Then Fredrickson’s [33] algorithm searches those vertices. The algorithm selects \( k \) maximum density segments in this way. The time at each vertex is blown up by a factor of \( \lg^2 m \).

We call the naive algorithm for expanding \( T \) as EXPAND. It is formally given in Algorithm 7. For each vertex \( c \) in \( C \) we store the entire \( lch \) of the set of points in its subtree in its associated array \( Q_c \). We do not need \( C^* \). Each vertex \( t \) of \( T \) has a pointer \( t_c \) to an associated vertex \( c \) in \( C \). EXPAND is called when Frederickson’s [33] algorithm wants to access the children of \( t \). It is not called from \( t \) if \( t_c \) points to a leaf vertex in \( C \). EXPAND calls the algorithm CONTACTPOINT in Algorithm 8 to find the tangent point on a hull from a right end point.

**Lemma 11** The naive algorithms described above selects the \( k \)-maximum density segments correctly.

**Proof.** We need to show that: (i) there is no duplication of vertices in expanded \( T \), (ii) the expanded \( T \) is a max heap and (iii) no vertex of the \( k \)-maximum density segment vertices are missed by Fredrickson’s [33] selection algorithm.

During construction of \( T \), the roots of \( T_j \) have been inserted in \( T \) in maximum heap order, and by construction, there is no duplicate root vertices \( T_j \)'s in \( T \).

Now we consider the vertices that are expanded as Fredrickson’s [33] algorithm wants to access them. These vertices are expanded from the roots of \( T_j \). Let us consider a tree rooted at \( T_j \) wrt a right end point \( p_j \). We shall show that for each point \( p_i \), only one vertex of \( T_j \) is associated with it. Let a vertex associated with \( p_i \) in \( T_j \) be \( t^j_i \). We shall show that
Algorithm 7 Algorithm for expanding the search tree

1: procedure EXPAND(t, c)
   Input: t is a vertex of search tree T. c is a vertex of LCH Tree C.
   Output: NULL
   ▷ When Frederickson’s [33] algorithm wants to access the children of t,
   this algorithm creates the children of t. t has a pointer to c. c stores the entire lch of
   the set of points in its subtree.
2: create a child vertex t' of t
3: p_j(t') ← p_j(t)
4: if lchild(c) is a leaf of C then
5:   Create a child vertex t'' of t
6:   p_j(t'') ← p_j(t)
7:   c(t') ← lchild(c)
8:   c(t'') ← rchild(c)
9:   p_i(t') ← p_i(lchild(c))
10:  p_i(t'') ← p_i(rchild(c))
11:  d(t') ← slope(p_i(t')p_j(t))
12:  d(t'') ← slope(p_i(t'')p_j(t))
13: else
14:   if x(t) > x_{b_l}(c) then
15:     c(t') ← lchild(c)
16:     CONTACTPOINT(t', lchild(c), b_l(c))
17:     EXPAND(t, rchild(c))
18:   else
19:     c(t') ← rchild(c)
20:     CONTACTPOINT(t', rchild(c), b_r(c))
21:     EXPAND(t, lchild(c))
22: end if
23: end if
24: end procedure

Algorithm 8 Algorithm for finding the tangent point on a lch.

1: procedure CONTACTPOINT(t', c', b)
   Input: t' is a vertex of search tree T. c' is a vertex of LCH Tree C. b is the end point
   of bridge.
   Output: NULL
2: binary search Q_c' to find the point of contact p'_i on the left over part of H_c' in Q_c'
   from p_j(t')
3:   if slope(p'_ip_j) < slope(b, p_j) then
4:     p'_i ← b
5:   end if
6:   p_i(t') ← p'_i
7:   d(t') ← slope(p_i(t')p_j(t'))
8: end procedure
no other vertex in the tree $T_j$ can be associated with $t^j_i$.

We note that a vertex is created in $T$ wrt only a point of contact on an $lch$. Let $t^j_i$ is created wrt the point of contact $p_h$ on the $lch H^\alpha$.

By construction, $t^j_i$ must have been created only when $p_h$ is found as the point of contact with the highest level super hull among all the super hulls of $p_h$, for which $p_h$ is the point of contact. So, $H^\alpha$ is the highest level super hull among all the super hulls of $p_h$ for which $p_h$ is the point of contact.

Again, we note that when a vertex $t^j_\beta$ wrt point of contact $p_i$ on $lch H^\beta$ is expanded, a child vertex of $t^j_\beta$ is created for each subhull [down to the single point subhull (that is associated with a leaf vertex of $C$)] of $H^\beta$ (on which $p^j_\beta$ does not lie). In other words, when a vertex $t^j_\beta$ wrt a point of contact $p^j_i$ on $lch H^\beta$ is expanded, a child vertex of $t^j_\beta$ is created for each subhull [down to the single point subhull (that is associated with a leaf vertex of $C$)] of $H^\beta$, whose brother subhull’s contour contains the point $p^\beta_i$.

Thus, $p_\alpha$ can be created only as a unique child vertex of $t^j_\beta$. Therefore, $t^j_\alpha$ is unique only if $t^j_\beta$ is unique. By induction on parent vertex and the fact that $T_j$ is a tree, it follows that $t^j_\alpha$ is unique.

Since $H^\alpha$ is a subhull of $H^\beta$, and $p_h$ and $p_i$ are points of contact of the tangents from $p_j$ to $H^\alpha$ and $H^\beta$ respectively, we have slope($p_hp_j$) $\leq$ slope($p_ip_j$). So, max heap order is preserved for creating $t^j_\alpha$ as a child of $t^j_\beta$.

To prove (iii) we assume that Fredrickson’s [33] algorithm has expanded the tree $T^j$
completely. Let \( p_h \) be an arbitrary left end point. We shall show that there exists a vertex in \( T^j \).

Let \( H^\alpha \) be the highest level superhull of \( p_h \) among all the super hulls of \( p_h \) which have \( p_h \) as the point of contact of the tangent from \( p_j \). Following the argument similar to the above we can show that a vertex wrt \( p_h \) must have been created in \( T^j \), if there exists a vertex \( t^j_\beta \) in \( T^j \) wrt to the point of contact \( p_i \) of the tangent from \( p_j \) to the highest level super hull of \( p_i \), among all the super hulls of \( p_i \) which have \( p_i \) as the point of contact of the tangent from \( p_j \) to them. The result follows by induction and the fact that each point has a super hull and that all the super hulls are composed hierarchically into the single highest level super hull \( H \).

\( \square \)

**Lemma 12** After constructing the LCH Tree \( C \) of a group \( G \) of size \( m \), the \( k \) maximum density segments can be found from \( G \) in \( O(k \lg^2 m) \) time.

From Lemmas 10 and 12 we have:

**Lemma 13** For a group \( G \) of size \( m \), the \( k \) maximum density segments can be found in \( O(m \lg m + k \lg^2 m) \) time and \( m \lg m \) space.

Let \( U - L + 1 = 2^s - 1 \). There will be \( \frac{U-L+2}{2^{i+1}} \) groups of size \( 2^i - 1 \). Total cost for the construction of \( lchs \) for the LR pass is

\[
\sum_{i=0}^{s-1} \frac{U - L + 2}{2^{i+1}} O(2^i \lg 2^i) = O((U - L) \lg^2(U - L))
\]
To find the $k$ maximum density segments for the LR pass the heap $T$ is constructed from all the groups. Then Fredrickson’s [33] algorithm is used to search the $k$ maximum density segments from it. The CH Trees of the groups are searched as described above. For each pass of each batch, the $k$ maximum density segments are updated using a linear time selection algorithm [12]. If $k > \theta(U - L)$ a single heap $T$ is constructed for all the passes and all the batch. There will be $(U - L) \log(U - L)$ vertices in the tree. Fredrickson’s [33] algorithm is used to search the $k$ maximum density segments from it as before. We have the following theorem:

**Theorem 14** For $k \in \omega(\log 2(U - L)) \cap o(n(U - L)/\log 2(U - L))$, there exists an algorithm for solving the $k$ maximum density segments problem in $O((n + k) \log 2(U - L))$ time.

We now improve the above algorithms for expanding $T$. Its asymptotic time complexity remains the same though. Each vertex $v_{ji}$ of $T$ is associated with a binary search tree $C^*_j$ which contains a copy of all the search paths in $C$ that have been traversed to date w.r.t. $p_j$. Let $v$ be any vertex of $T$. Let $p_i$ and $p_j$ be the left and right end points corresponding to $v$. Let $p_i$ be the tangent point on the $lch H^q_i$. Then $v$ will contain a pointer to $c^q_i$ and $c^*_r$, where $c^q_i$ represents the $lch H^q_i$ and $C^*_r$ is its corresponding vertex in $C^*$.

For each vertex $v_{j1}$ of $T$, $j \in I_r$, all the vertices of its middle subtrees as well as itself represent the segments for which right end points are the same point $p_j$. All these vertices are associated with the same binary search tree $C^*_j$.

For each vertex $v_{ji}$ of $T$, the keys at each vertex of associated tree $C^*$ stores the informa-
tion of the points that have already been selected as a left end point for a maximum density segment. This indicates that the points have been removed from contention w.r.t. $p_j$. But no point is deleted from the arrays associated with the vertices of $C$, because those points may be in contention w.r.t. right end points other than $p_j$. Since the points of any $lch H^i_j$ that are selected as left end points for maximum density segments must be consecutive on $H^i_j$, we store only the indices of $Q^i_j$ corresponding to the left end and the right points of this sequence of points as keys in the corresponding vertex of $C^*_j$. Let the names of these keys be $left$ and $right$.

For a point $p_j \in R$, as the search for a maximum density segment goes down a path in $C$ from the root towards a leaf and finds a maximum density segment at a vertex $u$ in $C$, a similar path from the root of $C^*$ as well as the bordering vertices are created or updated in $C^*$, if needed. The point selected from $u$ is removed from contention for $p_j$ and is recorded accordingly in the corresponding vertex in $C^*$ by updating either $left$ or $right$. When a new vertex of $C^*$ is created and no point is removed from contention from the corresponding hull, $left$ and $right$ are set to zero. Clearly, time for updating $C^*$ is $O(\lg m)$.

For searching on or interior to a $lch H^\alpha$, we always look for 3 adjacent points on it. Let the 3 points on it be $p^1_j$, $p^2_j$ and $p^3_j$ in order of increasing $x$-coordinates. The $x$-coordinate of the next maximum density segment must lie on or between the $x$-coordinates of $p^1_j$ and $p^3_j$. If we do not need to go into its interior, then we find the tangent point to it from $p_j$. The tangent point is obviously $p^2_j$. 

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Now consider the case when we need to go into its interior of $H^\alpha$. If $p^2_j$ is not one of the 2 bridge points on $H_\alpha$, then those 3 points must lie on one of the 2 constituent lchs of $H_\alpha$. We search that lch and the problem is the same as the original problem except for the change in lch.

Otherwise, let 1 point lies on the constituent lch $H^\alpha_1$ on the left and the rest 2 points lie on the constituent lch $H^\alpha_2$ on the right. For $H^\alpha_1$ we find the tangent point $p^4_j$ on the left over portion of it from $p_j$ using binary search on the array $Q^\alpha_1$ (which stores the leftover portion on the right of the lch) associated to $c^\alpha_1$ that represents $H^\alpha_1$. Then the tangent point on will be the one between $P^1_j$ and $p^4_j$ with the maximum slope. Corresponding to that tangent point we create a vertex in each of $C^*$ and $T$.

For $H^\alpha_2$, the two points are $p^2_j$ on the left and $p^3_j$ on the right. The point $p^5_j$ that is adjacent to $p^1_j$ on the left on $H^\alpha_2$ is the right most point on the remaining portion of $H^\alpha_2$. Consequently, it must be the current last point in the associated array $c^\alpha_j$. We select that point from the associated array $c^\alpha_2$. Corresponding to $p^5_j$ on $H^\alpha_2$, we create a vertex in each of $C^*$ and $T$. Now the 3 adjacent points on $H^\alpha_2$ are $p^5_j$, $p^2_j$ and $p^3_j$ in left-to-right order and the problem is similar to the original problem except for the change in lch and the 3 points.

The computation will by symmetric if one point lies on $H^\alpha_2$ and the rest 2 lies on $H^\alpha_1$.

The above algorithms can be easily modified, in a way similar to that in Section 2.3.7 for length constrained maximum density segment problem with non-uniform length, to solve the problem with non-uniform length. The algorithm is efficient when $k \in \omega(\lg^2(U - L)) \cap
$o(n(U - L)/\lg^2(U - L))$. When $k \in \Omega(n(U - L)/\lg^2(U - L)) \cap O(n(U - L))$ any brute force algorithm will be optimal as long as the selection is done in linear time in the number of feasible segments. We describe a brute force algorithm in the following subsection.

2.4.3 Algorithm for $k \in \Omega(n(U - L)/\lg^2(U - L)) \cap O(n(U - L))$

For $k \in \Omega(n(U - L)/\lg^2(U - L)) \cap O(n(U - L))$ a brute force algorithm as described in the following is optimal in time. From $O(n(U - L))$ number of all the possible feasible segments, $k$ maximum density segments are selected using linear time selection algorithm [12]. Its time complexity is clearly $O(n(U - L))$. To minimize space usage the sequence is scanned from left to right. For each element $a_j \in A$ all the feasible segments $A[i,j]$ with right end element being $a_j$ are considered. The segments are inserted into a set $D$ of candidate maximum density segments. As soon as $k$ new segments are inserted into $D$, $k$ number of maximum density segments are selected from it using linear time selection algorithm [12], and $D$ is updated with these $k$ maximum density segments. Its time complexity is clearly $O(n(U - L))$. We have the following theorem:

**Theorem 15** Given a sequence $A$ of $n$ real numbers, two integers $L$ and $U$ with $1 \leq L \leq U \leq n$, and one integer $k \in \Omega(n(U - L)/\lg^2(U - L)) \cap O(n(U - L))$, the above algorithm finds the $k$ maximum density segments of $A$ from among all the segments of $A$ of length at least $L$ and at most $U$ in $O(n(U - L))$ time and $O(k)$ space in an online manner.
2.5 Maximum Sum Segment

As before we solve the problem in batch mode with $U - L + 1$ elements in each batch. For each batch of elements we consider all the feasible segments with right end element in the batch. Analogous to Observation 1 for the maximum density segment problem, we have the following observation:

**Observation 3** For an element $a_j, U \leq j \leq n$, let $G^j$ be the set of the candidate left end elements $a_i$ of all feasible segments. If $G^j_1$ and $G^j_2$ are any 2 subsets of $G^j$ such that $G^j = G^j_1 \cup G^j_2$, then

$$\max_{a_i \in G^j} \sum_{t=i}^{j} a_t = \max \{ \max_{a_i \in G^j_1} \sum_{t=i}^{j} a_t, \max_{a_i \in G^j_2} \sum_{t=i}^{j} a_t \}.$$

For each batch of $U - L + 1$ number of right end elements we make 2 passes as before. Let the batch of elements be $a_b, ..., a_b + U - L$, where $b = U, U + (U - L + 1), U + 2(U - L + 1), ..., U + \lfloor \frac{n - U + 1}{U - L + 1} \rfloor (U - L + 1), b \geq U$.

First, we consider LR pass for the batch. For each right end element $a_j \in A[b, b + U - L]$ the feasible segments are $A[b - L + 1, j], A[b - L + 2, j], ..., A[j - L + 1, j]$. The set of sums of these segments is represented by $Q^j_{LR}$ and is defined as:

$$Q^j_{LR} = \{(i, j, s[i, j]) | i = b - L + 1, b - L + 2, ..., j - L + 1; s[i, j] = \sum_{t=i}^{j} a_t \}.$$
It can be incrementally defined as:

\[
Q^j_{\text{LR}} = \{(j - L + 1, j, s[j - L + 1, j])\} \cup \{(i, j, s + a_j)| (i, j - 1, s) \in Q^{j-1}_{\text{LR}}\}. \tag{2.1}
\]

The set \(Q^j_{\text{LR}}\) is constructed from \(Q^{j-1}_{\text{LR}}\) by adding \(a_j\) to each element of \(Q^{j-1}_{\text{LR}}\) and inserting an additional element. Since adding a constant number does not change the relative order of a set of numbers, the same element will be the maximum element for the set before and after the addition. We have

\[
\max Q^j_{\text{LR}} = \max \{\max Q^{j-1}_{\text{LR}} + a_j, s[j - L + 1, j]\}
\]

If we denote the maximum of \(Q^j_{\text{LR}}\) by \(M^j_{\text{LR}}, j = b, ..., b + U - L\), then the above relation becomes

\[
M^j_{\text{LR}} = \max \{M^{j-1}_{\text{LR}} + a_j, s[j - L + 1, j]\}
\]

To find the maximum value \(M^j_{\text{LR}}\) for \(Q^j_{\text{LR}}\) we need the information about the maximum element \(M^{j-1}_{\text{LR}}\) of \(Q^{j-1}_{\text{LR}}\). The algorithm, called MSS-LR, is given in Algorithm 9.

**Lemma 16** Given a sequence \(A\) of \(n\) real numbers and two real numbers \(L\) and \(U\) with \(1 \leq L \leq U \leq n\), MSS-LR finds the maximum sum segment of \(A\) from among all the segments of \(A\) of length at least \(L\) and at most \(U\) in \(LR\) pass for a batch of right end elements of size \(U - L + 1\) in \(O(U - L)\) time and \(O(U - L)\) space.

**Proof.** The array \(s\) of input prefix sum is computed in the preprocessing step in \(O(U - L)\)
Algorithm 9 Algorithm for LR pass for maximum sum segment problem

1: procedure MSS-LR(A, s, L, U, b)
   Input: A is the input sequence, s is the array of prefix sum of A, and L and U are respectively lower and upper bounds. b is the index of the first right end element of the current batch of right end elements.
   Output: Maximum sum M, and indices l and r of left and right elements of the maximum sum segment in LR pass for the current batch of right end elements.
2:   M ← s[b] − s[b − L] \text{▷} M is the current maximum sum.
3:   l ← b − L
4:   r ← b
5:   for \( t ← b + 1 \) to \( b + U − L \) do
6:     if \( A[t] > 0 \) then
7:       M ← M + A[t]
8:       r ← t
9:     end if
10:    if \( M < s[t] − s[t − L] \) then
11:       l ← t − L
12:      r ← t
13:      M ← s[t] − s[t − L]
14:    end if
15:  end for
16: return \((M, l, r)\)
17: end procedure

The time. Line 2 of MSS-LR initializes the maximum sum M in constant time. For each right end element \( a_t \), M is updated in lines 7-8 and/or 11-13 correctly in constant time. Thus, total time for the batch of \( U - L + 1 \) right end elements is \( O(U - L) \).

Updating in lines 7-8 corresponds to all the feasible segments with right end element \( a_t \) except the segment \( A[t - L + 1, t] \). Updating in lines 11-13 corresponds to the segment \( A[t - L + 1, t] \). Thus, the algorithm correctly finds the maximum sum segment among all the feasible segments in LR pass with a batch of \( U - L + 1 \) right end elements. \( \square \)

Now we consider RL pass for the batch. For each right end element \( a_j ∈ A[b, b + U - L] \) the feasible segments are \( A[i, j], i = b - L + 1, b - L, ..., j - U + 1 \). The set of sums of these
segments is represented by $Q^j_{LR}$ and is defined as:

$$Q^j_{RL} = \{(i, j, s[i, j]) | i = j - U + 1, j - U + 2, ..., b - U + 1; s[i, j] = \sum_{t=i}^{j} a_t\}.$$  

It can be incrementally defined as:

$$Q^j_{RL} = \{(j - U + 1, j, s[j - U + 1, j])\} \cup \{(i, j, s - a_{j+1})|(i, j + 1, s) \in Q^{j+1}_{RL}\}.$$  

(2.2)

The set $Q^j_{RL}$ is constructed from $Q^{j+1}_{RL}$ by subtracting $a_{j+1}$ from each element of $Q^{j+1}_{RL}$ and inserting an additional element. Since subtracting a constant number does not change the relative order of a set of numbers, the same element will be the maximum element for the set before and after the subtraction. We have

$$\max Q^j_{RL} = \max\{\max Q^{j+1}_{RL} - a_{j+1}, s[j - U + 1, j]\}$$

If we denote the maximum of $Q^j_{RL}$ by $M^j_{RL}$, $j = b + U - L, ..., b$, then the above relation becomes

$$M^j_{RL} = \max\{M^{j+1}_{RL} - a_{j+1}, s[j - U + 1, j]\}$$

To find the maximum value $M^j_{RL}$ for $Q^j_{RL}$ we need the information about the maximum element $M^{j+1}_{RL}$ of $Q^{j+1}_{RL}$. The algorithm, called MSS-RL, is given in Algorithm 10.

**Lemma 17** Given a sequence $A$ of $n$ real numbers and two real numbers $L$ and $U$ with $1 \leq L \leq U \leq n$, MSS-RL finds the maximum sum segment of $A$ from among all the
Algorithm 10 Algorithm for RL pass for maximum sum segment problem

1: procedure MSS-RL($A, s, L, U, b$)
   
   Input: $A$ is the input sequence, $s$ is the array of prefix sum of $A$, and $L$ and $U$ are respectively lower and upper bounds. $b$ is the index of the first right end element of the current batch of right end elements.

   Output: Maximum sum $M$, and indices $l$ and $r$ of left and right elements of the maximum sum segment in RL pass for the current batch of right end elements.

2: $M \leftarrow s[b + U - L] - s[b - L + 1]$ \Comment{$M$ is the current maximum sum.}
3: $l \leftarrow b - L + 1$
4: $r \leftarrow b + U - L$
5: for $t \leftarrow b + U - L - 1$ to $b$ do
6:   if $A[t] < 0$ then
7:      $M \leftarrow M - A[t]$
8:      $r \leftarrow t$
9:   end if
10: if $M < s[t] - s[t - U + 1]$ then
11:   $l \leftarrow t - U + 1$
12:   $r \leftarrow t$
13:   $M \leftarrow s[t] - s[t - U + 1]$ 
14: end if
15: end for
16: return $(M, l, r)$
17: end procedure

segments of $A$ of length at least $L$ and at most $U$ in RL pass for a batch of right end

elements of size $U - L + 1$ in $O(U - L)$ time and $O(U - L)$ space.

Proof. Similar to the proof of Lemma 16. \qed

By Lemmas 16 - 17 and Observation 3 we have

Theorem 18 Given a sequence $A$ of $n$ real numbers and two real numbers $L$ and $U$ with

$1 \leq L \leq U \leq n$, our algorithm as described above finds the maximum sum segment of $A$

from among all the segments of $A$ of length at least $L$ and at most $U$ in $O(n)$ time and

$O(U - L)$ space in online manner.
2.6 $k$ Maximum Sum Segments

We use a simple modification of Brodal and Jorgensen’s [13] method to solve the $k$ Maximum Sum Segments problem. As before we solve the problem in batch mode with $U - L + 1$ elements in each batch. For each batch of elements we consider all the feasible segments with right end elements in the batch. So, for each batch of $U - L + 1$ right end points we make 2 passes as before. For each pass we shall use Brodal and Jorgensen [13] algorithm to construct a partially persistent [29] max-heap ordered binary tree using vertex copying technique. The heap implicitly contains all the feasible segments with their respective sums. The heap is a modified version of the self adjusting heap (skew heap) of Sleator and Tarjan [64] such that it supports insertions in amortized constant time. Brodal and Jorgensen [13] called it $Iheap$. It will take $O(U - L)$ time and space to build it. From the heap the $k$ maximum sum elements are selected by using Frederickson’s [33] binary heap selection algorithm. It will take $O(k)$ time. For each pass of each batch of $U - L - 1$ right end elements we update the $k$ maximum sum segments by using a linear time selection algorithm [12].

First, we consider LR pass for the batch. For each right end element $a_j \in A[b, b + U - L]$ the set $Q_{LR}^j$ of sums of all the feasible segments are incrementally defined in equation (2.1). To avoid adding $a_j$ to each element of $Q_{LR}^{j-1}$ explicitly we represent the set of sums $Q_{LR}^j$ implicitly by a pair $\langle H_{LR}^j, j \rangle$, where $H_{LR}^j$ contains left end indices and $j$ is the right end index of the segments whose sums constitute $Q_{LR}^j$. Here $H_{LR}^j$ is a version of a partially
persistent Iheap representing all the segments whose sums constitute the set $Q^j_{LR}$. The right end index for the heap can be inserted to all the vertices of the heap by setting corresponding value of $j$. Then the set of sums $Q^j_{LR}$ can be computed using the prefix sums as follows:

$$Q^j_{LR} = \{(s_j - s_{t-1})|t \in H^j_{LR}\}. \text{ The pair } \langle H^j_{LR}, j \rangle \text{ is incrementally defined as follows:}$$

$$\langle H^b_{LR}, b \rangle = \{(b - L + 1), b\},$$

$$\langle H^{j+1}_{LR}, j + 1 \rangle = \langle H^j_{LR} \cup \{j - L + 2\}, j + 1 \rangle.$$ 

where $j = b, b+1, ..., b+U-L-1$.

To construct $\langle H^{j+1}_{LR}, j + 1 \rangle$ from $\langle H^j_{LR}, j \rangle$ a vertex with key value $j - L + 1$ is inserted into $H^j_{LR}$. Since the new version of the heap is constructed using partial persistence, $H^j_{LR}$ remains intact after this insertion. To evaluate the sum for a vertex in the version $H^{j+1}_{LR}$ of the Iheap the right end index $j + 1$ of all the segments in $H^{j+1}_{LR}$ are found from the access pointer to this version. Then the sum is evaluated by subtracting the corresponding prefix sums. Since the relative order of the sums in $Q^j_{LR}$ does not change in $Q^{j+1}_{LR}$ and since only one new element is inserted into $Q^{j+1}_{LR}$, the time to construct $H^{j+1}_{LR}$ from $H^j_{LR}$ is to insert the new element in $H^j_{LR}$. Figure 2.18 shows a partially persistent Iheap and its access pointers corresponding to the LR-pass of the sequence $(2, 5, -6, 3, -5, 2, 7, 3, 2, 4, -16, 6)$ with $L = 7$ and $U = 12$. The copy pointers, the inverse pointers and the version stamps of the vertices are not shown. The heap is constructed using vertex-copying technique.

From the heap the $k$ largest sums are selected by using Frederickson’s [33] binary heap
Figure 2.18: Partially persistent Iheap and its access pointers corresponding to the LR pass of the sequence \((2, 5, -6, 3, -5, 2, 7, 3, 2, 4, -16, 6)\) with \(L = 7\) and \(U = 12\). The copy pointers, the inverse pointers and the version stamps of the vertices are not shown.

selection algorithm. The algorithm visits vertices in top-down fashion. Before any vertex in a heap \(H^j_{LR}\) is visited by the algorithm it is explicitly constructed and the newly constructed vertex is visited. The right end element index \(j\) is moved downward and sum for the vertex is evaluated as the selection algorithm moves downward. For example, when Fredrickson’s [33] algorithm follows the access pointer corresponding to the 8th element \(a_8\) \((j = 8)\) (Figure 2.18), the root \(A^2\) of the 2nd version Iheap \(H^2\) is explicitly created. In this vertex the corresponding values of \(i\) and \(j\), i.e., 1 and 8 respectively, are stored as keys. The sum \(s[1, 8]\), i.e., 11, is stored as another key. If the search follows the right child of this vertex, i.e., \(A^2\) of version 2, then another vertex \(B^2\) is created as a right child of \(A^2\). In this
vertex 2 and 8 are stored as keys for the values of $i$ and $j$ respectively, and 9 is stored as a key for the value of the sum $s[2, 8]$. $B^2$ will be a leaf of $H$ since $B$ is a leaf in version 2.

A complete heap $H$ is constructed on top of all the heaps $H^j, j = b, \ldots, b + U - L$ [13], where the key values of all the top $U - L$ vertices have been set to $\infty$. Frederickson’s [33] algorithm starts from the root of $H$ and selects $U - L + k$ largest sum vertices in $H$. From them $k$ largest sum elements are selected using linear time selection algorithm [12].

Now we consider RL pass for the batch. This pass is similar to the LR pass except that the set $Q^j_{RL}$ of sums of all the feasible segments are defined in equation (2) and that $Q^j_{RL}$ is implicitly defined as follows:

$$\langle H^{b+U-L-1}_{RL}, b + U - L - 1 \rangle = \langle \{b - L\}, b + U - L - 1 \rangle,$$

$$\langle H^{j-1}_{RL}, j - 1 \rangle = \langle H^j_{RL} \cup \{j - U + 1\}, j - 1 \rangle.$$

where $j = b + U - L - 1, b + U - L - 2, \ldots, b + 1$. Here $H^j_{RL}$ contains all the left end indices and $j$ is the right end index of all the segments whose sums constitute $Q^j_{RL}$.

For each pass of every batch of points the $k$ maximum sum segments are updated using linear time selection algorithm [12] to provide the solution at the end. Thus, we have the following theorem:

**Theorem 19** Given a sequence $A$ of $n$ real numbers, two integers $L$ and $U$ with $1 \leq L \leq U \leq n$, and one integer $k \leq U - L$, there exists an algorithm to find the $k$ maximum sum segments of $A$ from among all the segments of $A$ of length at least $L$ and at most $U$ in $O(n)$
time and $O(U)$ space.

When $k \geq U - L$ we use the above algorithm for each group of $\lceil \frac{k}{U-L+1} \rceil$ number of batches of $U - L + 1$ consecutive elements of the sequence. For simplicity, let us assume that $n = U + m\lceil \frac{k}{U-L+1} \rceil(U - L + 1)$, where $m$ is an integer. We include all the feasible segments corresponding to the right end elements $a_j \in A[L, U]$ in the first group. For a group, we insert into $H$ all the feasible segments with right end points in that group. Select the $k$ maximum sum segments from the heap using Frederickson’s heap selection algorithm [33]. The set of $k$ maximum sum segments is updated using linear time selection algorithm [12]. Thus, we have the following theorem:

**Theorem 20** Given a sequence $A$ of $n$ real numbers, two real numbers $L$ and $U$ with $1 \leq L \leq U \leq n$, and one integer $k > U - L$, there exists an algorithm to find the $k$ maximum sum segments of $A$ from among all the segments of $A$ of length at least $L$ and at most $U$ in $O(n + k)$ time and $O(k)$ space.

### 2.7 Finding All the Segments with Some Content Requirement

In genomic sequence analysis at times it is necessary to find all the segments in a sequence with some user specified minimum sum or density requirements [44].
2.7.1 Finding All the Segments Satisfying a Sum Lower Bound

Let $\sigma$ be some user specified lower bound for sum. We use the algorithm in Section 4.1 to construct partially persistent $Iheap$ [13] and select largest value vertices from it using Frederickson’s heap selection algorithm [33]. The only change is that vertices are selected from the heap in iteration. In $t$-th iteration $2^t$ largest value vertices are selected and their minimum sum is found. The iteration stops when the minimum sum in an iteration is less than $\sigma$. Then all the segments with sum at least $\sigma$ are reported.

Constructing the heap takes $O(U - L)$ time. Let $h_b$ be the size of the output from the $b$-th batch and $2^{s-1} \leq h_b < 2^s$, for some $s \in I^+$. There will be $s$ number of iterations of the selection algorithm. The $b$-th iteration takes $O(2^b)$ time. Total time over all the iterations is $O(\sum_{t=1}^{s} 2^t) = O(2^{t+1}) = O(h_b)$, where $s$ is the number of iterations. Total time for $b$-th batch with $U - L + 1$ number of elements in the batch is $O(U - L + h_b)$. For $n$ inputs the time will be $O(n + h)$, where $h$ is the total number of outputs.

**Theorem 21** Given a sequence $A$ of $n$ real numbers, two integers $L$ and $U$ with $1 \leq L \leq U \leq n$, and one real number $\sigma$, All the segments of $A$ of length at least $L$ and at most $U$ and sum at least $\sigma$ can be found in $O(n + h)$ time and $O(U - L + h)$ space in an online manner, where $h$ is the number of output.
2.7.2 Finding All the Segments Satisfying a Density Lower Bound

Following Liu and Chao [51] we transfer the problem to the problem of finding segments satisfying a lower bound of 0 for sum. Let \( \delta \) be some user specified lower bound for the density. A segment \( A[i,j] \) has density of at least \( \delta \) iff \( \sum_{t=i}^{j} (a_t - \delta l_t) \geq 0 \), i.e., iff the sequence segment \( (a_i - \delta l_i, a_{i+1} - \delta l_{i+1}, ..., a_j - \delta l_j) \) has sum of at least 0. Then the modified problem is to find the length constrained segments of the sequence \( A' = ((a'_t, l_t) | a'_t = a_t - \delta l_t, (a_t, l_t) \in A) \) such that the sum is non-negative. Let us consider a batch of points \( (a'_{k}, l_k), (a'_{k+1}, l_{k+1}), ..., (a'_l, l_l) \) such that \( L \leq \sum_{t=k}^{l} l_t \leq U - L \). The new problem is solved by the algorithm of Section 2.7.1.

**Theorem 22** Given a sequence \( A \) of \( n \) real numbers, two integers \( L \) and \( U \) with \( 1 \leq L \leq U \leq n \), and one real number \( \delta \), all the segments of \( A \) of length at least \( L \) and at most \( U \) and density at least \( \delta \) can be found in \( O(n + h) \) time and \( O(U - L + h) \) space in an online manner, where \( h \) is the number of output.

2.8 Summary

In this chapter, some problems concerning the search for the interesting regions in a sequence are considered. We have presented linear time algorithms for both the problems of length-constrained maximum sum segments and length-constrained maximum density segments. The algorithms have been extended to find the \( k \) length-constrained maximum sum segments and \( k \) length-constrained maximum density segments problems. They have
also been extended to find all the segments satisfying a user specified sum or density lower bound in linear time. We indicate the extensions of our algorithms to higher dimensions. Our algorithms facilitate efficient solutions for all these problems in higher dimensions. All the algorithms can be extended in a straightforward way to solve the problems with non-uniform length.

The algorithms have applications in several areas of biomolecular sequence analysis including finding CG-rich regions, TA and CG-deficient regions, regions rich in periodic three-base pattern, post processing sequence alignment, annotating multiple sequence alignments and computing length constrained ungapped local alignment.

It would be interesting to study if there is any linear time algorithm for the $k$ length-constrained maximum density segments problem. It can also be investigated to find more efficient algorithms for the problems in higher dimensions. It remains open to improve the trivial lower bounds for these cases.
Chapter 3

Point Placement Problem:
Improved Algorithms

3.1 Introduction

3.1.1 The Problem

Let \( P = \{p_1, p_2, ..., p_n\} \) be a set of \( n \) distinct points on a line \( L \). In this chapter, we address the problem of determining a unique placement (up to translation and reflection) of the \( p_i \)'s on \( L \), by querying distances between some pairs of points \( p_i \) and \( p_j \), \( 1 \leq i, j \leq n \).

The resulting queries can be represented by a point placement graph (ppg, for short), \( G = (V, E) \), where \( V \) and \( E \) are the sets of vertices and edges respectively such that each point \( p_i \in P \) is represented as a vertex \( v_i \in V \) and each edge \( e \in E \) joins a pair of vertices \( v_i \) and \( v_j \) in \( V \) if the distance between the corresponding two points \( p_i \) and \( p_j \) on \( L \) is known. Each edge \( e \in E \) is assigned the length that is equal to the distance between its adjacent vertices. We shall use \( p_i \) to denote a point on \( L \) as well as a vertex of \( G \).

A ppg \( G \) is line rigid or just rigid if its vertices have a unique placement on a line. Thus, the original problem reduces to the construction of a rigid ppg. The density \( \rho \) of a ppg \( G \) is
defined as $\rho(G) = \frac{|E|}{|V|}$.

Let us take some simple examples to illustrate the ideas involved. Suppose we have just 3 points $\{p_1, p_2, p_3\}$ on a line whose positions we want to know. Three different ppigs, up to relabelling, are possible (omitting the trivial case when $E = \emptyset$) as shown in Figure 3.1 below.

Figure 3.1(a) corresponds to the situation when the distance between a pair of points, say $p_1$ and $p_2$, is known. For Figure 3.1(b), the distances between 2 pairs of points, say $\{p_1, p_2\}$ and $\{p_2, p_3\}$, are known. Figure 3.1(c) is the ppg when all the pairwise distances are known.

![Figure 3.1: Some point placement graphs for 3 points](image)

Clearly, for the ppg of Figure 3.1(a) a unique placement is not possible since the point $p_3$ can be anywhere relative to $p_1$ and $p_2$. The same is true of Figure 3.1(b): say, we place $p_1$ and $p_2$ first, but then the position of $p_3$ relative to $p_2$ is ambiguous. However, a unique placement is possible for the triangular ppg of Figure 3.1(c) as long as the length of one edge is the sum of or absolute difference between the lengths of the other two. Thus, if we first place $p_1$ and then place $p_2$ to $p_1$'s right, $p_3$ will be placed between $p_1$ and $p_2$ if the sum of its distances from $p_1$ and $p_2$ is $|p_1p_2|$, and to the left of $p_1$ or to the right of $p_2$ if the absolute difference between the distances is equal to $|p_1p_2|$. In other words, the ppg of Figure 3.1(c) is rigid.
The last case suggests a simple algorithm using triangle as the basic component of a ppg for the unique placement of n points. Query the distance between two points, say \( p_1 \) and \( p_2 \). The position of each of the remaining points \( p_i, i \geq 3 \) is determined by querying the distances from \( p_i \) to \( p_1 \) and \( p_2 \); \( p_i \) lies between \( p_1 \) and \( p_2 \) if the sum of the distances is equal to \( |p_1p_2| \), and to the left of \( p_1 \) or to the right of \( p_2 \) if the difference between the distances is equal to \( |p_1p_2| \). The corresponding ppg shown in Figure 3.2 is then rigid. The number of queries made is \( 2n - 3 \), which is of the form \( \alpha n + \beta \). Here \( \alpha(= 2) \) represents the asymptotic density of the ppg which is the limit of the number of edges per vertex as the number of vertices \( n \) goes to \( \infty \). However, the density of the triangle ppg is 1.

![Query graph using triangles](image)

Figure 3.2: Query graph using triangles

The principal goal is to make \( \alpha \) as small as possible. With this in mind, let us look at the more complicated and illuminating case when we have 4 points. Many different ppg’s are possible. We can dispense with those that have fewer than 4 edges since in these cases a unique placement is clearly not possible. Figure 3.3 below shows the possible ppg’s, up to relabelling, with 4 and 5 edges.

The ppg of Figure 3.3(a) is not rigid, for while the triangle formed by \( p_1, p_2 \) and \( p_3 \) is
Figure 3.3: Some point placement graphs for 4 points

rigid, the point \( p_4 \) can be placed to the left or right of \( p_3 \), making the placement non-unique.

The \( ppg \) of Figure 3.3(b) is interesting in that if the two pairs of opposite edges are equal then there is no unique placement. This is easily seen by drawing the \( ppg \) as a rectangle as shown in Figure 3.4(a) below and then giving a horizontal right shear to the top edge \( p_2p_3 \) so that \( p_2 \) and \( p_3 \) lie on the same line as \( p_1 \) and \( p_4 \), giving us the linear configuration shown in Figure 3.4(b). A horizontal left shear produces the linear configuration shown in Figure 3.4(c), which cannot be obtained from the linear configuration of Figure 3.4(b) by translation and/or reflection.

Figure 3.4: Point placement graph in the shape of a quadrilateral (a) with opposite edges being equal have 2 placements as shown in (b) and (c)

The \( ppg \) of Figure 3.3(c) is rigid since we have 2 triangles attached to the edge \( p_1p_3 \), each of which is rigid. Thus, it is the \( ppg \) of Figure 3.3(b) for which we have a structural rigidity condition, namely, \( |p_1p_2| \neq |p_3p_4| \) or \( |p_2p_3| \neq |p_1p_4| \) \[20\]. This means that if we want to
extend our previous algorithm for the unique placement of \( n \) points, by first placing two vertices, say, \( p_1 \) and \( p_2 \) on \( L \) and then building rigid quadrilaterals by querying distances from \( p_1 \) and \( p_2 \) with respect to two new vertices at a time, we must make sure that we meet the structural condition on the rigidity of each new quadrilateral.

If we try to construct a quadrilateral \( ppg \) (Figure 3.3(b)) in one round, its edges may not satisfy its rigidity condition, because all its 4 edges are connected and we cannot choose a suitable length for an edge to satisfy the rigidity condition on it. Suppose we want to satisfy the rigidity condition \(|p_2p_3| \neq |p_1p_4|\). This is possible if the edges \( p_2p_3 \) and \( p_1p_4 \) are not paired. Then we will have options for choosing some suitable length for either \( p_2p_3 \) or \( p_1p_4 \). Suppose we want to provide this option for \( p_2p_3 \). Then for each \( p_1p_4 \) we must have candidate edges for \( p_2p_3 \). If we do not query \( p_3p_4 \) in the first round, then \( p_2p_3 \) and \( p_1p_4 \) are not paired, and it is possible to provide candidate edges for \( p_2p_3 \) after the first round of query. It is to be noted that the quadrilateral will be rigid irrespective of the lengths of the edges \( p_1p_2 \) or \( p_3p_4 \). So, there is no problem in querying for the length of the edge \( p_1p_4 \) in the second round.

So, we need to query the lengths of the edges in 2 rounds to build a rigid \( ppg \) using quadrilateral as the basic component. After the first round of query, we can select \( p_2p_3 \) with a suitable length and can check that the rigidity condition on it is satisfied.

Here is a 2-round algorithm due to Damaschke [26]. Let the number of points be \( n = 2b + 4 \), where \( b \) is a positive integer. In the first round, we make \( 2b + 3 \) distance
queries represented by the edges in the graph in Figure 3.5. There are \( b \) leaf children \( p_i \) \((i = 3, \ldots, b+2)\) rooted at \( p_1 \) and \( b+2 \) leaf children \( p_j \) \((j = b+3, \ldots, 2b+4)\) rooted at \( p_2 \).

![Figure 3.5: Query graph for first round in a 2-round algorithm using quadrilaterals](image)

In the second round, for each edge \( p_1p_i \) \((i = 3, \ldots, b+2)\) we find an edge \( p_2p_j \) rooted at \( p_2 \) satisfying the rigidity condition \(|p_1p_i| \neq |p_2p_j|\). We can ensure this condition by having 2 extra edges at \( p_2 \), in view of the following basic observation [27]:

**Observation 4** *At most two equal length edges can be incident to any vertex in a ppg.*

By Observation 4, there are at most 2 edges \( p_2p_j \) such that \(|p_1p_i| = |p_2p_j|\). So, for each edge \( p_1p_i \), an edge \( p_2p_j \) will always be found such that \(|p_1p_i| \neq |p_2p_j|\). Then in the second round of query, for each \( i \) \((i = 3, \ldots, b+2)\), we query the distance \( p_ip_j \) to form a quadrilateral \( p_1p_ip_jp_2 \). It will be rigid since \(|p_1p_i| \neq |p_2p_j|\). It will fix the positions of \( p_i \) and \( p_j \) relative to \( p_1 \) and \( p_2 \). For each of the 2 unused leaves \( p_j \), the distance \( p_1p_j \) is queried in the second round to form the triangle \( p_1p_jp_2 \). It will fix the position of \( p_j \) relative to \( p_1 \) and \( p_2 \).

The number of queries made over the two rounds to construct this rigid ppg is \( 3b + 5 \), i.e., \( 3n/2 - 1 \). There are two noteworthy points: (a) the value of asymptotic density \( \alpha \) is reduced from 2 for the first algorithm to \( 3/2 \) for the second, and (b) there is a price for this - we have to query the edges in two rounds. It is interesting to note that the density of the
quadrilateral $ppg$ is 1, but it is not intrinsically rigid.

What if the number of points is greater than 6 but odd? Let $n = 2b + 5$, where $b$ is a positive integer. We make an unique placement of the first $2b + 4$ vertices using the above algorithm, and query the distances of the last odd vertex from any two vertices. Distance queries for this vertex can be made in either of the 2 rounds.

3.1.2 Motivation

The motivation for studying this problem stems from the fact that it arises in diverse areas of research such as computational biology, learning theory, computational geometry, etc.

In learning theory [26] this problem is one of learning a set of points on a line non-adaptively, when learning has to proceed based on a fixed set of given distances, or adaptively when learning proceeds in rounds, with the edges queried in one round depending on those queried in the previous rounds.

The version of this problem studied in computational geometry is known as the turnpike problem. The description is as follows. On an expressway stretching from town $A$ to town $B$ there are several gas exits; the distances between all pairs of exits are known. The problem is to determine the geometric locations of these exits. This problem was first studied by Skiena et al. [63] who proposed a practical heuristic for the reconstruction. A polynomial time algorithm was given by Daurat et al. [28].

In computational biology, it appears in the guise of the restriction site mapping problem. Biologists discovered that certain restriction enzymes cleave a DNA sequence at specific
sites known as restriction sites. For example, it was discovered by Smith and Wilcox [65] that the restriction enzyme Hind II cleaves DNA sequences at the restriction sites GTGCAC or GTTAAC. In lab experiments, by means of fluorescent in situ hybridization (FISH experiments), biologists are able to measure the lengths of such cleaved DNA strings. Given the distances (measured by the number of intervening nucleotides) between all pairs of restriction sites, the task is to determine the exact locations of the restriction sites.

The turnpike problem and the restriction mapping problem are identical, except for the unit of distance involved; in both of these we seek to fit a set of points to a given set of interpoint distances. As is well-known, the solution may not be unique and the running time is polynomial in the number of points. While the point placement problem, prima facie, bears a resemblance to these two problems it is different in its formulation - we are allowed to make pairwise distance queries among a distinct set of labeled points. It turns out that it is possible to determine a unique placement of the points up to translation and reflection in time that is linear in the number of points.

The 3-dimensional version of this problem has application in the area of molecular conformation. Often, the experimental data about the conformational state of molecules are available in terms of interatomic distances. Majority of energy functions can also be expressed in terms of interatomic distances. The problem is to determine the conformational space of a molecule from these distance data and chirality constraints.
3.1.3 Prior Work

Early research on this problem was reported in [58, 53]. In this chapter, our first principal reference is [26], where it was shown that both the jewel and $K_{2,3}$ are rigid, and also how to build large rigid ppg of density $8/5$ out of the jewel. A jewel is a graph with the set of vertices \{X, Y, Z, A, B, P, Q\} and the set of edges \{YZ,XA,AY,YB,BX,XP,PZ,ZQ,QX\} (see Figure 3.6). A $K_{2,3}$ is a graph with the set of vertices \{X,Y,Z,A,B\} and the set of edges \{XA,YA,ZA,XB,YB,ZB\}. In a subsequent paper, Damaschke [27] proposed a randomized 2-round strategy that needs $(1 + o(1))n$ distance queries with high probability and also showed that this is not possible with 2-round deterministic strategies.

Our second principal reference is the work of Chin et al. [20] who improved many of the results of [26]. Their principal contributions are the 2-round and 3-round construction of rigid graphs of density $4/3$ and $5/4$ using respectively 5-cycle and 6-cycle as the basic component, and a lower bound on the number of queries necessary in any 2-round algorithm. They also introduced the idea of a layer graph which is useful in finding the conditions for rigidity of a ppg. A layer graph is defined as follows:

**Definition 3.1.1** We first choose two orthogonal directions x and y (actually, any 2 non-
parallel directions will do). A graph $G$ admits a layer graph drawing if the following 4 properties are satisfied:

$P_1$ Each edge $e$ of $G$ is parallel to one of the two orthogonal directions $x$ and $y$.

$P_2$ The length of an edge $e$ is the distance between the corresponding points on $L$.

$P_3$ Not all edges are along the same direction (thus a layer graph has a two-dimensional extent).

$P_4$ When the layer graph is folded onto a line, by a rotation either to the left or to the right about an edge of the layer graph lying on this line, no two vertices coincide.

Chin et al. [20] proved the following result about a layer graph:

**Theorem 23** A ppg is rigid iff it cannot be drawn as a layer graph.

### 3.1.4 Contribution

In this chapter, we show how to construct in 2 rounds a rigid ppg on $n$ points, using an instance of a 5:5 jewel as the basic component. The number of edges queried during this construction is $10n/7 + O(1)$. We extend this result to 6:6 jewels, constructing in 2 rounds a rigid ppg with $4n/3 + O(1)$ queries. This improves the result in [20] for constructing a ppg with $4n/3 + O(\sqrt{n})$ queries in 2 rounds using 5-cycles. We also improve substantially the lower bound on any 2-round algorithm from $17n/16$ in [20] to $12n/11$. In Chapter 4 we improve the lower bound and upper bound to $9n/8$ and $9n/7 + O(1)$ respectively. The results are summarized in Table 3.1.
Table 3.1: Comparison of results for lower bound and upper bound

<table>
<thead>
<tr>
<th></th>
<th>No of rounds</th>
<th>Upper bound</th>
<th>Lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Damaschke [26, 27]</td>
<td>1</td>
<td>$8n/5 + O(1)$</td>
<td>$4n/3$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$3n/2 + O(1)$</td>
<td>$30n/29$</td>
</tr>
<tr>
<td>Chin et al. [20]</td>
<td>2</td>
<td>$4n/3 + O(\sqrt{n})$</td>
<td>$17n/16$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>$5n/4 + O(\sqrt{n})$</td>
<td></td>
</tr>
<tr>
<td>This dissertation</td>
<td>2</td>
<td>$10n/7 + O(1)$ (5:5 jewel) [5]</td>
<td>$9n/8$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$4n/3 + O(1)$ (6:6 jewel) [3]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$9n/7 + O(1)$ (3-path)</td>
<td></td>
</tr>
</tbody>
</table>

3.2 Generalized Jewels

The examples described in Section 3.1.1 demonstrates well how small ppg’s that are inherently rigid or rigid under some structural conditions can be glued together into a large rigid ppg. In this section we introduce a type of ppg, called an $m : n$ jewel, several copies of which we plan to glue together to form a large rigid ppg.

A generic $m : n$ jewel consists of an $m$-vertex cycle $C_1$ and another $n$-vertex cycle $C_2$ that are joined by a strut going between two vertices $Y$ (of $C_1$) and $Z$ (of $C_2$), and hinged at a third common vertex, $X$ (Figure 3.7). An instance of an $m : n$ jewel is obtained by the placement of the vertices that describe the cycles $C_1$ and $C_2$.

![Figure 3.7: A generic $m : n$ jewel](image)

To attain our goal we need to determine the structural conditions (if any) that make a chosen instance of the $m : n$ jewel rigid. In the next section, we obtain structural conditions
under which chosen instances of the \( m : n \) jewels remain rigid for small values of \( m \) and \( n \) by drawing them as layer graphs and applying Theorem 23. Before we do that, we establish a few useful facts about the generic \( m : n \) jewel. The first is as follows.

**Theorem 24** If cycles \( C_1 \) and \( C_2 \), consisting of \( m \) and \( n \) vertices respectively, are rigid then so is any \( m : n \) jewel made up of these two cycles.

**Proof.** Since \( C_1 \) and \( C_2 \) are rigid their respective vertices have unique linear layouts. Then in order for an \( m : n \) jewel to have a layer graph drawing these placements would have to be in the orthogonal directions \( x \) and \( y \). Suppose the vertex \( Y \) is placed on the \( x \)-axis and the vertex \( Z \) on the \( y \)-axis, then the edge \( YZ \) of the \( m : n \)-jewel is not parallel to either the \( x \) or the \( y \) direction. Hence the \( m : n \) jewel cannot be drawn as a layer graph and must, therefore, be rigid by Theorem 23. \( \square \)

As a direct consequence of the theorem we have the following corollary:

**Corollary 25** If an \( m : n \) jewel has a layer graph representation then in this representation at least one of \( C_1 \) or \( C_2 \) is a layer graph.

In order to obtain the structural conditions that make a cycle rigid, we draw all possible layer graph representations of it and find the structural conditions for the rigidity of each of these. The logical \( AND \) of all these conditions is our answer. The second corollary is this:
Corollary 26 The union of the set of all the structural conditions that make $C_1$ rigid with those that make $C_2$ rigid, constitute a sufficient set of structural conditions that make an $m : n$ jewel rigid.

We shall take this route in the next two sections to obtain the structural conditions for the rigidity of chosen instances of the $m : n$ jewels for some small values of $m$ and $n$.

It should be noted that a cycle with a fixed set of $n_x$ x-parallel edges and thus a fixed set of $n_y$ y-parallel edges can be drawn as a layer graph in different ways. They are all considered to be equivalent. For example, the three layer graph drawings of a 5-cycle in Figure 3.8 are considered to be equivalent. From now on, for an equivalent class of layer graphs we shall draw just one of them - not all. We shall not use the term class either. By a particular layer graph, we shall mean the class of layer graphs that are equivalent to it. Thus, two layer graph drawings of an $n$-vertex cycle are distinct from each other if at least one edge has different orientations in the two graphs.

![Figure 3.8: Equivalent layer graphs for a class of layer graphs of a 5-cycle](image)

As we shall resort to exhaustive enumerations of all the layer graph representations of a cycle, the following theorem [5] is useful for checking that we have the correct number.

Theorem 27 There are $2^{n-1} - \frac{n^2 - n + 2}{2}$ different layer graph representations of an $n$-vertex
cycle.

3.2.1 4:4 and 5:4 Jewels

The following observation is fundamental. A formal proof can be found in [26].

Observation 5 A 4-cycle XAYB is rigid if |XA| ≠ |YB| or |XB| ≠ |YA|.

The jewel in Figure 3.6 has two 4-cycles joined together. It is an instance of a generic 4:4 jewel.

To begin with, we prove the following theorem.

Theorem 28 The 4:4 jewel of Figure 3.6 is rigid.

Proof. We claim that cycles XAYB and XQZP are both rigid. Let the edge YZ is x-parallel. Three cases arise:

Case 1 The 4-cycle XAYB is rigid, while the 4-cycle XQZP has a layer graph representation.

Since XQZP is a 4-cycle evidently its layer graph can be a rectangle only. Let the vertices of the rigid 4-cycle XAYB lie on the x-parallel line through Y. Then for the rectangular layer graph XQZP the diagonally opposite vertices X and Z lie on an x-parallel line collinear with YZ. Consequently, XQZP cannot have a 2-dimensional extent. This violates property P1 of a layer graph. Thus the 4-cycle XQZP cannot be drawn as a layer graph.
To complete the argument assume that the vertices of the rigid 4-cycle $XAYB$ lie on the $y$-parallel line through $Y$. Then the only way we can draw the 4-cycle $XQZP$ as a layer graph such that $X$ and $Z$ are non-adjacent is to place one of the two vertices $P$ or $Q$ at $Y$. As this violates property P4 that a layer graph should have, the 4-cycle $XQZP$ can not be drawn as a layer graph.

Thus the 4-cycle $XQZP$ does not have a layer graph representation when the 4-cycle $XAYB$ is rigid.

**Case 2** An identical argument as in Case 1 proves that a layer graph representation of the 4-cycle $XAYB$ is impossible when the 4-cycle $XQZP$ is rigid.

**Case 3** Finally, assume both the 4-cycles have layer graph representations.

Evidently, each of these is a rectangle only. As $X$ and $Y$ are non-adjacent vertices, they are diagonally opposite vertices of the rectangle $XAYB$. Likewise, $X$ and $Z$ are diagonally opposite vertices of the rectangle $XQZP$.

The arguments adduced for Case 1 can once again be used to show that it is not possible to draw the 4-cycle $XAYB$ as a layer graph if $X$ lies on the $x$- or $y$-parallel lines passing through $Y$ or on any of the $x$- or $y$-parallel lines passing through $Z$.

Assume otherwise. Now, $X$ and $Y$ are diagonally opposite vertices of the rectangle $XAYB$ while $X$ and $Z$ are diagonally opposite vertices of the rectangle $XQZP$.

Therefore a vertex of the 4-cycle $XAYB$ must coincide with a vertex of the 4-cycle
$XQZP$ on an $x$-parallel line collinear with $YZ$. As this violates property P4 that a layer graph should have, the cycles $XQZP$ and $XQZP$ cannot have simultaneous layer graph representations.

Thus, none of the two 4-cycles of the jewel has a layer graph representation. By Theorem 23 both the cycles are rigid, and by Theorem 24 the 4:4 jewel is rigid. 

Unlike the 4:4 jewel of Figure 3.6, the 5:4 jewel of Figure 3.9 is not intrinsically rigid. As a prelude to our discussion in the following sections, it is interesting to find the structural conditions (or simply conditions) that make it rigid.

![Figure 3.9: An instance of a 5 : 4 jewel](image)

We first determine the conditions that make the cycle $XABYC$ rigid. By Theorem 27, there are five distinct layer graph representations of the 5-cycle $XABYC$, shown in Figure 3.10. As remarked earlier, each is a canonical representative of an entire class of layer graph representations; referring to Figure 3.10(a) for example, other representations can be obtained by varying the position of $A$ on the supporting line of $XB$.

It is impossible to extend the layer graph representations of the 5-cycle $XABYC$ shown in Figures 3.10(a) and 3.10(b) into a layer graph representation of the entire 5:4 jewel of Figure 3.9. without one of the vertices $P$ or $Q$ coinciding with one of the vertices $B$ or $C$. 

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However, it is possible to extend each of the layer graph representations of Figures 3.10(c) - 3.10(e) into a layer graph representation of our 5:4 jewel. The layer graph representations of Figures 3.10(c) - 3.10(e) can be prevented by insisting on the condition \( |XC| \neq |AB|, |XA| \neq |YB|, |YC| \neq |AB| \) respectively. By Theorem 23, these collectively constitute a set of sufficient conditions for the line rigidity of the 5-cycle \( XABYC \).

For the 4-cycle \( XPZQ \) the rigidity condition is \( |XP| \neq |ZQ| \) (Observation 5). Thus, by Corollary 26, the set of sufficient conditions for the rigidity of the 5:4 jewel of Figure 3.9 is

\[
\{ |XC| \neq |AB|, |XA| \neq |YB|, |YC| \neq |AB|, |XP| \neq |ZQ| \}.
\]

We note in passing that for each of the configurations in Figures 3.10(c) - 3.10(e), we have an alternate condition that prevents its drawing as shown. Thus for example \( |XA| \neq |CY| \pm |YB| \) also prevents the layer graph drawing of Figure 3.10(c). With the help of the label mapping \((X,C,Y,B,A)\) to \((p_3,p_4,p_5,p_1,p_2)\) we can see that this condition encapsulates the 3 different conditions corresponding to the 3 equivalent layer graph representations shown in Figure 3.8. In such situations, whenever possible, we choose the simpler condition, unless the other one is more useful for the construction of a \( ppg \).

**Theorem 29** The 5:4 jewel of Figure 3.9 is rigid if its edges satisfy the set of conditions

\[
\text{Theorem 29} \quad \text{The 5:4 jewel of Figure 3.9 is rigid if its edges satisfy the set of conditions}
\]
\{ |XC| \neq |AB|, |XA| \neq |YB|, |YC| \neq |AB|, |XP| \neq |ZQ| \}.

3.3 Algorithm Based on a 5:5 Jewel

We next consider the more complex case of the 5:5 jewel of Figure 3.11. From now on, we will refer to it simply as the 5:5 jewel. By Theorem 27 there are exactly 5 distinct layer graph representations of a 5-cycle (see Figure 3.10). Thus, the set of 5 distinct conditions in Lemma 30 are sufficient to ensure the rigidity of the 5-cycle \( XABYC \).

Lemma 30 A 5-cycle \( XABYC \) is rigid if its edges satisfy the following conditions:

\[
|XC| \neq |YB|, |XA| \neq |YC|, |XC| \neq |AB|, |XA| \neq |YB|, |YC| \neq |AB|
\] (3.1)

Proof. A formal proof appears in [20]. \qed

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{5_5_jewel.png}
\caption{An instance of the 5:5 jewel}
\end{figure}
\]

For the 5-cycle \( XPQZR \) these conditions are:

\[
|XR| \neq |ZQ|, |XP| \neq |ZR|, |XR| \neq |PQ|, |ZR| \neq |PQ|, |XP| \neq |ZQ|.
\]

By Corollary 4, these 10 conditions collectively constitute a sufficient set of conditions for the line-rigidity of the 5:5 jewel.
Our goal is to glue several copies of the 5:5 jewel of Figure 3.11 into a large ppg, as we did for the case of quadrilaterals in Section 3.1.1. All of these will have a common strut $YZ$. As each jewel will account for 7 new vertices in lieu of 10 new edge queries, we expect $\alpha$ to be $10/7$. This indeed turns out to be the case. The challenge here is to design the ppg in such a way that the rigidity conditions are satisfied for every jewel.

The rigidity conditions for a cycle, in their current form, involve all its edges. This requires to query the lengths of all of its edges in the first round to check if the rigidity conditions are satisfied. This does not provide us with the flexibility of choice that we need to meet the rigidity conditions in a 2-round algorithm. The edge lengths may not satisfy the conditions. If any condition is not satisfied then the cycle and thus the whole jewel may not be rigid because our set of conditions is sufficient (Theorem 24). Now, the 2-dimensional stretch of a layer graph gives a pointer - we can avoid involving one edge of a cycle from all the rigidity conditions for it. We shall avoid $AB$ and $PQ$ from the rigidity conditions for the two 5-cycles. Then the cycles will be rigid irrespective of the lengths of those edges. And the rigidity conditions for the cycles will involve all of their other edges. Again, in each rigidity condition we need to have at least one edge in it for which we can choose edge length, from among the options for edge lengths for that particular edge, that satisfies the condition. We shall provide options for choosing each of the edges $YB$ and $ZQ$.

There will be rigidity conditions of each cycle that will not involve these edges, i.e., $YB$ or $ZQ$. We cannot meet those rigidity conditions in a 2-round algorithm. We need to avoid
other edge(s) from the rigidity conditions of a cycle and/or provide options for choosing edge(s) for a cycle. We shall avoid $XC$ and $XR$ from the rigidity conditions for the two cycles. Then we shall have options for choosing edges $YC$ and $ZR$ to satisfy the rigidity conditions.

Thus, we shall avoid $AB$, $PQ$, $XC$ and $XR$ from the rigidity conditions. For each 5-cycle we shall replace each of its rigidity conditions that involve any of these edges. We shall replace that condition by a set of condition(s) that prevent the cycle from being drawn as the layer graph representation that corresponds to that condition.

Looking ahead slightly, Figure 3.18 describes the structure of our proposed ppg. It has a pool of edges hanging from each end of the strut $YZ$ and a set of 2-pronged subgraphs. The lengths of the edges of this ppg are queried in the first round. In the second round, we join each 2-pronged subgraph to a pair of edges incident to $Y$ and another pair of edges incident to $Z$ to form a 5:5 jewel, making sure that all the rigidity conditions satisfied.

Over the rest of this section we show how to replace the rigidity conditions of the 5-cycle $XABYC$ that involve $XC$ and/or $AB$ with rigidity conditions that exclude these edges. To replace a condition we shall find another set of conditions that prevents the drawing of the 5-cycle $XABYC$ as a layer graph in the configuration corresponding to that condition. For example, to replace the condition $|XC| \neq |YB|$, corresponding to the layer graph of Figure 3.10)(a), we shall find a set of conditions that prevent the drawing of the layer graph of the 5-cycle in the configuration of Figure 3.10)(a).
Our first attempt will be to use other edges in the layer graph drawing corresponding to a given rigidity condition involving $XC$ and/or $AB$. If this does not suit our purpose, the basic strategy will be to embed the layer graph drawing corresponding to such a rigidity condition into all possible layer graph drawings of the 5:5 jewel and derive a rigidity condition from each such embedding.

The rigidity conditions that we will consider for replacement are:

$$|XC| \neq |YB|, |XC| \neq |AB|, |YC| \neq |AB|$$

### 3.3.1 Replacing $|XC| \neq |AB|$  

This condition has been derived from the layer graph drawing shown in Figure 3.10(c). This figure shows that an alternate rigidity condition is

$$|XA| \neq ||YB| \pm |YC||,$$  

(3.2)

which we use to replace $|XC| \neq |AB|$.

### 3.3.2 Replacing $|XC| \neq |YB|$  

This rigidity condition corresponds to the layer graph drawing of Figure 3.10(a). $||XA| \pm |AB|| \neq |YC|$ is an alternate rigidity condition corresponding to the layer graph drawing in Figure 3.10a) of the 5-cycle $XABYC$. However, it involves the edge $AB$ that we wish to avoid. We shall find an alternate set of rigidity conditions. For this, we find all possible layer graph drawings of the 5:5 jewel in which the layer graph of Figure 3.10(a) is embedded.

Then we find conditions which prohibit those layer graph drawings. Consequently, those
conditions will replace $|XC| \neq |YB|$, because there will be no layer graph for the 5:5 jewel in which the layer graph of Figure 3.10(a) is embedded. We shall follow this method whenever we cannot use any rigidity condition for a 5-cycle $XABYC$ or $XPQZR$ that involves some edges of the corresponding cycle only. We have the following lemma for the replacement of the current condition:

**Lemma 31** The 5-cycle $XABYC$ of the 5:5 jewel of Figure 3.11 cannot be drawn as the layer graph of Figure 3.10(a) if the edges of the jewel satisfy the following conditions:

$$\{|ZR| \neq |YB|, |ZR| \neq |YC|\} \quad (3.3)$$

**Proof.** We argue below that there are exactly 4 possible layer graph drawings of the 5:5 jewel in which the layer graph of Figure 3.10(a) lies embedded. Two cases arise depending on the orientations of $YZ$:

- $YZ$ is horizontal (Figure 3.12)

  $Z$ is necessarily distinct from $C$, while $YZ$ and $YB$ are mutually perpendicular. Consider the edges on the path $XRZ$ of the 5:5 jewel. If $XR$ were vertical, then $ZR$ would have to be horizontal, forcing $R$ to coincide with $C$. Thus, $XR$ must be horizontal and consequently, $RZ$ must be vertical.

  Next, we consider the edges on the path $XPQZ$. $XP$ can be horizontal or vertical. If $XP$ is horizontal then $PQ$ must be vertical, else $Q$ and $R$ will coincide. This forces
$QZ$ to be horizontal giving us the layer graph of Figure 3.12(a).

If $XP$ is vertical, then $PQ$ must be horizontal; otherwise, $Q$ will coincide with $C$.

This forces $QZ$ to be vertical, giving us the layer graph of Figure 3.12(b).

In these layer graphs, the edges $YC$ and $YZ$ are on a horizontal line $CYZ$, and are parallel to $XR$. The vertical edges $XC$ and $ZR$ connect the parallel edges. So, we must have $|XC| = |ZR|$. Thus, these layer graphs are not possible if $|ZR| \neq |YB|$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3_12.png}
\caption{Replacing the condition $|XC| \neq |YB|$ when $YB$ and $YZ$ are mutually perpendicular}
\end{figure}

- $YZ$ is vertical (Figure 3.13)

Identical arguments as adduced for the case when $YZ$ was assumed horizontal, gives us the layer graph drawings of Figure 3.13(a) and Figure 3.13(b).

For both the configurations of Figure 3.13 the edges $XC$ and $XR$ are on a vertical line $XRC$, while the edges $YB$ and $YZ$ are on a vertical line $BZY$. The edge $YC$ is horizontal and connects those two parallel lines. The edge $ZR$ is horizontal and connects the two vertical lines $XRC$ and $BZY$. So, we must have $|ZR| = |YC|$. Thus, these layer graphs are not possible if $|ZR| \neq |YC|$.

It follows that there is no layer graph for the 5:5 jewel in which the layer graph in
Figure 3.10(a) of the 5-cycle \(XABYC\) is embedded if the edges of the jewel satisfy Eq. 3.3. Hence, the 5-cycle \(XABYC\) of the 5:5 jewel of Figure 3.11 cannot be drawn as the layer graph of Figure 3.10(a) if the edges of the jewel satisfy the conditions in Eq. 3.3.

Figure 3.13: Replacing the condition \(|XC| \neq |YB|\) when \(YB\) and \(YZ\) are collinear

3.3.3 Replacing \(|YC| \neq |AB|\)

This rigidity condition corresponds to the layer graph drawing of Figure 3.10(e). We argue below that there are exactly 12 possible layer graph drawings of the 5:5 jewel in which the layer graph of Figure 3.10(e) lies embedded. There are 2 main cases to consider.

- \(YZ\) is vertical and \(YB\) is orthogonal to it:

  The path \(XZR\) is made up of a vertical segment \(XR\), followed by a horizontal segment \(ZR\), else \(R\) will coincide with \(C\). If we consider the path \(XPQ\), by a similar argument when \(XP\) is horizontal \(PQ\) must be vertical. If \(QZ\) were vertical, then \(P\) would have to coincide with \(C\). Thus, \(QZ\) is horizontal. This gives us the layer graph drawing of Figure 3.14(a).
If $XP$ is vertical, we can argue similarly as in the last paragraph that $PQ$ must be horizontal and $QZ$ vertical. This gives us the layer graph drawing of Figure 3.14(b).

Figure 3.14: Replacing the condition $|YC| \neq |AB|$ when $YZ$ and $YB$ are perpendicular to each other. There is only one position for $R$.

$\{|YB| \neq ||XA| \pm |XC||\}$ is an alternate rigidity condition for the 5-cycle $XABYC$ with the layer graph drawing as in (Figure 3.10(e)). This condition however involves the edge $XC$ that we wish to avoid. For both the layer graph drawings of Figure 3.14, $YB$ and $YZ$ being mutually perpendicular, the edges $YC$ and $YZ$ are on a line $CYZ$, and they are parallel to $XR$. So, we must have $|XC| = |ZR|$. Using this, we get the replacement rigidity condition $\{|YB| \neq ||XA| \pm |ZR||\}$.

- $YB$ and $YZ$ are collinear:

3 subcases arise depending upon the orientations of $ZR$ and $XR$.

- $ZR$ is perpendicular to $YB$ and $YZ$, and $XR$ is perpendicular to $ZR$ (Figure 3.15):

In this case there are 4 distinct placements of the edges $XP$, $PQ$ and $QZ$ giving
rise to 4 distinct layer graph drawings of the 5:5 jewel (Figure 3.15(a)-(d)).

In all the 4 layer graph drawings the edges $YZ$ and $XR$ are horizontal and collinear, while the edge $ZR$ is vertical and connects those two parallel edges.

The edges $YB$ and $XA$ are horizontal and collinear, while the edge $AB$ is vertical and connects those two parallel edges. $YZ$ and $YB$ are collinear, and so are $XR$ and $XA$. Therefore, we must have $|AB| = |ZR|$ and the replacement rigidity condition for this subcase is $|YC| \neq |ZR|$.

![Figure 3.15](image)

Figure 3.15: Replacing the condition $|YC| \neq |AB|$ when $YB$ and $YZ$ are collinear, $ZR$ is perpendicular to $BYZ$ and $XR$ is perpendicular to $ZR$

- $ZR$ is perpendicular to $YB$ and $YZ$, and $XR$ and $ZR$ are collinear:

In this case $XP$, $PQ$ and $QZ$ can be placed in 2 distinct configurations (Figure 3.16). In these configurations of the jewel the 5-cycle $XABYC$ cannot be drawn as a layer graph in the present configuration if $||XA| \pm |XC|| \neq |YB|$. In both the configurations of the layer graph of the jewel $YC$ and $XRZ$ are parallel,
and both of $XC$ and $YZ$ connect them. We must have $|XC| = |YZ|$. We can rewrite the condition as $||XA| \pm |YZ|| \neq |YB|$ for this subcase.

![Figure 3.16: Replacing the condition $|YC| \neq |AB|$ when $YB$ and $YZ$ are collinear, and $ZR$ and $XR$ are perpendicular to $BYZ$](image)

- $ZR$ is collinear with $YB$ and $YZ$ (Figure 3.17):

  In this case, $XR$ is necessarily perpendicular to $ZR$, while $XP$, $PQ$ and $QZ$ can be in 4 distinct configurations. In all of these, the 5-cycle $XABYC$ cannot be drawn as a layer graph in the present configuration if $||XA| \pm |XC|| \neq |YB|$. Since $|XC| = ||YZ| \pm |ZR||$ in all 4 layer graphs, the condition can be replaced by $||XA| \pm |YZ| \pm |ZR|| \neq |YB|$ for this subcase.

Thus, we have the following lemma.

**Lemma 32** The 5-cycle $XABYC$ of the 5:5 jewel of Figure 3.11 cannot be drawn as a layer graph in the configuration of Figure 3.10(e) if the edges of the jewel satisfy the following conditions:

$$|YB| \neq ||XA| \pm |ZR||, |YC| \neq |ZR|, ||XA| \pm |YZ|| \neq |YB|, ||XA| \pm |YZ| \pm |ZR|| \neq |YB|.$$
3.3.4 Rigidity Conditions

From (1), (2), Lemma 31 and Lemma 32, we have the following result for the line-rigidity of the 5-cycle $XABYC$ of the 5:5 jewel:

**Lemma 33** The 5-cycle $XABYC$ of the 5:5 jewel $XABYCPQZR$ of Figure 3.11 is rigid if the edges of the jewel satisfy the following set of conditions:

$$\{|ZR| \neq |YB|, |ZR| \neq |YC|, |XA| \neq |YC|, |XA| \neq |YB| + |YC|, |XA| \neq |YB|, |ZR| \neq |YB| + |XA|, |ZR| \neq |YB| + |YA| + |YZ|, |YB| \neq ||XA| + |YZ||\}.$$ 

We thus have an amplified set of sufficient conditions to satisfy.

Similarly, we have the following result for the line-rigidity of the other 5-cycle $XPQZR$ of the 5:5 jewel:

Figure 3.17: Replacing the condition $|YC| \neq |AB|$ when $YB$ and $YZ$ are collinear and $ZR$ is collinear with them. $XR$ can only be perpendicular to $ZR$. 

\[\text{Figure 3.17: Replacing the condition } |YC| \neq |AB| \text{ when } YB \text{ and } YZ \text{ are collinear and } ZR \text{ is collinear with them. } XR \text{ can only be perpendicular to } ZR.\]
Lemma 34  The 5-cycle XPQZR of the 5:5 jewel XABYCPQZR of Figure 3.11 is rigid if the edges of the jewel satisfy the following set of conditions:

\[
\{ |YC| \neq |ZQ|, |YC| \neq |ZR|, |XP| \neq |ZR|, |XP| \neq ||ZQ| \pm |ZR||, |XP| \neq |ZQ|, \\
|YC| \neq ||ZQ| \pm |XP||, |YC| \neq ||ZQ| \pm |XP| \pm |YZ||, |ZQ| \neq ||XP| \pm |YZ|| \}.
\]

By Corollary 4, the union of the two sets of conditions in Lemmas 33 and 34 constitutes a set of sufficient conditions for the line rigidity of the 5:5 jewel of Figure 3.11. Taking care of one overlapping condition between the two sets of 8 conditions, we have 15 distinct conditions for the line-rigidity of the 5:5 jewel and hence the following lemma.

Lemma 35  The 5:5 jewel XABYCPQZR of Figure 3.11 is rigid if its edges satisfy the following set of conditions:

1. \[|YB| \notin \{|XA|, ||XA| \pm |YZ||\},\]

2. \[|YC| \notin \{|XA|, ||YB| \pm |XA||\},\]

3. \[|ZQ| \notin \{|XP|, |YC|, ||XP| \pm |YZ||, ||YC| \pm |XP||, ||YC| \pm |XP| \pm |YZ||\},\]

4. \[|ZR| \notin \{|XP|, |YB|, |YC|, ||YB| \pm |XA||, ||YB| \pm |XA| \pm |YZ||, ||ZQ| \pm |XP||\}.
\]

In the next section we show how to construct a composite ppg made up of 5:5 jewels such that all the 15 rigidity conditions listed above are satisfied for each one of these.
3.3.5 Algorithm

We use a pair of vertices \( \{Y, Z\} \) as reference vertices. We query the edge length \(|YZ|\) and the pairwise distances of some other suitable vertices in the first round. All the vertices will be placed relative to \( Y \) and \( Z \). Now we consider the second round. We select vertices in groups of 7 vertices each in such a way that the pairwise distances of the union of each group of vertices \( \{X, A, B, C, P, Q, R\} \) and \( \{Y, Z\} \) satisfy the conditions in Lemma 35. Then we query the remaining necessary pairwise distances of the union to form a 5:5 jewel. The jewel will be rigid by Lemma 35 irrespective of the lengths of the edges \( AB, CX, PQ \) and \( RX \), since no condition of the lemma involves any of these edges. The unused vertices are made rigid by using triangle as the \( ppg \).

Algorithm 3.1. First a bit of nomenclature. To indicate the affiliations of the vertices \( X, A, B, C, P, Q, R \) to different copies of a 5:5 jewel, we use the following indexing scheme:

\[
X \rightarrow X_i, A \rightarrow A_i, B \rightarrow B_j, C \rightarrow B_k, P \rightarrow P_i, Q \rightarrow Q_m \text{ and } R \rightarrow Q_l.
\]

Let the number of points be \( n = 7b + 30 \), where \( b \) is a positive integer. In the first round, we make \( 6b + 29 \) distance queries represented by the edges in the graph in Figure 4.7. There are \( 2b + 6 \) leaf children \( B_j(j = 1, ..., 2b + 6) \) rooted at \( Y \) and \( 2b + 22 \) leaf children \( Q_l(l = 1, ..., 2b + 22) \) rooted at \( Z \). The remaining \( 3b \) vertices are organized into groups of 3 as \( (A_i, X_i, P_i) \) \((i = 1, ..., b)\) and the distances \(|A_iX_i|\) and \(|X_iP_i|\), \((i = 1, ..., b)\) are queried.

In the second round, for each 2-link \((A_iX_i, X_iP_i)\) we find a pair of edges \( YB_j \) and \( YB_k \), rooted at \( Y \) satisfying Conditions 1 and 2 of Lemma 35; next, we find a pair of edges \( ZQ_m \).
Figure 3.18: Queries in the first round for 2-round algorithm using 5:5 jewel as the basic component and $ZQ_l$, rooted at $Z$ satisfying Conditions 3 and 4 of Lemma 35.

Then for each $i, (i = 1, ..., b)$, we query the distances $|A_iB_j|$, $|X_iB_k|$, $|X_iQ_l|$ and $|P_iQ_m|$ to form a 5:5 jewel $X_iA_iB_jYB_kP_iQ_mZQ_l$. Its edges will satisfy all the rigidity conditions of Lemma 35.

For each of the 6 unused leaves $B_j$ of the tree rooted at $Y$, we query the distance $|B_jZ|$ to form the triangle $YB_jZ$. Likewise, for each of the 22 unused leaves $Q_l$ of the tree rooted at $Z$ we query the distance $|Q_lY|$ to form the triangle $YQ_lZ$. □

The following theorem establishes the correctness of our algorithm.

**Theorem 36** The ppg constructed by Algorithm 3.1 is rigid.

**Proof.** Let us consider an arbitrary 2-link $(P_iX_i, X_iA_i)$. We show that the 5:5 jewel constructed by Algorithm 1 using the edges of this 2-link is rigid.

Let us consider the selection of the edge $YB_j$ for the jewel in the second round. From
Condition 1 of Lemma 35, $|YB_j|$ cannot be equal to $|X_iA_i|, ||X_iA_i|+|YZ||$ or $||X_iA_i|-|YZ||$. By Observation 4 there can be at most 2 edges rooted at $Y$ that are equal to a given length. Hence there are at most 6 edges rooted at $Y$ that do not qualify to be chosen as $YB_j$. By adding 6 extra leaves at $Y$ we provide the room needed to choose $YB_j$ for each of the 2-links $(P_iX_i, X_iA_i)$ with $i = 1, ..., b$, so that the rigidity conditions on this edge are satisfied.

An identical argument shows that the 6 additional leaves at $Y$ enables us to choose $YB_k$ in the second round so that the rigidity conditions on this edge are satisfied for each of the 2-links $(P_iX_i, X_iA_i)$ with $i = 1, ..., b$.

Consider next the selection of the edge $ZQ_m$ for the jewel in the second round. From Condition 3 of Lemma 35, $|ZQ_m|$ cannot be equal to $|X_iP_i|, |YB_k|, |X_iP_i|+|YZ|, ||X_iP_i|-|YB_k|+|YZ||, ||X_iP_i|-|YB_k|-|YZ||, ||X_iP_i|-|YB_k|-|YZ||$, $||X_iP_i|-|YB_k|-|YZ||$. Again from Observation 1 it follows that there are at most 20 edges rooted at $Y$ that do not qualify to be chosen as $ZQ_m$. Adding 22 extra leaves at $Z$ provides us with the room needed to choose $ZQ_m$ for each of the 2-links $(P_iX_i, X_iA_i)$ with $i = 1, ..., b$, so that the rigidity conditions on this edge are satisfied.

There will be at most 20 edges $ZQ_m$ rooted at $Z$ that do not satisfy the conditions on it as stated in Lemma 35 (by Observation 4). In addition to the $2b$ edges necessary to construct the $b$ jewels there are 22 extra edges rooted at $Z$. So, for each set of 2-link $(P_iX_i, X_iA_i)$ and 3-link $(B_jY, B_kY, YZ)$ with $i = 1, ..., b$ ($B_jY$ depends on $P_iX_i$ and $X_iA_i$, and $B_kY$ depends on $P_iX_i$, $X_iA_i$ and $B_jY$), we can always find an edge $YQ_m$ that satisfies
the condition on it as stated in Lemma 35.

Finally, consider the second-round selection of the edge $ZQ_l$ for the jewel. From Condition 4 of Lemma 35 there are 11 rigidity conditions on $|ZQ_l|$, and hence by Observation 4 will be at most 22 edges $ZQ_l$ rooted at $Z$ are not eligible to be chosen. In addition to the $2b$ edges necessary to construct the $b$ jewels there are 22 extra edges rooted at $Z$. So, for each set of 2-link $(P_iX_i, X_iA_i)$ and 4-link $(B_jY, B_kY, YZ, ZQ_l)$ with $i = 1, ..., b$ the 22 extra edges rooted at $Z$ provide us with the latitude to find an edge $ZQ_l$ always that satisfies the rigidity conditions on it.

So, for each 2-link $(A_iX_i, X_iP_i)$ we can always find edges $YB_j$, $YB_k$, $ZQ_l$ and $ZQ_m$ for the 5:5 jewel of Figure 3.11 such that the conditions for rigidity (Lemma 35) are satisfied. Each of the $b$ 5:5 jewels of Figure 3.11 with $YZ$ as an edge is constructed in the second round by satisfying the rigidity conditions of Lemma 35. So, they are rigid and, for each $i, (i = 1, ..., b)$, the positions of $X_i, A_i, B_j, B_k, P_i, Q_m$ and $Q_l$ are fixed relative to $Y$ and $Z$. Each of the remaining 6 leaves of $Y$ forms a triangle $(YB_j, B_jZ, YZ)$ with $YZ$ as an edge. So, their positions are fixed relative to $Y$ and $Z$. Each of the remaining 22 leaves of $Z$ forms a triangle $(ZQ_l, Q_lY, YZ)$ with $YZ$ as an edge. So, their positions are fixed relative to $Y$ and $Z$.

Hence, the whole $ppg$ is rigid. □

**Theorem 37** $10n/7 + 99/7$ queries are sufficient to place $n$ distinct points on a line in two rounds.
Proof. We need $6b + 29$ queries in the first round and $4b + 28$ queries in the second round. In total $10b + 57$ pairwise distances are to be queried for the placement of $7b + 30$ points. We have $10b + 57 = \frac{10}{7}(7b + 30) - 300/7 + 57 = \frac{10n}{7} + 99/7$. □

It is worth noting that our algorithm needs at least 37 points to work. When we have fewer points we can switch to the quadrilateral algorithm, described in the Introduction. The 2-round 5-cycle algorithm of Chin et al. [20] a total of $4/3n + 34/3\sqrt{n}$ queries for the placement of $n$ points. Thus our 5:5 jewel algorithm does better when $n \leq 4076$. This provides the motivation for considering 6:6 jewels, which we do next.

3.4 Algorithm Based on a 6 : 6 Jewel

The principal ideas underlying this algorithm are similar to the algorithm based on 5:5 jewel of the last section. So we will skip the repetitive details when there is no scope for confusion.

Figure 3.19 shows the ppg for an instance of the 6:6 jewel that we shall use in the construction of our composite ppg. For brevity we will refer to left cycle as $C_1$ and the right cycle as $C_2$, and by 6:6 jewel we will mean the instance shown.

![Figure 3.19: A 6 : 6 jewel](image)

By Theorem 27, the 6-cycle $XABYCD$ has 16 different layer graph representations
(Figure 3.20), giving us the following 16 conditions for its line-rigidity. The layer graphs can be grouped into 4 groups depending on the number of edges on each side:

1. $|YC| \neq |XD|, |YB| \neq |CD|, |YC| \neq |AB|, |YB| \neq |XA|, |XD| \neq |AB|, |XA| \neq |CD|$

2. $|YB| \neq |XD|, |AB| \neq |CD|, |YC| \neq |XA|$

3. $|YB| \neq |YC| \pm |XD|, |YB| \neq |XA| \pm |XD|, |YB| \neq |YC| \pm |CD|, |YB| \neq |XA| \neq |YC| \pm |XC|$

4. $|XA| \neq |YB| \pm |CD|$.

Figure 3.20: Different layer graph representations of a 6-cycle

Similarly, we have another set of 16 conditions for the line-rigidity of the cycle $C_2$, viz.,

1. $|ZR| \neq |XS|, |ZQ| \neq |RS|, |ZR| \neq |PQ|, |ZQ| \neq |XP|, |XS| \neq |PQ|, |XP| \neq |SR|$

2. $|ZQ| \neq |XS|, |ZR| \neq |XP|, |PQ| \neq |RS|$
3. $|ZQ| \neq |ZR| \pm |XS||, |ZR| \neq |XP| \pm |XS||, |ZQ| \neq |XS| \pm |RS||, |ZQ| \neq |XP| \pm |XS||.

4. $|XP| \neq |ZQ| \pm |RS||$.

By Corollary 26, the conjunction of these two sets of conditions constitutes a set of sufficient conditions for the line-rigidity of the 6:6 jewel above.

### 3.4.1 Replacing Conditions

We would like to make the 6:6 jewel rigid irrespective of the lengths of the edges $AB, CD, PQ$ and $RS$, as this allows us to query the remaining edges in such a way that the rigidity conditions are satisfied. Towards this goal, we reformulate 16 conditions (8 from each cycle) involving these edges with alternate sets of conditions, satisfying which we also satisfy the replaced ones.

We use the left cycle, $C_1 = XABYCD$, as a running example to demonstrate these replacements.

**Replacing** $|AB| \neq |CD|$:

The layer graph for the 6-cycle $C_1$ corresponding to this condition is shown in Figure 3.20(h).

From the figure it is evident that we can replace this with the condition

$$||(YB) \pm (YC)|| \neq ||XA| \pm XD||$$  \hspace{1cm} (3.4)

since this will also prevent the layer graph drawing of the cycle as in Figure 3.20(h).
Replacing $|XA| \neq |CD|$: 

The layer graph of $C_1$ corresponding to this condition is shown in Figure 3.20(f). To replace this condition we follow a similar strategy as for the 5:5 jewel, except for a small twist: we draw all possible layer graphs of the 6:6 jewel, excluding the chain $XSRZ$, in which the layer graph of Figure 3.20(f) is embedded. The condition $|XA| \neq |CD|$ is then amplified into the set of conditions that prevent the drawing of the layer graph representation of the 6-cycle corresponding to this condition (Figure 3.20(f)). Two cases arise, depending on whether $YZ$ is horizontal or vertical.

- $YZ$ is horizontal:
  
  Here $Z$ and $X$ have different $x$ and $y$ coordinates. $XP$, $PQ$ and $QZ$ can have 4 different orientations as shown in Figures 3.21(a) - 3.21(d). The following conditions will prevent the layer graph drawings of the 6-cycle $XABYCD$ in Figure 3.20(f), when $YZ$ is horizontal: $|ZQ| \neq ||XA| \pm |XP||$ (Figure 3.21(a)), $||YC| \pm |YZ|| \neq ||XD| \pm |XP||$ (Figure 3.21(b)), $|ZQ| \neq |XA|$ (Figure 3.21(c)) and $||ZQ| \pm |YC| \pm |YZ|| \neq ||XD| \pm |XP||$ (Figure 3.21(d)).

- $YZ$ is vertical and $|YZ| = |XA|$: 
  
  In this case only one layer graph is possible as shown in Figure 3.22. We can replace $|XA| \neq |CD|$ with $|YZ| \neq |XA|$. This will prevent the layer graph drawing of the 6-cycle $XABYCD$ in Figure 3.20(f) when $YZ$ is vertical and $|YZ| = |XA|$.
- YZ is vertical and \(|YZ| \neq |XA|\):

Here Z and X have different x and y coordinates. XP, PQ and QZ can have 6 different orientations as shown in Figure 3.23(a) - 3.23(f). These layer graphs give rise to the following set of conditions that prevents the layer graph drawing of the 6-cycle \(XABYCD\) as in Figure 3.20(f), when YZ is vertical and \(|YZ| \neq |XA|\):

\[||ZQ| \pm |YZ|| \neq |XA|\] (Figure 3.23(a)), \(|YC| \neq ||XD| \pm |XP||\) (Figure 3.23(b)),

\[||ZQ| \pm |YZ|| \neq ||XA| \pm |XP||\] (Figure 3.23(c)), \(|ZQ| \pm |YC|| \neq ||XD| \pm |XP||\) (Figure 3.23(d)), \(|ZQ| \pm |YC|| \neq |XD|\) (Figure 3.23(e)) and \(|YZ| \neq ||XA| \pm |XP||\) (Figure 3.23(f)).
Thus, we have the following lemma, the proof of which is similar to the proof of Lemma 31 and is omitted:

**Lemma 38** The 6-cycle $XABYCD$ of the 6:6 jewel of Figure 3.19 cannot be drawn as a layer graph in the configuration of Figure 3.20(f) if the edges of the jewel satisfy the following conditions:

$$\{|ZQ| \neq |XA| \pm |XP|, |YC| \pm |YZ| \neq |XD| \pm |XP|, |ZQ| \neq |YA|, |ZQ| \pm |YD| \neq |YP|, |ZQ| \pm |YD| \neq |YP|, |ZQ| \pm |YD| \neq |YP|, |ZQ| \pm |YD| \neq |YP|, |ZQ| \neq |YA|, |ZQ| \neq |YA|, |ZQ| \neq |YA|, |ZQ| \neq |YA|\}.$$ 

Replacing $|XD| \neq |AB|$:

The layer graph of the 6-cycle corresponding to this condition is as shown in Figure 3.20(e).

This layer graph is the same as that in Figure 3.20(f) if we interchange $A$ with $D$ and $B$
with $C$. By this interchange of the labels in Lemma 38 we have the following lemma for the replacement of condition:

**Lemma 39** The 6-cycle $XABYCD$ of the 6:6 jewel of Figure 3.19 cannot be drawn as a layer graph in the configuration of Figure 3.20(e) if the edges of the jewel satisfy the following conditions:

$\{ |ZQ| \neq |XD| \pm |XP|, |YB| \neq |YZ| \pm |XA| \pm |XP|, |ZQ| \neq |YB| \pm |YZ| \pm |XA| \pm |XP|, |ZQ| \neq |XD| \pm |XP|, |YB| \neq |YZ| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, |ZQ| \neq |YB| \pm |YA| \pm |XP|, \}.

**Replacing** $|YC| \neq |AB|:

The layer graph of the 6-cycle corresponding to this condition is as shown in Figure 3.20(c). Figure 3.24 shows all the possible layer graphs of the 6:6 jewel, excluding the chain $XSRZ$, in which the layer graph of Figure 3.20(c) is embedded (different configurations for $P$ and $Q$ are combined in the same figure). From Figure 3.24 we see that the condition $|YC| \neq |AB|$ can be replaced by the following conditions:

- $|ZQ| \neq |YC| \pm |XP|$, $|ZQ| \neq |YB| \pm |YZ| \pm |XA|$, $|YB| \neq |XY| \pm |XA| \pm |XP|$, $|ZQ| \neq |YB| \pm |YZ| \pm |XA| \pm |XP|$, $|ZQ| \neq |YB| \pm |YZ| \pm |XA| \pm |XP|$, (Figure 3.24(a)),

- $|YB| \neq |YZ| \pm |XA|$ (Figure 3.24(b)) and
\[ |ZQ| \neq |YB| \pm |XA||, \ |YC| \neq |YZ| \pm |XP||, \ |ZQ| \neq |YB| \pm |ZQ| \neq |YB| \pm |XA| \pm |XP|| \ (Figure \ 3.24(c)). \]

Thus, we have the following lemma, the proof of which is similar to the proof of Lemma 31 and is omitted:

**Lemma 40** The 6-cycle $XABYCD$ of the 6:6 jewel of Figure 3.19 cannot be drawn as a layer graph in the configuration of Figure 3.20(c) if the edges of the jewel satisfy the following conditions:

\[
\{|ZQ| \neq |YB| \pm |ZQ| \neq |YB| \pm |XA| \pm |XP||, \ |ZQ| \neq |ZQ| \neq |YB| \pm \}
\]

Replacing $|YB| \neq |CD|$: 

The layer graph of the 6-cycle corresponding to this condition is as shown in Figure 3.20(b).

This layer graph is the same as that in Figure 3.20(c) if we interchange $A$ with $D$ and $B$
with $C$. By this interchange of the labels in Lemma 40 we have the following lemma for the replacement of this condition:

**Lemma 41** The 6-cycle $XABYCD$ of the 6:6 jewel of Figure 3.19 cannot be drawn as a layer graph in the configuration of Figure 3.20(b) if the edges of the jewel satisfy the following conditions:

$$
\{|ZQ| \neq |YB| \pm |XP||, |ZQ| \neq |YC| \pm |YZ| \pm |XD||, |YB| \neq |XP|, |ZQ| \neq |YC| \pm |YZ| \pm |XD|, |YB| \neq |XP|, |ZQ| \neq |YB| \pm |YZ| \pm |XP|, |ZQ| \neq |YC| \pm |XD| \pm |XP||\}.
$$

**Replacing** $|YC| \neq |XA| \pm |CD|$:

The layer graph of the 6-cycle corresponding to this condition is as shown in Figure 3.20(o).

Figure 3.25 shows all the possible layer graphs of the 6:6 jewel, excluding the chain $XSRZ$, in which the layer graph of Figure 3.20(o) is embedded. From Figure 3.25 we see that the condition $|YC| \neq |XA| \pm |CD||$ can be replaced by the following conditions:

- $|ZQ| \neq |YZ| \pm |XD| \pm |XP||, |ZQ| \neq |XA|, |YB| \neq |XP|, |ZQ| \neq |YB| \pm |XA| \pm |XP||$ (Figure 3.25(a)),

- $|ZQ| \neq |YZ| \pm |XA| \pm |XP||, |ZQ| \neq |XD|, |YB| \neq |XP|, |ZQ| \neq |XD| \pm |XP||$ (Figure 3.25(b)).
Thus, we have the following lemma, the proof of which is similar to the proof of Lemma 31 and is omitted.

**Lemma 42** The 6-cycle $XABYCD$ of the 6:6 jewel of Figure 3.19 cannot be drawn as a layer graph in the configuration of Figure 3.20(o) if the edges of the jewel satisfy the following conditions:

$$\{|ZQ| \neq |YB| \pm |XZ| \pm |XP|, |ZQ| \neq |XA|, |YZ| \neq |XD| \pm |XP|, |ZQ| \neq || | \pm | |\}. $$

Replacing $|YB| \neq |XD| \pm |CD|$: 

The layer graph of the 6-cycle corresponding to this condition is as shown in Figure 3.20(l).

Figure 3.26 shows all the possible layer graphs of the 6:6 jewel, excluding the chain $XSRZ$, in which the layer graph of Figure 3.20(l) is embedded. From Figure 3.26 we see that the condition $|YB| \neq |XD| \pm |CD|$ can be replaced by the following conditions:

- $|ZQ| \neq |YB| \pm |YZ| \pm |XP|, |ZQ| \neq |YC|, |YB| \neq |YZ| \pm |XP|, |ZQ| \neq |YC| \pm |XZ| \pm |XP|, |ZQ| \neq || | \pm | |$. 

![Figure 3.25: Replacing condition $|YC| \neq |XA| \pm |CD|$.](image.png)
|XP| (Figure 3.26(a));

- |ZQ| ≠ ||YC| ± |YZ| ± |XP||, |ZQ| ≠ |YB|, |YC| ≠ ||YZ| ± |XP||, |ZQ| ≠ ||YB| ± |XP| (Figure 3.26(b)).

Therefore, we have the following lemma, the proof of which is similar to the proof of Lemma 31 and is omitted.

**Lemma 43** The 6-cycle $XABYCD$ of the 6:6 jewel of Figure 3.19 cannot be drawn as a layer graph in the configuration of Figure 3.20(l) if the edges of the jewel satisfy the following conditions:

$$\{|ZQ| ≠ ||YB| ± |YD| ± |CD||, |ZQ| ≠ ||YB| ± |CD||, |ZQ| ≠ ||YB| ± |CD||, |ZQ| ≠ ||YB| ± |CD||, |ZQ| ≠ ||YB| ± |CD||\}.$$  

Replacing $|YB| ≠ ||X| ± |CD||$

The layer graph of the 6-cycle corresponding to this condition is as shown in Figure 3.20(p).

Figure 3.27 shows all the possible layer graphs of the 6:6 jewel, excluding the chain $XSRZ$, 

---

(a) (b)
in which the layer graph of Figure 3.20(p) is embedded. From Figure 3.27 we see that the condition $|YB| \neq ||XA| \pm |CD||$ can be replaced by the following conditions:

- $|ZQ| \neq ||YC| \pm |XD| \pm |XP||$, $|ZQ| \neq ||YB| \pm |YZ| \pm |XA||$, $|YC| \neq ||XD| \pm |XP||$, $|ZQ| \neq ||YB| \pm |YZ| \pm |XA| \pm |XP||$ (Figure 3.27(a));

- $|YB| \neq ||YZ| \pm |XA||$ (Figure 3.27(b));

- $|YC| \neq ||YZ| \pm |XD||$ (Figure 3.27(c));

- $|YB| \neq ||XA| \pm |XP||$, $|ZQ| \neq ||YC| \pm |XD||$, $|ZQ| \neq ||YB| \pm |YA| \pm |XP||$, $|ZQ| \neq ||YB| \pm |YA| \pm |XP||$, $|ZQ| \neq ||YZ| \pm |YA| \pm |XP||$, $|ZQ| \neq ||YZ| \pm |YA| \pm |XP||$, $|ZQ| \neq ||YB| \pm |YA| \pm |XP||$, $|ZQ| \neq ||YB| \pm |YA| \pm |XP||$ (Figure 3.27(d)).

Figure 3.27: Replacing condition $|YB| \neq ||XA| \pm |CD||$
Thus, we have the following lemma, the proof of which is similar to the proof of Lemma 31 and is omitted.

**Lemma 44** The 6-cycle $XABYCD$ of the 6:6 jewel of Figure 3.19 cannot be drawn as a layer graph in the configuration of Figure 3.20(p) if the edges of the jewel satisfy the following conditions:

$$\{|ZQ| \neq |YC| \pm |XD| \pm |XP||, |ZQ| \neq |YB| \pm |YZ| \pm |XA| \pm |XP||, |ZQ| \neq |YC| \pm |XD||, |YB| \neq |YZ| \pm |XA| \pm |XP||, |ZQ| \neq |YB| \pm |XA| \pm |XP||, |ZQ| \neq |YB| \pm |XA| \pm |XP||, |ZQ| \neq |YB| \pm |XA| \pm |XP||, |ZQ| \neq |YB| \pm |XA| \pm |XP||, |ZQ| \neq |YB| \pm |XA| \pm |XP||, |ZQ| \neq |YB| \pm |XA| \pm |XP||, |ZQ| \neq |YB| \pm |XA| \pm |XP||, |ZQ| \neq |YB| \pm |XA| \pm |XP||\}.$$

### 3.4.2 Rigidity Conditions

From Eqs. (4)-(5) and Lemmas 38 - 44 we have the following lemma for the line-rigidity of the 6-cycle $XABYCD$ of the 6:6 jewel of Figure 3.19:

**Lemma 45** The 6-cycle $XABYCD$ of the 6:6 jewel $XABYCDPQZRS$ of Figure 3.19 is rigid if the edges of the jewel satisfy the following conditions:

1. $|YZ| \notin \{|XA|, |XD|, |XA| \pm |XP||, |XD| \pm |XP||\}$;

2. $|YB| \notin \{|XA|, |XD|, |XP|, |XA| \pm |XD||, |XA| \pm |XP||, |XA| \pm |YZ||, |XP| \pm |YZ||, |XA| \pm |XP| \pm |YZ||\}$;
Similarly, we have the following lemma for the line-rigidity of the other 6-cycle \( XPQZRS \) of the 6:6 jewel:

**Lemma 46** The 6-cycle \( XPQZRS \) of the 6:6 jewel \( XABYCDPQZRS \) of Figure 3.19 is rigid if the edges of the jewel satisfy the following conditions:

1. \( |YZ| \notin \{|XP|, |XS|, |XA| \pm |XP|, |XA| \pm |XS|\}; \)

2. \( |YB| \notin \{|XP|, |XS|, |XA| \pm |XP|, |XP| \pm |YZ|, |XS| \pm |YZ|, |XA| \pm |XS|, |XA| \pm |XP| \pm |YZ|, |XP| \pm |YZ|, |XS| \pm |YZ|\}; \)

3. \( |ZQ| \notin \{|XA|, |XS|, |YB|, |XP|, |XA| \pm |XP|, |XA| \pm |YZ|, |YB| \pm |YA|, |YB| \pm |XP|, |XP| \pm |YS|, |XP| \pm |YZ|, |XS| \pm |YS|, |XS| \pm |YZ|\}; \)

4. \( |ZR| \notin \{|XS|, |XP|, |YA|, |YB|, |ZQ| \pm |XP|, |ZQ| \pm |XS|, |XP| \pm |XS|, |YB| \pm |XS|\}; \)

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Lemma 47 for the line-rigidity of the 6:6 jewel and hence the following lemma: of overlapping conditions between the two sets of conditions, we have 74 distinct conditions for the line-rigidity of the 6:6 jewel and hence the following lemma:

Lemma 47 The 6:6 jewel XABYCDPQZRS of Figure 3.19 is rigid if its edges satisfy the following conditions:

1. \(|YZ| \notin \{|XA|, |XD|, |XP|, |XS|, |XA| \pm |XP|, |XD| \pm |XP|, |YA| \pm |XS|\};

2. \(|YB| \notin \{|XA|, |XD|, |XP|, |XS|, |YA| \pm |XD|, |XA| \pm |XP|, |YA| \pm |YZ|, |XP| \pm |YZ|, |XS| \pm |YZ|, |XA| \pm |XS| \pm |YZ|\};

3. \(|ZQ| \notin \{|XA|, |XS|, |XD|, |YB|, |XP|, |XA| \pm |XP|, |XD| \pm |XP|, |YA| \pm |YZ|, |YB| \pm |YA| \pm |YZ|, |XP| \pm |YZ|, |XD| \pm |XP| \pm |YZ|, |YB| \pm |YA| \pm |YZ|, |YB| \pm |XP| \pm |YZ|, |YB| \pm |YA| \pm |XP| \pm |YZ|\};

4. \(|YC| \notin \{|XD|, |XA|, |XP|, |ZQ|, |YB| \pm |YA|, |YB| \pm |XD|, |YA| \pm |XD|, |XD| \pm |XP|, |XD| \pm |YZ|, |XP| \pm |YZ|, |ZQ| \pm |XD|, |ZQ| \pm |XP|, |YB| \pm |YA| \pm |X|}.\)
In the next section we show how a composite ppg can be constructed by satisfying all the 74 conditions for each such jewel.

3.4.3 Algorithm

It is interesting to note that the substitution mechanism has generated rigidity conditions on the strut YZ (Condition 1 of Lemma 47). This implies that, unlike the case for a 5:5 jewel, we will need a pool of vertices S for which the pairwise distances of all the pairs of points corresponding to the vertices in S are known after the first round of query, and from which we choose the end vertices Y and Z of a strut YZ in order to meet the rigidity conditions on YZ. We make the vertices in S rigid in the first round. Then the pairwise distances of all the pairs of points corresponding to the vertices in S are known after the first round of query. We make the remaining 9 vertices of each 6:6 jewel rigid in the second round.

We have to choose the size of S carefully. Since there are 10 conditions on the length of
an $YZ$, from Observation 4 it follows that there must be at most 21 edges incident to the end vertex $Y$, when we are looking for the other end vertex $Z$ of a strut.

However, if we use $|S| = 22$ for the selection of $Z$ for a particular $Y$, it may happen that all the 6:6 jewels get attached to the same vertex $Z \in S$. This hinders our goal of obtaining a better value for $\alpha$ than previously known.

We need to attach 6:6 jewels evenly to all the vertices in $S$ so that the same number of edges can be attached to each of them in the first round, and all of those edges, except for a constant number, are used to attach 6:6 jewels. In other words, we need to attach the 6:6 jewels to the vertices in $S$ in such a way that the numbers of 6:6 jewels attached to any two vertices differ by at most a constant number.

To specify the number of basic components attached to a vertex in a rigid set $S$ in the first round we use the term *valence*. We denote the set of rigid vertices in round 1 with valence $d$ as $S_d$.

Now we describe our algorithm to select a pair of vertices $Y$ and $Z$ in $S$ to attach 6:6 jewels. To attach a 6:6 jewel we always select a vertex in $S$ with the lowest valence as the first vertex (say, $Y$). Of the remaining vertices in $S$, at most 20 vertices may not be acceptable for $Z$, because of the conditions on $YZ$ (Condition 1 of Lemma 47). From among the rest $|S| - 1$ vertices in $S$ that satisfy the conditions on $YZ$, we select the one that has the lowest valence, as $Z$. This method is followed to attach each 6:6 jewel to the vertices in $S$, while the 6:6 jewels are attached sequentially.
The following lemma tells us how big \( S \) must be.

**Lemma 48** A set \( S \) of 42 vertices is sufficient to ensure that the valences of two vertices in \( S \) differ by at most 1.

**Proof.** Initially, all the vertices in \( S \) have valence 0, i.e., \(|S_0| = 42 \) and \(|S_i| = 0\) for each \( i \).

Our algorithm selects pairs of vertices \((Y, Z)\) in \( S_0 \) to attach 6:6 jewels. To select a pair of vertices \((Y, Z)\) in \( S_0 \) by satisfying 10 different length restrictions (Condition 1 of Lemma 47) in all, we need a buffer of vertices in \( S_0 \) of size at least 20. Thus, our algorithm selects 11 pairs of vertices \((Y, Z)\) from \( S_0 \). This way our algorithm attaches the first 11 6:6 jewels to 22 vertices in \( S_0 \). Then, we have \(|S_0| = 20\) and \(|S_1| = 22\).

Next, for each new 6:6 jewel our algorithm selects at least 1 vertex (as \( Y \), say) from \( S_0 \) until it is empty. The valence of this vertex will be raised to 1. If the second vertex is not found in \( S_0 \), it will be selected from \( S_1 \), raising its valence to 2. Now, we have \(|S_0| = 0\), \(|S_1| \geq 22\) and the rest vertices are in \( S_2 \).

We note that when both the vertices of a pair are chosen from \( S_0 \) then \(|S_1| \) is increased by 2. Consequently, \(|S_1| \) remains even. Now we consider the case when only one vertex is chosen from \( S_0 \). Since \(|S_1| \geq 22\), we can choose the other vertex from \( S_1 \). This increases \(|S_1| \) by 1 and decreases it by 1 at the same time. Thus, in both the cases \(|S_1| \) will always be an even number.

If \(|S_1| > 20\) we choose pairs of vertices for \( Y \) and \( Z \) from \( S_1 \) until \(|S_1| = 20\). Eventually, we have \(|S_0| = 0\), \(|S_1| = 20\) and \(|S_2| = 22\). Thus, at some point of time all the vertices in \( S \)
have at most 2 consecutive valences, viz., 1 and 2. At any point of time before that, they may have at most 3 consecutive valences, viz., 0 - 2.

We shall show that our algorithm attach the 6:6 jewels in such a way that at any point of time the vertices in \( S \) will have at most 3 consecutive valences, and that at some point of time they will have at most 2 consecutive valences only. For this we use induction to show that if we start with vertices in \( S \) in the state of a valence distribution \((S_d, S_{d+1})\), and attach the basic components according to our algorithm, then at some point of time the state of the valences will be \((S_{d+1}, S_{d+2})\). We assume that \(|S_d| = 20\) and \(|S_{d+1}| = 22\).

We attach 6:6 jewels until \(|S_d| = 0\). For each new 6:6 jewel we choose at least 1 vertex from \( S_d \) and at most 1 vertex from \( S_{d+1} \) with a total of 2 vertices to form a 6:6 jewel. Then we have \(|S_{d+1}| \geq 22\). The rest vertices are of valence \( d + 2 \). We argue as above to show that \(|S_{d+1}| \) will always be even.

If \(|S_{d+1}| > 20\), then for each new 6:6 jewel our algorithm uses a pair of vertices in \( S_{d+1} \) as in the initial round above until \(|S_{d+1} = 20|\). The rest 22 will be of valence \( d + 2 \). Now the situation is the same as it was at the start of induction except that the levels of valences have been increased by 1. \( \square \)

The set \( S \) of 42 vertices can be set as the vertices of 8 4:4 jewels hanging from a common strut. Since each 4:4 jewel is rigid so is this configuration. edges of this ppg.

From Condition 2 of Lemma 47 we see that we need 48 extra edges for the selection of an \( YB \) that satisfies all the conditions on it as stated in the lemma. Similarly, by Conditions
3, 4 and 5 of Lemma 47 we need 98 extra edges for $ZQ$, 96 extra edges for $YC$ and 96 extra edges for $ZR$ respectively. Thus, 98 extra edges at $Y$ and $Z$ will suffice to satisfy all the conditions on these edges. In addition to these extra 98 edges we need 2 more edges to accommodate the difference of 1 6:6 jewel that can be attached to them. Thus, we need a total of 100 extra edges at each of the 42 vertices of $S$.

The main idea underlying the algorithm below is to construct multiple copies of a 6:6 jewel over two rounds to ensure their rigidity. We use the set of vertices $S$ as reference vertices. Any set of 42 vertices is chosen as $S$. The pair of vertices $\{Y,Z\}$ that make up the strut $YZ$ (see Figure 3.19) of a 6:6 jewel, is chosen from the set $S$. As part of the first round, a rigid layout of $S$ is fixed by attaching eight 4:4 jewels of Figure 3.6 from a common strut. The common strut of the 4:4 jewels joins two vertices of $S$. Pairwise distances of some other suitable vertices are also queried in the first round.

Now we consider the second round. Let $S' = V \setminus S$ be the complement of $S$. In the second round, the positions of all the vertices of $S'$ are fixed relative to the vertices in $S$ by first selecting groups of 9 vertices each from $S'$ and placing them relative to a pair of vertices $\{Y,Z\}$ of $S$. For this, we select a vertex $Y \in S$ which has the lowest valence of 6:6 jewel of Figure 3.19 and a 5-link $(X,A,D,P,S)$. Then we select a vertex $Z \in S$ such that it has the lowest valence of 6:6 jewel of Figure 3.19 and that $|YZ|$ satisfies all the conditions of rigidity on it as stated in Condition 1 of Lemma 47. Thereafter, the vertices $B,C,Q$ and $R$ of $S'$ are selected such that the conditions of rigidity on $|YB|$, $|ZQ|$, $|YC|$ and $|ZR|$ as
stated in respectively Conditions 2, 3, 4 and 5 of Lemma 47 are satisfied. Then we query the remaining necessary pairwise edge distances $|AB|, |CD|, |PQ|$ and $|RS|$ of the group to form a 6:6 jewel. The jewel will be rigid by Lemma 47 irrespective of the lengths of the edges $AB, CD, PQ$ and $RS$, since no condition of the lemma involves any of these edges.

The unused vertices of $S'$ are made rigid by using 4-cycle as the ppg.

**Algorithm 3.2.** As in Algorithm 3.1, we use the following indexing scheme: $X \rightarrow X_i$, $A \rightarrow A_i$, $B \rightarrow B_j$, $C \rightarrow B_k$, $D \rightarrow D_i$, $P \rightarrow P_t$, $Q \rightarrow Q_m$, $R \rightarrow Q_l$, $S \rightarrow S_i$, $Y \rightarrow Y_u$ and $Z \rightarrow Y_v$.

Let the total number of points be $n$. We attach $b$ 6:6 jewels (Figure 3.19) to each of 20 fixed vertices in $S$ and $b + 1$ to the remaining 22. This gives us a total of $21b + 11$ jewels.

In the first round, we make distance queries represented by the edges of the graph in Figure 3.28. All the vertices $Y_u (u = 1, ..., 42)$ (or, $Y_v, v = 1, ..., 42$) in the subgraph enclosed by the rectangle are made line rigid in the first round by using the 4:4 jewel of Figure 3.6 as the ppg. There are 8 4:4 jewels (Figure 3.6) attached to a common strut, 42 vertices and 65 edges in the subgraph. There are $2b + 100$ leaf children rooted at each of the vertices $Y_u (u = 1, ..., 42)$ (or, $Y_v, v = 1, ..., 42$) to attach $b$ or $b + 1$ 6:6 jewels (Figure 3.19). Since there will be $21b + 11$ 6:6 jewels we have $21b + 11$ groups of 5 vertices $(A_i, D_i, S_i, P_i, X_i)$ ($i = 1, ..., 21b + 11$). We query the distances $|A_iX_i|$, $|D_iX_i|$, $|S_iX_i|$ and $|P_iX_i|$, ($i = 1, ..., 21b + 11$) in the first round. We will make a total of $168b + 4309$ pairwise distance queries in the first round for the placement of $n = 189b + 4297$ points.
In the second round, for each 4-link \((A_i, D_i, S_i, P_i, X_i)\), \(i = 1, \ldots, 21b + 11\), we construct a 6:6 jewel (Figure 3.19), satisfying all its rigidity conditions as in Lemma 47. For each such 4-link we select a vertex \(Y_u\), from the subgraph of 42 fixed vertices \(Y_u/Y_v, (u, v = 1, \ldots, 42; u \neq v)\), that has the lowest valency of 6:6 jewel of Figure 3.19. Since all the 42 vertices \(Y_u, u = 1, \ldots, 42\), are fixed in the first round, for any pair of such fixed vertices \((Y_u, Y_v) (u, v = 1, \ldots, 42; u \neq v)\) we can find the distance \(|Y_uY_v|\). So, for each pair of vertices \((Y_u, Y_v) (u, v = 1, \ldots, 42; u \neq v)\), we shall use \((Y_u, Y_v)\) as an edge in the construction of the 6:6 jewel of Figure 3.19. Now from the subgraph of 42 fixed vertices we select another vertex \(Y_v (v \neq u)\) such that the length \(|Y_uY_v|\) satisfies all the conditions of rigidity on it as stated in Condition 1 of Lemma 47 and that it has the lowest valency of 6:6 jewel of Figure 3.19 among all such qualifying vertices. We note that we can always find such vertex \(Y_v\), because there will be at most 20 edges \(Y_uY_v\) whose length do not satisfy the rigidity conditions on it (Condition 1 of Lemma 47) whereas we have 41 vertices for choosing the vertex \(Y_v\).
Then we find an edge $Y_uB_j$ rooted at $Y_u$ satisfying the conditions of rigidity on it as stated in Condition 2 of Lemma 47, then we find another edge $Y_vQ_m$ rooted at $Y_v$ satisfying the conditions of rigidity on it as stated in Condition 3 of Lemma 47, then we find another edge $Y_uB_k$ rooted at $Y_u$ satisfying the rigidity conditions on it as stated in Condition 4 of Lemma 47 and, finally, we find another edge $Y_vQ_l$ rooted at $Y_v$ satisfying the rigidity conditions on it as stated in Condition 5 of Lemma 47. Then for each $i, (i = 1, \ldots, 21b+11)$, we query the distances $|A_iB_j|, |D_iB_k|, |S_iQ_l|$ and $|P_iQ_m|$ to form a 6:6 jewel $X_iA_iB_jY_uB_kD_iP_iQ_mY_vQ_lS_i$. Its edges will satisfy all the rigidity conditions of Lemma 47. Thus, all the $21b+11$ 4-links will be consumed to construct $21b+11$ jewels. For this $84b+44$ edges will be queried.

There will be unused leaves $B_j$ (or $Q_l$) numbering 100 for each of 20 fixed vertices $Y_u$ ($u = 1, \ldots, 42$) (or, $Y_v, v = 1, \ldots, 42$) and 98 for each of 22 fixed vertices $Y_u$ ($u = 1, \ldots, 42$) (or, $Y_v, v = 1, \ldots, 42$). The total number of such unused vertices is 4156. We use a 4-cycle $ppg$ to fix them in the second round. As before, for each pair of vertices $(Y_u, Y_v)(u, v = 1, \ldots, 42; u \neq v)$, we shall use $Y_uY_v$ as an edge in the construction of the 4-cycle. For each unused vertex $B_j$ rooted at $Y_u$ we find another vertex $Q_l$ rooted at $Y_v$ such that $|Y_uB_j| \neq |Y_vQ_l|$. Then the 4-cycle $B_jY_uY_vQ_l$ will be rigid (Observation 2). Then we query the distance $|B_jQ_l|$ to complete the 4-cycle.

Note that we can always find a vertex like $Q_l$. For after repeated selection of such matching pairs of edges there may remain at most 2 edges $Y_uB_j$ rooted at $Y_u$ of length
equal to that of the same number of edges rooted at $Y_v$ (Observation 4). In such a situation we switch the matching to match such edges rooted at $Y_u$ with edges other than those same length edge/s rooted at $Y_v$ - this is always possible because there are at most 2 edges rooted at $Y_v$ that have the same length (Observation 4).

For 4156 unused vertices (after the construction of the 6:6 jewel) there will be 2078 4-cycles, and 2078 edges will be queried to complete the 4-cycles. The total number of queries in the second round will be $(84b + 44) + 2078$, i.e., $84b + 2122$.

**Theorem 49** The ppg constructed by Algorithm 3.2 is rigid.

**Proof.** The proof is similar to that of Theorem 36 for the line rigidity of the ppg constructed by Algorithm 3.1. □

The number of queries in the first and second rounds are $168b + 4309$ and $84b + 2122$ respectively. Thus, in 2 rounds a total of $252b + 6431$ pairwise distances are to be queried for the placement of $189b + 4297$ points. It is interesting to note that our algorithm would need at least 4486 points to work, which makes it reasonably practical. When we have fewer points we can use Algorithm 3.1 instead.

Now, $252b + 6431 = (252/189) \times (189b + 4297) - (4/3) \times 4297 + 6431 = 4n/3 + (19293 - 17188)/3 = 4n/3 + 2105/3$. Thus, we have the following theorem:

**Theorem 50** $4n/3 + 2105/3$ queries are sufficient to place $n$ distinct points on a line in two rounds.
A consequence of the last theorem is that our 6:6 jewel algorithm is better than the 5-cycle algorithm of Chin et al. [20] for \( n \geq 11851 \).

### 3.5 Lower Bound for Two Rounds

In this section we revisit the adversarial argument given by [20] to establish a lower bound on 2-round algorithms. We show that a deeper analysis improves the lower bound substantially.

Let \( \mathcal{A} \) denote any 2-round algorithm and \( \mathcal{B} \) an adversary. The latter sets edge lengths for \( \text{ppg} \) in each of the 2 rounds and returns the distance between any two points queried by \( \mathcal{A} \). \( \mathcal{B} \) can also assign value to the distance between a pair of points \textit{not queried} by \( \mathcal{A} \). While \( \mathcal{A} \)'s goal is to make as few distance queries as possible, \( \mathcal{B} \) tries to maximize the density of the \( \text{ppg} \).

In the \textit{first round}, \( \mathcal{A} \) queries the distances between pairs of vertices corresponding to the edges \( E_1 \) of the \( \text{ppg} \), \( G_1 = (V, E_1) \). In response, \( \mathcal{B} \) returns queried edge-lengths consistent with the following 3-part strategy. We call a vertex of degree at least 3 in a \( \text{ppg} \) \( G \) as a \textit{heavy} vertex in \( G \).

\( S_1. \) \( \mathcal{B} \) \textit{fixes the layout of all heavy vertices in } \( G_1 \) \textit{ and sets the lengths of the edges in } \( G_1 \) \textit{ incident to these vertices.}

\( S_2. \) \textit{For each vertex of degree 2 in } \( G_1 \) \textit{ that is connected to a vertex of degree 1 in } \( G_1 \) \textit{, the length of one of the two edges incident to the degree 2 vertex is set to a fixed value } \( c > 0 \).
S3. Let \( P_k = < p_1, p_2, ..., p_k > \) be a maximal path of degree 2 vertices \( p_i, i = 1, \ldots, k \) in \( G_1 \). Let \( p_0 \) and \( p_{k+1} \) be non-degree 2 vertices in \( G_1 \) adjacent to \( p_1 \) and \( p_k \) respectively. First \( B \) sets \( |p_{i-1}p_i| = |p_{i+1}p_{i+2}| \) for \( i = 1 \pmod{3} \). If both \( p_0 \) and \( p_{k+1} \) are heavy vertices in \( G_1 \), then it sets \( |p_i p_{i+1}| = |p_{i-1} p_{i+2}| \) for \( i = 1 \pmod{3} \) and also fixes the layout of the vertices \( p_i, i = 0 \pmod{3} \). Otherwise, if at least one of them, say \( p_{k+1} \), is of degree one in \( G_1 \) \( B \) sets \( |p_k p_{k+1}| = |p_{k-2} p_{k-1}| \). Also, except for the edges whose length is \( c \), \( B \) sets the lengths of the rest of the edges to lie between \( 2c \) and \( 3c \).

**Lemma 51** Strategies \( S_2 \) and \( S_3 \) of \( B \) are mutually consistent.

**Proof.** Consider a path \( P_k \) of degree 2 vertices in \( G_1 \) such that both \( p_0 \) and \( p_{k+1} \) have degree 1. If \( k = 1 \), only \( S_2 \) comes into play and in this case \( B \) sets \( |p_1 p_2| = c \). For all \( k \geq 4 \), \( B \) sets \( |p_1 p_2| = c \), \( |p_{k-1} p_k| = c \) in accordance with \( S_2 \) and the lengths of all other edges in accordance with \( S_3 \). Figures 3.29(c) - 3.29(f) serve as examples of this length assignment since for any \( k \), the total number of edges is a multiple of 3 as in Figure 3.29(d), or a multiple of 3 plus 1 as in Figure 3.29(e) or a multiple of 3 plus 2 as in Figure 3.29(f).

For \( k = 2 \) and \( k = 3 \) \( B \) makes the length assignments as shown Figures 3.29(a) - 3.29(b), which are again consistent with \( S_2 \) and \( S_3 \).

If \( p_0 \) is heavy, then \( B \) does not have to set \( |p_1 p_2| \) to \( c \). \( \square \)

In the second round, \( A \) queries the distances between new pairs of vertices corresponding
Figure 3.29: The residual parts of maximal paths of degree 2 vertices that will satisfy $S2$

to the edges in $E_2$ of the ppg $G_2 = (V, E_1 \cup E_2)$. In response, $B$ returns queried edge lengths consistent with the following strategy:

**S4.** Let $P_k = < p_1, p_2, ..., p_k > (k \geq 2)$ be a maximal path of degree 2 vertices of length at least 2 in $G_1$. Let $p_0$ and $p_{k+1}$ be non-degree 2 vertices adjacent to $p_1$ and $p_k$ respectively.

If at least one of them, say $p_{k+1}$, is of degree 1 in the first round and if, for some $i$ with $i = 1 \pmod 3$ and $i < k$, no edge is connected to either $p_i$ or $p_{i+1}$ in the second round by the algorithm then $B$ sets $|p_i p_{i+1}| = |p_{i-1} p_{i+2}|$ for one of those values of $i$ in
the second round.

Or, if no edge is connected to either $p_{k-1}$ or $p_k$ in the second round by $A$, then $B$ sets

$$|p_{k-1}p_k| = |p_{k-2}p_{k+1}|.$$ 

An important observation is in order: the above strategies of $B$ do not prevent $A$ from making a linear placement of the vertices of a maximal path of degree 2 vertices that joins a heavy vertex to a vertex of degree 1 in distinct positions.

Let $p_0$ be a heavy vertex. Consider all the maximal paths $P_k$ of degree 2 vertices incident to $p_0$, whose other end is of degree 1. For each path, $B$ computes the sum of the lengths of all the edges in the path. Let $l_{\text{max}}$ be the maximum of all the sums. $B$ maintains an interval of this length on either side of $p_0$ free from the placement of the vertices that lie on a path $P_k$ incident to $p_0$ whose other end is a heavy vertex. This is ensured as follows:

1. The distance between $p_0$ and an adjacent heavy vertex is at least $l_{\text{max}}$.

2. Let $P_k = < p_0, p_1, p_2 >$. In this case, $B$ sets $|p_0p_1| > l_{\text{max}}$. If $P_k = < p_0, p_1, ..., p_{k+1} >$, where $k > 1$, $B$ sets $|p_0p_1| = |p_2p_3| > l_{\text{max}}$ and $|p_1p_2| > 2|p_0p_1|$. This ensures that all the vertices of the prefix segment $< p_0, p_1, p_2, p_3 >$ of the path is at a distance farther than $l_{\text{max}}$ away from $p_0$. Clearly the remaining vertices on $P_k$, however placed, will also be at a distance farther than $l_{\text{max}}$.

The strategies adopted by $B$ bound the lengths of maximal paths formed by degree 2 vertices in $G_2$. The precise results are given in the next 3 lemmas.
Lemma 52 In $G_2$, the length of a longest chain of consecutive edges from $E_1$ that terminate on a heavy vertex at each end of the chain is 4.

Proof. Let $p_0$ and $p_{k+1}$ be non degree 2 vertices adjacent to a maximal path $P_k =$

$<p_1, p_2, ..., p_k>$ (k \(\geq 2\)) of degree 2 vertices of length $k$ in $G_1$.

We first consider the case when both of $p_0$ and $p_{k+1}$ are heavy vertices of $G_1$.

Given strategy S3 of $B$, if for an $i < k$ with $i = 1$ (mod 3) $A$ attaches no edge to either $p_i$ or $p_{i+1}$ in the second round then their positions will be ambiguous. Thus, the lemma is settled for this case.

Consider the case when $p_{k+1}$ is of degree 1. In view of strategies S3 and S4 of $B$, $A$ must attach an edge at $p_i$ or $p_{i+1}$ in the second round, for $i < k$ and $i = 1$ (mod 3), to make the placements of these vertices unambiguous. Thus, the lemma is settled for this case also.

Lemma 53 A maximal path $P_k$ of degree 2 vertices in $G_2$ that contains at least one edge of $E_2$ can have at most 2 consecutive edges of $E_1$.

Proof. Let $P_k (k \geq 2)$ be a maximal path of degree 2 vertices in $G_1$, and $p_0$ and $p_{k+1}$ be non degree 2 vertices adjacent to $p_1$ and $p_k$ respectively, where one of $p_0$ and $p_{k+1}$ be of degree 1 in $G_1$.

Suppose $p_0$ is of degree 1 in $G_1$. In view of strategy S3 of $B$, if no edge is connected to either $p_i$ or $p_{i+1}$ for some $i = 1$ (mod 3) then following strategy S4, $B$ will set $|p_ip_{i+1}| =$
\[ |p_{i-1}p_{i+2}| \] for one of those values of \( i \) in the second round. Thus, there must be an edge connected to either \( p_i \) or \( p_{i+1} \) for all \( i = 1 \pmod{3} \). In particular, \( \mathcal{A} \) must add an edge to be incident to \( p_1 \) or \( p_2 \) (when \( i = 1 \)).

If \( p_{k+1} \) is of degree 1 then following strategy \( S3 \) the adversary sets \( |p_k p_{k+1}| = |p_{k-2} p_{k-1}| \) in the first round. If \( \mathcal{A} \) attaches no edge to either \( p_{k-1} \) or \( p_k \) in the second round, then following \( S4 \), \( \mathcal{B} \) sets \( |p_{k-1} p_k| = |p_{k-2} p_{k+1}| \). This makes the placements of the vertices \( p_{k-1} \) and \( p_k \) will ambiguous (Observation 2). Thus \( \mathcal{A} \) must attach an edge to \( p_{k-1} \) or \( p_k \) to preempt \( \mathcal{B} \).

Thus, for both the cases, there will be at most 2 vertices of degree at most 2 at an end of a path of degree 2 vertices of \( G_1 \), if the end vertex is of degree 1. The algorithm will place them in the second round by introducing edge/s to one or both of them. Thus, in a maximal path of degree 2 vertices in \( G_2 \) that contains at least one edge from \( E_2 \) there can be at most 2 consecutive edges from \( E_1 \).

\[ \square \]

**Lemma 54** *The number of vertices in any maximal path of degree 2 vertices in \( G_2 \) is at most 3.*

**Proof.** If a maximal path of degree 2 vertices of \( G_2 \) consists of edges from \( E_1 \) only then by Lemma 52 its length is at most 3.

Now we consider maximal path of degree 2 vertices of \( G_2 \) that contains at least one edge from \( E_2 \). In such a path there cannot be three consecutive edges from \( E_1 \) (Lemma 53). Suppose the number of degree 2 vertices in such a maximal path is 4. Let the vertices
be $p_1, p_2, p_3$ and $p_4$. Let $p_0$ and $p_5$ be heavy vertices adjacent to $p_1$ and $p_4$ respectively.

Since any maximal path of degree 2 vertices in $G_2$ can have at most 2 consecutive edges from $E_1$ the edges $p_0p_1$, $p_1p_2$, $p_2p_3$, $p_3p_4$ and $p_4p_5$ can be from $E_1$ or $E_2$ in the following 5 combinations:

1. $E_2, E_1, E_2, E_1, E_1$

2. $E_2, E_1, E_1, E_2, E_1$

3. $E_1, E_2, E_1, E_2, E_1$

4. $E_1, E_1, E_2, E_2, E_1$

5. $E_1, E_1, E_2, E_1, E_1$

For combination 1, $\mathcal{B}$ can set the length of the 2 edges in $E_2$ so that $|p_0p_5| = |p_1p_2| + |p_2p_3|$ and $|p_0p_1| = |p_4p_5| - |p_3p_4|$ (Figure 3.30). Then by Theorem 23 the 6-cycle $p_0p_1p_2p_3p_4p_5$ would not be rigid. Similarly, for the combinations 2-4 $\mathcal{B}$ can make the graph ambiguous.

As for combination 5, following $S2$ $\mathcal{B}$ can set $|p_1p_2| = |p_3p_4| = c$, and can set the length of $p_2p_3$ in the second round in such a way that $|p_2p_3| = |p_4p_5| + |p_5p_0| + |p_0p_1|$ (Figure 3.31). The 6-cycle $p_0p_1p_2p_3p_4p_5$ would not be rigid then (Theorem 23).

The density of a ppg, $G = (V, E)$ is defined as the ratio $|E|/n$, where $n = |V|$. We establish the following lower bound on the density of a ppg constructed by any 2-round algorithm.
Theorem 55

Any deterministic 2-round algorithm for solving the 1-dimensional point placement problem requires at least $12n/11$ queries in the worst case.

Proof. Let each edge of $G$ have weight 1, which we split evenly between the vertices in $V$ that define it. If $w_i$ is the accumulated weight of the $i$-th vertex, clearly $\sum_{i=1}^{n} w_i = |E|$ so that $n \cdot \min_i \{w_i\} \leq |E|$. Thus $\min_i \{w_i\}$ is a lower bound on the density.

We can get a more precise estimate. Observe that a ppg has 2 types of vertices, heavy ones (already defined before) and vertices lying on maximal paths of degree 2 vertices that we call light vertices. If an edge joins two light vertices or two heavy vertices then the edge weight is divided equally between the vertices. Otherwise, the light vertex gets $1/2 + g$ of the weight and the heavy vertex $1/2 - g$ of the weight, where $0 \leq g \leq 1/2$.

The density of a heavy vertex is at least $3(1/2 - g)$. As for light vertices, we note that by Lemma 54 each maximal path of degree 2 vertices has length $k$, where $k \leq 3$. The total edge weight of such a path is $2(1/2 + g) + (k - 1)$. Thus, the average density of each vertex in such a path is $1 + 2g/k$. It is minimum when $k = 3$. Thus, the density of a light vertex is at least $1 + 2g/3$. 
The minimum average density for all vertices in $G_2$ is thus

$$\max \min \{3/2 - 3g, 1 + 2g/3 \} = 12/11$$

when $g = 3/22$. □

### 3.6 Summary

In this chapter, 2-round algorithms based on 5:5 and 6:6 jewels have been presented and the lower bound have been improved from $17n/16$ to $12n/11$. The algorithms have been implemented and they correctly found the unique position of points in each case. In the next chapter, we further improve both the lower bound and the upper bound of the problem.
Chapter 4

Improved Algorithm and Lower Bound for Point Placement Problem

4.1 Introduction

In Chapter 3 we proposed a 2-round algorithm that query $4n/3 + O(1)$ edges to construct rigid $ppg$ on $n$ points using 6:6 jewels as the basic components. In this chapter, we present a 2-round algorithm that queries $9n/7 + O(1)$ edges to construct a rigid $ppg$ on $n$ points, using 3 paths of degree two vertices of length 2 each with a common vertex as the basic component, bettering a result of [20] that uses 5-cycles. More significantly, we improve the lower bound on any 2-round algorithm to $9n/8$.

4.2 A 2-round Algorithm Based on a 3-path Graph

In this section, we describe a 2-round algorithm that queries $9n/7 + O(1)$ edges to construct a rigid $ppg$ on $n$ points. We use the graph in Figure 4.1 with density 6/5 as the basic building block. It can be construed as 3 paths $p_1q_1r_1s$, $p_2q_2r_2s$ and $p_3q_3r_3s$ with a common
terminal vertex $s$. Hence we call it a 3-path graph and formally define it as $G_{3p} = (V_{3p}, E_{3p})$

where

$$V_{3p} = \{p_1, q_1, r_1, p_2, q_2, r_2, p_3, q_3, r_3, s\}$$ and

$$E_{3p} = \{p_1q_1, q_1r_1, r_1s, p_2q_2, q_2r_2, r_2s, p_3q_3, q_3r_3, r_3s, p_1p_2, p_2p_3, p_3p_1\}.$$

Figure 4.1: The 3-path graph

Since the $G_{3p}$ can be drawn as a layer graph (see Figure 4.4), by Theorem 23 it is not intrinsically rigid. Indeed, there does not exist an intrinsically rigid graph of density $6/5$ in view of the lower bound of $4/3$ on any rigid graph with the exception of the graphs $K_3, K^{-}_4, K_{2,3}$ and the jewel [26].

To find a set of conditions that make $G_{3p}$ rigid, first we fix the placements of $p_1, p_2$ and $p_3$. Next, we find conditions that make the 7-cycle $(p_1, q_1, r_1, s, r_2, q_2, p_2)$ rigid. Relative to the fixed placement of $p_3$ and $s$ we have a 4-cycle $(p_3, q_3, r_3, s)$ with a virtual edge between $s$ and $p_3$. Adding a condition that makes this 4-cycle rigid to the set of rigidity conditions of the 7-cycle, gives us a set of conditions that makes $G_{3p}$ rigid.

Let us describe how we construct a rigid $G_{3p}$. First, we make $p_1, p_2$ and $p_3$ rigid in the first round query. In the second round, let us first we make the 7-cycle rigid by choosing the edges of $G_{3p}$ in such a way that the rigidity conditions for it are satisfied. Then we make
the virtual 4-cycle rigid by choosing the edges of $G_{3p}$ in such a way that rigidity conditions for this cycle are satisfied. Clearly, the $G_{3p}$ constructed thus will be rigid. It is evident that this way of choosing the edges of $G_{3p}$ in the second round is equivalent to choosing its edges, in the second round, by satisfying the union of the conditions for line rigidity of the 2 cycles. We shall follow the latter method.

To find the rigidity conditions for the 7-cycle, we resort to exhaustive enumerations of all the layer graph representations of the cycle. By Theorem 27, a 7-cycle has 42 different layer graph representations. We find conditions that prohibit the drawing of any them. By Theorem 23, they constitute the set of conditions for the line rigidity of the 7-cycle. All of the 42 layer graphs for a 7-cycle can be grouped into 6 groups based on the number of edges on each side. For the 7-cycle $(p_1, q_1, r_1, s, r_2, q_2, p_2)$, a representative layer graph for each of those groups is shown in Figure 4.2.

![Figure 4.2: An example of each group of layer graph for the 7-cycle $(p_1, q_1, r_1, s, r_2, q_2, p_2)$](image)
From these layer graphs we deduce the conditions for line rigidity of this cycle. The result is summarized in the following lemma (Lemma 56). As an example, we consider the group of 7 layer graphs represented by Figure 4.2(a). Each layer graph in this group has 4 edges on one side of the layer graph. The layer graphs are shown in Figure 4.3. From these layer graphs we deduce the following rigidity conditions:

\[ |p_1q_1| \neq |r_1s|, |q_1r_1| \neq |sr_2|, |r_1s| \neq |r_2q_2|, |sr_2| \neq |q_2p_2|, |r_2q_2| \neq |p_2p_1|, |q_2p_2| \neq |p_1q_1| \text{ and } |p_2p_1| \neq |q_1r_1|. \]

Figure 4.3: A group of layer graphs for the 7-cycle \((p_1, q_1, r_1, s, r_2, q_2, p_2)\) where 4 edges lie on one edge of the layer graph

**Lemma 56** A 7-cycle \((p_1, q_1, r_1, s, r_2, q_2, p_2)\) is rigid if

1. \(|p_1p_2| \neq |q_2r_2|, |p_1p_2| \neq |q_1r_1|, |p_2q_2| \neq |r_2s|, |p_1q_1| \neq |r_1s|, |q_2r_2| \neq |r_1s|, |q_1r_1| \neq |r_2s|, |p_1q_1| \neq |p_2q_2| \).
2. \(|p_1p_2|\pm |p_2q_2| \neq |r_2s|, |p_2q_2|\pm |q_2r_2| \neq |r_1s|, |p_1p_2|\pm |p_1q_1|\pm |p_2q_2| \neq |r_1s|, |p_1q_1| \neq |r_1s|, |p_1p_2| \neq |r_1s|, |p_1q_1| \neq |r_1s|, |p_1q_1| \neq |r_1s|, |p_1q_1| \neq |r_1s|, |p_1p_2| \neq |r_2s|, |p_2q_2| \neq |r_2s|, |p_1p_2| \neq |r_2s|, |p_1q_1| \neq |r_2s|, |p_1q_1| \neq |r_2s|, |p_1p_2| \neq |r_2s|, |p_2q_2| \neq |r_2s|.

3. \(|p_1p_2|\pm |p_1q_1| \neq |r_1s|, \ |p_1q_1|\pm |q_1r_1| \neq |r_2s|, \ |p_1p_2|\pm |p_1q_1|\pm |p_2q_2| \neq |r_2s|, |p_2q_2| \neq |r_2s|, |p_1p_2| \neq |r_2s|, |p_2q_2| \neq |r_2s|, |p_1q_1| \neq |r_2s|, |p_1q_1| \neq |r_2s|, |p_1q_1| \neq |q_1r_1|, |q_1r_1| \neq |r_2s|.

4. \(|p_1p_2| \neq |r_2s|, |p_1p_2| \neq |r_1s|, |p_2q_2| \neq |r_1s|, |p_1q_1| \neq |r_2s|, |p_1q_1| \pm |p_2q_2| \pm |p_1p_2| \neq |r_1s| \pm |r_2s|, |p_2q_2| \neq |p_1p_2| \pm |r_1s| \pm |r_2s|, |p_1q_1| \neq |p_1p_2| \pm |r_1s| \pm |r_2s|, |p_2q_2| \neq |p_1p_2| \pm |r_1s| \pm |r_2s|, |p_2q_2| \neq |p_1p_2| \pm |r_1s| \pm |r_2s|, |p_1p_2| \neq |r_1s| \pm |r_2s|.

5. \(|p_2q_2| \neq |p_1p_2| \pm |r_1s|, |p_1q_1| \neq |p_1p_2| \pm |r_2s|, |p_1q_1| \neq |p_1p_2| \pm |r_2s|, |p_1p_2| \neq |r_1s| \pm |r_2s|, |p_1q_1| \neq |r_2s|, |p_2q_2| \neq |q_2r_2| \pm |r_2s|, |p_1q_1| \neq |q_1r_1| \pm |r_1s|, |p_1q_1| \neq |r_2s|, |p_1p_2| \neq |r_2s|.

6. \(|p_1q_1| \pm |q_1r_1| \neq |p_2q_2| \pm |r_2s|, |p_2q_2| \pm |q_2r_2| \neq |p_1q_1| \pm |r_2s|, |p_1q_1| \pm |p_2q_2| \pm |r_2s| \neq |q_1r_1| \neq |p_2q_2| \pm |r_2s|, |q_1r_1| \neq |p_2q_2| \pm |r_2s|, |q_2r_2| \neq |p_1q_1| \pm |r_2s|, |p_2q_2| \neq |p_1q_1| \pm |r_1s|, |p_1q_1| \neq |p_2q_2| \pm |r_2s|, |p_2q_2| \neq |p_1q_1| \pm |r_2s|.

**Proof.** The proof is similar to the proof of the corresponding lemma for 5-cycle given by Chin et al. [20], and is omitted. □

The above conditions involve all the edges of the 7-cycle. If we query the lengths of all the edges of the cycle in the first round of a 2-round algorithm, the edge lengths may not satisfy all the rigidity conditions. It is evident from the 2-dimensional stretch of layer graph that we can avoid the length of an edge from all the conditions of rigidity for a cycle.
We avoid \(q_1r_1\) and \(q_3r_3\) from the conditions of rigidity for the 7-cycle \((p_1, q_1, r_1, s, r_2, q_2, p_2)\) and the virtual 4-cycle \((p_3, q_3, r_3, s)\) respectively. Then the conditions for rigidity of each of these cycles will involve all the other edges in the corresponding cycle.

Again, for each rigidity condition we need to have at least one edge in the condition such that we can choose an edge with suitable length, that satisfies the rigidity condition, as that edge, from among the options for edges with different lengths for that particular edge. Thus, we need to have choices for edge lengths of some edges so that we can avoid some edge lengths for some edges according to the conditions for rigidity. We provide these choices for \(p_1q_1\) and \(p_3q_3\).

Since the rigidity conditions will involve neither \(q_1r_1\) nor \(q_3r_3\), \(G_{3p}\) will be rigid irrespective of the lengths of those edges. If we query \(q_1r_1\) and \(q_3r_3\) in the second round then we can create enough choices for \(p_1q_1\) and \(p_3q_3\) in the first round to satisfy any rigidity condition involving any of them.

There will be rigidity conditions for the 7-cycle that will not involve these edges, i.e., \(p_1q_1\) and \(p_3q_3\). We cannot meet those rigidity conditions in a 2-round algorithm. So, we need to avoid some other edge(s) from the rigidity conditions of the cycle and/or provide options for choosing some other edge(s) for the cycle. For the 7-cycle \((p_1, q_1, r_1, s, r_2, q_2, p_2)\) there will be rigidity conditions involving \(p_2q_2\) and \(r_2s\), \(p_2q_2\) and \(r_1s\), and all the 3 edges \(p_2q_2\), \(r_1s\) and \(r_2s\). We can meet all those conditions, if we provide sufficient options for choosing the length of \(p_2q_2\). We can provide choices for edge lengths of the edge \(p_2q_2\) in the
first round if we do not query the edge $q_2r_2$ of the 7-cycle in the first round.

Thus, we do not query the lengths of the edges $q_1r_1$, $q_2r_2$ and $q_3r_3$ in the first round. We query them in the second round. We find a set of sufficient conditions for rigidity for $G_{3p}$ that does not involve these edges. The 7-cycle have rigidity conditions involving either $q_1r_1$ or $q_2r_2$. We replace each of its rigidity conditions that involve any of these edges. We replace each such condition by a set of condition(s) that prevents the cycle from being drawn as the layer graph representation that corresponds to that condition. Then we can satisfy all the rigidity conditions irrespective of the lengths of these edges which will be reported in the second round.

Among the 42 conditions in Lemma 56 for line rigidity of the 7-cycle, 20 conditions involve either $q_1r_1$ or $q_2r_2$ of the 7-cycle that we want to avoid in the conditions. We replace each of these conditions by a set of conditions that prevents the 7-cycle from being drawn as the layer graph representation that corresponds to that condition. By Theorem 23, the set of all these new conditions and the ones that are not replaced will constitute the rigidity conditions for the 7-cycle. As stated before, if the 7-cycle is rigid then the $(p_3, q_3, r_3, s)$ will be a 4-cycle in the second round and it can be made rigid by imposing the condition [20]:

$$|p_3q_3| \neq |r_3s|.$$  \hspace{1cm} (4.1)

This condition together with the rigidity conditions for the 7-cycle will constitute the rigidity conditions for the whole $G_{3p}$. 

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In the next subsection, we show how to replace the above mentioned 20 conditions that involve the edges $q_1 r_1$ and $q_2 r_2$ of the 7-cycle $(p_1, q_1, r_1, s, r_2, q_2, p_2)$ with the ones that do not involve them. To this end, for each of these conditions, first we try to find use other edges of the cycle in the layer graph representation corresponding to that condition. If this fails then we embed the layer graph representation corresponding to that condition into all possible layer graph representations of the whole $G_{3p}$, and derive a rigidity condition from each such embedding.

4.2.1 Replacing Conditions

As an example of replacing conditions we shall replace the first condition, viz., $|p_1 p_2| \neq |q_2 r_2|$.

Replacing $|p_1 p_2| \neq |q_2 r_2|$

The rigidity condition $|p_1 p_2| \neq |q_2 r_2|$ corresponds to the layer graph of Figure 4.2(a). To replace this condition we find a set of conditions that prevent the drawing of layer graph of the 7-cycle $(p_1, q_1, r_1, s, r_2, q_2, p_2)$ in the configuration of Figure 4.2(a).

Lemma 57 The 7-cycle $(p_1, q_1, r_1, s, r_2, q_2, p_2)$ of $G_{3p}$ cannot be drawn as the layer graph of Figure 4.2(a) if the edges of $G_{3p}$ satisfy the following set of conditions:

$$\{|p_1 p_3| \neq |p_3 q_3| \pm |r_3 s|, |p_1 p_3| \neq |r_3 s|, |p_3 q_3| \pm |s r_2| \neq |p_2 q_2|, |p_3 q_3| \pm |s r_2| \pm |s r_3| \neq |p_2 q_2|\}.$$

Proof. We consider all possible layer graphs of $G_{3p}$ in which the 7-cycle appears in the
above fixed configuration. For each such layer graph of $G_{3p}$, we find the condition or set of conditions that prevents $G_{3p}$ from being drawn as a layer graph of that configuration and, a fortiori, the embedded 7-cycle $(p_1, q_1, r_1, s, r_2, q_2, p_2)$ in the configuration of Figure 4.2(a). This new set of conditions acts as a replacement for the condition $|p_1p_2| \neq |q_2r_2|$ since that set will prevent the drawing of the layer graph of the 7-cycle $(p_1, q_1, r_1, s, r_2, q_2, p_2)$ in the corresponding configuration in Figure 4.2(a).

Since $p_1$, $p_2$ and $p_3$ are made rigid in the first round, they must lie on a line and their positions must be unique (up to translation and reflection) after the first round. In the present configuration of the 7-cycle (Figure 4.2(a)), $p_1$, $q_1$, $r_1$, $s$ and $r_2$ are on the same side of the layer graph. Since $p_1$ and $s$ are collinear and they lie on a line perpendicular to the line through $p_1$, $p_2$ and $p_3$, the edges $p_3q_3$, $q_3r_3$ and $r_3s$ can have 4 distinct configurations giving rise to 4 distinct layer graph representations (Figure 4.4) of the whole $G_{3p}$ with the layer graph of the 7-cycle being in the configuration of Figure 4.2(a).

Thus, in order to be able to draw the layer graph of the 7-cycle $(p_1, q_1, r_1, s, r_2, q_2, p_2)$ in the configuration of Figure 4.2(a) the layer graph of the whole $G_{3p}$ must have one of the four distinct configurations as shown in Figure 4.4.

First, we consider the configuration where $p_3q_3$ and $r_3s$ are horizontal, and $q_3r_3$ is vertical (Figure 4.4a). The condition $|p_1p_2| \neq |q_2r_2|$ prevents the 7-cycle from being drawn as a layer graph of present configuration. However, it involves the edge $q_2r_2$ which we need to avoid. In the present configuration of the layer graph of the $G_{3p}$ $p_1, q_1, r_1, s$ and $r_2$ are on a line

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which is parallel to $p_2q_2$ and $q_3r_3$. So, we must have $|q_2r_2| = ||p_2p_3| \pm |p_3q_3| \pm |r_3s||$. Using this the condition becomes $|p_1p_2| \neq ||p_2p_3| \pm |p_3q_3| \pm |r_3s||$. Since $||p_1p_2| \pm |p_2p_3|| = |p_1p_3|$ the condition reduces to $|p_1p_3| \neq ||p_3q_3| \pm |r_3s||$. If we ensure this condition then we must have $|p_1p_2| \neq |q_2r_2|$ in the present configuration of $G_{3p}$. Thus, $G_{3p}$ in general and the 7-cycle in particular cannot be drawn as a layer graph in the present configurations of the 7-cycle and $G_{3p}$.

![Layer graphs](image)

Figure 4.4: Layer graphs of $G_{3p}$ when the layer graph of the 7-cycle $(p_1, q_1, r_1, s, r_2, q_2, p_2)$ has 4 edges $p_1q_1$, $q_1r_1$, $r_1s$ and $sr_2$ on one side

Now we consider the case when $p_3q_3$ and $q_3r_3$ are vertical, and $r_3s$ is horizontal (Figure 4.4b). In the present configuration of the layer graph of $G_{3p}$, $p_1, q_1, r_1, s$ and $r_2$ are on a line, and $p_3q_3$ and $q_3r_3$ are on a line. Those lines are parallel and they are parallel to $p_2q_2$. So, we must have $|q_2r_2| = ||p_2p_3| \pm |r_3s||$. Using this the condition becomes $|p_1p_2| \neq ||p_2p_3| \pm |r_3s||$. We have $||p_1p_2| \pm |p_2p_3|| = |p_1p_3|$. Using this the rigidity condition
\(|p_1 p_2| \neq |q_2 r_2|\) becomes \(|p_1 p_3| \neq |r_3 s|\).

Next, we consider the case when \(p_3 q_3\) is vertical, and \(q_3 r_3\) and \(r_3 s\) are horizontal (Figure 4.4c). The condition \(||p_1 q_1| \pm |q_1 r_1| \pm |r_1 s| \pm |s r_2|| \neq |p_2 q_2|\) prevents the 7-cycle from being drawn as a layer graph of present configuration. However, it involves the edge \(q_1 r_1\) which we need to avoid. In the present configuration of the layer graph of \(G_{3p}\) \(p_1, p_2\) and \(p_3\) are on a line, and \(q_3, r_3\) and \(s\) are on a line. The lines are parallel. So, we must have \(||p_1 q_1| \pm |q_1 r_1| \pm |r_1 s|\| = |p_3 q_3|\). Using this the condition becomes \(||p_3 q_3| \pm |s r_2|| \neq |p_2 q_2|\).

Finally, we consider the case when \(p_3 q_3\) is vertical, \(q_3 r_3\) is horizontal and \(r_3 s\) is vertical (Figure 4.4d). In the present configuration of the layer graph of \(G_{3p}\) \(p_1, p_2\) and \(p_3\) are on a line. The line is parallel to \(q_3 r_3\). So, we must have \(||p_1 q_1| \pm |q_1 r_1| \pm |r_1 s| \pm |s r_3|| = |p_3 q_3|\). Using this the rigidity condition \(||p_1 q_1| \pm |q_1 r_1| \pm |r_1 s| \pm |s r_2|| \neq |p_2 q_2|\) becomes \(||p_3 q_3| \pm |s r_3| \pm |s r_2|| \neq |p_2 q_2|\).

It follows that there is no layer graph for \(G_{3p}\) in which the layer graph in Figure 4.2(a) of the 7-cycle \((p_1, q_1, r_1, s, r_2, q_2, p_2)\) is embedded if the edges of \(G_{3p}\) satisfy the conditions in the statement of this lemma. So, the 7-cycle \((p_1, q_1, r_1, s, r_2, q_2, p_2)\) of \(G_{3p}\) cannot be drawn as the layer graph of Figure 4.2(a) if the edges of \(G_{3p}\) satisfy those conditions. \(\square\)

Similarly, we can replace the remaining 3 conditions corresponding to the layer graphs in group 1 and involving the edges \(q_1 r_1\) and \(q_2 r_2\). The result is summarized in the following lemma:

**Lemma 58** The 7-cycle \((p_1, q_1, r_1, s, r_2, q_2, p_2)\) of \(G_{3p}\) cannot be drawn as the layer graphs
corresponding to the conditions $|p_1p_2| \neq |q_1r_1|$, $|q_2r_2| \neq |r_1s|$ and $|q_1r_1| \neq |r_2s|$ if the edges of $G_{3p}$ satisfy the following conditions:

1. $||p_1p_3| \pm |r_3s|| \neq |p_3q_3|, |r_3s| \neq ||p_2p_3| \pm |r_2s| \pm |p_2q_2||$, $|p_3q_3| \neq |r_1s|$ and $|p_3q_3| \neq ||r_1s| \pm |r_3s||$.

2. $||p_2p_3| \pm |r_3s|| \neq |p_3q_3|, |r_3s| \neq ||p_1p_3| \pm |r_1s| \pm |p_1q_1||$, $|p_3q_3| \neq |r_2s|$ and $|p_3q_3| \neq ||r_2s| \pm |r_3s||$.

3. $||p_3q_3| \pm |r_3s|| \neq |p_2p_3|, |p_2p_3| \neq |r_2s|$, $|p_3q_3| \neq ||p_1q_1| \pm |r_1s|| and $|p_3q_3| \neq |r_3s|$ if the edges of $G_{3p}$ satisfy the following conditions:

Proof. Similar to the proof of Lemma 57.

Similarly, we can replace the 4 conditions corresponding to the layer graphs in group 2 and involving the edges $q_1r_1$ and $q_2r_2$. The result is summarized in the following lemma:

**Lemma 59** The 7-cycle $(p_1, q_1, r_1, s, r_2, q_2, p_2)$ of $G_{3p}$ cannot be drawn as the layer graphs corresponding to the conditions $||p_2q_2| \pm |q_2r_2|| \neq |r_1s|$, $|p_1p_2| \neq ||q_1r_1| \pm |r_1s||$, $||p_1q_1| \pm |q_1r_1|| \neq |p_2q_2|$ and $||p_1q_1| \pm |p_1p_2|| \neq |q_2r_2|$ if the edges of $G_{3p}$ satisfy the following conditions:

1. $||p_3q_3| \pm |r_3s|| \neq |p_2p_3| \pm |r_2s||$, $||p_2p_3| \pm |r_2s|| \neq |r_3s|$, $|p_3q_3| \neq |r_1s|$ and $|p_3q_3| \neq |r_1s| \pm |r_3s||$. 

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2. $|p_3q_3| ± |r_3s| \neq |p_2p_3|, |p_2p_3| \neq |r_3s|, |p_1q_1| \neq |r_1s|, |p_2q_2| \neq |r_2s|, |p_3q_3| \neq |p_1q_1| \text{ and } |p_3q_3| ± |p_1q_1| \neq |r_3s|.$

3. $|p_3q_3| ± |r_3s| ± |r_1s| \neq |p_1p_3|, |p_2p_2| \neq |r_2s|, |p_1p_3| \neq |r_1s| \pm |r_3s|, |p_2q_2| \neq |p_3q_3|, |p_1q_1| \neq |r_1s| \text{ and } |p_2q_2| \neq |p_3q_3| ± |r_3s|.$

4. $|p_3q_3| ± |r_3s| ± |p_1q_1| \neq |p_1p_3|, |p_2p_2| \neq |r_2s|, |p_1q_1| ± |p_1p_3| \neq |r_3s|, |p_2q_2| ± |r_2s| \neq |p_3q_3|, |p_1q_1| \neq |r_1s| \text{ and } |p_2q_2| ± |r_2s| \neq |p_3q_3| ± |r_3s|.$

**Proof.** Similar to the proof of Lemma 57. □

Replacing $|p_2q_2| ± |p_1p_2| \neq |q_1r_1|$

Now we replace the condition $|p_2q_2| ± |p_1p_2| \neq |q_1r_1|$ of group 3. Corresponding layer graph of the 7-cycle as well as all the possible configurations of $G_{3p}$ for this case are shown in Figure 4.5. From the figure we obtain the replacement conditions as before:

![Figure 4.5](image)

Figure 4.5: 2 edges are on one side of layer graph and 3 edges are on its adjacent side

**Lemma 60** The 7-cycle $(p_1, q_1, r_1, s, r_2, q_2, p_2)$ of $G_{3p}$ cannot be drawn as the layer graph of Figure 4.5 corresponding to the condition $|p_2q_2| ± |p_1p_2| \neq |q_1r_1|$ if the edges of $G_{3p}$
satisfy the following conditions:

\[ |p_3q_3| \pm |r_3s| \pm |p_2q_2| \neq |p_2p_3|, \ |p_1q_1| \neq |r_1s|, \ |p_2q_2| \pm |p_2p_3| \neq |r_3s|, \ |p_1q_1| \pm |r_1s| \neq |p_3q_3|, \]
\[ |p_2q_2| \neq |r_2s| \text{ and } |p_1q_1| \pm |r_1s| \neq |p_3q_3| \pm |r_3s|. \]

**Proof.** Similar to the proof of Lemma 57. \(\square\)

Similarly, we can replace the remaining 3 conditions corresponding to the layer graphs in group 3 and involving the edges \(q_1r_1\) and \(q_2r_2\). The result is summarized in the following lemma:

**Lemma 61** The 7-cycle \((p_1, q_1, r_1, s, r_2, q_2, p_2)\) of \(G_{3p}\) cannot be drawn as the layer graphs corresponding to the conditions \(||p_1q_1| \pm |q_1r_1|| \neq |r_2s|, \ |p_1p_2| \neq ||q_2q_2| \pm |r_2s|| \text{ and } ||p_2q_2| \pm |q_2r_2|| \neq |p_2q_2|\) if the edges of \(G_{3p}\) satisfy the following conditions:

1. \(||p_3q_3| \pm |r_3s| \pm |p_2q_2| \neq |p_2p_3|, \ |p_1q_1| \neq |r_1s|, \ |p_2q_2| \neq |r_2s| \pm |r_3s|, \ |p_1q_1| \neq |p_3q_3|, \]
\[ |p_2q_2| \neq |r_2s| \text{ and } |p_1q_1| \neq |p_3q_3| \pm |r_3s|. \]

2. \(||p_3q_3| \pm |r_3s|| \neq |p_1p_3|, \ |p_1p_3| \neq |r_3s|, \ |p_2q_2| \neq |r_2s|, \ |p_1q_1| \neq |r_1s|, \ |p_3q_3| \neq |p_2q_2| \text{ and } \]
\[ ||p_3q_3| \pm |p_2q_2| \neq |r_3s||. \]

3. \(||p_3q_3| \pm |r_3s|| \neq |p_1p_3| \pm |r_1s||, \ |p_1p_3| \pm |r_1s| \neq |r_3s|, \ |p_3q_3| \neq |r_2s| \text{ and } |p_3q_3| \neq \]
\[ |r_2s| \pm |r_3s||. \]

**Proof.** Similar to the proof of Lemma 57. \(\square\)
Similarly, we can replace the 2 conditions corresponding to the layer graphs in group 4 and involving the edges $q_1r_1$ and $q_2r_2$. The result is summarized in the following lemma:

**Lemma 62** The 7-cycle $(p_1,q_1,r_1,s,r_2,q_2,p_2)$ of $G_{3p}$ cannot be drawn as the layer graphs corresponding to the conditions $|p_2q_2| \neq |q_1r_1|$ and $|p_1q_1| \neq |q_2r_2|$. if the edges of $G_{3p}$ satisfy the following conditions:

1. $||p_3q_3| \pm |r_3s| \pm |r_1s|| \neq ||p_1p_3| \pm |p_1q_1||$, $||r_3s| \pm |r_1s|| \neq ||p_1p_3| \pm |p_1q_1||$, $|p_3q_3| \neq |p_2q_2|$, $||p_3q_3| \pm |r_3s|| \neq |p_2q_2|$ and $||p_1q_1| \pm |p_1p_3|| \neq |r_1s|$.

2. $||p_3q_3| \pm |r_3s| \pm |r_2s|| \neq ||p_2p_3| \pm |p_2q_2||$, $||r_3s| \pm |r_2s|| \neq ||p_2p_3| \pm |p_2q_2||$, $|p_3q_3| \neq |p_1q_1|$, $||p_3q_3| \pm |r_3s|| \neq |p_1q_1|$ and $||p_2q_2| \pm |p_2p_3|| \neq |r_2s|$.

**Proof.** Similar to the proof of Lemma 57.

Similarly, we can replace the 2 conditions corresponding to the layer graphs in group 5 and involving the edges $q_1r_1$ and $q_2r_2$. The result is summarized in the following lemma:

**Lemma 63** The 7-cycle $(p_1,q_1,r_1,s,r_2,q_2,p_2)$ of $G_{3p}$ cannot be drawn as the layer graphs corresponding to the conditions $|p_1q_1| \neq |q_2r_2| \pm |r_2s|$ and $|p_2q_2| \neq ||q_1r_1| \pm |r_1s|$ if the edges of $G_{3p}$ satisfy the following conditions:

1. $||p_2q_2| \pm |p_3q_3| \pm |p_2p_3|| \neq |r_3s|$, $||p_2q_2| \pm |p_2p_3|| \neq |r_3s|$, $|p_1q_1| \neq |p_3q_3|$ and $|p_1q_1| \neq $ $||p_3q_3| \pm |r_3s||$. 

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2. \(|p_1q_1| \pm |p_3q_3| \pm |p_1p_3| \neq |r_3s|, \ ||p_1q_1| \pm |p_1p_3| \neq |r_3s|, \ |p_2q_2| \neq |p_3q_3| \text{ and } |p_2q_2| \neq \ ||p_3q_3| \pm |r_3s||.

**Proof.** Similar to the proof of Lemma 57. \(\square\)

Replacing \(||p_1q_1| \pm |q_1r_1|| \neq ||p_2q_2| \pm |r_2s||\)

Now we replace the condition \(||p_1q_1| \pm |q_1r_1|| \neq ||p_2q_2| \pm |r_2s||\) of group 6. Corresponding layer graph of the 7-cycle as well as all the possible configurations of \(G_{3p}\) for this case are shown in Figure 4.6. From the figure we obtain the replacement conditions as before:

![Figure 4.6: The layer graph is staircase shaped with \(p_1q_1\) and \(q_1r_1\) on one side](image)

**Lemma 64** The 7-cycle \((p_1, q_1, r_1, s, r_2, q_2, p_2)\) of \(G_{3p}\) cannot be drawn as the layer graph of Figure 4.6 corresponding to the condition \(||p_1q_1| \pm |q_1r_1|| \neq ||p_2q_2| \pm |r_2s||\) if the edges of \(G_{3p}\) satisfy the following conditions:

\(||p_3q_3| \pm |r_3s| \pm |r_1s|| \neq |p_1p_3|, \ ||r_3s| \pm |r_1s|| \neq |p_1p_3|, \ |p_3q_3| \neq |p_2q_2| \pm |r_2s||, \ |p_3q_3| \neq ||p_2q_2| \pm |r_2s||\) and \(|p_1p_3| \neq |r_1s||.

**Proof.** Similar to the proof of Lemma 57. \(\square\)

Similarly, we can replace the remaining 3 conditions corresponding to the layer graphs in group 6 and involving the edges \(q_1r_1\) and \(q_2r_2\). The result is summarized in the following
Lemma 65 The 7-cycle \((p_1, q_1, r_1, s, r_2, q_2, p_2)\) of \(G_{3p}\) cannot be drawn as the layer graphs corresponding to the conditions \(|p_2q_2| \pm |q_2r_2| \neq |p_1q_1| \pm |r_1s|\), \(|q_1r_1| \neq |p_2q_2| \pm |r_2s|\) and \(|q_2r_2| \neq |p_1q_1| \pm |r_1s|\) if the edges of \(G_{3p}\) satisfy the following conditions:

1. \(|p_1q_1| \pm |p_1p_3| \neq |p_3q_3| \pm |r_1s| \pm |r_3s|\), \(|p_1q_1| \pm |p_1p_3| \neq |r_1s| \pm |r_3s|\), \(|p_3q_3| \neq |p_2q_2| \pm |r_2s|\), \(|p_3q_3| \neq |p_2q_2| \pm |r_2s| \pm |r_3s|\) and \(|p_1q_1| \neq |p_1p_3| \pm |r_1s|\).

2. \(|p_2q_2| \pm |p_2p_3| \neq |p_3q_3| \pm |r_2s| \pm |r_3s|\), \(|p_2q_2| \pm |p_2p_3| \neq |r_2s| \pm |r_3s|\), \(|p_3q_3| \neq |p_1q_1| \pm |r_1s| \pm |r_3s|\) and \(|p_2q_2| \neq |p_2p_3| \pm |r_2s|\).

3. \(|p_3q_3| \pm |r_3s| \pm |r_2s| \neq |p_2p_3|\), \(|r_3s| \pm |r_2s| \neq |p_2p_3|\), \(|p_3q_3| \neq |p_1q_1| \pm |r_1s|\), \(|p_3q_3| \neq |p_1q_1| \pm |r_1s| \pm |r_3s|\) and \(|p_2p_3| \neq |r_2s|\).

Proof. Similar to the proof of Lemma 57.

4.2.2 Rigidity Conditions

From Eq. 4.1 and Lemmas 56 - 65 we have the following lemma for the rigidity of the 7-cycle \((p_1, q_1, r_1, s, r_2, q_2, p_2)\) of \(G_{3p}\).

Lemma 66 The 7-cycle \((p_1, q_1, r_1, s, r_2, q_2, p_2)\) of \(G_{3p}\) is rigid if the edges of \(G_{3p}\) satisfy the following conditions:

1. \(|p_1p_2| \notin \{|r_1s|, |r_2s|, |r_1s| \pm |r_2s|\}\),

2. \(|p_2p_3| \notin \{|r_2s|, |r_3s|, |r_2s| \pm |r_3s|\}\),
3. \(|p_3p_1| \notin \{|r_3s|, |r_1s|, |r_3s| \pm |r_1s|\}|.

4. \(|p_1q_1| \notin \{|r_1s|, |r_2s|, |r_1s| \pm |r_2s|, |p_1p_2| \pm |r_1s|, |p_1p_2| \pm |r_2s|, |p_1p_3| \pm |r_1s|, |p_1p_3| \pm |r_3s|, |p_1p_2| \pm |r_2s|, |p_1p_3| \pm |r_3s|\}|.

5. \(|p_2q_2| \notin \{|r_1s|, |r_2s|, |p_1q_1|, |r_1s| \pm |r_2s|, |p_1p_2| \pm |r_1s|, |p_1p_2| \pm |r_2s|, |p_2p_3| \pm |r_2s|, |p_2p_3| \pm |r_3s|\}|.

6. \(|p_3q_3| \notin \{|r_1s|, |r_2s|, |r_3s|, |p_1q_1|, |p_2q_2|, |r_2s| \pm |r_3s|, |r_3s| \pm |r_1s|, |p_1p_3| \pm |r_3s|, |p_2p_3| \pm |r_3s|, |p_1q_1| \pm |r_3s|, |p_2p_3| \pm |r_3s|, |p_1q_1| \pm |r_3s|, |p_2q_2| \pm |r_3s|, |p_1q_1| \pm |r_3s|\}|.

The union of the two sets of conditions in Eq. 4.1 and Lemma 66 constitutes a set of sufficient conditions for the rigidity of \(G_{3p}\). Taking care of overlapping conditions between the two sets of conditions, we have 55 distinct conditions for the rigidity of \(G_{3p}\) and hence the following lemma:

**Lemma 67** The \(G_{3p}\) having the vertices \(p_1, p_2\) and \(p_3\) rigid in the first round, is rigid if its edges satisfy the conditions mentioned in Lemma 66.

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4.2.3 Algorithm

As mentioned before, we make triplet of vertices \((p_1, p_2, p_3)\) of each \(G_{3p}\) rigid in the first round. But we have rigidity conditions on the edges \(p_1p_2, p_2p_3\) and \(p_3p_1\) (Conditions 1-3 of Lemma 67). This implies that we need a pool of vertices, \(S\), for which the pairwise distances of all the pairs of points corresponding to the vertices in \(S\) are known after the first round of query, and from which we choose the triplet of vertices \((p_1, p_2, p_3)\) in order to meet the rigidity conditions on \(p_1p_2, p_2p_3\) and \(p_3p_1\). We make the vertices in \(S\) rigid in the first round. Then the pairwise distances of all the pairs of points corresponding to the vertices in \(S\) are known after the first round of query. We make the remaining 7 vertices of each \(G_{3p}\) rigid in the second round.

To select triplet of vertices in \(S\) as \((p_1, p_2, p_3)\) of a \(G_{3p}\), let us select any vertex of \(S\) as \(p_1\). Then let us find another vertex of \(S\), we denote it as \(p_2\), satisfying the conditions on the length \(|p_1p_2|\) mentioned in Condition 1 of Lemma 67. The length of \(p_1p_2\) cannot be equal to at most 4 different lengths. By Observation 4, each length can be attained by at most 2 edges incident on \(p_1\). Thus, at most 8 edges will not satisfy the conditions on \(|p_1p_2|\). We need at least 8 extra vertices, i.e., we need to have a total of at least 9 more vertices, other than \(p_1\), in \(S\) as candidate for \(p_2\).

After \(p_2\) is selected, let us find another vertex of \(S\), we denote it as \(p_3\), from the remaining vertices of \(S\) such that the conditions on \(|p_2p_3|\) in Condition 2 of Lemma 67 are satisfied. By Observation 4, at most 8 edges will not satisfy the conditions on \(|p_2p_3|\). This warrants the
set $S$ to have at least 8 extra vertices other than $p_1$, $p_2$ and $p_3$. The vertex $p_3$ selected this way by satisfying the conditions on $p_2p_3$ must also have to satisfy the conditions on $p_3p_1$ mentioned in Condition 3 of Lemma 67. By Observation 4, at most 8 edges will not satisfy the conditions on $|p_3p_1|$. This warrants the set $S$ to have at least 8 more extra vertices, i.e., a total of 16 extra vertices, other than $p_1$, $p_2$ and $p_3$. Then it is ensured that a triplet of vertices in $S$ can be found as $(p_1, p_2, p_3)$ of a $G_{3p}$.

But if $S$ has only 19 vertices for the selection of $p_i$s it may happen that all the $G_{3p}s$ are attached to the same triplets. This hinders our goal of obtaining a better value for $\alpha$ than previously known. We need to attach $G_{3p}s$ evenly to all the vertices of $S$ so that the same number of edges can be attached to each of them in the first round and all of those edges, except for a constant number, are used to attach the basic components. In other words, we need to attach the 3-paths to the vertices in $S$ in such a way that the numbers of $G_{3p}s$ attached to any two vertices differ by at most a constant number. To specify the number of $G_{3p}s$ attached to a vertex in $S$ we shall use the term valence. We denote the set of vertices with valence $d$ as $S_d$.

Now we describe our algorithm to select triplets of vertices in $S$ to attach $G_{3p}s$. To attach a $G_{3p}$ we always select a vertex in $S$ with the lowest valence as the first vertex (say $p_1$). Of the remaining vertices of $S$, at most 8 vertices may not be acceptable for the second vertex (say $p_2$), because of the conditions on $p_1p_2$. From among the rest $|S| - 1$ vertices that satisfy the conditions on $p_1p_2$ we select the one that has the lowest valence, as $p_2$. 

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Of the rest \(|S| - 2\) vertices of \(S\), at most 16 may not be acceptable for the last vertex, say \(p_3\), because of the conditions on \(p_2p_3\) and \(p_3p_1\). From among the rest vertices that satisfy the conditions on \(p_2p_3\) and \(p_3p_1\) we choose the one that has the lowest valence, as \(p_3\). This method is followed to attach each \(G_{3p}\) to the vertices in \(S\), while \(G_{3p}\)s are attached sequentially. The following lemma tells us how big \(S\) must be:

**Lemma 68** A set \(S\) of 35 vertices is sufficient to ensure that the valences of any two vertices in \(S\) differ by at most 2.

**Proof.** Initially, all the vertices in the pool have valence 0. To pick three vertices of minimum valence (0 in this case) to beat 16 different length restrictions in all, we need a buffer of size at least 16. Thus we can pick 6 triplets \((p_1, p_2, p_3)\) of valence 0, until the buffer limit is reached. At the end of this cycle, 18 vertices have valence 1 and 17 vertices have valence 0. The next cycle begins by picking pairs \((p_1, p_2)\) from the pool of 17 vertices of valence 0, as long as we have a buffer of size 8. Since we do not have enough vertices in the buffer to ensure that the third vertex \(p_3\) is from the valence 0 pool, we might have to pick these from the pool of valence 1 vertices. So up to 4 vertices can have valence 2. Thus, we have \(|S_0| \leq 9\) and \(|S_2| \leq 4\). The rest of the vertices are of valence 1, i.e., \(|S_1| \geq 22\).

Next we attach \(G_{3p}\)s until \(|S_0| \leq 2\). This will attach at most 7 \(G_{3p}\)s. Then we have \(|S_0| \leq 2\) and \(|S_2| \leq 18\), and consequently, \(|S_1| \geq 15\). Again, we attach \(G_{3p}\)s until \(|S_0| = 0\). This will attach at most 2 \(G_{3p}\)s. Then we have \(|S_0| = 0\) and \(|S_3| \leq 2\), and consequently, \(|S_1 \cup S_2| \geq 33\). Thus, at some point of time all the vertices in \(S\) have at most 3 consecutive
valences, viz., 1, 2 and 3. At any point of time before that, they may have at most 4 consecutive valences, viz., 0 - 3.

We shall show that at any point of time the vertices in $S$ will have at most 4 consecutive valences, and that at some point of time they will have at most 3 consecutive valences only. For this we use induction to show that if we start with vertices in $S$ in the state of a valence distribution $(S_d, S_{d+1}, S_{d+2})$, and attach $G_{3p}$s according to our algorithm, then at some point of time the state of the valences will be $(S_{d+1}, S_{d+2}, S_{d+3})$. We assume that $|S_d \cup S_{d+1}| \leq 18$. Otherwise, we attach $G_{3p}$s until $|S_d \cup S_{d+1}| \leq 18$.

First, we consider the cases for which $|S_d| \leq 9$. Then $|S_{d+1} \cup S_{d+2}| \geq 26$ with $|S_d \cup S_{d+1} \cup S_{d+2}| = 35$. We attach $G_{3p}$s until $|S_d| = 0$. For each new $G_{3p}$, at least 1 vertex of $S_d$ will be moved to $S_{d+1}$, and at most 2 vertices of $S_{d+2}$ will be moved to $S_{d+3}$. It is clear that at most 9 $G_{3p}$s will be attached, and that there will always be at least 19 vertices in $S_d \cup S_{d+1} \cup S_{d+2}$ until there is no vertex in $S_d$. We have $|S_d| = 0$, $|S_{d+1} \cup S_{d+2}| \geq 17$ and $|S_{d+3}| \leq 18$. Thus, the valences of all the vertices will become $d + 1$, $d + 2$ and $d + 3$.

Now we consider the worst case for which $|S_d| = 18$ and $|S_{d+1}| = 0$. They imply that $|S_{d+2}| = 17$. We attach $G_{3p}$s until $|S_d| \leq 10$. At most 4 $G_{3p}$s will be attached. We group all the possible situations into 2 subcases. First, we consider the subcase when 2 vertices are used from $S_d$ for each new $G_{3p}$. Exactly 4 $G_{3p}$s will be attached using 8 vertices from $S_d$. We have $|S_d| = 10$ and $|S_{d+1}| \geq 5$ with $|S_d \cup S_{d+1}| \geq 15$, and $|S_{d+3}| \leq 4$. After attachment of 1 more $G_{3p}$ we have $|S_d| \leq 8$ and $|S_{d+1}| \geq 6$ with $|S_d \cup S_{d+1}| \geq 14$, and $|S_{d+3}| \leq 5$. 

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Now we attach $G_{3p}$s until $|S_d| \leq 5$. Clearly, at most 3 $G_{3p}$s will be attached, and we have

$|S_d| \leq 5$ and $|S_{d+1}| \geq 3$ with $|S_d \cup S_{d+1}| \geq 8$ (because at most 6 valence $d + 1$ vertices will be raised to valence $d + 2$ vertices), and $|S_{d+3}| \leq 8$ (because at most 3 valence $d + 2$ vertices will be raised to valence $d + 3$ vertices). As long as there are at least 19 vertices in $S_d \cup S_{d+1} \cup S_{d+2}$, all the 3 vertices of a new $G_{3p}$ will be chosen from that union. No vertices will be used from $S_{d+3}$, and hence no vertex’s valence will be raised to $d + 4$. We attach $G_{3p}$s until $|S_d| = 0$. It is evident that at most 5 $G_{3p}$ will be attached, and we have $|S_d| = 0$, $|S_{d+1} \cup S_{d+2}| \geq 17$ and $|S_{d+3}| \leq 18$.

Now we consider the other subcase which consists of the remaining possible situations. For this case, 3 or 4 $G_{3p}$s will be attached. It can be easily seen that $|S_d| \leq 9$ and $|S_{d+1}| \geq 6$ with $|S_d \cup S_{d+1}| \geq 15$, and $|S_{d+3}| \leq 3$. We attach $G_{3p}$s until $|S_d| \leq 6$. It can be easily checked that at most 3 $G_{3p}$s will be attached, and we have $|S_d| \leq 6$ and $|S_{d+1}| \geq 3$ with $|S_d \cup S_{d+1}| \geq 9$, and $|S_{d+3}| \leq 6$. We attach $G_{3p}$s until $|S_d| = 0$. It is evident that at most 6 $G_{3p}$s will be attached, and we have $|S_d| = 0$, $|S_{d+1} \cup S_{d+2}| \geq 17$ and $|S_{d+3}| \leq 18$.

It can be easily shown that for all the other combinations of number of vertices in valences $d$ and $d + 1$ subject to a maximum of 18, all the vertices will be elevated to at most 3 consecutive valences $d + 1$, $d + 2$ and $d + 3$. The calculations will be similar to the above.

We make the above set $S$ of 35 vertices rigid in the first round by using jewels of Damaschke [26] as the ppg. We create 6 jewels hanging from a common strut that is
incident on 2 vertices of $S$. This will make 32 vertices rigid. For this we need to query the lengths of 49 edge. We make the remaining 3 vertices rigid by using triangle as the $ppg$. For each of these 3 vertices we query its distance from each of the pair of vertices that are incident on the strut. There will be 6 more queries for edge lengths. Thus, we shall query a total of 55 edges in the first round to make the 35 vertices of $S$ rigid in that round.

The conditions on $p_1q_1$, $p_2q_2$ and $p_3q_3$ in serial numbers respectively 3, 4 and 5 of Lemma 67 will not be satisfied by at most 40, 90 and 122 edges respectively (by Observation 4). In addition to the 122 extra edges needed at each of $p_i$’s to satisfy the conditions on $|p_1q_1|$, $|p_2q_2|$ and $|p_3q_3|$ we need 2 more extra edges incident on each of $p_i$ to accommodate the difference of 2 between the number of basic components that can be attached to the $p_i$’s. Thus, we need a total of 124 extra edges incident on each of the vertices $p_i$, $i = 1, ..., 35$ of $S$. We shall attach $3b$, $3b + 1$ or $3b + 2$ (where $b$ is a positive integer) number of $G_{3p}$s to each vertex in $S$. This requires us to have $3b + 124$ edges incident on each of $p_i$’s in $S$.

In the worst case there will be at most 18 vertices in $S$ with valence $3b$, no vertices in $S$ with valence $3b + 1$ and the remaining vertices with valence $3b + 2$. Thus, we shall be able to construct a total of at least $35b + 11$ number of $G_{3p}$s from the edges provided for $p_iq_i$ at all the $p_i$’s in $S$. Now we describe the algorithm to construct a composite $ppg$ made up of $G_{3p}$s such that all the rigidity conditions listed in Lemma 67 are satisfied for each of them.

**Algorithm 4.1.** Let the total number of vertices be $n = 245b + 4,419$, where $b$ is a positive integer. We attach at least $3b$ and at most $3b + 2$ numbers of $G_{3p}$s (Figure 4.1)
to each of 35 rigid vertices in $S$ subject to the condition that the total number of such components being $35b + 11$.

In the first round, we make distance queries represented by the edges of the graph in Figure 4.7. All the vertices $p_i$ ($i = 1, \ldots, 35$) in the subgraph enclosed by the rectangle are elements of $S$ and are made rigid in the first round by using the jewel of [26] as the ppg. There are 6 jewels attached to a common strut in the subgraph. Residual 3 vertices are made rigid by using triangle as the ppg. They are attached to the common strut. There are a total of 55 edges in the subgraph. There are $b + 124$ leaf children rooted at each of the vertices $p_i, p_j, \text{ or } p_k$ ($i, j, k = 1, \ldots, 35$) of $S$ to attach $3b$, $3b + 1$ or $3b + 2$ $G_{3p}$s (Figure 4.1). Since there will be $35b + 11$ $G_{3p}$s we make $35b + 11$ groups of 4 vertices $(r_{il}, r_{jl}, r_{kl}, s_l)$, $(l = 1, \ldots, 35b + 11)$. We query the distances $|r_{il}s_l|$, $|r_{jl}s_l|$ and $|r_{kl}s_l|$, $(l = 1, \ldots, 35b + 11)$ in the first round. We will make a total of $210b + 4,428$ pairwise distance queries in the first round for the placement of $n = 245b + 4,419$ vertices.

![Diagram of rigid subgraph of 35 vertices](image-url)

Figure 4.7: Queries in the first round for 2-round algorithm using $G_{3p}$ as the basic component
In the second round, for each 3-link \((r_{il}, r_{jl}, r_{kl}, s_l), l = 1, ..., 35b + 11\), we construct a \(G_{3p}\) (Figure 4.1), satisfying all its rigidity conditions as in Lemma 67. For each such 3-link we select a vertex \(p_i\), from the subgraph of 35 vertices of \(S\) that has the lowest valence of \(G_{3p}\) of Figure 4.1. Since all the 35 vertices \(p_i, i = 1, ..., 35\), are rigid in the first round, for any pair of such fixed vertices \((p_i, p_j) (i, j = 1, ... 35; i \neq j)\) we can find the distance \(|p_i p_j|\).

So, for each pair of vertices \((p_i, p_j) (i, j = 1, ..., 35; i \neq j)\), we shall use \((p_i, p_j)\) as an edge in the construction of the \(G_{3p}\) of Figure 4.1.

Now from the subgraph of 35 vertices of \(S\) we select another vertex \(p_j (j \neq i)\) such that the length \(|p_i p_j|\) satisfies all the 4 conditions of rigidity on it as stated in serial number 1 of Lemma 67 and that it has the lowest valence of \(G_{3p}\) of Figure 4.1 among all such qualifying vertices. We note that we can always find such vertex \(p_j\), because there will be at most 8 edges \((p_i p_j)\) whose lengths do not satisfy the rigidity conditions on it (Lemma 67) whereas we have 34 more vertices for choosing the vertex \(p_j\). Similarly, from the subgraph of 35 vertices of \(S\) we select another vertex \(p_k (k \neq i, k \neq j)\) such that the length \(|p_j p_k|\) satisfies all the 4 conditions of rigidity on it as stated in serial number 2 of Lemma 67 and the length \(|p_k p_i|\) satisfies all the 4 conditions of rigidity on it as stated in serial number 3 of Lemma 67, and that it has the lowest valency of \(G_{3p}\) of Figure 4.1 among all such qualifying vertices. We note that we can always find such vertex \(p_k\), because there will be at most 16 vertices \(p_k\) such that the lengths of the edges \(p_j p_k\) and \(p_k p_i\) do not satisfy the rigidity conditions on them (Lemma 67) whereas we have 33 more vertices for choosing the vertex \(p_k\).
Then we find an edge \( p_i q_{il} \) rooted at \( p_i \) satisfying the 20 conditions of rigidity on it as stated in serial no. 4 of Lemma 67, then we find another edge \( p_j q_{jl} \) rooted at \( p_j \) satisfying the 45 conditions on it as stated in serial no. 5 of Lemma 67 and finally, we find another edge \( p_k q_{kl} \) rooted at \( p_k \) satisfying the 61 conditions on it as stated in serial no. 6 of Lemma 67.

Then for each \( l, (l = 1, \ldots, 35b + 11) \), we query the distances \( |q_{il}r_{il}|, |q_{jl}r_{jl}| \) and \( |q_{kl}r_{kl}| \) to form a \( G_{3p} \) with vertices \( p_i, p_j, p_k, q_{il}, q_{jl}, q_{kl}, r_{il}, r_{jl}, r_{kl} \) and \( s_l \). Its edges will satisfy all the rigidity conditions of Lemma 67. Thus, all the \( 35b + 11 \) 3-links will be consumed to construct \( 35b + 11 G_{3p} \)s. For this \( 105b + 33 \) edges will be queried in the second round.

There will be unused leaves \( q_{il}/q_{jl}/q_{kl} \) numbering 4,307 in total for the 35 vertices of \( S \). We use a 4-cycle \( ppg \) [26] to fix 4,306 of them and a triangle \( ppg \) to fix the rest 1 vertex in the second round. As before, for each pair of vertices \( (p_i, p_j)(i, j = 1, \ldots, 35; i \neq j) \), we shall use \( (p_i, p_j) \) as an edge in the construction of the 4-cycle. For each unused vertex \( q_{il} \) rooted at \( p_i \) we find another vertex \( q_{jl} \) rooted at \( p_j \) such that \( |p_ip_{il}| \neq |p_jp_{jl}| \). Then the 4-cycle \( p_i q_{il} q_{jl} p_j \) will be rigid (Observation 5). Then we query the distance \( |q_{il}q_{jl}| \) in the second round to complete the 4-cycle. Note that we can always find a vertex like \( q_{jl} \). For, after repeated selection of such matching pairs of edges there may remain at most 2 edges \( p_i q_{il} \) rooted at \( p_i \) of length equal to that of the same number of edges rooted at \( p_j \) (Observation 4). In such a situation we switch the matching to match such edges rooted at \( p_i \) with edges other than those same length edge\( s \) rooted at \( p_j \) - this is always possible because there are at most 2 edges rooted at \( p_j \) that have the same length (Observation 4). To make the remaining
1 leave vertex rigid we query in the second round its distance from any vertex of \( S \) other than its parent vertex.

For 4,307 unused vertices (after the construction of the \( G_{3,p} \)) 2,153 4-cycles and 1 triangle will be constructed. 2,153 edges will be queried to complete the 4-cycles and 1 edge will be queried to construct the triangle. The total number of queries in the second round will be \((105b + 33) + 2,153 + 1\), i.e., \(105b + 2,187\).

**Theorem 69** The ppg constructed by Algorithm 4.1 is rigid.

**Proof.** The proof is similar to that of Theorem 36 for the line rigidity of the ppg constructed by Algorithm 3.1. 

The number of queries in the first and second rounds are \(210b + 4,428\) and \(105b + 2,187\) respectively. Thus, in 2 rounds a total of \(315b + 6,615\) pairwise distances are to be queried for the placement of \(245b + 4,419\) points. Now, \(315b + 6,615 = (315/245) \times (245b + 4419) - (9/7) \times 4419 + 6615 = 9n/7 + (46305 - 39771)/7 = 9n/7 + 6534/7\). Thus, we have the following theorem:

**Theorem 70** \(9n/7 + 6534/7\) queries are sufficient to place \(n\) distinct points on a line in two rounds.

### 4.3 An Improved Lower Bound for Two Rounds

In this section we improve the lower bound for a 2-round algorithm to \(9/8\). Our argument is adversarial as in [27, 20]. Let \(A\) denote any 2-round algorithm. We imagine that an
adversary $B$ sets edge lengths for $ppg$ in each of the 2 rounds with the intention of maximizing its density and returns the distance between any two points queried by $A$. Let the set of edges queried in the first and second round be $E_1$ and $E_2$ respectively. Then $G_1 = (V, E_1)$ is the query graph for the first round, while $G_2 = (V, E_1 \cup E_2)$ is the $ppg$ $G$.

Let $<p_1, p_2, ..., p_k>$ denote a path of distinct degree 2 vertices in $G_i$. We call it simply as degree 2 path in $G_i$. We note that each vertex of a degree 2 path in $G_i$ is of degree 2 in $G_i$. Let $p_0$ and $p_{k+1}$ be the vertices adjacent to $p_1$ and $p_k$ respectively. If both $p_0$ and $p_{k+1}$ are not of degree 2, then the path is a maximal path of degree 2 vertices in $G_i$, and $p_0$ and $p_{k+1}$ are called start and end vertices respectively of the path. We call a maximal path of degree 2 vertices in $G_i$ simply as degree 2 maximal path in $G_i$. We call a vertex of degree at least 3 in $G_i$ as a heavy vertex in $G_i$. If both the start and end vertices of a degree 2 maximal path in $G_1$ are heavy in $G_1$ then we call the maximal path as class A path. We define the length of a degree 2 path in a graph $G_i$ as the number of vertices in the path in $G_i$. We note that an edge has a path length of 0. We use $P_k$ to denote a degree 2 path of length $k$.

A connected subgraph $H$ of $G_i$ ($i = 1, 2$) is called a handle [27] in $G_i$ if the layout of $H$ is ambiguous in the $i$-th round, though the layout of all the remaining vertices of $G_i$ are fixed in round $i$. For example, the subgraphs ($\{p_1\}, \emptyset$), ($\{p_1, p_2\}, \{p_1 p_2\}$) and ($\{p_1, p_2, p_3\}, \{p_1 p_2, p_2 p_3\}$) are handles in the the graphs whose layer graphs are shown in
Figures 4.8(a), 4.8(b) and 4.8(c) respectively. In the rest of the discussion of this section, handles play a critical role.

![Figure 4.8: Three different handles](image)

**Lemma 71**  (a) For each handle $H$ in $G_1$, the algorithm must insert an edge incident to at least one vertex of $H$ in round 2. (b) For each potential handle $H$ in $G_2$, the algorithm must insert an edge incident to at least one vertex of $H$ in round 2.

**Proof.** For each of the cases (a) and (b), suppose that the algorithm does not insert any edge at some vertices of $H$ in round 2, then the layout of $H$ will remain ambiguous in round 2. This contradicts the fact that $G_2$ is rigid.

Let us consider round 1. An algorithm constructs a $G_1$ and submits it to $B$. $B$ assigns lengths to the edges of $G_1$ by creating handles in $G_1$ and scopes for potential handles in $G_2$, with the intention of forcing the algorithm to insert as many edges as possible in round 2; and returns it to the algorithm. $B$ assigns lengths to the edges according to the following strategies. The algorithm is oblivious of the strategies.

**S1.** The adversary fixes the layout of all heavy vertices except the following 3 types of degree 3 vertices. Let $p_0$ be a vertex of degree 3. The exceptions are:

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4Heavy vertices are circled in the figures of this section
(1) The length of each of the 3 degree 2 maximal paths in $G_1$ connected to $p_0$ is at most 1 and the other terminals of the path are not heavy. (Figure 4.9)

(2) The vertex $p_0$ is connected to exactly one heavy vertex by a degree 2 maximal path of length 1 in $G_1$ and the length of each of the remaining 2 degree 2 maximal paths in $G_1$ connected to $p_0$ is at most 1 and remaining two terminals not heavy. (Figure 4.10)

(3) The vertex $p_0$ is adjacent either to exactly one heavy vertex in $G_1$, or to the start or end vertex of a class A path; and the length of each of the remaining 2 degree 2 maximal paths in $G_1$ connected to $p_0$ is exactly 1. (Figure 4.11)

We call the vertex $p_0$ of exception (3) as *specialOne* vertex and its adjacent vertex in $G_1$ as *specialTwo* vertex if it is heavy in $G_1$.

The motivation for these exceptions is to provide scope to create handles in round 2 where the vertex set of the handle includes the degree 3 vertex $p_0$ or some vertices of some degree 2 paths in $G_2$ attached to $p_0$. In Figures 4.9, 4.10 and 4.11 the vertex $p_0$ is an example of exceptions (1), (2) and (3) respectively for a degree 3 vertex whose layout is not fixed by $B$ in round 1. Exception (1) is used in Figures 4.19 and 4.20 to create the handle $\{(p_1, p_2), \{p_1p_2\}\}$ and $\{(p_2, p_3), \{p_2p_3\}\}$ respectively. Exception (2) is used in Figure 4.18 to create the handles with vertex set $\{p''_2, p''_1, p_0, p'_1, p'_2\}$ and edge set $\{p''_2p''_1, p''_1p_0, p_0p'_1, p'_1p'_2\}$.

**S2.** For all degree 2 vertices, if one of the incident edges is also incident on a degree 1 vertex, the adversary sets the length of one of the two incident edges to be the same,
say $c$, over all these degree 2 vertices.

The aim of this strategy is to provide scope to create a handle in each degree 2 path of length exactly 4 in $G_2$ having an edge in $E_2$ (see Figure 4.12). Then by Lemma 71, the algorithm must insert an edge at a vertex of the handle. Consequently, the path will be divided into 2 smaller paths of degree 2 vertices.

In Figure 4.12, $p_1$ and $p'_1$ are degree 2 vertices in $G_1$ and their incident edges $p_1p_2$ and $p'_1p'_2$ are incident to the degree 1 vertices $p_2$ and $p'_2$ respectively in $G_1$. For the degree 2 path ($p_1$) of length 1, $B$ sets either $|p_0p_1| = c$ or $|p_1p_2| = c$ in round 1 as per $S2$. In Figure 4.12 $B$ sets $|p_1p_2| = c$ as per $S2$. Similarly, for the degree 2 path ($p'_1$) of length 1, $B$ sets $|p'_1p'_2| = c$ in round 1 as per $S2$. In round 2, if the algorithm inserts an edge $p_2p'_2$, then the adversary sets $|p_2p'_2| = |p_1p'_1|$. This creates a handle ($\{p_2, p'_2\}, \{p_2p'_2\}$) in $G_2$. By
Lemma 71, the algorithm must insert an additional edge at either $p_2$ or $p'_2$. This will split the potential degree 2 path $< p_1, p_2, p'_2, p'_1 >$ of length 4 in $G_2$ into degree 2 paths, each of length at most 2.

![Figure 4.12: The subgraph $\{p_2, p'_2\}, \{p_2p'_2\}$ is a potential handle in $G_2$](image)

**S3.** For each degree 2 maximal path $P_k = < p_1, p_2, \ldots, p_k >$, $k \geq 2$, of length at least 2 in $G_1$, let $p_0$ and $p_{k+1}$ be non-degree 2 vertices in $G_1$ adjacent to $p_1$ and $p_k$ respectively in $G_1$. (a) The adversary sets $|p_{i-1}p_i| = |p_{i+1}p_{i+2}|$ for $i = 1 \pmod{3}$. In addition, (b) if $P_k$ is a class A path then $B$ fixes the layout of $p_i, i = 0 \pmod{3}$ and sets $|p_ip_{i+1}| = |p_{i-1}p_{i+2}|$ for $i = 1 \pmod{3}$, with the exception that if $p_0$ is a special vertex then $B$ keeps option for potentially fixing $p_0$ such that $|p_1p_2| = |p_0p_3|$, and that if $p_k+1$ is a special vertex and $k+1 = 0 \pmod{3}$ then $B$ keeps option for potentially fixing $p_{k+1}$ such that $|p_{k-1}p_k| = |p_{k-2}p_{k+1}|$; finally, (c) if at least one of them, say $p_{k+1}$, is of degree one the adversary sets the lengths of alternate edges equal.

Strategies $S3(a)$ and $S3(b)$ aim to create handles $\{p_i, p_{i+1}\}, \{p_ip_{i+1}\}$ for $i = 1 \pmod{3}$ in $G_1$ in each class A path (Figure 4.13). Then by Lemma 71, the algorithm must insert an edge at a vertex of each of the handles in round 2. The path will be divided into smaller degree 2 paths of length at most 3. We have the following lemma:
Lemma 72  [20] For each class A path, say \( < p_1, p_2, ..., p_k > \), \( k \geq 2 \), there exists at least one edge in \( E_2 \) incident to either \( p_i \) or \( p_{i+1} \) for \( i = 1 \) (mod 3) in \( G_2 \).

S4. (1) If a vertex, say \( p_0 \), of degree 3 has 2 degree 2 maximal paths the other ends of which are not attached to any heavy vertex, and if \( p_0 \) is incident on only one degree 2 maximal path of length 1 of which the other end is incident on a heavy vertex, then set the length of one of the edges of this third path as \( c \).

(2) If 2 specialOne vertices \( p_0 \) and \( p'_0 \) are adjacent in \( G_1 \) then set \( |p_0p'_0| = c \). If a specialTwo vertex \( p'_0 \) of degree 3 in \( G_1 \) has exactly 2 adjacent vertices of type specialOne then \( B \) sets the length of the edge incident to \( p'_0 \) and one of the specialOne vertices adjacent to \( p'_0 \) as \( c \). Let \( p_0 \) be any specialOne vertex and the 2 degree 2 paths of length 1 attached to it be \( < p_0, p_1, p_2 > \) and \( < p_0, p'_1, p'_2 > \). Then \( B \) sets \( |p_1p_2| = |p'_1p'_2| = c \).

Below we show that the application of \( S^2 \) and \( S^3 \) keeps edge lengths consistent:

Lemma 73 Strategies \( S^2 \) and \( S^3 \) of \( B \) are mutually consistent.

Proof. Consider a path \( P_k \) of degree 2 vertices in \( G_1 \) such that both \( p_0 \) and \( p_{k+1} \) have degree 1. If \( k = 1 \), only \( S^2 \) comes into play and in this case \( B \) sets \( |p_1p_2| = c \). For all \( k \geq 4 \),
\( B \) sets \( |p_1p_2| = c, |p_{k-1}p_k| = c \) in accordance with S2 and the lengths of all other edges in accordance with S3. Figures 4.14(c)-(f) serve as examples of this length assignment since for any \( k \), the total number of edges is a multiple of 3 as in Figure 4.14(d), or a multiple of 3 plus 1 as in Figure 4.14(e). or a multiple of 3 plus 2 as in Figure 4.14(f). For \( k = 2 \) and \( k = 3 \), \( B \) makes the length assignments as shown in Figures 4.14(a)-(b), which are again consistent with S2 and S3.

If one of \( p_0 \) and \( p_{k+1} \), say \( p_0 \), is heavy, then \( B \) does not have to set \( |p_1p_2| \) to \( c \). On the other hand, if only \( p_{k+1} \) is heavy then the length assignment is symmetrically reversed, i.e., starts from \( p_{k+1} \).

\[ \square \]

Figure 4.14: The residual parts of maximal paths of degree 2 vertices that will satisfy S2

Now we consider round 2. The algorithm completes the construction of \( G_2 \) by disam-
biguatiing existing handles in $G_1$ and potential handles in $G_2$ with the insertion of edges into $G_1$ so that $G_2$ becomes rigid and submits it to $B$; $B$ assigns lengths to all the edges of $E_2$ with an intention to make $G_2$ ambiguous and returns it to the algorithm.

For a maximal path of degree 2 vertices in $G_2$, as a consequence of $S_3$ there are limits on: (1) the maximum number of edges from $E_1$ if the path consists of edges from $E_1$ only (Figure 4.15 shows a degree 2 maximal path $<p_1, p_2, p_3, p_4, p_5, p_6>$ in $G_1$ with both the vertices $p_0$ and $p_7$ adjacent to start and end vertices $p_1$ and $p_{k+1}$ respectively being heavy), and (2) the maximum number of consecutive edges from $E_1$ if it contains at least one edge from $E_2$ (Figure 4.14 shows some degree 2 maximal paths in $G_1$ with none of the end vertices being heavy).

If both of $p_0$ and $p_{k+1}$ are heavy in $G_1$, then $B$ sets the above layout in such a way that if, for any $i$ with $i = 1 \pmod{3}$ and $i < k$, no edge is attached to either $p_i$ or $p_{i+1}$ in the second round their positions will be ambiguous. Thus, for this case the length of a maximal path of degree 2 vertices in $G_2$ containing only the edges in $E_1$ can be at most 3.

If at least one of $p_0$ and $p_{k+1}$, say $p_{k+1}$, is of degree one in $G_1$, then $B$ sets the above layout in such a way that if, for any $i$ with $i = 1 \pmod{2}$ and $i < k$, no edge is attached to either $p_i$ or $p_{i+1}$ in the second round, they can be made ambiguous by setting $|p_i p_{i+1}| = |p_{i-1} p_{i+2}|$ in the second round. Thus, for this case the length of a maximal path of degree 2 vertices in $G_2$ containing only the edges in $E_1$ can be at most 2.

If $p_{k+1}$ is of degree 1 in $G_1$ and no edge is attached to either $p_{k-1}$ or $p_k$ in the second
round, then the positions of \( p_{k-1} \) and \( p_k \) can be made ambiguous by setting \( |p_{k-1}p_k| = |p_{k-2}p_{k+1}| \) in that round. The algorithm must attach an edge in \( G_2 \) to \( p_{k-1} \) or \( p_k \). Still then there will be a handle with at most 2 vertices at an end of a path of degree 2 vertices if the end vertex is of degree 1. The algorithm must make them rigid in the second round by attaching an edge in \( E_2 \) to at least \( p_{k+1} \). Thus, we have the following lemma:

**Lemma 74** In a degree 2 maximal path in \( G_2 \) that contains at least one edge from \( E_2 \), there can be at most 2 consecutive edges from \( E_1 \).

![Diagram](image)

Figure 4.15: \(< p_1, p_2, p_3, p_4, p_5, p_6 > \) is a degree 2 maximal path in \( G_1 \) with both the end vertices being heavy. In the second round, the algorithm has to introduce edges at \( p_1 \) or \( p_2 \) to make them unambiguous, and at \( p_4 \) or \( p_5 \) to make them unambiguous. This will reduce the length of the degree 2 maximal path in \( G_2 \).

The above results together with \( S_2 \) and \( S_3 \) imply that Theorem 54 holds for the \( ppg \) [3].

The following theorem establishes the lower bound on the density of a \( ppg \) for any 2-round algorithm.

**Theorem 75** Any deterministic 2-round algorithm for solving the 1-dimensional point placement problem requires at least 9\( n/8 \) queries in the worst case.

**Proof.** We determine the average density in \( G_2 \) for each type of vertices in \( V \). For this, we categorize the vertices in \( V \) into two types: A and B as described below. For density
calculation of the vertices in $V$, each edge in $E_1$ and $E_2$ is split into two fractional edges.

The two incident vertices of the edge owns the two fractions. Each of the following edges of $E_1$ is split into $\frac{5}{8}$ and $\frac{3}{8}$ fractional edges:

(1) For a degree 3 specialTwo vertex $p_0$ in $G_1$ that is adjacent to 2 specialOne vertices in $G_1$, one of the incident edges between $p_0$ and its adjacent specialOne vertices.

(2) For a degree 3 specialTwo vertex $p_0$ in $G_1$ that is adjacent to 3 specialOne vertices in $G_1$, all the incident edges between $p_0$ and its adjacent specialOne vertices.

(3) For a specialTwo vertex $p_0$ of degree at least 4 in $G_1$, each of the incident edges between $p_0$ and its adjacent specialOne vertices.

For each of the above 3 cases the incident specialTwo vertex $p_0$ owns $\frac{3}{8}$ and the incident specialOne vertex owns $\frac{5}{8}$. Each of the remaining edges in $E_1$ and $E_2$ is divided into 2 equal halves. Each of its 2 incident vertices owns $1/2$ of the edge. If a vertex $p_0$ is divided into 2 equal halves, each half owns one half of the total number of edges owned by $p_0$.

A. Vertices in class A paths

For each class A path we compute the average density of its vertices (see Figure 4.16).

The density of a class A path is the ratio of the edges owned by all the vertices in the path and the number of vertices in the path. The minimum of the densities of all the class A paths will be the minimum density of all the vertices of type A.

For a class A path of length $k$, there must be at least one edge in $E_2$ incident to
Figure 4.16: Type A vertices are on the path \(< p_1, p_2, p_3, p_5, p_6, p_7 >\) of degree 2 vertices in \(G_1\). They are enclosed by a dash dotted polygon.

A vertex in each pair of vertices \(p_i, p_{i+1}\) \((i = 1 \pmod{3})\), (by Lemma 72), and

\[
\rho = \frac{1}{k}[k+\left\lfloor \frac{k+1}{3} \right\rfloor \times \frac{1}{2}] \geq \frac{1}{k}[k+\left\lfloor \frac{k+1}{3} - \frac{2}{3} \right\rfloor \times \frac{1}{2}] \geq \frac{9}{8} \text{ if } k \geq 4.
\]

For \(k = 2\), \(\rho = \frac{1}{2}(2 + \frac{1}{2}) = \frac{5}{4} > \frac{9}{8}\), for \(k = 3\) \(\rho = \frac{1}{3}(3 + \frac{1}{2}) = \frac{7}{6} > \frac{9}{8}\). We note that no specialOne or specialTwo vertex is of type A, because each of them is a heavy vertex in \(G_1\) but a type A vertex is a degree 2 vertex in \(G_1\). Thus, the minimum average density of type A vertices is \(\frac{9}{8}\).

B. All the remaining vertices

To compute the minimum density of this type of vertices we group these vertices and their adjacent edges into neighbourhoods of heavy vertices in \(G_2\) of this type and evaluate the densities of these groups. Their minimum will be the minimum density for this type of vertices.

We call each of the following two as a class B path: (1) the maximal path of degree 2 vertices in \(G_2\) that is not a part of any class A path, and (2) an edge in \(G_2\) that is incident to at least one heavy vertex of type B. We note that all the vertices in a class
B path are of type B. There are 2 types of groups around the heavy vertices of type B based on whether the heavy vertex is connected to a vertex of type A or a heavy vertex of type B by a class B path.

Accounting for this type of vertices is as follows. (1) If class B path is attached to two heavy vertices \( v_1 \) and \( v_2 \) of type B then the path (i.e., the vertices and the edges owned by them) is divided equally, and each of the two groups around \( v_1 \) and \( v_2 \) own one half of the path. (2) Now we consider the case where the two ends of a class B path are attached to two types of vertices, say \( v_1 \) of type A and \( v_2 \) of type B. Clearly, one half of the edge incident to \( v_1 \) is owned by \( v_1 \). All the vertices and the remaining edges of the path are owned by the group of vertices around \( v_2 \).

We consider the two types of groups of type B vertices separately.

(a) **Group of type B vertices centered at a heavy vertex of type B in \( G_2 \) that is connected to heavy vertices of type B in \( G_2 \) only, by class B paths**

First, we consider group B(a) vertices that are not centered at specialOne or specialTwo vertices. The average density of a group B(a) vertices decreases as the length of any of the class B paths attached to the heavy vertex of the group increases (Figure 4.17 shows a group B(a) vertices). Consequently, for a group of vertices around a heavy vertex the contribution of average density for the group
from an attached path of degree 2 vertices decreases as the length of the path increases. Thus, the density contribution from a class B path will be minimum if the path length is the maximum. By Lemma 54 the maximum length of a path of degree two vertices is 3. The minimum density contribution from a class B path will be from a path of length 3.

Figure 4.17: A group of vertices of type B(a) around a heavy vertex $p_0$ in $G_2$. They are enclosed by a dash dotted circle. If a vertex is on the circle then its one half belongs to this group.

There are 4 edges and 3 vertices in such a path. The density of a half of this path is $\frac{4}{2^2} = \frac{4}{4} > \frac{9}{8}$. So, the minimum density of one half of a class B path is greater than $\frac{9}{8}$. We note that one half of a class B path is owned by a B(a) group. Consequently, class B paths will not contribute to reduce the average density of a B(a) group around a heavy vertex of type B to lower than $\frac{9}{8}$. So, we only consider the groups around the heavy vertices of this group each of which has the least number of class B paths attached to the heavy vertex, i.e., which has exactly 3 class B paths attached since degree of a heavy vertex is at least 3.

Let the total number of vertices in the 3 class B paths be $m$. Then average
density for the group around the heavy vertex is \( d = \frac{m+3}{2m+1} = 1 + \frac{1}{m+2} \geq \frac{9}{8} \) for \( m \leq 6 \). Thus, for the groups with paths having total number of degree 2 vertices at most 6 the minimum average density will be \( \frac{9}{8} \). It remains to consider the groups with total number of degree 2 vertices 7, 8 and 9, since there can be at most 3 degree 2 vertices in a path by Lemma 54, and at most 9 degree 2 vertices in the 3 paths.

For the group with 2 class B paths of length 3 and 1 class B path of length 1, placement will not be unique due to \( S_2 \) and \( S_4 \) (Figure 4.18). For path \( < p_0, p_1, p_2 > \) either \( |p_0p_1| = c \) or \( |p_1p_2| = c \) by \( S_4 \). For path \( < p_0, p'_1, p'_2, p'_3, p'_4 > \), among the edges \( p'_1p'_2 \) and \( p'_2p'_3 \) the one in \( E_1 \) will have length \( c \) by \( S_2 \). Similarly, for the path \( < p_0, p''_1, p''_2, p''_3, p''_4 > \) either \( |p''_1p''_2| = c \) or \( |p''_2p''_3| = c \). One more edge must be attached to fix the placements of the points. Then total number of vertices for the paths at the heavy vertex \( p_0 \) will be at most 5, which is less than 6. Consequently, the density for the group around \( p_0 \) will be at least \( \frac{9}{8} \).

Next we consider the group with all the 3 paths of length 3. For this case the placement will not be unique (Figures 4.19 and 4.20). For Figure 4.19 there must be an edge at \( p_1 \) or \( p_2 \) of the path \( < p_0, p_1, p_2, p_3, p_4 > \) to make \( p_1 \) and \( p_2 \)
unambiguous (by Lemma 71). For Figure 4.20 there must be an edge at $p_2$ or $p_3$ to make $p_2$ and $p_3$ unambiguous (by Lemma 71). Similarly, there must be an extra edge for each of the other 2 paths $<p_0, p'_1, p'_2, p'_3, p'_4>$ and $<p_0, p''_1, p''_2, p''_3, p''_4>$. Thus, the reduced group consists of 3 degree 2 paths each of maximum length 2.

There will be at most 6 degree 2 vertices in the degree 2 maximal paths at the heavy vertex $p_0$ and the average density for the group around $p_0$ will be at least $\frac{9}{8}$.

Similarly, for the group with 2 paths of length 3 and 1 path of length 2, and the group with 1 path of length 3 and 2 paths of length 2 we can show that there must be edges at the vertices of the paths that will make the total number of vertices in the degree 2 paths around $p_0$ at most 6. So, the minimum average density for the group around $p_0$ will be $\frac{9}{8}$.

Figure 4.19: Point placement graph and layer graph of heavy vertex $p_0$ of group B(a) with 3 degree 2 maximal paths of length 3. For a degree 2 maximal path $<p_1, p_2, p_3>$ attached to $p_0$, there are two consecutive edges $p_0p_1$ and $p_1p_2$ in $E_1$ at $p_0$. 181
Figure 4.20: Point placement graph and layer graph of heavy vertex $p_0$ of group B(a) with 3 degree 2 maximal paths of length 3. For a degree 2 maximal path $<p_1, p_2, p_3>$ attached to $p_0$, there is only one consecutive edge $p_0p_1$ in $E_1$ at $p_0$.

Thus, the minimum of the averages for this type of group is $\frac{9}{8}$.

It is found above that all the type $B$ paths attached to a degree 3 vertex $v$ of type $B(a)$ cannot have the maximum possible length of 3, and that the maximum of the total number of degree 2 vertices in the 3 type $B$ paths attached to $v$ is at most 6. And for a total of at most 6 such vertices the density of the group around $v$ is at most $\frac{9}{8}$.

So, we need to check the type $B(a)$ groups around degree 4 vertices in $G_2$. If all the 4 type $B$ paths attached to a degree 4 vertex $v$ of type $B(a)$ in $G_2$ are of maximum possible length 3, then the density of the group around $v$ is $\rho = \frac{4 \times 2}{4 \times 1 + 1} = \frac{8}{7} > \frac{9}{8}$. So, all the groups around degree 4 vertices of type $B(a)$
must have density greater than \( \frac{9}{8} \), since the density of a shorter group \( B \) path is shorter than the density of a longer group \( B \) path. Again, since a group \( B \) path does not help reduce the density of a \( B(a) \) group, density of a \( B(a) \) group around a \( B(a) \) vertex of degree greater than 4 must be greater than \( \frac{9}{8} \).

It can be readily checked that the minimum densities of the groups of type \( B(a) \) around specialOne or specialTwo vertices is \( \frac{9}{8} \). We shall check these results for the more restrictive case when they are of type \( B(b) \).

(b) **Group of type \( B \) vertices centered at a heavy vertex of type \( B \) in \( G_2 \) that is connected by at least one class \( B \) path to at least one type \( A \) heavy vertex**

First, we consider group \( B(a) \) vertices that are not centered at specialOne or specialTwo vertices. It is shown above that class \( B \) path attached to 2 type \( B \) heavy vertices does not contribute to reduce the average density of a group to lower than \( \frac{9}{8} \).

Now we consider a class \( B \) path between a type \( B \) heavy vertex and a type \( A \) vertex. A heavy vertex of type \( B \) in \( G_2 \) can be connected to a vertex of type \( A \) by a maximal path of degree 2 vertices in \( G_2 \) in two ways based on whether the edge of the path incident on the type \( A \) vertex is in \( E_1 \) or \( E_2 \). If the edge is in \( E_1 \) the length of the degree 2 path is 0, because type \( A \) vertices are found only in the maximal paths of degree 2 vertices in \( G_1 \) where each end of a path is connected
to a heavy vertex of type B in \( G_1 \) by an edge from \( E_1 \). For this case one half of each end edge is counted towards the density of its adjacent vertex of type B. Clearly, this path will not contribute to reduce the density of the corresponding neighbourhood of type B vertices.

For the second case, i.e., if the edge is in \( E_2 \), the maximum length of the maximal path of degree 2 vertices in \( G_2 \) is 2 since one end of the maximal path is connected to a heavy vertex in \( G_2 \) by an edge from \( E_2 \) and since there can be at most 1 edge from \( E_2 \) and at most 2 consecutive edges from \( E_1 \) in a maximal path of degree 2 vertices in \( G_2 \) containing edges from \( E_1 \) as well as \( E_2 \) (by Lemma 74).

The minimum average density of the vertices of this path is \( \frac{1}{2}(2 + \frac{1}{2}) = \frac{5}{4} > \frac{9}{8} \).

Also this path will not contribute to reduce the density of its corresponding neighbourhood of type B vertices to lower than \( \frac{9}{8} \).

So, we consider the heavy vertices of this group each of which has exactly 3 paths of degree 2 vertices in \( G_2 \). If the group of vertices around a heavy vertex of type B(b) has 2 degree 2 paths of length 3 attached to heavy vertices of type B and 1 path of degree 2 vertices attached to a heavy vertex of type A by an edge from \( E_2 \) then each of the 3 paths will have an edge from \( E_2 \). In a way similar to the case of group B(a) vertices consisting of 3 degree 2 paths (Figures 4.19 and 4.20) it can be shown that the reduced group will have density of at least \( \frac{9}{8} \).

For the group with 2 paths of length 3 being attached to heavy vertex of type B
and the third path of length 0 being attached to a vertex of type A by an edge from $E_1$, then the total number of vertices in all the 3 class B paths around the vertex is 6. It can be shown in a way similar to the discussion of group B(a) that such a group's density is at least $\frac{9}{8}$. It can be easily checked that for all other combinations of 3 maximal paths the minimum average density for the groups of vertices will be at least $\frac{9}{8}$.

Now for the reason similar to group B(a), we consider the group of vertices around a heavy vertex $v$ of type B(b) where $v$ is attached 3 degree 2 paths of length 3 the other ends of which are adjacent to heavy vertices of type B, and 1 path of degree 2 vertices of length 2 the other end of which is incident to a heavy vertex of type A. Density of the group around $v$ is $\rho = \frac{3 \times 2 + 2 \times 2}{3 \times 1 + 2 + 1} > \frac{9}{8}$.

Let us consider groups of vertices of type B(b) around specialOne vertices. We note that for a group of type B(b) around a specialOne vertex $p_0$, either there is a handle by strategies S1(3) and S4(2) or $p_0$ owns $\frac{5}{8}$ edge count of the edge incident to $p_0$ and its adjacent specialTwo vertex. For the former case the group around $p_0$ becomes smaller and the density of the reduced group is at least $\frac{9}{8}$.

For the latter case $\rho = \frac{2 \times 2 + 2 \times 2}{2 \times 2 + 0 + 1} = \frac{9}{8}$.

Now we consider the specialTwo vertex $p'_0$ of degree 3 in $G_1$ adjacent to $p_0$. If $p_0$ is the only specialOne vertex adjacent to $p'_0$ then either the density of the group around $p'_0$ is at least $\frac{9}{8}$ or $p'_0$ too is a specialOne vertex and a handle must
have been created by $B$ according to strategies $S1(3)$ and $S4(2)$. The reduced group around $p'_0$ will have density at least $\frac{9}{8}$. If there are 2 specialOne vertices adjacent to $p'_0$ then $\rho = \frac{1 \times 2 + \frac{1}{2} + \frac{3}{8}}{1 + 2 + 2 \times 0} = \frac{9}{8}$. For 3 specialOne vertices adjacent to $p'_0$, $\rho = \frac{3 \times \frac{3}{8}}{3 \times 0 + 1} = \frac{9}{8}$. It can be easily checked that if $p'_0$ is of degree at least 4 then the minimum density of the group around it will be at least $\frac{9}{8}$.

Thus, the minimum average density for all vertices in $G_2$ will be $\frac{9}{8}$. □

4.4 Summary

We have presented a 2-round algorithm for the point placement problem and improved the upper bound for a 2-round algorithm from $4n/3 + O(\sqrt{n})$ to $9n/7 + O(1)$. Its worst-case lower bound for 2-round algorithm is improved from the existing best $17n/16$ to $9n/8$. 

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Chapter 5

Detection of Potential Ligand Binding Sites

5.1 Introduction

The biological functions of proteins are the result of their interactions with other molecules such as other proteins, nucleic acids, substrates\(^1\), coenzymes, etc [73]. These interactions generally occur in the concave regions on the surfaces of proteins. The concave regions on the outer and inner surfaces of proteins are called pockets and cavities respectively. It is important to identify the pockets and cavities in proteins. The region of a protein that binds to another molecule is called binding site (Figure 5.1). A ligand is a small molecule that binds with a protein to modulate its function. The binding occurs by intermolecular forces. Ligand binding sites are often found in the largest pockets on protein surfaces [52, 54]. Information about a ligand binding site provides valuable information about protein-ligand docking and the structure of the ligand. This helps design small molecules which can control protein functions [14, 67]. Comparative analysis of ligand binding sites helps to understand

\(^1\)Substrates are the molecules that are bound and acted upon by enzymes.
the protein-ligand binding specificity [15]².

Figure 5.1: Ligand binding site on a protein [figure of Wikipedia: http://en.wikipedia.org/wiki/Ligand_binding ]

Binding ability depends on the tertiary structure of the protein and the chemical properties of the surrounding amino acids side chains. Tertiary structure of the protein defines the pocket. Ligand binding sites are often located at the largest pockets on the protein surface.

In this chapter, we present a modification of a geometric method called MSPocket [75] for finding pockets on protein surface.

5.2 Prior Work

Many computational search methods have been proposed for the identification of ligand binding sites. They can be classified into geometric approach and comprehensive approach based on the type of information used to characterize the pocket [75]. Geometric approaches

²Specificity is the ability to identify negative results.
use only geometric properties of proteins. A comprehensive approach, on the other hand, uses other properties of proteins as well, viz., evolutionary information, interaction energy, chemical properties of proteins, etc.

According to Zhu and Pisabarro [75], among the methods in current use, the three best ones are Fpocket [47], VICE [72] and Roll [73]. VICE [72] uses a 3D rectangular grid to represent protein (Figure 5.2). The resolution of the grid is adjustable (1 Å is appropriate in most cases). Grid points occupied by protein atoms are assigned 0, while the rest are assigned 1. For each of the latter grid points, the algorithm scans in 30 directions represented by vectors of specified length and passing through the grid points. The vectors are grouped into 3 shells. In Figure 5.2 shell 1, 2 and 3 vectors are shown in red, green and blue respectively. For a grid point, if at least half of the scan directions are blocked then it is inside a pocket. Figure 5.3 shows search for 3 grid points numbered 1, 2 and 3 along shell 1 vectors (1, 0), (1, 1), (0, 1), (−1, 1), (−1, 0), (−1, −1), (0, −1) and (1, −1). Black vectors are blocked by protein and are classified as blocked. Green vectors are not blocked and have clear paths to the edges of the grid. They are classified as clear. Pink vectors are not blocked, but they do not reach the grid edge because of finite length. They are classified as stalled. Point 1 has more clear than blocked vectors, and is outside the pocket. Point 2 has more blocked than clear vectors, and is inside the pocket. Point 3 is ambiguous. It needs further examination along shell 2 vectors.

Roll [73] uses the atomic coordinates of protein. A probe of radius 2 Å – 8 Å (2 Å is
appropriate in most cases) is rolled on each slice of a 3D grid representation of a protein.

The regions between probe and the protein surface are pockets. An implementation of Roll is called POCASA [73]. Fpocket [47] is a comprehensive approach. It uses geometric criteria, and uses electronegativity of protein atoms for pruning. It uses geometric objects
alpha sphere [49] to fill the space in protein. Each alpha sphere is defined by 4 atoms. Pockets correspond to ensembles of alpha spheres of intermediate radii. They must be apolar also.

Zhu and Pisabarro [75] proposed MSPocket. They claim that its performance is at par with the above 3 methods. MSPocket is a purely geometric approach to find ligand binding site on protein solvent excluded surface (SES). It does not use cubic grid representation of protein. So, it is not protein orientation dependent. In Section 5.3 we propose a modification of it. Here we describe MSPocket in detail.

Input to the MSPocket algorithm is the SES of a protein which in turn is generated by MSMS, a widely used tool for computing molecular surfaces [60]. The SES of a molecule consists of zero or more internal components and one external component of SES of a molecule. The components are reported as triangulated meshes. MSPocket processes only the external component of protein SES.

Let $G = (V,E)$ be the graph corresponding to the mesh, where $V$ is the set of vertices in the mesh and $E$ is the set of edges in it. Vertex normals to the mesh are used as features to identify pockets. For each vertex $v$, MSPocket calculates the average angle of deviation $\theta$ of the normal vectors of the adjacent vertices of $v$ with respect to the normal vector at $v$, and assigns it to $v$. Then it selects representative vertices $v'$ in ascending order of $\theta$. When a vertex is selected its adjacent vertices are removed from contention. This reduces the number of vertices to about 25%.
Let $V'$ be the set of representative vertices. Then a new graph $G' = (V', E')$ is constructed using $V'$ as its set of vertices. Two vertices are adjacent in $G'$ if they are adjacent in $G$, or one of them is adjacent to a neighbour of the other in $G$, or their neighbours are adjacent in $G$. For each pair of vertices in $G'$, if their mutual distance is within some distance cutoff $d_p$ (the authors of MSPocket used $d_p = 8\text{Å}$), then they are called close vertex pairs (ClsVP). Each ClsVP is a candidate for pocket vertex pair (PktVP) or protrusion vertex pair (PtrVP). A ClsVP is selected as a PktVP if the distance between them decreases by a value greater than $0.2r$, where $r$ is a user specified distance change ratio parameter (the authors of MSPocket suggested $r = 1.3$), as the pair is moved along their normal by a short distance ($0.2\text{Å}$). The vertices in a PktVP are potential pocket vertices. A PktVP is considered as a PtrVP if their normals are inclined at an angle larger than some user specified parameter $a_p$ (the authors of MSPocket used $a_p = 200^\circ$).

Finally, pocket outlier vertices are pruned from the pocket and missed vertices at the pocket bottom are included into the pocket using some neighbourhood conditions.

All the parameters are adjustable by the user. This makes it very flexible and can be adjusted according to the demand of the input. The authors claim that its performance is comparable to the existing best performing methods.
5.3 Contribution

In MSPocket each vertex of a ClsVP \((A, B)\) is moved along its respective normal by a small distance 0.2 Å. Let the new positions of the pair be \(A'\) and \(B'\). The pair is considered as a PktVP if \(d_{AB} - d_{A'B'} > 0.2r\), where \(r\) is a distance change ratio parameter. The vertices in a PktVP are potential pocket vertices. The larger the value of \(r\), the more the vertex normals are pointing towards each other. We replace this constraint by the angles of normals at \(A\) and \(B\) with \(\overrightarrow{AB}\) and \(\overrightarrow{BA}\) respectively. They are more closely related to the angle of inclination of the two planes passing through the vertices and perpendicular to the two vertex normals. Let the angles be \(\theta_1\) and \(\theta_2\) respectively. We require that \(\theta_1 < 70^\circ\), \(\theta_2 < 70^\circ\) and \(\theta_1 + \theta_2 < \theta\), where \(\theta\) is a parameter that depends on \(\theta_1\) and \(\theta_2\). The value of \(\theta\) is adjustable by users depending on the values of \(\theta_1\) and \(\theta_2\). In this work, we have set its value as follows: (i) for \(\theta_1 < 70^\circ\) and \(\theta_2 < 70^\circ\), \(\theta_1 + \theta_2 < 85^\circ\); (ii) for \(\theta_1 < 65^\circ\) and \(\theta_2 < 65^\circ\), \(\theta_1 + \theta_2 < 90^\circ\); (iii) for \(\theta_1 < 60^\circ\) and \(\theta_2 < 60^\circ\), \(\theta_1 + \theta_2 < 95^\circ\); and (iv) for \(\theta_1 < 55^\circ\) and \(\theta_2 < 55^\circ\), \(\theta_1 + \theta_2 < 100^\circ\). The remaining parameters of MSPocket are set to the same values that are used in MSPocket. Parameter \(r\) is not used in our method. We call this method as Modified MSP.

5.3.1 Experimental Results

We have used the same 48 bound benchmark dataset [43] that is used by MSPocket. Table 5.1 shows the success rates of top 1 and top 3 pockets that are identified by Modified
Table 5.1: Comparison of success rates in detection of ligand binding sites

<table>
<thead>
<tr>
<th>Method</th>
<th>Top 1 (%)</th>
<th>Top 3 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modified MSP</td>
<td>79</td>
<td>90</td>
</tr>
<tr>
<td>MSPocket</td>
<td>77</td>
<td>90</td>
</tr>
</tbody>
</table>

MSP and MSPocket. For top 1 pocket Modified MSP has a slightly better success rate than that of MSPocket, while for top 3 pockets the success rates are the same.

5.4 Summary

We have modified MSPocket algorithm to make the angle constraint more closely related to the concavity of the ligand binding sites. The algorithm is tested on a small dataset. The results are encouraging. The method can be further extended and/or refined to make it more effective.
Chapter 6

Conclusions

In this dissertation, we have investigated 3 problem areas of computational biology where geometry is involved. First, we have considered the length-constrained sums/densities of DNA sequence. An optimal algorithm has been presented for the length-constrained maximum density segment problem. Experiments show that it runs significantly faster than an existing optimal algorithm [21]. It has been extended to solve the $k$ length-constrained maximum density segments problem in $O(nk), O((n + k) \log^2(U - L))$ and $O(n(U - L))$ time when $k \in O(\log^2(U - L)), k \in \omega(\log^2(U - L)) \cap o(n(U - L)/\log^2(U - L))$ and $k \in \Omega(n(U - L)/\log^2(U - L)) \cap O(n(U - L))$ respectively. Previously there was no non-trivial solution for this problem. The method has been extended to solve also the length-constrained maximum sum segment problem and the $k$ length-constrained maximum sum segments problem in optimal time. We have also presented optimal algorithms to find all the length-constrained segments satisfying a sum or density lower bound. We have indicated the extensions of our algorithms to higher dimensions. Our algorithms facilitate efficient solutions for all these problems in higher dimensions. All the algorithms can
be extended in a straightforward way to solve the problems with non-uniform length.

It would be interesting to study if there is any linear time algorithm for the $k$ length-constrained maximum density segments problem. It can also be investigated to find more efficient algorithms for the problems in higher dimensions. It remains open to improve the trivial lower bounds for these cases.

Second, we have explored the point placement problem on a line. The 3-dimensional version of it has application in the area of molecular conformation. 2-round algorithms based on 5:5 and 6:6 jewels have been presented. We have presented a 2-round algorithm based on 3-path and improved the upper bound for a 2-round algorithm from $4n/3 + O(\sqrt{n})$ to $9n/7 + O(1)$. Worst-case lower bound for 2-round algorithm for the problem is improved from the existing best $17n/16$ to $9n/8$. It is challenging to decrease the gap between the upper and lower bounds for two rounds even further. One can consider 7:7 and 8:8 jewels as basic component for algorithm. Improving the upper bound of $5n/4$ for three rounds [20] can also be investigated. One can study the relation between this problem and the restriction site mapping problem. Another interesting line of work is to generalize this problem to higher dimensions.

Third, we have studied the problem of detection of ligand binding sites on protein surface. We have proposed a modification of a geometric method called MSPocket [75] for detection of ligand binding sites on protein surface. We have replaced a constraint for the angular inclination of a pair of vertices in a pocket. Experimental comparison of our method
with MSPocket using benchmark dataset of 48 bound protein structures shows encouraging result. It can be investigated to extend/improve this method by incorporating interaction energy, evolutionary information, chemical properties of proteins, etc.
Bibliography


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