1-1-1967

The response of a uniformly wound coil to a step voltage.

Brij Krishen Bhat

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THE RESPONSE OF A UNIFORMLY WOUND
COIL TO A STEP VOLTAGE

by

BRIJ KRISHEN BHAT

A Thesis

Submitted to the Faculty of Graduate Studies through the
Department of Electrical Engineering in Partial Fulfilment
of the requirements of the degree of
Master of Applied Science at the
University of Windsor
Windsor, Ontario
1967
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ABSTRACT

The response of a uniformly wound coil with constant inductance and capacitance parameters to a step voltage has been considered from the point of view of stresses on major and minor insulation. The effect of mutual inductance between the various turns of the coil has been taken into consideration. Expressions for voltage and voltage gradient at any section have been developed for grounded-neutral and isolated-neutral types of coils.
ACKNOWLEDGEMENTS

The author wishes to thank Dr. P.A.V. Thomas, Head of the Department of Electrical Engineering for the opportunity afforded to the author to work at the University of Windsor. Thanks are also expressed to Dr. A.H. Qureshi of the Department of Electrical Engineering for his guidance and to N.R.C. of Canada for the financial assistance.

The author also thanks Mrs. S. A. Ouellette for the typing of this manuscript.
LIST OF SYMBOLS

L
Axial length of the winding.
e
Instantaneous value of voltage at any point.
E
Peak value of the voltage surge.
a, b, A, B
Constants
K
Inter-turn capacitance per unit axial length of the winding.
C
Capacitance to ground per unit axial length of the winding.
α
\( \sqrt{C/K} \)
x, y, z
Variable points on the winding.
μ
Mutual inductance between two unit-length coils distant
x and y from the line end.
λ
Self inductance per unit length of the winding.
i
Current through the winding at any point.
i_k
Current through inter-turn capacitance at any point.
i_c
Current through ground capacitance at any point.
I
Total current at any point.
H
Magnetic Potential at any point Z.
R
Reluctance of Magnetic Core per unit length.
1/G
Reluctance of surrounding medium per unit length.
ϕ
Magnetic flux crossing the Core Section at Z.
ϕ_o
Total flux set by the current in the turn at x.
g_o
Axial voltage gradient at any point at t = 0.
g_o(0)
Value of g_o at x = 0.
g_o(L)
Value of g_o at x = L.

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<tr>
<td>$g_{\text{max}}$</td>
<td>Maximum stress.</td>
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<td>$e_0$</td>
<td>Initial voltage distribution.</td>
</tr>
<tr>
<td>$e_f$</td>
<td>Final voltage distribution.</td>
</tr>
<tr>
<td>$g_f$</td>
<td>Final axial voltage gradient at any point.</td>
</tr>
<tr>
<td>$g_{fo}$</td>
<td>Final axial voltage gradient at $x = o$.</td>
</tr>
<tr>
<td>$g_{ff}$</td>
<td>Final axial voltage gradient at $x = \ell$.</td>
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<tr>
<td>$C_{\text{eff}}$</td>
<td>Initial effective capacitance.</td>
</tr>
<tr>
<td>$V$</td>
<td>Velocity of propagation of a harmonic.</td>
</tr>
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<td>$e_n$</td>
<td>Oscillatory component of the voltage at any point.</td>
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INTRODUCTION

In the design of large electric machines, economic utilization of insulating materials is one of the prime concerns of the designer. The problem becomes more important for high voltage machines for which the cost of insulation is a considerable portion of the total cost of the machine. The insulation must withstand the voltage stresses under normal working conditions of the equipment and also under abnormal conditions, e.g., under a lightning stroke or a switching surge. A lot of experimental data about the distribution of voltage along a uniformly wound winding is available. Analytical investigations are too unreliable because of the effect of stray capacitance and mutual inductance between various turns of the winding. If the effect of mutual inductance is ignored, the experimental results do not agree with theoretical results within experimental errors.

An attempt has been made in the present work to include approximately the effect of mutual inductance and expressions for voltage gradient and voltage at any point along the winding have been developed. It is shown that higher harmonics in the surge are relatively less important for calculating major insulation but for calculating minor insulation, the voltage stress due to higher harmonics is also important.

It is known that on the basis of neglecting the effect of mutual inductance between various turns of the winding, there is a definite value of frequency, called critical frequency, above which the winding behaves as an open circuit and all harmonics above the critical harmonic are reflected by the winding. The reflection gives rise to over-voltage at (viii)
the line end. In fact, one of the suggestions made by R. Rudenburg\textsuperscript{(1)} for improving the performance of the coil is to increase its critical frequency.

Inclusion of the effect of mutual inductance shows that there is no frequency at which the coil behaves as an open circuit and all frequencies penetrate the winding and that the unevenness of voltage distribution is caused only by capacitive and inductive coupling of the various turns and by ground capacitance.
CHAPTER I

1.1 Wave Shape of the Surge

Analysis of the response of a uniformly wound finite winding is the first step towards the study of surge voltage distribution on actual windings - those of reactors, transformers and rotating machines. The wave shape of the surge that arises out of a lightning stroke to or near a transmission line or a substation is a very uncertain factor. The surge has usually a very steep front and a long tail. Most lightning waves have been found to be approximately representable by the difference of two exponentials -

$$e = A (e^{-at} - e^{-bt}), \ b >> a$$

where $A$, $a$ and $b$ are constants. These constants vary for different waves and depend upon polarity of the surge, front time and tail time.

The time that the surge takes to attain maximum value is called wave front time and the time that the surge takes to attain half the peak value after the peak is called wave tail time. Both times are measured from the origin and are measured in microseconds. Fig. 1 shows a typical lightning wave and the method of specifying it.

The steeper the wave front the more uneven the initial voltage distribution and the longer the tail, the greater are the oscillations subsequent to initial distribution and before steady conditions. Since both wave front and wave tail are uncertain parameters analysis has been made for the response of the coil to a step voltage which is the most severe voltage - having steepest front and longest tail, Fig. 2.
At normal power frequency operation, the voltage distribution in a coil from line end to the other end is solely determined by the common flux and is uniformly distributed which requires uniform axial or minor insulation (turn insulation) and a uniformly graded transverse or major insulation (turn to core insulation). This presents no difficulties to a designer. But when a surge enters a winding the voltage distribution is no longer uniform. More voltage concentrates on a few initial turns or the line end where excessive axial stress is developed. The unevenness of voltage distribution increases with the steepness of wave. Moreover, after initial uneven distribution oscillations are set up in the coil which are sustained by the tail of the wave. The energy of the surge is partly dissipated as heat in the resistance of the winding, as dielectric loss in the insulation and also as magnetic loss in the core. The final steady state conditions are reached when the transient oscillations completely die down due to these losses. In the case of grounded-neutral winding, some energy of the surge passes to ground through grounding impedance. The exchange of energy between inductance and capacitance is responsible for the setting up of oscillations. The knowledge of maximum voltage and maximum voltage gradient is necessary for the efficient design of insulation. The maximum voltage is necessary for calculating major insulation and maximum voltage gradient is used for calculating minor insulation. The zero-front, infinite-tail surge assumed for the purpose of analysis corresponds to \( a = 0 \) and \( b = \infty \) in the expression

\[
e(t) = A \left( e^{-at} - e^{-bt} \right)
\]

1.2 Equivalent Circuit

To represent a winding by an equivalent circuit presents the following difficulties.
1. The capacitance between two adjacent turns can be easily accounted for but the capacitance of one turn with all other turns particularly the end turns presents great difficulty and is completely neglected in the present work.

2. As has already been indicated, the energy of the surge is partly dissipated as dielectric loss, magnetic loss and ohmic loss. Individual representation of these losses in the equivalent circuit has not been attempted but their effect is included by allowing a general damping factor for the oscillations.

3. During the infinitely short wave-front of the surge the flux in the Core can not penetrate any considerable depth and the core may offer greater reluctance to the establishment of Common flux than the non magnetic surrounding. Fig. 3 shows the assumed equivalent circuit of the winding in which:

\[ K = \text{Inter-turn capacitance per unit axial length of the winding} \]
\[ C = \text{Capacitance to ground (Core, tank) per unit axial length of the winding} \]
\[ x, y = \text{Two points distant } x \text{ and } y \text{ from the line end, } x = 0. \]
\[ \mu = \text{Mutual inductance between two unit-length-coils distant } x \text{ and } y \text{ respectively from the line end.} \]
\[ \lambda = \text{Self inductance per unit length of the winding} \]
\[ i = \text{Current through the winding at any point} \]
\[ i_k = \text{Current through inter-turn capacitance at any point} \]
\[ i_c = \text{Current through ground capacitance} \]
\[ I = \text{Total current at } x = i + i_k \]

In the above definitions of various symbols:

\[ K = \text{Constant} \]
\[ C = \text{Constant} \]
\[
e = e(x,t) \\
\lambda = \text{Constant} \\
\mu = \mu(x,y) \\
i = i(x,t) \\
i_k = i_k(x,t) \\
I = I(x,t)
\]

1.3 Mutual Inductance Function

Figure 4 shows a magnetic core of length \( l \) on which are wound two separate turns at distances \( x \) and \( y \) respectively from one end. Let the turn at \( x \) carry a current to set up flux in the core. The second turn carries no current.

Let:

\[ H = \text{M.M.F. at any point } Z. \]
\[ R = \text{Reluctance of magnetic core per unit length.} \]
\[ \frac{1}{G} = \text{Reluctance of surrounding medium per unit length.} \]
\[ \phi = \text{Magnetic flux crossing the section at } Z. \]
\[ \phi_o = \text{Total flux set up by the current in the turn at } X. \text{ This flux crosses the core section at } x. \]

The following equations can be written for magnetic flux and potential:

\[
H + \frac{dH}{dZ} dZ - H = - \phi R dZ
\]

or,
\[
\frac{dH}{dZ} = - \phi R \quad (1.1)
\]

and
\[
\frac{d\phi}{dZ} dZ = - H G dZ
\]

or,
\[
\frac{d\phi}{dZ} = - H G \quad (1.2)
\]
Eliminating $H$ from equations 1.1 and 1.2 we get

$$\frac{d^2 \phi}{d z^2} - R G \phi = 0$$

This equation has a solution

$$\phi = A e^{\sqrt{R G} z} + B e^{-\sqrt{R G} z} \quad (1.3)$$

Where $A$ and $B$ are constants of integration.

To calculate the values of constants, the following boundary conditions are plugged in:

$$\phi = \phi_0 \text{ at } z = x$$

$$\phi = 0 \text{ at } z = \ell$$

giving

$$A = -\frac{\phi_0 - \sqrt{R G} \ell}{e^{\sqrt{R G} (\ell - x)} - e^{-\sqrt{R G} (\ell - x)}}$$

$$B = \frac{\phi_0 e^{\sqrt{R G} \ell}}{e^{\sqrt{R G} (\ell - x)} - e^{-\sqrt{R G} (\ell - x)}}$$

which when substituted in equation 1.3, gives

$$\phi = \phi_0 \left[ \frac{\sqrt{R G} (\ell - z) - e^{-\sqrt{R G} (\ell - z)}}{e^{\sqrt{R G} (\ell - x)} - e^{-\sqrt{R G} (\ell - x)}} \right] \quad (1.4)$$

at $Z = y$

$$\phi_y = \phi_0 \left[ \frac{\sqrt{R G} (\ell - y) - e^{-\sqrt{R G} (\ell - y)}}{e^{\sqrt{R G} (\ell - x)} - e^{-\sqrt{R G} (\ell - x)}} \right]$$

For $x$ and $y$ not close to the ends of the core

$$e^{\sqrt{R G} (\ell - y)} \gg e^{-\sqrt{R G} (\ell - y)}$$

and

$$e^{\sqrt{R G} (\ell - x)} \gg e^{-\sqrt{R G} (\ell - x)}$$
Therefore, neglecting $e^{-\sqrt{RG} (l - y)}$ and $e^{-\sqrt{RG} (l - x)}$
we get:

$$\phi_y = \phi_0 e^{-\sqrt{RG} (y - x)} \quad \text{for } y > x$$

$$\phi_y = \phi_0 e^{-\sqrt{RG} (x - y)} \quad \text{for } y < x$$

because maximum flux crosses the core at $x$. The mutual inductance between
turns at $x$ and $y$ can, therefore, be written as

$$\mu = \lambda e^{-m |x - y|} \quad (1.5)$$

where $m = \sqrt{RG}$ and $\lambda$ is a constant. All turns are supposed to be wound
in the same direction making $\mu$ always positive. When $x = y$, we have
$\mu = \lambda$, the self inductance per unit length of the coil.

Variation of $\mu$ with $y$ is shown in Fig. 5.
CHAPTER 2

2.1 Fundamental Relations

Referring to the equivalent circuit of Fig. 3

Let \( Q_K = \text{charge on the capacitor } K/dx \)

\[
\frac{d}{dx} \left[ e - \frac{3e}{2} \frac{dx}{dx} - e \right] = -K \frac{3e}{3x3t}
\]

\[
i_k = \frac{\partial Q_k}{\partial t} = -K \frac{\partial^2 e}{\partial x^2 \partial t}
\]  \hspace{1cm} (2.1)

Similarly

\[
I + \frac{\partial I}{\partial x} dx - I = -C \frac{\partial e}{\partial t}
\]

or,

\[
\frac{\partial I}{\partial x} = -C \frac{\partial e}{\partial t}
\] \hspace{1cm} (2.2)

\[
I = i + i_k
\] \hspace{1cm} (2.3)

\[
i = i - K \frac{\partial^2 e}{\partial x^2 \partial t}
\] \hspace{1cm} (2.4)

Differentiating (2.4) twice, once w.r.t. \( x \) and once w.r.t. \( t \) we get:

\[
\frac{\partial^2 I}{\partial x^2 \partial t} = \frac{\partial^2 I}{\partial x^2 \partial t} - K \frac{\partial^4 e}{\partial x^2 \partial t^2}
\] \hspace{1cm} (2.5)

Also differentiating (2.2) w.r.t. \( t \)

\[
\frac{\partial^2 I}{\partial x^2 \partial t} = -C \frac{\partial^2 e}{\partial t^2}
\] \hspace{1cm} (2.6)

2.2 Rate of Change of Voltage Along the Winding

The potential difference between points P and Q (Fig.3) is caused by the inductive effects of current in the winding. Then,

\[
e - (e + \frac{3e}{2} \frac{dx}{dx}) = \int_0^2 \frac{\partial I}{\partial t} dy \, dx, \quad i = i(y, t)
\]
\[- \frac{\partial e}{\partial x} = \int_{0}^{x} \lambda e^{-m(x-y)} \frac{\partial i}{\partial t} \, dy + \int_{y}^{x} \lambda e^{-m(y-x)} \frac{\partial i}{\partial t} \, dy \]

\[= \lambda e^{-mx} \int_{0}^{x} \frac{\partial i}{\partial t} e^{-my} \, dy + \lambda e^{-mx} \int_{y}^{x} \frac{\partial i}{\partial t} e^{-my} \, dy \quad (2.7)\]

To get a relation between \(e\) and \(i\) we make the following substitutions:

\[\phi(x, t) = \int \frac{\partial i}{\partial t} e^{-mx} \, dx \quad (2.8)\]

\[\psi(x, t) = \int \frac{\partial i}{\partial t} e^{mx} \, dx \quad (2.9)\]

Differentiating (2.8) and (2.9) twice w.r.t. \(x\)

\[\phi'(x, t) = \frac{\partial i}{\partial t} e^{-mx} \quad (2.10)\]

\[\phi''(x, t) = -m \frac{\partial i}{\partial t} e^{-mx} + e^{-mx} \frac{\partial^2 i}{\partial x \partial t} \quad (2.11)\]

and

\[\psi'(x, t) = \frac{\partial i}{\partial t} e^{mx} \quad (2.12)\]

\[\psi''(x, t) = m \frac{\partial i}{\partial t} e^{mx} + e^{mx} \frac{\partial^2 i}{\partial x \partial t} \quad (2.13)\]

With substitutions (2.8) and (2.9), equation (2.7) becomes

\[- \frac{\partial e}{\partial x} = \lambda e^{mx} \left[ \phi(y, t) \right]_{x}^{x} + \lambda e^{-mx} \left[ \psi(y, t) \right]_{0}^{x} \]

or,

\[- \frac{1}{\lambda} \frac{\partial e}{\partial x} = e^{mx} \left[ \phi(x, t) - \phi(x, t) \right] + e^{-mx} \left[ \psi(x, t) - \psi(x, t) \right] \quad (2.14)\]

\[- \frac{1}{\lambda} \frac{\partial^2 e}{\partial x^2} = m \left[ \phi(x, t) - \phi(x, t) \right] - \phi'(x, t) e^{mx} \]

\[- m \left[ \psi(x, t) - \psi(x, t) \right] + \psi'(x, t) e^{-mx} \quad (2.15)\]

Differentiating (2.15) w.r.t. \(x\) we get
\[-\frac{1}{\lambda}\frac{\partial^3 e}{\partial x^3} = m^2 e \max \left[ \phi(l, t) - \phi(x, t) \right] - m e \phi'(x, t) - m e \phi''(x, t) + m^2 e \max \left[ \psi(t) - \psi(o, t) \right] \]

\[-m e \phi'(x, t) - e \phi''(x, t) + m^2 e \max \left[ \psi(t) - \psi(o, t) \right] \]

\[-m e \psi'(x, t) - m e \psi'(x, t) + -m e \psi''(x, t) \quad (2.16) \]

Re-arranging equations (2.14), (2.15) and (2.16)

\[\phi(l, t) e - \psi(o, t) e = -\frac{1}{\lambda}\frac{\partial e}{\partial x} + m e \phi(x, t) - e \psi(x, t) \quad (2.17)\]

\[m \max \left[ \phi(l, t) e + \psi(o, t) e \right] = -\frac{1}{\lambda}\frac{\partial^2 e}{\partial x^2} + m e \phi(x, t) + e \phi'(x, t) + m e \psi(x, t) - m e \psi'(x, t) \]

\[m^2 \max \left[ \phi(l, t) e - \psi(o, t) e \right] = -\frac{1}{\lambda}\frac{\partial^3 e}{\partial x^3} + m^2 e \phi(x, t) + 2 m e \phi'(x, t) + e \phi''(x, t) - m^2 e \psi(x, t) + 2 m e \psi'(x, t) \]

\[-e \psi''(x, t) \quad (2.19)\]

To eliminate \(\phi(l, t)\) and \(\psi(o, t)\) we multiply (2.17) by \(m^2\) and equate to (2.19), giving:

\[-\frac{m^2}{\lambda}\frac{\partial e}{\partial x} + m^2 e \phi(x, t) - m^2 e \psi(x, t) = \frac{1}{\lambda}\frac{\partial^3 e}{\partial x^3} + m^2 e \phi(x, t) + 2 m e \phi'(x, t) + e \phi''(x, t) - m^2 e \psi(x, t) + 2 m e \psi'(x, t) \]

or,

\[-\frac{1}{\lambda}\frac{\partial^3 e}{\partial x^3} + \frac{m^2}{\lambda}\frac{\partial e}{\partial x} + 2 m e \phi'(x, t) + 2 m e \psi'(x, t) \]

\[e \phi''(x, t) - e \psi''(x, t) = 0. \]
Substituting for \( \phi'(x,t) \), \( \phi''(x,t) \), \( \psi'(x,t) \) and \( \psi''(x,t) \) from equations (2.10), (2.11), (2.12) and (2.13) respectively we get:

\[
- \frac{1}{\lambda} \frac{\partial^3 e}{\partial x^3} + \frac{m^2}{\lambda} \frac{\partial e}{\partial x} + 2m e \frac{\partial^2 i}{\partial t \partial x} - \frac{e}{\lambda} \frac{\partial^3 e}{\partial x^3} + e \left[ -m \frac{\partial^2 i}{\partial t^2} + e \frac{\partial^2 e}{\partial x^2} \right] = 0
\]

or,

\[
- \frac{1}{\lambda} \frac{\partial^3 e}{\partial x^3} + \frac{m^2}{\lambda} \frac{\partial e}{\partial x} + 2m \frac{\partial^2 i}{\partial x \partial t} = 0
\]  

(2.20)

Relations expressed by equations (2.5), (2.6) and (2.20) are fundamental relations from which the differential equation governing voltage distribution is developed.

2.3 The Differential Equation

Differentiating (2.20) w.r.t. \( x \) we get:

\[
- \frac{1}{\lambda} \frac{\partial^4 e}{\partial x^4} + \frac{m^2}{\lambda} \frac{\partial^2 e}{\partial x^2} + 2m \frac{\partial^2 i}{\partial x \partial t} = 0
\]

using relations (2.5) and (2.6) we get

\[
- \frac{1}{\lambda} \frac{\partial^4 e}{\partial x^4} + \frac{m^2}{\lambda} \frac{\partial^2 e}{\partial x^2} + 2m \left[ \frac{\partial^2 I}{\partial x \partial t} + K \frac{\partial e}{\partial x^2 \partial t^2} \right] = 0
\]

or,

\[
- \frac{1}{\lambda} \frac{\partial^4 e}{\partial x^4} + \frac{m^2}{\lambda} \frac{\partial^2 e}{\partial x^2} + 2m \left[ -C \frac{\partial^2 e}{\partial t^2} + K \frac{\partial^4 e}{\partial x^2 \partial t^2} \right] = 0
\]

i.e.

\[
- \frac{1}{\lambda} \frac{\partial^4 e}{\partial x^4} + 2mK \frac{\partial^4 e}{\partial x^2 \partial t^2} + \frac{m^2}{\lambda} \frac{\partial^2 e}{\partial x^2} - 2mC \frac{\partial^2 e}{\partial t^2} = 0.
\]  

(2.21)

This is the basic equation governing voltage distribution along the winding.
Bewley (2) developed a relation for voltage distribution by ignoring the effect of mutual inductance. His expression for axial rate of change of voltage, i.e.

\[ \frac{\partial e}{\partial x} = -L \frac{\partial i}{\partial t} \]

where \( L \) is inductance per unit length, can be obtained from (2.20) by letting \( \lambda \) and \( m \) approach infinity and keeping the ratio \( \frac{\lambda}{m} \) constant.

In the relation

\[ \frac{m^2}{\lambda} \frac{\partial e}{\partial x} + 2m \frac{\partial i}{\partial t} = 0. \]

If we divide by \( m \) and let \( \frac{2\lambda}{m} = L \), we get

\[ \frac{\partial e}{\partial x} = -L \frac{\partial i}{\partial t} \]

The differential equation (2.21) takes the form

\[ K \frac{\partial^4 e}{\partial x^2 \partial t^2} + \frac{1}{L} \frac{\partial^2 e}{\partial x^2} - c \frac{\partial^2 e}{\partial t^2} = 0 \]  

(2.22)
CHAPTER 3

INITIAL AND FINAL VOLTAGE DISTRIBUTION

AND VOLTAGE GRADIENT

3.1 Grounded Neutral

Since the equivalent circuit of the winding is capacitively continuous, it is obvious that initial voltage distribution should be determined by capacitances only. Also because the capacitance distribution assumed in Fig. 3 is the same as assumed by Bewley (2), identical expressions for initial voltage distribution and gradient are expected.

If we write equations (2.21) and (2.22) in the operational form, we have

\[ -\frac{1}{\lambda} \frac{d^4 e}{dx^4} + 2mK p^2 \frac{d^2 e}{dx^2} + \frac{m^2}{\lambda} \frac{d^2 e}{dx^2} - 2mC p^2 e = 0 \]

and

\[ K p^2 \frac{d^2 e}{dx^2} + \frac{1}{L} \frac{d^2 e}{dx^2} - C p^2 e = 0, \text{ where } p = \frac{9}{8t} \]

Both these relations reduce to

\[ K \frac{d^2 e}{dx^2} - C e = 0. \quad (3.0) \]

when divided by \( p^2 \) and letting \( p \to \infty \)

The solution to this equation is:

\[ e_o = A e^{\alpha x} + B e^{-\alpha x} \quad (3.1) \]

where \( \alpha = \sqrt{\frac{C}{K}} \) and A and B are constants of integration and \( e_o \) is the voltage distribution at \( t = 0 \).

The constants A and B depend upon terminal conditions. For grounded neutral (Fig. 6)
$e_0 = 0$ at $x = l$
$e_0 = E$ at $x = o$

Substituting these values in (3.1) we get

$$o = A e^{\alpha l} + B e^{-\alpha l}$$

and

$$E = A + B$$

Hence,

$$A = \frac{E e^{-\alpha l}}{e^{\alpha l} - e^{-\alpha l}}$$

and

$$B = \frac{E e^{\alpha l}}{e^{\alpha l} - e^{-\alpha l}}$$

or,

$$e_0 = \frac{E \sinh \alpha (l-x)}{\sinh (\alpha l)} \quad (3.2)$$

3.2 Dependence of $e_0$ on $\alpha$

To see how, for a particular value of $x$, $e_0$ depends on $\alpha$ we differentiate (3.2) w.r.t. $\alpha$ giving

$$\frac{de_0}{d\alpha} = E \left[ \frac{(l-x) \cosh \alpha (l-x) \sinh (\alpha l) - l \cosh (\alpha l) \sinh \alpha (l-x)}{\sinh^2 (\alpha l)} \right] \quad (3.3)$$

which vanishes for $x = o$ and $x = l$, showing that for different values of $\alpha$, $e_0$ has constant values for $x = o$ and $x = l$, which is to be expected because the potentials at these points are those of the surge and ground respectively. In other words, all curves of $e_0$ for various values of $\alpha$ start and end at $x = o$ and $x = l$ respectively. For values of $x$ in the range $0 < x < l$ the sign of $\frac{de_0}{d\alpha}$ is important and is now investigated.

The numerator of the R.H.S. of (3.3) without the constant term $E$

simplifies to $l \sinh (ax) - x \cosh \alpha (l-x) \sinh (\alpha l)$. Since the slope of $\sinh(x)$ is everywhere greater than Unity for $x < c$, we have
\[
\frac{\sinh (ax)}{x} > \frac{a}{x} \quad \text{for } 0 < x < \ell
\]

or,
\[
\ell \sinh (ax) < x \sinh (a\ell)
\]
\[
\ell \sinh (ax) - x \cosh a (\ell - x) \sinh (a\ell)
\]
\[
< x \sinh (a\ell) - x \cosh a (\ell - x) \sinh (a\ell)
\]
\[
= x \sinh (a\ell) \left[ 1 - \cosh a (\ell - x) \right]
\]

The term within the brackets is always negative for \(0 < x < \ell\).

Hence, increasing the value of \(a\) brings the curve for \(e_o\) more and more below the curve corresponding to \(a = 0\). Fig. 6 shows the dependence of \(e_o\) on \(a\).

The major insulation has to withstand a voltage \(e_o\) whereas the minor insulation has to withstand a gradient \(\frac{de_o}{dx}\).

If the voltage were uniformly distributed in the winding, the voltage distribution would be given by the line \(E\ell\) Fig. 6 and axial stress everywhere would be \(E/\ell\). This corresponds to the value of \(a = 0\). Due to the presence of ground capacitance, therefore, the voltage is unevenly distributed. Part of the winding is overstressed and a part understressed if the winding is insulated uniformly. The axial stress at any point is given by the slope of \(e_o\) at that point. Let \(P\) be the point where the tangent \(TT\) is parallel to the line \(E\ell\) (Fig. 7). To the left of \(P\) the slope is greater than that of this line whereas to the right of \(P\) the slope is less. The portion \(OA\) of the winding is, therefore, overstressed and the portion \(\ell\) understressed. This conclusion can also be drawn by plotting the gradient \(g_o = -\frac{de_o}{dx}\) as in Fig. 8.
From 3.2

\[ \frac{de_0}{dx} = \frac{E \alpha \cosh \alpha (l-x)}{\sinh (\alpha \ell)} = g_0 \tag{3.4} \]

\[ g_{oo} = -\frac{de_0}{dx} \text{ at } x = 0 \]

\[ = E \alpha \coth (\alpha \ell) \]

and

\[ g_{\ell\ell} = -\frac{de_0}{dx} \text{ at } x = \ell \]

\[ = \frac{E\alpha}{\sinh (\alpha \ell)} \]

OA is then given by the value of x

Satisfying the relation

\[ \frac{E \alpha \cosh \alpha (l-x)}{\sinh (\alpha \ell)} = \frac{E}{\ell} \]

or,  

\[ OA = \ell - \frac{1}{\alpha} \cosh^{-1} \left( \frac{\sinh (\alpha \ell)}{\alpha \ell} \right) \]

3.3 Initial Voltage Gradient

Initial voltage gradient = \[ \frac{de_0}{dx} \]

\[ = \frac{E \alpha \cosh \alpha (l-x)}{\sinh (\alpha \ell)} \]

\[ - \frac{d^2e_0}{dx d\alpha} = \frac{E}{\sinh^2 (\alpha \ell)} \left[ \sinh (\alpha \ell) \{ \cosh \alpha (l-x) + \alpha(l-x) \sinh \alpha(l-x) \} - \alpha \ell \cosh (\alpha \ell) \cosh \alpha (l-x) \right] \]

By the reason similar to that used in proving the dependence of \( e_0 \) on \( \alpha \) it is shown that the above expression is always positive. Hence, initial voltage gradient increases with increase in \( \alpha \).

In other words, larger the value of \( \alpha \) the more convex upwards the curve of \( e_0 \) versus \( x \).
From (3.4) maximum gradient occurs at \( x = 0 \) and is given by,

\[ g_{\text{max}} = \frac{E \alpha \cosh (\alpha \ell)}{\sinh (\alpha \ell)} = g_{\infty}. \]

If the voltage were uniformly distributed, the stress everywhere would be

\[ \lim_{\alpha \to 0} \frac{E \alpha \cosh (\alpha (L-x))}{\sinh (\alpha \ell)} = E/L \]

The ratio of \( g_{\text{max}} \) to this value is

\[ \frac{g_{\text{max}}}{E/L} = \frac{\alpha \cosh (\alpha \ell)}{L \sinh (\alpha \ell)}. \]

For a coil of axial length of one meter and \( \alpha = 15 \), this ratio would be about 15.

### 3.4 Final Voltage Distribution

For a rectangular wave of infinite duration, the winding will attain a steady state voltage distribution given by relations (2.21) and (2.22) when \( p = \frac{\partial}{\partial t} \) approaches zero. Relation (2.22) corresponds to neglecting mutual inductance and (2.21) takes mutual inductance into account. Equation (2.22) reduces to:

\[ \frac{d^2e}{dx^2} = 0 \quad \text{(3.5)} \]

which has the solution

\[ e_f = Ax + B \quad \text{(3.6)} \]

where \( e_f \) is the final voltage distribution and \( A \) and \( B \) are constants of integration.

**Boundary conditions:**

\[ e_f = E \text{ at } x = 0 \quad \text{(3.7)} \]

\[ e_f = 0 \text{ at } x = L \quad \text{(3.8)} \]

or,

\[ e_f = E (1 - x/L) \quad \text{(3.9)} \]
The voltage finally distributes uniformly throughout the winding. That portion of the winding which was overstressed at \( t = 0 \) experiences stress relaxation and that portion which was understressed experiences an increase in stress.

Maximum decrease in stress = \( \frac{E \alpha \cosh(\alpha x)}{\sinh(\alpha x)} - \frac{E}{x} = \delta_{oo} - E/x \)

Maximum increase in stress = \( \frac{E}{x} - \frac{E \alpha \cosh(\alpha x)}{\sinh(\alpha x)} = \frac{E}{x} - \delta_{ol} \)

Similarly, equation (2.21) reduces to

\[
\frac{1}{\lambda} \frac{d^4e}{dx^4} - \frac{m^2}{\lambda} \frac{d^2e}{dx^2} = 0
\]

which reduces to (3.5) for \( m \) and \( \lambda \rightarrow \infty \) while the ratio \( \frac{\lambda}{m} \) is maintained constant.

This equation has the solution:

\[
e_f = A e^{mx} + B e^{-mx} + C_1x + C_2
\]

where \( A, B, C_1 \) and \( C_2 \) are constants of integration and \( e_f \) is the final voltage distribution.

Boundary conditions

\[
\begin{align*}
e_f &= E \text{ at } x = 0 \\
e_f &= 0 \text{ at } x = L
\end{align*}
\]

These are the only well defined boundary conditions. However, four boundary conditions are needed to determine the values of the constants. If we assume that the second derivative of voltage at \( x = 0 \) and first derivative at \( x = L \) are both zero, two more boundary conditions are obtained. This assumption is equivalent to assuming that the curvature and slope at \( x = 0 \) and \( x = L \) respectively, are zero. We, therefore, have:

\[
o = Am^2 + Bm^2
\]
\[ o = A_m e^{m^2} - B_m e^{-m^2} + C_1 \quad (3.15) \]

Equations (3.12), (3.13), (3.14) and (3.15) give

\[ A = \frac{E}{2 \left[ m^2 \cosh (m^2) - \sinh (m^2) \right]} \]
\[ B = -\frac{E}{2 \left[ m^2 \cosh (m^2) - \sinh (m^2) \right]} \]
\[ C_1 = -\frac{m^2 C \cosh (m^2)}{m^2 \cosh (m^2) - \sinh (m^2)} \]
\[ C_2 = E \]

and finally

\[ e_f = E \left[ 1 + \frac{\sinh (mx)}{m^2 \cosh (m^2) - \sinh (m^2)} \right. \]
\[ - \frac{\sinh (mx)}{m^2 \cosh (m^2) - \sinh (m^2)} \left] \quad (3.16) \right. \]

This is not a straight line relation as obtained when mutual inductance was neglected.

However, \[ \lim_{m \to \infty} e_f = \lim_{m \to \infty} E \left[ 1 + \frac{\sinh (mx)}{m^2 \cosh (m^2) - \sinh (m^2)} \right. \]
\[ - \frac{\sinh (mx)}{m^2 \cosh (m^2) - \sinh (m^2)} \left] \quad (3.16) \right. \]

\[ = E \left( 1 - \frac{x}{\lambda} \right) \text{ same as (3.9)} \]

The maximum difference between \( e_f \) as calculated from (3.9) and (3.16) is given by the intercept PM in Fig. 9, where the tangent TT is parallel to \( E \lambda \), for then,

\[ \frac{d}{dx} \left[ E \left( 1 - \frac{x}{\lambda} \right) - e_f \right] = 0 \]

or,

\[ - \frac{de_f}{dx} = E/\lambda \quad (3.17) \]

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To calculate the distance along the winding where this holds, we substitute (3.17) in (3.16) giving

\[
\frac{E}{L} = E \left[ \frac{m \cosh (mx)}{m \cosh (mL) - \sinh (mL)} - \frac{m \cosh (mL)}{m \cosh (mL) - \sinh (mL)} \right]
\]

or,

\[
x = \frac{1}{m} \cosh^{-1} \left[ \frac{1}{m} \left\{ \frac{m \cosh (mL) - \sinh (mL)}{m \cosh (mL) - \sinh (mL)} \right\} \left\{ \frac{m \cosh (mL) - \frac{1}{L}}{m \cosh (mL) - \sinh (mL)} \right\} \right]
\]

= a, say. This gives the distance at which maximum reduction in major insulation could be effected. The maximum difference in voltage

\[
= E \left[ \frac{m \cosh (mL) a}{m \cosh (mL) - \sinh (mL)} - \frac{\sinh (ma)}{m \cosh (mL) - \sinh (mL)} - \frac{a}{L} \right].
\]

Final voltage gradient calculated from (3.16) is given by

\[
- \frac{dE}{dx} = E \left[ \frac{m \cosh (mL) - m \cosh (mx)}{m \cosh (mL) - \sinh (mL)} \right]
\]

which is maximum at \( x = 0 \) and minimum at \( x = L \)

Let \( E_{f0} \) = final gradient at \( x = 0 \)

and \( E_{fL} \) = final gradient at \( x = L \)

Then, \( E_{f0} = \frac{E \cosh (mL) - m}{m \cosh (mL) - \sinh (mL)} \)

and \( E_{fL} = 0. \)

Also, \( E_{oo} = E \alpha \cosh (\alpha L) \)

and, \( E_{oL} = \frac{E \alpha}{\sinh (\alpha)} \)

In order to satisfy the boundary conditions, the curves \( g_f \) and \( g_o \) must cross at two points \( P \) and \( Q \) (Fig. 10).

The portion \( OA \) of the winding experiences stress relaxation, the portion \( OB \) experiences stress increase and the portion \( BC \) experiences stress relaxation from initial to final conditions. \( OA \) and \( OB \) are given by the two roots of the equation.
\[ \frac{m \cosh(mx) - m \cosh(mx)}{m \cosh(mx) - \sinh(mx)} = \frac{\alpha \cosh(\beta - x)}{\sinh(\beta)} \]

in the range \(0 < x < \ell\). (Fig. 11).

\((g_o - g_f)\) plotted against \(x\) for the two cases are shown in Figs. 11 and 12 respectively.
CHAPTER 4

INITIAL AND FINAL VOLTAGE DISTRIBUTION AND VOLTAGE GRADIENT FOR ISOLATED-NEUTRAL WINDING

4.1 Initial Voltage Distribution

The general differential equation for voltage distribution developed for grounded neutral winding holds for the present case also. The constants of integration will be evaluated from the relevant boundary conditions:

\[ e_o = E \text{ at } x = 0 \quad (4.1) \]
\[ i_k = 0 \text{ at } x = \ell \quad (4.2) \]

From 2.1

\[ i_k = -k_e \frac{de}{dx} = 0 \]

hence,

\[ \frac{de}{dx} = 0 \text{ at } x = \ell \quad (4.3) \]

From 3.1

\[ \frac{de}{dx} = A e^{\alpha x} - B e^{-\alpha x} \]

with these two conditions we have

\[ A = \frac{e^{-\alpha \ell}}{e^{\alpha \ell} + e^{-\alpha \ell}} \]
and

\[ B = \frac{e^{\alpha \ell}}{e^{\alpha \ell} + e^{-\alpha \ell}} \]

giving

\[ e_o = E \frac{\cosh \alpha (\ell - x)}{\cosh (\alpha \ell)} \quad (4.4) \]

\[ e_{oo} = E \]
and

\[ e_{o\ell} = \frac{E}{\cosh (\alpha \ell)} \]
By reasoning similar to that for the grounded neutral winding, it is seen that the curves of $e_o$ for various values of $\alpha$ are similar in nature to those for grounded neutral winding except that in the present case, the curves terminate at different points at $x = l$. These points have coordinates $\left( l, \frac{E}{\cosh (\alpha l)} \right)$ for various values of $\alpha$ as shown in Fig. 13.

For $\alpha = 0$, $e_o = E$ for all values of $x$.

4.2 Initial Voltage Gradient

From 4.4, \[
\frac{de_o}{dx} = - \frac{E \alpha \sinh \alpha (l-x)}{\cosh (\alpha l)}
\]

$e_o = - \frac{de_o}{dx} = \frac{E \alpha \sinh \alpha (l-x)}{\cosh (\alpha l)}$ (4.5)

$e_{o \text{max}} = e_{oo} = \frac{E \alpha \sinh (\alpha l)}{\cosh (\alpha l)}$

$e_o$ at $x = 0$

In the case of grounded neutral winding $\alpha = 0$ gives uniform voltage distribution along the winding, but in the present case, this value of $\alpha$ gives constant voltage for the entire winding showing that the winding acts as an open circuit.

4.3 Final Voltage Distribution

When mutual inductance is not taken into account, the final voltage distribution is given by equation 3.6 with the following boundary conditions:

\[
e_f = E \text{ at } x = 0
\]

\[
\frac{de_f}{dx} = 0 \text{ at } x = l
\]
Substitution of these conditions in (3.6) give

\[ A = 0 \]
\[ B = E \]

which gives

\[ e_f = E. \] (4.6)

When mutual inductance is taken into account, the boundary conditions become:

\[ e_f = E \text{ at } x = 0 \]
\[ \frac{d^2 e_f}{dx^2} = 0 \text{ at } x = 0 \]
\[ \frac{d e_f}{dx} = 0 \text{ at } x = \ell \]
\[ e_f = E \text{ at } x = \ell \]

These conditions when substituted in equation (3.11) give

\[ E = A + B + C_2 \]
\[ o = A m^2 + B m^2 \]
\[ o = A e^{-m \ell} - B e^{-m \ell} + C_1 \]
\[ E = A e^{m \ell} + B e^{m \ell} + C_1 \ell + C_2 \]

These relations give

\[ A = 0 \]
\[ B = 0 \]
\[ C_1 = 0 \]
\[ C_2 = E \]

and finally

\[ e_f = E \]
The entire winding is at the potential of the surge and there is no axial stress. Voltage across the major insulation is throughout equal to the surge voltage.
This expression is now evaluated for the two cases.

Neutral grounded:

\[ C_{\text{eff}} = -\frac{K}{E} \frac{de}{dx} \text{ at } x = 0 \]

\[ = K\text{coth } (\alpha z). \]

for \( \alpha z > 3 \text{ coth } (\alpha z) = 1 \), then

\[ C_{\text{eff}} = K\alpha = K \frac{\sqrt{C/K}}{\sqrt{K}} \]

Isolated Neutral:

\[ C_{\text{eff}} = K\alpha \text{ tanh } (\alpha z) = K\alpha = \sqrt{\frac{C}{K}} \]

This effective capacitance gets charged to the initial voltage distribution exponentially and voltage \( v \) after time \( t \) is given by

\[ t = \sqrt{\frac{C}{K}} Z \log_e \left( \frac{E}{E - v} \right) \]

where \( Z \) is the surge impedance of the line connected to the winding and may be 300 to 400 ohms for overhead lines and 50 ohms for cables. If the value of \( t \) for \( v = \text{ about 95% of } E \) is less than the wave front of the incoming wave, the analysis will hold closely to when the wave front is infinitesimal.

Since \( Z \) differs markedly for overhead lines, and cables, it seems that the winding will behave differently when connected to a cable or an O.H. line.
CHAPTER 5

INITIAL EFFECTIVE CAPACITANCE

Equation (3.0) suggests that the initial voltage distribution is determined by C and K alone. The winding behaves as a pure capacitance $C_{\text{eff}}$ at $t = 0$. The equivalent circuit for $t = 0$ is shown in Fig. 14:

$C_{\text{eff}} = \text{Initial effective capacitance of the winding.}$

The capacitance when looking into terminals ab or cd in Fig. 14 is

$C_{\text{eff}} = \frac{(C_{\text{eff}} + Cdx)K}{(C_{\text{eff}} + Cdx + \frac{K}{dx})dx}$

or $C_{\text{eff}}^2 + C_{\text{eff}}dx - CK = 0$

$Lt C_{\text{eff}}^2 = CK$

$dx \to 0$

$C_{\text{eff}} = \sqrt{CK}$ (5.1)

From the point of view of total current drawn by the winding at $t = 0$, we have

$I = C_{\text{eff}} \frac{3e}{3t} \bigg|_{t = 0, x = 0} = p C_{\text{eff}} E$

also at $t = 0$

$I = i_k = -K \frac{3^2 e}{3x3t} = -pK \frac{3e}{3x} \bigg|_{t = 0, x = 0}$

Therefore, $C_{\text{eff}} E = -K \frac{3e}{3x} \bigg|_{x = 0}$ (5.2)
TRANSITION FROM INITIAL TO FINAL DISTRIBUTION

The transition from initial to final distribution takes place through oscillation.

First the case of neglecting the effect of mutual inductance is considered. It is shown that the conclusions based on neglecting mutual inductance are different from those obtained by taking mutual inductance into account.

6.1 Mutual Inductance Neglected.

The solution to equation \( (2.22) \) can be written as:

\[
e = N e^{j\omega t} e^{j\beta x}
\]

\( N, \omega \) and \( \beta \) take all possible values

and \( \omega = 2\pi f \)

\( f = \) frequency of a particular harmonic

\( \beta = \frac{2\pi}{\lambda} \)

\( \lambda = \) wave length

Then,

\[
\frac{\partial^2 e}{\partial x^2} = -N\beta^2 e^{j\omega t} e^{j\beta x}
\]

\( \frac{\partial^2 e}{\partial t^2} \)

\[
= -N\omega^2 e^{j\omega t} e^{j\beta x}
\]

\[
\frac{\partial^4 e}{\partial x^2 \partial t^2} = N\omega^2 \beta^2 e^{j\omega t} e^{j\beta x}
\]

Substituting in \( (2.22) \) we get

\[
K\omega^2 \beta^2 - \frac{1}{L} \beta^2 + C\omega^2 = 0
\]
giving \( \omega = \frac{\beta}{\sqrt{LC + KL \beta^2}} \) \hspace{1cm} (6.6)

or, \( \beta = \frac{\omega \sqrt{C}}{\sqrt{\frac{1}{L} - K\omega^2}} \) \hspace{1cm} (6.7)

For very low frequencies \( \omega^2 \) may be neglected giving
\[ \beta = \omega \sqrt{LC} \]

The wave density \( \beta \) increases rapidly with frequency and becomes infinite at \( \omega_c \) called critical frequency.

\[ \frac{1}{L} - K\omega_c^2 = 0 \]

or, \[ \omega_c = \frac{1}{\sqrt{LK}} \] \hspace{1cm} (6.8)

Figure 15 shows the variation of \( \beta \) with \( \omega \).

If \( M \) is the number of full waves over the length \( L \) then \( n \) the wave length is given by \( \frac{L}{n} \) and
\[ \beta = \frac{2\pi}{n} = \frac{2\pi M}{L} \]

Equation (6.6) becomes:
\[ \omega = \frac{2\pi M}{L \sqrt{LC + \frac{4\pi^2 M^2 KL}{\beta^2}}} \] \hspace{1cm} (6.9)

\( M \) can take the following values:
\[ M = \frac{L}{\beta} \times 1, 3, 5, 7, \ldots \]
or \[ M = \frac{L}{\beta} \times 2, 4, 6, 8, \ldots \]

depending upon whether the neutral is isolated or grounded.

For grounded neutral winding even multipliers apply and for isolated neutral winding odd multipliers apply. This is so because in the former case, the potential of the neutral is not free to oscillate and must
form the node of the wave and in the latter case, the potential of the neutral is free to oscillate and must form an antinode.

As M increases, the interval between successive natural frequencies decreases. The natural frequencies are crowded at a single frequency \( \omega_c \). The frequency spectrum is shown in Fig. 16.

If the solution to the equation is written as a series of travelling waves of the form

\[
e = N e^{j\omega(t - \frac{x}{v})}
\]

where \( v \) is the velocity of propagation of the particular harmonic.

\[
\frac{\partial^2 e}{\partial x^2} = -\frac{\omega^2}{v^2} N e^{j\omega(t - \frac{x}{v})} \tag{6.10}
\]

\[
\frac{\partial^2 e}{\partial t^2} = -\omega^2 N e^{j\omega(t - \frac{x}{v})} \tag{6.11}
\]

\[
\frac{\partial^4 e}{\partial x^2 \partial t^2} = \frac{\omega^4}{v^2} N e^{j\omega(t - \frac{x}{v})} \tag{6.12}
\]

Substituting in equation (2.22) we get

\[
K \frac{\omega^4}{v^2} - \frac{1}{L} \frac{\omega^2}{v^2} + C \omega^2 = 0.
\]

or

\[
v = \sqrt{\frac{K \omega^2 - \frac{1}{L}}{-C}} \tag{6.13}
\]

which is imaginary for values of \( \omega > \omega_c = \frac{1}{\sqrt{LC}} \).

This result shows that exciting frequencies less than \( \omega_c \) propagate through the winding as travelling waves and undergo reflection at the neutral end if it is not properly grounded. Frequencies greater than or equal to \( \omega_c \) do not propagate through the winding and must consequently be reflected at the winding giving rise to over voltage. It also implies that by increasing \( \omega_c \) a broader spectrum of frequencies will penetrate the winding and will, at the most, be reflected at the neutral. But, since
the wave will be considerably smoothed by the capacitance of the winding
before reaching the neutral, the voltage rise there may be only normal.

6.2 Mutual Inductance Included

Assuming the solution expressed by equation (6.1),

$$\frac{\partial^4 e}{\partial x^4} = N \beta^4 e^{j\beta x} e^{j\omega t}$$  \hspace{1cm} (6.14)

Substituting (6.2), (6.3), (6.4) and (6.14) in (2.21) we get:

$$\beta^4 - 2mk \beta^2 \omega^2 + m^2 \beta^2 - 2mc \lambda \omega^2 = 0$$  \hspace{1cm} (6.15)

and

$$\beta = \frac{1}{\sqrt{2}} \left[ \sqrt{\sqrt{(m^2 - 2mk \omega^2)^2 + 8mc \lambda \omega^2} - (m^2 - 2mk \lambda \omega^2)} \right]$$

which is always positive

or,

$$\omega = \frac{\beta \sqrt{m^2 + \beta^2}}{\sqrt{2m \lambda (C + K\beta^2)}}$$  \hspace{1cm} (6.16)

This shows that there is no finite value of \( \omega \) for which \( \beta \) is infinite.

In order to see how spatial wave density \( \beta \) varies with time frequency

\( \omega \) equation (6.15) is differentiated w.r.t. \( \omega \) giving

$$4\beta^3 \frac{d\beta}{d\omega} - 2mk \lambda \left[ 2\omega^2 \beta \frac{d\beta}{d\omega} + 2 \beta^2 \omega \right]$$

$$+ 2m^2 \beta \frac{d\beta}{d\omega} - 4mc \lambda \omega = 0$$

The plot of \( \beta \) versus \( \omega \) will have a maxima or a minima if \( \frac{d\beta}{d\omega} = 0 \)

for \( \omega > 0 \)

i.e. if

$$- 4mk \lambda \omega \beta^2 - 4mc \lambda \omega = 0$$  \hspace{1cm} (6.17)

or \( \beta^2 = - 4C \). This is not tenable for \( \beta \geq 0 \), therefore the curve has

no maxima or minima. Since \( \beta = 0 \) when \( \omega = 0 \) and \( \beta \) is always positive, it

follows that \( \beta \) increases continuously with \( \omega \).
This conclusion is quite contrary to that obtained when distributive component of inductance is neglected.

If the solution is written in the form

\[ e = N e^{j \omega(t - x/v)} \]

where \( v \) is the velocity of propagation of a particular harmonic.

\[
\frac{\partial^4 e}{\partial x^4} = \frac{N \omega^4}{v^4} e^{j \omega(t - x/v)} \quad (6.18)
\]

Substituting (6.2), (6.3), (6.4) and (6.18) in (2.21) we get

\[
- \frac{1}{\lambda} \frac{\omega^4}{v^4} + 2mK \frac{\omega^4}{v^2} - \frac{m^2 \omega^2}{\lambda} + 2mC \omega^2 = 0.
\]

for \( \omega \neq 0 \).

\[
\omega^2 = \frac{m^2 v^2 - 2mC \lambda v^4}{2mK \lambda v^2 - 1}
\]

for very high frequencies

\[
2mK \lambda v^2 - 1 = 0 \quad (6.19)
\]

or,

\[
v = \frac{1}{\sqrt{2mK \lambda}}
\]

and for very low frequencies

\[
m^2 v^2 - 2mC \lambda v^4 = 0.
\]

or,

\[
v = \sqrt{\frac{m}{v^2 C \lambda}} \quad (6.20)
\]

6.3 Oscillatory Component of the Voltage

An expression for the oscillatory component \( e_n \) of the complete solution of the equation can be written as:

\[
e_n = \sum_{n=1}^{\infty} A_n \cos (\omega t) \sin (\beta x).
\]

This assumes no damping. In the presence of damping, due to di-electric loss, resistance of the winding and magnetic losses the oscillatory component would be written as:
\[ e_n = \sum_{n=1}^{\infty} A_n \cos(\omega_n t) \sin(\beta_n x) e^{-a_n t} \]

where \( a_n \)'s are positive constants. The complete solution then becomes:

\[ e = e_n + e_f = \sum_{n=1}^{\infty} A_n \cos(\omega_n t) \sin(\beta_n x) e^{-a_n t} + e_f \quad (6.21) \]

as \( t \to \infty, \) \( e \to e_f \)

and at \( t = 0, \) \( e = \sum_{n=1}^{\infty} A_n \sin(\beta_n x) + e_f = e_0 \)

Hence, \( e_0 - e_f = \sum_{n=1}^{\infty} A_n \sin(\beta_n x) \quad (6.22) \)

To calculate the values of \( A_n \) and \( \beta_n \) for the cases of grounded neutral and isolated neutral windings \( (e_o - e_f) \) is expanded as half range sine series in the interval \((0, \ell)\).

Grounded Neutral:

\[ e_o - e_f = E \left[ \frac{\sinh(\alpha(\ell-x))}{\sinh(\alpha x)} - 1 - \frac{\sinh(mx) - mx \cos(m\ell)}{m\ell \cosh(m\ell) - \sinh(m\ell)} \right] \]

\[ = \sum_{n=1}^{\infty} A_n \sin(\beta_n x) \]

clearly \( \beta_n = \frac{n\pi}{\ell} \)

and \( A_n = \frac{2E}{\ell} \int_0^\ell \left( e_o - e_f \right) \sin \left( \frac{n\pi x}{\ell} \right) dx \)

\[ = \frac{2E}{\ell} \int_0^\ell \left[ \frac{\sinh(\alpha(\ell-x))}{\sinh(\alpha x)} - 1 - \frac{\sinh(mx) - mx \cos(m\ell)}{m\ell \cosh(m\ell) - \sinh(m\ell)} \right] \sin \left( \frac{n\pi x}{\ell} \right) dx \]

\[ = \frac{2E}{\ell} \left[ \frac{n\pi}{n^2\pi^2 + \ell^2 \alpha^2} + \frac{\cos(n\pi)}{n\pi} \right] - \frac{n\pi \sinh(m\ell) \cos(n\pi)}{(n^2\pi^2 + \ell^2 \alpha^2)(m\ell \cosh(m\ell) - \sinh(m\ell))} \]

\[ + \frac{\ell m \cos(n\pi) \cosh(m\ell)}{n\pi m\ell \cosh(m\ell) - \sinh(m\ell)} \quad (6.23) \]
and
\[ e = 2E \sum_{n=1}^{\infty} \left\{ \frac{n \pi}{n^2 \pi^2 + \xi^2} + \frac{\cos(n \pi) - 1}{n \pi} \right\} \]
\[ + \frac{n \pi \sinh(m \xi) \cos(n \pi)}{(n^2 \pi^2 + m^2 \xi^2) (m \xi \cosh(m \xi) - \sinh(m \xi))} \]
\[ + \frac{\xi m}{n \pi} \frac{\cos(n \pi) \cosh(m \xi)}{m \xi \cosh(m \xi) - \sinh(m \xi)} \left( e^{-a t \cos(\omega t) \sin(n \pi / \xi)} \right) \]
\[ + E \left[ 1 + \frac{\sinh(m \xi) - m \xi \cosh(m \xi)}{m \xi \cosh(m \xi) - \sinh(m \xi)} \right] \]

(6.25)

\[ \frac{de}{dx} = -g = 2E \sum_{n=1}^{\infty} \left\{ \frac{n^2 \pi^2}{\xi (n^2 \pi^2 + \xi^2)} + \frac{\cos(n \pi) - 1}{\xi} \right\} \]
\[ - \frac{n^2 \pi^2 \sinh(m \xi) \cos(n \pi)}{\xi (n^2 \pi^2 + m^2 \xi^2) (m \xi \cosh(m \xi) - \sinh(m \xi))} \]
\[ + \frac{m}{m \xi \cosh(m \xi) - \sinh(m \xi)} \left( \cos(\omega t) \cos(n \pi / \xi) e^{-a t} \right) \]
\[ + E \left[ \frac{m \cosh(m \xi)}{m \xi \cosh(m \xi) - \sinh(m \xi)} - \frac{m \cosh(m \xi)}{m \xi \cosh(m \xi) - \sinh(m \xi)} \right] \]

(6.26)

It is clear from (6.25) that, for major insulation, to calculate the voltage at any point of the winding higher harmonics may be ignored because \( n \) appears as second power in the denominator, but only as first power in the numerator. But, for purposes of calculating minor insulation to withstand the gradient \( \frac{de}{dx} \), higher harmonics cannot be ignored because \( n \) appears as second power in both numerator and denominator.

Neutral Isolated

\[ e_o - e_f = E \left[ \frac{\cosh(a (l-x))}{\cosh(\alpha l)} -1 \right] = \sum_{n=1}^{\infty} A_n \sin \beta_n x \]

as before, \( e_o - e_f \) is expanded as half range sine series in the interval \((0, l)\) we have \( \beta_n = \frac{n \pi}{l} \).
and

$$A_n = \frac{2E}{\lambda} \int_0^\lambda \left[ \frac{\cosh \alpha (\lambda-x) - 1}{\cosh (\alpha x)} \right] \sin \frac{n\pi x}{\lambda} \, dx$$

$$= \frac{2En\pi}{(n^2\pi^2 + \lambda^2\alpha^2) \cosh \alpha \lambda} - \frac{2E}{n\pi} \left( 1 - \cos n\pi \right)$$

giving

$$e = 2E \sum_{n=1}^\infty \left\{ \frac{n\pi (\cosh (\alpha \lambda) - \cos (n\pi))}{(n^2\pi^2 + \lambda^2\alpha^2) \cosh (\alpha \lambda)} \right\}$$

$$- \frac{1 - \cos (n\pi)}{n\pi} \int \cos \omega t \sin \frac{n\pi x}{\lambda} \, e^{-\alpha t} \right\} + E$$

for this case also higher harmonics may be ignored while calculating major insulation but for minor insulation higher harmonics can not be ignored.

An approximate estimate of maximum voltage and gradient can be obtained by assuming that the maxima of various harmonics occur at the same time. If damping is neglected these quantities for the two cases become:

Neutral grounded

maximum voltage =

$$= 2E \sum_{n=1}^\infty \left\{ \frac{n\pi}{(n^2\pi^2 + \lambda^2\alpha^2)} \right\}$$

$$+ \frac{\cos (n\pi) - 1}{n\pi} - \frac{n\pi \sinh (m\lambda) \cos (m\lambda)}{(n^2\pi^2 + m^2\lambda^2) (m\lambda \cosh m\lambda - \sinh m\lambda)}$$

$$+ \frac{\frac{m\lambda \cos (n\pi) \cosh (m\lambda)}{n\pi (m\lambda \cosh m\lambda - \sinh m\lambda)}} \sin \frac{n\pi x}{\lambda} \right\}$$

$$+ E \left[ 1 + \frac{\sinh mx - mx \cosh (m\lambda)}{m\lambda \cosh (m\lambda) - \sinh (m\lambda)} \right]$$

and maximum gradient

$$= 2E \sum_{n=1}^\infty \left\{ \frac{n^2\pi^2}{\lambda (n^2\pi^2 + \lambda^2\alpha^2)} + \frac{\cos (n\pi) - 1}{\lambda} \right\}$$

$$- \frac{n^2\pi^2 \sinh (m\lambda) \cos (n\pi)}{\lambda (n^2\pi^2 + m^2\lambda^2) (m\lambda \cosh m\lambda - \sinh m\lambda)}$$

$$+ \frac{m \cos (n\pi) \sinh (m\lambda)}{m\lambda \cosh (m\lambda) - \sinh (m\lambda)} \cos \frac{n\pi x}{\lambda} \right\}$$

(continued)
Neutral Isolated:

\[
\text{maximum voltage} = 2E \sum_{n=1}^{\infty} \left\{ \left[ \frac{n\pi \left( \cosh (n\pi) - \cos (n\pi) \right)}{(n^2 \pi^2 + \ell^2 \alpha^2) \cosh (n\alpha)} \right] \sin \left( \frac{n\pi x}{\ell} \right) \right\} + E
\]

and Maximum gradient

\[
= 2E \sum_{n=1}^{\infty} \left\{ \left[ \frac{n^2 \pi^2 \left( \cosh (n\pi) - \cos (n\pi) \right)}{\ell (n^2 \pi^2 + \ell^2 \alpha^2) \cosh (n\alpha)} \right] \cos \left( \frac{n\pi x}{\ell} \right) \right\}
\]
CONCLUSION

When a voltage surge enters a uniformly wound winding, the voltage is not evenly distributed along its axis. A large proportion of the total voltage appears across a few initial turns of the winding. The initial and final voltage distribution along the winding depend upon the conditions at the neutral. If mutual inductance between various turns of the winding is neglected, the winding behaves as an open circuit for frequencies above the critical frequency. If mutual inductance is taken into account an altogether different conclusion is arrived at: that there is no frequency at which the winding acts as an open circuit and all frequencies travel through the winding with different velocities.

The voltage distribution in the coil settles down to the final steady state through oscillations.

To calculate the amount of major insulation at any section the total voltage there may be calculated by neglecting higher harmonics. In the case of grounded-neutral winding, the major insulation could also be graded. For isolated-neutral winding the major insulation cannot be graded. For purposes of calculating minor insulation the axial voltage gradient must include higher harmonics also.

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FIGURE 1
A TYPICAL LIGHTNING WAVE OF + POLARITY SPECIFIED AS +E/t_1/ t_2
FIGURE 2

STEP VOLTAGE

e = E \quad \text{FOR} \quad t > 0

= 0 \quad \text{FOR} \quad t < 0
FIGURE 4
A MAGNETIC CORE WOUND WITH TWO SEPARATE TURNS

\[ H + \frac{dH}{dz} \, dz, \, \phi + \frac{d\phi}{dz} \, dz \]
FIGURE 5
MUTUAL INDUCTANCE BETWEEN TWO TURNS AT x AND y
FIGURE 6
DEPENDENCE OF $e_0$ ON $\alpha$

$\alpha_1 < \alpha_2 < \alpha_3 \ldots$
FIGURE 7
PARTS OF THE WINDING WHICH ARE OVERSTRESSED AND UNDERSTRESSED AT $t = 0$
FIGURE 8
INITIAL VOLTAGE GRADIENT
(NEUTRAL EARTHED)
(1) NEGLECTING MUTUAL INDUCTANCE
(2) INCLUDING MUTUAL INDUCTANCE

FIGURE 9
FINAL VOLTAGE DISTRIBUTION

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FIGURE 10
INITIAL AND FINAL VOLTAGE GRADIENT

(1) FINAL VOLTAGE GRADIENT
(2) INITIAL VOLTAGE GRADIENT
FIGURE II

INITIAL—FINAL GRADIENT
(MUTUAL INDUCTANCE INCLUDED)

THIS FIGURE IS DERIVED FROM FIG 10

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FIGURE 12
INITIAL–FINAL GRADIENT
(MUTUAL INDUCTANCE NEGLECTED)
THIS FIGURE IS DERIVED FROM FIG 3

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FIGURE 13
INITIAL VOLTAGE DISTRIBUTION FOR ISOLATED NEUTRAL WINDING
\[ 0 < \alpha_1 < \alpha_2 \ldots < \alpha_n \]
FIGURE 14
EQUIVALENT CIRCUIT AT \( t = 0 \)
FIGURE 15
VARIATION OF WAVE DENSITY WITH FREQUENCY
FIGURE 16.
SPECTRUM OF NATURAL FREQUENCIES

2M = 1 2 3 ∞
0 0.1 0.2 0.3 1.0

ω/ω₀