Synthesis of a digital computer controlled optimal and adaptive control system.

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SYNTHESIS OF A DIGITAL COMPUTER CONTROLLED
OPTIMAL AND ADAPTIVE CONTROL SYSTEM

by

Adam Baziw

A Thesis
Submitted to the Faculty of Graduate Studies Through the
Department of Electrical Engineering in Partial Fulfillment
of the Requirements for the Degree of
Master of Applied Science at
University of Windsor

Windsor, Ontario
1966
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ABSTRACT

This thesis describes the synthesis of a novel digital method for optimization and adaption of certain second order systems which may be represented in the phase plane.

A digital controller had been partially constructed when the project was begun, therefore, only details of the modifications and additions to the controller are given in this report.

The plant includes a large load mass to approximate a pure inertia system. It is believed that the method of applying control to the armature of a d.c. motor is novel.

In the control scheme, the optimal switching curve is stored in memory at the locations given by the velocity which is approximated by a difference of error readings over a constant sampling time. A new optimal switching curve is continually stored during each control cycle.

The program was written with the limited number of instruction steps, possible on the controller, in mind. It was found that the program could not be accommodated on the present controller. To carry out meaningful studies, the controller must be expanded or a controller with greater input and memory facilities used.
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1 INTRODUCTION

The main purpose of this study is to describe the synthesis of a control scheme which was proposed in a previous study [1]. However, some of the statements made in the previous study might lead us to believe that the control scheme proposed there, and synthesized here, is more general than it actually is.

In reference 1, we are told that, "The study proposes a digital computer controller (DCC) method of optimization and adaption of systems representable in the phase plane". Actually the system proposed there, when suitably modified, is more accurately described as follows: "A digital computer controller is proposed for second order systems which can have varying parameters, but the variations must be slow compared to the shorter "control periods, so that, for a particular control period, we may assume that the plant has constant coefficients. Furthermore, the two "poles"* of the second order system must always be real-valued. The system is optimal in the sense that the desired state of the controlled variables (position and velocity) is reached in minimum time, with amplitude constraints on the controlling function. The control scheme is adaptive in the sense that the controller adjusts itself to slowly varying parameters of the system". This modified statement shall be justified

* By "control period" we mean one cycle of the control operation. Thus, once the controller "locks" onto the optimal switching curve, it is able to satisfactorily follow this curve, which of course changes as the parameters of the system being controlled.

* The reason for the quotations around the poles is that in this study we shall consider systems with two poles, but as pointed out in section 11-C, the control scheme could work for a restricted class of first and second order non linear plants and some higher order plants.
on a theoretical basis in the following sections.

Most of the analysis for the problem we are considering has been done in the 1950's, for example, Refs. 2,3,4,5,6.

A. Control Scheme

The control scheme applied in this study [1] is applicable to a certain class of second order systems (plants); in particular, those plants which have an optimal switching characteristic (OSC) which is of the form shown in Figs. 1-1 and 1-2. To be more precise, Fig. 1-2 is used to define certain regions of the phase plane. Using the symbols in Ref. 7, where much of this section is covered, we see in Fig. 1-1 that $u=+1$ for $(x_1,x_2) \in R_+ \cup \gamma_+$, and that $u=-1$ for $(x_1,x_2) \in R_- \cup \gamma_-$

In order to describe clearly the control scheme which is used, phase trajectories typical of the class of systems being considered, are shown in Fig. 1-2. The solid lines are the phase trajectories for the

Legend

$\gamma_-$ curve is the locus of all points which can be forced to the origin by the control $u=-1$.

$\gamma_+$ curve is the locus of all points which can be forced to the origin by the control $u=+1$.

$\gamma$ is the optimal switching characteristic and is given by $=\gamma_+ U \gamma_-$.

$R_-$ is the set of all points to the right of $\gamma$.

$R_+$ is the set of all points to the left of $\gamma$.

Fig. 1-1 The optimal switching characteristic typical of plants considered in this study.

$\dagger$ The symbol $A \cup B$ means the union of the two sets of points $A,B$, and the symbol $(x_1,x_2) \in C$ means the point $(x_1,x_2)$ is a member of the set $C$. 

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control \( u=+1 \), while the dashed lines are the phase trajectories for \( u=-1 \).

Next the control scheme used in this study is described. Referring to Fig. 1-3, consider an initial point in the phase plane \((x_{10}, x_{20})\).

Without any previous knowledge of the systems switching characteristic \( Y \), the \( x_2 \)-axis is used as a switching curve. Thus, if \( x_1 \geq 0 \), we apply the control \( u=-1 \), and if \( x_1 < 0 \) we apply the control \( u=+1 \). At \( S_1 (x_1=0) \), the control is switched from \( u=-1 \) to \( u=+1 \). From \( S_1 \) to \( P_1 (x_2=0) \), this portion of the trajectory is stored in the memory of the computer. It is noted that the trajectory from \( P_0 (x_2=0, x_1>0) \) to \( S_1 \) is not stored. During this portion of the trajectory, the values of \( x_1 \) are compared with values of \( x_1 \) in the memory at the addresses given by the corresponding value of \( x_2 \). During the first pass, these values are zero. Now the assumption is made that the curve from \( S_1 \) to \( P_1 \) is similar to \( Y^+ \), so if \( S_1-P_1 \) is shifted to the right by the distance \( \Delta x_1 \), then a portion of \( Y^+ \) will have been approximated. Next, it is assumed that \( Y_-=-Y^+ \) (which is the same as assuming that the optimal switching characteristic \( Y \) is an odd function). We denote the approximating trajectories to \( Y \) by \( Y' \), so that, \( Y'=Y^+_1U^-_1 \). Now if the points along the trajectory from \( P_1 \) to \( S_2 \) are compared to the corresponding points along \( Y' \) (i.e., for the same value of \( x_2 \)) and the control switched from \( u=+1 \) to \( u=-1 \) when the two
initial $u = 4 - 1$ is applied here

This portion of trajectory is stored in memory $S_1$

(a) At start of operation, $x_1 = 0$ is the switching curve.

(b) After one cycle of operation, $\gamma_1 \cup Y_1$ is used to approximate the OSC, $Y$.

(c) Target zone $D$, in vicinity of the origin.

(c)

Fig. 1-3 Schematic of the Proposed Control Scheme

curves intersect, the system will move along the $Y_1$ curve toward the origin.

In an actual system, it is practically impossible to reach the
origin exactly. However, if a region $D$ is specified by

$$D = \{(x_1, x_2); -M < x_1 < M, -N < x_2 < N\}$$

where $M$ and $N$ are prespecified constants depending on the accuracy requirement of the system, then, the desired final state will have been reached when $(x_1, x_2) \in D$. If the system misses or overshoots region $D$, the logic will sense the error and switch again toward $Y$.

At this first time the system is not time optimal. If the system parameters do not change, the system will move to the origin with only one switch \[i.e., \text{at } S_0 \text{ if } (x_{10}, x_{20}) \text{ is in region } R_- \text{ or at } S_2 \text{ if } (x_{10}, x_{20}) \text{ is in region } R_+\] and the system will be essentially time optimal.

B. The Digital Control Computer

In 1963, before an actual control scheme for any particular problem had been proposed, the Electrical Engineering Department at the University of Windsor began developing a digital control computer [8]. At the time, the design was to "incorporate into this computer the desired flexibility. This flexibility would include an extensive list of instructions, a suitable means of data storage, and special input and output facilities".†

The computer uses NAND logic throughout, and the digits ZERO and ONE are represented by voltage levels, 0 volts and -6 volts, respectively [10].

The sign of the binary number is denoted by the most significant digit of the 12 bit word: ZERO and ONE represent the positive and negative signs, respectively. For positive numbers, the number digits

†The quotation is from Reference 8.
represent the magnitude, while for negative numbers the number digits are in 2's complement. The signed 2's complement representation of a binary number gives the actual value of the binary number if the value of the sign digit is regarded as a negative number [9]. By considering a decimal point after the sign digit the number system is:

- $0.00000000000$ to $0.01111111111$ ($0 \leq x < \frac{1}{2}$)
- $1.10000000000$ to $1.11111111111$ ($-\frac{1}{2} \leq x < 0$)

so that any number is represented by a fraction.
II. CALCULATIONS OF THE OPTIMAL SWITCHING CHARACTERISTIC

A. Physical Problem Considered

In this study we consider applying the control scheme to an armature controlled d.c. motor. The quantity to be controlled is the output angle (or position) $\theta$ of a Shaft Encoder, which is coupled by gears to the motor shaft. A schematic representation of the problem is shown in Fig. 2-1.

In Fig. 2-1, $A, J_L(t), B_L(t)$ are the voltage gain, load inertia and load damping coefficients, respectively. The torque due to the load is

$$T_L = B_L(t)\dot{\theta} + \frac{d}{dt} \left[ J_L(t)\dot{\theta} \right]$$

$$= J_L(t)\ddot{\theta} + \left[ B_L(t) + J_L(t) \right] \dot{\theta}$$

(2-1)
Next, as pointed out in the Introduction, $A, B_L(t)$ and $J_L(t)$ are assumed to be essentially constant for the relatively short control periods.

Now, let $J_M(t)$ and $B_M(t)$ be the inertia and viscous damping of the motor. These quantities are not known in general, but are assumed to vary only slightly for a particular control period. Then, dropping $t$ from the terms, we may let

$$J = J_L + J_M$$
$$B = B_L + B_M$$

(2-2)

In appendix A, the transfer function of the motor is shown to be

$$\frac{\delta(s)}{Au(s)} = \frac{K_T/J_M}{s + \frac{1}{J_M} (B_M + K_4 K_T)}$$

(2-3)

For the motor and load the transfer function is

$$\frac{\delta(s)}{Au(s)} = \frac{K_T/(J_L + J_M)}{s + (B_L + B_M + K_4 K_T)/(J_L + J_M)}$$

(2-4)

or

$$\frac{\delta(s)}{u(s)} = \frac{K''/J}{s + B'/J}$$

(2-5)

where

$$K'' = \frac{A K_T}{J'}$$
$$B' = B_L + B_M + K_4 K_T$$

(2-6)

If $J \gg B'$, then (2-5) becomes

$$\frac{\delta(s)}{u(s)} = \frac{K''/J}{s}$$

(2-7)

In any practical problem, we must consider constraints on the amount of control available. It is intuitively plausible that we could make $u$ (which is a voltage in this problem) arbitrarily large in order to drive the motor to the reference position in minimum time (the voltage on the coils of the motor would be a very large value for some time interval, and then a very large value of the opposite sign for the rest of the
time until $\dot{x}=0$ and $x=\theta_{\text{ref}}$, i.e., we accelerate the motor using very large voltages). Arbitrarily large voltages are very hard on the motor and associated electronics, and would not be a desirable feature of the speed control because the large accelerations would damage the controlled object. So we can assume

$$|u'(t)| = |f'\dot{x},x| \leq M, \text{ a known constant.} \quad (2-8)$$

Then we can summarize the problem as follows:

Given the physical system shown in Fig. 2-1, having the mathematical description of Eqns. (2-5) or (2-7), where the parameters of the system are as specified by Eqns. (2-2) and (2-6), find the control $u(t) = f(\dot{x},x)$ such that $\dot{x} = 0$ and $x = \theta_{\text{ref}}$ in minimum time, where the control is subject to the constraint

$$|u'(t)| = |f(\dot{x},x)| \leq M, \text{ a known constant.}$$

We write $u(t) = f'\dot{x},x$ to indicate that the control is to be based on observing the velocity, $\dot{\theta}$, and the position, $\theta$, of the motor (or load) multiplied by the gear ratio, $n$ (a constant), i.e., $x = n\theta$ and $\dot{x} = n\dot{\theta}$.

**B. Calculation of the Optimal Switching Characteristic**

From the discussion in the previous section, we see that there is no loss in generality if we consider the two problems shown in Fig. 2-2.

(a) Problem 1; plant $K/s(s+a)$  \hspace{2cm} (b) Problem 2; plant $K/s^2$

$$|u| \leq 1 \hspace{5cm} |u| \leq 1$$

Fig. 2-2 The physical plants considered in the analysis.
In fig. 2-2 \[ K' = \frac{MK^2}{J} = \frac{MAK_T}{(J_L + J_M)} \]  
and \[ a = \frac{B'}{J} = \frac{(B_L + B_M + K_4K_T)}{(J_L + J_M)} \]

B. 1 **Optimal Switching Characteristic for the Plant \( K/s(s+a) \)**

The analysis of the section follows the techniques outlined in chapter 7 of Ref. 7. Details are not given here as they are very clearly explained in the reference.

If we analyze the plant \( K/s(s+a) \), \( K = nK \), with the constraint \(|u| \leq 1\), in order to determine the OSC, we proceed, as in reference 7, by first forming the Hamiltonian.\(^\dagger\) The differential equation for the plant may be written as

\[ \dot{\theta} + a\theta = K'u \]  

Letting

\[ x_1(t) = n(\theta(t) - \theta_{\text{ref}}) \]  
\[ x_2(t) = n\delta(t), \]

then Eqn. (2-5) becomes

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -ax_2 + K'u \]  

The Hamiltonian for this system is

\[ H = 1 + p_1x_2 - ap_2x_2 + p_2K'u \]

where \( p_1 \) and \( p_2 \) are the costate variables.\(^\dagger\) The differential equations for the costate variables are given by

\[ \dot{p}_1 = -\partial H/\partial x_1 = 0 \]
\[ \dot{p}_2 = -\partial H/\partial x_2 = -p_1 + ap_2 \]  

Let the initial condition for the costate variables be

\[ p_1(0) = \pi_1 \text{ and } p_2(0) = \pi_2 \]

The solutions to Eqn (2-15) are

\(^\dagger\) See reference 7.
\[ p_1(t) = c = \pi_1 \quad (c = \text{a constant}) \]

\[ p_2(t) = p_{2h}(t) + p_{2p}(t) \]

\[ = e^{at} \pi_2 + \frac{\pi_1 - \pi_1}{a} e^{at} \]

\[ = \pi_1/a + e^{at}(\pi_2 - \pi_1/a) \]  

(2-17)

Next, according to the Minimum Principle, we must minimize the Hamiltonian \( H \), which is the same as minimizing \( p_2^2 u \). Thus

\[ u = -\text{sgn } p_2 \quad \text{(since } |u| \leq 1) \]  

(2-18)

Since \( p_2(t) = e^{at}(\pi_2 - \pi_1/a) + \pi_1/a \), we see that it is either equal to a constant \( (\pi_2 = \pi_1/a) \), monotonically increasing \( (\pi_2 > \pi_1/a) \), or monotonically decreasing \( (\pi_2 < \pi_1/a) \). This implies that the possible control sequences are \( \{-1\}, \{+1\}, \{-1,+1\} \) or \( \{+1,-1\} \). This is illustrated in Fig. 2-3.

---

**Fig. 2-3 Possible Optimal Control Functions for the plant \( K/s(s+a) \).**
Next, it is necessary to solve explicitly for \( x_1 \) and \( x_2 \) in order to obtain the OSC. To do this, we let \( u=\Delta \), where \( \Delta = +1 \) or \(-1\), in Eqn. (2-13) which gives

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -ax_2 + \Delta' \quad (\Delta'=\pm \Delta)
\end{align*}
\]

and \( x_1(0) = \xi_1 \) and \( x_2(0) = \xi_2 \)

The solutions to equations (2-19) are

\[
\begin{align*}
x_2(t) &= (\xi_2-\Delta'/a) e^{-at} + \Delta'/a \\
x_1(t) &= \xi_1 + \Delta't/a + \frac{1}{a^2} (\Delta'-a\xi_2) e^{-at} + \frac{1}{a} (\xi_2-\Delta'/a)
\end{align*}
\]

(2-20)

Note \( x_1(t) = \int x_2(t) \)

If \( u=+1 \), and time is eliminated from Eqn. (2-20), we obtain

\[
x_1 = \xi_1 + \frac{K}{a^2} \ln \frac{a\xi_2-K}{a\xi_2+K} + \frac{1}{a} (\xi_2-x_2)
\]

(2-21)

Similarly, if \( u=-1 \), we obtain

\[
x_1 = \xi_1 - \frac{K}{a^2} \ln \frac{a\xi_2+K}{a\xi_2-K} + \frac{1}{a} (\xi_2-x_2)
\]

(2-22)

a) The \( T^+(\xi_1, \xi_2, x_1, x_2) \) trajectories  
b) The \( T^-(\xi_1, \xi_2, x_1, x_2) \) trajectories

Fig. 2-4 Optimal Phase Trajectories for the plant \( K/s(s+a) \), \( K = 1 \), \( a=1 \)
In order to make the definition of the phase trajectories more systematic, we let \( T^+(\xi_1,\xi_2,x_1,x_2) \) denote the trajectory from \((\xi_1,\xi_2)\) to \((x_1,x_2)\) using the control \( u=+1 \), and \( T^-(\xi_1,\xi_2,x_1,x_2) \) denote the trajectory from \((\xi_1,\xi_2)\) to \((x_1,x_2)\) using the control \( u=-1 \). These trajectories are plotted in Fig. 2-4.

In order to obtain the OSC, i.e., the curve which divides the \( x_1-x_2 \) plane into the two regions where \( u=+1 \) is applied and \( u=-1 \) is applied, respectively, we need only superimpose the \( T^+ \) and \( T^- \) trajectories onto one diagram. By elementary logical reasoning the OSC shown in Fig. 2-5 is obtained. It should be noted that we have already proven (c.f. Fig.2-3) that there can be at most one switching of the control. Furthermore, the bang-bang type of control was shown to be optimal. (Eqn. 2-17).

![Fig. 2-5 OSC for the plant \( K/s(s+a) \), \( a=1, K=1 \).](image)

C. Optimal Switching Characteristics for Additional Second Order Plants

1. The Double Integral Plant \( G(s) = K/s^2 \)

Systems which can be considered as purely inertial may be represented by the transfer function \( K/s^2 \). If we go through the same analysis as in the previous sections, we obtain the phase trajectories with OSC.
shown in Fig. 2-6

![Diagram](image)

a) Trajectories for \( u = -1 \)

b) Trajectories for \( u = +1 \)

c) Optimal trajectories for \( u = +1 \) and \( u = -1 \).

**Fig. 2-6 Optimal Switching Characteristics for the double integral plant, \( K/s^2, K=1 \)**

The equation for the phase trajectories, \( T^\Delta(\xi_1, \xi_2, x_1, x_2) \) is given by

\[
x_1 = \xi_1 + \frac{1}{2\Delta'} x_2^2 - \frac{1}{2\Delta'} \xi_2^2, \quad (\Delta = +1 \text{ or } -1)
\]

\[
t = \frac{1}{\Delta'} (x_2 - \xi_2)
\]

2. **Optimal Switching Characteristic for a Plant with Two Time Constants \( K/(s+a)(s+b) \)**

For the case of a plant \( K/(s+a)(s+b), a>0, b>0, a\neq b \), an optimal switching characteristic of the same general shape which can be handled by the synthesized control scheme is obtained. Fig. 2-7 shows the OSC in the \( z_1-z_2 \) plane, where \( z_1, z_2 \) are related to \( x_1, x_2 \) by the transformations
\[ z_1 = \frac{1}{K}(-abx_1+bx_2) \]
\[ z_2 = \frac{1}{K}(-abx_1-bx_2) \]  \hspace{1cm} (2-23)

Fig. 2-7 Optimal Switching Characteristic for a plant with two time constants, \( K / (s+a)(s+b) \).

This transformation is used to simplify computation of the phase trajectories by uncoupling the original state variables \( x_1 \) and \( x_2 \).

3. Optimal Switching Characteristic for the Harmonic Oscillator, \( K / (s^2+\omega^2) \)

This example is given to show that the proposed control scheme will not work for all second order plants representable in the phase plane.

An oscillator may be represented by the equation.

\[ \ddot{x} + \omega^2 x = K u, \]  \hspace{1cm} (2-24)

(assume \(|u| \leq 1, K > 0\))

whose transfer function is

\[ G(s) = \frac{K}{s^2 + \omega^2} \]  \hspace{1cm} (2-25)

If we let \( x_1 = x \) and \( x_2 = \dot{x} \), and use the transformations

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\[ y_1 = \frac{\omega}{K} x_1 \]
\[ y_2 = \frac{1}{K} x_2, \]

then equation (2-24) becomes
\[ \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} \]
(2-26)

In this case the Optimal Switching Characteristic has the form shown in Fig. 2-8.

Fig. 2-8 Optimal Switching Characteristic for the harmonic oscillator, \( K / (s^2 + \omega^2) \).

It is obvious from this figure that more than one switching of the control \( u \) is required. Thus the proposed control scheme would not work here. Actually Theorem 6-8 of reference[7] gives the criterion for the number of switchings required for linear constant systems with negative, distinct eigenvalues (or poles) of the plant, namely, there are no more than \( n-1 \) switchings required for an \( n^{th} \) order system.

4. Optimal Switching Characteristic for a Second Order Nonlinear Plant.

Next a nonlinear plant is considered to demonstrate that the proposed control scheme would work for a certain class of nonlinear plants.
Consider the system (see section 7-11, [7]).

\[
\ddot{y} + y|\dot{y}| = u, \quad |u| \leq 1
\]

(2-27)

[Actually a more general form of (2-27) is

\[
\ddot{y} + f(\dot{y}) = K u, \quad |u| \leq 1,
\]

(2-28)

with certain conditions on \( f(\dot{y}) \), is given in reference 7 which would give an OSC of a form to which the proposed control scheme could be applied].

The OSC and typical phase trajectories for the plant (2-27) are shown in Fig. 2-9.

\[\text{Fig. 2-9 The Optimal Switching Characteristic and phase trajectories for the plant } \ddot{y} + y|\dot{y}| = u, \quad |u| \leq 1.\]

D. Analysis of Control Relay Dead-Time

Due to the physical properties of the components used for switching the control voltage to the armature of the motor, a finite length of time is required to switch the control from +1 to -1, and from -1 to +1. Typical relay switching characteristics are shown in Fig. 2-10. Details of the relays are given in section IV-C-2.

In order to be precise in the discussion of this section, the following notation is used:

- \( t_{c}^{+} \) is the time that the control is commanded to switch from -1 to +1,
- \( t_{c}^{-} \) is the time that the control is commanded to switch from +1 to -1,
Fig. 2-10 Typical Control Relay Switching Characteristics.

t$^{+}$,0 is the actual time that the relays switch from +1 to 0,
t$^{-}$,0 is the actual time that the relays switch from -1 to 0,
t$^{0}$,- is the actual time that the relays switch from 0 to -1,
t$^{0}$,+ is the actual time that the relays switch from 0 to +1,

\[ \delta_{h}^{+,-} = t^{+},0 - t_{c}^{+} \]
\[ \delta_{d}^{+,-} = t^{0},- - t^{+},0 \]
\[ (2-29a) \]

\[ \delta_{h}^{-,+} = t^{-},0 - t_{c}^{-} \]
\[ \delta_{d}^{-,+} = t^{0},+ - t^{-},0 \]
\[ (2-29b) \]

Fig. 2-11 Effect of $\delta_{h}$ and $\delta_{d}$ on the Phase Trajectories
It can be readily seen that $\delta_h$ and $\delta_d$ represent a kind of "hysteresis-time" and "dead-time", respectively. The effect of $\delta_h$ and $\delta_d$ on the phase trajectories will be typically as shown in Fig. 2-11. In the figure $(\xi_1'', \xi_2'')$ is the state of the system at $t^+\gamma$, $(\xi_1', \xi_2')$ is the state of the system at $t^-\gamma$, and $(\xi_1, \xi_2)$ is the state of the system at $t^0\gamma$.

In order to illustrate the method of analysis for the effect of the "hysteresis-time", $\delta_h$, and the "dead-time", $\delta_d$, the two plants $K/s^2$ and $K/s(s+a)$ shall be considered only. A similar analysis would be used for other plants.

1. Effect of $\delta_h$ and $\delta_d$ for the Plant $K/s^2$

For the plant $K/s^2$, we have

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \Delta' \\
    &= (\Delta' = K\Delta, \Delta = \pm 1)
\end{align*}$$

(2-30)

The solution to (2-30) is

$$\begin{align*}
x_2 &= \xi_2 + \Delta't \\
x_1 &= \xi_1 + \xi_2't + \frac{1}{2}\Delta' t^2.
\end{align*}$$

(2-31)

Using the notation specified above, at $t^+\gamma$ or at $t^-\gamma$ we have

$$\begin{align*}
\xi_2' &= \xi_2'' + \Delta'\delta_h \\
\xi_1' &= \xi_1'' + \xi_2''\delta_h + \frac{1}{2}\Delta'\delta_h^2,
\end{align*}$$

(2-32)

and at $t^0\gamma$ or at $t^0\gamma$, we have (i.e., $\Delta = 0$ in (2-30)),

$$\begin{align*}
\xi_2 &= \xi_2' \\
\xi_1 &= \xi_1' + \xi_2'\delta_d.
\end{align*}$$

(2-33)

Eliminating $\xi_1'$ and $\xi_2'$ from (2-32) and (2-33) gives

$$\begin{align*}
\xi_2 &= \xi_2'' + \Delta'\delta_h \\
\xi_1 &= \xi_1'' + \xi_2''\delta_h + \frac{1}{2}\Delta'\delta_h^2 + \xi_2\delta_d.
\end{align*}$$

(2-34)

Thus we switch if

$$\begin{align*}
\xi_1 &= \xi_1'' + \xi_2''(\delta_h + \delta_d) + \Delta K\delta_h(\delta_h/2 + \delta_d)
\end{align*}$$

(2-35)
compares to the value of $\xi_1$ which is stored in the memory of the computer. We see that (2-35) is of the form

$$\xi_1 = \xi_1'' + c_1 \xi_2'' + c_2 K \Delta$$  \hspace{1cm} (2-36)

The constants $c_1$ and $c_2$ (defined by 2-35) are known, and depend only on the physical properties of the relays which are used. If the constant $K$ were known (it depends on the plant, whose parameters are assumed to be unknown) eqn.(2-36) could be inputted to the computer program and the effects of $\delta_d$ and $\delta_h$ would be accounted for in the switching logic. However, if $c_1$ and $c_2 K$ are assumed unknown they could be estimated from (2-33), (2-34), and (2-35) by observing the state ($x_1, x_2$) during the first control period as shown below.

**Fig. 2-12** Effect of $\delta_h$ and $\delta_d$ during first control period

Then from (2-34),

$$K \delta_h = a_1 = (\xi_2'' - \xi_2')/\Delta \hspace{1cm} (\Delta = \pm 1)$$  \hspace{1cm} (2-37)

and from (2-33)

$$\delta_d = a_2 = (\xi_1'' - \xi_1')/\xi_2'$$  \hspace{1cm} (2-38)

and from (2-35)

$$\delta_h = a_3 = (\xi_1''(1 + a_2) - \Delta a_1 a_2)/(\xi_2'' + \Delta a_1/2).$$  \hspace{1cm} (2-39)
Then
$$c_1 = a_3 + a_2$$

and
$$c_2 = a_1(ka_3 + a_2),$$

(2-40)

and (2-36) could be used to account for $\delta_h$ and $\delta_d$. Logic must be provided in the computer program to account for the case $\xi_2' = 0$ and/or $(\xi_2'' + \Delta a_1/2) = 0$. Actually $(\xi_1'', \xi_2'')$, $(\xi_1', \xi_2')$, and $(\xi_1, \xi_2)$ could be determined during the first control period by physically sensing whether $u$ is positive, negative, or zero, and observing the states $(x_1, x_2)$ at these times. Practical considerations and topics for further study in regards to $\delta_h$ and $\delta_d$ are discussed in section VII.

2. Effect of $\delta_h$ and $\delta_d$ for the Plant $K/s(s+a)$

For the plant $K/s(s+a)$, we have

$\dot{x}_1 = x_2$

$\dot{x}_2 = -ax_2 + \Delta' \quad (\Delta' = KA)$

(2-41)

and from section 11-B, the optimum phase trajectories are specified by

$$x_2 = (\xi_2' - \Delta'/a)e^{-at} + \Delta'/a$$

$$x_1 = \xi_1 + \Delta't/a + \frac{1}{a^2}(\Delta' - a\xi_2)e^{-at} + \frac{1}{a}(\xi_2' - \Delta'/a)$$

(2-42)

Using the notation specified above, at $t^+, o$ or at $t^-, o$ we have

$$\xi_2' = (\xi_2'' - \Delta'/a)e^{-a\delta_h} + \Delta'/a$$

$$\xi_1' = \xi_1'' + \Delta'\delta_h/a + \frac{1}{a^2}(\Delta' - a\xi_2'')e^{-a\delta_h} + \frac{1}{a}(\xi_2'' - \Delta'/a)$$

(2-43)

and at $t^0, -$ or at $t^0, +$, we have (i.e., $\Delta = 0$ in Eqn. 2-41)

$$\xi_2 = e^{-a\delta_d} \xi_2'$$

$$\xi_1 = \frac{\xi_2'}{a}(1 - e^{-a\delta_d}) + \xi_1'$$

(2-44)

Thus

$$\xi_2 = e^{-a(\delta_h + \delta_d)}\xi_2'' + \frac{\Delta K}{a}(1 - e^{-a(\delta_h + \delta_d)})$$

$$\xi_1 = \xi_1'' + \frac{1}{a}(1 - e^{-a(\delta_h + \delta_d)})\xi_2'' + \frac{\Delta K}{a}(\delta_h + 1)(e^{-a\delta_d} - 1)$$

(2-45)

If $a, K, \delta_h$ and $\delta_d$ were all known quantities, equations (2-45) could be
used to compensate for $\delta_h$ and $\delta_d$ by looking at the present state $(\xi_1'', \xi_2'')$, computing the projected state $(\xi_1, \xi_2)$, and then comparing $(\xi_1, \xi_2)$ with that stored in the computer's memory.

If these parameters are not known then we might proceed as in the previous section and use equations (2-43) to (2-45) to determine the unknown coefficients in (2-45) in terms of the observed states $(\xi_1'', \xi_2'')$, $(\xi_1', \xi_2')$ and $(\xi_1, \xi_2)$ during the first control period.
III. DIGITAL CONTROL COMPUTER MODIFICATIONS

Although a large part of the digital control computer had been constructed, considerable changes had to be made to both the design and wiring. Because these modifications and additions were quite extensive, the author felt a section should be devoted to show these changes. For the future, a complete set of logic diagrams will be available.

As closely as possible, the order of the diagrams and explanations will be the same as those used by the original designer [8]. In this section, the title and page number in brackets on each figure, are the same as those used in Ref. 8. Therefore, if the original design is used to augment the explanations, the reader should find no difficulty in following this section.

Multiplication Order

The computer uses a serial repeated addition and subtraction method. The algorithm for multiplication in the signed 2's complement form is shown in Fig. 3-1.

<table>
<thead>
<tr>
<th>Q</th>
<th>Q</th>
<th>K</th>
<th>Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Neither add nor subtract, shift right twice</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>Add, then shift right twice</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>Shift right once, add, then shift right again</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>Shift right once, subtract, then shift right again</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>Subtract, then shift right twice</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>Neither add nor subtract, shift right twice</td>
</tr>
</tbody>
</table>

Fig. 3-1 Algorithm for Multiplication
Eight phases are required to carry out the multiplication order. The operation is as follows: during the first phase, the contents of the Accumulator (A-Register) are shifted into the Q-Register. For the following six phases, three digits (the two least significant digits of the Q-Register and a bit held in the K-flip-flop) are examined and the contents of the appropriate registers are operated upon according to the algorithm in Fig. 3-1. In the eighth phase, the contents of the Q-Register are shifted into the A-Register.

A. Phase Counter

Fig. 3-2 shows the Phase Counter. $T_{14}$ is used instead of $T_{15}$ because in the second phase of the multiplication order a preshift may be required. Preshift occurs at $T_{15}$, so the next phase must begin before the preshift occurs (i.e., if $T_{14}$ sets the Counter, $T_{15}$ is assured to occur during the next phase). Also, $T_{14}$ or $T_{15}$ may only be used here because at $T_{13}$ the AxH order is completed and a noise "spike" occurred on the "Right Shift Line" to the A and Q-Registers at $T_{13}$. This spike caused an "attempt" to right-shift the A and Q-Registers after Q*A occurred in the eighth phase.

During the eighth phase, right-shifting pulses are inhibited by $\overline{0123}$.

Fig. 3-2 Phase Counter

(Phase Counter, fig. 30, pg. 55)
at gate C12A of the "Right Shift Accumulator" shown in Fig. 3-9. At the same time, \( P_0 P_1 P_2 P_3 \) at gate C13C, in conjunction with \( T_{13} \) at C8B determine \( \text{NEW} \). \( \text{NEW} \) in turn, resets the Phase Counter. The change had to be made in order that during the eighth phase no right-shift information occurred until after the AxH order had been completed when it no longer mattered. Fig. 3-3 shows the waveforms produced by the Phase Counter during an AxH order.

![Waveform Diagram](image)

Fig. 3-3 Multiplication Order Waveforms

Initially the Phase Counter is set at 0000 and counts up to 0111 (7). Then \( \text{NEW} \) resets it to 0000 again. The original counter was set to count six phases (i.e., 10+15, 10+15, ...). Gate C4B has been left because it does not affect the AxH order. However, in the future when the divider is made operational, it will have to be changed or removed entirely.

B. Multiplication Unit

Fig. 3-4 shows a block diagram of the Multiplication Unit. Gate D17A eliminates \( \text{ADD} \) or \( \text{SUBT} \) signal from being generated during the eighth phase (Q+\( \overline{A} \)) of the AxH order. The flip-flops D27B and D27C are required to hold the generated signal (i.e., \( \text{ADD} \) or \( \text{SUBT} \)) which would be changed
Fig. 3-4 Multiplication Unit

by a change in \( Q_1 \), \( Q_0 \) or \( K \) at gates D18H or D17E respectively during any phase of the AxH order.

The multiplication operation can best be explained by an example. Two examples are shown so that the handling of a negative number can also be illustrated.

In signed 2's complement

\[
\begin{align*}
7 &= 0,00000000111 \text{ (initially in A-Register)} \\
3 &= 0,00000000011 \text{ (initially in an H-Register)}
\end{align*}
\]
### Fig. 3-5a Multiplication of 7x3

In signed 2's complement:
\[ -8 = \overline{11111111000} \text{ (initially in A-Register)} \]
\[ 3 = 0,0000000011 \text{ (initially in an H-Register)} \]

### Fig. 3-5b Multiplication of -8x3

\[ 7x3 = 16 + 4 + 1 = 21 \]

### C. Delay Line Memory

Fig. 3-6 shows a block diagram of the Delay Line Memory. Information is continuously being circulated. Input/output to/from the delay line memory occurs only during COMPARE or MOD.OP. operations.

In the original circuit, the output of A5B would be at "0" if no
S-order signal at A4A and COMPARE signal at A5A occurred. Hence, stored information would not circulate. At the same time, information at ENTER would be allowed to enter when it is not desired to do so.

![Diagram](image)

Fig. 3-6 Delay Line Memory

**D. Gating of Accumulator**

Fig. 3-7 shows a block diagram of the Accumulator Gating. The changes here are quite extensive. In the original design, gates C10C, C11G, and C12C were used to restore the sign bit in the Accumulator after the first phase of the AxH order. There is no point in doing this. A review of the multiplication operation will clearly show that this recirculation is not required. Gates D16D, D15E, and D17G were used in the first phase of the AxH order to recirculate the contents of the Accumulator. Since the multiplication operation does not require this recirculation, the gates have been removed.

The COMPL. signal at gate A2G is used during the |A|→A order when 2's complementing a negative number is required. Gate C15D is used to restore the sign bit during the AxH order. It is restored in the eighth phase (i.e., Q→A). This is necessary because in left shifting there is no connection between A₁₀ and A₁₁ (i.e., no information can be left-shifted into
Fig. 3-7 Gating of Accumulator

the sign bit location \(A_{11}\) of the Accumulator.

E. Accumulator

Fig. 3-8 shows a block diagram of the Accumulator (A-Register), Q-Register and the input from the Shaft Position Encoder. The circuit for resetting the Q-Register and K-flip-flop (C14C and D15H) is necessary because if there were any digits left in the Q-Register from the previous multiply order or from initial turn-on of the system, during the first phase \((A\times Q)\) the unwanted digits will move into positions \(Q_1, Q_0,\) or \(K\) and set ADD or SUBT signal which gates the ENTER to the Accumulator. Therefore, unwanted arithmetic is performed and enters the Accumulator.
From shaft position encoder

Output

Cannon connector KD2-16-10PN

+12V

-12V

Encoder

Interrogate pulses

Clears Q-Register in 1st phase

Accumulator, fig. 18, pg. 38

Fig. 3-8 Accumulator

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The signal marked Z, at the output of flip-flop D27D, allows information to flow from the A-Register to the Q-Register between times $T_{13}$ and $T_1$, during the AxH order, and is used to gate the ADD or SUBT signals. This is necessary because while the information is being shifted to the right in both registers, at times $T_{13}$, $T_{15}$, or $T_0$, the digits in $Q_1$, $Q_0$, or $K$ may become such combinations as to generate an ADD or SUBT signal which would enter information to the Accumulator from the Arithmetic Unit by the ENTER path.

Because the Optical Shaft Position Encoder has serial output information with most significant digit first, and, since the computer INPUT information is moved around least significant digit first, during the order, ten (Interrogate) left-shifting pulses $(T_1 - T_{10})$ serially move information into the least significant position of the A-Register. Left-shifting pulses $(T_1 - T_{12})$ are already available, so, rather than generate pulses $(T_1 - T_{10})$, INPUT* signal opens gates C25E and C25B during only ten pulses.

Originally, the Accumulator could be cleared manually, only. However, during the INPUT order, only ten left-shifting pulses are used, so that, if the least significant digit ($A_0$) was present before the INPUT order was called, it would be shifted to the overflow position ($A_{10}$). Gate A25B is used to clear the ACCUMULATOR just prior to accepting information from the Shaft Position Encoder as well as for manual clearing. A25D acts as an inverter.

F. Right and Left Shifting the Accumulator

Fig. 3-9 shows a block diagram of the logic for providing right and left-shifting pulses. Here changes were considerable but the logic is easy to understand, therefore, only the less obvious changes will be
Fig. 3–9 Right and Left-Shifting the Accumulator

described. Gate C14A provides a "1" at C15C, C11H, and C11G for the seven last phases of the AxH order. For the first phase (A→Q) it inhibits T15 to T13.

The signal marked X is required to prevent right shift pulses during the eighth phase (i.e., Q→A) of the AxH order. Gate B3E has been used to replace C15G in the original design because three inputs were required. The INPUT* signal allows the first 10 shift pulses of (T1-T12) which are

(Right and Left Shift Accumulator, figs.20 & 21, pgs. 40 & 41)
required as Interrogate pulses for the Optical Shaft Position Encoder (see "Accumulator" Fig. 3-8).

G. I-Registers

Fig. 3-10 shows a block diagram of the two Index Registers. In the original design, no provisions were made to either select the proper register or circulate its contents during the modification operation (MOD.OP.). An added order, A→I is also shown. For the A→I order, the network shown was necessary to provide both positive and negative going shift pulses at the I-Registers because the I-Register shift pulses are positive-going while the Accumulator (A-Register) uses negative-going pulses.
H. Inversion Gating

Fig. 3-11 shows a block diagram of the Inversion Gating. Changes here are few. Several new orders have been added and the flip-flops required resetting in order that no order involving the memory was activated during initial turn on of the computer.

Fig. 3-11 Inversion Gating

I. Adder/Subtractor

Fig. 3-12 shows a block diagram of the Adder/Subtractor and its inputs. In the original design no provision was made to clear the CARRY flip-flop prior to an arithmetic order. Initial turn-on of the computer or a previous arithmetic order could leave it in either state.

The logic consisting of gates B3A, B8H, and B3H are used because
during a modification order leading edge positive-going pulses are required while H-Registers use trailing edge negative-going pulses. Gate B15F acts as an OR gate, therefore, negative-going pulses \((T_1-T_{12})\) or positive-going pulses \((T_1-T_6)\) are available at the CARRY flip-flop.

Gates B3D, B3F, and flip-flop B1B are part of the complementing order, \(A+|A|\). During this order, at time \(T_0\), the sign bit of the Accumulator, \(A_{11}\), is sensed. If \(A_{11}\) is "0" (i.e., positive value), the contents of the Accumulator are merely recirculated (see Fig. 3-8). But, if \(A_{11}\) is "1" (i.e., negative number), the subtraction of the contents of the Accumulator from ZERO is performed, which is the same as taking the 2's complement of the number in the Accumulator.

J. Bit Counter and Timing Unit

Fig. 3-13 shows a block diagram of the Clock, Bit Counter, and Timing Unit. The maximum serial read-out rate for the Optical Shaft Position Encoder is only 100 kHz. The Computer has a clock rate of 200 kHz. A simple means of producing INTERROGATE pulses at the lower rate was to reduce the Clock speed by half. Normally gate CIA is open and C1B closed by the INPUT signal: the output of C1B is at "1" so the pulses at clock rate of 200 kHz are provided by power amplifier C24D.

Flip-flop B1A continuously divides the Clock rate by two. When the Shaft Position Encoder is interrogated during the INPUT order, gate CIA is closed and C1B is open to allow pulses at half the clock rate to be present at C24D. C1C resets B1A before the INPUT order but not during: this is important for pulse shaping.

In generating INPUT* signal (see Fig. 3-8) \(T_{11}\) was required. Since \(T_9\) was not necessary in the computer, gates D12D and D12H were used to save time and space.
Fig. 3-13 Bit Counter and Timing Unit

(Bit Counter, fig. 33, pg. 59)
(Timing Unit, fig. 32, pg. 58)
Power amplifier D14A has been used for signal $T_{14}$ which is required in the LOCATION ENCODER (see Fig. 3-17). At the time it was the only power amplifier available. The power requirements for $T_{14}$ are low, so, D1A was used.

The TEST MOD signal was necessary at gates D15C and D15D to inhibit ($T_1 - T_6$) during the TEST MODIFIER operation (i.e., this order is two word lengths in duration).

Not shown in Fig. 3-13, are two gates in cascade between B3G and C1B. Gates D5H and C10C, were required to provide enough "delay" in turning off the INPUT signal at time $T_{13}$, to prevent a "spike" when the pulses, provided by the divider BlA, are inhibited and taken from the clock (at this time, the two signals were at the same levels). This "spike" was of sufficient duration to add an extra "count" in the BIT COUNTER.

K. (Control) N-Register

Fig. 3-14 shows a block diagram of the Control Register. The original design shows that the contents of this register is recirculated for each of the following orders: I+N+I, I-N+I, and N+I. There is no purpose in

![Control Register Diagram](Control Register, Fig.34,p.60)

Fig. 3-14 Control Register
this since the number N is obtained from the "LOCATION ENCODER", shown in Fig. 3-17. N is inserted into the Control Register at time $T_{14}$ (i.e., one clock pulse after the beginning at $T_{13}$ of any of the above orders). The other changes are minor.

L. **Compare Unit**

The Compare Unit is shown in Fig. 3-15. This unit compares the numbers in the WORD COUNTER and in the (Control) N-REGISTER. If the numbers are

![Diagram of Compare Unit](Compare Unit, fig. 36, pg. 62)
the same, the COMPARE flip-flop is set. The next word to come out of
the DELAY LINE (memory) is then the required word.

This unit is used during the modification (MOD. OP.) operations as well
as during the S-orders. During an address modification operation, at
which time the numbers in the WORD COUNTER and CONTROL REGISTER are the
same, the COMPARE Signal is produced at time $T_0$, at which time information
may be inserted into, or retrieved from, the circulating memory at the
location in memory given by the number in the Control Register.

M. Word Counter

Fig. 3-16 shows the WORD COUNTER. The WORD COUNTER assigns a "count"
or number to each word in the memory. Only 60 words can be stored in the
delay line memory, therefore, this counter advances from 61 to 63.

During the INPUT order the clock is slowed down (see Fig. 3-13) but
the information flow in the delay line is not affected: hence, one word
count would be lost. Gate A25G is used to add an extra "count" at $T_0$,
during the INPUT order, to "keep up" with the information flow in the delay line. $T_0$ was chosen because it is the only signal which was available without adding an inverter (see Fig. 3-13).

N. Location Encoder

Fig. 3-17 shows a block diagram of the Location Encoder. The COMPARE signal resets flip-flop D27A which in turn gates D1B to allow $T_{14}$ to insert

(Location Encoder, fig. 40, pg. 68)
the number N from the Location Encoder into the Control Register
(see Fig. 3-14). The modifications were necessary because according to
the original design there would be an attempt to insert a number into the
Control Register at every $T_{14}$.

For S-orders, if MOD.OP. is true, the order is prohibited until
modification is completed.

0. Increase I.C. Gating

The block diagram for the "Increase Instruction Counter Gating" is
shown in Fig. 3-19. This works on the prohibition principle. That is,
the Instruction Counter is advanced at every $T_{13}$ by the INCREASE I.C.
signal, unless this signal is prohibited. Prohibition occurs if:

1. an S-order is incomplete
2. a multiplication (AxH) order is incomplete
3. a JUMP signal has been produced
4. a TEST signal has been produced
5. the START flip-flop has been reset.

Flip-flop D26C inhibits the instruction counter from advancing on the
first count, during an S-order. Fig. 3-18 showing a time diagram, illus­
trates how the Instruction Counter is inhibited for two orders, mentioned
in the list above. The second order (JUMP) follows the first (TEST MOD.)
by two word lengths in this example only.

Fig. 3-18 Waveform diagrams illustrating the "Inhibition" principle
The NEW instruction flip-flop is required in the multiplication (and division) order. It is used to gate the operation of the first phase (see Fig. 3-2).

Fig. 3-19 Increase Instruction Counter Gating

P. Jump/Test Logic and Instruction Counter

Fig. 3-20 shows a block diagram of the Jump/Test Logic and Instruction Counter. Most of the changes are minor. The important change is that INCREASE I.C. instead of INCREASE is used to advance the Counter. The modification (MOD. OP.) operation requires two word lengths. The first phase is a compare phase and in the second the actual order is executed.
Fig. 3-20 Jump/Test Logic and Instruction Counter

(Jump/Test Logic, fig. 37, pg. 63)

/Instruction Counter, fig. 38, pg. 65)
IV. DESCRIPTION OF HARDWARE

A. General Description

The actual system to be controlled was a 1/15 hp-dc motor connected through a set of gears to an optical shaft position encoder. A differential gear was used to set the initial position manually. Connected directly to the shaft of the motor was a 35 lb.-4 inch radius, steel disc. This large load was used in order that the overall system could be regarded as approximately purely inertial and in order that the system moved slowly enough for the digital controller. Fig. 4-1 shows a block diagram of the set-up. Referring to Fig. 4-1, the vertical input of an oscilloscope was connected at a and the horizontal input was connected at b.

Fig. 4-1 Block diagram of the hardware set-up

Since their effects are small in comparison to those of the load, an idealized version of the differential gear (with no appreciable

† see reference 11.
moment of inertia or friction on the input or output shaft) and an ideal gear train are assumed since their effects are relatively negligible and to eliminate the mathematical problems involved [13]. Fig. 4-2 shows an oscillograph of the actual system trajectory. Here, the results were obtained by manually setting the appropriate values for switching in the Accumulator. Three trajectories are shown: number 1 is the trace ABCO, number 2 is the trace ADO and number 3 is the trace AFO.

Fig. 4-2 Oscillograph showing system response

B. Input

During the INPUT order, a 3M Model 203524, Serial no. 136, Optical Shaft Position Transducer provides a serial conventional binary output representing the shaft angle being measured [11].

The conversion of shaft angle to a digital number is obtained using a code pattern wheel, incandescent light source, and photo diodes. The
Encoder divides each revolution into 1024 parts allowing digital measurements to 21.1 minutes accuracy. Light, from a lamp, passing through a code pattern wheel determines which of the ten silicon photo diodes will be energized. The output of each photo diode is amplified, and upon receiving a SAMPLE & HOLD pulse, the pattern is stored as digital information in a register consisting of ten flip-flops. Ten INTERROGATE pulses then shift the information serially out of the register, most significant digit first, into the least significant end of the A-Register.

The maximum rate of read-out for the Encoder is 100 kHz. The Digital Computer has a clock rate of 200 kHz. To make the two systems compatible, the clock rate is reduced to 100 kHz during the INPUT order (see Fig. 3-14). The method of accomplishing this speed reduction is described in the section on the Accumulator.

C. Output

1. Logic

The output logic is shown in Fig.4-5. During the OUTPUT order the digits in the Accumulator are sensed and the proper control is applied according to the algorithm of Fig.4-3.

<table>
<thead>
<tr>
<th>Sign bit $A_{11}$</th>
<th>Most significant bits $A_{10} \ldots A_{4}$</th>
<th>Drive $u(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 0</td>
<td>all 0's</td>
<td>no drive</td>
</tr>
<tr>
<td>0, 1</td>
<td>not all 0's</td>
<td>+ve drive</td>
</tr>
<tr>
<td>1, 0</td>
<td>all 1's</td>
<td>no drive</td>
</tr>
<tr>
<td>1, 1</td>
<td>not all 1's</td>
<td>-ve drive</td>
</tr>
</tbody>
</table>

Fig. 4-3 Algorithm for proper control.

Fig.4-5 shows the logic diagram for the output. Gate A3C sets flip-flop A1A which in turn operates the solenoid driver B2A. B2A operates the coil of relay 1 to apply negative torque to the motor. Similarly, gate A3D, flip-flop A1B and solenoid driver A2B energize relay 2 which applied positive torque to the motor. A2F resets both flip-flop before the test for proper control is
made. This simplifies turn-off. Some examples showing all possible condition for switching are shown in Fig. 4-4

<table>
<thead>
<tr>
<th>Accumulator state sign bit</th>
<th>Number A11</th>
<th>Inputs An</th>
<th>Nand outputs A3A A3C</th>
<th>Control u(t)=+1</th>
<th>Inputs An</th>
<th>Nand outputs A3B A3D</th>
<th>Control u(t)=-1</th>
<th>Rotor motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0000000000 000</td>
<td>0</td>
<td>0 all 0's 1 1</td>
<td>no all 1's 0 1</td>
<td>no</td>
<td>standstill</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 0000000000 111</td>
<td>&lt;ε</td>
<td>0 all 0's 1 1</td>
<td>no all 1's 0 1</td>
<td>no</td>
<td>standstill</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 111111111 001</td>
<td>ε</td>
<td>1 all 1's 0 1</td>
<td>no all 0's 1 1</td>
<td>no</td>
<td>standstill</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 0000000001 000</td>
<td>+8</td>
<td>0 0's&amp;1 1 1</td>
<td>no 0's&amp;1 1 1</td>
<td>yes</td>
<td>-ve torque</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 111111111 010</td>
<td>large</td>
<td>0 all 1's 0 1</td>
<td>no all 0's 1 1</td>
<td>yes</td>
<td>-ve torque</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 1111111111 000</td>
<td>-8</td>
<td>1 all 1's 0 1</td>
<td>no all 0's 1 1</td>
<td>no</td>
<td>standstill</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 111111111 111</td>
<td>-9</td>
<td>1 1's&amp;0 1 0</td>
<td>yes 0's&amp;1 1 1</td>
<td>no</td>
<td>standstill</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 0000000000 011</td>
<td>large</td>
<td>1 all 0's 1 1</td>
<td>yes all 1's 0 1</td>
<td>no</td>
<td>+ve torque</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

† 0<ε<8

Fig. 4-5  Examples showing conditions for switching

---

Fig. 4-5  Output Logic and armature control

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2. **Relays**

An ideal switching device, a relay with states (+1, -1) as shown in Fig. 4-6 is impractical. A device with a dead-band D must therefore be used.

![Fig. 4-6 Relay characteristics (a) ideal (b) with dead-band](image)

Because finances did not permit sophisticated switching systems and an inexpensive set of relays was already available, the arrangement shown in Fig. 4-5 was used. It was felt that the plant was slow enough so that the long pull-in and drop-out times of the relays would not require special programming. These delays are described in another section.

Type 200-12D "GUARDIAN Universal Series 200 Relays" were used. The DC coils were 12 volts at 240 ma. The drop-out time for these relays is approximately 30 milliseconds and pull-in time is approximately 17 msec. The measured dead-times for the control $u$, are shown in Fig. 4-7

<table>
<thead>
<tr>
<th>Control, $u(t)$</th>
<th>Time (msec)</th>
<th>Time (msec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u=0 \rightarrow u=-1$</td>
<td>17</td>
<td>30</td>
</tr>
<tr>
<td>$u=0 \rightarrow u=+1$</td>
<td>17</td>
<td>30</td>
</tr>
<tr>
<td>$u=-1 \rightarrow u=0$</td>
<td>17</td>
<td>17</td>
</tr>
<tr>
<td>$u=+1 \rightarrow u=0$</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$u=+1 \rightarrow u=-1$</td>
<td>30</td>
<td>17</td>
</tr>
<tr>
<td>$u=-1 \rightarrow u=+1$</td>
<td>17</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 4-7 Relay Operation Time.
each pair of contacts is a 1 mfd 600 v capacitor in series with a 470 ohm resistor to reduce the large inductive voltage which results when the armature circuit is suddenly broken.

3. **Contact Protection**

When any current carrying circuit is made or broken, sparking occurs and power is dissipated as heat. In an inductive circuit, the back emf, developed when the circuit is broken, may be several times the applied voltage. Energy stored is equal to \( \frac{1}{2}LI^2 \), where \( L \) is the inductance and \( I \) is the current flowing before the circuit is broken. When the contacts separate an arc is formed. This arc increases in length with the contact separation until at some point the arc is extinguished. The extinguishing of the arc causes a sharp voltage spike across the inductive circuit. The damage to contacts is increased if there is any contact bounce. Electro-static attraction \([12]\) also affects efficient contact closure.

If a capacitor is connected across the contacts, or across the load, energy can be absorbed when the contacts open. Resistance is necessary to limit the current flow into the capacitor when connected across the load, or out of the capacitor, when connected across the contacts.

The resistor-capacitor combination for arc suppression was determined by choosing a capacitor as large as practical and the resistor was then chosen to keep the current peak at a safe level. The most efficient suppression circuit, the one with the lowest peak, was determined by using an oscilloscope connected across the combination. The circuit is shown in Fig. 4-5.
V. EXTENSION TO HIGHER ORDER SYSTEMS

A useful topic for future study would be to extend the control scheme discussed in this thesis to systems of order higher than two. In this section we wish to point out briefly an approach which might be taken. This approach is theoretically justified in chapter 7 of reference 7 for constant coefficient plants (with negative real poles) whose parameters are all known. Of course we have been assuming that the plants have unknown parameters and that the control scheme must "adapt" to the plant.

A. Optimal Switching Characteristic for the Plant $K/s^3$.

This plant is considered because it is simple enough to illustrate the ideas without complicated mathematics. Proceeding as in section II-B let

\[
\begin{align*}
\dot{x}_1 &= \dot{x} = x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= Ku.
\end{align*}
\] (5-1)

The Hamiltonian is

\[
H = 1 + p_1x_2 + p_2x_3 + p_3Ku,
\] (5-2)

which implies that

\[
u = -\text{sgn } p_3
\]

The costate variables satisfy the equations

\[
\begin{align*}
\dot{p}_1 &= 0 \\
\dot{p}_2 &= -p_1 \\
\dot{p}_3 &= -p_2,
\end{align*}
\] (5-3)

which imply that
\[ p_1 = \pi_1 \]
\[ p_2 = -\pi_1 t + \pi_2 \]
\[ p_3 = \pi_1 t^2/2 - \pi_2 t + \pi_3 \]  
(5-4)

Thus
\[ u(t) = -\text{sgn} \ p_3(t) = -\text{sgn}(\pi_1 t^2/2 - \pi_2 t + \pi_3) \]  
(5-5)

Since \( p_3(t) \) is a quadratic equation in \( t \), it can be seen that \( p_3(t) \) will change sign two times at most. Thus, the possible control sequences are
\[ \{+1\}, \{-1\}, \{-1,+1\}, \{+1,-1\}, \{-1,+1,-1\}, \{1,-1,1\}. \]

Next we let \( K_u = K_d = K' \) in equations (5-1), and obtain
\[ x_3 = \Delta't + \xi_3 \]
\[ x_2 = \Delta't^2/2 + \xi_3 t + \xi_2 \]
\[ x_1 = \Delta't^3/6 + \xi_3 t^2/2 + \xi_2 t + \xi_1 \]  
(5-6)

Eliminating \( t \) from the above equations, gives
\[ t = (x_3 - \xi_3)/\Delta' \]  
(5-7)
\[ x_2 = x_2^2/2\Delta' - \xi_3^2/2\Delta' + \xi_2 \]  
(5-8)
\[ x_1 = (x_3^3 - \xi_3^3)/6K^2 + \xi_2(x_3 - \xi_3)/K\Delta - \xi_3^2x_3/2K^2 + \xi_1 \]  
(5-9)

In order to determine the time optimal switching surface, we consider the sets of states \( V^+_2 \) and \( V^-_2 \) from which the origin may be reached using the control \{+1\} and \{-1\}, respectively. This is a curve which is obtained by letting \( \xi_1 = \xi_2 = \xi_3 = 0 \) in (5-8) and (5-9), that is,
\[ V_2 = \{(x_1, x_2, x_3) : x_2 = x_2^2/2K\Delta, x_1 = x_3^3/6\}, \]  
(5-10)
and the control is given by \( u = -\text{sgn} \ x_2 \). Next the sets of states \( V^+_1 \) and \( V^-_1 \), from which the origin may be reached using the control \{+1,-1\} and \{-1,+1\} are considered. Obviously, if \( u = \{-1,+1\} \) is used, the phase trajectory must go to the origin along \( V^+_2 \) (and along \( V^-_2 \) when \( u = \{+1,-1\} \) is used). This surface is obtained by letting
(x_1, x_2, x_3) = (x_{12}, x_{22}, x_{32}) where x_{12}, x_{22}, x_{32} satisfy (5-10), and letting (\xi_1, \xi_2, \xi_3) = (x_1 x_2, x_3) in (5-8) and (5-9). In this way the surface \( V_1 \) is obtained which divides the state space into two parts and \( V_1 \) contains \( V_2 \). For those states above \( V_1 \), \( u = -1 \) is applied, and for those states below \( V_1 \), \( u = +1 \) is applied. Thus, all of the possible control sequences have been considered.

B. Control Surface for the Plant \( K/s^2(s+a), |u| \leq 1 \)

As pointed out in section 7-4 of reference 7, the plant \( K/s^2(s+a) \) may be used to represent many physical systems, in particular, a field controlled motor with constant armature current (c.f. Appendix A).

In order to avoid a complicated switching surface a transformation of variables is required. However, this transformation requires a previous knowledge of the system parameters \( K \) and \( a \). If these parameters could be determined from the first control period, the results outlined below, and worked out in complete detail in reference 7, could be used for extending this control scheme to a third order system.

Let

\begin{align*}
\dot{x}_1 &= x_1, \quad \dot{x}_2 = x_2, \quad \dot{x}_3 = \ddot{x} \\
\end{align*}

(5-11)

then

\begin{align*}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} &=
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & -a
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
K
\end{bmatrix} u.
\end{align*}

(5-11)

or

\begin{align*}
\dot{x} &= Ax + bu
\end{align*}

(5-12)

Using the transformation

\begin{align*}
x &= Py
\end{align*}

(5-13)

where

\begin{align*}
P &=
\begin{bmatrix}
1 & 0 & 1/a^2 \\
0 & 1 & -1/a \\
0 & 0 & 1
\end{bmatrix}
\end{align*}

(5-14)

(which may be obtained by considering the eigenvalues \( \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -a \), of the system matrix), gives

\begin{align*}
\dot{y} &= P^{-1}APy + P^{-1}bu
\end{align*}

(5-15)
or
\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & -a
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix} +
\begin{bmatrix}
-1/a^2 \\
1/a \\
1
\end{bmatrix} K u
\]
(5-16)

Next let
\[
z_1 = a^3 y_1/K, \quad z_2 = a^2 y_2/K, \quad z_3 = a y_3/K,
\]
(5-17)
so that the original equations become
\[
\begin{align*}
\dot{z}_1 &= az_2 - au \\
\dot{z}_2 &= au \\
\dot{z}_3 &= az_3 + au
\end{align*}
\]
(5-18)

The relationship between the original variables and the above is shown in Fig. 7-17 [7].

If the Hamiltonian, \( H \), is found and then minimized, and the costate variables, \( p_1, p_2, p_3 \), solved for (as in previous sections), the following equations are obtained:
\[
\begin{align*}
z_1 &= \xi_1 + \xi_2 at + \Delta a^2 t^2/2 - \Delta at \\
z_2 &= \xi_2 + \Delta at \\
z_3 &= (\xi_3 - \Delta)e^{-at} + \Delta
\end{align*}
\]
(5-19)

Eliminating time from these equations gives
\[
\begin{align*}
t &= (z_2 - \xi_2)/\Delta a \\
z_1 &= \xi_1 - (z_2 - \xi_2) + \Delta(z_2^2 - z_1^2)/2 \\
z_3 &= (\xi_3 - \Delta)e^{-\Delta(z_2 - \xi_2)} + \Delta
\end{align*}
\]
(5-20) (5-21) (5-22)

In these equations the \( \xi_i \) are the initial conditions on the \( z_i \). From (5-21) it is seen that the phase trajectories are independent of \( z_3 \) and \( \xi_3 \), and from (5-22) it is seen that the phase trajectories are independent of \( z_1 \) and \( \xi_1 \). Thus (5-21) represents the projection of the phase trajectories onto the \( z_1-z_2 \) plane and (5-22) represents the projection of the phase trajectories on the \( z_2-z_3 \) plane. The first set of trajectories are parabolas similar to those for the plant \( 1/s^2 \) (Fig. 2-6), and the second set of trajectories are similar to those for the plant \( 1/(s(s+1)) \) (Fig. 2-4). The two sets are illustrated in Fig. 7-18 and 7-19 of reference 7.
In order to obtain the switching surface for this system, the set of states from which the origin can be reached using the control \{+1\}, and \{-1\} are considered first. These states are obtained by setting \( z_1 = z_2 = z_3 = 0 \) in (5-21) and (5-22). This results in a curve \( V_2 \) in the state space, whose projections onto the \( z_1-z_2 \) and \( z_2-z_3 \) planes are as shown in Fig. 7-20 and 7-21, respectively, of reference 7. It should be noted that these curves are of the same shape as those considered in this thesis. The control used when the state is on this curve is given by

\[
u = \Delta^* = -\text{sgn} \ z_2. \tag{5-23}\]

Next the set of states \( V_1 \) from which the origin may be reached using the control \{-1,+1\} or \{+1,-1\} are considered. It is clear that when the control \{-\Delta^*,\Delta^*\} switches, \((z_1,z_2,z_3) \in V_2\). This surface is defined by (7-158) and (7-159), [7]. The control \{-\Delta^*,\Delta^*\} used is given by

\[
\Delta^* = \text{sgn}(z_1 + \frac{1}{2}\text{sgn} \ z_2 |z_2|^2 + z_2) \tag{5-24}\]

The projection of the trajectories which belong to \( V_2 \) onto the \( z_1-z_2 \) plane are shown in Fig. 7-22,[7]. The surface \( V_1 \) divides the state space into two parts, \( V_2 \) is contained in \( V_1 \), and \( V_1 \) is symmetric about the origin. If an arbitrary state \((z_1,z_2,z_3)\) is above this surface, then \( u=1 \) is applied to drive this state to \( V_2 \). If \((z_1,z_2,z_3)\) lies below this surface, then \( u=+1 \) is used to drive this state to the origin. Thus the control sequence \{-1,+1,-1\} drives a state which is above \( V_1 \) to the origin, and the control sequence \{+1,-1,+1\} drives a state which is below \( V_1 \) to the origin. This takes care of all the possible control sequences. Typically and would be as shown Fig. 5-1.

Methods for determining \( V_1 \) and \( V_2 \), and for calculating the systems parameters \( K \) and \( \alpha \) efficiently could be fruitful areas for future research.
C. Control Hypersurfaces for Plants With N Real Poles

For systems with the transfer function

\[ G(s) = \frac{K}{s(s+s_1)(s+s_2)\ldots(s+s_N)} \]  

(5-25)

where it is assumed that the \( s_i \) are real and that

\[ 0 < s_1 < s_2 < \ldots < s_N, \]

reference 7 details the method for obtaining the time optimal switching hypersurfaces. This method requires that the \( s_i \) and \( K \) be known exactly, so that the state equations for (5-25) could be written as

\[ \dot{x}_i = -s_ix_i + s_iu, \quad i = 1,\ldots,N. \]  

(5-26)

\[ |u| < 1. \]

A sequence of sets of points \( V_{N-1} \subseteq V_{N-2} \subseteq \ldots \subseteq V_1 \) are defined, and the determination of these sets is quite involved. Upon reading this section in reference 7, it will be clear that challenging problems could be considered for Nth order systems.

Fig. 5-1 Typical Switching Surface \( V_1 \) and Switch Curve \( V_2 \) for the Plant \( K/s(s+a) \).
VI. RESULTS AND DISCUSSION

A. Description of Computer Program

This section gives a physical description of the computer program and the reasoning for choosing the various routines. These routines depend on the physical system being controlled, which in this case is a d.c. armature controlled motor with an approximately pure inertia load (described in section II-A).

1. Determination of Initial Control and Input

Fig. 6-1 describes the routine used to test which control is required. The actual step numbers are the numbers to the left of Fig. 6-1.

![Flowchart of the control determination process]

Fig. 6-1 Test Which Control is Required.
Fig. 6-2 describes the input routine. Timing is used to ensure that the input routine begins at the same time relative to the circulating memory. Time is marked so that a meaningful calculation of \( x_2 \) is obtained. That is, \( x_2 \) is obtained from
\[
\frac{x_2 - x_1(t_1)}{\Delta},
\]
where \( \Delta = t_2 - t_1 \). If \( x_1 \) does not change significantly in the time interval \( \Delta \), then \( x_1(t_2) - x_1(t_1) \) could be an integer, like 0 or 1.

---

**Input routine**

9

a. For timing. Begins each input routine at the same time relative to the circulating memory.

b. Clear \( H_2 \) for new encoder reading

c. Take previous reading and store in \( H_1 \)

d. Set index register \( (I_1 = 60) \) and do 60 loops using the memory to mark time (=0.3 sec) while relays operate and \( x_2 \) changes noticeably. (used to vary sample time \( \Delta \))

e. Read encoder.
   calc \( x_1 \) and store in \( H_2 \)

f. Calculate \( x_2 = n \) and store in \( H_3 \)

---

Fig. 6-2 Input Routine
2. Relay Compensation and Compare Modes

The relay compensation is used to account for the "hysteresis" and "dead" times of the relays which are discussed in sections II-D and IV-C. The compensation used here is based on an average value of $\delta_d + \delta_h = T$ (section II-D), and the assumption that the plant is $K/s^2$. The correction used for the position $x_1$ is

$$x_1 = \xi_1 + \xi_2 T,$$

and no correction is made for $x_2$. The true correction is given by equations 2-34 and 2-35 (or equations 2-45 if the plant is of the form $K/s(s+a)$). However, the computer does not have sufficient capacity to implement these equations. To determine $x_1^5$, the value of $x_1$

---

36 Set address in I, \(x_1 = 0\) determines OSC

Switching Delay Compensation For $x_1$

37 a. Borrow $H_4$ temporarily, and place \(T(=22) \to H_4\)

b. Calculate $x_2 T$ and store in $H_4$

c. Calculate $x_1 - x_2 T$ and store in $S_{55}$

a. Determination of $x_1^5$

i. If $\Delta x_1 < 0$ take -1 x value of $x_1$ in memory.
   If $\Delta x_1 > 0$ take value of $x_1$ in memory.

ii. If $x_2 < 0$ take -1 x value of $x_1$ in memory modified by (i).
   If $x_2 > 0$ take value of $x_1$ in memory modified by (i).

b. Compare compensated $x_1$ with $x_1^5$,

$$x_1 + x_2 T - x_1^5 + \Delta x_1 = \delta x_1$$

Fig. 6-3 Relay Compensation and Compare Modes.
stored in the computer memory, the logic in steps a(i) and a(ii) is used. This is required because the memory's address, $x_2$, can only be a positive quantity.

In Fig. 6-4 the present state $(x_1,x_2)$ is checked to see if it is in region D (see section I-A) by checking if $|x_1|<M(=8)$ and $|x_2|<N(=8)$. If $(x_1,x_2) ∈ D$, the relays turn off the control. If $(x_1,x_2) ∉ D$, then the sign of $\delta x_1$ is checked. If $\delta x_1>0$ ($\delta x_1<0$) the control $u=-1$ ($u=+1$) is switched on. The computer is then placed in the store mode.

### Check if Control is to be Removed or Switch on OSC

a. Check if $(x_1,x_2) ∈ D$.

i. Is $|x_1|<M(=8)$?

ii. Is $|x_2|<N(=8)$?

If these are both true, then the relays switch to $u=0$, and return to location B. If they are not both true, then continue.

b. If $\delta x_1<0$, relays switch $u=+1$.
   If $\delta x_1>0$, relays switch $u=-1$.

### Set index register ($I_1=0$) to begin store mode.

### Restore value in $H_4$.

---

**Fig. 6-4 Switch on OSC or Turn-off Control**

3. **Store Mode**

Once the control changes sign, flag $I_1$ is set for store mode ($I_1=0$). Then $|x_2|$ is checked to see if it has crossed the $x_1$-axis. If it has not, the values of $x_2$ are put into address register ($I_2$) and the corresponding values of $x_1$ are stored in the memory. As soon as
$|x_2|<N(=8)$, the value of $x_1$, called $\Delta x_1$, is stored in memory as the offset for the switching curve.

**Fig. 6-5 Store Mode**

24

a. Is $|x_2|<N(=8)$?
   - If yes, go to compare mode, location D.
   - If no, go to store mode.

b. Store $x_2$ in the address register,
   - Store $x_1$ in the memory.

c. When $|x_2|<N(=8)$, $x_1 + \Delta x_1$
   - is the offset for the switch curve.

d. Set flag $I_1\#0$, to get out of store mode.

e. Continue.
B. **Control Computer Program**

Manually store the following

\[
M(=8) \rightarrow S_{58} \\
x_1 = n^0_{ref}(=512) \rightarrow S_{60} \\
T(=22) \rightarrow S_{56} \\
(-1) \rightarrow S_{54} \\
(3) \rightarrow S_{51}
\]

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<tr>
<td>2</td>
<td>14</td>
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<td>60</td>
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<tr>
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Control Computer Program (continued)

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Control Computer Program (continued)

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C. Discussion

The project was begun for several reasons: the computer was developed mainly for the purpose of studying control systems, and secondly, practical experience could be obtained. When a control scheme had been proposed, again practical experience could be obtained in implementing the theory. These objects have been achieved.

The computer is operational and does all that was expected of it. Also for study, a second order position control system was built. An actual trajectory for the system was not obtained because the program board could not accommodate the program. However, it is felt that if the program board were extended, one more order of magnitude (128 steps),
and the number of times certain orders (e.g., A-S, |A|→A, A→H, etc.) may be used is increased, results for any system of the form treated in this report, could be obtained.

When the program capacity has been increased, consideration must be given to the particular quadrant of the phase plane in which the control mode occurs. The program should compensate for the different delay times (see section IV-C-2) in the two quadrants.

Calculations show that the present program has a maximum sampling period of approximately 70 milliseconds. This is due mainly to the waiting time for the circulating memory. The present program could be refined to be a minimum-access program for shorter sampling periods. With no waiting time for the memory, the sampling time would be approximately 3 milliseconds.

If a more sophisticated controller were available some topics for future study could be: (a) Study, in more detail, the effect of the Hysteresis and Dead Time in the control relays as discussed in Sect. II-D. (b) Make a study of how fast the plants' parameters may vary for a given computer speed, (i.e., time to go through one cycle). (c) Make a study of how fast the system, with reference to the input, may vary with the computer's cycling time. (d) Using a digital computer with a large memory, the possibility of extending the control scheme to higher order systems could be considered. In the extension to higher order systems (e.g., K/s^2(s+a)) the controller must be able to calculate K and a, during the first cycle, in order to make transformations to avoid complicated switching surfaces (i.e., to uncouple the equations). In the second order system the control sequences are obvious. However, in a higher order system the control sequences are not as obvious. (e) In this study
position control of the motor is considered only. It is possible that the control scheme might be modified so that speed control or torque control be achieved. For example, if some material is being wound into a roll (paper, cloth, etc.,) it is important that \( v \) be held as closely constant as possible to avoid breakage in the material and to minimize imperfections of the material being processed. The radius of the roll is known as a function of time because the amount of material on the roll could be computed as a function of time. In this case we let

\[
x_2 = \dot{x} = n\{\delta(t)r(t)-V_0\}
\]

and

\[
x_1 = x = n\int_{t_0}^{t} \delta(t)r(t)dt - nV_0(t-t_0)
\]

Then the states \((x_1, x_2)\) are observed and they are required to drive this state to the origin in minimum time. Sometimes it may not be important that \(x_1\) be zero, then we observe \((x_1, x_2)\) and drive \(x_2\) to zero in minimum time.
APPENDIX A: Motor Transfer Function

Fig. A-1 shows a simplified schematic diagram of a d-c motor. Two transfer functions are possible because speed can be controlled by 1) keeping \( i_a \) constant and varying \( i_f \), and 2) keeping \( i_f \) constant and varying \( i_a \). Refer-

\[ e_a \] voltage applied to armature, volts  
\[ i_a \] armature current, amperes  
\[ e_f \] voltage applied to field, volts  
\[ i_f \] field current, amperes  
\[ \Phi \] air-gap flux, webers  
\[ T_M \] torque developed at shaft by e.m.f., newton-meters  
\[ T_L \] load torque, newton-meters  
\[ J_M \] moment of inertia of rotor, kilogram-meter\(^2\)  
\[ B_M \] viscous friction coefficient, newton-meters per radian per second  
\[ \theta \] angular velocity of rotor, radians per second

Fig. A-1  Schematic diagram for a d-c motor

ing to figure A-1, and considering the second case, for a linear magnetic circuit,

\[ \Phi = K_1 i_f \]  

(A-1)

where \( K_1 \) is the constant of proportionality in webers per ampere. Torque developed on the shaft is

\[ T_M = K_2 \Phi i_a \]  

(A-2)

where \( K_2 \) has the dimensions newton-meters per weber-ampere. Only viscous friction exists on the rotor. If this coefficient of friction is \( D \), and, if \( i_f \) is constant, then

\[ T_M = K_{av} i_a \]  

(A-3)

where \( K_{av} = K_1 K_2 i_f \) newton-meters per ampere.  

(A-4)
Writing the voltage loop equations for the armature circuit,

\[ e_a = R_a i_a + L_a \frac{di_a}{dt} + K_4 \dot{\theta} \]  \hspace{1cm} (A-5)

where \( R_a \) is the armature resistance, \( L_a \) is the armature inductance, \( K_4 \dot{\theta} \) is the back e.m.f. induced in the armature circuit. \( K_4 \) has dimensions, volts per radian per second.

Assuming \( L_a \) is small, we may neglect it and write Eqn. (A-5) as

\[ e_a = R_a i_a + K_4 \dot{\theta} \]  \hspace{1cm} (A-6)

Substituting eqn. (A-3) into eqn. (A-6),

\[ e_a = K_T T_M + K_4 \dot{\theta} \]  \hspace{1cm} (A-7)

where \( K_T = \frac{R_a}{K_{av}} \) volts per newton-meter.

The equation which describes the dynamic behavior of the rotor may be expressed by the differential equation

\[ T_M = J_M \frac{d\dot{\theta}}{dt} + B \dot{\theta} + T_L \]  \hspace{1cm} (A-8)

If \( T_L = 0 \), and if we combine eqns. (A-7) and (A-8) to eliminate \( T_M \), then

\[ e_a = K_T \left( J_M \frac{d\dot{\theta}}{dt} + B_M \dot{\theta} \right) + K_4 \dot{\theta} \]  \hspace{1cm} (A-9)

Writing the Laplace transform of the transfer function and assuming zero initial velocity,

\[ \frac{\delta(s)}{E_a(s)} = \frac{\delta(s)}{u(s)A} = \frac{K_T}{J_M s^2 + B_M + K_4 K_T} \]  \hspace{1cm} (A-10)

where \( K_T = \frac{1}{K_T} \) newton-meters (called the torque constant of the motor),\( A \) is the gain of the relays and \( u \) is the control function, \( u(t) = \pm 1 \)

Finally,

\[ \frac{\delta(s)}{Au(s)} = \frac{K_T/J_M}{s + \frac{1}{J_M} \left( B_M + K_4 K_T \right)} \]  \hspace{1cm} (A-11)

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### APPENDIX B: Control Computer Functions

<table>
<thead>
<tr>
<th>Funct No.</th>
<th>Instr.</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>H → A</td>
<td>Place contents of A-reg. into H-reg. and clear A-reg.</td>
</tr>
<tr>
<td>1</td>
<td>A → H</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>A + H</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>A - H</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>A × H</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>A ÷ H</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>spare</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>S → H</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>H → S</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>S → A</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>A → S</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>A + S &amp; CL</td>
<td>Place contents of A-reg. in memory then clear A-reg.</td>
</tr>
<tr>
<td>13</td>
<td>A + S</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>A - S</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>spare</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>Right sh A</td>
<td>Right shift contents of a A-reg. one position to the right.</td>
</tr>
<tr>
<td>17</td>
<td>Left sh A</td>
<td>Left shift contents of A-reg. one position to the left.</td>
</tr>
<tr>
<td>18</td>
<td>Jump (A&lt;0)</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>Jump (A≠0)</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>N → I</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>I + N → I</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>I - N → I</td>
<td></td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>Funct No.</th>
<th>Instr.</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>H → I</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>Test Mod</td>
<td>Jump if I ≠ 0</td>
</tr>
<tr>
<td>25</td>
<td>spare</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>Input</td>
<td>Place Encoder reading into A-reg.</td>
</tr>
<tr>
<td>27</td>
<td>Output</td>
<td>Test Contents of A-reg. and apply proper control.</td>
</tr>
<tr>
<td>28</td>
<td>spare</td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>U Jump</td>
<td>Unconditional jump.</td>
</tr>
<tr>
<td>30</td>
<td>Halt</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>spare</td>
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</tr>
</tbody>
</table>
REFERENCES


