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ULTIMATE STRENGTH AND BEHAVIOR OF
REINFORCED CONCRETE SLABS

A Dissertation
submitted to the Faculty of Graduate Studies
in partial fulfilment of the requirements
for the Degree of Doctor of Philosophy
in Civil Engineering

by
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1971
ABSTRACT

This investigation considers the behavior of reinforced concrete slabs from initial to collapse load. A yield criterion is developed considering the equality of normal resisting moment with normal applied moment at the yield line for which all orientations of the yield line are examined. It is shown that along a yield line applied and resisting values of twisting moments are also equal. The yield criterion is expressed in simple form analytically as well as graphically.

Utilising this yield criterion complete solution of a reinforced concrete slab is obtained based on small deflection bending theory of orthotropic plates. Finite difference method is used to approximate the differential equations. Solution is carried out in incremental form since the governing differential equations in both the elastic and the plastic range are linear. First the equilibrium equations are written in terms of moments, then the elastic moment-deflection relations are substituted in these and the first stage elastic solution is carried out. The moment-deflection relations for the yielded points are then modified to take into account the yielding and second stage solution is similarly obtained. The entire procedure of second stage is continued till enough points in the slab are yielded to convert it into a collapse mechanism.

The method enables one to obtain the complete load-deflection characteristic of the slab and the ultimate load is found without recourse to the bound theorems. The sequence of yielding of points and thus the yield line pattern is also traced.

An experimental study was performed on 32 reinforced concrete slabs
tested up to failure. Twenty six tests were carried out to verify the yield criterion and the remaining six tests were performed to confirm the elastic-plastic solution. The results obtained from the tests are found to be in satisfactory agreement with the theoretical solution.
ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude to Dr. J.B. Kennedy, Professor and head of Civil Engineering Department, for his guidance, encouragement and valuable suggestions throughout the preparation of this thesis.

Thanks are also due to the staff of the Computer Centre where all numerical calculations were performed and to the staff of the Structural Engineering laboratories for their assistance in the experimental part of this investigation.

This study was made under a scholarship programme and the author is thankful to the Indian Ministry of Education and the Canadian Commonwealth Scholarship and Fellowship Administration for making a Commonwealth Scholarship award to him and to the University of Roorkee for granting study leave for the duration of this study.

Thanks are also due to Miss Colleen Hodgins for typing the manuscript.
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<tr>
<td>a</td>
<td>side of a square slab or the width of a rectangular slab</td>
</tr>
<tr>
<td>A</td>
<td>constant, defined where used</td>
</tr>
<tr>
<td>(A_s)</td>
<td>area of steel reinforcement per unit width in one layer for positive bending</td>
</tr>
<tr>
<td>(A'_s)</td>
<td>area of steel reinforcement per unit width in one layer for negative bending</td>
</tr>
<tr>
<td>b</td>
<td>length of a rectangular slab</td>
</tr>
<tr>
<td>c_n</td>
<td>normal force in concrete per unit width</td>
</tr>
<tr>
<td>d</td>
<td>distance from extreme compression fiber to centroid of reinforcement for first layer</td>
</tr>
<tr>
<td>d_1</td>
<td>distance from extreme compression fiber to centroid of reinforcement for second layer</td>
</tr>
<tr>
<td>D</td>
<td>flexural rigidity of an isotropic plate</td>
</tr>
<tr>
<td>(D_x, D_y)</td>
<td>flexural rigidities of an orthotropic plate in x and y directions respectively</td>
</tr>
<tr>
<td>(D_t)</td>
<td>flexural rigidity of an orthotropic plate in t direction</td>
</tr>
<tr>
<td>(D_{xy}, D_{yx})</td>
<td>torsional rigidities</td>
</tr>
<tr>
<td>(E)</td>
<td>modulus of elasticity of an isotropic material</td>
</tr>
<tr>
<td>(E_x, E_y)</td>
<td>moduli of elasticity in x and y directions respectively</td>
</tr>
<tr>
<td>(E_c, E_s)</td>
<td>moduli of elasticity of concrete and steel reinforcement respectively</td>
</tr>
<tr>
<td>F</td>
<td>constant, defined where used</td>
</tr>
<tr>
<td>(f'_c)</td>
<td>compressive cylinder strength of concrete</td>
</tr>
<tr>
<td>(f_{cr})</td>
<td>cracking stress of concrete</td>
</tr>
<tr>
<td>(f_r)</td>
<td>modulus of rupture of concrete</td>
</tr>
<tr>
<td>(f_y)</td>
<td>yield stress of steel reinforcement</td>
</tr>
<tr>
<td>(G_{xy}, G_{yx})</td>
<td>moduli of shear</td>
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h non-dimensional grid spacing when equal in x and y directions

$h_x$, $h_y$ non-dimensional grid spacings in x and y directions respectively

H constant, defined and used in Eq. 4.8.

$I_{cx}$, $I_{cy}$ moments of inertia of concrete slab about the neutral axis in sections perpendicular to the x and y axes respectively

$I_{sx}$, $I_{sy}$ moments of inertia of steel reinforcement about the neutral axis in sections perpendicular to the x and y axes respectively.

$k_1$ ratio of average stress to maximum stress in concrete compressive stress diagram

$k_2$ ratio of depth of resultant of compressive force to the depth of neutral axis.

$k_3$ ratio of maximum stress in concrete compressive stress diagram to cylinder strength $f_c'$. 

$k_u$ ratio of depth of neutral axis to the effective depth at failure

$m$ modular ratio $= E_s/E_c$

$M_1$, $M_2$ applied external principal moments

$M_x$, $M_y$ bending moments per unit length of sections of a plate perpendicular to x and y axes respectively

$M_{xy}$, $M_{yx}$ twisting moments per unit length of sections of a plate perpendicular to x and y axes respectively

$\Delta M_x$, $\Delta M_y$, $\Delta M_{xy}$ incremental values of the moments

$M_p$ resisting moment per unit length for positive bending in an isotropic slab.

$M_{px}$, $M_{py}$ resisting moments per unit length in x and y directions respectively for positive bending

$M'_p$ resisting moments per unit length in x and y directions respectively for negative bending

$M_n$, $M_t$, $M_{nt}$ external applied bending and twisting moments per unit length in n-t system of axes
<table>
<thead>
<tr>
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<th>Definition</th>
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<td>( M_{pn} ), ( M_{pt} ), ( M_{pnt} )</td>
<td>resisting moments per unit length in ( n-t ) system of axes for positive bending</td>
</tr>
<tr>
<td>( M_{cr} )</td>
<td>cracking moment of concrete section</td>
</tr>
<tr>
<td>( P )</td>
<td>constant, defined where used</td>
</tr>
<tr>
<td>( P_u )</td>
<td>ultimate load</td>
</tr>
<tr>
<td>( q )</td>
<td>intensity of uniform load per unit area</td>
</tr>
<tr>
<td>( \Delta q )</td>
<td>incremental value of the load ( q )</td>
</tr>
<tr>
<td>( Q )</td>
<td>constant, defined where used</td>
</tr>
<tr>
<td>( Q_x, Q_y )</td>
<td>shearing forces parallel to ( z )-axis per unit length of section of a plate perpendicular to ( x ) and ( y ) axes respectively</td>
</tr>
<tr>
<td>( t )</td>
<td>thickness of the plate</td>
</tr>
<tr>
<td>( w )</td>
<td>displacement along the ( z )-axis</td>
</tr>
<tr>
<td>( \Delta w )</td>
<td>incremental value of the displacement along the ( z )-axis</td>
</tr>
<tr>
<td>( x, y, z )</td>
<td>rectangular coordinates</td>
</tr>
<tr>
<td>( \beta )</td>
<td>ratio of moment ( M_1 ) to moment ( M_{px} )</td>
</tr>
<tr>
<td>( \gamma_{xy} )</td>
<td>shearing strain</td>
</tr>
<tr>
<td>( \varepsilon_x, \varepsilon_y )</td>
<td>normal strains in ( x ) and ( y ) directions respectively</td>
</tr>
<tr>
<td>( \varepsilon_c )</td>
<td>strain in concrete</td>
</tr>
<tr>
<td>( \varepsilon_s )</td>
<td>strain in steel</td>
</tr>
<tr>
<td>( \theta )</td>
<td>angle between the ( x ) and ( n ) directions</td>
</tr>
<tr>
<td>( \mu )</td>
<td>coefficient of non-isotropy for positive bending moment</td>
</tr>
<tr>
<td>( \mu' )</td>
<td>coefficient of non-isotropy for negative bending moment</td>
</tr>
<tr>
<td>( \mu_1 )</td>
<td>ratio of reinforcement area in ( y ) direction to the reinforcement area in ( x ) direction</td>
</tr>
<tr>
<td>( \nu )</td>
<td>Poisson's ratio for isotropic material or Poisson's ratio for concrete in section 4.3</td>
</tr>
<tr>
<td>( \nu_x, \nu_y )</td>
<td>Poisson's ratios in an orthotropic material in the ( x ) and ( y ) directions respectively</td>
</tr>
<tr>
<td>( \sigma_x, \sigma_y )</td>
<td>normal components of stress parallel to the ( x ) and ( y ) axes respectively</td>
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\( T_{xy}, T_{yx} \) shearing stresses

\( \phi \) angle between x direction and the direction of principal moment \( M_1 \)

\( \psi \) angle between the direction of principal moment \( M_1 \) and the direction of normal to yield line

\( \omega \) ratio of applied principal moments = \( \frac{M_2}{M_1} \)
CHAPTER I

Introduction

1.1 Description of the Problem

The methods of plastic analysis of structures which are quite well established at this time depend upon the two fundamental theorems, the upper and lower bound theorems. These enable the reserve strength beyond the elastic limit load to be utilised. However, since the magnitude of the deflections are not predicted, these theorems present a serious drawback particularly where a deformation criterion governs. Furthermore, the exact limit load will not be obtained unless the upper and lower bounds coincide. For the solution of a plate problem, the upper bound method (yield line theory) requires the assumption of a yield pattern while for the lower bound solution an acceptable moment field must be assumed.

For reinforced concrete slabs the problem is even more difficult due to the absence of a generally accepted yield criterion. Such questions as the strength of concrete under biaxial bending and kinking of reinforcing steel at cracks, which have arisen, have weakened the faith in the Johansen's yield criterion for reinforced concrete slabs.

It is established that for inelastic beams and frames the overall non-linear response of the structure may well be represented by a series of incremental linear solutions, which form the basis of the plastic hinge method. This method has been applied to the reinforced concrete slabs in this study since the governing equations in incremental form are linear in both the elastic and plastic ranges.

1.2 Object and Scope

The overall objective of this investigation had been to obtain a
better understanding into the ultimate strength and behavior of reinforced concrete slabs at all stages of loading. With this in mind the specific objectives are:

(1) To develop a simple and rational approach to the yield criterion applicable to both isotropically and non-isotropically reinforced slabs under any given combination of applied moments.

(2) To develop a method of complete elastic-plastic solution of slab bending problem where the behavior of the slab at any stage of loading from initial to collapse can be predicted.

(3) To verify experimentally both the yield criterion and the complete elastic-plastic solution.
CHAPTER 2

Historical Review

2.1 Introductory Remarks

This chapter presents the background for the present investigation. The development of upper bound (yield line theory) and lower bound methods for plastic analysis of reinforced concrete slabs is briefly described, and their limitations pointed out. The application of any elastic-plastic solution to slab structure is dependent on the strength and behavior of slab elements under a given loading condition. This has also been the basis of yield line theory and is termed as 'yield criterion'. Some of the problems associated with yield criterion are also described.

2.2 Yield Line Theory

The yield line theory is a theory of plasticity modified to apply to reinforced concrete slabs. It considers the pattern of yield lines which are developed as a result of applied moment at certain sections approaching the yield moment. These yield lines are the line plastic hinges and divide the slab into portions which rotate about the supports. This converts the slab into a mechanism resulting in its collapse.

Ingerslev (1)* formulated and used this theory in some analyses of the strength of reinforced concrete slabs in 1921. He introduced the concept of yield lines with constant bending moment over the entire length and assumed that no twisting moments can exist along the yield lines. Thus his method could not be applied to cover all cases of slabs. In

* Numbers in paranthesis refer to the list of references given at the end.
1943 Johansen (2,3) advanced the more comprehensive theory which is applicable to every case. It differed from Ingerslev's theory essentially in that the twisting moments along the yield lines were not assumed equal to zero. His most important contribution was the introduction of work equation for determination of upper bound solutions; a contribution that has been of significance, in general, in the plastic theory of structures. Johansen introduced the well known nodal forces which actually represent twisting moments and shears. He was then able to develop the theory on mathematical grounds.

In 1953 Hognestad (4) summarised the development of the theory mainly based on Johansen's work (2, 3) and later in 1962 Johansen's work was translated in English (5). An up-to-date development of the theory is available in a more recent work of Jones and Wood (6).

2.3 Limitations of Yield Line Theory

The yield line theory has the following limitations:

(1) It gives an upper bound solution to the collapse load as the assumed yield pattern is not necessarily the most critical mechanism.

(2) It gives no information about the behavior of the slab prior to collapse i.e. moment - curvature relation cannot be obtained.

(3) It does not provide sufficient information regarding the distribution of bending and twisting moments in the interior of the slab away from the yield lines and thus the distribution of reinforcement becomes difficult.

(4) It is an ultimate strength theory and does not in itself guarantee satisfactory performance of the slab at working loads. It does not give any information about the deflections, which are shown to be
2.4 Lower Bound Solutions

Several efforts have been made to find the lower bound solutions. Circular plates of various types have been analysed extensively for the rigid-plastic and elastic-plastic cases. Hodge (8) gives a summary of the limit analysis theory pertaining to the rectangular slabs. Using trial distribution of moments and deflections and using equilibrium and energy approaches respectively, he arrives at lower and upper bound solutions.

Hillerborg (9, 10, 11) has developed a 'strip method' based on equilibrium theory. He chooses two of the three unknowns in the plate equilibrium equation and lets the equilibrium condition determine the third one. Thus he assumes simplified moment fields and gives a simple method to find the solution. But if the assumption of moments is not realistic, the lower bound value of the load obtained by this method may be significantly below the correct value. Crawford (12) has presented a review of this method and has compared it with the upper bound yield line theory. More detailed account of it is presented by Wood and Armer (13, 14).

Nielsen (15) follows the Prager's theory of plasticity and finds both upper and lower bound solutions of some typical cases. Where these two solutions coincide, the result is exact. If the solutions do not coincide, the lower bound solution will give some indication as to how well is the yield line theory solution in that particular case. Lower bound solutions of certain slab problems are also given by Wood (16) and Kemp (17).

Even if the upper and lower bound solutions are close to each other
or are coincident, these cannot be used to determine the behavior of the slab. In other words, the progression of yielding and the corresponding deflections of the slab until the formation of a collapse mechanism are still not known. The knowledge of the behavior of the slab is important from the design point of view since the ultimate load divided by an arbitrary load factor gives the working load and this working load must not fall in post elastic range otherwise it will result in unsatisfactory cracks and deflections of the slab. Thus the knowledge of the load deformation characteristic will guide in arriving at a suitable load factor. Therefore it seems desirable to find a method which will give both the strength and behavior of a slab without the necessity of assuming the yield pattern or assuming the distribution of moments.

2.5 Yield Criterion

In spite of its limitations as pointed out in section 2.3, yield line theory has proved to be very useful for finding the ultimate load carrying capacity of slabs. However, Johansen's assumption of the underlying yield criterion has not been fully confirmed by experiments. In the research work carried out on this criterion questions have arisen concerning the strength of a slab element when stressed in more than one direction, a problem associated with the yield criterion of plane concrete under biaxial stresses -- as well as questions concerning the reorientation of the reinforcement bars at a yield line.

Although the conventional yield line theory may be on the unsafe side, experimental results show generally a higher carrying capacity of the slab than predicted by the theory. This has led to the examination of membrane forces action. It has also been pointed out that reinforcement inclined to the yield line may be bent across the crack in the concrete
and consequently increase the yield moment. Wood (16) reports that in tests carried out at the Building Research Station in the United Kingdom the yield moment capacity for slabs with reinforcement inclined at $\pm 45^\circ$ with the span direction was up to 16% higher as compared to that of a slab with reinforcement parallel to the span direction. Nielsen's experiments (24) where the reinforcing bars were also inclined at $45^\circ$ to the yield lines indicated no significant effect of the kinking of reinforcing bars.

Baus and Tolaccia (18) claim that there is an increase in the yield moment when a slab element is subjected to biaxial stresses. They noticed this increase in tests on isotropic slabs with principal moments not in the steel directions and attributed it to the combined effects of biaxial compression of concrete and of kinking of reinforcing bars.

Kwiecinski (19, 20, 21) adopted the idea that reinforcement inclined to the crack is partly kinked across it and along with the Ingerslev's assumption of no twisting moments in the yield lines, he developed an intermediate approach which he calls 'Partial Kinking' criterion.

Lenschow (22, 23) carried out extensive investigation of the yield criterion and on the basis of his theoretical and experimental results he has shown that there is no increase in yield moment as a result of biaxial stresses and that the kinking of a bar across a yield line is so small that increase in moment capacity is negligible. Morley (25, 26) on the basis of his experimental investigation at the University of Cambridge has also concluded that for practical amounts of reinforcement there is no significant kinking of steel across the cracks.

Various other papers on this subject with new ideas and critical remarks are available, but the yield line theory in practical use is the
one developed by Johansen.

2.6 Concluding Remarks

The yield line theory is generally believed to be a powerful method for the solution of reinforced concrete slabs, and though it may have influenced the building codes in various countries in one way or the other, it has not had the corresponding popularity as a design method in spite of its simplicity. One reason for this might be that it remained virtually unknown to the English speaking world for quite a long time. First detailed account of the theory was published in English in 1953 (4) and more comprehensive works (5, 6, 16) have been available only in the sixties. However, one of the main reasons seems to be due to the inherent limitations of the theory as outlined in section 2.3. No information about the deflections of the slab at any stage can be obtained which is more serious where a deformation criterion governs. One such case is discussed in reference (7). Lack of information required for efficient distribution of reinforcement is also a serious drawback. Another major cause seems to be the absence of a generally accepted yield criterion, on which any solution of a reinforced concrete slab problem will depend.
3.1 Introduction

The yield criterion defines the behavior of the slab element under a given loading condition. On this is based the formation of yield lines. Mathematically it will relate the resisting and applied moment components at the formation of yield line. In order to apply any yield criterion correctly, it is necessary that resisting moment components of the slab along an expected orientation of yield line be estimated with a fair degree of accuracy. In this chapter, the problems concerning the calculations of these components are first analysed and then a more rational approach to the yield criterion applicable for both isotropically and non-isotropically reinforced slabs is developed.

3.2 Resisting Moments at a Yield Line

For a yield line which crosses the reinforcement at a certain angle the moment components are calculated by the 'stepped' criterion proposed by Johansen (5) and later developed and explained by Jones and Wood (6).

Consider a slab element shown in Fig. 3.1 where reinforcement is provided in two mutually perpendicular directions. For each band of reinforcement taken on its own the yield line is considered to be divided into small steps parallel to, and at right angles to, the reinforcing bars; and, in addition, the reinforcement is assumed to remain in its original straight line when the adjoining parts of the slab rotate. For a yield line, normal to which makes an angle \( \theta \) with the x-axis, as shown in Fig. 3.1, the resisting moment components are as follows:
\[ M_{pn} = M_{px} \cos^2 \theta + M_{py} \sin^2 \theta \]

\[ M_{pt} = M_{px} \sin^2 \theta + M_{py} \cos^2 \theta \]

\[ M_{pnt} = (M_{px} - M_{py}) \sin \theta \cos \theta \]  \hspace{1cm} (3.1)

where \( M_{px} \) and \( M_{py} \) are the resisting moment components corresponding to the steel parallel to \( x \) and \( y \) directions respectively, and \( M_{pn}, M_{pt} \) and \( M_{pnt} \) are respectively the normal, tangential and twisting moments for the yield line direction.

These expressions indicate that moment components can be determined from \( M_{px} \) and \( M_{py} \) by using Mohr's circle for moments. With isotropic reinforcement \( (M_{px} = M_{py} = M_p) \), the twisting moment \( M_{pnt} \) becomes zero and \( M_{pn} = M_{pt} = M_p \).

The stepped criterion as described above is the most commonly used for reinforced concrete slabs; however, there are some problems associated with it, on which the correctness of expressions 3.1 will depend. These are discussed below.

(a) **Change of Moment Arm**

The resisting moment components along a yield line are affected by the change of moment arm which takes place if yield line is rotated from direction \( x \) to \( y \). This effect will not be present if the resisting moment arm in the section perpendicular to \( x \)-axis is the same as the resisting moment arm in the section perpendicular to \( y \)-axis. These two moment arms, however, will not be the same because of the following:

(1) The centre of gravity of the steel areas in two perpendicular directions is at different levels since one layer of steel must be placed over the other.
Unless the amount of steel in two layers is the same and both the layers are yielding, the equilibrium of forces on the section dictates that the compression force in the concrete in the two directions is different, and hence the point of application of the two forces is at different levels.

If we consider a slab element reinforced in two orthogonal directions at the bottom only and assuming that the reinforcement in both layers of steel is yielding at the instant of the formation of yield lines, then using Hognestad's concrete compressive stress block (27), Fig. 3.2, the resisting moment per unit length for a section perpendicular to the direction is:

\[
M_{px} = A_s f_y (d - \frac{k_2 A f_y}{k_1 k_3 f_c})
\]  \hspace{1cm} (3.2)

where according to Hognestad (27),

\[
\frac{k_2}{k_1 k_3} = \frac{1600 + 0.46 f'_c - f'^2/80000}{3900 + 0.35 f'_c}
\]  \hspace{1cm} (3.3)

Similarly the resisting moment per unit length for a section perpendicular to the direction is:

\[
M_{py} = \mu_1 A_s f_y (d_1 - \frac{k_2 \mu_1 A f_y}{k_1 k_3 f_c})
\]  \hspace{1cm} (3.4)

It may be noted here that both the terms of expression 3.2 affecting the moment arm of the section are different from those of expression 3.4.

For computing resisting moment per unit length for a section perpendicular to the direction, the total tensile force per unit length
of section, $T_n$, will be made up of the components from both layers of steel. Thus,

$$T_n = A_f y \cos^2 \theta + \mu A_f y \sin^2 \theta$$

and for equilibrium, force is concrete, $C_n$, will be given by

$$C_n = T_n$$

The normal moment will then be given by

$$M_{pn} = A_s y \cos^2 \theta (d - \frac{t}{2}) + \mu A_s y \sin^2 \theta (d_1 - \frac{t}{2}) + C_n \left(\frac{t}{2} - \frac{k_c n}{k_f f_c}\right)$$

Here it is assumed that yield criterion for plane concrete is of square yield type, that is compressive stress in one direction is independent of the stress in other direction.

The error introduced by using expression 3.1 for $M_{pn}$ instead of the exact expression 3.7 is shown in Fig. 3.3 for the following case:

$$t = 4 \text{ in.}, \quad d = 0.85t, \quad d_1 = 0.75t$$

$$f_c' = 3000 \text{ psi}, \quad f_y = 40,000 \text{ psi}$$

$$A_s = 1.0 \% \text{ of effective concrete area}$$

It is seen that error in $M_{pn}$ increases with decrease in the value of $\mu_1$. For $\mu_1 = 1.0$ the error is zero. Also the error increases with the
increase in the reinforcement ratio for under reinforced sections as shown in Fig. 3.4 for the same case. In all cases the exact value of the normal moment obtained from expression 3.7 is greater than that obtained from the first of expressions 3.1. For practical percentages of reinforcement and \( \mu_1 \), the maximum error in \( M_{pn} \) for any orientation of yield line is less than 2 \%

(b) Kinking of Reinforcement

Expressions 3.1 are based on the assumption that steel bars carry only uniaxial stresses in their original direction. Wood \( (16) \) has suggested that under certain circumstances it is possible that bars are dragged out of their original direction and locally become perpendicular to the direction of an opening crack which makes an angle with the steel bars as shown in Fig. 3.5. If this is true then expressions 3.1 will underestimate the resisting moments across a yield line. This phenomenon can be explained by considering Fig. 3.6 (a) where the reinforcing bar is shown inclined to the local direction of the crack and as the crack opens, the bar becomes perpendicular to the crack. The crushing of the concrete at the edges of the crack would, however, reduce the extent of kinking. In fig. 3.6 (b) this case is shown in more realistic proportions, which indicates that kinking of reinforcement may not have any significant effects. It is easy to show that with complete kinking of reinforcement, Fig. 3.5, the normal moment for the yield line is:

\[
M_{pn} = M_{px} \cos \theta + M_{py} \sin \theta
\]

(3.8)
For an isotropically reinforced slab where \( M_{px} = M_{py} = M_p \), the variation of \( M_{pn} \) with \( \theta \) is shown in Fig. 3.7. For a complete kinking case the maximum moment \( M_{pn} \) occurs for \( \theta = 45^\circ \) where \( M_{pn} = 1.414 M_p \).

Kwiecinski (19, 20) adopted an intermediate approach and advanced a partial kinking criterion. He found experimentally the value of a coefficient which he calls \( \mu \), defined by expression 3.9, to be 1.188.

\[
1.0 < \mu = \frac{M_{pn}(\theta = 45^\circ)}{M_p} < 1.414
\]  

(3.9)

Recent investigations by Lenschow (23) and Morley (26) have attempted to conclude that there is no kinking of steel bars across the cracks at any stage and the reinforcement can be assumed to carry only the uniaxial stress in its original direction. The absence of kinking is explained by Morley by considering the geometry of the crack as follows:

As shown in Fig. 3.8 a crack AB is assumed to run generally in the direction \( t \) with a crack width \( \delta \). If, as at D, the sides of the crack are everywhere parallel to the t-axis, points D and D' initially coincident must be separated by a distance \( \delta \) in the n direction and if local crushing of concrete does not occur, all steel bars must kink through an angle \( \theta \). If, however, the cracks are not quite straight but include zig-zag portions as at C, with cracks running for a short distance 1 along the steel, then for a displacement \( \delta \) the bar will be bent through an angle \( \tan^{-1} \left[ (\delta/1) \sin \theta \right] \). In this case kinking increases gradually as the crack widens but can be easily reduced to an insignificant amount not only due to local crushing of concrete, but due to the fact that usually 1 will be large enough in comparison with \( \delta \). For \( 1 = 20 \delta \) and for \( \theta = 45^\circ \), kinking = 2°. Thus for \( \delta = 0.05 \) in., the crack need
only run along the steel bar for about one inch to reduce kinking to an insignificant value.

In the present investigation seven experiments were planned to obtain direct observation on the question of kinking and all the remaining tests also provided indirect observation as will be explained in section 6.1; no evidence of kinking was observed in any of the tests.

3.3 Johansen's Yield Criterion

This is a principal stress yield criterion and the most commonly used one for reinforced concrete slabs. If either or both of the principal stresses reach the yield stress, it is then assumed that the material is in the plastic state at that point. In terms of moments it states that the yield moment in one direction is independent of the moment in the perpendicular direction. The plastic strains are perpendicular to the yield locus (or yield lines). For different types of reinforcing schemes the yield criterion can be expressed as follows:

(a) Isotropic Reinforcement

The yield criterion is shown graphically in Fig. 3.9 (a). Because of its shape it is often called the 'square' yield criterion. The resisting moments are equal in all directions and no twisting moments are produced along the yield lines. If the slab has unequal reinforcement at top and bottom but each level still contains isotropic reinforcement then the yield criterion is as shown in Fig. 3.9 (b).

(b) Non-isotropic Reinforcement

In case the directions of reinforcements and the principal moments coincide, the yield criterion can still be represented graphically in two dimensions.
Fig. 3.9(c) shows the case when the reinforcement is different in two directions but is identical at the top and bottom.

A general case is shown in Fig. 3.9 (d) where the reinforcement is different in two directions and also is not identical at the top and bottom. The ratios $\mu$ and $\mu'$ also need not be the same.

If the directions of reinforcements and applied principal moments are not coincident, twisting moments will be present and the representations of Fig. 3.9 do not apply. This is the most general case and is discussed in detail in the following section.

### 3.4 Formation of Yield Lines - General Case

Consider a reinforced concrete slab element as shown in Fig. 3.10 in which the reinforcement is placed in two orthogonal directions $x$ and $y$ for both positive and negative bending, and the corresponding unit resisting moments are $M_x^p$ and $M_y^p$ for positive bending and $M_x^{p'}$ and $M_y^{p'}$ for negative bending. $M_1$ and $M_2$ are the applied principal moments which depend upon the loading, boundary conditions and physical dimensions of the slab. It is assumed that the $M_x^p$ and $M_y^p$ are the governing resisting moments when the element is subjected to applied moments $M_1$ and $M_2$, or, in other words the element fails in positive bending. As seen in Fig. 3.10 moments $M_1$ and $M_2$, make an angle $\phi$ with the $x$ - $y$ system. If the loading on the slab is gradually increased, $M_1$ and $M_2$ increase proportionately and a stage will be reached when the yield line forms; it is assumed that the yield line is parallel to the direction $t$, the normal to which makes an angle $\Psi$ from the direction of $M_1$.

For the $n$ and $t$ system of axes, the resisting moment components are:
\[ M_{pn} = M_{px} \cos^2 (\bar{\phi} + \psi) + M_{py} \sin^2 (\bar{\phi} + \psi) \]

\[ M_{pt} = M_{px} \sin^2 (\bar{\phi} + \psi) + M_{py} \cos^2 (\bar{\phi} + \psi) \]  

\[ M_{pnt} = (M_{px} - M_{py}) \sin (\bar{\phi} + \psi) \cos (\bar{\phi} + \psi) \]  

and the components of the applied moments are:

\[ M_n = M_1 \cos^2 \psi + M_2 \sin^2 \psi \]

\[ M_t = M_1 \sin^2 \psi + M_2 \cos^2 \psi \]  

\[ M_{nt} = (M_1 - M_2) \sin \psi \cos \psi \]  

The variation of these resisting and applied moment components for various inclinations of yield line is shown in Fig. 3.11. Yield will occur corresponding to point A in Fig. 3.11 (a) where the normal applied moment curve becomes tangential to normal resisting moment curve. This condition is obtained when the following requirements are satisfied.

\[ M_{pn} = M_n \]  

(3.12)

and

\[ \frac{\partial M_{pn}}{\partial \psi} = \frac{\partial M_n}{\partial \psi} \]  

(3.13)
Also, in accordance with the principle of minimum resistance, it is necessary to specify that $M_n$ is nowhere greater than $M_{pn}$; this means that applied moment curve in Fig. 3.11 (a) should touch the resisting moment curve from below and not from above. When expressed mathematically this condition requires:

$$\frac{\partial}{\partial \psi} (M_n - M_{pn}) = 0 \quad (3.14)$$

provided that

$$\frac{\partial^2}{\partial \psi^2} (M_n - M_{pn}) < 0 \quad (3.15)$$

Condition 3.14 is identical to 3.13; and the proof of condition 3.15 is given in Appendix 'A'.

We now proceed with expressions 3.12 and 3.13 to express the yield criterion. Using expressions 3.10 and 3.11 in 3.12 yields

$$M_1 \cos^2 \psi + M_2 \sin^2 \psi = M_{px} \cos^2 (\phi + \psi) + M_{py} \sin^2 (\phi + \psi) \quad (3.16)$$

Similarly 3.13 yields

$$(M_1 - M_2) \sin \psi \cos \psi = (M_{px} - M_{py}) \sin (\phi + \psi) \cos (\phi + \psi) \quad (3.17)$$

Expression 3.17 signifies that along a yield line, which is the line of least resistance, the resisting twisting and applied twisting moments...
are also equal. This is also evident from Fig. 3.11 (c).

Thus expressions 3.16 and 3.17 describe the yield criterion completely in a general case. This is essentially a normal bending moment criterion where at any point all orientations of yield line are examined and not only the principal moment direction (28). The yield line will form at a location where the external applied normal moment will become equal to the resisting normal moment of the slab regardless of the absolute values of these moments.

Introducing

\[ \mu = \frac{M_{py}}{M_{px}} \]

\[ \omega = \frac{M_2}{M_1} \]

and \[ \beta = \frac{M_1}{M_{px}} \]

expression 3.16 can be simplified as:

\[ \beta \cos^2 \psi + \omega \beta \sin^2 \psi = \cos^2 (\tilde{\phi} + \psi) + \mu \sin^2 (\tilde{\phi} + \psi) \]

or \[ \frac{\beta}{2} (1 + \cos 2\psi) + \frac{\omega \beta}{2} (1 - \cos 2\psi) = \frac{1}{2} [1 + \cos 2(\tilde{\phi} + \psi)] + \frac{\mu}{2} [1 - \cos 2(\tilde{\phi} + \psi)] \]

or \[ \beta(1+\omega) + \beta(1-\omega)\cos 2\psi = (1+\mu) + (1-\mu) \cos 2(\tilde{\phi} + \psi) \] (3.19)

Similarly using expressions 3.18, expression 3.17 can be simplified as:
\[
\beta(1 - \omega) \sin \psi \cos \psi = (1 - \mu) \sin (\bar{\phi} + \psi) \cos (\bar{\phi} + \psi)
\]
\[\text{or} \quad \beta(1 - \omega) \sin 2\psi = (1 - \mu) \sin 2(\bar{\phi} + \psi) \quad (3.20)\]

Multiplying expression 3.20 by \(\frac{\cos 2\psi}{\sin 2\psi}\) and subtracting it from expression 3.19 yields

\[
\beta(1 + \omega) = (1 + \mu) + (1 - \mu) [\cos 2(\bar{\phi} + \psi) - \sin 2(\bar{\phi} + \psi) \frac{\cos 2\psi}{\sin 2\psi}]
\]
\[\text{or} \quad \beta(1 + \omega) = (1 + \mu) - (1 - \mu) \frac{\sin 2\bar{\phi}}{\sin 2\psi} \quad (3.21)\]

From expression 3.20

\[
\beta(1 - \omega) \sin 2\psi = (1 - \mu) (\sin 2\bar{\phi} \cos 2\psi + \cos 2\bar{\phi} \sin 2\psi)
\]

Rewriting,

\[
\sin 2\psi \left[ \beta(1 - \omega) - (1 - \mu) \cos 2\bar{\phi} \right] = (1 - \mu) \sin 2\bar{\phi} \cos 2\psi
\]
\[\text{or} \quad \tan 2\psi = \frac{(1 - \mu) \sin 2\bar{\phi}}{\beta(1 - \omega) - (1 - \mu) \cos 2\bar{\phi}} \quad (3.22)\]

and

\[
\sin 2\psi = \frac{(1 - \mu) \sin 2\bar{\phi}}{[(1 - \mu)^2 + \beta^2(1 - \omega)^2 - 2\beta(1 - \mu)(1 - \omega) \cos 2\bar{\phi}]^{1/2}} \quad (3.23)
\]

Using expression 3.23 in 3.21 yields

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\[
\beta(1 + \omega) = (1 + \mu) - [(1 - \mu)^2 + \beta^2(1 - \omega)^2 - 2\beta(1 - \mu)(1 - \omega) \cos 2\phi]^{1/2}
\]

Squaring both sides and rearranging,

\[
\beta^2(1+\omega)^2 + (1+\mu)^2 - 2\beta(1+\omega)(1+\mu) = (1-\mu)^2 + \beta^2(1-\omega)^2 - 2\beta(1-\mu)(1-\omega)\cos 2\phi
\]

or \( \omega \beta^2 - \beta \left[ (\sin^2 \phi + \mu \cos^2 \phi) + \omega(\cos^2 \phi + \mu \sin^2 \phi) \right] + \mu = 0 \) \hspace{1cm} (3.24)

Introducing \( P = \sin^2 \phi + \mu \cos^2 \phi \)

and \( Q = \cos^2 \phi + \mu \sin^2 \phi \),

yields the following roots for equation 3.24:

\[
\beta_1, \beta_2 = \frac{(P + Q) \pm \sqrt{(P + Q)^2 - 4\mu \omega}}{2\omega} \hspace{1cm} (3.26)
\]

It has been shown in Appendix 'B' that the portion under the square root sign in expression 3.26 is greater than or equal to zero; in other words, both these values of \( \beta \) are real. The smaller value \( \beta_2 \) corresponds to the yield represented by the point A in Fig. 3.11 (a) while the value \( \beta_1 \) corresponds to the yield in a perpendicular direction. When \( \omega = \mu = 1.0 \), then for any value of \( \phi \), \( \beta_1 = \beta_2 \) and this condition is similar to that at the centre of a square isotropically reinforced slab subjected to uniform load. Thus the required solution of equation 3.24 is taken as:

\[
\beta = \frac{(P + Q) - \sqrt{(P + Q)^2 - 4\mu \omega}}{2\omega} \hspace{1cm} (3.27)
\]
Having found $\beta$, the value of $\psi$ can be obtained from expression 3.22.

3.5 Particular Cases

(a) Isotropically Reinforced Slab

The expressions for such a slab can be simplified by substituting $\mu = 1.0$ in expressions 3.22 and 3.27, which yields

$$\beta = 1.0$$  \hspace{1cm} \text{(3.28)}

and

$$\psi = 0^\circ$$  \hspace{1cm} \text{(3.29)}

This indicates that the yield line will form perpendicular to the principal moment direction and at yielding the magnitude of $M_1$ will be equal to $M_{px}$ (also equal to $M_{py}$ since $\mu = 1.0$). An isotropically reinforced slab has the same resisting capacity at any orientation of the yield line and so the above results are as expected.

(b) Non-isotropic Slab Subjected to Uniaxial Moments

Simplified expressions for this case can be obtained by the substitution of $\omega = 0$ in the expressions 3.22 and 3.27 to yield

$$\beta = \frac{\mu}{\sin^2 \phi + \mu \cos^2 \phi}$$  \hspace{1cm} \text{(3.30)}

and

$$\tan 2\psi = \frac{(1 - \mu) \sin 2\phi}{\beta \cdot (1 - \mu) \cos 2\phi}$$  \hspace{1cm} \text{(3.31)}
(c) **Applied Principal Moments Coinciding with the Directions of Reinforcements**

In this case $\psi = 0$; the substitution of this value of $\psi$ in expressions 3.25 and 3.27 gives

$$\beta = \frac{(\mu + \omega) - \sqrt{(\mu + \omega)^2 - 4\mu_\omega}}{2\omega}$$

or

$$\beta = \frac{(\mu + \omega) - \sqrt{(\mu - \omega)^2}}{2\omega}$$

(3.32)

and from expression 3.22, one obtains

$$\tan 2\psi = 0$$

or

$$\psi = 0^\circ \text{ or } 90^\circ$$

(3.33)

From expressions 3.32 and 3.33 the following three cases will arise depending upon the relative values of $\mu$ and $\omega$.

(i) If $\mu > \omega$; $\beta = 1.0$ and $\psi = 0^\circ$, in this case the yield line will form parallel to the y-axis.

(ii) If $\mu < \omega$; $\beta = \frac{\mu}{\omega}$ and $\psi = 90^\circ$, in this case the yield line will form parallel to the x-axis.

(iii) If $\mu = \omega$; the applied moments will be equal to the resisting moments in all directions at the same time. This will result in a number of yield lines in all directions. This moment state corresponds to the point 'A' of Johansen's yield criterion (Fig. 3.9).

The following two examples will now illustrate the use of the yield criterion.
Example (1) A non-isotropically reinforced slab subjected to uniaxial moment as shown in Fig. 3.12.
Assume $\phi = 30^\circ$ and $\mu = 0.5$.
From expressions 3.30 and 3.31, one obtains
$\beta = 0.80$, and
$\psi = 19.1^\circ$
Thus the slab will fail when the value of uniaxial applied moment will reach 0.80 times $M_{px}$ and the yield line will form at $19.1^\circ$ with the centre line of the slab as shown in Fig. 3.12. If the yield line was assumed to be perpendicular to the principal moment direction, the resisting capacity would have been $0.875 M_{px}$ — an overestimate of about 10%.

Example (2). A non-isotropically reinforced slab subjected to biaxial bending moments as shown in Fig. 3.13.
Assume $\phi = 45^\circ$, $\mu = 0.8$, and $\omega = 0.5$
Substitution of these values in expressions 3.25, 3.27 and 3.22 yields

$P = Q = 0.90$

$\beta = 0.875$, and

$\psi = 12.3^\circ$

This indicates that yield will occur when $M_1$ reaches a value of $0.875 M_{px}$ and the normal to the yield line will make an angle of $12.3^\circ$ with the axis of $M_1$ as shown in Fig. 3.13.
3.6 Discussion

Expressions 3.22 and 3.27 completely define the yield criterion at any point of an isotropically or non-isotropically reinforced slab. Usually the values of $M_{px}$ and $M_{py}$ are known from the thickness of the slab and the amount of reinforcement provided in the two directions. $\xi$ and $\omega$ which depend upon the applied moments are known from the geometry and physical dimensions of slab, boundary conditions and the type of applied loading; thus $\beta$ can be obtained from expression 3.27. This implies that at increasing loads, yield will occur when the value of moment $M_1$ at the point in question will reach a value equal to $\beta \times M_{px}$. The direction of the yield line will then be obtained from expression 3.22.

Figures 3.14 to 3.16 represent the yield conditions graphically of an isotropically reinforced slab subjected to different combinations of applied moments. Such a slab has the same resisting capacity at any orientation, so the yield line is always perpendicular to the applied principal moment irrespective of the value of $\xi$.

Figure 3.17 shows a non-isotropically reinforced slab subjected to uniaxial moments. In this case the yield line is not perpendicular to the direction of $M_1$ as was explained in Example (1) of the last section. Here the yield occurs corresponding to the angle $(\xi + \psi)$ where the applied and resisting moment curves become tangential.

The slab represented in Fig. 3.18 is non-isotropically reinforced subjected to pure torsional moments. Formation of yield lines is governed by expressions 3.22 and 3.27. If the top and bottom reinforcements are identical then simultaneous negative and positive yield will occur only for $\xi = 45^\circ$ as indicated by curve 2 of Fig. 3.18.
Figure 3.19 represents a slab non-isotropically reinforced and subjected to a combination of bending and twisting moments. If the applied moments correspond to curve 1, positive yield occurs at point A. For certain combination of applied moments simultaneous negative and positive yield will occur as in the case represented by curve 2 where yield occurs at points B and C.

Figures 3.20 to 3.23 represent variation of angle $\Psi$ with angle $\hat{\phi}$ for different types of slabs. The curves are plotted based on expressions 3.22 and 3.27. In fig. 3.21, where $\omega = 0.5$ and $\mu = 0.5$, for $\hat{\phi} = 0$, the moment state corresponds to the point 'A' of Johansen's yield criterion (Fig. 3.9) which indicates that at this point the yield line can form in in-numerable directions. This case which was explained in section 3.5 (c), case (iii), will occur when $\omega = \mu$ and $\hat{\phi} = 0^\circ$. A similar situation will also exist for an isotropically reinforced slab when the ratio of applied moments, $\frac{M_1}{M_2} = \omega = 1.0$, for any value of $\hat{\phi}$ as shown in Fig. 3.22.

These principles of formation of yield lines are used in carrying out a complete elastic - plastic solution of slabs in chapter 4, and an experimental investigation of these yield conditions is described in chapter 5.
4.1 Introduction

In this chapter the method used to obtain the elastic-plastic solution is developed. Small deflection theory of bending of orthotropic plates is used and it is assumed that reinforcement in the slab is provided in two orthogonal directions. Furthermore, the formation of yield lines is assumed to be governed by the principles established in section 3.4.

4.2 Theoretical Considerations

In the small deflection theory of laterally loaded plates (29), the following differential equation in terms of bending and twisting moments is derived from a consideration of equilibrium of forces on a plate element (Fig. 4.1):

\[ \frac{\partial^2 M_x}{\partial x^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yx}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q(x, y) \]  

(4.1)

where \( M_x \) and \( M_y \) are the bending moments per unit length acting on the edges parallel to the \( y \) and \( x \) axes respectively; \( M_{xy} \) denotes the twisting moment per unit length tending to rotate the element about an axis parallel to the \( x \)-axis and \( M_{yx} \) is the twisting moment per unit length to rotate the element about an axis parallel to the \( y \)-axis. \( q(x, y) \) is the intensity of the continuously distributed load.

It may be noted that in Fig. 4.1, the moments are represented by double headed vectors and the \( x \)-\( y \) plane is in the middle surface of the plate element. The directions in which moments and forces are considered positive are those shown in Fig. 4.1.

The equilibrium equation 4.1 is valid for the elastic as well as for plastic stage since no material properties have been considered in its derivation.

Considering a plate made of orthotropic material having axes of orthotropy corresponding to \( x \) and \( y \) directions, the stress-strain relations can be expressed in the following forms:
\[
\sigma_x = \frac{E_x}{(1-\nu_\nu)} \left( \varepsilon_x + \nu \varepsilon_y \right)
\]
\[
\sigma_y = \frac{E_y}{(1-\nu_\nu)} \left( \varepsilon_y + \nu \varepsilon_x \right)
\]

\[
\tau_{xy} = G_{xy} \gamma_{xy}
\]

\[
\tau_{yx} = G_{yx} \gamma_{xy}
\]

If plane section remains plane before and after bending, the following relations are valid:

\[
\varepsilon_x = -z \frac{\partial^2 w}{\partial x^2}
\]
\[
\varepsilon_y = -z \frac{\partial^2 w}{\partial y^2}
\]

\[
\gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y}
\]

Combining expressions 4.2 and 4.3, the following stress-displacement relations are obtained

\[
\sigma_x = -z \frac{E_x}{1-\nu_\nu} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)
\]
\[
\sigma_y = -z \frac{E_y}{1-\nu_\nu} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right)
\]

\[
\tau_{xy} = -2z G_{xy} \frac{\partial^2 w}{\partial x \partial y}
\]
\[ \tau_{yx} = -2z \frac{\partial^2 w}{\partial x \partial y} \]

With these expressions for stresses, the bending and twisting moments can be expressed as:

\[ M_x = -(D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}) \]

\[ M_y = -(D_y \frac{\partial^2 w}{\partial y^2} + D_2 \frac{\partial^2 w}{\partial x^2}) \]

\[ M_{xy} = +2 D_{xy} \frac{\partial^2 w}{\partial x \partial y} \]

\[ M_{yx} = -2 D_{yx} \frac{\partial^2 w}{\partial x \partial y} \]

Various flexural and torsional rigidities in expressions 4.5 are defined as follows:

\[ D_x = \frac{E_t}{12(1-v_x v_y)} t^3 \]

\[ D_y = \frac{E_t}{12(1-v_x v_y)} t^3 \]

\[ D_{xy} = G_{xy} \frac{t^3}{12} \]

\[ D_{yx} = G_{yx} \frac{t^3}{12} \]

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In order to express the governing differential equation of an orthotropic plate in terms of transverse displacement \( w \) of the middle surface, expressions 4.5 are substituted into equation 4.1 and the following equation is obtained:

\[
\frac{D}{\partial x^4} + 2 \frac{H}{\partial x^2 \partial y^2} + \frac{D}{\partial y^4} w = q(x, y) \quad (4.7)
\]

in which

\[
2H = D_1 + D_2 + 2D_{xy} + 2D_{yx} \quad (4.8)
\]

This is the general differential equation of the orthotropic plate, deduced by Huber and known as Huber's equation.

In the case of isotropic medium, no more than two elastic constants are present in the stress-strain laws. These constants, modulus of elasticity \( E \), and Poisson's ratio \( \nu \), are related to the other constants as:

\[
E_x = E_y = E \quad (4.9)
\]

\[
\nu_x = \nu_y = \nu
\]
\[ G_{xy} = G_{yx} = \frac{E}{2(1 + \nu)} \quad (4.9) \]

Expressions 4.6 then reduce to

\[ D_x = D_y = \frac{Et^3}{12(1 - \nu^2)} = D \]

\[ D_1 = D_2 = \nu D \quad (4.10) \]

\[ D_{xy} = D_{yx} = \frac{1 - \nu}{2} D \]

It follows further that expressions 4.5 reduce to

\[ M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \]

\[ M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (4.11) \]

\[ M_{xy} = M_{yx} = +D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \]

Also \[ 2H = D_1 + D_2 + 2D_{xy} + 2D_{yx} = 2D \] and equation 4.7 reduces to the form:

\[ D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = q(x,y) \quad (4.12) \]

A solution of equation 4.7 (for an orthotropic plate) or its reduced form equation 4.12 (for an isotropic plate) satisfying the appropriate boundary conditions constitutes the complete solution to an elastic

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plate bending problem under the assumptions of the small deflection theory. This solution is valid for all points of the plate within the elastic range.

4.3 Application to Reinforced Concrete Slabs

The various flexural and torsional rigidities of the reinforced concrete slab may be approximated as follows (29):

Let $E_s$ be the modulus of elasticity of steel, $E_c$ that of concrete assumed equal in all directions, $\nu$ Poisson's ratio for concrete, also assumed equal in all directions; and $m$ is the ratio of $E_s$ to $E_c$, then

$$D_x = \frac{E_c}{1-\nu^2} \left[ I_{cx} + (m-1) I_{sx} \right]$$

$$D_y = \frac{E_c}{1-\nu^2} \left[ I_{cy} + (m-1) I_{sy} \right]$$

$$D_1 = \nu D_x$$

$$D_2 = \nu D_y$$

$$D_{xy} = D_{yx} = \frac{1-\nu}{2} \sqrt{D_x D_y}$$

where $I_{cx}$ and $I_{sx}$ are moments of inertia of the concrete and of the steel reinforcement respectively about the neutral axis, in the section parallel to the y-axis. $I_{cy}$ and $I_{sy}$ are the respective values for the section parallel to the x-axis. The values of $I_{cx}$ and $I_{cy}$ are based on the effective (uncracked) area of concrete, which is dependent on the depth of the crack. The value of the modulus of elasticity of concrete is taken as the secant modulus* corresponding to the point on the stress-strain diagram of concrete (42) representing the maximum compressive stress at the point in question. Also if the concrete has cracked at the level of the steel,

* Since the stress at the section varies from zero at the neutral axis to the maximum at the extreme fibre, the secant modulus is believed to give better approximation than the tangent modulus.
(m-1) in the expressions 4.13 is replaced by m.

Substituting $D_{xy} = D_{yx}$, as from expressions 4.13, the equation of equilibrium, Eq. 4.1, becomes:

$$\frac{\partial^2 M}{\partial x^2} - 2\frac{\partial^2 M}{\partial x \partial y} + \frac{\partial^2 M}{\partial y^2} = -q(x, y)$$

(4.14)

4.4 Moment - Deflection Relations for a Yielded Point

Let it be assumed that at a yielded point, the yield line forms along the direction $t$, the normal to which makes an angle $\theta$ with the $x$ direction; then, from the first two of expressions 4.5, one can write

$$\frac{\partial^2 w}{\partial y^2} = - \frac{M_y}{D_y} \frac{D_y}{D_x} \frac{u M_x}{x}$$

(4.15)

which can further be written as:

$$M_y = -D_y (1-u^2) \frac{\partial^2 w}{\partial y^2} + \frac{D_y}{D_x} u M_x$$

(4.16)

Applying expression 4.16 to the $n$-$t$ system of axes at the instant of yielding, one can write

$$M_t = -D_t (1-u^2) \frac{\partial^2 w}{\partial t^2} + \frac{D_t}{D_n} u M_{pn}$$

(4.17)

where $D_t$ and $D_n$ are the values of the flexural rigidities of the slab in the $t$ and $n$ directions respectively.

At the instant of yielding $M_n = M_{pn}$ and $M_{nt} = M_{pnt}$ from

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expressions 3.16 and 3.17 respectively. Thus for further increase of
load, increase in $M_n$ and $M_{nt}$ at this point will be zero. The last term
in expression 4.17 depends upon $M_{pn}$ which is constant for a particular
point so this term will contribute nothing to the increase in $M_t$. Thus
for additional loads the incremental values of moments are given by the
followings:

$$\Delta M_n = 0$$

$$\Delta M_t = -D_t(1-u^2) \frac{\partial^2(\Delta w)}{\partial t^2}$$

$$\Delta M_{nt} = 0$$

where $\Delta w$ is the incremental value of the deflection and $D_t$ is the flexural
rigidity of the slab in the direction $t$. From expressions 4.18 the
moments in $x$-$y$ system can be obtained by using the Mohr's circle for
plane stresses.

According to the principles of formation of yield lines established
in section 3.4, there can be four types of yielding in the slab. The
corresponding simplified moment-deflection relations for each type are
described below.

**Type I**

The first type of yielding will occur at a point where the follow-
ing conditions exist.

$$M_{xy} = 0, \quad M_x = M_{px}, \quad M_y < M_{py}$$

(4.19)
In this case $M_x$ is the maximum principal moment and yielding is parallel to $y$-axis. A further increment of loading will result in an increase in $M_y$ which can still increase until its value reaches the plastic moment $M_{py}$. However, since changes in the curvature in the $x$ direction will not affect $M_y$ because of yielding, Poisson's ratio for this point for subsequent increments of load must be taken as zero. The following moment-deflection relations can be used for this point in incremental form:

$$\Delta M_x = 0$$

$$\Delta M_y = -D_y(1-\nu^2) \frac{\partial^2(\Delta w)}{\partial y^2}$$

Here it may be mentioned that $D_y(1-\nu^2) = E I_y$, which is the value of flexural rigidity with $\nu = 0$.

**Type II**

The second type of yielding occurs at a point where the following conditions are present:

$$M_{xy} = 0, \quad M_x < M_{px}, \quad M_y = M_{py}$$

In this case yielding is parallel to the $x$ direction. $M_x$ can still increase in magnitude for subsequent increase in load until its value reaches $M_{px}$. Here again Poisson's ratio will be taken as zero, giving the following moment-deflection relations in incremental form:
Type III

The third type of yielding will occur at a point where \( M_{xy} \neq 0 \) and principal moments do not coincide with the x-y directions. In this case yielding will occur at an angle \( \psi \) given by the expression 3.22. It may be pointed out that this yielding will be perpendicular to the direction of maximum principal moment only in an isotropically reinforced slab but will be in a different direction, depending upon the principle of least resistance, in a non-isotropically reinforced slab. The moment deflection relations for such a point in the n-t system of axes are given by the expressions 4.18. From Fig. 3.10, the angle which the normal to the yield line will make with the x direction is given by \( \theta = \varphi + \psi \). Therefore the moment-deflection relations in the x-y system of axes for this point can be written as follows in incremental form:

\[
\Delta M_x = -D_x (1-v^2) \frac{\partial^2(\Delta w)}{\partial x^2} \sin^2 \theta - \frac{\partial^2(\Delta w)}{\partial x \partial y} \sin 2\theta + \frac{\partial^2(\Delta w)}{\partial y^2} \cos^2 \theta \sin^2 \theta
\]

\[
\Delta M_y = -D_x (1-v^2) \frac{\partial^2(\Delta w)}{\partial x^2} \sin^2 \theta - \frac{\partial^2(\Delta w)}{\partial x \partial y} \sin 2\theta + \frac{\partial^2(\Delta w)}{\partial y^2} \cos^2 \theta \cos^2 \theta
\]

\[
\Delta M_{xy} = -D_t (1-v^2) \frac{\partial^2(\Delta w)}{\partial x^2} \sin^2 \theta - \frac{\partial^2(\Delta w)}{\partial x \partial y} \sin 2\theta + \frac{\partial^2(\Delta w)}{\partial y^2} \cos^2 \theta \sin \theta \cos \theta
\]
It can be readily observed that types I and II are special cases of type III yield when $\theta = 0^\circ$ and $90^\circ$ respectively.

**Type IV**

The fourth type of yielding will occur at a point which has previously yielded in accordance to any of the first three types of yields and, after subsequent increase of load, now yields in a direction perpendicular to that of the previous yield. At this stage the resisting capacity of this point may be assumed to have been completely exhausted and no increase in any moment can now take place at this point for further increase of load. The moment field is now represented by the corner point 'A' of Fig. 3.9. Hence for this point in subsequent stages:

\[ \Delta M_x = 0 \]

\[ \Delta M_y = 0 \]  \hspace{1cm} (4.24)

\[ \Delta M_{xy} = 0 \]

4.5 **Outline of the Method**

Expression 4.1 represents the equilibrium equation and is valid for both elastic and elastic-plastic states. Expressions 4.5 give moment deflection relations in the elastic stage only. Substitution of expressions 4.5 into expression 4.1 gives the governing differential equation, Eq. 4.7, or its reduced form, Eq. 4.12 (for the isotropic case), in terms of unknown deflections and applied loadings. Solution of this equation with appropriate boundary conditions will constitute the complete solution of the elastic...
plate problem.

Although a governing equation in terms of deflections such as expression 4.7 may be derived for the elastic state, there exists no such simple relationship for the plastic state. Therefore, in this study the complete solution is obtained in a series of linear incremental solutions which approximate the overall behavior of the elastic-plastic plate. At the elastic limit load the first point in the plate yields. For the sake of mathematical representation if the plate is replaced by a number of discrete mass points, then beyond the elastic limit load the elastic equations govern at all points except the one that has yielded. Moment-deflection relations for this yielded point are modified according to section 4.4. These new moment-deflection relations are now substituted in the equilibrium equation and the second stage solution is obtained. The structure acts linearly during the load increment required to cause the next point to yield. The incremental load must be so adjusted that yielding just starts at the next highly stressed point. This will require an iteration procedure since the total moments at any point are not linear with the incremental value of the load. The total deflections and moments will now be the sum of the elastic deflections and moments, and their increments due to incremental load in the second stage. At this load level the next point yields whose moment-deflection relations will have to be modified before the third stage solution can be carried out. Thus even though the overall behavior of the structure is non-linear it may be adequately expressed by a series of linearly incremental solutions.

It should be noted that in certain stages of loading, more than one point of the slab may yield simultaneously and if so, moment deflection relations of all such points must be modified before the
next stage solution is attempted.

The ultimate load is obtained when sufficient points have yielded
to convert the slab into a collapse mechanism which is indicated when
deflections become large.

4.6 Numerical Procedure

The consideration and inclusion of non-linear material behavior
into the problem necessitates a numerical technique of solution. Finite
difference method is employed in this study to approximate the differential equations. This requires writing the equations at discrete points
on the slab using finite difference approximations to represent the
partial differential equations; this reduces the problem to a system of
algebraic equations. The following standard finite difference operators
(32) are used:

\[
\frac{\partial}{\partial x} \approx \frac{1}{2h_x} \begin{bmatrix} -1 & 0 & +1 \end{bmatrix}
\]

\[
\frac{\partial^2}{\partial x^2} \approx \frac{1}{(h_x)^2} \begin{bmatrix} +1 & -2 & +1 \end{bmatrix} \tag{4.25}
\]

\[
\frac{\partial^2}{\partial x \partial y} \approx \frac{1}{4h_x h_y} \begin{bmatrix} +1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & +1 \end{bmatrix}
\]

where \( h_x \) and \( h_y \) are the non-dimensional spacings between grid points in
x and y directions respectively.

The boundary conditions along the simply supported edge \( x = 0 \) or
\( x = a \) are:
\[ M_x = 0 \]

\[ w = 0 \]

Since \( M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \) and since there is no curvature in the y direction, i.e. \( \frac{\partial^2 w}{\partial y^2} = 0 \), then the zero moment condition reduces to \( \frac{\partial^2 w}{\partial x^2} = 0 \). Similarly at the edge \( y = 0 \) or \( y = b \), the boundary conditions are:

\[ w = 0 \]

\[ \frac{\partial^2 w}{\partial y^2} = 0 \quad (4.26) \]

Since both curvatures are zero, the only moment component present at the simply supported edge is the twisting moment.

The boundary conditions along a fixed edge \( x = 0 \) or \( x = a \) are:

\[ w = 0 \]

\[ \frac{\partial w}{\partial x} = 0 \quad (4.27) \]

and along the edge \( y = 0 \) or \( y = b \)
\[ w = 0 \quad (4.28) \]

\[ \frac{\partial w}{\partial y} = 0 \]

To apply any of the applicable finite difference operator for expressions 4.5 at the edge points, fictitious points outside the edge must be established. It has been shown above that both curvatures along a simply supported edge are zero. Writing this for point 3 in Fig. 4.2:

\[ w_7 - 2w_3 + w_{19} = 0 \]

Since the deflection at the edge is zero, therefore

\[ w_{19} = -w_7 \quad (4.29) \]

Fictitious points established in the same way are used in applying the finite difference approximations at the edge points. At the corner

\[ w_{16} = w_6 \]

has been used.

For a fixed edge, the slope perpendicular to the edge is zero. Applying this for point 3 in Fig. 4.2 yields:

\[ w_{19} = w_7 \quad (4.30) \]

and at the corner \( w_{16} = w_6 \)

In this study square and rectangular slabs subjected to uniform
load and with either simply supported or fixed edges were analysed.

Due to the symmetry of the problem, therefore, only one octant of the
slab was considered for square slab and one quadrant for the rectangular
slab. The solution procedure is described below and a typical example
is given in Appendix 'C'.

The finite difference approximation of Eq. 4.14 at all interior
grid points gives a set of linear equilibrium equations. These equations
may be written in matrix form as follows:

\[ [AA] \{ M_x \} + [AB] \{ M_{xy} \} + [AC] \{ M_y \} = \{ q \} \tag{4.31} \]

where \( \{ M_x \} \), \( \{ M_y \} \) and \( \{ M_{xy} \} \) are the column matrices of unknown moments
at all grid points and \( \{ q \} \) is the column matrix of known applied loads.
\([AA], [AB] \) and \([AC] \) are the matrices corresponding to the first, second
and third term of Eq. 4.14 obtained by applying the appropriate finite
difference operators at those grid points whose deflections are unknown.
Expressions 4.5 represent the moment-deflection relations in the elastic
stage and finite difference approximations of these expressions gives the
following relations in matrix form:

\[ \{ M_x \} = [B] \{ w \} \]
\[ \{ M_y \} = [C] \{ w \} \tag{4.32} \]
\[ \{ M_{xy} \} = [D] \{ w \} \]

where \( \{ w \} \) is a column matrix of unknown deflections, and \([B], [C] \) and
[D] are the corresponding matrices obtained by applying the appropriate finite difference operators to expressions 4.5.

Substitution of expressions 4.32 into 4.31 gives a set of simultaneous equations for the unknown deflections of the form given below:

\[ [A] \{w\} = \{q\} \quad (4.33) \]

where,

\[ [A] = [AA] [B] + [AB] [D] + [AC] [C] \]

Here [A] is a square matrix of the order n x n while n is the number of unknown deflections. At this stage it may be noted that equations of the form of 4.33 may also be obtained by direct substitution of finite difference approximations into the differential equation 4.7 but this has not been done here for the simple reason that although a governing equation in terms of deflections such as Eq. 4.7 may be obtained for the elastic stage, there exists no such simple relationship for the plastic stage.

Solution of Eqs. 4.33 yields the deflection surface and the moments can then be obtained from expressions 4.32. This is the elastic solution in terms of load intensity q. The principal moments at each point are calculated according to the usual combined moment relationships (Mohr's circle) and each point is examined for yield according to the yield criterion of expression 3.27. The value of the load to cause each point to yield is calculated separately and the minimum load intensity which will cause the most highly stressed one or more points to yield is obtained. This is the elastic load and is termed as the first stage solution. All values of solution vectors, deflections and moments, are

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recorded.

After the initial yielding of a point the moment-deflection relations for that point are modified to account for yielding, i.e. the elastic equations are replaced according to the type of yield described in section 4.4. This requires changing only the corresponding rows in matrices \([B]\), \([C]\) and \([D]\) in Eqs. 4.32. This procedure is followed for all points that yield simultaneously in the first stage. Substituting these new moment deflection relations into Eq. 4.31 gives a new set of simultaneous equations having a revised matrix \([A]\). The solution of this set of simultaneous equations yields the second stage solution. An arbitrary load increment \(\Delta q\) is applied and the resulting values of moments are added to the previous values. New principal moment values are calculated at each point and each point is checked for yielding. The increment of load for the second stage must be such that yielding just occurs at the most highly stressed point in accordance with any one of the four types of yield established in section 4.4. Since the yield criterion is not linearly dependent upon the moment increment, proportioning of the required increment of load with respect to the arbitrary increment \(\Delta q\) was not as simple. Method of iteration was employed each time using the newly established value of load increment, computing the principal moments and checking for the yield at every individual point.

The incremental values of deflections and moments are then added to the existing values to provide new total values at every point at the end of each increment of loading, i.e.

\[
[w]_{i+1} = [w]_i + [\Delta w]_{i+1}
\]
The computed value of deflection will depend upon the values assumed for the various stiffnesses. Generally the stiffness computations in case of reinforced concrete slabs are based either on the gross uncracked section or on the cracked transformed section. In reality the stiffness of the reinforced concrete section depends upon the level of the load, since the extent of cracking at any point is dependent on the magnitude of moment, and moment of inertia must be calculated corresponding to the effective transformed section, and also the value of the modulus of elasticity of concrete is dependent upon the concrete stress at the point in question.

The stiffness in the present case is the stiffness of the unyielded parts of the slab. However the unyielded zone near the yielded area will be cracked whereas the zone further away from the yielded area may still be relatively uncracked. In this study the value of the stiffness based on the effective transformed section has been used at every stage of loading at different points. The extent of cracking at any loading stage at every grid point is obtained from the magnitudes of the moments at the end of the preceding stage.

The complete procedure of the second stage is continued until sufficient points of the slab have yielded to convert it into a collapse.
mechanism, which is indicated by the deflections suddenly becoming large. At this stage the solution is terminated.

The entire computations of forming the required matrices, solution of the equilibrium equations, calculation of principal moments, checking for yield, proportioning the load increment by iteration, adding increments to previous values, changing the stiffnesses of the points and modifying the matrix to account for yielded points were performed on the IBM system 360/50 computer at the University of Windsor; a comprehensive computer program was written in the Fortran language and a sample appears in Appendix 'E'.

4.7 Presentation and Comparison of Results

(a) Simply Supported Square Slab

A square slab isotropically reinforced, simply supported on all four sides and uniformly loaded was analysed for elastic-plastic behavior, using a grid size of a/16 where 'a' represents the side of the slab. Because of the symmetry only one octant of the slab was considered. The elastic solution was first obtained and each point was examined for yielding. The minimum load to initiate yielding at any point was calculated. This occurred at the centre point at a load value of $q = 22.57 \frac{M_p}{a^2}$ for $\nu = 0.2$. This value of load was $q = 20.845 \frac{M_p}{a^2}$ for $\nu = 0.3$, which compares favourably with the value of 20.877 given by Timoshenko (29). Other values obtained for the initial elastic solution are compared with the exact values obtained from reference (29) in Table 4.1. The error for the central deflection and bending moments was less than 0.5% and for twisting moments was less than 2%. Convergence of the elastic solution was verified by considering the convergence of deflections and moments obtained using decreasing mesh size. Fig. 4.3 shows three
solutions plotted against $h^2$ where 'h' is the non-dimensional grid spacing. It can be observed that extrapolating the elastic quantities to the limiting mesh size, $h = 0$ leads to the values obtained by Timoshenko.

For further stages of loading yielding progressed outwards from the centre along the diagonal and then spread from the diagonal. The ultimate load was obtained when the slab was converted into mechanism and exhibited large deflections. Fig. 4.4 gives the load-deflection diagram for the complete elastic-plastic solution. Typical moments and deflections at the first yield and at ultimate load are shown in Fig. 4.5. Fig. 4.6 presents the plot of load versus centre point deflection of the slab for four different mesh sizes. The difference between the curves decreases with decreasing mesh size, indicating the convergence of the solution.

The ultimate load obtained in this investigation was $q = 24.29 M_p/a^2$ for a mesh size of $a/16$, and $q = 24.41 M_p/a^2$ for a mesh size of $a/12$. The values of the collapse load obtained by limit analysis are given in Table 4.2. Based on Johansen's yield criterion, the upper and lower bound solutions coincide for this case and give a value of $q = 24.0 M_p/a^2$. Thus the value of ultimate load obtained in the present method was 1.2% higher than the exact value which may be attributed to the fact that the present method is a numerical procedure. It may be interesting to note that in reference (33), a similar problem is attempted by finite element method using a square yield criterion and the solution obtained for an element size of $a/12$ is $q = 24.51 M_p/a^2$. 

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### Table 4.1

**Simply supported square slab under uniform load**

\( v = 0.3 \)

<table>
<thead>
<tr>
<th>Grid size ( a/8 )</th>
<th>Central Deflection</th>
<th>Maximum ( M_x )</th>
<th>Maximum ( M_{xy} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00416</td>
<td>0.0485</td>
<td>0.0306</td>
<td></td>
</tr>
<tr>
<td>0.00411</td>
<td>0.0482</td>
<td>0.0315</td>
<td></td>
</tr>
<tr>
<td>0.00408</td>
<td>0.0480</td>
<td>0.0319</td>
<td></td>
</tr>
<tr>
<td>Exact Solution (29)</td>
<td>0.00406</td>
<td>0.0479</td>
<td>0.0325</td>
</tr>
<tr>
<td>multipliers</td>
<td>( qa^4/b )</td>
<td>( qa^2 )</td>
<td>( qa^2 )</td>
</tr>
</tbody>
</table>

### Table 4.2

**Simply supported square slab under uniform load**

<table>
<thead>
<tr>
<th>source</th>
<th>values of ultimate load*</th>
</tr>
</thead>
<tbody>
<tr>
<td>yield criterion</td>
<td></td>
</tr>
<tr>
<td>Johansen</td>
<td>Wood (16)</td>
</tr>
<tr>
<td>Tresca</td>
<td>Huan and Shull (34)</td>
</tr>
<tr>
<td>Von mises</td>
<td>Hodge (8)</td>
</tr>
<tr>
<td>Lower bound</td>
<td>24.0</td>
</tr>
<tr>
<td>upper bound</td>
<td>24.0</td>
</tr>
</tbody>
</table>

*multiplier = \( M_p/a^2 \)
(b) **Square slab with fixed edges**

The solution for an isotropically reinforced square slab with fixed edges and subjected to uniform load was obtained using a grid size of $a/16$ where '$a'$ is the side of the slab. The values obtained in the first stage elastic solution are compared with exact values in Table 4.3. Here the error in central deflection was 4.7% while the errors in the maximum positive and negative moments were 1.3% and 0.8% respectively. Similar to the simply supported case the convergence of the elastic solution was verified by considering the convergence of deflections and moments obtained by using decreasing mesh size. This is shown in Fig. 4.7. First yielding occurred at the middle points of the edges in negative bending at a load value of $q = 19.62 \, M_p/a^2$. The negative moment at the edges is independent of the value of Poisson's ratio since the curvature parallel to the edges remains zero. Therefore the first yield load also does not depend on Poisson's ratio in this case. This value of load is only 0.8% higher than the exact value given by Timoshenko (29).

For further stages of loading yielding progressed along the fixed edges in negative bending. At a later stage the centre point of the slab yielded in positive bending from where yielding progressed outwards along the diagonals, and then spread from the diagonals. The load-deflection diagram for the complete elastic-plastic solution is shown in Fig. 4.8 and typical moment and deflection diagrams are shown in Fig. 4.9. The convergence of the solution with decreasing mesh size is verified from Fig. 4.10 where load-deflection diagrams for decreasing mesh sizes are shown. The value of ultimate load obtained was $q=40.77 \, M_p/a^2$. The upper and lower bounds obtained by Koopman and Lance (35) are: lower bound (Tresca) = $38.3 \, M_p/a^2$, upper bound (Tresca) = $41.1 \, M_p/a^2$. 

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The solution based on the yield line theory (upper bound) yields \( q = 48.0 \, M_p / a^2 \). The value obtained in reference (36) using a square yield criterion with a mesh size of \( a/10 \) is \( q = 41.84 \, M_p / a^2 \).

Table 4.3

Square slab with fixed edges under uniform load

\( \nu = 0.3 \)

<table>
<thead>
<tr>
<th>Grid size ( a/8 )</th>
<th>Central deflection</th>
<th>+ve ( M_{\text{max}} )</th>
<th>-ve ( M_{\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a/12 )</td>
<td>0.00137</td>
<td>0.0237</td>
<td>-0.0505</td>
</tr>
<tr>
<td>( a/16 )</td>
<td>0.00132</td>
<td>0.0234</td>
<td>-0.0509</td>
</tr>
<tr>
<td>Exact solution (29)</td>
<td>0.00126</td>
<td>0.0231</td>
<td>-0.0513</td>
</tr>
<tr>
<td>multipliers</td>
<td>( qa^4/D )</td>
<td>( qa^2 )</td>
<td>( qa^2 )</td>
</tr>
</tbody>
</table>

(c) Rectangular Slabs

Using this method a rectangular slab with simply supported or fixed edges and with any given value of non-isotropy index may be analysed in the same manner as a square slab. The only difference will be in the numbering of grid points since in the case of a rectangular slab one fourth of the slab will have to be considered with two lines of symmetry being present, compared to one octant of slab which was considered for the square slab. The case chosen for illustration in this study was a rectangular slab simply supported along four sides with a length to width ratio equal to 1.5. This case was analysed for a grid size of \( a/12 \) where 'a' is the width of the rectangular slab. The first elastic
solution is compared with the exact one in Table 4.4 where it is observed that the error in the central deflection is less than 1 % and the error in the moments is almost negligible.

The complete elastic-plastic solution was obtained for four different values of non-isotropy index. The results are shown in Table 4.5, where these are compared with other known solutions. These four cases have been studied experimentally as well and the results are presented in Table 5.3. The theoretical load-deflection curves are given in Figs. 5.17 to 5.20 and compared with the experimental curves. The values of moments developed at various points of the slab are shown in Figs. 5.21 and 5.22.

<p>| Table 4.4 |
| Simp\ily supported rectangular slab under uniform load $u=0.3$, $\frac{b}{a}=1.5$, $\mu=1.0$ |</p>
<table>
<thead>
<tr>
<th>Grid size $a/12$</th>
<th>central deflection</th>
<th>$\text{max. } M_x$</th>
<th>$\text{max. } M_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact solution ($\mathcal{B}$)</td>
<td>0.00772</td>
<td>0.0812</td>
<td>0.0498</td>
</tr>
<tr>
<td>multipliers</td>
<td>$qa^4/\mathcal{B}$</td>
<td>$qa^2$</td>
<td>$qa^2$</td>
</tr>
</tbody>
</table>

4.8 Concluding Remarks

A numerical procedure is presented in this chapter for the complete elastic-plastic solution of reinforced concrete slabs. The entire load deflection characteristic is obtained along with the ultimate load. The magnitudes of moments at different points of the slab at any load level may also be computed, which is a valuable information for the distribution
of reinforcement. The method has the added advantage that progression of yielding of points and thus the yield line pattern is also obtained, which in the case of yield line theory has to be assumed. The experimental verification of the method is presented in the next chapter.

Table 4.5

Simply supported rectangular slab under uniform load

\[ \frac{b}{a} = 1.5 \]

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>values of ultimate load*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>yield line theory(5)</td>
</tr>
<tr>
<td>1.0</td>
<td>16.97</td>
</tr>
<tr>
<td>0.8</td>
<td>15.73</td>
</tr>
<tr>
<td>0.6</td>
<td>14.38</td>
</tr>
<tr>
<td>0.4</td>
<td>12.98</td>
</tr>
</tbody>
</table>

\* multiplier = \( \frac{M_p x}{a^2} \)
CHAPTER 5
Experimental Investigation

5.1 Scope of experimental programme

The experimental programme consisted of 32 tests on reinforced concrete slabs with different loading and boundary conditions. Seven tests were aimed at obtaining direct observation on the kinking of reinforcing steel across the cracks; it should be mentioned that all the remaining tests provided an indirect observation on this question also. Twenty six tests were carried out to verify the yield criterion established in chapter 3. The ultimate load and the orientation of yield lines were observed and compared with the predicted analytical results. The remaining six tests were performed to confirm the elastic-plastic solution; load versus deflection behavior, distribution of moments at different loading stages and the values of ultimate load were observed and compared with the analytical solutions.

5.2 Materials

(a) Cement

High early strength Portland cement manufactured by Canada Cement Company was used in all specimens.

(b) Aggregates

Since the cover to the steel reinforcing bars was 3/16 in., the maximum size of aggregate was restricted to 1/8 in. The combined aggregate was prepared by mixing 85% coarse sand and 15% fine sand. This gave a well graded combined aggregate with a Fineness modulus equal to 2.75.

(c) Reinforcement

Plain mild steel bright basic rods of 1/8 in. and 3/16 in.
diameter were used as reinforcement. When received, these rods were rather brittle and resembled typical cold worked steel. Therefore it was decided to anneal these bars so as to simulate their behavior with that of a typical mild steel used as reinforcement. The annealing was done in pieces of 6 ft. length in an industrial air circulation oven at 1200°F for two hours. The annealing process served not only to give the correct yield point but also produced a horizontal part on stress-strain diagram. Typical stress-strain curves for a 3/16 in. diameter rod before and after annealing are shown in Fig. 5.1.

To improve the bond characteristic of these plain steel rods, they were put in a humid room after annealing for one week to accelerate rusting. All loose rust was removed before they were used. Also hooks were made at both ends of each reinforcing rod. Results of pull-out tests on annealed rods with hooks embedded in concrete are shown in Fig. 5.2.

5.3 Concrete mix

The concrete used in all specimens was a small aggregate mix having a compressive strength of about 4700 psi. Water cement ratio was 0.65 and the aggregate cement ratio was 4.5, both by weight. These ratios were maintained constant for all batches. Mixing was done in one cubic foot capacity mixer of tilting drum type and only one batch of concrete was required for each specimen.

5.4 Description of specimens

A total of 32 specimens were tested in addition to the four initial pilot tests. All specimens were 1.5 in. thick and in each case the first layer of steel was placed at 3/16 in. clear cover, and the second layer,
orthogonal to the first, was placed just over the first layer. Principal variables were the non-isotropy index, \( \mu \), the inclination of the first layer of steel with the applied principal moment direction, angle \( \phi \), and the type of applied moments. Properties of these specimens are given in tables 5.2 and 5.3. These can be classified broadly into five series as follows:

1. **Series A**: There were four specimens in this series. All were isotropically reinforced with the first layer of steel making angles of 0°, 15°, 30° and 45° respectively with the direction of applied uniaxial moment, and the second layer of steel was orthogonal to the first one. These specimens were 34 in. x 18 in. in plan and were subjected to uniaxial moment as shown in Fig. 5.3.

2. **Series B**: The dimensions of all the eight specimens of this series were identical to those of series A. These specimens were also subjected to the uniaxial moment but all specimens in this series were non-isotropically reinforced. The variables were the value of \( \mu \) and the inclination of first layer of steel with the direction of applied uniaxial moment, angle \( \phi \).

3. **Series C**: All eight specimens of this series were square in plan with overall dimensions 24 in. x 24 in. as indicated in Fig. 5.4. These specimens were subjected to pure torsional moments and the variables were \( \mu \) and angle \( \phi \). All specimens of this series were doubly reinforced with identical steel at top and bottom.

4. **Series D**: All six specimens of this series were rhomboidal in plan with the ratio of smaller diagonal to the bigger one being 3:4. This ratio was arbitrarily chosen so that the ratio of the applied
principal moments in the test area will fall in the range \(-1 < \frac{M_2}{M_1} < 0\), thereby producing a combination of bending and twisting moments.

With the dimensions of the specimen shown in Fig. 5.5, the ratio \(\frac{M_2}{M_1} = -0.5626\). All specimens of this series were also doubly reinforced with identical steel at top and bottom and the variables were \(\mu\) and angle \(\phi\).

(5) **Series E**: All six specimens in this series were rectangular in plan with overall dimensions 40 in. \(\times\) 28 in. These specimens were reinforced on the bottom tension face only and were tested simply supported on all edges with uniformly distributed load applied on top. The variables were \(\mu\) and angle \(\phi\). This series was planned mainly to verify the results of the elastic-plastic solution.

5.5 **Casting and curing**

All concrete was mixed in a tilting drum type mixer of one cubic foot capacity and only one batch of concrete was needed for each specimen. All specimens were cast in forms made of plywood. The bottom surface of the form was covered with a thin metal sheet. The sides were replaced by new ones as soon as they appeared to be worn.

The reinforcement was placed in the forms after being instrumented with electric strain gages. The bottom layer which is called the first layer was supported by small pieces of cement mortar of 3/16 in. thickness which were specially made for this purpose. The second layer of steel was supported directly over the first layer. In specimens of series C and D which were doubly reinforced with identical reinforcement on top, steel bars for these top layers, called third and fourth layers, were placed on chairs which also provided a clear cover of 3/16 in. from the top. All reinforcing bars were hooked at both ends.
To determine the compressive strength of concrete, three 3 in. x 6 in. cylinders were cast with each specimen. In addition control strips were also cast with some of the specimens. All control strips were 24 in. long and 1.5 in. thick, while the width was varied as whole multiple of bar spacings and ranged from 6.6 in. to 9.0 in. One control strip was made for every spacing and effective depth of steel in the test specimens. Thus 16 control strips reproduced the 16 different combinations of bar spacings and effective depth which occurred in the test specimens. Modulus of rupture specimens were 1 in. x 1 in. in cross section and 6 in. long. These were cast in metal forms.

All concrete in the test specimens and in the control specimens was vibrated on a high frequency vibration table. The top surface of the specimens was troweled smooth. The test and control specimens were cured in water for three days and then were allowed to dry for at least two days before instrumenting the test specimens. Control cylinders were capped before testing.

5.6 Instrumentation

(a) Electric strain gages on reinforcement

One reinforcing bar in each layer of reinforcement was instrumented with electric strain gages placed approximately in the middle of the test area. One reinforcing bar in each control strip was also similarly instrumented. The gages used were BEAN metal foil gages of type BAE-06-125BB-120. This had a nominal gage length of 0.125 in. and a grid width of 0.06 in.

The surface of the reinforcing bar was prepared by cleaning it using fine silicon carbide paper and acetone. The gage was mounted
using Eastman 910 cement as the bonding agent applied according to the manufacturer's recommendations. The lead wires were then soldered to the gage and it was water-proofed with a plasticised epoxy resin system 'Bean Gagekot No. 7'. After curing for 24 hours at room temperature a layer of wax was applied on the gage and lead wires for further protection from the concrete.

Steel reinforcement for specimens E2 and E6 was instrumented at a number of selected points and the locations of gages are indicated in Figs. 5.6 and 5.7.

(b) Electric strain gages on concrete

The concrete strains were measured on the top face of the specimens of series A to D. BEAN type BAE-250 RA, 45° rosette electric strain gages were used. Each of the three elements of the rosette gage had a nominal gage length of 0.25 in. Each specimen was instrumented at the middle point of the test area.

The concrete surface at the location of gage was smoothed using fine sand paper, all dust was removed using compressed air and then the surface was cleaned with acetone. Surface cavities were then filled by applying an epoxy resin cement at the gage location. After the surface was dry, it was again smoothed with fine silicon carbide paper and the gage was applied using Eastman 910 cement. The lead wires were soldered and the gage was moisture-proofed using BEAN Gagekot No. 3.

In the early stages of the experiments some gages were mounted using RTC Epoxy resin (BEAN) and then curing it for 24 hours at room temperature. Other gages were mounted using Eastman 910 cement; no difference was noted in the performance of the gages with either of
the two cementing agents.

(c) **Mechanical dial gages**

The deflections were measured using mechanical dial gages with a sensitivity of 0.001 in. The locations of the dial gages are shown in Figs. 5.3 to 5.5, and are seen in position in Figs. 5.8 and 5.9. The majority of the gages were placed on the top surface of the specimens but some were placed on the bottom surface depending on the ease of handling. In series E all gages were placed on the bottom surface. In series C and D, a dial gage could not be placed at the centre point and so the deflection of this point was obtained by means of another dial gage employing a lever system.

5.7 **Experimental setup and test procedure**

Specimens of series A and B were supported on 1 in. diameter rollers placed at 30 in. apart. Line loads were applied at 6 in. from each support line through similar rollers so as to give a central 18 in. length of specimen under constant uniaxial moment. A sponge rubber of 1/4 in. thickness was used between the rollers and specimens to obtain uniform distribution of load. A central concentrated load was transferred to two uniform line loads through a system of distribution beams. Two lower beams of this distribution system were instrumented with electric strain gages so as to check at every stage of loading that load is uniformly distributed along the 18 in. length of rollers. This setup is shown in Fig. 5.8.

Specimens of series C and D were supported at two diagonally opposite corners with downward concentrated loads applied at the other two corners. Each concentrated load was applied as uniformly distributed
on a circular area of 1.75 in. diameter, and was transferred to the concrete through a 1/4 in. sponge rubber to ensure uniform distribution. A central concentrated load was transferred to two equal concentrated loads as the reactions of a simply supported beam. To minimize on the effects of the membrane forces free lateral movement of corners of slab in the direction of diagonals was accommodated by inserting 1/2 in. diameter steel rollers. This setup is shown in Fig. 5.9.

One specimen from each of series C and D was tested with electric strain gages attached to each reinforcing bar at the middle of the test area. The strains so measured were interpreted in terms of developed moments by the procedure explained in Appendix 'D'. This was performed to verify whether the distribution of moments in the test area is as intended. Figs. 5.10 and 5.11 show the variation of the developed moments at various stages of loading. Near the ultimate stage the variation of moments is quite uniform as seen in these figures. At the initial stages of loading, however, the distribution is not exactly uniform but the difference is well below 5 %.

In each case the central concentrated downward load was applied by means of a mechanical loading frame available in the structural laboratory as shown in Figs. 5.8 and 5.9. This had the added advantage that there was no loss of pressure during the time of recording. The top member of this frame was of solid 3 in. x 6 in. steel section that housed a 2 in. diameter loading screw in a threaded hole at the centre. A load cell (type strain insert, flat universal) was attached at the lower end of the loading screw that acts through the threaded hole in the top member. The compressed load cell which was calibrated before hand,
was coupled with the portable strain indicator (Budd model P-350) which indicated the strain in the load cell.

During the tests strains were measured by means of two switch and balance units (model C1OLC, Budd Instruments Division), a digital strain indicator and an automatic print-out unit as shown in Fig. 5.12.

Specimens of series E were subjected to uniform load which was applied by pumping compressed air into a rubber membrane for which a special setup was designed and fabricated. The rubber membrane was in contact with the top surface of the specimen applying a downward pressure on it. Care was taken to insure that the membrane touched the specimen over the entire area intended to be loaded, and that the conditions at the edges of the specimen remained simply supported. Pressure of the compressed air in the membrane was measured by means of a test gage and also a mercury filled U-tube manometer. Fig. 5.13 shows this test setup with the specimen unloaded, and the details of the arrangement are shown in Fig. 5.14.

All control strips were tested in one way bending at 20 in. span with line loads applied at 6 in. from each support. Modulus of rupture specimens were tested in flexural bending at an effective span of 5 in. under a central concentrated load.

5.8 Experimental results

Experimental results of all control and test specimens are presented in Tables 5.1 to 5.3 and in Figs. 5.15 to 5.22, where these results are also compared with the theoretical results. Yield line patterns of some typical cases are shown in Fig. 5.23.
These results and their comparison with the theoretical values are discussed in chapter 6.
# TABLE 5.1

## PROPERTIES AND TEST RESULTS OF CONTROL STRIPS

<table>
<thead>
<tr>
<th>Mark</th>
<th>tested with slab mark</th>
<th>layer</th>
<th>effective thickness in.</th>
<th>width in.</th>
<th>$f_c'$ p.s.i.</th>
<th>Dia. of bars in.</th>
<th>No. of bars</th>
<th>spacing of bars in.</th>
<th>computed ultimate load lb.</th>
<th>test ultimate load lb.</th>
<th>$Pu(test)$</th>
<th>$Pu(calc.)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CS1</td>
<td>A4</td>
<td>1</td>
<td>1.220</td>
<td>7.71</td>
<td>4535</td>
<td>0.1875</td>
<td>3</td>
<td>2.57</td>
<td>1007</td>
<td>1002</td>
<td>0.996</td>
<td></td>
</tr>
<tr>
<td>CS2</td>
<td>A4</td>
<td>2</td>
<td>1.030</td>
<td>8.60</td>
<td>4535</td>
<td>0.1875</td>
<td>4</td>
<td>2.15</td>
<td>1115</td>
<td>1106</td>
<td>0.992</td>
<td></td>
</tr>
<tr>
<td>CS3</td>
<td>B2</td>
<td>2</td>
<td>1.025</td>
<td>8.25</td>
<td>4580</td>
<td>0.1875</td>
<td>3</td>
<td>2.75</td>
<td>842</td>
<td>849</td>
<td>1.008</td>
<td></td>
</tr>
<tr>
<td>CS4</td>
<td>B7</td>
<td>2</td>
<td>1.065</td>
<td>9.00</td>
<td>4740</td>
<td>0.1250</td>
<td>4</td>
<td>2.25</td>
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<td>617</td>
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<td>463</td>
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<td>9.00</td>
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<td>0.1250</td>
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<td>1543</td>
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<td>8.00</td>
<td>4700</td>
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<td>7.50</td>
<td>4650</td>
<td>0.1250</td>
<td>3</td>
<td>2.50</td>
<td>600</td>
<td>617</td>
<td>1.028</td>
<td></td>
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<td>C7</td>
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<td>1.220</td>
<td>7.50</td>
<td>4560</td>
<td>0.1875</td>
<td>4</td>
<td>1.875</td>
<td>1417</td>
<td>1517</td>
<td>1.071</td>
<td></td>
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<td>C8</td>
<td>2</td>
<td>1.060</td>
<td>7.00</td>
<td>4710</td>
<td>0.1250</td>
<td>2</td>
<td>3.50</td>
<td>400</td>
<td>437</td>
<td>1.093</td>
<td></td>
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<td>5</td>
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<td>1374</td>
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<td>1.060</td>
<td>6.60</td>
<td>4370</td>
<td>0.1250</td>
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CS7 to CS12 (both inclusive) were doubly reinforced with identical steel at top and bottom; others were singly reinforced at bottom only.
### TABLE 5.2

PROPERTIES AND TEST RESULTS OF SPECIMENS

<table>
<thead>
<tr>
<th>Mark</th>
<th>Total thickness in.</th>
<th>$f_c'$ psi</th>
<th>steel layer 1</th>
<th>steel layer 2</th>
<th>angle $\phi$ deg.</th>
<th>$\frac{M_{Py}}{N_{P_k}}$</th>
<th>$\frac{M_2}{M_1}$</th>
<th>orientation of yield line</th>
<th>ultimate load ($P_u$)</th>
<th>$P_u$(meas)</th>
<th>$P_u$(compu)</th>
</tr>
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<tr>
<td>A1</td>
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<td>0.1875</td>
<td>2.15</td>
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<td>0</td>
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<td>0.995</td>
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<td>$M_{Py}$</td>
<td>$M_x$</td>
<td>orientation of yield line</td>
<td>ultimate load $(P_u)$</td>
<td>$R_u$(meas.)</td>
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All slabs in series C and D were doubly reinforced with identical steel at top and bottom.
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<th>steel layer 1 spacing in.</th>
<th>steel layer 2 Dia. in.</th>
<th>steel layer 2 spacing in.</th>
<th>angle $\theta$ deg.</th>
<th>$\frac{M_p}{M_p}$</th>
<th>ultimate load $P_u$ (meas.) lb/sq.ft</th>
<th>ultimate load $P_u$ (comp.) lb/sq.ft</th>
<th>$\frac{P_u (meas.)}{P_u (comp.)}$</th>
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CHAPTER 6

Discussion of Results

6.1 Kinking of reinforcement

In the present investigation specimens A1 to A4, and E1, E3 and E4 gave direct observation on the question of kinking of reinforcement, while all the other tests also provided indirect observation on this phenomenon.

Fig. 5.15 shows the moment-deflection relationships for four isotropically reinforced specimens A1 to A4. The main reinforcement in specimen A1 was parallel with the span direction while the reinforcements in A2, A3 and A4 were inclined to the span direction with angles $\theta$ equal to $15^\circ$, $30^\circ$ and $45^\circ$, respectively. The amount of reinforcement was the same in each case. From Fig. 5.15, it should be observed that if any appreciable kinking of bars had taken place, the specimens with the larger values of $\theta$ would have sustained a higher load, but as can be seen the value of the ultimate moment in each case is the same.

The differences in the initial slopes of these curves can be explained by the fact that if the steel is not in line with the principal curvature, a greater rotation and hence a greater deflection will result before yielding of the steel.

Similarly, Fig. 5.16 shows the load-deflection relationships for specimens E1, E3 and E4. The reinforcement in E1 was parallel to the edges while the reinforcements in E3 and E4 made angles $22.5^\circ$ and $45^\circ$ respectively with the edges. All three specimens were isotropically reinforced with identical amounts of steel. Since all three specimens developed the same resisting moment, it may be concluded that there is
no evidence of kinking of reinforcement.

The analytical values of ultimate load for all the 32 specimens are based on the assumption of no kinking of reinforcement; it can be readily seen from Tables 5.2 and 5.3 that there is good agreement between the analytical and experimental values of ultimate load---an observation which could be considered as proof of the validity of the above assumption. Thus it may be said that the modified yield line theories based on the assumption of kinking may be inaccurate and the explanation for the absence of kinking given in section 3.2 could be considered as valid.

Results of the present investigation with regard to kinking of reinforcement support the results of tests carried out by Nielsen (24), Lenschow and Sozen (23), Morley (26) and Muspratt (38). However, present results are opposed to the findings of Baus and Tolaccia (18), and Kwieciński (19, 20). In the test series of Baus and Tolaccia, no specimen was tested under uniaxial moment, a load condition which gives direct observation on the kinking of reinforcements, while in the test series of Kwieciński a bond failure was encountered at the edges and to take its account, the contribution of some of the bars to ultimate moment was ignored arbitrarily, assuming that these bars would have slipped. To avoid any bond failures in the present investigation both ends of the steel bars were hooked. Morley (39) has pointed out that if kinking as proposed by Kwieciński does take place, it should leave noticeable permanent deformation of steel. This was carefully noted in the present investigation and no deformation of this type was noticed. Wood in a later paper (40) has also proposed to ignore the effect of kinking.
6.2 Yield criterion

Twenty six specimens of series A, B, C and D were planned to obtain direct information about the formation of yield lines. Twelve of these specimens were subjected to uniaxial moment, eight to pure torsional moments and six to a combination of bending and twisting moments. The principal variables were the non-isotropy index and inclination of main reinforcing steel from the applied principal moment direction. The manner in which these parameters were varied is given in Table 5.2.

In the initial stages of loading in all specimens regardless of their properties, the cracks were formed perpendicular to the applied principal moment direction as the moment approached the cracking moment. However, further development of cracks depended on the direction of reinforcement and manner of loading. The theoretical value of angle $\Psi$, the angle which the normal to the yield line makes with the applied principal moment direction, for each case is calculated based on expression 3.22 and is compared with the observed value in Table 5.2.

It should be noted that in reality the yield line is not a single straight line whose inclination may be easily measured; rather, it is a yielding band of certain width made of a number of cracks all of which may not be exactly parallel. Thus some error may have been introduced in estimating the observed value of $\Psi$. Every effort was made to ascertain that the measured value corresponds to the general trend of cracks in the yielding band.

The theoretical and experimental values of the ultimate load are compared in Table 5.2, the theoretical values being calculated from
expression 3.27. The experimental values are on an average 3% higher than the predicted theoretical values. This may be due to the effect of strain hardening in the steel and/or due to the assumed properties of the concrete compressive stress block, and is consistent with the observations in regard to the control strips (Table 5.1).

The generally good agreement between the theoretical and the experimental values of ultimate load and angle $\psi$ in Table 5.2 confirms the validity of the yield criterion of expressions 3.22 and 3.27 as applied to reinforced concrete slabs.

6.3 Elastic-Plastic Solution

The theoretical results of this investigation were compared with the other known solutions in section 4.7 and a good agreement was observed. Specimens E1, E2, E5 and E6 were planned to verify these results experimentally. Figs. 5.17 to 5.20 present the comparison of theoretical and experimental load-deflection characteristics of these specimens, where the load is plotted against the central deflection of the slab. The theoretical and experimental curves are almost coincident up to the first yielding in the slab, after which the experimental value is about 20% to 30% greater than the theoretical value. Since the change in the stiffness of the concrete section as a result of cracking has been accounted for in arriving at the theoretical deflections, this discrepancy may be attributed to the inherent variability of the material.

The experimental values of ultimate load compare reasonably well with the theoretical values in Table 5.3, the former being on the average 4% higher than the later; this may be attributed either to the effect of strain hardening in the steel and/or the membrane action in the slab.
Since the average experimental value of ultimate load is 3% higher in the case of control strips too (Table 5.1), the 4% discrepancy is believed to be due to strain hardening of the steel and/or to the assumed properties of the concrete compressive stress block rather than to the membrane action.

The experimental and theoretical values of the moments at ultimate load and at a load level below the first yield are compared in Fig. 5.21 for slab E2 and in Fig. 5.22 for slab E6. The experimental values of the moments are obtained from the measurements of steel strains at selected points. The relationships between steel strains and developed moments are obtained analytically and also experimentally from the test results of control strips following the procedure used by Guralnick (41). This procedure is described in detail in Appendix 'D'. Figs. 5.21 and 5.22 show a good agreement between the experimental and theoretical values of the moments.

Therefore these experimental observations confirm the validity of the method of elastic-plastic solution developed in chapter 4.

6.4 Sources of error

The possible sources of error in the analytical solution may be one or more of the following arising either due to the assumptions made or due to the numerical approach adopted for the solution:

1) The method of solution is based on the small deflection theory of flexure of plates, where the effect of normal stresses in the direction transverse to the plate are disregarded.

2) In the yielded regions, the material is assumed to behave in a purely plastic manner ignoring the effect of strain.
hardening. The experimental values of the ultimate load were consistently 3% to 4% higher than the theoretical values; this may be attributed partly to the strain hardening.

3) Although a relatively fine grid was used in this study, a better accuracy could have been obtained by decreasing the mesh size.

4) Due to a large number of operations involved in the computer program, the round off error tends to become large. However, it is believed that this error was insignificant for an Engineering problem.

The possible sources of error in the experimental results may be one or more of the following:

1) The strength of 3 in. x 6 in. cylinders was used as a measure of concrete quality. From each batch of concrete three cylinders were made, the average strength of which is believed to represent the true strength of the batch. Even though each slab and corresponding cylinders were fully compacted by vibration, it is reasonable to assume that a difference in strength, due to different degrees of compaction, was present. However, in under-reinforced slabs the effect of variation of concrete strength on ultimate load is quite insignificant.

2) The yield points of reinforcement were determined by means of 12 samples for each size of bar where deviations from the average value of the order of 3% were present. The ultimate
loads of the under-reinforced slabs are almost proportional to the yield point of the tension reinforcement; experimental results will therefore be influenced by such variations.

(3) The variation in the thickness of the slab specimen as well as in the position of reinforcing bar in the section affects the internal moment arm of the section. These were carefully measured at the location of yield lines in all control strips and were found to vary between ± 0.02 in., which corresponds to a variation of the ultimate load of the order of 1%.

(4) The value of angle ψ was estimated so as to correspond to the general trend of cracks along a yield line; some personal judgement error may have been introduced in this estimation.

(5) Some inaccuracy in the measurement of the ultimate load may have been introduced.

(6) Some inaccuracy in the measurement of strains may have been introduced either due to improper alignment of gage or due to inaccuracy of the strain measuring device.

(7) Any deformation of the loading frame will introduce some error but it could not have contributed much considering the excessive rigidity of the loading frame.
CHAPTER 7

Summary and Conclusions

The overall objective of this study had been to obtain a better understanding into the behavior of reinforced concrete slabs at all stages of loading. To this end, first of all a simple approach was followed to develop a yield criterion applicable for both isotropically and non-isotropically reinforced slabs subjected to any given combination of applied moments; and second, a numerical method is developed to obtain the complete elastic-plastic solution which gives both the strength and behavior of a slab without the necessity of assuming a yield pattern or a distribution for moments.

The established yield criterion in any general case can be defined completely by the expressions 3.22 and 3.27. These expressions are sufficient to find two unknown quantities $\beta$ and $\psi$, which will give the magnitude of yield moment and the orientation of yield line in any given case. It is shown that yield will occur perpendicular to the applied principal moment direction only in isotropically reinforced slab and that in non-isotropically reinforced slab the orientation of yield line is governed by the principle of least resistance. This yield criterion has been experimentally verified and it is concluded that there is no increase in moment capacity at the yield line as a result of biaxial compression of concrete or due to the kinking of reinforcing bars. No evidence of kinking, of the bars at any stage of loading was evident.

The numerical method of complete elastic-plastic solution developed in this study can be used to predict the behavior of the slab at any stage of loading from initial to collapse stage. The solution consists
of the sum of a series of incrementally linear solutions, which when added together, represent the non-linear response of the slab. The entire response is obtained including the ultimate load. The method is independent of the bound theorems and there is no necessity of assuming the yield patterns or the distribution of moments. It provides the exact value of the ultimate load within the limits of accuracy of the numerical procedure and the assumptions made. Magnitudes of moments away from the yield lines are also obtained which is a valuable information for the distribution of reinforcement. Progression of yielding of points and thus the yield line pattern is traced during the solution procedure which is an added advantage to the method. Theoretical solutions obtained by this method compare favourably with other known solutions and with experimental results.
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FIG. 3.1 THE STEPPED YIELD CRITERION
FIG. 3.2 STRESSES AND STRAINS IN CONCRETE AT ULTIMATE LOAD
FIG. 3.3 PERCENTAGE ERROR IN NORMAL MOMENT
FOR VARIOUS INCLINATIONS WITH 1.0% STEEL
FIG. 3.4 PERCENTAGE ERROR IN NORMAL MOMENT FOR VARIOUS REINFORCEMENT RATIOS, $\theta = 45^\circ$
FIG. 3.5 KINKING OF REINFORCEMENT ON AN INCLINED YIELD LINE
(a) FULL KINKING

(b) NO KINKING

FIG. 3.6 KINKING OF A REINFORCING BAR
FIG. 3.7 EFFECT OF KINKING ON NORMAL MOMENT
FIG. 3.8 CRACK GEOMETRY
Fig. 3.9 Johansen's yield criteria

(a) $\mu = \mu' = 1.0$

$M_{px} = M_{px}$

(b) $\mu = \mu' = 1.0$

$M_{px} = M_{px}$

(c) $\mu = \mu' \leq 1.0$

$M_{px} = M_{px}$

(d) $\mu \leq 1.0, \mu' \leq 1.0$

$M_{px} = M_{px}$
FIG. 3.10 ORIENTATION OF YIELD LINE-GENERAL CASE
FIG. 3.11 VARIATION OF APPLIED AND RESISTING MOMENTS
FIG. 3.14 ISOTROPICALLY REINFORCED SLAB SUBJECTED TO UNIAXIAL MOMENTS
FIG. 3.15 ISOTROPICALLY REINFORCED SLAB SUBJECTED TO PURE TORSIONAL MOMENTS
FIG. 3.16 ISOTROPICALLY REINFORCED SLAB SUBJECTED TO A COMBINATION OF BENDING AND TWISTING MOMENTS
FIG. 3.17 NON-ISOTROPICALLY REINFORCED SLAB SUBJECTED TO UNIAXIAL MOMENTS
FIG. 3.18 NON-ISOTROPICALLY REINFORCED SLAB SUBJECTED TO PURE TORSIONAL MOMENTS
FIG. 3.19  NON-ISOTROPICALLY REINFORCED SLAB SUBJECTED TO A COMBINATION OF BENDING AND TWISTING MOMENTS
Fig. 3.20 Variation of $\psi$ with $\phi$ for slabs subjected to uniaxial moments ($\omega = 0$).
FIG. 3.21 VARIATION OF $\psi$ WITH $\phi$ FOR SLABS SUBJECTED TO BIAXIAL MOMENTS ($\omega = 0.5$)
FIG. 3.22 VARIATION OF $\psi$ WITH $\phi$ FOR SLABS SUBJECTED TO BIAXIAL MOMENTS ($\omega = 1.0$)

FOR $\mu = 1.0$, $\psi$ CAN HAVE ANY VALUE
FIG. 3.23 VARIATION OF $\psi$ WITH $\phi$ FOR SLABS SUBJECTED TO BIAXIAL MOMENTS ($\omega = -0.5$)
FIG. 4.1 FORCES ON A PLATE ELEMENT
FIG. 4.2 NUMBERING OF GRID POINTS NEAR THE CORNER
FIG. 4.3 CONVERGENCE OF ELASTIC SOLUTION FOR SIMPLY SUPPORTED SQUARE SLAB
FIG. 4.4 LOAD VERSUS CENTRE DEFLECTION DIAGRAM FOR SIMPLY SUPPORTED SQUARE SLAB.
FIG. 4.5 TYPICAL MOMENT AND DEFLECTION DIAGRAMS FOR SIMPLY SUPPORTED SQUARE SLAB

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FIG. 4.6 LOAD-DEFLECTION CURVES FOR SIMPLY SUPPORTED SQUARE SLAB FOR DIFFERENT GRID SIZES

1 — \( a/10, \frac{q a^2}{M_p} = 24.56 \text{ Mpa} \)
2 — \( a/12, \frac{q a^2}{M_p} = 24.41 \text{ Mpa} \)
3 — \( a/14, \frac{q a^2}{M_p} = 24.31 \text{ Mpa} \)
4 — \( a/16, \frac{q a^2}{M_p} = 24.29 \text{ Mpa} \)
FIG. 4.7 CONVERGENCE OF ELASTIC SOLUTION FOR SQUARE SLAB WITH FIXED EDGES
Fig. 4.8 Load versus centre deflection diagram for square slab with fixed edges.
Fig. 4.9 Typical moment and deflection diagrams for square slab with fixed edges.
FIG. 4.10 LOAD-DEFLECTION CURVES FOR SQUARE SLAB WITH FIXED EDGES FOR DIFFERENT GRID SIZES

1—\(a/10, q = 41.81 \text{ M}_p/a^2\)
2—\(a/12, q = 41.22 \text{ M}_p/a^2\)
3—\(a/14, q = 40.90 \text{ M}_p/a^2\)
4—\(a/16, q = 40.77 \text{ M}_p/a^2\)
FIG. 5.1 STRESS STRAIN CURVE FOR BB STEEL RODS $\frac{3}{16}$" DIAMETER

- UNTREATED ROD
- ANNEALED ROD

$E = 29.0 \times 10^6$ PSI

$\sigma_y = 31,900$ PSI

ULT. ELONG. = 27%
FIG. 5.2 RESULTS OF PULL-OUT TEST ON 3/16 INCH DIAMETER BAR WITH AND WITHOUT HOOKS
FIG. 5.3 LOADING ARRANGEMENT FOR TEST SPECIMENS OF SERIES A AND B.

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FIG. 5.4 PLAN VIEW OF TYPICAL SLAB OF SERIES C
FIG. 5.5 PLAN VIEW OF TYPICAL SLAB OF SERIES D
FIG. 5.6 LOCATION OF STRAIN GAGES ON SPECIMEN E2
FIG. 5.7 LOCATION OF STRAIN GAGES ON SPECIMEN E6
Fig. 5.8 Loading arrangement for Series A and B

Fig. 5.9 Loading arrangement for Series C and D

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FIG. 5.10 DEVELOPED MOMENTS IN SPECIMEN C1

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FIG. 5.11 DEVELOPED MOMENTS IN SPECIMEN D1
Fig. 5.12 Digital strain indicator and printout unit

Fig. 5.13 Specimen of Series E before loading
FIG. 5.14 LOADING ARRANGEMENT FOR SPECIMENS OF SERIES E
FIG. 5.15 COMPARISON OF MOMENT-DEFLECTION DIAGRAMS FOR SPECIMENS A1 TO A4

- A1, $\theta = 0^\circ$
- A2, $\theta = 15^\circ$
- A3, $\theta = 30^\circ$
- A4, $\theta = 45^\circ$

MOMENT AT MID SPAN, \( \text{lb} \cdot \text{in} \)
CENTRE DEFLECTION, IN.
FIG. 5.16 COMPARISON OF LOAD-DEFLECTION DIAGRAMS FOR SPECIMENS E1, E3 AND E4
Fig. 5.17 Theoretical and Experimental Load-Deflection Diagrams for Slab E1.
FIG. 5.18 THEORETICAL AND EXPERIMENTAL LOAD-DEFLECTION DIAGRAMS FOR SLAB E2
FIG. 5.19 THEORETICAL AND EXPERIMENTAL LOAD-DEFLECTION DIAGRAMS FOR SLAB E5
FIG. 5.21 COMPARISON OF MEASURED AND THEORETICAL MOMENTS FOR SPECIMEN E2

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FIG. 5.22 COMPARISON OF MEASURED AND THEORETICAL MOMENTS FOR SPECIMEN E6
Fig. 5.23 Typical yield line patterns
APPENDIX A

Proof of the expression 3.15

It is shown below that the condition represented by expression 3.15 is satisfied, that is

\[ \frac{\partial^2}{\partial \psi^2} (M_n - M_{pn}) \leq 0 \]

Let \( F = (M_n - M_{pn}) \) \hspace{1cm} (A.1)

Using expression 3.10 and 3.11 in expression A.1 yields

\[
F = (M_1 \cos^2 \psi + M_2 \sin^2 \psi) - [M_{px} \cos^2 (\phi + \psi) + M_{py} \sin^2 (\phi + \psi)]
\]

\[ \frac{\partial F}{\partial \psi} = -(M_1 - M_2) \sin 2\psi + (M_{px} - M_{py}) \sin 2(\phi + \psi) \]

\[ \frac{\partial^2 F}{\partial \psi^2} = -2(M_1 - M_2) \cos 2\psi + 2(M_{px} - M_{py}) \cos 2(\phi + \psi) \] \hspace{1cm} (A.2)

Using expressions 3.18 in expression A.2 yields

\[ \frac{\partial^2 F}{\partial \psi^2} = -2\beta M_{px} (1-\omega) \cos 2\psi + 2M_{px} (1-\mu) \cos 2(\phi + \psi) \] \hspace{1cm} (A.3)

From expression 3.22

\[ \sin 2\psi = \frac{(1-\mu) \sin 2\phi}{\sqrt{D_1}} \]
\[
\cos 2\psi = \frac{\beta(1-\omega) - (1 - \mu) \cos 2\phi}{\sqrt{D_1}} 
\] (A.4)

where \( D_1 = [(1-\mu)^2 + \beta^2(1-\omega)^2 - 2\beta(1-\mu)(1-\omega) \cos 2\phi] \)

Using expression A.4 in A.3, one finds that

\[
\frac{\partial^2 F}{\partial \psi^2} = \frac{2M}{\sqrt{D_1}} \left[ -\beta(1-\omega) \left\{ \beta(1-\omega) - (1-\mu) \cos 2\phi \right\} 
+ (1-\mu) \cos 2\phi \left\{ \beta(1-\omega) - (1-\mu) \cos 2\phi \right\} 
- (1-\mu) \sin 2\phi \left\{ (1-\mu) \sin 2\phi \right\} \right] 
\]

On simplification it yields

\[
\frac{\partial^2 F}{\partial \psi^2} = \frac{2M}{\sqrt{D_1}} \left[ \beta^2(1-\omega)^2 - 2\beta(1-\omega)(1-\mu) \cos 2\phi + (1-\mu)^2 \right] 
\] (A.5)

In expression A.5, \( \frac{2M}{\sqrt{D_1}} \) is a positive quantity since the positive sign will be taken before the square root in \( \sqrt{D_1} \). Also the first and third terms in the square bracket on the right hand side are positive, while the second term which is negative will have a maximum negative value for \( \phi = 0 \) at which \( \cos 2\phi = 1.0 \). Thus the least value of the expression in the square bracket on the right hand side will be \( \left\{ \beta(1-\omega) - (1-\mu) \right\}^2 \).

This proves that the value of \( \frac{\partial^2 F}{\partial \psi^2} \) as given by expression A.5 will always be less than or equal to zero, that is

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\[ \frac{\partial^2}{\partial \psi^2} (H_n - H_{pn}) \leq 0 \]
APPENDIX B

To show that the values of $\beta$ given by expression 3.26 are real.

In order to show that values of $\beta$ given by expression 3.26 are real, it must be proved that the portion under the square root sign in the expression 3.26 is greater than or equal to zero. Let this portion be denoted by $S$ as follows:

$$S = (P + \omega Q)^2 - 4\mu \omega$$

(B.1)

Using expression 3.25 and rearranging the terms

$$S = [(1 + \mu \omega) \sin^2 \phi + (\mu + \omega) \cos^2 \phi]^2 - 4\mu \omega$$

$$= [(1 + \mu \omega) \sin^2 \phi + (\mu + \omega) \cos^2 \phi]^2 - 4\mu \omega(\sin^2 \phi + \cos^2 \phi)^2$$

collecting the terms

$$S = [(1 + \mu \omega)^2 - 4\mu \omega] \sin^4 \phi + [(\mu + \omega)^2 - 4\mu \omega] \cos^4 \phi$$

$$+ 2 [(1 + \mu \omega)(\mu + \omega) - 4\mu \omega] \sin^2 \phi \cos^2 \phi$$

$$S = [(1 - \mu \omega) \sin^2 \phi + (\mu - \omega) \cos^2 \phi]^2 - 2(1 - \mu \omega)(\mu - \omega) \sin^2 \phi \cos^2 \phi$$

$$+ 2[(1 + \mu \omega)(\mu + \omega) - 4\mu \omega] \sin^2 \phi \cos^2 \phi$$
Rearranging,

\[ S = \left[(1-\mu\omega)\sin^2\phi + (\mu-\omega)\cos^2\phi\right]^2 + 4\omega(1-\mu)^2\sin^2\phi \cos^2\phi \]  \hspace{1cm} (B.2)

Now since the value of \(\omega\) varies from -1.0 to +1.0, the following two cases are considered:

(1) \(\omega \geq 0\)

It is evident from expression B.2 that for this case always:

\[ S \geq 0 \]

(ii) \(\omega < 0\)

For negative values of \(\omega\), the value of \(S\) as given by expression B.2 will be minimum for \(\mu = 0\), since \(\mu\) varies from 0 to +1.0. Substitution of \(\mu = 0\) in expression B.2 yields

\[ S = \left[\sin^2\phi + \omega \cos^2\phi\right]^2 \]  \hspace{1cm} (B.3)

Since this minimum value of \(S\) as given by expression B.3 is positive, it implies that for all values of \(\omega < 0\), it can be concluded that:

\[ S \geq 0 \]

It is therefore, proved that for all possible values of \(\omega\) and \(\mu\), the value of \(S\) as given by expression B.1 is greater than or equal to zero.
APPENDIX C

Illustrative example

The following example illustrates the method of solution outlined in section 4.6. A simply supported square slab with uniform load is analysed. For the sake of keeping the sizes of the matrices small, in this illustration, a relatively coarse grid size of $a/6$ is chosen where 'a' is the side of the square. The value of Poisson's ratio is assumed to be equal to 0.2. Numbering of the grid points is shown in Fig. C.1.

![Diagram of a square slab with grid points numbered 1 to 16.](image)

**FIG. C.1**

There are only six unknown deflections of the points 1 to 6. Applying the equilibrium equation 4.11 to the points 1 to 6, the following equation is obtained:
\[
[AA] \begin{bmatrix} M_{x_1} \\ M_{x_2} \\ \vdots \\ M_{x_{16}} \end{bmatrix}_{i=1,16} + [AB] \begin{bmatrix} M_{xy_1} \\ M_{xy_2} \\ \vdots \\ M_{xy_{16}} \end{bmatrix}_{i=1,16} + [AC] \begin{bmatrix} M_{y_1} \\ M_{y_2} \\ \vdots \\ M_{y_{16}} \end{bmatrix}_{i=1,16} = \{q_i\}_{i=1,6}
\]

(C.1)

where \(\{M_{x_i}\}, \{M_{y_i}\}\) and \(\{M_{xy_i}\}\) are the column matrices of unknown moments of the form

\[
\begin{bmatrix} M_{x_1} \\ M_{x_2} \\ \vdots \\ M_{x_{16}} \end{bmatrix}, \begin{bmatrix} M_{y_1} \\ M_{y_2} \\ \vdots \\ M_{y_{16}} \end{bmatrix}, \begin{bmatrix} M_{xy_1} \\ M_{xy_2} \\ \vdots \\ M_{xy_{16}} \end{bmatrix}
\]

\(\{q_i\}\) is the column matrix of applied loads; assuming a unit value for the uniform load considered in this example it has the value:

\[
\{q\} = \begin{bmatrix} -1.0 \\ -1.0 \\ -1.0 \\ -1.0 \end{bmatrix}
\]
and \([\text{AA}], [\text{AB}], [\text{AC}]\) are the resulting matrices obtained by applying appropriate finite difference operators to the first, second and third terms respectively of Eq. 4.14. For the present example these matrices have the values shown in Fig. C.2.

Finite difference approximations of Eqs. 4.11, which are the moment deflection relations in the elastic range, gives the following expressions:

\[
\begin{align*}
[M_{x_i}]_{i=1,16} &= [B] \{w_i\}_{i=1,6} \\
[M_{y_i}]_{i=1,16} &= [C] \{w_i\}_{i=1,6} \\
[M_{xy_i}]_{i=1,16} &= [D] \{w_i\}_{i=1,6}
\end{align*}
\]

where \(\{w\}\) is a column matrix of 6 unknown deflections of points 1 to 6, and matrices \([B], [C]\) and \([D]\) are as given below:

\[
[B] = \frac{D}{h^2} \times \begin{bmatrix}
2.40 & -1.20 & 0.0 & 0.0 & 0.0 & 0.0 \\
-1.00 & 2.40 & -1.00 & -0.20 & 0.0 & 0.0 \\
0.0 & -2.00 & 2.40 & 0.0 & -0.20 & 0.0 \\
0.0 & -1.20 & 0.0 & 2.40 & -1.20 & 0.0 \\
0.0 & 0.0 & -0.20 & -2.00 & 2.40 & -0.20 \\
0.0 & 0.0 & 0.0 & 0.0 & -2.40 & 2.40 \\
0.0 & 0.0 & -1.00 & -0.40 & 2.40 & -1.00 \\
0.0 & -0.40 & 2.40 & 0.0 & -1.00 & 0.0 \\
-0.20 & 2.40 & -0.20 & -1.00 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0
\end{bmatrix}
\]
\[
[C] = \frac{p}{h^2} x
\]

\[
\begin{bmatrix}
 2.40 & -1.20 & 0.0 & 0.0 & 0.0 & 0.0 \\
-0.20 & 2.40 & -0.20 & -1.00 & 0.0 & 0.0 \\
0.0 & -0.40 & 2.40 & 0.0 & -1.00 & 0.0 \\
0.0 & -1.20 & 0.0 & 2.40 & -1.20 & 0.0 \\
0.0 & 0.0 & -1.00 & -0.40 & 2.40 & -1.00 \\
0.0 & 0.0 & 0.0 & 0.0 & -2.40 & 2.40 \\
0.0 & 0.0 & -0.20 & -2.00 & 2.40 & -0.20 \\
0.0 & -2.00 & 2.40 & 0.0 & -0.20 & 0.0 \\
-1.00 & 2.40 & -1.00 & -0.20 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -0.20 & 0.0 & 0.0 & 0.20 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -0.40 & 0.0 & 0.40 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.40 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.40 & 0.0 & 0.0 & 0.0 & 0.0 \\
-0.40 & 0.0 & 0.40 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
\end{bmatrix}
\]

\[
[D] = \frac{p}{h^2} x
\]

\[
\begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.20 & 0.0 & 0.0 \\
0.0 & -0.20 & 0.0 & 0.0 & 0.20 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.20 & 0.0 & -0.40 & 0.0 & 0.0 & 0.20 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -0.20 & 0.0 & 0.0 & 0.20 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-0.40 & 0.0 & 0.40 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.40 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.40 & 0.0 & 0.0 & 0.0 & 0.0 \\
-0.40 & 0.0 & 0.40 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
-2.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
1.0 & -2.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 2.0 & -2.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & -2.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 2.0 & -2.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -2.0 & 2.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.0 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & -0.5 & 0.0 & 0.5 \\
0.0 & 0.0 & 0.0 & 0.0 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.5 & 0.0 & 0.0 & -0.5 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & -2.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-2.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -2.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\
0.0 & 0.0 & -2.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\
0.0 & 1.0 & 0.0 & -2.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 1.0 & 0.0 & -2.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 2.0 & -2.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0
\end{bmatrix}
\]

**FIG. C.2 SHOWING MATRICES [AA], [AB] and [AC]**

---

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Substitution of Eqs. C.2 into C.1 yields a set of 6 simultaneous equations of the form:

\[ [A] \{w\} = \{q\} \]  \hspace{1cm} (C.3)

where,

\[ [A] = [AA] \times [B] + [AB] \times [D] + [AC] \times [C] \]  \hspace{1cm} (C.4)

\([A]\) is a square matrix of order 6 x 6 and is given below:

\[
[A] = \frac{D \times (6)^4}{a^4} \begin{bmatrix}
-12.50 & 9.60 & -1.40 & -0.40 & 0.0 & -0.10 \\
4.80 & -14.30 & 4.80 & 4.80 & -1.30 & 0.0 \\
-1.40 & 9.60 & -13.40 & -0.80 & 4.80 & -0.80 \\
-0.40 & 9.60 & -0.80 & -15.70 & 9.60 & -0.40 \\
0.0 & -2.60 & 4.80 & 9.60 & -15.60 & 4.80 \\
-0.40 & 0.0 & -3.20 & -1.60 & 9.20 & -14.00
\end{bmatrix}
\]

From Eq. C.3

\[ \{w\} = [A]^{-1} \{q\} \]

Carrying out the inversion of matrix \([A]\) yields the following values of unknown deflections.

\[ \{w\} = \begin{bmatrix}
0.00118 \\
0.00196 \\
0.00223 \\
0.00328 \\
0.00374 \\
0.00427
\end{bmatrix} \times q \frac{a^4}{D} \]
The values of moments are then obtained from Eqs. (C.2)

Testing for yield at different points the following values of first yield load are obtained:

<table>
<thead>
<tr>
<th>Point</th>
<th>Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24.56 x M_p/a^2</td>
</tr>
<tr>
<td>2</td>
<td>26.52</td>
</tr>
<tr>
<td>3</td>
<td>33.90</td>
</tr>
<tr>
<td>4</td>
<td>22.49</td>
</tr>
<tr>
<td>5</td>
<td>23.73</td>
</tr>
<tr>
<td>6</td>
<td>21.97</td>
</tr>
<tr>
<td>7</td>
<td>23.73</td>
</tr>
<tr>
<td>8</td>
<td>33.90</td>
</tr>
<tr>
<td>9</td>
<td>26.52</td>
</tr>
<tr>
<td>10</td>
<td>will not yield</td>
</tr>
<tr>
<td>11</td>
<td>66.18</td>
</tr>
<tr>
<td>12</td>
<td>35.48</td>
</tr>
<tr>
<td>13</td>
<td>29.51</td>
</tr>
<tr>
<td>14</td>
<td>35.48</td>
</tr>
<tr>
<td>15</td>
<td>66.18</td>
</tr>
<tr>
<td>16</td>
<td>will not yield</td>
</tr>
</tbody>
</table>

Since the minimum load to start yield is at point 6, the yielding starts at this point at a load value of q=21.97 M_p/a^2. This yielding is according to type IV yield. Final values of deflections and moments are obtained by multiplying the values due to unit load by the actual value of the load.

To modify the moment-deflection relations, the 6th row in matrices [B], [C] and [D] must now be changed. Since all moment increments are zero after type IV yield, each element of 6th row is made equal to zero. The stiffness values of all the grid points are also modified. Substituting the modified values of [B], [C] and [D] in Eq. (C.4) gives a
revised matrix \([A]\). After inversion of this matrix, incremental values of moments and deflections due to a load \(\Delta q = 1.00\) are computed. Each point is tested for yield and the true value of \(\Delta q\) for this stage is obtained by iteration; corresponding values of moments and deflections are added to the first stage solution. The procedure of second stage is repeated till the slab is converted into mechanism. For the present example, the following values of load versus central deflection were obtained.

<table>
<thead>
<tr>
<th>Load (21.97 \times M_p/a^2)</th>
<th>Central deflection (0.0937 \times M_p a^2/D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>22.32</td>
<td>0.0962</td>
</tr>
<tr>
<td>23.32</td>
<td>0.1095</td>
</tr>
<tr>
<td>24.31</td>
<td>0.1312</td>
</tr>
<tr>
<td>24.96</td>
<td>0.1529</td>
</tr>
<tr>
<td>25.39</td>
<td>0.1790</td>
</tr>
</tbody>
</table>

Large deflections of the order > 500.0 for unit load.
APPENDIX D

Relationship between moments and reinforcement strains

The straight line cracked section analysis of a reinforced concrete section which is widely used in design can not be utilised to relate the measured strain with moment because the tensile forces in the concrete are ignored. This assumption may not lead to large errors with high steel ratios but in slabs where reinforcement ratios are quite low, tensile forces in the concrete must be considered. Thus the moment strain diagram will consist of -

(i) a part in which concrete remains uncracked.
(ii) a part in which the concrete has cracked but the strain in the steel is below the yield point, and
(iii) a part in which the steel strain is at or beyond the yield point.

These moment versus steel strain relationships were obtained both experimentally and analytically. Experimental moment strain curves were obtained from tests of the control strips. Only live load strain in the steel were measured during the tests and the curves were extrapolated to zero to include dead load effects. In this extrapolation the steel strains due to dead weight of the strip and loading equipment were computed based on the transformed section.

For the purpose of obtaining the analytical moment strain curves, the moment at flexural cracking was computed on the basis of the transformed section of concrete. The average measured value of the modulus of rupture was 530 psi. The reinforcement present in the section restrained the normal free shrinkage of concrete and this restraint induced the tensile stresses in the concrete. The maximum value of this
tensile stress was computed to be 130 psi, based on the procedure given by Large and Chen*. Reducing the measured modulus of rupture by this amount, the cracking stress in the concrete was thus assumed to be 400 psi. The cracking moment per unit width of the strip, $M_{cr}$, is then obtained as 165 in. lb/in. This corresponds to a steel strain, $\varepsilon_{cr}$, of 64 micro in./in. in the control strip. Thus the first point on the analytical curve is defined by $M_{cr}$ and $\varepsilon_{cr}$.

The coordinates of the second point on the analytical curve are the moment and strain at the yield of reinforcement. The yield strain, $\varepsilon_y$, as measured experimentally was 1100 micro in./in. The yield moment of concrete section was computed using Hognestad's concrete stress block (Eq. 3.2).

Straight lines were drawn joining the origin, the point of flexural cracking and the yield point. For strains greater than yield strain, the moment is considered to remain constant and the moment strain diagram becomes horizontal. Typical analytical and experimental curves obtained in this manner are shown in Fig. D.1.

FIG. D.1 TYPICAL MOMENT - STEEL STRAIN RELATIONSHIPS
APPENDIX E

Sample Computer Program
Start

Read data, initialise and compute constants

Number the grid points

Generate matrices AA, AB, AC (Eq. 4.31)

Generate matrices B, C, D (Eq. 4.32)

Substitute Eq. 4.32 into Eq. 4.31 and compute matrix A.

compute deflections by inversion of matrix A

If deflections very large

calculate deflections at all grid points

Find minimum load for any point to yield

Add the incremental load, moments and deflections to the existing values

Modify matrices B, C, D according to type of yield

Modify stiffness values at all grid points

Print out Summary of the results

STOP

Flow Diagram for Main Computer Program
ELASTIC-PLASTIC SOLUTION OF SIMPLY SUPPORTED RC SLAB
WITH ISO OR NON-ISO REINFORCEMENT USING ORTHOTROPIC
PLATE THEORY, SLAB MAY BE SQUARE OR RECT, MAIN PROGRAM.

DIMENSION IP(10,7), IPE(12,9), DIF(70)
DIMENSION AA(54,70), AB(54,70), AC(54,70)
DIMENSION B(70,54), C(70,54), D(70,54)
DIMENSION LZ(54), MZ(54), A(54,54), BI(70,108)
DIMENSION AW(54), AMX(70), ANY(70), AMXY(70)
DIMENSION BW(54), BMX(70), BMY(70), BMXY(70)
DIMENSION AMI(70), AM2(70), K(70), KK(70), FIR(70)
DIMENSION DELTA(7J), FLOAD(70), AQYLD(70), BETA(70)
DIMENSION DX(70), DY(70), DXY(70), SIR(70), YLD(70)
COMMOM /C20/IR1, IC1, IRE, ICE, IRE1, ICE1, MNE, ISQREC/C10/SI

READ INPUT DATA AND INITIALISE

IR=10
IC=7
MS=54
ANU=0.200
ISQREC=2
MN=IR*IC
IR1=IR-1
IC1=IC-1
IRE=IR+2
ICE=IC+2
IRE1=IRE-1
ICE1=ICE-1
MNE=IRE*ICE
ANX=12.0
ANY=18.0
SIDEX=1.00
SIDEY=1.500
HX = SIDEX/ANX
HY = SIDEY/ANY
RHS = -1.00
PI = 3.14159265
CALL MTS (FC,WI,EFFDL,EFFDU,H,EC,TYL,TYU,ATL,ATU,ES,AM,
1CMAXL,CMAXU,EBSIZ,ANJ,DXA,DYA,DXYA,AMCL,AMCU,AMYL,
2AMYU,AMUL,AMUU,AMRL,AMU)
PAA = -2.0/(HX**2)
QAA = +1.0/(HX**2)
PAB = -2.0/(HY**2)
QAB = +1.0/(HY**2)
TAC = -1.0/(4.0*HX*HY)
UAC = +1.0/(4.0*HX*HY)
PB = +2.0*((1.0/HX**2)+(ANU/HY**2))
QB = -1.0/(HX**2)
SB = -ANU/(HY**2)
PC = +2.0*((1.0/HY**2)+(ANU/HX**2))
QC = -ANU/(HX**2)
SC = -1.0/(HY**2)
TD = -1.0/(2.0*HX*HY)
UD = +1.0/(2.0*HX*HY)
READ 100, ((IP(M,N), N=1,IC), M=1,IR)
DO 700 I=1,MN

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\( K(I) = MN + 10 \)
\( KK(I) = MN + 20 \)
\( \Delta(I) = 0.0 \)
\( FLOAD(I) = 0.0 \)
\( DX(I) = DXA \)
\( DY(I) = DYA \)
\( DXY(I) = DXYA \)

700 CONTINUE
C
NUMBER THE GRID POINTS AND PRINT OUT
DO 10 M=2,IRE1
DO 10 N=2,ICE1
IPE(M,N)=IP(M-1,N-1)
10 CONTINUE
DO 11 N=1,ICE
IPE(1,N)=MN+N
IPE(IRE,N)=MN+ICE+N
11 CONTINUE
DO 12 M=2,IRE1
IPE(M,1)=MN+(2*ICE)+M-1
IPE(M,ICE)=MN+(2*ICE)+IR+M-1
12 CONTINUE
PRINT 390
DO 550 M=1,IR
PRINT 301, (IP(M,N), N=1,IC)
550 CONTINUE
PRINT 391
DO 560 M=1,IRE
PRINT 301, (IPE(M,N), N=1,ICE)
560 CONTINUE
C
GENERATE REQUIRED MATRICES AND PRINT OUT
CALL AAMAT (PAA,QAAB,IR,IC,MS,MN)
CALL ABMAT (PAAB,QAAB,IR,IC,MS,MN)
CALL ACMAT (TAC,UAC,IR,IC,MS,MN)
CALL BCMAT (P3,Q3,SB,IR,IC,MS,MN)
DO 701 I=1,MN
DO 701 J=1,MS
B(I,J) = B(I,J)
701 CONTINUE
CALL BCMAT (PC,QC,SC,IR,IC,MS,MN)
DO 702 I=1,MN
DO 702 J=1,MS
C(I,J) = B1(I,J)
702 CONTINUE
CALL DMAT (TD,UD,IR,IC,MS,MN)
PRINT 401, ((AA(I,J), J=1,MN), I=1,MS)
PRINT 402, ((AB(I,J), J=1,MN), I=1,MS)
PRINT 403, ((AC(I,J), J=1,MN), I=1,MS)
PRINT 404, ((B(I,J), J=1,MS), I=1,MS)
PRINT 405, ((C(I,J), J=1,MS), I=1,MS)
PRINT 406, ((D(I,J), J=1,MS), I=1,MS)
C
FIRST STAGE SOLUTION - FRAME EQUILIBRIUM EQUATIONS AND
C
SOLVE FOR DEFLECTIONS
DO 705 I=1,MS
DO 705 J=1,MS
A(I,J) = 0.0
DO 705 N=1,MN
A(I,J) = A(I,J)+(AA(I,N)*B(N,J)*DX(N)/DXA)+(AB(I,N)*C(N,J)
1*DY(N)/DXA)-(2*AC(I,N)*D(N,J)*DXY(N)/DXA)
705 CONTINUE
DO 706 I=1,MS
DO 706 J=1,MS

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A(I,J) = A(I,J)*(HX**4)

CONTINUE
PRINT 407, ((A(I,J), J=1,MS), I=1,MS)
CALL MINV (A,MS,DTRM,LZ,MZ)
DO 707 I=1,MS
DO 707 J=1,MS
A(I,J) = A(I,J)*(HX**4)

CONTINUE
PRINT 411, ((A(I,J), J=1,MS), I=1,MS)
PRINT 413, DTRM
CALL DEFLEC (AW,A,RHS,MS)
C CALCULATE MOMENTS AT ALL GRID POINTS AND COMPLETE
C FIRST STAGE SOLUTION
CALL MXYXY (AW,AMX,AMY,AMXY,MS,MN,1,DX,DY,DXY,DXA)
CALL M1M2FI (AMX,AMY,AMXY,AM1,AM2,FIR,MS)
CALL BETASI (AM1,AM2,FIR,AMU,BETA,MN)
DO 710 I=1,MN
IF (AM1(I) .EQ. 0.0) GO TO 708
AQYLDCI(1) = BETACI(1) / AM1(I)
GO TO 710
708 AQYLDCI(1) = 5000.0
710 CONTINUE
PRINT 415
PRINT 450, (I, AQYLDC(I), I=1,MN)
SFIN = 1100.00
CALL LEAST (AQYLDC, MN,KY,QMINI,KYY, Q2, SFIN)
K(I) = KY
KK(I) = KYY
PRINT 418, (K(I), I=1,1)
ALOAD = QMINI
FLOAD(1) = QMINI
DO 711 I=1,MS
AW(I) = AW(I)*QMINI
711 CONTINUE
DELTA(I) = AW(MS)
DO 712 I=1,MN
AMX(I) = AMX(I) * QMINI
AMY(I) = AMY(I) * QMINI
AMXY(I) = AMXY(I) * QMINI
FID = FIR(I)*180.0/PI
712 CONTINUE
C START SECOND AND SUBSEQUENT STAGES SOLUTION AND REPEAT
C TILL THE SLAB IS CONVERTED INTO COLLAPSE MECHANISM
DMAX = 10.0 * AW(MS)
DO 800 IS=2,MN
IF (AW(MS) .GE. DMAX) GO TO 810
CHANGE MOMENT DEFLECTION RELATIONS FOR YIELDED POINTS
AND FRAME NEW EQUILIBRIUM EQUATIONS
CALL MWREL (AMX,AMY,AMXY,IR,IC,MS,MN,KY,AMU,FIR,IR,IC,MS,MN,KY,AMU,FIR,IR,IC,MS,MN)
IF (KKY .GT. MN) GO TO 720
CALL MWREL (AMX,AMY,AMXY,IR,IC,MS,MN,KKY,AMU,FIR,IR,IC,MS,MN,KKY,AMU,FIR,IR,IC,MS,MN)
720 CONTINUE
CALL UPDATE (AMX,FC,1,EFFDL,H,EC,TYL,ATL,ES,AM,CMAXL,
1 EBSIZ,MN,AMCL,AMY,AMUL,DX,ANU,AMRL)
CALL UPDATE (AMY,FC,1,EFFDU,H,EC,TYU,ATU,ES,AM,CMAXU,
1 EBSIZ,MN,AMCU,AMY,AMUU,DY,ANU,AMRL)
DO 850 I=1,MN
DXY(I) = (1.0-ANU) * (SORT(DX(I))*DY(I))/2.0
850 CONTINUE

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DO 721 I=1,MS
DO 721 J=1,MS
A(I,J) = 0.0
DO 721 N=1,MN

DO 721 CONTINUE

A(I,J) = A(I,J) * (HX**4)

DO 715 I=1,MS
DO 715 J=1,MS
A(I,J) = A(I,J) * (HX**4)

DO 715 CONTINUE

PRINT 407, ((A(I,J), J=1,MS), I=1,MS)
CALL MINV (A,MS,DTRM,LZ,MZ)
DO 716 I=1,MS
DO 716 J=1,MS
A(I,J) = A(I,J) * (HX**4)

PRINT 411, ((A(I,J), J=1,MS), I=1,MS)
PRINT 413, DTRM
IF (DTRM .EQ. 0.0) GO TO 810
CALL DEFLEC (BW,A,RHS,MS)
CALL MXYXY (BW,BMX,BMY,BMXY,MS,MN,IS,DX,DY,DXY,DXA)
C CALCULATE LOAD INCREMENT FOR THIS STAGE
CALL ORTHOS (AMX,AMY,AMXY,BMX,BMY,BMXY,ALOAD,AMU,SFIN,MN,K,KK)
DO 900 I=1,MN
AMX(I) = AMX(I) + BMX(I) * SFIN
AMY(I) = AMY(I) + BMY(I) * SFIN
AMXY(I) = AMXY(I) + BMXY(I) * SFIN

DO 900 CONTINUE
CALL M1MF21 (AMX,AMY,AMXY,AM1,AM2,FIR,MIN)
CALL BETAS1 (AM1,AM2,FIR,AMU,BETA,MIN)

DIF(I) = BETA(I) - AM1(I)
GO TO 920

IF (BMX(I) .EQ. 0.0) GO TO 910
DIF(I) = 1.00 - AMX(I)
GO TO 920

IF (BMY(I)) 912, 911, 912

IF (I .EQ. K(J)) GO TO 910
IF (I .EQ. KK(J)) GO TO 910

DIF(I) = BETA(I) - AM1(I)
GO TO 920

DIF(I) = 1.00 - AMX(I)
GO TO 920

DIF(I) = 500.00
GO TO 920

DIF(I) = AMU - AMY(I)
GO TO 920

DIF(I) = 100.00

DO 721 I=1,MS
AW(I) = AW(I) + BW(I) * SFIN

DELTA(IS) = AW(MS)

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DO 724 I = 1, MN
FID = FID(1) * 180.0 / PI
724 CONTINUE

800 CONTINUE

C PRINT OUT SUMMARY OF RESULTS

810 DO 730 J = 1, 3
PRINT 455, ANU, AMU, AMRL, DXA
PRINT 390
DO 860 N = 1, 1R
PRINT 301, (IP(M,N), N = 1, IC)
860 CONTINUE
PRINT 460
DO 730 I = 1, MN
IF (DELTAL(I) .EQ. 0.0) GO TO 730
IF (I .EQ. 24) PRINT 470
GO TO 730
732 PRINT 462, I, K(I), FLOAD(I), DELTAL(I)
730 CONTINUE

301 FORMAT (/12X, 1514)
390 FORMAT (/7X, @MESH POINT NUMBERS@/)
401 FORMAT (/7X, @POINTS IN EXTENDED MESH@/)
402 FORMAT (/7X, @MATRIX AA IS@ // (9X, 15F8.2))
403 FORMAT (/7X, @MATRIX AC IS@ // (9X, 15F8.2))
404 FORMAT (/7X, @MATRIX B IS@ // (9X, 15F8.2))
405 FORMAT (/7X, @MATRIX C IS@ // (9X, 15F8.2))
406 FORMAT (/7X, @MATRIX D IS@ // (9X, 15F8.2))
407 FORMAT (/7X, @MATRIX A IS@ // (9X, 10F11.2))
411 FORMAT (/7X, @INVERSE OF MATRIX A IS@ // (9X, 10F11.5))
413 FORMAT (/7X, @DETERMINANT IS@ // E20.8)
415 FORMAT (/7X, @FIRST GYLDIS ARE@ // (5(1X, @POINT@, 3X, @GYLD@)@/)
418 FORMAT (/7X, @SEQUENCE OF YIELDIS IS@ // (12X, 2015@/)
424 FORMAT (12X, 13, 5F12.4, F12.2)
449 FORMAT (/7X, @VALUES OF DIFS ARE@ // (5(9X, @POINT@, 7X, @DIF@)@/)
450 FORMAT (5(10X, 14, F11.4))
455 FORMAT (@1@// 12X, @ELASTIC-PLASTIC SOLUTION OF R C SLAB@/
1 14X, @SQUARE OR RECT, S S EDGES@, U D L@//
2 21X, @ANU@, 5X, @AMU@, 8X, @AMRL@, 12X, @DXA@ // (16X, 2F8.2, 2E16.6)
460 FORMAT (/ 8X, @NO@, 6X, @K@, 6X, @KK@, 4X, @LOAD@, 7X, @DELTA@@/)
461 FORMAT (2X, 318, F9.2, F12.4)
462 FORMAT (2X, 218, 8X, F9.2, F12.4)
470 FORMAT (@1@)
100 FORMAT (714)

STOP
END

SUBROUTINE PROGRAM TO COMPUTE RESISTING MOMENT VALUES
SUBROUTINE MTS (FC, B, EFFDL, EFFDU, H, EC, TYL, TYU, ATL, ATU, 3 ES, AM, CMAXL, CMAXU, ESIZ, ANU, DXA, OYA, OXYA, AMCL, AMCU, 2 AMYL, AMYU, AMMU, AMRL, AMMU, AMRL, AMMU)
DIMENSION AL(4), BL(4), AU(4), UU(4), RR(3), RI(3)
H = 1.500
B = 12.00
ES = 29.0 * 10.0 ** 6
COVER = 3.0 / 16.0
PI = 3.14159265
TYL = 32000.0
TYU = 32000.0
READ 100, FC, DIAL, SPL, DIAU, SPU
PRINT 200, FC, DIAL, SPL, DIAU, SPU

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DO 400 I = 1, 3
   IF (RI(I) .NE. 0.0) GO TO 400
   IF (RR(I) .GT. CMAXL) GO TO 400
   IF (RR(I) .LT. 0.0) GO TO 400
   C = RR(I)
400 CONTINUE
   IF (C .GT. H) GO TO 405
   AMYL = ATL*TYL*(EFFDL-(3.0*C/8.0))
   GO TO 409
405 PRINT 210, C
410 CONTINUE
   IF (C .GT. H) GO TO 415
   AMU = ATU*TYU*(EFFDU-(3.0*C/8.0))
   GO TO 414
415 PRINT 210, C
419 BLOCK = (1600.0 + (0.46*FC) - (FC*FC/80000.0))
1   (3900.0 + (0.35*FC))
   AMUL = ATL*TYL * (EFFDL-(BLOCK*ATL*TYL/(FC*12.0))
   AMUU = ATU*TYU * (EFFDU-(BLOCK*ATU*TYU/(FC*12.0)))

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AM U = AMUU/AMUL
PRINT 220, AMCL, AMCU
PRINT 220, AMYL, AMYU
PRINT 220, AMUL, AMUU, AMU
220 FORMAT (/ 1UX, 3F10.2 /)
DXA = EC*AIL/(1.0-ANU**2)
DYA = EC*AIU/(1.0-ANU**2)
DXYA = (1.0-ANU)*(SQRT(DXA*DYA))/2.0
AMRL = AMUL
RETURN
END

C SUBROUTINE PROGRAM TO CHANGE THE STIFFNESS VALUES IN
C EVERY STAGE OF LOADING
SUBROUTINE UPDATE (AMX, FC, B, D, H, EC, FY, AS, ES, AM, CMAX,
1 EBSIZ, MN, AM, AMY, AMU, DX, ANU, AMRL)
DIMENSION AMX(70), DX(70)
DIMENSION A(5), Z(5), RR(4), RI(4), P(6), Q(6), RRP(5), RIP(5), EB(5)
C HOME AMU IS THE VALUE OF ULTIMATE MOMENT
DO 1000 I=1,MN
AMN = AMX(I)*AMRL
IF (AMN .LE. AMC) GO TO 400
IF (AMN .GT. AMC .AND. AMN .LE. AMY) GO TO 500
IF (AMN .GT. AMY .AND. AMN .LT. AMU) GO TO 600
IF (AMN .GE. AMU) GO TO 721
400 GO TO 1000
500 A(1) = -3.40*FC*D*AS*AM/(3.0*AM)2
A(2) = +3.40*19.0*FC*AS*AM/(2.0*D)
A(3) = + (6.80*FC*AS*AM*AS*AM)/(D*D)
A(4) = + (6.80*FC*AS*AM*AM)/(2.0*AM)
A(5) = 1.7*FC*AS*AM/(2.0*D)
CALL POLRT (A, 2, 4, RR, RI, IER)
C = 10.0
DO 501 J=1,4
IF (RI(J) .NE. 0.0) GO TO 501
IF (RR(J) .GT. CMAX) GO TO 501
IF (RR(J) .LT. 0.0) GO TO 501
C = RR(J)
501 CONTINUE
IF (C .GT. H) GO TO 510
FS = AMN/(AS*(D-(3.0*C)/8.0))
FCC = AS*FS*1.50/(B*C)
EBSIC = FS/C/(ES*(D-C))
GO TO 999
510 PRINT 520, C
520 FORMAT (/7X, @VALUE OF C IS®, F8.2 /)
GO TO 1000
600 C = (8.0/3.0)*(D-(AMN/(AS*FY))
FCC = 1.5*AS*FY/(B*C)
P85FC = 0.85*FC
IF (FCC .GT. P85FC) GO TO 700
AA = 0.85*FC
AB = -1.7*FC*EBSIZ
AC = FCC*(EBSIZ**2)
EBSIC1 = (-AB+SQRT((AB**2)-(4.*AA*AC)))/2.0
EBSIC2 = (-AB-SQRT((AB**2)-(4.*AA*AC)))/2.0
IF (EBSIC1 .GT. 0.0 .AND. EBSIC1 .LT. EBSIZ) GO TO 610
IF (EBSIC2 .GT. 0.0 .AND. EBSIC2 .LT. EBSIZ) GO TO 611
610 EBSIC = EBSIC1
GO TO 999
611 EBSIC = EBSIC2

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GO TO 999

700 TANTH = (0.15*0.85*FC)/(0.0038-(1.70*FC/EC))

A1 = -TANTH
A2 = (0.85*FC)+(1.70*FC*TANTH/EC)
B1 = AS*FY
B2 = 1.70*FC*3*EBSIZ/3.0
B3 = B*(((0.85*FC)+A2)/2.0
J4 = 9*A1/2.0
J5 = EBSIZ
C1 = B1
C2 = B2-(B3*B5)
C3 = B3-(B4*B5)
C4 = B4
D1 = A2+(1.70*FC)
D2 = A2+(0.85*FC)
E1 = -AMN*A1
E2 = -AMN*D2
E3 = (C2)**2
E4 = (C3)**2
E5 = (C4)**2
E6 = 2.0*C2*C3
E7 = 2.0*C3*C4
E8 = 2.0*C2*C4
F1 = B2*C1*A1
F2 = B2*C1*D2
F3 = (D*C2)+(5.0*EBSIZ*C1/8.0)
F4 = (D*C3)-C1
F5 = D*C4
G1 = D*C1
G2 = C2*D2
G3 = (A1*C2)+(C3*D2)
G4 = (A1*C3)+(C4*D2)
G5 = A1*C4
G6 = B3-(B4*B5)
G7 = B4
G8 = -B3*B5
H1 = -(C1**2)/3.0
H2 = D1-(A1*EBSIZ)
H3 = A1
H4 = -D1*EBSIZ
P(1) = (E2*E3)+(F2*F3)+(G1*G8*G2)+(H1*G8*H2)
P(2) = (E1*E3)+(E2*E6)+(F1*F3)+(F2*F4)+(G1*G6*G2)+(G1*G8*G3)
      +(H1*G6*H4)+(H1*G8*H2)
P(3) = (E1*E6)+(E2*E4)+(E2*E8)+(F1*F41)+(F2*F5)+(G1*G6*G3)
      +(G1*G7*G2)+(G1*G8*G4)+(H1*G6*H2)+(H1*G7*G4)+(H1*G8*H3)
P(4) = (E1*E4)+(E1*E8)+(E2*E7)+(F1*F5)+(G1*G6*G4)
      +(G1*G7*G3)+(G1*G8*G5)+(H1*G6*H3)+(H1*G7*G3)
P(5) = (E1*E7)+(E2*E5)+(G1*G6*G5)+(G1*G7*G4)+(H1*G7*H3)
P(6) = (E1*E5)+(G1*G7*G5)
CALL POLRT (P1,0.5,RRP,RIP,IERP)
DO 701 J=1,5
    EB(J) = 8.30
    IF (RRP(J) .NE. 0.0) GO TO 701
    IF (RRP(J) .GT. 0.00380) GO TO 701
    IF (RRP(J) .LT. EBSIZ) GO TO 701
    EB(J) = RRP(J)
    IF (EB(J) .LT. EBSIZ) EBSIC=EB(J)
DO 705 J=2,5
    IF (EB(J) .LT. EBSIC) EBSIC=EB(J)
701 CONTINUE
705 CONTINUE

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IF (CE3SIC > 1.0) GO TO 721
FCC = (0.35*FC + (1.70*FC*TANTH/EC) - (TANTH*E3SIC)
C = (C1*E3SIC)/(C2 + (C3*E3SIC) + (C4*E3SIC**2))
GO TO 999
721 EBSIC = 0.003000
FCC = (0.85*FC + (1.70*FC*TANTH/EC) - (TANTH*E3SIC)
C = (C1*E3SIC)/(C2 + (C3*E3SIC) + (C4*E3SIC**2))
999 EUP = FCC/EBSIC
AMUP = ES/EUP
AIUP = (B*C*C*C/3.0) + AS*AMUP*(D-C)**2
DX(I) = EUP*AIUP/(1.0-ANU**2)
1000 CONTINUE
RETURN
END

C SUBROUTINE PROGRAM TO GENERATE MATRIX AA
SUBROUTINE AAMAT (P, Q, IR, IC, MS, MN)
DIMENSION IP(10, 7), IPE(12, 9), AA(54, 70)
COMMON /C1/IP, IPE/C2/AA
COMMON /C20/IR1, IC1, IRE, ICE, IRE1, ICE1, MNE, ISQREC
DO 25 I=1, MS
DO 25 J=1, MN
AA(I, J)=0.0
25 CONTINUE
IF (IC .LE. 3) GO TO 100
DO 30 M=2, IR
DO 30 N=2, IC
IF (IP(M+N) .GT. MS) GO TO 30
AA(IP(M, N), IP(M, N)) =P
AA(IP(M, N), IP(M, N-1))=Q
AA(IP(M, N), IP(M, N+1))=Q
30 CONTINUE
100 DO 32 M=2, IR
N=IC
AA(IP(M, N), IP(M, N)) =P
AA(IP(M, N), IP(M-1, N))=Q+Q
32 CONTINUE
RETURN
END

C SUBROUTINE PROGRAM TO GENERATE MATRIX AB
SUBROUTINE ABMAT (P, Q, IR, IC, MS, MN)
DIMENSION IP(10, 7), IPE(12, 9), AB(54, 70)
COMMON /C1/IP, IPE/C3/AA
COMMON /C20/IR1, IC1, IRE, ICE, IRE1, ICE1, MNE, ISQREC
DO 25 I=1, MS
DO 25 J=1, MN
AB(I, J)=0.0
25 CONTINUE
IF (IR .LE. 3) GO TO 100
DO 30 M=2, IR
DO 30 N=2, IC
IF (IP(M+N) .GT. MS) GO TO 30
AB(IP(M, N), IP(M, N)) =P
AB(IP(M, N), IP(M-1, N))=Q
AB(IP(M, N), IP(M+1, N))=Q
30 CONTINUE
100 DO 31 N=2, IC
M=IR
IF (IP(M, N) .GT. MS) GO TO 31
AB(IP(M, N), IP(M, N)) =P
AB(IP(M, N), IP(M-1, N))=Q+Q
31 CONTINUE
SUBROUTINE ACMAT (T,U,IR,IC,MS,MN)
DIMENSION IP(10,7), IPE(12,9), AC(54,70), ACI(70,108)
COMMON /C1/IP, IPE /C4/AC /C10/ACI
COMMON /C2/IR1, IC1, IC, IRE, ICE, IRE1, ICE1, MNE, ISQREC
DO 20 I = 1, MN
DO 20 J = 1, MNE
ACI(I,J) = 0.0
20 CONTINUE
DO 21 M = 3, IRE1
DO 21 N = 3, ICE1
IF (IPE(M,N) .GT. MS) GO TO 21
AC(IPE(M,N), IPE(M-1,N-1)) = U
AC(IPE(M,N), IPE(M-1,N+1)) = T
AC(IPE(M,N), IPE(M+1,N-1)) = T
AC(IPE(M,N), IPE(M+1,N+1)) = U
21 CONTINUE
DO 30 I = 1, MS
DO 30 N = 1, ICE
30 CONTINUE
RETURN
END

SUBROUTINE BCMAT (P,Q,S,IR,IC,MS,MN)
DIMENSION IP(10,7), IPE(12,9), BI(70,108)
COMMON /C1/IP, IPE /C10/BI
COMMON /C2/IR1, IC1, IC, IRE, ICE, IRE1, ICE1, MNE, ISQREC
DO 20 I = 1, MN
DO 20 J = 1, MNE
BI(I,J) = 0.0
20 CONTINUE
DO 21 M = 2, IRE1
DO 21 N = 2, ICE1
BI(IPE(M,N), IPE(M,N)) = P
BI(IPE(M,N), IPE(M,N-1)) = Q
BI(IPE(M,N), IPE(M,N+1)) = Q
BI(IPE(M,N), IPE(M-1,N)) = S
BI(IPE(M,N), IPE(M+1,N)) = S
21 CONTINUE
DO 30 I = 1, MN
DO 30 N = 1, ICE
30 CONTINUE
RETURN
END
CONTINUE
IF (ISQREC .EQ. 2) GO TO 45
DO 40 I=1,MN
DO 40 M=2,IR1
IP1 = M+1
DO 40 N=MP1,IC
BI(I,IP(M,N)) = BI(I,IP(M,N)) + BI(I,IP(N,M))
40 CONTINUE
RETURN
END

C SUBROUTINE PROGRAM TO GENERATE MATRIX D
SUBROUTINE DMAT (TU,IR,IC,MS,MN)
DIMENSION IP(70,9), D(70,108)
COMMON /C1/IP, IPE /C7/D /C10/DI
COMMON /C2U/IR1, IC1, IRE, ICE, IRE1, ICE1, MNE, ISQREC
DO 20 I=1,MN
DO 20 J=1,MNE
DI(I,J)=0.0
20 CONTINUE
DO 21 M=2,IR1
DO 21 N=2,IC1
DI(IPE(M,N)) = U
DI(IPE(M,N)) = T
DI(IPE(M,N)) = T
DI(IPE(M,N)) = U
21 CONTINUE
DO 30 I=1,MN
DO 30 N=1,IC1
DI(I,J)=DI(I,J)
30 CONTINUE
RETURN
END

C SUBROUTINE PROGRAM TO COMPUTE AND PRINT THE DEFLections
C OF ALL GRID POINTS
SUBROUTINE DEFLEC (XW, A, RHS, N)
DIMENSION XW(N), A(N,N)
DO 70 I=1,N
XW(I) = 0.0
DO 70 J=1,N
XW(I) = XW(I) + (A(I,J)*RHS)
70 CONTINUE
PRINT 30
30 FORMAT ('DEFLECTIONS ARE @', (5(1X,'POINT ',3X,'DEFLECT@'))/)

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SUBROUTINE PROGRAM TO CALCULATE AND PRINT THE MOMENTS
AT ALL GRID POINTS
SUBROUTINE MXXYXY (XW, XMX, XMY, XMXY, N, NP, IS, DX, DY, DXY, DXA)
DIMENSION XW(N), XM(NP), XMY(NP), XMXY(NP)
DIMENSION B(70,54), C(70,54), D(70,54)
DIMENSION DX(70), DY(70), DXY(70)
COMMON /C5/B /C6/C /C7/D
DO 10 I = 1, NP
  XMX(I) = 0.0
  XMY(I) = 0.0
  XMXY(I) = 0.0
DO 10 J = 1, N
  XMX(I) = XMX(I) + B(I,J)*DX(I)*XW(J)/DXA
  XMY(I) = XMY(I) + C(I,J)*DY(I)*XW(J)/DXA
  XMXY(I) = XMXY(I) + D(I,J)*DXY(I)*XW(J)/DXA
10 CONTINUE
IF (IS .GT. 1) GO TO 60
PRINT 50
50 FORMAT (///12X,©POINT®, 6X, ©MX©, 10X, ©MY©, 9X, ©MXY©/)
  DO 31 I = 1, NP
    PRINT 51, I, XMX(I), XMY(I), XMXY(I)
  31 CONTINUE
60 RETURN
SUBROUTINE PROGRAM TO CHOOSE MINIMUM VALUE OF LOAD
INCREMENT TO CAUSE YIELDING AT ANY POINT
SUBROUTINE LEAST (XQYLD, NP, K*, QMINI, KK, Q2, SFIN)
DIMENSION XQYLD(NP)
QMINI = XQYLD(1)
K = 1
DO 70 I = 2, NP
  IF (XQYLD(I) .GE. QMINI) GO TO 70
  QMINI = XQYLD(I)
  K = I
70 CONTINUE
Q2 = 3000.0
KK = NP + 10
DO 101 I = 1, NP
  IF (I .EQ. K) GO TO 101
  IF (QMINI .NE. XQYLD(I)) GO TO 101
  Q2 = XQYLD(I)
  KK = I
101 CONTINUE
PRINT 30
30 FORMAT (///7X,©YIELDING WILL START AT©/)
  IF (SFIN .GT. 1000.0) GO TO 60
  QMINI = SFIN
60 PRINT 31, K, QMINI
31 FORMAT (12X, ©POINT©, 13, 8X, ©AT LOAD©, F10.5)
  IF (KK .EQ. (NP+10)) GO TO 105
  PRINT 135, KK
135 FORMAT (///12X, ©AT SAME LOAD YIELDING ALSO STARTS AT POINT©,14)
  GO TO 200
105 PRINT 140
140 FORMAT (///12X, ©NO OTHER POINT YIELDS IN THIS STAGE© )
200 RETURN
SUBROUTINE PROGRAM TO MODIFY MATRICES A, C AND D TO ACCOUNT FOR CHANGE IN MOMENT DEFLECTION RELATIONS

SUBROUTINE MWREL (AMX, AMY, AMXY, IR, IC, MS, MN, KY, AMU,
                  1          FIR, SIR, YLD, AM1, AM2, AMPX, ANU, DX, DY, HX, HY)

DIMENSION AMX(MN), AMY(MN), AMXY(MN), T(9), X(9), Y(9), Z(9),
                  1          FIR(MN), SIR(MN), YLD(MN), AM1(MN), AM2(MN)
DIMENSION IP(10,7), IPE(12,9), BI(70,108), DX(70), DY(70)
DIMENSION B(70,54), C(70,54), D(70,54)
COMMON /C1/IP, IPE/C5/B/ /C6/C/C7/Y/C10/BI
COMMON /C2/G/IR1, IR2, ICE1, ICE2, ICE3, MNE, ISQREC

PRINT 400, KY, AMX(KY), AMY(KY), AMXY(KY)
400 FORMAT (/10X, 13, 3F16.8)

TH = FIR(KY) + SIR(KY)
AMPX1 = AMPX
AMPY = AMU * AMPX
IF (AMXY(KY) .EQ. 0.0) GO TO 200
PMDIF = ABS(AM1(KY)) - ABS(AM2(KY))
IF (PMDIF .GT. 0.005) GO TO 600
DO 601 J = 1, MS
   B(KY, J) = 0.0
   C(KY, J) = 0.0
   D(KY, J) = 0.0
601 CONTINUE
GO TO 300
600 S2 = ((SIN(TH))**2)
C2 = ((COS(TH))**2)
SC = (SIN(TH)) * (COS(TH))
32T = SIN(2.0*TH)
T(1) = -S2T/(4.0*HX*HY)
T(2) = +C2/HY**2
T(3) = +S2T/(4.0*HX*HY)
T(4) = +S2/(HX**2)
T(5) = -(2.*S2/1HX*HY**2)+(2.*C2/HY**2))
T(6) = +S2/(HX**2)
T(7) = +S2T/(4.0*HX*HY)
T(8) = +C2/HY**2
T(9) = -S2T/(4.0*HX*HY)
DO 602 I = 1,9
   X(I) = -T(I) * S2 * (1.0-ANU**2)
   Y(I) = -T(I) * C2 * (1.0-ANU**2)
   Z(I) = -T(I) * SC * (1.0-ANU**2)
602 CONTINUE
DO 605 M = 2, IRE1
   DO 605 N = 2, ICE1
      IF (IPE(M, N) .EQ. KY) GO TO 610
605 CONTINUE
610 M1 = M
   N1 = N
   DO 700 J = 1, MNE
      BI(KY, J) = 0.0
700 CONTINUE
   BI(KY, IPE(M1, N1)) = X(5)
   BI(KY, IPE(M1, N1-1)) = X(4)
   BI(KY, IPE(M1, N1+1)) = X(6)
   BI(KY, IPE(M1-1, N1-1)) = X(1)
   BI(KY, IPE(M1-1, N1)) = X(2)
   BI(KY, IPE(M1-1, N1+1)) = X(3)
   BI(KY, IPE(M1+1, N1-1)) = X(7)
   BI(KY, IPE(M1+1, N1)) = X(8)
   BI(KY, IPE(M1+1, N1+1)) = X(9)
DO 701 N=1,ICE
BI(KY,IP(M,N))=BI(KY,IP(M,N))-BI(KY,IP(M+1,N))
BI(KY,IP(M+1,N))=BI(KY,IP(M+1,N))+BI(KY,IP(M+1,N))

DO 702 M=2,IRE1
BI(KY,IP(M,N))=BI(KY,IP(M,N))-BI(KY,IP(M,N+1))
BI(KY,IP(M,N+1))=BI(KY,IP(M,N))+BI(KY,IP(M,N))

BI(KY,IP(M,N))=BI(KY,IP(M,N))-BI(KY,IP(M,N+1))
BI(KY,IP(M+1,N))=BI(KY,IP(M+1,N))+BI(KY,IP(M+1,N))

DO 706 J=1,MS
BI(KY,J)=BI(KY,J)

DO 707 J=1,MS
BI(KY,J)=0.0

BI(KY,IP(M,N))=BI(KY,IP(M,N))-BI(KY,IP(M,N+1))
BI(KY,IP(M+1,N))=BI(KY,IP(M+1,N))+BI(KY,IP(M+1,N))

DO 708 J=1,MS
C(KY,J)=BI(KY,J)

DO 709 J=1,MS
C(KY,J)=0.0

DO 710 J=1,MS
C(KY,J)=BI(KY,J)

DO 711 J=1,MS
C(KY,J)=0.0

DO 712 M=2,IRE1
BI(KY,IP(M,N))=BI(KY,IP(M,N))-BI(KY,IP(M,N+1))
BI(KY,IP(M+1,N))=BI(KY,IP(M+1,N))+BI(KY,IP(M+1,N))

DO 716 J=1,MS
C(KY,J)=BI(KY,J)

DO 715 M=2,IRE1
MP1=M+1
DO 715 N=MP1,ICE
BI(KY,IP(M,N))=BI(KY,IP(M,N))+BI(KY,IP(M,N))

DO 716 J=1,MS
C(KY,J)=BI(KY,J)

DO 720 J=1,MS
C(KY,J)=0.0

DO 721 N=1,ICE
BI(KY,IP(M,N))=BI(KY,IP(M,N))-BI(KY,IP(M,N+1))
BI(KY,IP(M+1,N))=BI(KY,IP(M+1,N))+BI(KY,IP(M+1,N))

DO 722 M=2,IRE1
DO 725 M = 2 , IM1
MPI = M+1
DO 725 N = MPI , IC
BI(KY, IP(M, N)) = BI(KY, IP(M, N)) + BI(KY, IP(N, M))
725 CONTINUE
DO 726 J = 1 , MS
C(KY, J) = BI(KY, J)
726 CONTINUE
GO TO 300
200 IF (AMX(KY) .GE. AMPX) GO TO 210
250 IF (AMY(KY) - AMPY) 300 , 302 , 302
302 DO 10 J = 1 , MS
C(KY, J) = 0.0
10 CONTINUE
IF (AMX(KY) .GE. AMPX) GO TO 300
PC = +2.0*(1.0-ANU**2)/(HX**2)
QC = 0.0
SC = -1.0*(1.0-ANU**2)/(HX**2)
DO 20 J = 1 , MS
B(KY, J) = 0.0
20 CONTINUE
GO TO 300
210 PC = +2.0*(1.0-ANU**2)/(HY**2)
QC = 0.0
SC = -1.0*(1.0-ANU**2)/(HY**2)
DO 30 J = 1 , MS
B(KY, J) = 0.0
30 CONTINUE
CALL BCMAT (PC, QC, SC, IR, IC, MS, MN)
DO 40 J = 1 , MS
C(KY, J) = 0.0
40 CONTINUE
GO TO 250
300 CONTINUE
AMPX = AMPX1
RETURN
END

SUBROUTINE PROGRAM TO COMPUTE PRINCIPAL MOMENTS AND THEIR
INCLINATIONS AT ALL GRID POINTS
SUBROUTINE M1M2FI (XMX, XMY, XMXY, XM1, XM2, FIR, MN)
DIMENSION XMX(MN), XMY(MN), XMXY(MN)
DIMENSION XM1(MN), XM2(MN), FIR(MN)
PI = 3.14159265
DO 80 I = 1 , MN
IF (XMX(I) .EQ. 0.0) GO TO 51
AAA = XMX(I) + XMY(I)
ABB = ((XMX(I) - XMY(I))**2) + (4.0*(XMXY(I)**2))
XM1(I) = 0.5 * (AAA + SQRT(ABB))
XM2(I) = 0.5 * (AAA - SQRT(ABB))
ANUMFI = 2.0 * XMXY(I)
IF (ANUMFI .EQ. 0.0) GO TO 97
DENOFI = XMY(I) - XMX(I)
IF (DENOFI .EQ. 0.0) GO TO 95
TTFI = ANUMFI / DENOFI
FIR(I) = 0.5 * ATAN(TTFI)
IF (FIR(I) .GT. 0.0) FIR(I) = FIR(I) - PI/2.0
GO TO 96
C  SUBROUTINE PROGRAM TO COMPUTE VALUES OF BETA AND SI
C  SUBROUTINE BETASI (XM1,XM2,FIR,AMU,BETA2,MN)
DIMENSION XM1(MN), XM2(MN), FIR(MN), BETA2(MN)
PI=3.14159265
DO 80 I=1*MN
 IF (XM1(I) .EQ. 0.00) GO TO 86
 OMEGA = XM2(I) / XM1(I)
 IF (OMEGA .LT. 0.00) GO TO 85
 P = ((SIN(FIR(I)))**2) + (AMU*((COS(FIR(I)))**2))
 Q = ((COS(FIR(I)))**2) + (AMU*((SIN(FIR(I)))**2))
 AE = OMEGA
 BE = -(P+(OMEGA*Q))
 CE = AMU
 ARGU = (BE**2) - (4.0*AE*CE)
 IF (ARGU .LT. 0.00) ARGU=0.00
 BETA2(I)=(-E+SQRT(ARGU)) / (2.0*AE)
 ANUM = (1.0-AMU) * (SIN(2.0*FIR(I)))
 DENO = BETA2(I) * (1.0-OMEGA) - (1.0-AMU) * (COS(2.0*FIR(I)))
 IF (DENO .EQ. 0.00) GO TO 90
 TTSI = ANUM / DENO
 SIR = 0.5 * (ATAN(TTSI))
 SID = SIR * 180.0 / PI
 GO TO 91
90 TTSI = 9999.0
 SID = 45.0
 GO TO 91
85 P = ((SIN(FIR(I)))**2) + (AMU*((COS(FIR(I)))**2))
 BETA2(I) = AMU / P
 ANUMZ = (1.0-AMU) * (SIN(2.0*FIR(I)))
 DENOZ = BETA2(I) * (1.0-OMEGA) - (1.0-AMU) * (COS(2.0*FIR(I)))
 TTSI = ANUMZ / DENOZ
 SIR = 0.5 * (ATAN(TTSI))
 SID = SIR * 180.0 / PI
91 GO TO 80
86 BETA2(I) = 5000.0
80 CONTINUE
RETURN
END

C  SUBROUTINE PROGRAM TO COMPUTE LOAD INCREMENT TO BE APPLIED
C FOR THE NEXT HIGHLY STRESSED POINT TO YIELD.
C  SUBROUTINE ORTHOS (AMX,AMY,AMXY,BMX,BMY,BMXY,ALOAD,AMU,SFIN,1MN,KK)
DIMENSION AMX(MN), AMY(MN), AMXY(MN)

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DIMENSION BMX(MN), BMY(MN), SMXY(MN)
DIMENSION K(MN), KK(MN)
DIMENSION SMX(70), SMY(70), SMXY(70), SM1(70), SM2(70), SFIR(70), 1
SDIF(70)
PI = 3.14159265
S1 = 1.000
S = S1
50 SLOAD = ALOAD + S
DO 10 I = 1, MN
   SMX(I) = AMX(I) + BMX(I) * S
   SMY(I) = AMY(I) + BMY(I) * S
   SMXY(I) = AMXY(I) + BMXY(I) * S
10 CONTINUE
DO 20 I = 1, MN
   IF (SMX(I) .EQ. 0.0) GO TO 21
25 SAA = SMX(I) + SMY(I)
   SBB = ((SMX(I) - SMY(I)) ** 2) + (4.0 * (SMXY(I) ** 2))
   SM1(I) = 0.5 * (SAA + SORT(SBB))
   SM2(I) = 0.5 * (SAA - SORT(SBB))
   SANUMF = 2.0 * SMXY(I)
   IF (SANUMF .EQ. 0.0) GO TO 24
   SDSNOF = SMY(I) - SMX(I)
   IF (SDSNOF .EQ. 0.0) GO TO 26
   STTFI = SANUMF / SDSNOF
   SFIR(I) = 0.5 * ATAN(STTFI)
   IF (SFIR(I) > 0.0) SFIR(I) = SFIR(I) - PI / 2.0
   GO TO 20
24 IF (SMX(I) .GE. SMY(I)) SFIR(I) = 0.0
   IF (SMX(I) .LT. SMY(I)) SFIR(I) = PI / 2.0
   GO TO 20
26 SFIR(I) = -PI / 4.0
   GO TO 20
21 IF (SMY(I) .NE. 0.0) GO TO 25
   IF (SMXY(I) .NE. 0.0) GO TO 25
   SM1(I) = 0.0
   SM2(I) = 0.0
   SFIR(I) = 0.0
20 CONTINUE
DO 30 I = 1, MN
   IF (SM1(I) .EQ. 0.0) GO TO 31
   SOMEGA = SM2(I) / SM1(I)
   IF (SOMEGA .EQ. 0.0) GO TO 32
   SP = ((SIN(SFIR(I))) ** 2) + (AMU * ((COS(SFIR(I))) ** 2))
   SQ = ((COS(SFIR(I))) ** 2) + (AMU * ((SIN(SFIR(I))) ** 2))
   SAE = SOMEGA
   SBE = -(SP + (SOMEGA * SQ))
   SCE = AMU
   SARGU = (SBE ** 2) - (4.0 * SAE * SCE)
   IF (SARGU .LT. 0.0) SARGU = 0.0
   SBETA = (-SBE - (SORT(SARGU))) / (2.0 * SAE)
   GO TO 35
32 SP = ((SIN(SFIR(I))) ** 2) + (AMU * ((COS(SFIR(I))) ** 2))
   SBETA = AMU / SP
35 DO 51 J = 1, MN
   IF (I .EQ. K(J)) GO TO 55
   IF (I .EQ. KK(J)) GO TO 55
51 CONTINUE
SDIF(I) = SBETA - SM1(I)
   GO TO 30
55 IF (BMX(I) .EQ. 0.0) GO TO 58
   SDIF(I) = 1.00 - SMX(I)

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GO TO 30
58 IF (BMX(I)) 72, 70, 72
70 SDIF(I) = 500.00
GO TO 30
72 SDIF(I) = AMU - SMY(I)
GO TO 30
31 SDIF(I) = 100.00
30 CONTINUE
SDIFM = SDIF(I)
DO 40 I=2,MN
IF (SDIF(I) .GE. SDIFM) GO TO 40
SDIFM = SDIF(I)
40 CONTINUE
IF (SDIFM) 41, 42, 43
41 S = S - SI
IF (S1 .LT. 0.005) GO TO 82
IF (S1 .LT. 0.050) GO TO 81
IF (S1 .LT. 0.500) GO TO 80
S1 = 0.100
GO TO 43
80 S1 = 0.0100
GO TO 43
81 S1 = 0.001
GO TO 43
82 SFIN = S
GO TO 60
42 SFIN = S
GO TO 60
43 S = S + SI
GO TO 50
60 PRINT 99, SFIN
99 FORMAT ('/7X, @SFIN = @, F9.4)
RETURN
END
Computer output - Sample for one Stage of Loading

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**YIELDING WILL START AT**

**POINT 65 AT LOAD 0.0600**

**NO OTHER POINT YIELDS IN THIS STAGE**

**SEQUENCE OF YIELDING IS**

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### SUMMARY OF RESULTS

ELASTIC-PLASTIC SOLUTION OF R C SLAB
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VITA

Satish Chand Jain was born on June 20, 1938 at Bagpat, U.P., India. He received elementary education in his home town and secondary and intermediate education at D. Jain College, Baraut, U.P. In 1961 he graduated from the University of Roorkee, Roorkee, U.P. with a Bachelor of Engineering (Civil) degree with Honours. Thereafter he worked for one year as lecturer at D.J. Polytechnic Institute, Baraut, U.P. In 1962 he joined the University of Roorkee for graduate studies and obtained the Master of Engineering degree in 1964. In August 1963, he was employed as lecturer in Civil Engineering Department of the University of Roorkee where he was made Reader in Feb. 1967. Later in that year he was selected for a Commonwealth Scholarship award to pursue further graduate studies at the University of Windsor which he joined in Sept. 1967.